

Convex Choice^{*}

Navin Kartik[†]

Andreas Kleiner[‡]

June 27, 2024

Abstract

For multidimensional Euclidean type spaces, we study convex choice: from any choice set, the set of types that make the same choice is convex. We establish that, in a suitable sense, this property characterizes the sufficiency of local incentive constraints. Convex choice is also of interest more broadly. We tie convex choice to a notion of directional single-crossing differences (DSCD). For an expected-utility agent choosing among lotteries, DSCD implies that preferences are either one-dimensional or must take the affine form that has been tractable in multidimensional mechanism design.

^{*}Zhihui Wang and Yangfan Zhou provided excellent research assistance. Kleiner acknowledges financial support from the German Research Foundation (DFG) through Germany's Excellence Strategy - EXC 2047/1 - 390685813, EXC 2126/1-390838866 and the CRC TR-224 (Project B01).

[†]Department of Economics, Columbia University. Email: nkartik@gmail.com.

[‡]Department of Economics, University of Bonn. Email: andykleiner@gmail.com.

1. Introduction

A fundamental result in one-dimensional signaling and mechanism design is that the Spence-Mirrlees single-crossing property reduces incentive compatibility (IC) to local IC. Local IC requires only that “nearby” types have no incentive to mimic each other. Sufficiency of local IC is central to the tractability, and hence success, of the aforementioned paradigms, particularly for finding optimal mechanisms.

Our paper is concerned with multidimensional environments. Consider an agent with utility function $u(a, \theta)$, where $a \in A$ is an allocation or choice variable and $\theta \in \Theta$ is the agent’s type or preference parameter. Let $\Theta \subset \mathbb{R}^n$ be convex. We study a property we call *convex choice*: for any choice set $B \subset A$ and any allocation $a \in B$, the set of types for which a is (uniquely) optimal is convex.

Convex choice is, in our view, a compelling desideratum. There is a sense in which convex choice characterizes the sufficiency of local IC. More precisely, absent indifferences, convex choice characterizes when IC between any two types is equivalent to local IC along the line segment between those two types ([Proposition 1](#)). The necessity of convex choice for such “integration up” of local IC suggests that absent convex choice, a mechanism design model is unlikely to be tractable. Convex choice is also relevant for reasons beyond the sufficiency of local IC. For instance, it offers a natural generalization of “interval equilibria” in multidimensional versions of [Crawford and Sobel’s \(1982\)](#) famous cheap-talk model.

[Proposition 2](#) shows that convex choice is essentially equivalent to a form of single crossing that we term *directional single-crossing differences* (DSCD). Modulo details, DSCD requires that for any two (undominated) allocations, the set of types that prefer one to the other is defined by a half-space. Importantly, the orientation of the half-space can vary with the allocation pair. When $A \subset \mathbb{R}^n$, leading examples of DSCD are the quadratic-loss utility $u(a, \theta) = -\|a - \theta\|^2$ and the constant-elasticity-of-substitution utility $u(a, \theta) = \left(\sum_{i=1}^d (a_i)^r \theta_i\right)^s$ with parameters $r \in \mathbb{R}$ and $s > 0$.

Studies of multidimensional mechanism design frequently assume the *affine form* $u(a, \theta) = v(a) \cdot \theta + w(a)$ (e.g., [Armstrong, 1996](#); [Rochet and Choné, 1998](#); [Manelli and Vincent, 2007](#); [Kleiner, 2022](#)). Our approach delivers a novel perspective on this specification. [Propositions 3 and 4](#) show that if the convex choice/DSCD requirement is strengthened to hold for expected utility over allocation lotteries, then—subject to some regularity conditions—either the setting is effectively one-dimensional, or preferences can be represented in the affine form. A provocative suggestion, then, is that when allocation lotteries are important, multidimensional mechanism design is unlikely to be tractable beyond the affine form. Lotteries may be

inescapable when stochastic mechanisms are allowed or there are other agents whose private information also influences a deterministic mechanism.

Related Literature. Convex choice is closely related to a notion of [Grandmont \(1978\)](#), as elaborated below. Our [Proposition 2](#) is thus also similar to his characterization Proposition (p. 322). Beyond that, our work is distinct; in particular, his interest was in social choice, he did not relate convex choice to the sufficiency of local IC or to applications like cheap talk, and he had no analog to our characterization of DSCD over lotteries.

Our work also relates to [Carroll \(2012\)](#), [Kartik, Lee, and Rappoport \(2023\)](#), and [McAfee and McMillan \(1988\)](#). We detail these connections subsequently.

2. Convex Choice and Incentive Compatibility

There is an agent with type $\theta \in \Theta \subset \mathbb{R}^n$, where Θ is convex.¹ We write $\theta_{(i)}$ for the i -th coordinate of θ , and $\theta > \theta'$ if $\theta_{(i)} \geq \theta'_{(i)}$ for all i with strict inequality for some i . We denote the *line segment* between types θ' and θ'' by

$$\ell(\theta', \theta'') := \{\theta : \exists \lambda \in [0, 1] \text{ with } \theta = \lambda\theta' + (1 - \lambda)\theta''\}.$$

The agent must take an action, or choose an allocation, $a \in A$. The agent's preferences are given by the utility function $u : A \times \Theta \rightarrow \mathbb{R}$.

Definition 1. u has *convex choice* if for all $B \subset A$ and $a \in B$,

$$\left\{ \theta : \{a\} = \arg \max_{b \in B} u(b, \theta) \right\} \text{ is convex.}$$

In other words, convex choice requires that from any choice set, the set of types that find an action uniquely optimal is convex. It is equivalent to only consider all binary choice sets, as the intersection of convex sets is convex. Convex choice implies that the preferences of any type $\theta \in \ell(\theta', \theta'')$ are “between” those of θ' and θ'' , in the sense of [Grandmont \(1978\)](#).²

Convex choice is related to but different from [Kartik et al.'s \(2023\)](#) “interval choice”. They require that if $\theta'' > \theta' > \theta$ and some action is optimal for θ and θ'' , then it is also

¹In this paper, the “ \subset ” symbol means “weak subset”.

²[Grandmont](#)'s betweenness notion is stated for binary relations that need not be transitive; so, in his setting, choice from non-binary choice sets may not be well-defined. His notion also imposes some requirements concerning indifference. Ignoring indifferences and assuming transitivity, convex choice is equivalent to the preferences of any $\theta \in \ell(\theta', \theta'')$ being between those of θ' and θ'' .

optimal for θ' .³ Absent indifferences, convex and interval choice are equivalent when $\Theta \subset \mathbb{R}$, but they are incomparable when Θ is multidimensional.⁴ Convex sets are, of course, an (alternative) salient generalization of one-dimensional intervals to multiple dimensions. We discuss yet another alternative, connected sets, in the [Conclusion](#).

To define our notions of incentive compatibility, let $N_\theta \subset \Theta$ denote an open neighborhood—simply neighborhood hereafter—of type θ in the relative topology of Θ . Throughout this paper, we focus on direct mechanisms $\Theta \rightarrow A$. Stochastic mechanisms are subsumed by taking A to be a lottery space.

Definition 2. A mechanism $m : \Theta \rightarrow A$ is

1. *incentive compatible* (IC) if $u(m(\theta), \theta) \geq u(m(\theta'), \theta)$ for all $\theta, \theta' \in \Theta$;
2. *locally IC* if for each $\theta \in \Theta$ there is a neighborhood $N_\theta \subset \Theta$ such that

$$\forall \theta' \in N_\theta : u(m(\theta), \theta) \geq u(m(\theta'), \theta) \text{ and } u(m(\theta'), \theta') \geq u(m(\theta), \theta'). \quad (\text{LIC})$$

IC is a standard and fundamental property. Our formulation of local IC follows [Carroll \(2012\)](#): for every type θ , there is a set of nearby types for which (i) θ cannot profitably mimic those types and (ii) those types cannot profitably mimic θ . Requirement (ii) here cannot be dispensed with.⁵ Plainly, IC implies local IC (because Θ is a neighborhood of every type). The converse is not true:

Example 1. $\Theta = [0, 1]$, $A = \{a', a''\}$, $u(a', \cdot) = 0$, $u(a'', \theta) = 1$ if $\theta \neq 1/2$, and $u(a'', 1/2) = 0$. Note that convex choice fails. The mechanism $m(\theta) = a'$ if $\theta \leq 1/2$ and $m(\theta) = a''$ if $\theta > 1/2$ is not IC. But it is locally IC: for every $\theta \neq 1/2$, there a neighborhood of θ on which $m(\cdot)$ is constant; whereas for type $1/2$, neither does it prefer to mimic any type nor does any type prefer to mimic it.

