Strong duality in monopoly pricing

Andreas Kleiner and Alejandro Manelli*

Department of Economics,
W.P. Carey School of Business
Arizona State University
Tempe, Az 85287

March 18, 2019

Abstract

The main result in Daskalakis, Deckelbaum, and Tzamos (2017) establishes strong duality in the monopoly problem with an argument based on transportation theory. We provide a short, alternative proof using linear programming.

The question of how a monopoly should sell its wares to maximize profit is, mathematically, an optimization program in infinite dimensions because the optimization variable, the selling mechanism, is a function of the buyer's valuations, a continuum. When private information is one-dimensional, the answer was provided, famously, by Myerson (1981): the form of the optimal mechanism is independent of the seller's prior distribution of buyer's valuations. When private information is multi-dimensional, as it is usual when the monopoly has multiple objects to sell, the form of the optimal mechanism depends fundamentally on the prior distribution (Manelli and Vincent 2007). This highlights the need for methods to identify the optimal mechanism for classes of prior distributions. Daskalakis, Deckelbaum, and Tzamos (2017), henceforth DDT, is an important contribution in this regard. Their main result, Theorem 2, establishes that the value of the primal program equals the value of its dual; that

*We are grateful to Daniel Vincent for bringing to our attention the relationship between duality in transportation and in linear programming, and to John Thannasoulis and the referees for useful comments on a previous draft.

is to say strong duality holds. Their proof is complex and long; its outline follows the proof of the Monge-Kantorovich duality presented in Villani (2009), with technical variations due to convexity constraints.

We provide an alternative proof of strong duality as a direct consequence of duality in linear programming. To this end we use a result in Gretsky, Ostroy, and Zame (2002). In the rest of this essay we present the economic environment, introduce preliminaries, set the primal and dual programs, state and prove our theorem, and conclude with remarks.

First, we offer a sketch of the economic question and its formalization. A fuller treatment can be found in McAfee and McMillan (1988) and in Rochet and Choné (1998). The N-vector of buyer's valuations $x \in X = [0,1]^N$, one valuation per available good, is private information of the buyer and distributed according to a density function f that is continuous and differentiable with bounded derivative. The buyer's preferences are linear in value and money. The seller maximizes her expected revenue. A direct revelation mechanism is a pair of functions specifying probabilities of trade and a transfer payment for every possible buyer's report such that the buyer's optimal choice is to report valuations truthfully. Using a well-known characterization of incentive compatibility (for instance Rochet (1987), Rochet and Choné (1998)), a mechanism may be defined by a continuous, convex utility function $u : X \to \mathbb{R}$ whose gradient ∇u, where defined, is the vector of probabilities of trade for each of the N goods. As a vector of probabilities, $\mathbf{0} \leq \nabla u \leq \mathbf{1}$. Hence u is non-decreasing and satisfies a 1-Lipschitz constraint, $\forall x, y \in X,\ u(x) - u(y) \leq \|x - y\|_1$. The function $u(x)$ is the buyer's expected utility when her valuation is x. Individual rationality, the requirement of a non-negative utility for every type of buyer, is captured by the equation $u(\mathbf{0}) = 0$ which also implies that no unnecessary surplus will be left for the buyer. The seller's revenue when the buyer reports x is $\nabla u(x) \cdot x - u(x)$.

The seller's problem is

$$
\begin{aligned}
\sup_{u} \quad & \int_X [\nabla u(x) \cdot x - u(x)] f(x)\, dx \\
\text{s.t.} \quad & u \text{ is continuous, non-decreasing, convex} && (1) \\
& \forall x, y \in X, u(x) - u(y) \leq \|x - y\|_1 && (2) \\
& u(\mathbf{0}) = 0, && (3)
\end{aligned}
$$

where the objective function is the seller's expected revenue; the first constraint follows from incentive compatibility; the second one is feasibility, the 1-Lipschitz constraint; and the last one corresponds to individual rationality. (The optimization corresponds to equation (1) in DDT with the immaterial difference that DDT, while noting that $u(\mathbf{0}) = 0$ suffices to guarantee individual rationality, write $u(x) - u(\mathbf{0})$ in the objective function and maximize over all continuous, convex, non-decreasing functions u. We chose instead to include $u(\mathbf{0}) = 0$ in the feasible set.) Per DDT equations (2), (3), and DDT Theorem 1, there is a signed Radon measure μ on X such that $\mu(X) = 0$ and for all u satisfying (1)-(3),

$$\int_X u \, d\mu = \int_X [\nabla u(x) \cdot x - u(x)] f(x) \, dx. \tag{4}$$

(Expressions related to (4) were used in similar contexts by McAfee and McMillan (1988) and Rochet and Choné (1998).) This new form of the objective function is more convenient than the original one because incentive compatibility is mainly expressed as the convexity of u and because μ is a bounded linear functional on the space of continuous functions to which u belongs.

