The Extreme Points of Fusions

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Abstract

Our work explores fusions, the multidimensional counterparts of mean-preserving contractions and their extreme and exposed points. We reveal an elegant geometric/combinatorial structure for these objects. Of particular note is the connection between Lipschitz-exposed points (measures that are unique optimizers of Lipschitz-continuous objectives) and power diagrams, which are divisions of a space into convex polyhedral "cells" according to a weighted proximity criterion. These objects are frequently seen in nature–in cell structures in biological systems, crystal and plant growth patterns, and territorial division in animal habitats–and, as we show, provide the essential structure of Lipschitz-exposed fusions. We apply our results to several questions concerning categorization.

> To comport oneself with perfect propriety in
> Polygonal society, one ought to be a Polygon oneself.
>
> Edwin Abbott Abbott

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1 Introduction

We study the extreme points of the set of fusions of a given measure, the multidimensional analogues of extreme mean-preserving contractions. The set of fusions of a given measure μ–taken here to be absolutely continuous with respect to the Lebesgue measure–is the set of probability distributions that are dominated by μ in the convex stochastic order, and are thus "less variable" than μ.

The burgeoning literature on information design and Bayesian persuasion highlights the importance of fusions for economic applications: by a famous result that goes back to the work of Blackwell (1953) and Strassen (1965), the set of fusions of a given measure μ is also the set of distributions of posterior means that can be induced by some "signal" (or "experiment") starting from a prior belief μ about the state of the world. Thus, optimization over the set of feasible signals of an objective function that depends linearly or convexly on the posterior mean can be reduced to an optimization over the set of fusions of μ. By Bauer's Theorem, the optimum in such an exercise–representing an optimal way to reveal information about the state–will be achieved at an extreme point of the set of fusions. This feature constitutes the primary motivation for our study.

The case where the underlying state of world is unidimensional has been studied along the above lines by Kleiner et al. (2021) and Arieli et al. (2023): in that case, each extreme measure is characterized by a collection of (convex) intervals where either the mass put by the prior on the interval remains in place (this corresponds to full information revelation of these states in a Bayesian-persuasion problem), or where this mass is contracted into a measure with at most two elements in its support (this corresponds to pooling of states where information is only partially or not-at-all revealed). Here we extend these insights to the multidimensional case.

Our first main result, Proposition 1 in Section 2 focuses on finitely supported, extreme fusions of a measure μ defined on a convex, compact subset X of $\mathbb{R}^n$. It shows that, necessarily, each such extreme fusion ν induces a partition of X in convex sets such that the support of ν on each element of the partition is an affinely independent set, and such that the restriction of ν on each element of the partition is itself a fusion of the restriction of the prior μ on that element. The analogy to the unidimensional result sketched above becomes clear by noting that the maximal cardinality of an affinely independent set in $\mathbb{R}^n$ is $n+1$ and this equals two for states on the real line.

Our second main result, Proposition 2 in Section 3, offers a necessary and sufficient condition for a finitely supported measure ν to be the unique maximizer (on the set of fusions of a given measure μ) of a linear functional induced by a Lipschitz-continous objective function. We call such maximizers *Lipschitz-exposed* since in convex analysis *exposed* points are those extreme points that are unique maximizers of some linear functional.

Our result characterizes finitely supported Lipschitz-exposed fusions in terms of very particular convex partitions that form a *power diagram*. A power diagram in $\mathbb{R}^n$ is generated by a set of *sites* and by a set of *weights*, one attached to each site. It is the partition of $\mathbb{R}^n$ into (necessarily convex) subsets such that each element in the partition represents the set of points that are closer in Euclidean distance (modulo the linearly additive weight) to a given site than to any other site.

In particular, the somewhat better known *Voronoi diagrams* are power diagrams where all weights attached to sites are equal. In a power diagram the boundary between two elements of a partition is necessarily perpendicular to the line connecting the two respective sites. Modifying the weights while allowing them to differ in value induces parallel shifts of this boundary, whereas the perpendicular boundary is completely fixed by the location of sites in any Voronoi diagram[1].

The above emergence of an elegant combinatorial/geometrical structures in our study is rather surprising–it cannot be gleaned from the unidimensional environment because there each convex partition is trivially a power diagram. The proof of Proposition 2 combines insights from duality developed by Dworczak and Martini (2019) and Dworczak and Kolotilin (2024) with a fundamental result that establishes the equivalence between

[1]A somewhat related appearance of power diagrams is the one in the theory of *semi-discrete optimal transport* (see for example the textbook Galichon (2018)): when a continuous measure μ is optimally transported to another one with a finite support ν, the sets of points that are transported to each point in the support of ν form a power diagram.

power diagrams and regular polyhedral divisions due to Aurenhammer (1987).

Recall that in the unidimensional case an extreme fusion allows mass to stay in place on some intervals (corresponding to full revelation of information). For the multidimensional case, Corollary 1 (to Proposition 2) allows mass to stay in place within some partition elements of a power diagram and offers a sufficient condition for obtaining a Lipschitz-exposed point that may not have finite support. The proof of this Corollary requires a non-trivial result from the theory of the Monge-Ampere partial differential equation. It is still an open question whether **all** Lipschitz-exposed points are indeed characterized by a power diagram where mass in each element of the partition either stays in place or is fused to an affinely independent set of points.

In the remainder of Section 3 we look at the gap between the necessary conditions for finitely supported extreme points and the sufficient conditions for finitely supported Lipschitz-exposed points. In Section 4 we study *convex-partitional* extreme points where the mass in each element of a partition is fused into a single point and their application to canonical solutions to moment persuasion. These special fusions correspond to monotone-partitional optimal signals that were analyzed in the unidimensional case by several authors, e.g., Ivanov (2021). In Section 5 we apply our results to the study of categorization. An Appendix (A) contains the proofs.

1.1 Preliminaries and Notation

We begin by introducing the notation used below, and a few important concepts. Let X be a compact and convex subset of a finite-dimensional Euclidean space. For $A \subseteq X$, $\operatorname{int} A$ denotes the (relative) interior of A, and $\operatorname{ch} A$ denotes the convex hull of A. For a measure[2] μ on X, $\operatorname{supp} \mu$ denotes its support and for measurable $A \subseteq X$, $\mu|_A$ denotes the restriction of μ to A. δ_x denotes a Dirac measure concentrated on $x \in X$.

The *barycenter* of a measure μ on X is defined by

$$r_X(\mu) = \frac{1}{\mu(X)} \int_X x d\mu(x).$$

[2]Throughout, we use the term measure for what sometimes is called signed measure.

Note that the vector integral in the above definition is in the Bochner sense. This means that, for any linear function V on X one has

$$V(r_X(\mu)) = \frac{1}{\mu(X)} \int_X V(x) d\mu(x).$$

Definition 1.1. For measures μ and ν, we say that μ *dominates* ν *in the convex order* (or that ν is a *fusion* of μ), denoted by $\mu \geq \nu$, if $\int \phi \, \mathrm{d}\mu \geq \int \phi \, \mathrm{d}\nu$ for all convex functions ϕ such that both integrals exist. We write $\mu > \nu$ if μ dominates ν in the convex order and $\mu \neq \nu$. We denote by $F_\mu = \{\nu : \nu \leq \mu\}$ denote the set of fusions of a given measure μ.

A finite collection of vectors $V = \{x_1, \ldots, x_k\}$ in $\mathbb{R}^n$ is *affinely independent* if the unique solution to $\sum_{i=1}^k \lambda_i x_i = 0$ and $\sum_{i=1}^k \lambda_i = 0$ is $\lambda_i = 0$, $i = 1, \ldots, k$. Recall also the well-known equivalence: $x_1, \ldots, x_k$ are affinely independent if and only if $x_2 - x_1, \ldots, x_k - x_1$ are linearly independent. Linear independence implies affine independence, but not vice-versa. In particular, the maximal number of affinely independent vectors in $\mathbb{R}^n$ is $n+1$. A finite collection of vectors $V = \{x_1, \ldots, x_k\}$ in $\mathbb{R}^n$ is *convexly independent* if no element $x_i \in V$ lies in $\mathrm{ch}(V \setminus \{x_i\})$.

An *extreme* point of a convex set X is a point $x \in X$ that cannot be represented as a convex combination of two other points in X.[3] The usefulness of extreme points for optimization stems from *Bauer's Maximum Principle*: a convex, upper-semicontinuous functional on a non-empty, compact and convex set X of a locally convex space attains its maximum at an extreme point of X. The *Krein–Milman Theorem* states that any convex and compact set X in a locally convex space is the closed, convex hull of its extreme points. In particular, such a set has extreme points.

An element x of a convex set X is *exposed* if there exists a linear functional that attains its maximum on X uniquely at x.[4] Every exposed point is extreme, but the converse is not true in general.

[3] Formally $x \in A$ is an extreme point of A if $x = \alpha y + (1-\alpha)z$, for $z, y \in A$ and $\alpha \in [0,1]$ imply together that $y = x$ or $z = x$.

[4] Formally, x is exposed if there exists a supporting hyperplane H such that $H \cap A = \{x\}$.

2 The Finitely Supported Extreme Points of F_μ

Our first main result offers a necessary condition in order for a finitely supported measure ν to be an extreme point of F_μ. A key feature of an extremal fusion is the partition of the domain X into convex sets P such that all the original mass $\mu|_P$ remains within P and is fused into $\nu|_P$ whose support is an affinely independent set of points.

Proposition 1. *Let $X \subseteq \mathbb{R}^n$ be compact and convex, and let μ be an absolutely continuous probability measure on X. Suppose that ν is an extreme point of F_μ that is finitely supported. Then there exists a partition $\mathcal{P}$ of X into convex sets such that, for each $P \in \mathcal{P}$, $\nu|_P \preceq \mu|_P$ and $\nu|_P$ has affinely-independent support.*

In the finite-state Bayesian persuasion problem introduced by Kamenica and Gentzkow (2011), there is an unknown state of the world $\theta \in \Theta$ ($\Theta = n \in \mathbb{N}$) distributed according to some full-support prior $\mu_0 \in \Delta(\Theta)$, about which a principal can commit to an information structure (stochastic map). The principal's preferences over the agent's action induce a reduced-form value function $w\colon \Delta(\Theta) \to \mathbb{R}$, and the principal's problem reduces formally to $\max_F \int_{\Delta(\Theta)} w(\mu) dF(\mu)$, subject to the Bayes-plausibility (martingality) constraint $\int_{\Delta(\Theta)} \mu dF(\mu) = \mu_0$.

Kamenica and Gentzkow (2011) use the Fenchel-Bunt extension of Carathéodory's Theorem (Fenchel (1929), Bunt (1934)) to show that (under a regularity specification on the objective) there exists an optimal solution to the principal's problem in which the optimal distribution has support on at most n points. An alternative way of obtaining this finding is by observing that Winkler (1988)'s main result can be used to argue that the extreme points of distributions supported on the $(n-1)$-simplex with a specified barycenter have affinely independent support. This observation provides intuition for the structure of the necessary condition given by this Proposition. There must always be a local-global decomposition of a fusion of probability measure in which locally the support is affinely independent–if a measure is an extreme point there is always a decomposition that makes it akin to a collection of probability measures with specified barycenters.

3 Exposed Fusions

In this Section we prove our main result, a characterization of those finitely supported fusions of a given distribution that are unique maximizers of linear functionals. First, however, let us briefly discuss finitely supported *extreme* points.

Definition 3.1. A measure $\nu \in F_\mu$ is a *Lipschitz-exposed* point of F_μ if there exists a Lipschitz-continuous function $u\colon X \to \mathbb{R}$ such that ν is the unique solution to the problem

$$\max_{\lambda \in F_\mu} \int u(x)\,\mathrm{d}\lambda(x)$$

In order to characterize the Lipschitz-exposed points of the set of fusions we need the following concept:

3.1 Power Diagrams

Definition 3.2. Consider a finite collection of *sites* $S = \{s_1, \ldots, s_k\}$ in $\mathbb{R}^n$ and a finite collection of *weights* $W = \{w_1, \ldots, w_k\}$ in $\mathbb{R}_+$. For each site s_i and weight w_i define the shifted distance function

$$g_i(x) = \| x - s_i \|^2 - w_i$$

where $\| \cdot \|$ denotes the Euclidean norm. For each site s_i define the *cells* $P_i^{S,W}(x) = P_i(x) \subseteq \mathbb{R}^n$ as following:

$$P_i^{S,W}(x) = \{x : \ \forall j,\ g_i(x) \leq g_j(x)\}$$

The collection of cells $\mathcal{P} = \{P_1, P_2, .., P_k\}$ is the *power diagram* generated by (S, W)[5].