We will see that, in a suitable sense, convex choice is necessary for local IC to imply

³ Those authors use weak optimality whereas we use strict optimality; this difference is minor.

⁴ For example, let $A = \{a', a''\}$ and denote $D(\theta) := u(a', \theta) - u(a'', \theta)$. If $\Theta = \{\theta \in \mathbb{R}^2 : \theta_1 + \theta_2 = 1\}$, then interval choice trivially holds for any $D(\cdot)$ because no two types are ordered; whereas convex choice may fail. On the other hand, if $\Theta = [0, 1]^2$ and both $D((0, 0)) > 0$ and $D((1, 1)) > 0$, then interval choice requires $D(\cdot) \geq 0$; whereas convex choice holds if $D(\theta) > 0$ for $\theta_1 \geq \theta_2$ and $D(\theta) < 0$ otherwise.

⁵ Here is an example. $\Theta = [0, 1]$, $A = \{a', a''\}$, $u(a', \cdot) < u(a'', \cdot)$, and $m(\theta) = a'$ if and only if $\theta < 1/2$. For every type $\theta < 1/2$, there is $\varepsilon > 0$ such that all types in $[0, \theta + \varepsilon)$ receive the same allocation a' . Hence, for every θ , there is a neighborhood in which θ has no incentive to mimic any type in that neighborhood. Local IC fails, however, because any neighborhood of type $1/2$ contains some types strictly below, all of which would profitably mimic $1/2$. If we did not rule this out in local IC, then the example would contradict [Proposition 1](#). [Carroll \(2012, p. 669\)](#) makes an analogous point in his framework.

IC. While it is also sufficient when preferences are strict, the next example shows that indifferences are a threat.

Example 2. $\Theta = [0, 1]$, $A = \{a', a''\}$, and $u(a, \theta) = \mathbb{1}\{a = a'', \theta = 1\}$. Despite convex choice, the mechanism $m(\theta) = a''$ if and only if $\theta \leq 1/2$ is not IC. Yet it is locally IC: only type $\theta = 1$ can profitably mimic any other type, but $m(\cdot)$ is constant on $(1/2, 1]$.

The problem in [Example 2](#) is “thick” (but not total) indifferences. The next definition allows us to pinpoint the issue.

Definition 3. u has *regular indifferences* if for all $a', a'' \in A$ and $\theta', \theta'' \in \Theta$ and $\theta \in \ell(\theta', \theta'') \setminus \{\theta', \theta''\}$,

$$\left[u(a', \theta') > u(a'', \theta') \text{ and } u(a', \theta'') = u(a'', \theta'') \right] \implies u(a', \theta) > u(a'', \theta).$$

An implication of regular indifferences is that for any two actions and any line segment of types, either all those types are indifferent or at most one type in the (relative) interior is indifferent. Plainly, regular indifferences holds when there are no indifferences, whereas [Example 2](#) is a violation.

Proposition 1. *The following are equivalent:*

1. u has convex choice and regular indifferences;
2. for any line segment $\Theta' \subset \Theta$, if mechanism $m : \Theta' \rightarrow A$ is locally IC, then it is IC.

(All proofs are in the appendices.)

The proposition says that convex choice and regular indifferences jointly characterize when IC between any two types θ and θ' can be determined by just checking local IC along the line segment $\ell(\theta, \theta')$. Such “integration up” is a common strategy used to verify IC.

A corollary of [Proposition 1](#) is that convex choice and regular indifferences imply that on the full type space, local IC is sufficient for IC.⁶ The necessity of convex choice in [Proposition 1](#) owes to our seeking sufficiency of local IC on not only the full type space, but also lower-dimensional subsets—in particular, all line segments. We find this a natural desideratum: a “tractable” problem must remain tractable when restricted to lower dimensions. [Figure 1](#) illustrates why lower-dimensional problems are essential to [Proposition 1](#). The figure’s example violates convex choice, yet it can be shown that on the full type space, any

⁶ For, local IC on the full type space implies local IC on every line segment. Formally, if $m : \Theta \rightarrow A$ is locally IC, then for every $\ell(\theta, \theta')$, the mechanism $m' : \ell(\theta, \theta') \rightarrow A$ defined by $m'(\theta) = m(\theta)$ is locally IC.

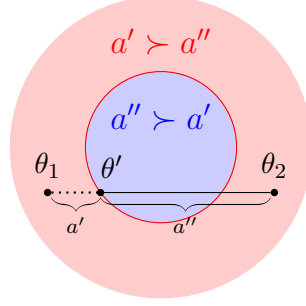


Figure 1: Θ is the large disc and $A = \{a', a''\}$. Blue (inner disc) types prefer a'' to a' ; red (outer region) types prefer a' to a'' . A mechanism that allocates a' on the dotted line segment (including to θ') and a'' on the solid one is locally IC, but not IC, on the line segment from θ_1 to θ_2 . Any such mechanism is not locally IC on Θ .

locally IC mechanism is IC (see [Proposition 8](#) in the Supplementary Appendix). However, as shown in the figure, there are mechanisms in which local IC holds along the line segment $\ell(\theta_1, \theta_2)$ even though θ_2 would mimic θ_1 .

Examples 1 and 2 are illustrative of the general argument for why both convex choice and regular indifferences are necessary for [Proposition 1](#)'s second statement. Here is a heuristic sketch for why convex choice is sufficient when there are no indifferences. Take any θ and θ' . Consider a fine grid of types $\theta = \theta_1, \dots, \theta_m = \theta'$ traversing the line segment $\ell(\theta, \theta')$. We can regard local IC as implying $u(m(\theta), \theta) \geq u(m(\theta_2), \theta)$, $u(m(\theta_2), \theta_2) \geq u(m(\theta_3), \theta_2)$, and $u(m(\theta_3), \theta_3) \geq u(m(\theta_2), \theta_3)$. Absent indifferences, all those inequalities are strict. Convex choice then implies $u(m(\theta), \theta) > u(m(\theta_3), \theta)$; otherwise, both θ and θ_3 strictly (absent indifferences) prefer $m(\theta_3)$ to $m(\theta_2)$, hence so must θ_2 , a contradiction. Iterating this logic, using the combination of local IC and convex choice each time, yields $u(m(\theta), \theta) > u(m(\theta_i), \theta)$ for all $i = 2, \dots, m$. Consequently, $u(m(\theta), \theta) > u(m(\theta'), \theta)$.

We can now compare our approach with that of [Carroll \(2012\)](#) in his “cardinal type space” analysis.⁷ [Proposition 1](#) subsumes [Carroll](#)'s Proposition 1. To see why, recall that [Carroll](#) assumes each $a \in A \subset \mathbb{R}^n$ is a probability vector over a finite set of n outcomes, and he associates types with their utility vectors over those outcomes. (He does not distinguish between types with the same preferences.) We subsume his framework by setting $u(a, \theta) = a \cdot \theta$. Our notion of local IC (and IC) is then equivalent to his. [Carroll](#)'s Proposition 1 says that given convexity of Θ , local IC implies IC. The conclusion follows from our [Proposition 1](#) because u in this specification has convex choice and regular indifferences. Our result is more general; for instance, we do not require finite outcomes. But more importantly, we view our perspective as complementary to [Carroll](#)'s, and perhaps more familiar: rather than equating

⁷ He also considers “ordinal type spaces”, which are less comparable to our framework.

types with preferences and analyzing properties of the type space, we fix an abstract (convex Euclidean) type space and seek properties of the utility function to guarantee sufficiency of local IC.