Second, we introduce the relevant spaces, an order on each space, and functions to represent the constraints. Let $C(X)$ be the space of continuous, real-valued functions on X with the sup norm $\|\cdot\|$. Let $\Gamma(X)$ be its dual, the space of signed Radon measures with the variation norm $\|\cdot\|_{TV}$. Bilinear forms are represented by $\langle \cdot, \cdot \rangle$.

To express the inequality constraints, we use order relations that are preserved under scalar multiplication and vector addition. Any such order defines a pointed convex cone, and any pointed convex cone defines such an order (Aliprantis and Border 2006, pages 312-313). We use the pointed, convex cones below as the non-negative cones of $C(X)$, $C(X \times X)$, $\Gamma(X)$, and $\Gamma(X \times X)$ respectively,

$$\begin{aligned}
\mathcal{U} &= \{u \in C(X) \mid u \text{ is non-decreasing and convex}; u(\mathbf{0}) = 0\} \\
\mathcal{V} &= \{g \in C(X \times X) \mid \forall (x, y) \in X \times X, g(x, y) \geq 0\} \\
\mathcal{U}^* &= \{\mu \in \Gamma(X) \mid \forall u \in \mathcal{U},\ \langle u, \mu \rangle \geq 0\} \\
\mathcal{V}^* &= \{\gamma \in \Gamma(X \times X) \mid \forall g \in \mathcal{V},\ \langle g, \gamma \rangle \geq 0\}.
\end{aligned}$$

With a slight abuse of notation, we use the symbol "$\geq$" to represent the various orders; the meaning is determined by the elements being compared. Thus we write $u \geq 0$ if $u \in \mathcal{U}$, $\mu \geq 0$ if $\mu \in \mathcal{U}^*$, $g \geq 0$ if $g \in \mathcal{V}$ and $\gamma \geq 0$ if $\gamma \in \mathcal{V}^*$. (In contrast to DDT's use of the same symbol, our definition of $\mathcal{U}$ includes the constraint $u(\mathbf{0}) = 0$. This difference is immaterial in the seller's problem and we will show later that it is equally inessential in the dual program.)

We use the two functions below to write the constraints cleanly, $A : C(X) \to$

$C(X \times X)$ and $b : X \times X \to \mathbb{R}$ defined by

$$Au(x,y) = u(x) - u(y) \tag{5}$$

$$b(x,y) = \|x-y\|_1. \tag{6}$$

Third, we state the primal and dual programs and verify that they coincide with the corresponding programs in DDT. The primal program is

$$\begin{aligned} \sup_{u\geq 0} \quad & \langle \mu, u\rangle \\ \text{s.t.} \quad & Au \leq b. \end{aligned} \tag{7}$$

The objective function is (4), $u \geq 0$ means $u \in \mathcal{U}$ and captures (1) and (3); and (7) is the 1-Lipschitz constraint (2). Thus this program is the seller's problem and corresponds to DDT-Theorem 1-(4).

Gretsky, Ostroy, and Zame (2002) define the dual

$$\begin{aligned} \inf_{\gamma\geq 0} \quad & \langle \gamma, b\rangle \\ \text{s.t.} \quad & A^*\gamma \geq \mu, \end{aligned} \tag{8}$$

where $A^* : \Gamma(X \times X) \to \Gamma(X)$ is the adjoint of A. This is the dual in DDT, Theorem 2. To see this consider first the objective function. Recall that $\gamma \geq 0$ means $\gamma \in \mathcal{V}^*$. The set $\mathcal{V}^*$ is the set of positive measures (Dunford and Schwartz 1988, page 262) which DDT denote by $\Gamma_+(X \times X)$. The function b is defined in (6). Thus our objective function is the function in the right side of DDT-Theorem 2-(5). We now show that (8) is equivalent to the convex domination constraint in DDT. For any $u \in C(X), \gamma \in \Gamma(X \times X)$,

$$\begin{aligned} \langle u, A^*\gamma\rangle &= \langle Au, \gamma\rangle \\ &= \int [u(x) - u(y)]\ d\gamma \\ &= \int u(x)\, d\gamma_1 - \int u(y)\, d\gamma_2 \\ &= \int u(x)\, d(\gamma_1 - \gamma_2) \\ &= \langle u, \gamma_1 - \gamma_2\rangle, \end{aligned}$$

where the first line is the definition of the adjoint; the second line uses the definition of Au in (5); the third line follows when γ_1 and γ_2 are the marginal distributions of