For each $i, j, i \neq j$, the set $P_i \cap P_j$ in a power diagram is contained in a hyperplane that is perpendicular to the line connecting the sites s_i and s_j. It is clear then that each cell of a power diagram is a polyhedron. Moreover, the intersection of any two cells in a power diagram is a common face. Every power diagram therefore corresponds to a polyhedral subdivision:

[5]Voronoi diagrams are those power diagrams where all weights are equal, e.g. $w_i = 0$ for all i. In contrast to Voronoi diagrams, in a power diagram a cell may be empty, and a site may not belong to its cell.

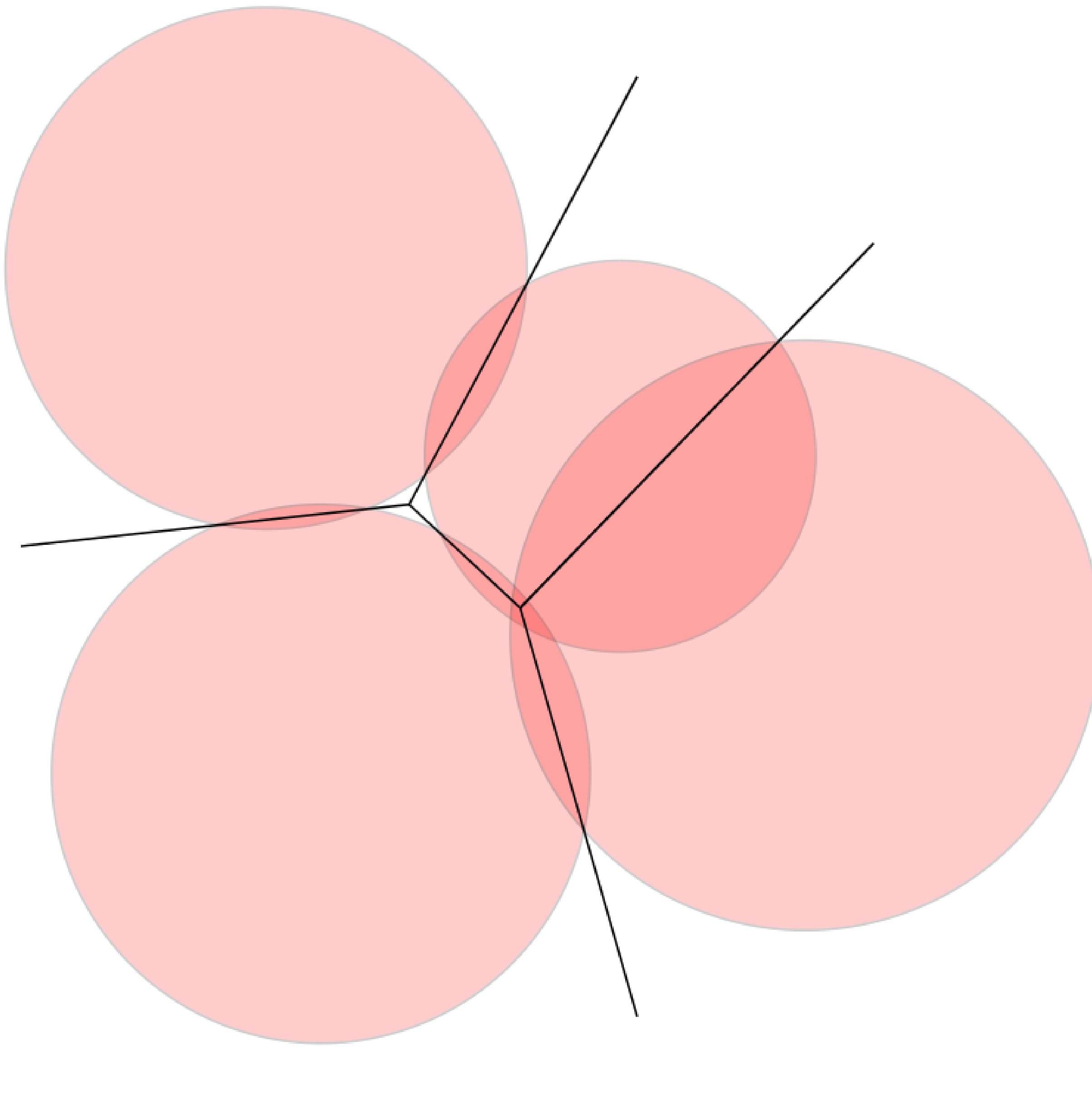

Figure 1: A power diagram (Eppstein (2014))

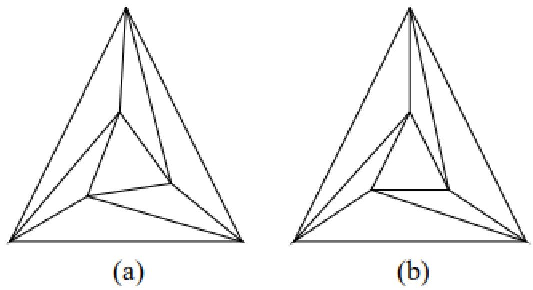

Figure 2: A regular (a) and irregular (b) subdivision (Lee and Santos (2017)).

Definition 3.3. A *polyhedral subdivision* T is a finite collection of polyhedra K_i such that:

- $\cup K_i = K$.
- For any K_i in T, all faces of K_i are in T.
- The intersection of any two polyhedra K_i and K_j in T is a face of both.

A polyhedral subdivision T of K in $\mathbb{R}^n$ is *regular* if, for each point $x \in K$ there is a *height* $\alpha_x \geq 0$ such that T is isomorphic to the set of lower faces[6] of the polyhedron *ch* $\{(x, \alpha_x) \in \mathbb{R}^{n+1}, x \in K\}$. Equivalently, for any regular polyhedral subdivision T there is a convex function that is affine on each $K_i \in T$ and, if the function is affine on a set B, then $B \subseteq K_i$ for some $K_i \in T$. See Figure 2 for an example of a polyhedral subdivision that is not regular.

A fundamental result due to Aurenhammer (1987) establishes the equivalence between power diagrams and regular polyhedral subdivisions. To see that each power diagram $\mathcal{P}$ corresponds to a regular polyhedral subdivision, consider the function

$$p(x) := -\min_i \{g_i(x) - \|x\|^2\} = -\min_i \{\|s_i\|^2 - 2x \cdot s_i - w_i\}.$$

This function is convex and its restriction to any cell of the power diagram is affine. Moreover, if p is affine on $B \subseteq X$ then there is $P \in \mathcal{P}$ with $B \subseteq P$. Therefore, each power diagram corresponds to a regular polyhedral subdivision and we will use this observation below.

[6] Lower faces are in directions with a negative $(d+1)$–th coordinate.

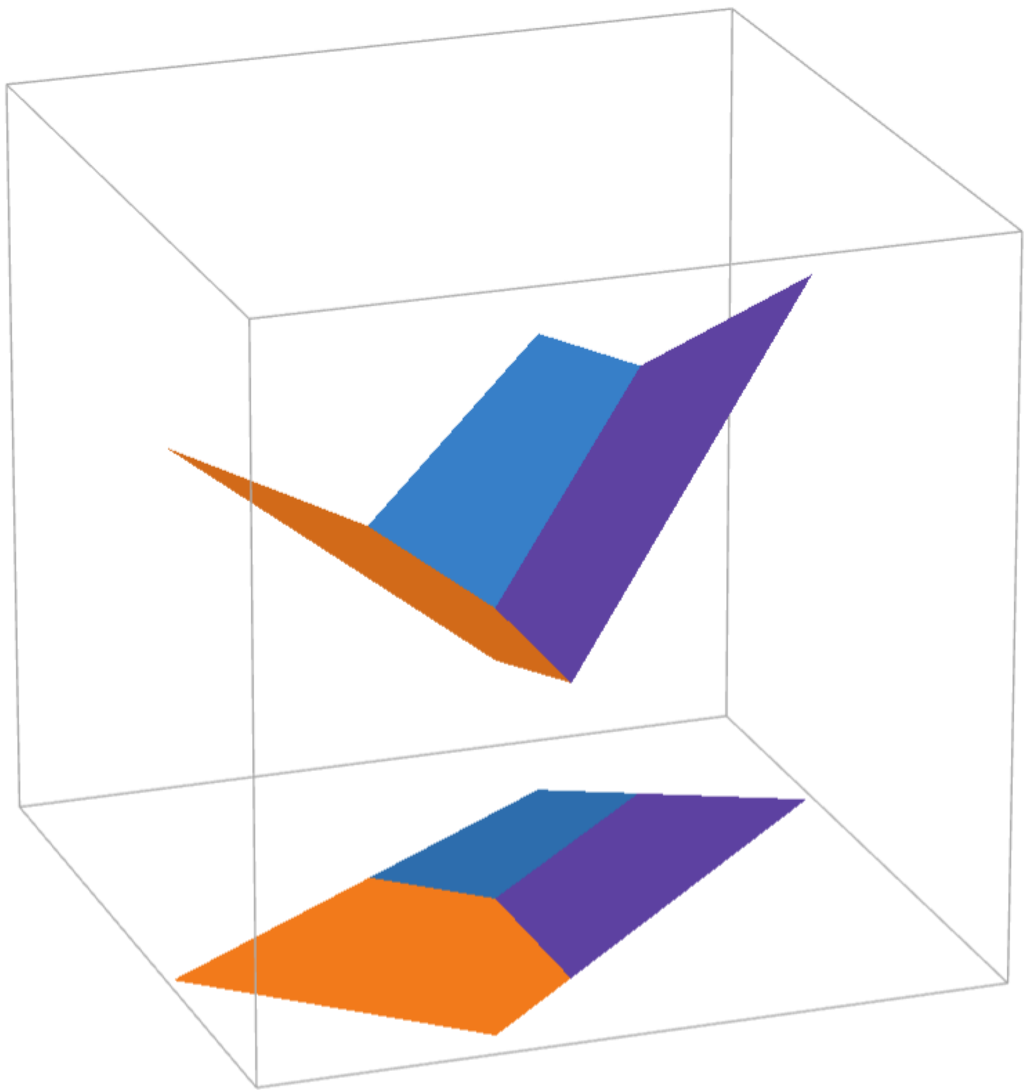

Figure 3: The lower faces of a polyhedron in $\mathbb{R}^3$ and the corresponding polyhedral subdivision.

3.2 The Main Result

Proposition 2. *Let $X \subseteq \mathbb{R}^n$ be compact and convex, and let μ be a probability measure with full support on X that is absolutely continuous with respect to the Lebesgue measure.*

Let $\nu \in F_\mu$ have finite support. Then ν is a Lipschitz-exposed point of F_μ if and only if there exists a power diagram $\mathcal{P}$ of X such that $\lambda|_P \preceq \mu|_P$ for all $P \in \mathcal{P}$ and $\operatorname{supp}(\lambda) \subseteq \operatorname{supp}(\nu)$ jointly imply $\lambda = \nu$.

In particular, $\nu \in F_\mu$ is a Lipschitz-exposed point if there is a power diagram such that, for each cell P, $\nu|_P \preceq \mu|_P$ and the support of $\nu|_P$ is affinely independent: in that case, since $\lambda_P \preceq \mu|_P$ implies that $\nu|_P$ and $\lambda|_P$ have the same barycenter, affine independence of the support implies that $\nu|_P$ and $\lambda|_P$ must be equal.

Proof of Proposition 2. ($\Leftarrow$) Since $\mathcal{P}$ is a power diagram, there exists a continuous convex function $p \colon X \to \mathbb{R}$ such that for each $P \in \mathcal{P}$ the restriction of p to P is affine, and if p is affine on $B \subseteq X$ then there is $P \in \mathcal{P}$ with $B \subseteq P$.

Define a Lipschitz-continuous function $u \colon X \to \mathbb{R}$ by

$$u(x) = p(x) - \inf_{y \in \operatorname{supp}(\nu)} \|x - y\|.$$

Note that, by definition, $u(x) = p(x)$ if $x \in \operatorname{supp}(\nu)$ and $u(x) < p(x)$ for $x \in X \setminus \operatorname{supp}(\nu)$. We now claim that ν is the unique solution to

$$\max_{\lambda \in F_\mu} \int u(x) d\lambda(x) \tag{L}$$

Step 1: *ν is a solution to problem* (L).

By construction, $\int u(x) d\nu(x) = \int p(x) d\nu(x)$. Moreover, for each $P \in \mathcal{P}$, $\int_P p(x) d\nu = \int_P p(x) d\mu$ since $\nu|_P \in F_{\mu|_P}$ and p is affine on P. Therefore,

$$\int u(x) d\nu(x) = \int p(x) d\mu(x). \tag{1}$$

For any $\lambda \in F_\mu$ we obtain

$$\int u(x) d\lambda(x) \leq \int p(x) d\lambda(x) \leq \int p(x) d\mu(x),$$

where the first inequality follows from $u \leq p$ and the second follows by the definition of the convex order. We conclude that ν is a solution to (L).