Remark 1. Although we emphasize its connection with the sufficiency of local IC, convex choice is a notable property for other reasons as well. For instance, in cheap-talk or costly-signaling applications, it is natural to ask whether, in equilibrium, the set of types that choose a particular signal is convex. Modulo details about indifferences, that is equivalent to the equilibrium having convex choice for the sender.⁸

3. Directional Single Crossing

To unpack which preferences have convex choice, we next develop a connection with single crossing.

Definition 4. A function $f : \Theta \rightarrow \mathbb{R}$

1. is *directionally single crossing (DSC)* if there is $\alpha \in \mathbb{R}^n \setminus \{0\}$ such that for all $\theta, \theta' \in \Theta$,

$$(\theta - \theta') \cdot \alpha \geq 0 \implies \text{sign}(f(\theta)) \geq \text{sign}(f(\theta'));$$

2. *strictly violates DSC* if for all $\alpha \in \mathbb{R}^n \setminus \{0\}$, there are $\theta, \theta', \theta'' \in \Theta$ with $\alpha \cdot \theta < \alpha \cdot \theta' < \alpha \cdot \theta''$ such that

$$\min\{f(\theta), f(\theta'')\} > 0 > f(\theta') \quad \text{or} \quad \min\{f(\theta), f(\theta'')\} < 0 < f(\theta').$$

DSC says that for any θ and θ' such that θ is to the right of the hyperplane passing through θ' in the direction of α (i.e., with normal vector α), we have $\text{sign}(f(\theta)) \geq \text{sign}(f(\theta'))$. Hence, we will sometimes say more explicitly that a function is DSC in the direction α . DSC is equivalent to the sets $\{\theta : f(\theta) < 0\}$ and $\{\theta : f(\theta) > 0\}$ being either empty, full (i.e., all of Θ), or half-spaces defined by parallel (possibly identical) hyperplanes. See [Figure 2](#): the left panel depicts a typical DSC function with a “thin” zero set, illustrating the geometry of the

⁸In their study of multidimensional cheap talk, [Levy and Razin \(2004\)](#), a working paper version of [Levy and Razin \(2007\)](#), assume $\Theta = A = \mathbb{R}^2$ and $u(a, \theta) = \left(\sum_{i \in \{1,2\}} \alpha_i |a_{(i)} - (b_{(i)} + \theta_{(i)})|^p\right)^{\frac{1}{p}}$ for some parameters $\alpha, b \in \mathbb{R}^2$. It can be checked that this specification violates convex choice unless $p = 2$. Consistent with that, [Levy and Razin \(2004\)](#) do not show that all equilibria are “convex partitional”; rather, in their Section 4.1, they identify parameters under which they can construct some such equilibria. In other working paper versions, the authors assume $p = 2$ and show that in this case all equilibria are convex partitional.

definition; the middle panel depicts a DSC function with a “thick” zero set; and the right panel depicts a violation of DSC.

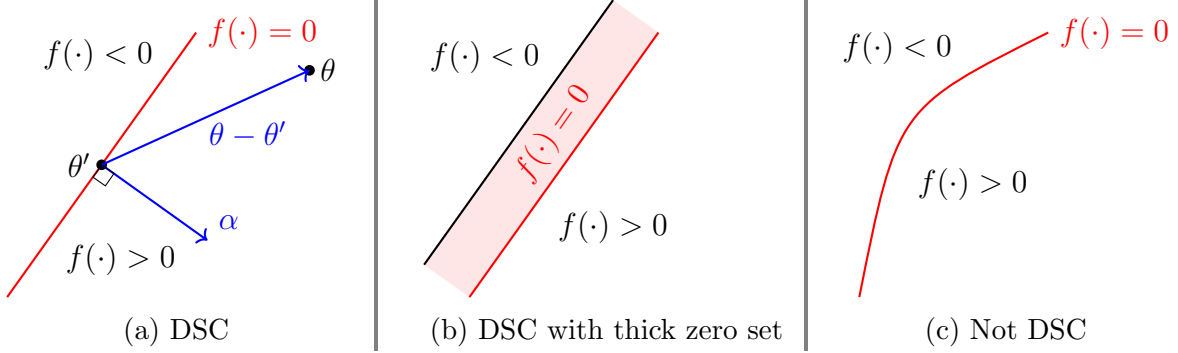


Figure 2: Directional single crossing with $\Theta \subset \mathbb{R}^2$.

Regarding part 2 of Definition 4, note that a strict violation of DSC is slightly stronger than just violating DSC. To illustrate, consider $\Theta = [-1, 1]$ and $f(\theta) = \text{sign}(|\theta|)$. This function is not DSC because its sign is not monotonic. But it does not strictly violate DSC. If, instead, $f(0) < 0$ (maintaining $f(\theta) > 0$ for $\theta \neq 0$), then it does strictly violate DSC. The violation of DSC in Figure 2c is also a strict violation.

Definition 5. The utility function u

1. has *directionally single-crossing differences (DSCD)* if for all $a, a' \in A$, the difference $D_{a,a'}(\theta) := u(a, \theta) - u(a', \theta)$ is DSC;
2. *strictly violates DSCD* if there are $a, a' \in A$ such that $D_{a,a'}(\theta)$ strictly violates DSC.

DSCD can be viewed as saying that for any pair of actions a and a' , the strict-preference sets $\{\theta : u(a, \theta) > u(a', \theta)\}$ and $\{\theta : u(a, \theta) < u(a', \theta)\}$ are parallel half-spaces, each either open or closed, intersected with the type space. Importantly, the direction of the hyperplanes defining these half-spaces can vary across action pairs.

Here are two leading families of DSCD when $A \subset \mathbb{R}^n$: (i) weighted Euclidean preferences, where $u(a, \theta)$ is any decreasing function of $(a - \theta) \cdot W \cdot (a - \theta)$ with W a $n \times n$ symmetric positive definite matrix; and (ii) constant-elasticity-of-substitution (CES) preferences, where $A, \Theta \subset \mathbb{R}_+^n$ and $u(a, \theta) = \left(\sum_{i=1}^n (a_{(i)})^r \theta_{(i)}\right)^s$ with parameters $r \in \mathbb{R}$ and $s > 0$.⁹ In either case, adding a type-independent function $w(a)$ preserves DSCD.¹⁰ With that addition, the

⁹ Family (i) satisfies DSCD with the direction $\alpha = W \cdot (a - a')$ and family (ii) with $\alpha = a^r - (a')^r$.

¹⁰ More generally, DSCD may not be preserved by such addition. But the current utility families don't just have DSCD, they have directionally monotone differences: for any pair of actions, the utility difference is monotonic—rather than just single crossing—in the relevant direction.

CES family subsumes $u(a, \theta) = a \cdot \theta + w(a)$, which is frequently used in multidimensional mechanism design, and which we will return to.

Proposition 2. *If u has DSCD, then u has convex choice. If u strictly violates DSCD, then u does not have convex choice.*

In light of its half-spaces interpretation, DSCD implying convex choice is straightforward. Conversely, a separating hyperplane theorem yields that under convex choice there is no strict violation of DSCD. The upshot is that DSCD “almost” characterizes convex choice.¹¹

Proposition 2 is closely related to a result of [Grandmont \(1978, p. 322\)](#). His condition (H.2) is a little stronger than imposing both convex choice and regular indifference. Combined with a continuity condition—[Grandmont’s](#) (H.1)—and an assumption that the convex type space is open, he obtains a characterization that is similar to, but more restrictive than, DSCD. See [Appendix B.4](#) for details. The leading families of DSCD given above satisfy [Grandmont’s](#) characterization. A one-dimensional example of DSCD that does not is $\Theta = (0, 1)$, $A = \{a', a''\}$, $u(a', \cdot) = 0$, $u(a'', \theta) = -1$ if $\theta < 1/2$ and $u(a'', \theta) = 1$ if $\theta \geq 1/2$; [Grandmont’s](#) continuity requirement fails here.