γ, that is for any measurable $B \subseteq X$, $\gamma_1(B) = \gamma(B \times X)$ and $\gamma_2(B) = \gamma(X \times B)$; and the last two lines are notation. We conclude that $A^*\gamma = \gamma_1 - \gamma_2$ and hence (8) can be written as $\gamma_1 - \gamma_2 \geq \mu$. By definition of the relevant order, this means

$$\forall u \in \mathcal{U}, \int u \, d(\gamma_1 - \gamma_2) \geq \int u \, d\mu. \tag{9}$$

Inequality (9) holds also for any constant u: $\forall a \in \mathbb{R}$, $\int a \, d(\gamma_1 - \gamma_2) = \int a \, d\mu$ because $\gamma_1(X) = \gamma_2(X) = \gamma(X \times X)$ and $\mu(X) = 0$. Since (9) holds for any $u \in \mathcal{U}$ and for any constant function u, it holds for any continuous, non-decreasing, convex u even if $u(\mathbf{0}) \neq 0$. This is precisely $\gamma_1 - \gamma_2 \succeq_{cvx} \mu$ in DDT's notation.

Fourth, we state and prove our theorem.

Theorem. *The value of the dual equals the value of the primal and both primal and dual have an optimal solution.*

To prove the theorem, we parametrize the primal. For $g \in C(X \times X)$, define $F(g) = \{u \in \mathcal{U} : Au \leq g\}$ and $V(g) = \sup\{\langle \mu, u \rangle : u \in F(g)\}$. Thus $F(b)$ and $V(b)$ are the primal's feasible set and value.

We establish first that $V(b)$ is finite and there exists $u \in F(b)$ such that $\langle \mu, u \rangle = V(b)$. The set $F(b)$ is non-empty since it contains $u \equiv 0$. It is equicontinuous and uniformly bounded because every $u \in F(b)$ is 1-Lipschitz by equation (7) and $u(\mathbf{0}) = 0$. It is closed because uniform convergence preserves continuity, convexity, and monotonicity. The Arzelà-Ascoli Theorem (Dunford and Schwartz 1988, Theorem IV.6.7) implies that it is compact. Since the objective function is continuous, it attains a maximum.

It remains to prove that there is no duality gap and that the dual has an optimal solution. Theorem 1 in Gretsky, Ostroy, and Zame (2002) establishes that this is so if and only if the subdifferential of $-V(b)$ at b is non-empty. (Gretsky, Ostroy, and Zame define the primal as a minimization, and thus its value function is convex. We chose to represent the primal as a maximization given its underlying economic interpretation. It is therefore $-V(b)$ that is convex and that must have a non-empty subdifferential at b.) The lemma completes the proof of the theorem.

Lemma. *The subdifferential of $-V(b)$ at b is non-empty.*

Proof of Lemma. It suffices to show that $\frac{V(g)-V(b)}{\|g-b\|}$ is bounded above (Condition 3, page 266 in Gretsky, Ostroy, and Zame 2002). To this end, for any $\bar{u} \in F(g)$ we construct $u \in F(b)$ such that $\|\bar{u} - u\| \leq \|g - b\|$.

Given $g \in C(X \times X)$ and $\bar{u} \in F(g)$, define for $x \in X$

$$u(x) = \inf_{z \in X} \{\bar{u}(z) + \|z - x\|_1\} \tag{10}$$

We verify that $u \in F(b)$. By Rockafellar (1974) Theorem 1, u is convex because $\bar{u}(z) + \|z - x\|_1$ is convex in (z, x).

For the remainder of the proof, fix any $x \in X$. Since $\bar{u}$ is continuous, the infimum is attained in (10). Therefore, pick $z \in X$ so that $u(x) = \bar{u}(z) + \|z - x\|_1$. Then

$$\forall x' \in X,\ u(x') - u(x) \leq \bar{u}(z) + \|z - x'\|_1 - \bar{u}(z) - \|z - x\|_1 \leq \|x' - x\|_1 = b(x', x)$$

where the first inequality follows by (10) for $u(x')$, and the second one by the triangle inequality. This shows that $Au \leq b$.

To see that u is non-decreasing, note that for any $x' \leq x$

$$u(x') \leq \bar{u}(x' \wedge z) + \|x' - (x' \wedge z)\|_1 \leq \bar{u}(z) + \|x - z\|_1 = u(x)$$

where $x' \wedge z$ is the componentwise minimum of x' and z. The first inequality follows by (10) for $u(x')$. The second inequality follows because $\bar{u}$ is non-decreasing and because $\|x' - (x' \wedge z)\|_1 \leq \|x - z\|_1$ (if $x'_i \leq z_i$, $x'_i - (x'_i \wedge z_i) = 0$; if $x'_i > z_i$, then $x'_i - z_i \leq x_i - z_i$ since $x' \leq x$).

Since $u(\mathbf{0}) = 0$, we have verified that $u \in F(b)$.