Step 2: *There is no other solution to problem* (L).[7]

Let $\lambda \in F_\mu$ solve problem (L). First, we claim that $\operatorname{supp}\lambda \subseteq \operatorname{supp}\nu$. If not, $\int u(x)d\lambda(x) < \int p(x)\,d\lambda(x)$ since $u(x) < p(x)$ for all $x \notin \operatorname{supp}\nu$. This implies

$$\int u(x)d\lambda(x) < \int p(x)\,d\lambda(x) \leq \int p(x)\,d\mu(x) = \int u(x)d\nu(x),$$

where the second inequality follows because p is convex and because $\lambda \in F_\mu$, and where the equality follows from (1). We conclude that λ does not solve problem (L), a contradiction.

We now establish that $\lambda|_P \leq \mu|_P$ for all $P \in \mathscr{P}$: By Bauer's Theorem, we can assume that λ is an extreme point of F_μ. By Proposition 1, there exists a finite collection $\mathcal{Q} = \{Q_j | j \in J\}$ of convex sets such that, for all $Q \in \mathcal{Q}$, $\mu(Q) > 0$, $\lambda|_Q \prec \mu|_Q$ and the support of λ on Q is affinely independent. Moreover, we can assume that the partition $\mathcal{Q}$ cannot be further refined. Note then that

$$\int p(x)\,d\lambda \leq \int p(x)\,d\mu = \int u(x)d\nu = \int u(x)d\lambda \leq \int p(x)\,d\nu,$$

where:

1. the first inequality follows since p is convex and since $\lambda \in F_\mu$,
2. the first equality follows from (1),
3. the second equality follows since λ achieves, by assumption, the same objective value as ν
4. the final inequality follows since $u \leq p$.

Therefore, both inequalities are, in fact, equalities. In particular, for each $Q \in \mathcal{Q}$, $\int_Q p(x)\,d\lambda(x) = \int_Q p(x)\,d\mu(x)$. Since Q has positive μ-measure and since the partition $\mathcal{Q}$ cannot be refined, the function p must be affine on $\operatorname{ch}(\operatorname{supp}(\mu|_Q))$.

[7] For a similar argument, see the proof of Theorem 2 in Arieli et al. (2023).

It follows that for each $Q \in \mathcal{Q}$ there is $P \in \mathcal{P}$ such that $\text{ch}(\text{supp}(\lambda|_Q)) \subseteq \text{ch}(\text{supp}(\mu|_Q)) \subseteq P$. Therefore, $\lambda|_P \prec \mu|_P$ for each $P \in P$, and our hypothesis implies that $\lambda = \nu$.

($\Rightarrow$) Let ν be the unique solution to (L), where u is Lipschitz-continuous. Proposition 4 in Dworczak and Kolotilin (2024) implies that the associated moment persuasion problem with prior μ is regular. It follows from Theorem 3 in Dworczak and Kolotilin (2024) that there is a convex function $p : X \to \mathbb{R}$ with $p \geq u$ that satisfies

$$\int u(x) d\nu(x) = \int p(x) d\mu(x). \tag{2}$$

This implies, in particular, that

$$u(x) = p(x) \text{ for all } x \in \text{supp}(\nu). \tag{3}$$

Since ν is the unique solution to problem (L), Bauer's Theorem implies that ν is an extreme point of F_μ. Proposition 1 implies that there exists a partition $\mathcal{P}$ of X such that, for each $P \in \mathcal{P}$, P is convex, P has positive μ-measure, $\nu|_P \prec \mu|_P$, and $\text{supp}(\nu|_P)$ is affinely independent. Assume that $\mathcal{P}$ is a finest partition with these properties.

We claim that, for each $P \in \mathcal{P}$, p is affine on P. Let $\{x_1, ..., x_m\}$ be the support of $\nu|_P$ and let $\mu|_P = \sum_i \mu_i$ be such that $\nu(\{x_i\}) = \mu_i(X)$ and $r(\mu_i) = x_i$ (see Lemma 2). Note that since p is convex and since $\int p(x) d\nu|_P = \int p(x) d\mu|_P$, we obtain that $\int p(x) d\mu_i(x) = p(x_i)$. Jensen's inequality implies that $\int p(x) d\mu_i(x) = p(x_i)$ if and only if p is affine on $\text{ch}(\text{supp}(\mu_i))$. We conclude that p is affine on P since otherwise we could refine the partition $\mathcal{P}$, contradicting our assumption.

The above argument implies that p is a piecewise affine, convex function. Let $\mathcal{Q}$ denote the power diagram corresponding to the lower envelope of this function (recall Aurenhammer's fundamental equivalence Theorem). Suppose that there is $\lambda \neq \nu$ satisfying $\lambda|_Q \prec \mu|_Q$ for all $Q \in \mathcal{Q}$ and $\text{supp}(\lambda) \subseteq \text{supp}(\nu)$. Using (3), this yields $\int u(x) d\lambda(x) = \int p(x) d\lambda(x)$. Since p is affine on Q and since $\lambda|_Q \prec \mu|_Q$, we obtain for any $Q \in \mathcal{Q}$ that $\int p(x) d\lambda|_Q(x) = \int p(x) d\mu|_Q(x)$. Putting these observations together and using (2), we finally obtain

$$\int u(x) d\lambda(x) = \int p(x) d\lambda(x) = \int p(x) d\mu(x) = \int u(x) d\nu(x)$$

This contradicts the hypothesis that ν is the unique solution to (L), and therefore we conclude that $\lambda|_Q \prec \mu|_Q$ for all $Q \in \mathcal{Q}$ and $\mathrm{supp}(\lambda) \subseteq \mathrm{supp}(\nu)$ imply $\lambda = \nu$. ∎

The result above treats extreme fusions with finite support. It is clear that the original measure μ is itself an extreme point of $F_\mu -$ in a persuasion problem this corresponds to the signal that perfectly reveals to the receiver each realized state. Perfect revelation can be optimal in some cases, i.e., if the the sender's utility function is convex on a certain domain. The next Corollary shows that a combination of perfect revelation and a finitely supported fusion can also constitute a Lipschitz-exposed point. For its proof we first need a Lemma whose proof uses the theory of the Monge-Ampere equation:

Lemma 1. *For any compact and convex set $X \subset \mathbb{R}^n$ there is a Lipschitz-continuous and convex function $f : X \to \mathbb{R}$ that is strictly convex on the interior of X and that satisfies $f(x) = 0$ for all x in the boundary of X.*

Proof. It follows from Theorem 1.1 in Hartenstine (2006) that the Dirichlet problem of the Monge-Ampere equation $detD^2 f = \mu$ subject to the constraint $f = 0$ on the boundary of X has a continuous convex solution f for any finite Borel measure μ. By Corollary 5.2.2 in Gutierrez (2016), this solution is strictly convex on the interior of X whenever μ is strictly positive. Moreover, f is Lipschitz-continuous by Theorem 5.4.8 in Gutierrez (2016). ∎

Corollary 1. *Let $X \subseteq \mathbb{R}^n$ be compact and convex, and let μ be a probability measure with full support on X that is absolutely continuous with respect to the Lebesgue measure. Let v be a fusion of μ and suppose that there exists a power diagram $\mathcal{P}$ of X such that, for each $P \in \mathcal{P}$, either $\nu|_P \preceq \mu|_P$ and $\nu|_P$ has affinely independent support, or $\nu|_P = \mu|_P$. Then ν is an Lipschitz-exposed point of F_μ.*

The proof of this Corollary (in the Appendix) follows the proof of one direction in Proposition 2, but adjusts the function p that we construct to be strictly convex on some cells using Lemma 1.

One can construct Lipschitz objective functionals in which the unique optimal fusion is induced by a partition of X into an uncountable collection of convex sets, on each of which μ is collapsed to its barycenter. A natural open question, therefore, is whether all Lipschitz-exposed points take the form of a (possibly infinite) collection $\mathscr{P}$ of convex sets (satisfying some regularity condition) on each of which either $\nu|_P \preceq \mu|_P$ and $\nu|_P$ has affinely independent support or equals $\nu|_P = \mu|_P$. One challenge is in the precise formalization of these objects, as the mathematical literature primarily studies the case where $\mathscr{P}$ has countably many elements. Perhaps a clever approximation approach would work.

3.3 The Gap Between Necessity and Sufficiency

We now note that there is a small gap between the necessary condition for extreme points identified in Proposition 1 and our sufficient condition for Lipschitz-exposed points. To that end, we present an example illustrating that the necessary condition cannot be strengthened to the partition being a power diagrams. We also present an example of an extreme point that is not Lipschitz-exposed. One perspective on the gap is that it is not too large: Straszewicz's Theorem (Theorem 18.6 in Rockafellar (1970)) and Klee's Theorem (Klee Jr (1958)) state that the set of exposed points of a compact convex set is, under mild conditions, dense in the set of extreme points; and by convolving a Bochner-integrable function with the appropriate smooth mollifier, one can approximate a Bochner-integrable function with a Lipschitz-continuous one.

Our first example shows that there are Lipschitz-exposed points (which are therefore extreme points) for which there is no power diagram such that the support on each cell is affinely independent. This shows that the necessary conditions derived in Proposition 1 cannot be strengthened to require that the partition $\mathscr{P}$ of X into convex sets is a power diagram.

Example 3.4. Let μ be the uniform distribution on the rectangle $[0,2]\times[0,1]$, and let ν be the fusion obtained by contracting μ on $[1,2]\times[0,1]$ to $\left(\frac{3}{2},\frac{1}{2}\right)$; on $[0,1]\times\left[0,\frac{1}{2}\right]$ to $\left(\frac{3}{8},\frac{1}{4}\right)$ and $\left(\frac{5}{8},\frac{1}{4}\right)$; and on $[0,1]\times\left[\frac{1}{2},1\right]$ to $\left(\frac{3}{8},\frac{3}{4}\right)$ and $\left(\frac{5}{8},\frac{3}{4}\right)$. This is depicted in Figure 4.

Observe that the partition of X shown in Figure 4 is not a polyhedral subdivision, as the intersection of the small rectangle with the large rectangle is not a common face. Moreover, there is no polyhedral subdivision $\mathscr{P}$ of X such that, for each $P \in \mathscr{P}$, $\nu|_P \preceq \mu|_P$

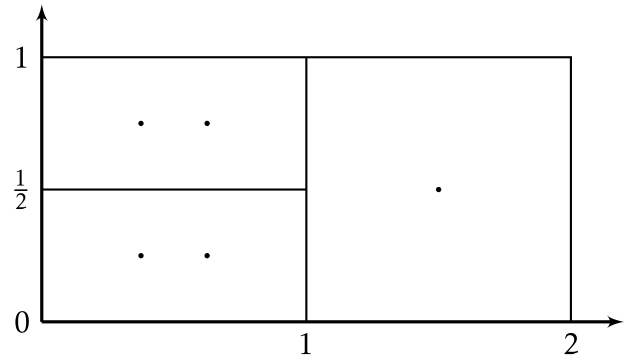

Figure 4: Example 1

and the support of $\nu|_P$ is affinely independent.

Nonetheless, one can verify using Proposition 2 that ν is a Lipschitz-exposed point of F_μ.[8]

We now derive weaker conditions that are necessary for a fusion to be the unique solution.

Corollary 2. *Let μ be absolutely continuous and have full support. Suppose ν has finite support and is a Lipschitz-exposed point of F_μ. Then there is a Power diagram $\mathcal{Q}$ such that the support of $\nu|_Q$ is convexly independent for all $Q \in \mathcal{Q}$.*

We use this result in the following example to illustrate a fusion ν that is an extreme point of F_μ but not an exposed point.

Example 3.5. Let μ be the uniform distribution on the rectangle $[0,2]\times[0,1]$, and let ν be the fusion obtained by contracting μ on $[1,2]\times[0,1]$ to $\left(\frac{3}{2},\frac{1}{2}\right)$; on $[0,1]\times\left[0,\frac{1}{2}\right]$ to $\left(\frac{1}{2},\frac{1}{4}\right)$; and on $[0,1]\times\left[\frac{1}{2},1\right]$ to $\left(\frac{1}{2},\frac{7}{10}\right)$ and $\left(\frac{1}{2},\frac{8}{10}\right)$. This is depicted in Figure 5.

[8]To see this, let $p\colon [0,2]\times[0,1]\to\mathbb{R}$ be given by $p(x)=0$ if $x_1\le 1$ and $p(x)=x_1-1$ if $x_1>1$. Let

$$u(x)=p(x)-\inf_{y\in\operatorname{supp}(\nu)}\|x-y\|.$$

Since $\int u(x)\,\mathrm{d}\nu(x)=\int p(x)\,\mathrm{d}\mu(x)$, ν solves L. Finally, there is no other distribution λ such that $\operatorname{supp}\lambda\subseteq\operatorname{supp}\nu$ and such that $\lambda|_{[0,1]\times[0,1]}\preceq\mu|_{[0,1]\times[0,1]}$ and $\lambda|_{[1,2]\times[0,1]}\preceq\mu|_{[1,2]\times[0,1]}$. Suppose there was such a λ. By symmetry of the left part of the figure, we could obtain another fusion by mirroring λ along a horizontal or a vertical line. The convex combination of the four fusions we obtain this way would equal ν, implying that ν is not an extreme point. This contradicts the conclusion of Lemma 6.