If $\Theta \subset \mathbb{R}$, then modulo indifference, DSCD is equivalent to [Kartik et al.’s \(2023\)](#) “single-crossing differences” and [Proposition 2](#) is equivalent to part 1 of their Theorem 1. But in multiple dimensions the notions are distinct, and their result is not about convex choice. Similarly, [Milgrom and Shannon’s \(1994\)](#) single-crossing preferences do not guarantee convex choice when Θ is multidimensional.¹²

[McAfee and McMillan \(1988\)](#) present a notion of “generalized single crossing”. Their condition is formulated for a Euclidean allocation space and twice-differentiable utilities. For differentiable mechanisms, they establish that their condition implies that checking local IC along line segments is sufficient. [Appendix B.5](#) provides an example demonstrating that [McAfee and McMillan’s](#) condition does not imply convex choice, and hence does not imply DSCD. Consequently, as displayed explicitly in our example, there are non-differentiable mechanisms for which their condition does not yield sufficiency of local IC.

¹¹ [Appendix B.3](#) gives examples showing that even absent indifference, neither does convex choice imply DSCD nor does a failure of convex choice imply a strict violation of DSCD.

¹² Neither does [Barthel and Sabarwal’s \(2018\)](#) “ i -single crossing property”. Their notion is motivated by [Quah’s \(2007\)](#) weakening of [Milgrom and Shannon’s \(1994\)](#) comparisons of constraint/choice sets. Convex choice considers all choice sets.

3.1. Convex Environments

In various settings, the agent may be choosing among lotteries. For instance, a mechanism designer may consider stochastic mechanisms; indeed, the revelation principle in general requires allowing for those. Or, the mechanism may take inputs from other agents with private information, so that from the interim point of view of one agent, her report induces a lottery. Alternatively, in a cheap-talk application, the receiver's decision may be determined by both the sender's message and the receiver's private preference type; Kartik et al. (2023) detail this application with $\Theta, A \subset \mathbb{R}$.

Following Kartik et al. (2023), we say that the environment (A, Θ, u) is *convex* if

$$\text{the set of functions } \{u(a, \cdot) : \Theta \rightarrow \mathbb{R}\}_{a \in A} \text{ is convex.} \quad (\star)$$

That is, a convex environment has the property that for any pair of actions and any weighting, there is a third action that replicates the weighted sum of utilities of the original actions. Note that convexity is a property of the utility function rather than of A . However, if A is convex, then it is straightforward that (\star) is assured by linearity of u in its first argument. Expected utility over a convex set of lotteries (e.g., all lotteries) is thus a convex environment. In fact, rank-dependent expected utility also produces a convex environment (Kartik et al., 2023, Example 2). An example without lotteries is $A = [\underline{a}_1, \bar{a}_1] \times \dots \times [\underline{a}_k, \bar{a}_k] \subset \mathbb{R}^k$ and $u(a, \theta) = \sum_{i=1}^k g_i(a_{(i)}) f_i(\theta)$ with arbitrary functions $(f_i)_{i=1}^k$ and continuous functions $(g_i)_{i=1}^k$.

The following concepts are useful to elucidate the implications of DSCD in convex environments. We say that the utility function $\tilde{u}(a, \theta)$ *represents* $u(a, \theta)$ if $\tilde{u}(a, \theta) = h(u(a, \theta), \theta)$ for some function h such that for all θ , $h(\cdot, \theta) : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing. This is simply the standard notion of (type-dependent) preference representation. An *affine representation* is any representation that is affine in θ , i.e., has the form $v(a) \cdot \theta + w(a)$ for some $v : A \rightarrow \mathbb{R}^n$ and $w : A \rightarrow \mathbb{R}$. We say that u is *one-dimensional* if there are $\alpha \in \mathbb{R}^n \setminus \{0\}$ and $\tilde{u} : A \times \mathbb{R} \rightarrow \mathbb{R}$ such that $u(a, \theta) \geq (>) u(a', \theta)$ if and only if $\tilde{u}(a, \alpha \cdot \theta) \geq (>) \tilde{u}(a', \alpha \cdot \theta)$.¹³ In other words, any type θ 's preferences are fully determined by the one-dimensional sufficient statistic $\alpha \cdot \theta$. A type θ is *totally indifferent* if $u(a, \theta) = u(a', \theta)$ for all $a, a' \in A$.

Regardless of a convex environment, any utility function with an affine representation satisfies DSCD. For, any affine utility clearly satisfies DSCD, and DSCD is an ordinal property that is preserved by any representation. If preferences are one-dimensional, then even in a convex environment there are DSCD utilities that do not have an affine rep-

¹³ For a DSCD preference, this is equivalent to the utility difference between every pair of actions satisfying DSC in the common direction α .

representation: for example, $\Theta = [1, 2]$, $A = \Delta(\{x_0, x_1, x_2\})$, and for any lottery $a \in A$, $u(a, \theta) = \sum_{i=0}^2 a(x_i)v(x_i, \theta)$ with $v(x_0, \theta) = 0$, $v(x_1, \theta) = \theta$ and $v(x_2, \theta) = \theta^2$.¹⁴ The following result says that under some additional assumptions, these two cases—one-dimensional preferences and those with an affine representation—exhaust DSCD in a convex environment.

Proposition 3. *Assume $\Theta = \mathbb{R}^n$, $u(a, \theta)$ is differentiable in θ , and no type is totally indifferent.¹⁵ If the environment is convex and u satisfies DSCD, then either u is one-dimensional or it has an affine representation.*

Proof sketch. For convenience, we pick an arbitrary action a^* and normalize $u(a^*, \theta) = 0$ for all θ . Analogous to Kartik et al. (2023), DSCD then implies that u is linear-combinations DSC-preserving: for all finite index sets I , $\{a_i\}_{i \in I} \subset A$, and $\{\lambda_i\}_{i \in I} \subset \mathbb{R}$, it holds that $\sum_{i \in I} \lambda_i u(a_i, \cdot)$ is DSC.

We first observe that if preferences are not one-dimensional, then there are actions a_0 and a_1 such that $u(a_0, \cdot)$ and $u(a_1, \cdot)$ are not DSC in a common direction. We show that this implies that the zero sets of the two functions are non-parallel hyperplanes. Consequently, since $\Theta = \mathbb{R}^n$, there is θ_0 such that $u(a_0, \theta_0) = u(a_1, \theta_0) = 0$. Moreover, we can assume $u(a_0, \cdot)$ is affine: because it is DSC with zero set a hyperplane, there is a representation in which it is affine.¹⁶

For any θ_1 with $u(a_0, \theta_1) \neq 0$ and $u(a_1, \theta_1) \neq 0$, there is a nontrivial linear combination of $u(a_0, \cdot)$ and $u(a_1, \cdot)$ that is zero at θ_1 ; the choice of θ_0 and DSC then imply that the line segment $\ell(\theta_0, \theta_1)$ is in the zero set of the linear combination. Since $u(a_0, \cdot)$ is affine, this implies that $u(a_1, \cdot)$ is affine on $\ell(\theta_0, \theta_1)$. As θ_1 is largely arbitrary, we can use the differentiability of $u(a_1, \cdot)$ to further establish that it must in fact be globally affine.¹⁷

Finally, consider any other action a_2 ; for simplicity, suppose it is neither dominated by nor dominates a_0 , i.e., $u(a_2, \theta) > 0$ for some θ and $u(a_2, \theta') < 0$ for some θ' . Then $u(a_2, \cdot)$ is either not DSC in a common direction with $u(a_0, \cdot)$ or not DSC in a common direction with

¹⁴ DSCD holds because the utility difference between any pair of lotteries is a linear combination of θ and θ^2 , which has at most one root in Θ . To see that there is no affine representation, observe that because u is an expected utility, any representation \tilde{u} must be a type-dependent positive affine transformation of u , i.e., have the form $\tilde{u}(a, \theta) = b(\theta)u(a, \theta) + c(\theta)$ with $b(\cdot) > 0$. But no functions b and c can make both $b(\theta)\theta + c(\theta)$ and $b(\theta)\theta^2 + c(\theta)$ affine.