Finally, $0 \leq \bar{u}(x) - u(x) = \bar{u}(x) - \bar{u}(z) - \|z - x\|_1 \leq g(x, z) - b(x, z)$ where the first inequality follows by (10) for $u(x)$ and the second one because $\bar{u} \in F(g)$ and $b(x, z) = \|z - x\|_1$. This shows that $\|\bar{u} - u\| \leq \|g - b\|$. Since μ is bounded, $\langle \mu, \bar{u} \rangle - \langle \mu, u \rangle \leq \|\mu\|_{TV} \|g - b\|$ and thus $V(g) - V(b) \leq \|\mu\|_{TV} \|g - b\|$. Hence, $\frac{V(g) - V(b)}{\|g - b\|}$ is bounded above and the subdifferential of $-V$ is non-empty at b. $\square$

Lastly, we conclude with some comments.

A well known sufficient condition for strong duality is the existence of $u \in \mathcal{U}$ such that $b - Au$ lies in the interior of $\mathcal{V}$ (see, for instance, Luenberger 1969, Theorem 1, page 217). In some programs, this interiority condition fails because the analogue of the cone $\mathcal{V}$ has an empty interior. In our application $\mathcal{V}$ has a non-empty interior but the condition still fails because the operator A places Au on the boundary of $\mathcal{V}$: $\forall u \in \mathcal{U}, \forall x, y \in X \times X$ with $x = y$, $Au(x, y) = u(x) - u(y) = 0$.

Some of the ideas in Gretsky, Ostroy, and Zame (2002) appear in Rockafellar (1974), Theorems 7 and 16 which characterize strong duality in general convex programs using variational methods, and in Theorem 18 which states sufficient conditions for strong duality. Mitter (2008) also considers general convex programs and provides, among other things, sufficient conditions for strong duality based on perturbations along feasible directions.

With the inclusion of $u(\mathbf{0}) = 0$ in the definition of $\mathcal{U}$, $\mathcal{U}$ becomes a pointed cone, and therefore Gretsky, Ostroy, and Zame (2002) may be applied directly. Shapiro (2010), Proposition 2.5, obtains an analogous result to Gretsky, Ostroy, and Zame's for linear programs with cone constraints where the cones need not be pointed.

The measure μ in the objective function of the primal was derived from economic primitives. Our theorem applies to any signed Radon measure μ, not just those obtained from economic primitives.

References

Aliprantis, Charalambos D. and Kim C. Border (2006): Infinite dimensional analysis, a hitchhiker's guide. Third edition. Berlin-Heidelberg: Springer-Verlag.

Daskalakis, Constantinos, Alan Deckelbaum, and Christos Tzamos (2017): Strong Duality for a Multiple-Good Monopolist, *Econometrica* 85.3, pp. 735–767.

Dunford, Nelson and Jacob T. Schwartz (1988): Linear Operators, Part 1: General Theory. John Wiley and Sons.

Gretsky, N. E., J. M. Ostroy, and William R. Zame (2002): Subdifferentiability and the Duality Gap, *Positivity* 6, pp. 261–274.

Luenberger, David G. (1969): Optimization by Vector Space Methods. New York: John Wiley & Sons.

Manelli, Alejandro M. and Daniel R. Vincent (2007): Multidimensional mechanism design: Revenue maximization and the multiple good monopoly, *Journal of Economic Theory* 137, pp. 153–185.

McAfee, R. Preston and John McMillan (1988): Multidimensional incentive compatibility and mechanism design, *Journal of Economic Theory* 106, pp. 335–354.

Mitter, Sanjoy K. (2008): Convex Optimization in Infinite Dimensional Spaces. In: *Recent Advances in Learning and Control*. Ed. by Vincent D. Blondel, Stephen P. Boyd, and Hidenori Kimura. Springer London, pp. 161–179.

Myerson, Roger (1981): Optimal auction design, *Mathematics of Operations Research* 6, pp. 58–73.

Rochet, Jean-Charles (1987): A necessary and sufficient condition for rationalizability in a quasi-linear context, *Journal of Mathematical Economics* 16.2, pp. 191–200.

Rochet, Jean-Charles and Philippe Choné (1998): Ironing, Sweeping, and Multidimensional Screening, *Econometrica* 66.4, pp. 783–826.

Rockafellar, R. Tyrrell (1974): Conjugate Duality and Optimization. CBMS-NSF Regional Conference Series in Applied Mathematics. Philadelphia: SIAM.

Shapiro, Alexander (2010): On duality theory of conic linear problems. In: *Semi-Infinite Programming: Recent Advances*. Ed. by M. A. Goberna and M. A. Lopez. Springer, U.S.A.

Villani, C. (2009): Optimal Transport: Old and New. Springer-Verlag, Berlin, Heidelberg.