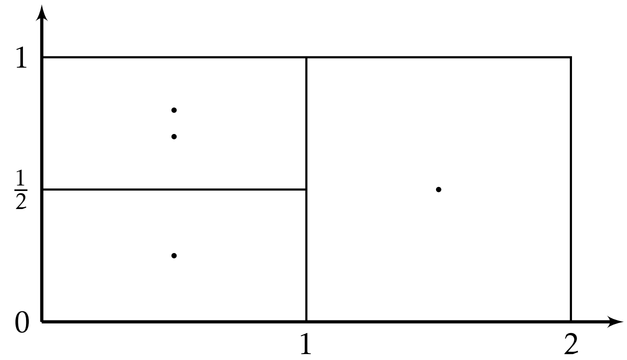

Figure 5: Example 2

The only power diagram such that $\nu|_P \prec \mu|_P$ on each cell P is the trivial power diagram with only one cell $P = X$. However, since the support of ν is not convexly independent, Corollary 2 implies that ν is not a Lipschitz-exposed point. But it follows from Lemma 6 that it is an extreme point.

4 Convex partitional fusions and moment persuasion

We say that a fusion ν of μ is *convex partitional* if there is a partition of X into convex sets such that, for each cell P, the support $\nu|_P$ is a singleton and $\nu|_P \prec \mu|_P$. An extreme point of F_μ need not be convex partitional, but convex partitional fusions are special in the sense that they are the finest fusions among those with a fixed number of support points. The next Proposition generalizes the one-dimensional result (Lemma 1) of Ivanov (2021):

Proposition 3. *Suppose that ν is a fusion of μ with K points in its support. Then there exists a convex partitional measure λ with at most K points in its support that satisfies $\nu \preceq \lambda \preceq \mu$. If ν is not convex partitional, then $\nu \prec \lambda$.*

In Section 5, we apply this proposition to categorization. Namely, we discuss that under processing or memory constraints in which a decision maker is constrained to arrange data in a number of finite "bins," there always exists an optimal arrangement in which the world is categorized according to convex partitions of the state space.

4.1 Moment Persuasion

A special class of multidimensional Bayesian persuasion problems are those in which the sender's payoff can be written in a reduced-form way as a function of the receiver's vector of posterior means. That is, given $G \in \Delta(\Theta)$ with

$$\int_{\Theta_1} a dG_1(a) = x_1, \int_{\Theta_2} a dG_2(a) = x_2, \ldots, \int_{\Theta_d} a dG_d(a) = x_d,$$

there exists a measurable function $V : \Theta \to \mathbb{R}$ such that the sender's payoff is $V(x)$, where $x = (x_1, \ldots, x_d)$. We impose that V is Lipschitz on Θ.

Given this reduced-form payoff V, the receiver solves

$$\max_{\nu \in F_\mu} \mathbb{E}_\nu[V].$$

If the receiver has only finitely many actions, i.e., if $|A| = m \in \mathbb{N}$, then there is a solution to the persuasion problem where the optimal distribution of posterior means ν^* has at most m points in its support.

Proposition 1 then suggests the following interpretation of an optimal solution to a persuasion problem. For any finitely supported extreme point ν of F_μ, we define $\hat{\nu}$, the *convex-partitional coarsening* of ν, to be the fusion of ν obtained by taking each element $P \in \mathscr{P}$ - the underlying partition supporting ν (see Proposition 1) - and collapsing μ on it to its barycenter, $r_P(\mu)$. By construction, $\hat{\nu} \in F_\nu \subseteq F_\mu$ since the convex order is transitive. Moreover, $\nu = \hat{\nu}$ if and only if ν is itself convex partitional.

Let $D_{\mu,P}$ denote the set of mean-preserving spreads of $\delta_{r_P(\mu)}$ that are supported on P. The *unconstrained persuasion problem* on P is

$$\max_{\nu \in D_{\mu,P}} \mathbb{E}_\nu V.$$

We say that an unconstrained persuasion problem is *standard* if P is a simplex. As any simplex is homeomorphic to the standard simplex, any standard unconstrained persuasion problem is equivalent to a problem of the class studied in Kamenica and Gentzkow (2011).

We define the upper concave envelope of V on P, $\text{cav}_P V$, to be the smallest concave function that lies pointwise above V on P. Alternatively, we call this the concavification of V on P. We say that a solution to the persuasion problem, ν^*, is *canonical* if on each cell, P, of the corresponding partition $\mathscr{P}^*$, $\nu^*|_P$ is the solution of an unconstrained persuasion problem on P. Moreover, the sender's payoff is

$$\sum_{P\in\mathscr{P}^*} \mu(P)\operatorname{cav}_P V(r_P(\mu))$$

Proposition 4. *There exists a canonical solution to the persuasion problem.*

Proof. By Bauer's maximum principle, there is a solution to the persuasion problem, ν^*, that is an extreme point. As the receiver only has finitely many actions, ν^* can be taken to have finitely many support points. Let $\mathscr{P}^*$ be a convex partition of Θ corresponding to ν^* such that there is no strictly finer partition corresponding to ν^*.

Suppose for some $P \in \mathscr{P}^*$ $\nu^*|_P$ has multiple points of support (or else we are done). As there is no strictly finer partition corresponding to ν^*, by Proposition 3, there exists a $\lambda^*|_P$ such that $\nu^*|_P \prec \lambda^*|_P \preceq \mu|_P$. Moreover, as $\operatorname{supp}\nu^*|_P$ is affinely independent, and as $\nu^*|_P \prec \lambda|_P$ implies that $\operatorname{conv}\operatorname{supp}\nu^*|_P \subseteq \operatorname{conv}\operatorname{supp}\lambda^*|_P$, Proposition 3 implies that $\operatorname{supp}\lambda^*|_P$ is affinely independent.

By construction, $\nu^*|_P$ is a solution to $\max_{\nu\in F_{\lambda^*|_P}} \mathbb{E}_\nu V$. This is a standard unconstrained persuasion problem, whose solution is given by the concavification of the persuader's value function on $\operatorname{conv}\operatorname{supp}\lambda^*|_P$ (Kamenica and Gentzkow (2011)). Let $\mathscr{H}^*(x)$ be the set of hyperplanes supported on $\operatorname{supp}\nu^*|_P$ that lie weakly above V on $\operatorname{conv}\operatorname{supp}\lambda^*|_P$.

Suppose for the sake of contradiction that there exists an $x \in P$ for which $H^*(x) < V(x)$ for all $H^*(x) \in \mathscr{H}^*$. By Theorem 5 in Dworczak and Kolotilin, strong duality holds for multi-dimensional moment persuasion and as ν^* is an optimal fusion of μ on Θ, $\nu^*|_P$ is an optimal fusion of $\mu|_P$ (on P). This implies that there exists a convex piecewise-affine function q_P that lies everywhere above V on P and such that $q_P(y) = V(y)$ for all $y \in \operatorname{supp}\nu^*|_P$. However, $\int q_P d\mu|_P = \int V d\nu^*|P$ only if q_P is affine on P, by our assumption that there is no strictly finer partition, a contradiction. ■

This result indicates that the solution to a moment persuasion problem (in which finitely many different actions are induced) can be decomposed into two parts. First, in the **global** portion of the problem, the domain is partitioned into finitely many convex sets. Second, in the **local** portion of the problem, the persuader solves an unconstrained problem on each of these convex sets, whose solution is given by the concavification of the value function. As Proposition 4 states, there always exists such a canonical solution to a moment-persuasion problem.

This result also elucidates the bi-pooling result of Kleiner et al. (2021) and Arieli et al. (2023): any mean-persuasion problem with an interval state space admits a bi-pooling solution. When the optimal solution has a finite support, this means that the interval is partitioned into finitely many intervals such that, on each of them, the prior is either collapsed to its barycenter, or is fused into two points. This is precisely the single-dimensional version of Proposition 1.

5 An Application to Categorization

Categorization, or the arrangement of objects according to some rule, is a central part of our interaction with the world. An influential work that studies this formally is Gärdenfors (2004). There, a central idea is the notion of a conceptual space–in economics, a state space–and its division into convex sets, each containing a prototype, i.e., a central object. For example, one can think of colors: "red" describes a whole family of colors, as does "blue" and so on. Gärdenfors notes several ways of producing this phenomenon: one can define a *natural property* as a convex region of the conceptual space, in which case the prototype in each region is the central object in each region. He also notes that one can instead start with a finite set of prototypes then partition the conceptual space by grouping points together that are closest to each prototype. When the notion of closeness is the Euclidean distance, this corresponds to a particular type of convex partition of the conceptual space, a Voronoi tessellation.

These constructions seem reasonable and realistic, yet the central properties are exogenous. That is, convexity of the regions, or existence of the prototypes, is/are assumed. Here, we use our earlier results to argue that these properties are natural outcomes of a decision-maker's (DM's) optimization when she faces limits about the amount of information she can acquire or process. We provide two different micro-foundations for categorization. In both the outcome is a convex partition of the state space in which there exists a single representative object (prototype) per partition element. In the first derivation, we show that this categorization emerges as information becomes sufficiently cheap when a decision maker (DM) faced with a decision problem acquires information flexibly. In the second derivation, we show that this phenomenon emerges when a DM must instead store information in finitely many bins. We show that the optimal way of doing so is precisely categorization.

5.1 Decision Problems with Flexible Information Acquisition

Let the state of the world $x \in X \equiv [0,1]^d$ be distributed according to absolutely continuous μ. There is a risk-neutral DM with a bounded utility function, who must choose action a from some finite set of undominated actions A, $|A| = t$ ($\infty > t \geq 2$). As the maximum of Lipschitz-continuous functions is Lipschitz-continuous, the agent's decision problem induces a Lipschitz-continuous (piecewise-affine) reduced-form value function, V, of the vector of posterior means.

Prior to taking her decision, the DM acquires (or processes) information. We assume that this can be modeled as follows. There is a cost functional $C\colon F_\mu \to \mathbb{R}$, where

$$C(\nu) = \kappa \int c d\nu \ ,$$

with $\kappa > 0$ and c some strictly convex, Lipschitz-continuous, twice continuously differentiable function whose partial derivatives are bounded on X; i.e., acquiring any fusion $\nu \in F_\mu$ costs the DM $C(\nu)$. All in all, the DM solves

$$\max_{\nu \in F_\mu} \int (V - \kappa c) d\nu.$$

Remark 5.1. For any decision problem, cost function, and prior of the form described above, there exists a $\bar{\kappa} > 0$ such that if the cost parameter $\kappa \leq \bar{\kappa}$, there exists a solution to the DM's information acquisition problem that corresponds to categorization with a single prototype per category.

Proof. Fix a decision problem, cost function, and prior of the form described above. Let

c_i denote the partial derivative of c in x_i: $c_i(x) \equiv \frac{\partial}{\partial x_i} c$. Define

$$\alpha \equiv \max_{x,y \in [0,1]^d; i=1,\ldots,d} \left| c_i(x) - c_i(y) \right|,$$

i.e., α is the upper bound for the largest possible (absolute) difference between marginal costs of beliefs. Because the partial derivatives of c are bounded and c is strictly convex, $\alpha \in \mathbb{R}_{++}$. Similarly, define

$$\beta \equiv \min_{x,y \in [0,1]^d; i=1,\ldots,d} \left\{ \left| V_i(x) - V_i(y) \right| \ : \ \left| V_i(x) - V_i(y) \right| > 0 \right\}.$$

Because there are only finitely many actions and all are undominated, $\beta \in \mathbb{R}_{++}$. Finally, let $\kappa < \beta/\alpha$.

By Proposition 4, there exists a canonical solution, ν, to the persuasion problem. Suppose for the sake of contradiction that on one of the elements $P \in \mathcal{P}$, ν has multiple ($m \geq 2$) points of support. This collection of points $(x_1, \ldots, x_m)$ must be such that for each $j = 1, \ldots, d$, and for any $i, k \in \{1, \ldots, m\}$,

$$V_j(x_i) - V_j(x_k) = \kappa \left(c_j(x_i) - c_j(x_k) \right).$$

By the optimality of ν and the strict convexity of c, because ν has multiple support points on P, the DM must take at least two different actions with strictly positive probability. Therefore, for at least one trio j, i, k the left-hand side of this equation is nonzero and so

$$\kappa = \frac{V_j(x_i) - V_j(x_k)}{c_j(x_i) - c_j(x_k)} \geq \frac{\beta}{\alpha},$$

a contradiction. ■

5.2 Decision Problems with Finite Memory

We now drop the assumption that the DM's problem induces a mean-measurable value function[9] and also jettison the specification that the cost of information acquisition is smooth. Instead, the DM has access to $K \in \mathbb{N}$ "bins" or categories, to which she can costlessly assign arbitary points in the state space. Importantly, we do not assume that

[9]By Stone-Weierstrass, this assumption earlier is innocuous.