¹⁵ As explained in the proof, instead of no totally indifferent type, it is enough to assume that for any type θ that is totally indifferent, there are two actions a' and a'' such that $\nabla[u(a', \theta) - u(a'', \theta)] \neq 0$.

¹⁶ Our assumption that no type is totally indifferent ensures that we can find such a representation that remains differentiable in θ .

¹⁷ Dealing with the zero sets of either $u(a_0, \cdot)$ and $u(a_1, \cdot)$ requires some care, as θ_1 cannot be in either zero set. Moreover, because we changed the representation of u to make $u(a_0, \cdot)$ affine, it is not guaranteed that $u(a_1, \cdot)$ in the chosen representation is differentiable; the formal proof must handle this obstacle.

$u(a_1, \cdot)$. So the same argument as in the previous paragraph—with a_2 in place of a_1 , and a_1 in place of a_0 if necessary—can be used to conclude that $u(a_2, \cdot)$ is affine. \square

An expected utility is affine or one-dimensional if and only if its generating von Neumann-Morgenstern (vNM) utility is, respectively, affine or one-dimensional. Hence, for convex environments induced by lotteries and expected utility, [Proposition 3](#) can be equivalently stated in terms of either the vNM or the expected-utility function.

Recall that DSCD essentially characterizes convex choice ([Proposition 2](#)), and we have argued that convex choice is crucial for tractability ([Proposition 1](#)) and appealing for other reasons ([Remark 1](#)). We thus view [Proposition 3](#) as suggesting that if one cannot substantially restrict the set of lotteries to consider, then in many contexts multidimensional expected-utility preferences without an affine representation will be unwieldy.

[Appendix B.2](#) shows that the assumption of $\Theta = \mathbb{R}^n$ cannot be dropped from [Proposition 3](#). But one can obtain the result with alternative assumptions; what is important is some richness, beyond the convexity requirement (\star), in terms of the family $\{u(a, \cdot)\}_{a \in A}$ when preferences are not one-dimensional. We offer one such variant below. Say that an action $a \in A$ *strictly dominates* $a' \in A$ if $u(a, \theta) > u(a', \theta)$ for all $\theta \in \Theta$. We also say that u is *minimally rich* if there are no actions $a_0, a_1, a_2 \in A$ and functions $\lambda_0, \lambda_1 : A \rightarrow \mathbb{R}$ such that for all $a \in A$ and $\theta \in \Theta$,

$$u(a, \theta) - u(a_0, \theta) = \lambda_0(a)[u(a_1, \theta) - u(a_0, \theta)] + \lambda_1(a)[u(a_2, \theta) - u(a_0, \theta)].$$

That is, a failure of minimal richness means that after normalizing $u(a_0, \cdot) = 0$, all types' utilities from any action a are a linear combination of their utilities from actions a_1 and a_2 .

Proposition 4. *Assume $u(a, \theta)$ is differentiable in θ , minimally rich, has regular indifference curves, and there is an action that strictly dominates another. If the environment is convex and u satisfies DSCD, then either u is one-dimensional or it has an affine representation.*

[Proposition 4](#) substitutes [Proposition 3](#)'s assumptions of $\Theta = \mathbb{R}^n$ and no total indifference with minimal richness, regular indifference curves, and strict dominance between some pair of actions. Observe that the dominance assumption is satisfied whenever there is one component of the action space over which all types have strictly monotonic preferences; in particular, it holds in the quasi-linear environments common in mechanism design.

To illustrate how the “one-dimensional or affine representation” conclusion of [Propositions 3](#) and [4](#) is useful, we return to the CES preferences discussed earlier. Consider $X, \Theta \subset \mathbb{R}_+^n$ each with nonempty interior, and the (generalized) CES utility $v(x, \theta) =$

$(\sum_{i=1}^n (x_{(i)})^r \theta_{(i)})^s + w(x)$ with $s > 0$. Although v satisfies DSCD, does the corresponding expected-utility function $u(a, \theta)$ over the lottery space $A = \Delta X$? In one dimension, $n = 1$, yes: for example, in that case any linear combination $\lambda v(x, \theta) + \lambda' v(x', \theta) = (\lambda x^{rs} + \lambda' (x')^{rs}) \theta^s + (\lambda w(x) + \lambda' w(x'))$ is monotonic in θ , hence (directionally) single crossing. But in multiple dimensions, $n > 1$, [Proposition 4](#) implies that u has DSCD if and only if $s = 1$; for, if $s \neq 1$, there is no affine representation of u .¹⁸

Remark 2. Although one-dimensional preferences with DSCD in a convex environment need not have an affine representation, their structure can be characterized using [Kartik et al.’s \(2023\)](#) results. If Θ is compact, then all types’ preferences must be a convex combination of two “extreme” types’, with an ordered weighting. More precisely, there must be a representation $\lambda(\alpha \cdot \theta) \bar{u}(a) + (1 - \lambda(\alpha \cdot \theta)) \underline{u}(a)$, where $\alpha \in \mathbb{R}^n \setminus \{0\}$ is the direction in which preferences are one-dimensional, $\lambda : \mathbb{R} \rightarrow [0, 1]$ is an increasing function, and \bar{u} and \underline{u} represent the preferences of the types that respectively maximize and minimize $\alpha \cdot \theta$.

4. Conclusion

We believe convex choice is a valuable property. We have shown that (i) in a suitable sense, it characterizes the sufficiency of local IC ([Proposition 1](#)), (ii) it is essentially equivalent to a form of single crossing with a simple geometric interpretation ([Proposition 2](#)), and (iii) in convex environments satisfying some regularity conditions, it reduces to “one-dimensional or affine representation” ([Propositions 3 and 4](#)).

An alternative notion—equivalent only in one dimension—is connected choice: the set of types that find an action optimal is connected. [Appendix B.6](#) shows that, modulo details, this property characterizes the sufficiency of local IC in a somewhat stronger sense than convex choice: connected choice is necessary even without considering lower-dimensional type subsets. Nevertheless, we find convex choice more appealing for three (related) reasons. First, it ensures that IC between any two types can be verified via local IC along that line segment; this is more tractable than checking local IC along all paths connecting the two

¹⁸ The assumptions on Θ and X ensure that the assumptions of [Proposition 4](#) are satisfied, and that preferences are not one-dimensional when $n > 1$. To see that there is no affine representation if $s \neq 1$, assume $0 \in X$ (to simplify the argument), fix some action $x_0 \neq 0$, and consider the representation

$$\tilde{v}(x, \theta) = [(x^r \cdot \theta)^s + w(x) - w(x_0)] \frac{x_0^r \cdot \theta}{(x_0^r \cdot \theta)^s}. \quad (1)$$

The function \tilde{v} is affine in θ for $x = x_0$, but not for general x . For, if x is not a scalar multiple of x_0 , then changing θ on a hyperplane orthogonal to x_0 does not change the fraction in (1), and since $(x^r \cdot \theta)^s$ is not affine when $s \neq 1$, $\tilde{v}(x, \cdot)$ is not affine. Since any representation of v is a type-dependent positive affine transformation of \tilde{v} , there is no affine representation.

types. Second, convex choice sets are typically more economically meaningful; for instance, in multidimensional cheap talk, equilibria that merely partition the sender’s type space into connected sets would be less satisfactory. Third, convex choice ties to directional single crossing and its implications/tractability in a way that connected choice has no analog, to our knowledge.

References

- ARMSTRONG, M. (1996): “Multiproduct Nonlinear Pricing,” *Econometrica*, 64, 51–75.
- BARTHEL, A.-C. AND T. SABARWAL (2018): “Directional Monotone Comparative Statics,” *Economic Theory*, 66, 557–591.
- CARROLL, G. (2012): “When Are Local Incentive Constraints Sufficient?” *Econometrica*, 80, 661–686.
- CRAWFORD, V. AND J. SOBEL (1982): “Strategic Information Transmission,” *Econometrica*, 50, 1431–1451.
- GRANDMONT, J.-M. (1978): “Intermediate Preferences and the Majority Rule,” *Econometrica*, 46, 317–330.
- KARTIK, N., S. LEE, AND D. RAPPOPORT (2023): “Single-Crossing Differences in Convex Environments,” *Review of Economic Studies*, forthcoming.
- KLEINER, A. (2022): “Optimal Delegation in a Multidimensional World,” Unpublished.
- LEVY, G. AND R. RAZIN (2004): “On the Limits of Communication in Multidimensional Cheap Talk,” Working Paper, <http://eprints.lse.ac.uk/545/>.
- (2007): “On the Limits of Communication in Multidimensional Cheap Talk: A Comment,” *Econometrica*, 75, 885–893.
- MANELLI, A. M. AND D. R. VINCENT (2007): “Multidimensional Mechanism Design: Revenue Maximization and the Multiple-good Monopoly,” *Journal of Economic Theory*, 137, 153–185.
- MCAFEE, R. P. AND J. MCMILLAN (1988): “Multidimensional Incentive Compatibility and Mechanism Design,” *Journal of Economic Theory*, 46, 335–354.
- MILGROM, P. AND C. SHANNON (1994): “Monotone Comparative Statics,” *Econometrica*, 157–180.
- QUAH, J. K.-H. (2007): “The Comparative Statics of Constrained Optimization Problems,” *Econometrica*, 75, 401–431.
- ROCHET, J.-C. AND P. CHONÉ (1998): “Ironing, Sweeping, and Multidimensional Screening,” *Econometrica*, 66, 783–826.

Appendix A: Proofs

A.1. Proof of Proposition 1

Proof that statement 1 \implies statement 2. Assume convex choice and regular indifference. Fix an arbitrary line segment $\Theta' \subset \Theta$ and let $m : \Theta' \rightarrow A$ satisfy local IC. Fix any two distinct types in Θ' ; we can label the line segment between these types as $[0, 1]$, corresponding to the convex combinations of the two types. Our goal is to show that $u(m(0), 0) \geq u(m(1), 0)$, which implies IC on Θ' .

Local IC implies that there is an open cover of $[0, 1]$ such that each element of the cover, $N_\theta \subset [0, 1]$ with $\theta \in [0, 1]$, satisfies condition (LIC). As $[0, 1]$ is compact, there is a finite subcover, which we index by $\theta_1 < \dots < \theta_m$. So $0 \in N_{\theta_1}$ and $1 \in N_{\theta_m}$. Local IC implies

$$u(m(0), 0) \geq u(m(\theta_1), 0). \quad (2)$$

Now choose any $\tilde{\theta}_1 \in N_{\theta_1} \cap N_{\theta_2}$ with $\tilde{\theta}_1 > \theta_1$. By local IC,

$$u(m(\theta_1), \theta_1) \geq u(m(\tilde{\theta}_1), \theta_1), \quad (3)$$

$$u(m(\tilde{\theta}_1), \tilde{\theta}_1) \geq u(m(\theta_1), \tilde{\theta}_1). \quad (4)$$

We claim that

$$u(m(\theta_1), 0) \geq u(m(\tilde{\theta}_1), 0). \quad (5)$$

To prove inequality (5), suppose to the contrary $u(m(\tilde{\theta}_1), 0) > u(m(\theta_1), 0)$. Then inequality (3) implies $\theta_1 > 0$. If inequality (4) holds strictly, then convex choice implies $u(m(\tilde{\theta}_1), \theta_1) > u(m(\theta_1), \theta_1)$, contradicting (3). If (4) holds with equality, then since $\theta_1 \in (0, \tilde{\theta}_1)$, it follows from regular indifference that $u(m(\tilde{\theta}_1), \theta_1) > u(m(\theta_1), \theta_1)$, contradicting (3).

Combining (2) and (5), $u(m(0), 0) \geq u(m(\tilde{\theta}_1), 0)$. Continuing in the same fashion, we obtain $u(m(0), 0) \geq u(m(1), 1)$, as desired. Q.E.D.

Proof that statement 2 \implies statement 1. We prove the contrapositive, first for convex choice and then for regular indifference.

If u violates convex choice, then there are $a', a'' \in A$ and $\theta_1, \theta_2 \in \Theta$ and $\theta' \in \ell(\theta_1, \theta_2)$ such that $u(a'', \theta_i) > u(a', \theta_i)$ for $i = 1, 2$ and $u(a'', \theta') \leq u(a', \theta')$. Let $\Theta' = \ell(\theta_1, \theta_2)$ and let $a^*(\theta)$ be an arbitrary selection from $\arg \max_{a \in \{a', a''\}} u(a, \theta)$. Consider the mechanism

$m : \Theta' \rightarrow \{a', a''\}$ given by

$$m(\theta) = \begin{cases} a^*(\theta) & \text{if } \theta \in \ell(\theta_1, \theta') \setminus \{\theta'\} \\ a' & \text{otherwise.} \end{cases}$$

This mechanism is locally IC because all types in $\ell(\theta_1, \theta')$ are getting an optimal action from $\{a', a''\}$ and the mechanism is constant on $\ell(\theta', \theta_2)$. But IC fails because type θ_2 can profitably mimic θ_1 .

If u violates regular indifference, then there are $a', a'' \in A$ and $\theta_1, \theta_2 \in \Theta$ and $\theta' \in \ell(\theta_1, \theta_2)$ such that $u(a', \theta_1) = u(a'', \theta_1)$, $u(a'', \theta_2) > u(a', \theta_2)$, and $u(a', \theta') \geq u(a'', \theta')$. Consider the same mechanism m defined in the previous paragraph, except that $m(\theta_1) = a''$. It is locally IC but not IC by the argument given in the previous paragraph. Q.E.D.

A.2. Proof of Proposition 2

Recall that $D_{a,a'}(\theta) \equiv u(a, \theta) - u(a', \theta)$.

For the proposition's first statement, suppose u has DSCD. It is sufficient to show convex choice for all binary choice sets. So consider any $A' = \{a', a''\} \subset A$. Let $\theta', \theta'' \in \{\theta : \{a'\} = \arg \max_{a \in A'} u(a, \theta)\}$. We have $D_{a',a''}(\theta') > 0$ and $D_{a',a''}(\theta'') > 0$. Since u has DSCD, there is $\alpha \in \mathbb{R}^n \setminus \{0\}$ such that $D_{a',a''}(\cdot)$ is DSC in direction α . This implies $D_{a',a''}(\theta) > 0$ for all $\theta \in \ell(\theta', \theta'')$, as required.

For the proposition's second statement, we prove the contrapositive. Suppose u has convex choice. Fix arbitrary $a', a'' \in A$ and let

$$\Theta' := \{\theta : D_{a',a''}(\theta) > 0\},$$

$$\Theta'' := \{\theta : D_{a',a''}(\theta) < 0\}.$$

It is straightforward there is not a strict violation of DSCD if either of these sets is empty. So assume $\Theta' \neq \emptyset$ and $\Theta'' \neq \emptyset$. Then Θ' and Θ'' are nonempty, convex (by convex choice), and disjoint sets. A standard separating hyperplane theorem implies that there is $\alpha \in \mathbb{R}^n$ such that $\alpha \cdot \theta' \leq \alpha \cdot \theta''$ for all $\theta' \in \Theta'$ and $\theta'' \in \Theta''$. Hence, there are no $\theta, \theta', \theta'' \in \Theta$ with $\alpha \cdot \theta < \alpha \cdot \theta' < \alpha \cdot \theta''$ and $\min\{D_{a_1,a_2}(\theta), D_{a_1,a_2}(\theta'')\} > 0 > D_{a_1,a_2}(\theta')$, i.e., there is not a strict violation of DSCD. Q.E.D.