K corresponds to a partition, much less a convex one. For instance, the DM could assign any $x \in X$ to one of the bins (uniformly) at random.

Nevertheless, as information is always beneficial to a DM, Proposition 3 implies the following:

Remark 5.2. There is an optimal categorization that is convex partitional. If the number of undominated actions in the decision problem is weakly greater than the number of possible categories, any optimal categorization must be convex partitional.

5.3 Decision-Making With Complexity Constraints

Suppose there is a finite state space Θ. There is a DM with a compact set of actions A and a continuous utility function $u\colon A \times \Theta \to \mathbb{R}$. The Bayesian DM has a prior, $\eta_0 \in int\Delta(\Theta)$, and observes information before taking a decision. Formally, she observes the realization of a signal $\pi\colon \Theta \to \Delta(S)$, where S is a compact set of signal realizations, before choosing an action $a \in A$. We let $\mu \in \Delta\Delta(\Theta)$ denote the distribution over posteriors η induced by the prior and the signal.

We specify that the DM is simplicity constrained in the following sense: there exists some $K \in \mathbb{N}$ and the DM is constrained to optimize over decision rules $\sigma\colon S \to \Delta(A)$ with support of at most K. We call such a decision rule a simplicity-constrained decision rule. It is immediate that this is equivalent to the DM being restricted to choose a fusion of μ, ν, that is supported on at most K points before choosing an arbitrary (unconstrained) decision rule. We say that a simplicity-constrained decision rule is convex partitional if ν is a convex-partitional fusion of μ. Then, Proposition 3 gives us

Remark 5.3. There is an optimal simplicity-constrained decision rule that is convex partitional. If the number of undominated actions in the decision problem is weakly greater than the number of support points of μ, any optimal simplicity-constrained decision rule must be convex partitional.

A Omitted Proofs

A.1 Proof of Proposition 1

We first need several auxiliary Lemmas:

Lemma 2. *(Cartier et al. (1964)): Let μ, ν be positive measures on a compact, convex set X such that $\nu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ where $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i > 0$ for each i. Then $\nu \preceq \mu$ if and only if there exist positive measures $\mu_1, \ldots, \mu_n$ such that, for all i, the barycenter of μ_i equals x_i, and such that $\mu = \sum_{i=1}^{n} \alpha_i \mu_i$ (in particular $\delta_{x_i} \preceq \mu_i$ for all i).*

For any two measures μ and ν on X, $\mu \leq \nu$ denotes the pointwise order defined by $\mu(A) \leq \nu(A)$ for all measurable $A \subseteq X$.

Lemma 3. *Let μ be a positive measure supported on X and let $y \in \operatorname{int}(\operatorname{ch}(\operatorname{supp}(\mu)))$. Then there exists a positive measure π such that $\pi \leq \mu$, $\pi(X) > 0$, and $r(\pi) = y$.*

Proof. Choose a finite set $\{y_1, \ldots, y_k\} \subseteq \operatorname{supp} \mu$ such that y lies in the interior of the convex hull of $\{y_1, ..., y_k\}$. Since the convex hull operator is continuous (e.g., Sertel (1989)), there is $\varepsilon > 0$ such that for all $z_i \in B_\varepsilon(y_i)$, $y \in \operatorname{ch}(z_1, ..., z_k)$. Choosing $\varepsilon > 0$ small enough, we can assume that $B_\varepsilon(y_i)$ is disjoint from $B_\varepsilon(y_j)$ whenever $i \neq j$.

For each i, $\mu(B_\varepsilon(y_i)) > 0$ since $y_i \in \operatorname{supp}(\mu)$. Moreover, the barycenter of $\mu|_{B_\varepsilon(y_i)}$ is in $B_\varepsilon(y_i)$. Therefore, there are $\lambda_i \geq 0$ with $\sum_i \lambda_i = 1$ such that the barycenter of $\pi := \sum_i \lambda_i \mu|_{B_\varepsilon(y_i)}$ is y. Since $\pi \leq \mu$ by construction, the claim follows. ∎

Remark A.1. Observe that the measure π constructed in the proof of the above Lemma satisfies $\pi(X) \geq \min_i \{\mu(B_\varepsilon(y_i))\}$.

Lemma 4. *Let μ_1, μ_2 be positive measures on X satisfying*

$$C = \operatorname{int}(\operatorname{ch}(\operatorname{supp}(\mu_1))) \cap \operatorname{int}(\operatorname{ch}(\operatorname{supp}(\mu_2))) \neq \emptyset.$$

For all $d \in \mathbb{R}^n$ there is $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$ and $a \in [-\varepsilon, \varepsilon]$ there is a measure π with $\mu_1 + \pi \geq 0$, $\mu_2 - \pi \geq 0$, $\pi(X) = a$ and $\int x \, \mathrm{d}\pi = \varepsilon d$.

The Lemma implies that for any positive measures whose supports intersect nontrivially, and for any vector $d \in \mathbb{R}^n$, we can shift mass from one measure to the other such that the barycenters of the measures moves in direction d and $-d$ respectively, while the measures remain positive.

Proof. Let $\delta > 0$ and $z \in C$ be such that $B_{2\delta}(z) \subseteq C$. Note that by Lemma 3 and the following remark, there is $M > 0$ such that for all $z' \in B_\delta(z)$ there are positive measures $\xi \leq \mu_1$ and $\zeta \leq \mu_2$ with barycenter $r(\xi) = r(\zeta) = z'$ and $\xi(X), \zeta(X) \geq M$.

Let $\bar{\varepsilon} = \min\left\{\frac{M}{2}, \frac{\delta M}{\|d\|}, \frac{\delta M}{\|z\|+\delta}\right\} > 0$ and fix $\varepsilon \in (0, \bar{\varepsilon})$. Suppose $a \in [0, \varepsilon]$ (the arguments for $a \in [-\varepsilon, 0)$ are analogous). Since $\frac{\varepsilon}{M}\|d\| < \delta$, the preceding paragraph implies that there is a positive measure $\pi_1 \leq \mu_2$ with barycenter $z + \frac{\varepsilon}{M}d$ and mass $\pi_1(X) \geq M$. By multiplying π_1 with a positive constant less than 1, we can assume that $\pi_1(X) = M$. Analogously, since $\left(\frac{M}{M-a} - 1\right)\|z\| < \delta$, there is a positive measure $\pi_2 \leq \mu_1$ with barycenter $\frac{M}{M-a}z$ and mass $\pi_2(X) \geq M$. By multiplying π_2 with a positive constant less than 1, we can assume that $\pi_2(X) = M - a > 0$.

Now define $\pi := \pi_1 - \pi_2$. Then

$$\mu_1 + \pi = \mu_1 + \pi_1 - \pi_2 \geq \pi_1 \geq 0,$$

where the first inequality follows from $\pi_2 \leq \mu_1$, and the second holds since π_1 is positive. Similarly, because $\pi_1 \leq \mu_2$ and π_2 is positive, we have $\mu_2 - \pi \geq 0$. Moreover, $\pi(X) = a$ and

$$\int x \,\mathrm{d}\pi = \int x \,\mathrm{d}\pi_1 - \int x \,\mathrm{d}\pi_2 = M\left(z + \frac{\varepsilon}{M}d\right) - (M-a)\frac{M}{M-a}z = \varepsilon d,$$

which establishes the claim. ∎

Corollary 3. *Let μ_1, μ_2 be positive measures with barycenters x_1 and x_2, respectively. If*

$$C = \mathrm{int}(\mathrm{ch}(\mathrm{supp}(\mu_1))) \cap \mathrm{int}(\mathrm{ch}(\mathrm{supp}(\mu_2))) \neq \emptyset,$$

1. *there is a measure π satisfying $\pi(X) > 0$, $\mu_1 + \pi \geq 0$, $\mu_2 - \pi \geq 0$ and such that the barycenter of $\tilde{\mu}_1 := \mu_1 + \pi$ is x_1.*

2. *there is a measure π satisfying $\pi(X) < 0$, $\mu_1 + \pi \geq 0$, $\mu_2 - \pi \geq 0$ and such that the barycenter of $\tilde{\mu}_1 := \mu_1 + \pi$ is x_1.*

The Corollary establishes that if two positive measures have supports that intersect nontrivially, we can move mass from one measure to the other without changing the barycenters of the measures.

Proof. By Lemma 4, for $\varepsilon > 0$ small enough there exists a measure π with $\mu_1 + \pi \geq 0$, $\mu_2 - \pi \geq 0$, $\pi(X) = \varepsilon$ and $\int x \, d\pi = \varepsilon x_1$. It follows that $r(\tilde{\mu}_1) = \frac{1}{\mu_1(X)+\varepsilon}[\mu_1(X)x_1 + \varepsilon x_1] = x_1$. The argument for the second part is analogous. ■

For a set $A \subseteq \mathbb{R}^n$, let $\operatorname{aff} A$ denote the affine span of A:

$$\operatorname{aff} A = \{\sum_{i=1}^{k} \alpha_i x_i : k > 0, x_i \in A, \alpha_i \in \mathbb{R}, \sum_{i=1}^{k} \alpha_i = 1\}.$$

Lemma 5. *Let $\{\mu_i | i \in I\}$ be a finite collection of positive measures with $r(\mu_i) = x_i$ and let $\mu = \sum_{i \in I} \mu_i$. Let $V = \{J_1, ..., J_k\}$ be a partition of I.*

Consider the graph G whose vertex set is V and which contains an edge between J and J' if

$$\operatorname{int}(\operatorname{ch}(\operatorname{supp}(\sum_{j \in J} \mu_j))) \cap \operatorname{int}(\operatorname{ch}(\operatorname{supp}(\sum_{j \in J'} \mu_j))) \neq \emptyset.$$

Suppose that G is connected, and let λ be a measure such that $\mu + \lambda$ is positive and $r(\mu + \lambda) \in \operatorname{aff}\{x_i | i \in I\}$.

Then, for all $\varepsilon > 0$ small enough we can decompose $\tilde{\mu} := \mu + \varepsilon\lambda$ into positive measures $\tilde{\mu}_J$ with $\sum_{J \in V} \tilde{\mu}_J = \tilde{\mu}$ and $r(\tilde{\mu}_J) \in \operatorname{aff}\{x_j | j \in J\}$.

Proof. Let $\mu_J = \sum_{i \in J} \mu_i$, and let $\{\lambda_J\}_{J \in V}$ be such that $\lambda = \sum_{J \in V} \lambda_J$, $\hat{\mu}_J := \mu_J + \varepsilon\lambda_J \geq 0$, and $\operatorname{supp} \mu_J \subseteq \operatorname{supp} \hat{\mu}_J$.[10] Since $r(\mu), r(\mu + \lambda) \in \operatorname{aff}\{x_i | i \in I\}$, there are $\gamma_i \in \mathbb{R}$ such that $\int x \, d\lambda = \sum_{i \in I} \gamma_i x_i$ and $\sum_{i \in I} \gamma_i = \lambda(X)$.[11]

Consider a spanning tree T of G. For each leaf J of T, define a weight $w_J = \sum_{i \in J} \gamma_i - \lambda_J(X)$ and let

$$\tilde{\mu}_J = \hat{\mu}_J + \pi_J,$$

[10] Such a decomposition of λ exists whenever ε is small enough: Since $\mu + \lambda$ is positive, $\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\mu + \varepsilon\lambda)$ for $\varepsilon > 0$ small enough. For each $J \in V$, let $\lambda_J := \frac{1}{\varepsilon|V|}(\mu + \varepsilon\lambda - |V|\mu_J)$. Then $\hat{\mu}_J = \mu_J + \varepsilon\lambda_J = \frac{1}{|V|}(\mu + \varepsilon\lambda) \geq 0$ and $\operatorname{supp}(\mu_J) \subseteq \operatorname{supp}(\mu) \subseteq \operatorname{supp}(\hat{\mu}_J)$.

[11] Let α_i satisfy $r(\mu + \lambda) = \sum_i \alpha_i x_i$ and $\sum_i \alpha_i = 1$ and let β_i satisfy $r(\mu) = \sum_i \beta_i x_i$ and $\sum_i \beta_i = 1$. Then $\gamma_i := (\alpha_i - \beta_i)\mu(X) + \alpha_i \lambda(X)$ satisfy these properties.

where π_J is a measure chosen such that $\pi_J(X) = \varepsilon w_J$, $\tilde{\mu}_J \geq 0$, $\pi_J \leq \frac{1}{|I|+1}\hat{\mu}_{J'}$ (where J' is the neighbor of J in T), and

$$\varepsilon \int x \, \mathrm{d}\lambda_J + \int x \, \mathrm{d}\pi_J = \varepsilon \sum_{i \in J} \gamma_i x_i.$$

Such π_J exist by Lemma 4 whenever ε is chosen small enough.

Proceeding inductively, consider the tree obtained by deleting all leaves from the previous tree. For each leaf J of the new tree, denote by $L(J)$ the set of leaves of the previous tree that were connected to J, and note that for each leaf J of the new tree, $\hat{\mu}_J - \sum_{K \in L(J)} \pi_K \geq 0$ and $\operatorname{supp}(\hat{\mu}_J) \subseteq \operatorname{supp}(\hat{\mu}_J - \sum_{K \in L(J)} \pi_K)$. Define the weight $w_J = \sum_{K \in L(J)} w_K + \sum_{i \in J} \gamma_i - \lambda_J(X)$ for each leaf J of the new tree. As long as there are at least 3 vertices remaining, define

$$\tilde{\mu}_J = \hat{\mu}_J - \sum_{K \in L(J)} \pi_K + \pi_J$$

where the π_J's are chosen so that $\pi_J(X) = \varepsilon w_J$, $\tilde{\mu}_J \geq 0$, $\pi_J \leq \frac{1}{|I|+1}\hat{\mu}_{J'}$ (where J' is the neighbor of J in the remaining tree), and

$$\varepsilon \int \mathrm{x} d\lambda_J - \sum_{K \in L(J)} \int x \, \mathrm{d}\pi_K + \int x \, \mathrm{d}\pi_J = \varepsilon \sum_{i \in J} \gamma_i x_i. \tag{A1}$$

Such a choice exists by Lemma 4 whenever $\varepsilon > 0$ is small enough. We repeat this procedure until we obtain a tree with at most two vertices.

If only two nodes are left in the remaining tree (denote them by $\hat{J}$ and $\bar{J}$), define

$$\tilde{\mu}_{\hat{J}} = \hat{\mu}_{\hat{J}} - \sum_{K \in L(\hat{J})} \pi_K + \pi_{\hat{J}}$$
$$\tilde{\mu}_{\bar{J}} = \hat{\mu}_{\bar{J}} - \sum_{K \in L(\bar{J})} \pi_K - \pi_{\hat{J}}$$

where $\pi_{\hat{J}}$ is chosen so that $\pi_{\hat{J}}(X) = \varepsilon w_{\hat{J}}$, $\tilde{\mu}_{\hat{J}} \geq 0$, $\tilde{\mu}_{\bar{J}} \geq 0$, and

$$\varepsilon \int x \, \mathrm{d}\lambda_{\hat{J}} - \sum_{K \in L(\hat{J})} \int x \, \mathrm{d}\pi_K + \int x \, \mathrm{d}\pi_{\hat{J}} = \varepsilon \sum_{i \in \hat{J}} \gamma_i x_i.$$

If only one node, denoted by $\bar{J}$, is left in the remaining tree, define

$$\tilde{\mu}_{\bar{J}} = \mu_{\bar{J}} + \varepsilon \lambda_{\bar{J}} - \sum_{K \in L(\bar{J})} \pi_K.$$

In either case, it follows that $\sum_{J\in V}\tilde{\mu}_J = \sum_{J\in V}\hat{\mu}_J$ by construction. Moreover, for all $J \neq \bar{J}$, the barycenter of $\tilde{\mu}_J$ is in $\mathrm{aff}\{x_i | i \in J\}$.[12]

It remains to verify that the barycenter of $\tilde{\mu}_{\bar{J}}$ is in $\mathrm{aff}\{x_i | i \in \bar{J}\}$. For $J \in V$, let $b_J = \int x \,\mathrm{d}\pi_J$. For any node $J \neq \bar{J}$ we obtain from ($A1$) that

$$b_J = \varepsilon \sum_{i\in J} \gamma_i x_i - \varepsilon \int x \,\mathrm{d}\lambda_J + \sum_{K\in L(J)} b_K, \tag{A2}$$

where $L(J) = \emptyset$ if J is a leaf of T. Therefore

$$\begin{aligned}\left[\mu_{\bar{J}}(X) + \varepsilon\lambda_{\bar{J}}(X) - \varepsilon \sum_{K\in L(\bar{J})} w_K - \varepsilon w_{\hat{J}}\right] r(\tilde{\mu}_{\bar{J}}) &= r(\tilde{\mu}_{\bar{J}})\tilde{\mu}_{\bar{J}}(X) \\ &= r(\mu_{\bar{J}})\mu_{\bar{J}}(X) + \varepsilon\lambda_{\bar{J}}(X) r(\lambda_{\bar{J}}) - \sum_{K\in L(\bar{J})} b_K \\ &= r(\mu_{\bar{J}})\mu_{\bar{J}}(X) + \varepsilon \int x \,\mathrm{d}\lambda - \varepsilon \sum_{i\notin \bar{J}} \gamma_i x_i,\end{aligned}$$

where the first equality follows from the definition of π_K and the third equality follows from using ($A2$) repeatedly.

Suppose $\hat{J}$ and $\bar{J}$ are the remaining vertices. Since $\sum_{K\in L(\bar{J})} w_K + w_{\hat{J}} = \sum_{i\notin\bar{J}} \gamma_i - \sum_{J\neq\bar{J}} \lambda_J(X)$, and $\int x\,\mathrm{d}\lambda = \sum_i \gamma_i x_i$, this implies

$$r(\tilde{\mu}_{\bar{J}}) = \frac{1}{\mu_{\bar{J}}(X) + \varepsilon \sum_{i\in\bar{J}} \gamma_i}\left[\mu_{\bar{J}}(X) r(\mu_{\bar{J}}) + \varepsilon \sum_{i\in\bar{J}} \gamma_i x_i\right].$$

Since the barycenter of $\mu_{\bar{J}}$ lies in $\mathrm{aff}\{x_i | i \in \bar{J}\}$, we conclude that the barycenter of $\tilde{\mu}_{\bar{J}}$ lies in $\mathrm{aff}\{x_i | i \in \bar{J}\}$.

If $\bar{J}$ is the only remaining vertex, then $\sum_{K\in L(\bar{J})} w_K = \sum_{i\notin\bar{J}} \gamma_i - \sum_{J\neq\bar{J}} \lambda_J(X)$, which again implies

$$r(\tilde{\mu}_{\bar{J}}) = \frac{1}{\mu_{\bar{J}}(X) + \varepsilon \sum_{i\in\bar{J}} \gamma_i}\left[\mu_{\bar{J}}(X) r(\mu_{\bar{J}}) + \varepsilon \sum_{i\in\bar{J}} \gamma_i x_i\right].$$

It follows that the barycenter of $\tilde{\mu}_{\bar{J}}$ lies in $\mathrm{aff}\{x_i | i \in \bar{J}\}$. ■

[12] This holds because $r(\mu_J) \in \mathrm{aff}\{x_i | i \in J\}$, $\tilde{\mu}_J(X) = \mu_J(X) + \varepsilon\sum_{i\in J}\gamma_i$ and $\int x\,\mathrm{d}\tilde{\mu}_J = \mu_J(X) r(\mu_J) + \varepsilon \sum_{i\in J} \gamma_i x_i$.

Proof of Proposition 1. Let $\mathcal{P}$ be a partition of X such that each $A \in \mathcal{P}$ is convex, satisfies $\mu(A) > 0$, and $\nu|_A \prec \mu|_A$. Moreover, assume there is no finer partition satisfying these properties.[13]

Suppose that for some $A \in \mathcal{P}$, the support of $\nu|_A$ is not affinely independent. Let $\{x_i | i \in I\}$ denote the support of $\nu|_A$. Also, let μ_i satisfy $r(\mu_i) = x_i$, $\mu_i(X) = \nu(\{x_i\})$, and $\mu|_A = \sum_i \mu_i$ (such μ_i exist by Lemma 2).

Construct a graph G_1 with vertex set I and an edge between i and j if

$$\mathrm{int}(\mathrm{ch}(\mathrm{supp}(\mu_i))) \cap \mathrm{int}(\mathrm{ch}(\mathrm{supp}(\mu_i))) \neq \emptyset.$$

Recursively, given a graph G_n construct a new graph G_{n+1} with vertex set being the set of connected components in G_n (for each connected component in G_n there is one vertex in G_{n+1}; each vertex is labeled by a subset of I corresponding to the set of vertices it represents). Add an edge between vertices C and C' in G_{n+1} if

$$\mathrm{int}(\mathrm{ch}(\bigcup_{i \in C} \mathrm{supp}(\mu_i))) \cap \mathrm{int}(\mathrm{ch}(\bigcup_{i \in C'} \mathrm{supp}(\mu_i))) \neq \emptyset.$$

Stop this procedure of constructing graphs once it yields a graph G_N with a vertex C such that $\{x_i | i \in C\}$ is affinely dependent.

We first claim that the procedure always stops with such a graph G_N.Suppose the procedure does not stop. Up to some iteration M, each iteration reduces the number of vertices by at least one; after that, the number of vertices stays constant, which implies that there are no edges. Let G_M denote this graph without edges. We claim that G_M has exactly one vertex. Suppose not and let $J_1, \ldots, J_m$ with $m \geq 2$ denote its vertices. For $k = 1, \ldots, m$, $A_k := \mathrm{ch}(\bigcup_{i \in J_k} \mathrm{supp}(\mu_i))$ is convex and satisfies $\mu(A_k) > 0$. Since there are no edges in G_M, $\mathrm{int}(A_j) \cap \mathrm{int}(A_k) = \emptyset$ for $j, k = 1, \ldots, m$ with $j \neq k$. Since every hyperplane has μ-measure 0, this implies $\nu|_{A_k} \prec \mu|_{A_k}$. This implies that the collection $\{A_1, \ldots, A_m\}$ gives rise to a partition of $\mathcal{P}'$ of X that is finer than $\mathcal{P}$, contradicting our initial hypothesis. We conclude that G_M has one vertex; since $\{x_i | i \in I\}$ is affinely dependent, this contradicts the hypothesis that the procedure did not stop.

[13] The trivial partition $\mathcal{P} = \{X\}$ satisfies all conditions. Moreover, there is an upper bound on the number of partition elements in any such partition since the support of ν is finite. Therefore, there is a partition that satisfies all conditions and cannot be refined.

This implies that graph G_{N-1} has a connected component corresponding to vertex C in graph G_N. Let T denote a spanning tree of this connected component. We can assume that T is minimal in the sense that for any leaf J of T, $\{x_i | i \in J\}$ and $\{x_i | i \in C \setminus J\}$ are affinely independent: Indeed, for any leaf J of T, $\{x_i | i \in J\}$ is affinely independent since the procedure would have stopped with graph G_{N-1} otherwise. If $\{x_i | i \in C \setminus J\}$ is affinely dependent, consider the graph T' obtained by deleting vertex J from graph T. For any leaf J' of T', either $\{x_i | i \in J'\}$ and $\{x_i | i \in C \setminus (J \cup J')\}$ are affinely independent, or we can reduce the tree T' further until we obtain a tree with the desired properties.

Now let $\{J_1, \dots, J_m\}$ denote the vertices of T, choose a leaf J of T (without loss of generality, assume $J = J_1$), and let $\mu_{C \setminus J} = \sum_{i \in C \setminus J} \mu_i$. Since $\{x_i | i \in C\}$ is affinely dependent, there is β_i such that $\sum_{i \in C} \beta_i = 0$ and $\sum_{i \in C} \beta_i x_i = 0$. This implies

$$d := \frac{\sum_{i \in J} \beta_i x_i}{\sum_{i \in J} \beta_i} = \frac{\sum_{i \in C \setminus J} \beta_i x_i}{\sum_{i \in C \setminus J} \beta_i}.$$

Lemma 4 implies that for any $\varepsilon > 0$ small enough, there is a measure π with $\mu_J + \pi \geq 0$, $\mu_{C \setminus J} - \pi \geq 0$, $\pi(X) = \varepsilon$ and $\int x \, \mathrm{d}\pi = \varepsilon d$. It follows that

$$\begin{aligned} r(\mu_J + \pi) &= \frac{1}{\mu_J(X) + \varepsilon}\left[\int x \, \mathrm{d}\mu_J + \int x \, \mathrm{d}\pi\right] = \frac{1}{\mu_J(X) + \varepsilon}\Big[\mu_J(X) r(\mu_J) + \varepsilon d\Big] \\ r(\mu_{C \setminus J} - \pi) &= \frac{1}{\mu_{C \setminus J}(X) - \varepsilon}\left[\int x \, \mathrm{d}\mu_{C \setminus J} - \int x \, \mathrm{d}\pi\right] = \frac{1}{\mu_{C \setminus J}(X) - \varepsilon}\Big[\mu_{C \setminus J}(X) r(\mu_{C \setminus J}) - \varepsilon d\Big]. \end{aligned}$$

Therefore, $r(\mu_J + \pi) \in \operatorname{aff}\{x_i | i \in J\}$ and $r(\mu_{C \setminus J} - \pi) \in \operatorname{aff}\{x_i | i \in C \setminus J\}$. Lemma 5 then implies that whenever ε is small enough, we can decompose $\mu_{C \setminus J} - \pi$ into positive measures $\{\tilde{\mu}_{J_k}\}_{k=2}^m$ with $\sum_{k=2}^m \tilde{\mu}_{J_k} = \mu_{C \setminus J} - \pi$ and $r(\tilde{\mu}_{J_k}) \in \operatorname{aff}\{x_i | i \in J_k\}$. We can decompose $\mu_J + \pi$ in a similar manner. Applying these decompositions repeatedly to graphs G_{N-2}, G_{N-3}, etc., this yields measures $\{\tilde{\mu}_i\}_{i \in C}$ with $\sum_{i \in C} \tilde{\mu}_i = \sum_{i \in C} \mu_i$ and $r(\tilde{\mu}_i) = x_i$ that satisfy $\tilde{\mu}_i(X) \neq \mu_i(X)$ for some $i \in C$ (since $\sum_{i \in J} \tilde{\mu}_i(X) - \mu_i(X) = \pi(X) > 0$). We can then define $\tilde{\nu}$ to be a positive measure with the same support as $\nu|_A$ that satisfies $\tilde{\nu}(\{x_i\}) = \tilde{\mu}_i(X)$ for $i \in C$ and $\tilde{\nu}(\{x_i\}) = \nu(\{x_i\})$ for $i \in I \setminus C$. This is a fusion of $\mu|_A$ by Lemma 2 and it satisfies $\tilde{\nu} \neq \nu|_A$.