A.3. Proof of Proposition 3

For this proof, we fix an arbitrary action $a^* \in A$ and work with the function

$$f(a, \theta) := u(a, \theta) - u(a^*, \theta).$$

Since we are interested in representations of u , we say that a function \tilde{f} is a *representative* of f if there is a representation \tilde{u} of u such that $\tilde{f}(a, \theta) = \tilde{u}(a, \theta) - \tilde{u}(a^*, \theta)$. Note that any type-dependent positive linear transformation of f (i.e., $g(\theta)f(a, \theta)$ with $g(\cdot) > 0$) is a representative of f . We say that f is *DSC-preserving* if for all finite index sets I , $\{a_i\}_{i \in I} \subset A$, and $\{\lambda_i\}_{i \in I} \subset \mathbb{R}$, it holds that $\sum_{i \in I} \lambda_i f(a_i, \cdot)$ is DSC.

Proof of Proposition 3. Let $A^* := A \setminus \{a^*\}$. Analogous to Kartik et al. (2023, proof of Theorem 2), for every $\mu : A^* \rightarrow \mathbb{R}$ with finite support A_μ , there exist probability distributions P, Q on $A_\mu \cup \{a^*\}$ such that $\sum_{a \in A_\mu} f(a, \theta)\mu(a)$ is a multiple of

$$\sum_{a \in A_\mu \cup \{a^*\}} u(a, \theta)P(a) - \sum_{a \in A_\mu \cup \{a^*\}} u(a, \theta)Q(a). \quad (6)$$

Since the environment is convex and u has DSCD, the function (6) is DSC, which implies that $\sum_{a \in A_\mu} f(a, \theta)\mu(a)$ is also DSC. Hence, $f(a, \theta)$ is DSC-preserving. Proposition 5 below implies that f has an affine representative or is one-dimensional. It follows that u has an affine representation or is one-dimensional. Q.E.D.

Accordingly, the key to Proposition 3 is the following result.

Proposition 5. *Assume $\Theta = \mathbb{R}^n$ and $f(a, \theta)$ is DSC-preserving and differentiable in θ . If no type is totally indifferent, then f is either one-dimensional or has an affine representative.*

The proof of Proposition 5 requires a few lemmas.

Lemma 1. *Assume $\Theta = \mathbb{R}^n$, f is DSC-preserving, and for some $a_1, a_2 \in A$, $f(a_1, \cdot)$ and $f(a_2, \cdot)$ are not DSC in a common direction. Then, for $i = 1, 2$, $\{\theta : f(a_i, \theta) = 0\}$ is a hyperplane or empty.*

Proof. For $i = 1, 2$, it is sufficient to prove that the zero set of $f(a_i, \cdot)$ is contained in a hyperplane, since $f(a_i, \cdot)$ being DSC then implies that its zero set is a hyperplane or empty. Moreover, there is no loss of generality in proving it for just $f(a_1, \cdot)$.

Accordingly, suppose towards contradiction that the zero set of $f(a_1, \cdot)$ is not contained in a hyperplane. Since $f(a_1, \cdot)$ and $f(a_2, \cdot)$ are not DSC in a common direction, there exist

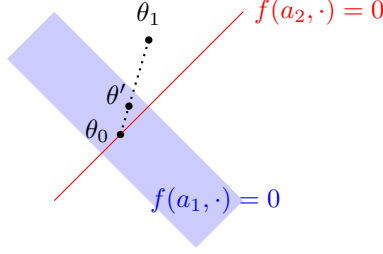


Figure 3: Illustration for Lemma 1

θ_0 and θ_1 with $f(a_i, \theta_0) = 0$ and $f(a_i, \theta_1) \neq 0$ for $i = 1, 2$. Moreover, one can choose θ_0 and θ_1 with these properties such that there is $\theta' \in \ell(\theta_0, \theta_1)$ with $f(a_1, \theta') = 0 \neq f(a_2, \theta')$; see Figure 3 for an illustration. Then the linear combination $f_3(\theta) := f(a_1, \theta) - \frac{f(a_1, \theta_1)}{f(a_2, \theta_1)} f(a_2, \theta)$ satisfies $f_3(\theta_0) = f_3(\theta_1) = 0 \neq f_3(\theta')$. Since $\theta' \in \ell(\theta_0, \theta_1)$, the function f_3 is not DSC, which contradicts f being DSC-preserving. Q.E.D.

Lemma 2. Let f be DSC-preserving. Let $a_0, a_1 \in A$ be such that $f(a_0, \cdot)$ and $f(a_1, \cdot)$ are not DSC in a common direction and suppose θ_0 satisfies $f(a_0, \theta_0) = f(a_1, \theta_0) = 0$.

If $f(a_0, \cdot)$ and $f(a_1, \cdot)$ are differentiable at θ_0 and $\nabla f(a_0, \theta_0) \neq 0$, then for all θ with $f(a_0, \theta), f(a_1, \theta) \neq 0$, it holds that

$$f(a_1, \theta) = f(a_0, \theta) \frac{\nabla f(a_1, \theta_0) \cdot (\theta - \theta_0)}{\nabla f(a_0, \theta_0) \cdot (\theta - \theta_0)}.$$

Proof. Let $f_i(\theta) := f(a_i, \theta)$ for all θ and $i = 0, 1$. Fix arbitrary $\theta \in \Theta$ with $f_0(\theta) \neq 0$ and $f_1(\theta) \neq 0$, and consider the function

$$f_2(\theta') := f_0(\theta') - \frac{f_0(\theta)}{f_1(\theta)} f_1(\theta').$$

Since f is DSC-preserving, f_2 is DSC. Moreover, $f_2(\theta_0) = f_2(\theta) = 0$ and, therefore,

$$\nabla f_2(\theta_0) \cdot (\theta - \theta_0) = \left[\nabla f_0(\theta_0) - \frac{f_0(\theta)}{f_1(\theta)} \nabla f_1(\theta_0) \right] \cdot (\theta - \theta_0) = 0.$$

Moreover, $\nabla f_0(\theta_0) \cdot (\theta - \theta_0) \neq 0$ (because $\nabla f_0(\theta_0) \neq 0$ and $f_0(\theta) \neq 0$), hence $\nabla f_1(\theta_0) \cdot (\theta - \theta_0) \neq 0$. Therefore, for all θ with $f_0(\theta) \neq 0$ and $f_1(\theta) \neq 0$,

$$f_1(\theta) = f_0(\theta) \frac{\nabla f_1(\theta_0) \cdot (\theta - \theta_0)}{\nabla f_0(\theta_0) \cdot (\theta - \theta_0)}. \quad \text{Q.E.D.}$$

Lemma 3. Assume f is DSC-preserving and, for all $a \in A$, $f(a, \cdot)$ is differentiable with a

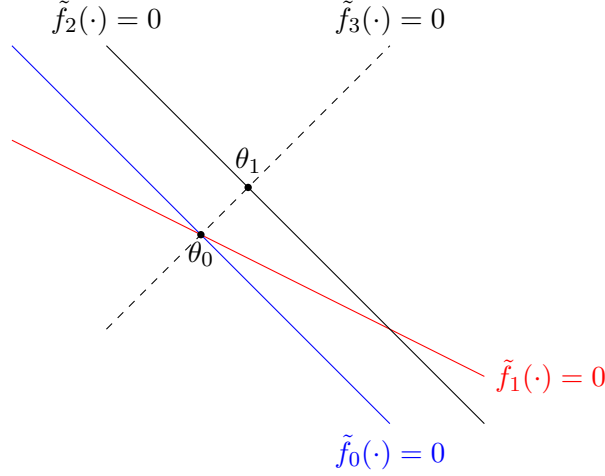


Figure 4: A step in the proof of Lemma 3.

zero set that is either a hyperplane (intersected with Θ), Θ , or empty.