Repeating the procedure with $-d$ in place of d yields another fusion $\hat{\nu} \neq \nu|_A$. We claim that $\nu|_A = 1/2(\tilde{\nu} + \hat{\nu})$: Note that

$$\sum_{i \in J} \nu(\{x_i\})x_i = \int x \,\mathrm{d}\mu_J = 1/2\Big[\int x \,\mathrm{d}\mu_J + \varepsilon d + \int x \,\mathrm{d}\mu_J - \varepsilon d\Big]$$
$$= 1/2 \sum_{i \in J} [\tilde{\nu}(\{x_i\}) + \hat{\nu}(\{x_i\})]x_i, \tag{A3}$$

where the first equality follows since $\sum_{i\in J} \nu(\{x_i\})$ is a fusion of μ_J and the final equality follows since, by construction of $\tilde{\nu}$ and $\hat{\nu}$, $\int x \,\mathrm{d}[\sum_{i\in J} \tilde{\nu}(\{x_i\})] = \int x \,\mathrm{d}\mu_J + \varepsilon d$ and $\int x \,\mathrm{d}[\sum_{i\in J} \hat{\nu}(\{x_i\})] = \int x \,\mathrm{d}\mu_J - \varepsilon d$. Since $\{x_i | i \in J\}$ is affinely independent, (A3) implies that for each $i \in J$, $\nu(\{x_i\}) = 1/2[\tilde{\nu}(\{x_i\}) + \hat{\nu}(\{x_i\})]$. Analogous arguments apply for $i \in C \setminus J$, which establishes the claim.

This implies that ν is not an extreme point of F_μ, contradicting our initial hypothesis. We conclude that $\mathrm{supp}(\nu|_A)$ is affinely independent for each $A \in \mathcal{P}$. ■

A.2 Proof of Proposition 3

Proof. Let G denote the set of probability measures that are fusions of μ and have at most K points in their support.We show first using Zorn's lemma that there is a maximal measure in G according to the convex order: The convex order is a partial order. Let $\{g_i\}_{i\in I}$ be a totally ordered subset of G, which can be viewed as a net by setting $i < j$ if $g_i \prec g_j$. Since G is compact (e.g., in the weak*-topology), there is subnet with limit g_∞. Then $g_i \prec g_\infty$ because for any convex continuous function f and $i < j$, $\int f \,\mathrm{d}g_i \le \int f \,\mathrm{d}g_j$. Therefore, every totally ordered subset of G has an upper bound in G. It follows from Zorn's lemma that there is a maximal element in G.

Let λ denote the maximal element of G, let $\{x_1, \dots, x_K\}$ denote its support points, and suppose λ is not convex partitional. Using Lemma 2 there is a decomposition of μ into positive measures $\{\lambda_i\}$ such that $\mu = \sum_{i=1}^K \lambda_i$, $r(\lambda_i) = x_i$, and $\lambda_i(X) = \lambda(\{x_i\})$. Since λ is not convex partitional, for some i, j, $\mathrm{int}(\mathrm{ch}(\mathrm{supp}(\lambda_i))) \cap \mathrm{int}(\mathrm{ch}(\mathrm{supp}(\lambda_j))) \neq \emptyset$. It follows from Lemma 4 that there exist a measure α and $\varepsilon > 0$ such that $\alpha(X) = 0$, $-\lambda_j \le \alpha \le \lambda_i$, and $\int x \mathrm{d}\alpha_i = \varepsilon(x_j - x_i)$. If we fuse for each $k \neq i, j$ the measure λ_k to its barycenter and fuse $\lambda_i - \alpha_i \ge 0$ and $\lambda_j + \alpha_i/2 \ge 0$ to their barycenters (if two of these measures have the same barycenter, decrease the value of ε slightly), we obtain an element of G that is larger than

λ in the convex order, a contradiction. ■

Proof of Proposition 5. 1 Assume ν is not an extreme point. Then there are $\nu_1, \nu_2 \in F_\mu$ such that $\nu_1 \neq \nu_2$ and $\nu = \frac{1}{2}\nu_1 + \frac{1}{2}\nu_2$. Let $\{x_1, \ldots, x_m\}$ denote the support of ν and let μ_j^i be positive measures such that, for $i = 1, 2$, $\mu = \sum_j \mu_j^i$, $\nu_i(\{x_j\}) = \mu_j^i(X)$, and $r(\mu_j^i) = x_j$ (such measures exist by Lemma 2). Define $\mu_P^i := \sum_{j: x_j \in P} \mu_j^i$ and $u_P := \frac{1}{2}\mu_P^1 + \frac{1}{2}\mu_P^2 - \mu|_P$. It follows that $\sum_P u_P = 0$. Moreover, u_P is positive on $X \setminus P$ and, since $\mu_P^i \leq \mu$, it satisfies (1). Analogous arguments establish (2). Finally, $u_P(X) = 0$ and $\int x d u_P = 0$. Since $\nu_1 \neq \nu \neq \nu_2$, $u_P \neq 0$ for some P.

2 Conversely, suppose there is a solution $\{u_P\}$ to (1)–(5) satisfying $u_Q \neq 0$ for some $Q \in \mathcal{P}$. Enumerate the elements of $\mathcal{P}$ as $\mathcal{P} = \{P_1, \ldots, P_m\}$. Consider a fusion ν of μ that satisfies $\nu_P \prec \mu|_P$, the support of $\nu|_P$ is affinely independent and spans X, all points in the support of $\nu|_P$ are contained in a δ-ball around the barycenter of $\nu|_P$.

Choose $\varepsilon \in (0, 1)$ small enough and define, for $i = 1, \ldots, m$,

$$\mu_{P_i}^i := \mu|_{P_i} + \varepsilon u_{P_i}|_{X \setminus P_i}$$
$$\mu_P^i := \mu|_P - \varepsilon u_{P_i}|_P \text{ for } P \neq P_i$$

By construction, μ_P^i is a positive measure for all $P \in \mathcal{P}$ and $\sum_{P \in \mathcal{P}} \mu_P^i = \mu$. If we define ν_P^i to have support $\mathrm{supp}(\nu) \cap P$ and the same barycenter and mass as μ_P^i, then ν_P^i is a fusion of μ_P^i whenever $\varepsilon > 0$ is chosen small enough (if it wasn't a fusion, we could choose the δ-ball above smaller and redo the exercise). Consequently, $\nu^i := \sum_{P \in \mathcal{P}} \nu_P^i$ is a fusion of μ and $\nu^i \neq \nu$ for some i.

We claim that $\frac{1}{m}\sum_{i=1}^m \nu^i = \nu$, which implies that ν is not an extreme point. To establish the claim, we argue that $\frac{1}{m}\sum_{i=1}^m \nu^i(P) = \nu(P)$ and $\frac{1}{m}\sum_{i=1}^m \int_P x d\nu^i = \int_P x d\nu$ for all $P \in \mathcal{P}$. Since the support of ν^i and ν coincides on each P and is affinely independent, this implies that $\frac{1}{m}\sum_{i=1}^m \nu^i = \nu$. That is,

$$
\begin{aligned}
\frac{1}{m}\sum_{i=1}^{m}\nu^i(P_j) &= \mu|_{P_j} + \frac{1}{m}\varepsilon u_{P_j}|_{X\setminus P_j}(X) - \frac{1}{m}\varepsilon \sum_{i:i\neq j} u_{P_i}|_{P_j}(X) \\
&= \mu|_{P_j} + \frac{1}{m}\varepsilon u_{P_j}|_{X\setminus P_j}(X) + \frac{1}{m}\varepsilon u_{P_j}|_{P_j}(X) \text{ (since } \sum_i u_{P_i}|_{P_j} = 0 \text{)} \\
&= \mu|_{P_j} + \frac{1}{m}\varepsilon u_{P_j}(X) \\
&= \nu(P_j)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\frac{1}{m}\sum_{i=1}^{m}\int_{P_j} x d\nu^i &= \frac{1}{m}\sum_{i=1}^{m}\int_{P_j} x d\mu^i \\
&= \int_{P_j} x d\mu + \varepsilon/m \int_{X\setminus P_j} x d u_{P_i} - \varepsilon/m \sum_{i:i\neq j}\int_{P_j} x d u_{P_i} \\
&= \int_{P_j} x d\mu + \varepsilon/m \int_{X\setminus P_j} x d u_{P_i} + \varepsilon/m \int_{P_j} x d u_{P_j} \\
&= \int_{P_j} x d\nu.
\end{aligned}
$$

■

Proof of Corollary 1. Let $\mathscr{P}_1 \subseteq \mathscr{P}$ be a maximal subset such that for all $A \in \mathscr{P}_1$, $\mu|_A = \nu|_A$ and let $\mathscr{P}_2 := \mathscr{P} \setminus \mathscr{P}_1$.

Since $\mathscr{P}$ is a power diagram, there is a convex function $p\colon X \to \mathbb{R}$ such that for each the restriction of p to A is affine and if p is affine on $B \subseteq X$ then there is $A \in \mathscr{P}$ with $B \subseteq A$. We first adjust p to obtain another convex function q as follows. For any $A \in \mathscr{P}_1$, choose a Lipschitz-continuous function c_A that is strictly convex on the interior of A and equals 0 outside the interior of A (see Lemma 1). Then define

$$
q(x) := p(x) + \sum_{A \in \mathscr{P}_1} k_A c_A(x)
$$

where the constants k_A are chosen such that q is convex.

Now we define a Lipschitz-continuous function $u\colon X \to \mathbb{R}$ by

$$
u(x) := q(x) - \inf_{y \in \operatorname{supp}(\nu)} \|x - y\|.
$$

Note that by definition $u(x) = q(x)$ if $x \in \operatorname{supp}(\nu)$ and that $u(x) < p(x)$ for $x \notin \operatorname{supp}(\nu)$.

We claim that ν is the unique solution to

$$\max_{\lambda \in F_\mu} \int u(x)\, \mathrm{d}\lambda(x) \tag{L}$$

Step 1: *ν is a solution to problem* (L).

By construction, $\int u(x)\, \mathrm{d}\nu(x) = \int q(x)\, \mathrm{d}\nu(x)$. Moreover, $\int q(x)\, \mathrm{d}\nu = \int q(x)\, \mathrm{d}\mu$ since for each $A \in \mathscr{P}$, either $\mu|_A = \nu|_A$ or $\nu|_A \in F_{\mu|_A}$ and p is affine on A. Therefore,

$$\int u(x)\, \mathrm{d}\nu(x) = \int q(x)\, \mathrm{d}\mu(x). \tag{$A4$}$$

On the other hand, since q is convex and $q \geq u$, for any $\lambda \in F_\mu$ we obtain

$$\int u(x)\, \mathrm{d}\lambda(x) \leq \int q(x)\, \mathrm{d}\lambda(x) \leq \int q(x)\, \mathrm{d}\mu(x),$$

where the second inequality follows from Jensen's inequality. We conclude that ν is a solution to (L).

Step 2: *There is no other solution to problem* (L).

Let $\lambda \in F_\mu$ solve problem (L).

First, we establish that $\lambda|_P \preceq \mu|_P$ for all $P \in \mathscr{P}$: Suppose not and let D_x denote the dilations that carry λ to μ. Then there is a set of positive ν–measure B such that the dilations D_x for all $x \in B$ have the property that $\operatorname{supp} D_x$ is not contained in an element of $\mathscr{P}$. Therefore, q is not affine on $\operatorname{supp} D_x$, and we obtain $\int q(x)\, \mathrm{d}\lambda < \int \int q(y)\, \mathrm{d}D_x\, \mathrm{d}\lambda(x) = \int q(x)\, \mathrm{d}\mu(x)$. Therefore, λ is not a solution, a contradiction.

The same argument establishes $\lambda|_A = \mu|_A$ for all $A \in \mathscr{P}_1$ since q is strictly convex on the interior of A.

Next, we claim that $\operatorname{supp} \lambda \subseteq \operatorname{supp} \nu$. If not, there is $A \in \mathscr{P}_2$ and $x \in A \cap \operatorname{supp} \nu$ with $x \notin \operatorname{supp} \lambda$. Then $\int u(x)\, \mathrm{d}\lambda(x) < \int q(x)\, \mathrm{d}\lambda(x)$ since $u(x) < q(x)$ for all $x \notin \operatorname{supp} \nu$. This implies

$$\int u(x)\, \mathrm{d}\lambda(x) < \int q(x)\, \mathrm{d}\lambda(x) \leq \int q(x)\, \mathrm{d}\mu(x) = \int u(x)\, \mathrm{d}\nu(x), \tag{$A5$}$$

where the second inequality follows since p is convex and $\lambda \in F_\mu$, and the equality follows from ($A4$). We conclude that λ does not solve problem (L), a contradiction.

Finally, since the support of ν is affinely independent on each $A \in \mathscr{P}_2$, this yields $\nu = \lambda$. ∎

B Sufficient conditions for extremal fusions

We next explore sufficient conditions for a finitely supported measure ν to be an extreme point of F_μ. We use these conditions to verify that a particular fusion that is not Lipschitz-exposed is an exposed point.

Definition B.1. Given a positive measure μ and a partition $\mathscr{P}$ of X into finitely many convex subsets, we say that a collection of measures $\{u_P\}_{P\in\mathscr{P}}$ is a *feasible flow for $\mathscr{P}$* if it satisfies

$$0 \le u_P|_{X\setminus P} \le \mu|_{X\setminus P} \tag{1}$$

$$-\mu|_P \le u_P|_P \le 0 \tag{2}$$

$$u_P(X) = 0 \tag{3}$$

$$\int_X x\, du_P = 0 \tag{4}$$

$$\sum_P u_P = 0 \tag{5}$$

Proposition 5. *Fix an absolutely continuous measure μ and let $\mathscr{P}$ be a partition of X into finitely many convex sets.*

1. *Suppose that $\nu \in F_\mu$ satisfies $\nu|_P \preceq \mu_P$ for all $P \in \mathscr{P}$ and that the support of $\nu|_P$ is affinely independent. If $u_P \equiv 0$ for all $P \in \mathscr{P}$ is the unique feasible flow for $\mathscr{P}$ then ν is an extreme point of F_μ.*

2. *Suppose there exists a non-zero feasible flow for $\mathscr{P}$. Then there exists a fusion $\nu \in F_\mu$ that is not an extreme point of F_μ and that satisfies*

   (a) *For each $P \in \mathscr{P}$, $\nu|_P \preceq \mu|_P$; and*

   (b) *The support of $\nu|_P$ is affinely independent.*

Proof of Proposition 5. Part 1: Assume ν is not an extreme point. Then there are $\nu_1, \nu_2 \in F_\mu$ such that $\nu_1 \neq \nu_2$ and $\nu = \frac{1}{2}\nu_1 + \frac{1}{2}\nu_2$. Let $\{x_1, \ldots, x_m\}$ denote the support of ν and let μ_j^i be positive measures such that, for $i = 1, 2$, $\mu = \sum_j \mu_j^i$, $\nu_i(\{x_j\}) = \mu_j^i(X)$, and $r(\mu_j^i) = x_j$ (such measures exist by Lemma 2). Define $\mu_P^i := \sum_{j:x_j\in P} \mu_j^i$ and $u_P := \frac{1}{2}\mu_P^1 + \frac{1}{2}\mu_P^2 - \mu|_P$. It follows that $\sum_P u_P = 0$. Moreover, u_P is positive on $X \setminus P$ and, since $\mu_P^i \le \mu$, it satisfies (1). Analogous arguments establish (2). Finally, $u_P(X) = 0$ and $\int x \, \mathrm{d}u_P = 0$. Since $\nu_1 \ne \nu \ne \nu_2$, $u_P \ne 0$ for some P. Hence, there is a non-zero feasible flow.

Part 2: Conversely, suppose there is a solution $\{u_P\}$ to (1)–(5) satisfying $u_Q \ne 0$ for some $Q \in \mathscr{P}$. Enumerate the elements of $\mathscr{P}$ as $\mathscr{P} = \{P_1, ..., P_m\}$. Consider a fusion ν of μ that satisfies $\nu_P \prec \mu|_P$, the support of $\nu|_P$ is affinely independent and spans X, all points in the support of $\nu|_P$ are contained in a δ-ball around the barycenter of $\nu|_P$.

Choose $\varepsilon \in (0,1)$ small enough and define, for $i = 1, ..., m$,

$$\mu_{P_i}^i := \mu|_{P_i} + \varepsilon u_{P_i}|_{X \setminus P_i}$$
$$\mu_P^i := \mu|_P - \varepsilon u_{P_i}|_P \text{ for } P \ne P_i$$

By construction, μ_P^i is a positive measure for all $P \in \mathscr{P}$ and $\sum_{P \in \mathscr{P}} \mu_P^i = \mu$. If we define ν_P^i to have support $\mathrm{supp}(\nu) \cap P$ and the same barycenter and mass as μ_P^i, then ν_P^i is a fusion of μ_P^i whenever $\varepsilon > 0$ is chosen small enough (if it wasn't a fusion, we could choose the δ-ball above smaller and redo the exercise). Consequently, $\nu^i := \sum_{P \in \mathscr{P}} \nu_P^i$ is a fusion of μ and $\nu^i \ne \nu$ for some i.

We claim that $\frac{1}{m}\sum_{i=1}^m \nu^i = \nu$, which implies that ν is not an extreme point. To establish the claim, we argue that $\frac{1}{m}\sum_{i=1}^m \nu^i(P) = \nu(P)$ and $\frac{1}{m}\sum_{i=1}^m \int_P x d\nu^i = \int_P x d\nu$ for all $P \in \mathscr{P}$. Since the support of ν^i and ν coincides on each P and is affinely independent, this implies that $\frac{1}{m}\sum_{i=1}^m \nu^i = \nu$. That is,

$$\begin{aligned}
\frac{1}{m}\sum_{i=1}^m \nu^i(P_j) &= \mu|_{P_j} + \frac{1}{m}\varepsilon u_{P_j}|_{X \setminus P_j}(X) - \frac{1}{m}\varepsilon \sum_{i:i \ne j} u_{P_i}|_{P_j}(X) \\
&= \mu|_{P_j} + \frac{1}{m}\varepsilon u_{P_j}|_{X \setminus P_j}(X) + \frac{1}{m}\varepsilon u_{P_j}|_{P_j}(X) \text{ (since } \sum_i u_{P_i}|_{P_j} = 0 \text{)} \\
&= \mu|_{P_j} + \frac{1}{m}\varepsilon u_{P_j}(X) \\
&= \nu(P_j)
\end{aligned}$$

Also,

$$
\begin{aligned}
\frac{1}{m}\sum_{i=1}^{m}\int_{P_j} x d\nu^i &= \frac{1}{m}\sum_{i=1}^{m}\int_{P_j} x d\mu^i \\
&= \int_{P_j} x d\mu + \varepsilon/m \int_{X\setminus P_j} x d u_{P_i} - \varepsilon/m \sum_{i:i\neq j}\int_{P_j} x d u_{P_i} \\
&= \int_{P_j} x d\mu + \varepsilon/m \int_{X\setminus P_j} x d u_{P_i} + \varepsilon/m \int_{P_j} x d u_{P_j} \\
&= \int_{P_j} x d\nu.
\end{aligned}
$$

■

Lemma 6. *Suppose that $P \in \mathcal{P}$ is the intersection of a half-space and X, i.e., $P = H \cap X$ for some half-space of $\mathbb{R}^n$. If $\{u_P\}$ is a feasible flow then $u_P|_{P'} = u_{P'}|_P = 0$ for all P'.*

If u is a feasible flow then $\int x \, \mathrm{d}u_P = 0$ and therefore $\int x \, \mathrm{d}[-u_P|_P] = \sum_{P' \neq P} \int x \, \mathrm{d}[u_P|'_P]$. The result follows since if $u_P|_{P'} \neq 0$ then the barycenter of $-u_P|_P$ lies in the interior of P, hence in the interior of the half-space H, whereas the barycenter of $\sum_{P' \neq P} u_P|_{P'}$ lies in the complement of this half-space. This violates $\int x \, \mathrm{d}u_P = 0$, therefore $u_P|_{P'} = 0$.

Proposition 5 and Lemma 6 imply that the fusion we construct in our second example is an extreme point.

Lemma 7. *Let μ be an absolutely continuous measure with full support on X.*

Let $\mathcal{P}$ be a partition of X into finitely many convex sets and suppose there is a non-zero feasible flow $\{u_P\}_{P\in\mathcal{P}}$ for $\mathcal{P}$. Let $m_{PQ} := u_P|_Q(X)$ and $x_{PQ} := \int x d u_P|_Q$. For all $P, Q \in \mathcal{P}$ with $m_{PQ} > 0$, let $b_{PQ} := \frac{x_{PQ}}{m_{PQ}}$.

If $\mathcal{P}'$ is a partition of X that approximates $\mathcal{P}$ in the sense that for each $P \in \mathcal{P}$ there is $P' \in \mathcal{P}'$ such that $P' \subseteq P$ and, for all $Q \in \mathcal{P}$, $b_{QP} \in \operatorname{int} P'$ then there is a non-zero feasible flow for $\mathcal{P}'$.

This result implies that if $\nu \in F_\mu$ is obtained from a partition $\mathcal{P}$ by collapsing the mass on each cell to its barycenter and if ν is not an extreme point, then any fusion ν' obtained from a closeby partition $\mathcal{P}'$ is also not an extreme point.

Proof. Let k denote the number of partition elements in $\mathscr{P}$ and fix $\delta > 0$ small enough. For all $P \in \mathscr{P}$ and all $Q \in \mathscr{P} \setminus \{P\}$, let P' and Q' be the corresponding partition elements in $\mathscr{P}'$ that approximate P and Q, respectively. Let $\tilde{u}_{Q'P'}$ be a non-negative measure with $\tilde{u}_{Q'P'} \leq \frac{1}{k}\mu|_{P'}$, $\tilde{u}_{Q'P'}(X) = \delta m_{QP}$, and barycenter $\frac{1}{\delta m_{QP}} \int x d\tilde{u}_{Q'P'} = b_{QP}$ (if $m_{QP} > 0$). Such measures exist by Lemma 3 whenever $\delta > 0$ is chosen small enough. For all $P', Q' \in \mathscr{P}$ for which $\tilde{u}_{P'Q'}$ has not been defined this way, let $\tilde{u}_{P'Q'}$ be the zero measure.

For any $P' \in \mathscr{P}'$, define $\tilde{u}_{P'P'} = -\sum_{Q' \in \mathscr{P}' \setminus \{P'\}} \tilde{u}_{Q'P'}$ and define $\tilde{u}_{P'} = \sum_{Q' \in \mathscr{P}'} \tilde{u}_{P'Q'}$.

Then $\sum_{P' \in \mathscr{P}'} \tilde{u}_{P'} = \underline{0}$ by construction. Moreover, since $\sum_{Q \in \mathscr{P}} u_Q|_P$ is the zero measure, we obtain for any $P \in \mathscr{P}$ and its approximating partition cell $P' \in \mathscr{P}'$,

$$\delta u_P|_P(X) = -\delta \sum_{Q \in \mathscr{P} \setminus \{P\}} u_Q|_P(X) = -\sum_{Q' \in \mathscr{P}' \setminus \{P'\}} \tilde{u}_{Q'P'}(X) = \tilde{u}_{P'P'}(X).$$

This yields

$$\tilde{u}_{P'}(X) = \sum_{Q' \in \mathscr{P}'} \tilde{u}_{P'Q'}(X) = \sum_{Q \in \mathscr{P}} \delta u_P|_Q(X) = \delta u_P(X) = 0.$$

Analogous arguments establish that $\int x d\tilde{u}_{P'} = 0$. Since $0 \leq \tilde{u}_{P'}|_{X \setminus P'} \leq \mu|_{X \setminus P'}$ and $-\mu|_{P'} \leq \tilde{u}_{P'}|_{P'} \leq 0$, it follows that $\{\tilde{u}\}$ is a non-zero feasible flow for $\mathscr{P}'$. ■

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