If there are $a_0, a_1, a_2 \in A$ and $\theta_0 \in \Theta$ such that $f(a_0, \theta_0) = f(a_1, \theta_0) = 0 \neq f(a_2, \theta_0)$ and $f(a_0, \cdot)$ and $f(a_1, \cdot)$ are not DSC in a common direction, then f has an affine representative.

Proof. Let $f_i(\theta) := f(a_i, \theta)$ for $i = 0, 1$.

Case 1: Suppose $\nabla f_0(\theta_0) \neq 0$.

By Lemma 2, for all θ with $f_0(\theta) \neq 0$ and $f_1(\theta) \neq 0$, we have

$$f_1(\theta) = f_0(\theta) \frac{\nabla f_1(\theta_0) \cdot (\theta - \theta_0)}{\nabla f_0(\theta_0) \cdot (\theta - \theta_0)}. \quad (7)$$

For each $i = 0, 1$, the set $\{\theta : f_i(\theta) = 0\}$ is neither empty (as it contains θ_0) nor Θ (as f_0 and f_1 are not DSC in a common direction), and therefore, by hypothesis, is a hyperplane. Moreover, for each $i = 0, 1$, since f_i is DSC and $\nabla f_i(\theta_0) \neq 0$, it follows that $\text{sign}([f_i(\theta)]) = \text{sign}(\nabla f_i(\theta_0) \cdot (\theta - \theta_0))$. Therefore, we can define a representative \tilde{f} as follows:

$$\tilde{f}(a, \theta) := \begin{cases} f(a, \theta) \frac{\nabla f_0(\theta_0) \cdot (\theta - \theta_0)}{f_0(\theta)} & \text{if } f_0(\theta) \neq 0 \\ f(a, \theta) \frac{\nabla f_1(\theta_0) \cdot (\theta - \theta_0)}{f_1(\theta)} & \text{if } f_0(\theta) = 0 \neq f_1(\theta) \\ kf(a, \theta) & \text{if } f_0(\theta) = f_1(\theta) = 0, \end{cases} \quad (8)$$

for some $k \in \mathbb{R}$ that remains to be specified. It follows from (7) and (8) that for each $i = 0, 1$, the function $\tilde{f}(a_i, \cdot)$ is affine, as it is 0 on $\{\theta : f_0(\theta) = f_1(\theta) = 0\}$ and $\nabla f_i(\theta_0) \cdot (\theta - \theta_0)$ otherwise.

Consider any $a_2 \in A$ and let $f_2 := f(a_2, \cdot)$ and $\tilde{f}_2 := \tilde{f}(a_2, \cdot)$. We will argue that \tilde{f}_2 is affine, which proves the lemma.

- First, suppose $\tilde{f}_2(\theta_0) \neq 0$ and \tilde{f}_2 has a nonempty zero set. By DSC, the zero set of \tilde{f}_2 contains a hyperplane (that does not pass through θ_0). Since the zero sets of \tilde{f}_0 and \tilde{f}_1 are hyperplanes (that pass through θ_0), there is θ_1 with $\tilde{f}_2(\theta_1) = 0$, $\tilde{f}_0(\theta_1) \neq 0$ and $\tilde{f}_1(\theta_1) \neq 0$. Moreover,

$$\tilde{f}_3(\theta) := \tilde{f}_0(\theta) - \frac{\tilde{f}_0(\theta_1)}{\tilde{f}_1(\theta_1)} \tilde{f}_1(\theta)$$

is affine as a linear combination of affine functions and satisfies $\tilde{f}_3(\theta_1) = \tilde{f}_3(\theta_0) = 0$. Since $\tilde{f}_2(\theta_1) = 0 \neq \tilde{f}_2(\theta_0)$, \tilde{f}_2 and \tilde{f}_3 are not DSC in a common direction. See [Figure 4](#). Since $\tilde{f}_0(\theta_1) \neq 0$ and \tilde{f}_0 is continuous, we observe from (8) that in a neighborhood of θ_1 ,

$$\tilde{f}_2(\theta) = \tilde{f}_3(\theta) \frac{\nabla \tilde{f}_3(\theta_1) \cdot (\theta - \theta_1)}{\nabla \tilde{f}_3(\theta_1) \cdot (\theta_0 - \theta_1)}.$$

Since \tilde{f}_3 and \tilde{f}_0 are differentiable, \tilde{f}_2 is differentiable at θ_1 as a composition of differentiable functions. Also, \tilde{f} is DSC-preserving because f is DSC-preserving and \tilde{f} is a type-dependent positive affine transformation of f .¹⁹ By [Lemma 2](#),

$$\tilde{f}_2(\theta) = \tilde{f}_3(\theta) \frac{\nabla \tilde{f}_3(\theta_1) \cdot (\theta - \theta_1)}{\nabla \tilde{f}_3(\theta_1) \cdot (\theta_0 - \theta_1)}$$

on the set $\{\theta : \tilde{f}_2(\theta) \neq 0 \text{ and } \tilde{f}_3(\theta) \neq 0\}$. Since \tilde{f}_3 is affine, it follows that \tilde{f}_2 is affine on that set. [Lemma 1](#) implies that the zero set of \tilde{f}_2 is a hyperplane; hence, \tilde{f}_2 is affine except possibly on $\{\theta : \tilde{f}_2(\theta) = 0\}$ (which is a hyperplane) and one can check that there is a choice of k in (8) that makes \tilde{f}_2 affine everywhere.

- Next, suppose $\tilde{f}_2(\theta_0) \neq 0$ and \tilde{f}_2 is strictly positive (or strictly negative) on Θ . Fix θ_2 with $\tilde{f}_0(\theta_2) \neq 0$ and define $\tilde{f}_3(\theta) := \tilde{f}_2(\theta) - \frac{\tilde{f}_2(\theta_2)}{\tilde{f}_0(\theta_2)} \tilde{f}_0(\theta)$. Then $\tilde{f}_3(\theta_2) = 0$. If \tilde{f}_3 vanishes everywhere, \tilde{f}_2 is affine. Otherwise, the arguments in the previous bullet point imply that \tilde{f}_3 is affine. Since \tilde{f}_0 is affine, it follows that \tilde{f}_2 is affine.
- It remains to consider $\tilde{f}_2(\theta_0) = 0$. [Lemma 2](#) yields that for all θ with $\tilde{f}_0(\theta) \neq 0$ and

¹⁹ Indeed, fix arbitrary $a, a' \in A$ and $\lambda, \lambda' \in \mathbb{R}$. Then $\lambda f(a, \theta) + \lambda' f(a', \theta)$ is DSC and, for all θ ,

$$\text{sign}(\lambda f(a, \theta) + \lambda' f(a', \theta)) = \text{sign}(\lambda \beta(\theta) f(a, \theta) + \lambda' \beta(\theta) f(a', \theta)) = \text{sign}(\lambda \tilde{f}(a, \theta) + \lambda' \tilde{f}(a', \theta)),$$

where $\beta(\theta) > 0$ as defined in (8). Therefore, $\lambda \tilde{f}(a, \theta) + \lambda' \tilde{f}(a', \theta)$ is DSC. We remark that in general, type-dependent monotonic transformations (rather than positive affine transformations) of f need not be DSC-preserving.

$f_2(\theta) \neq 0$, we have

$$f_2(\theta) = f_0(\theta) \frac{\nabla f_2(\theta_0) \cdot (\theta - \theta_0)}{\nabla f_0(\theta_0) \cdot (\theta - \theta_0)}$$

and, therefore, $\tilde{f}_2(\theta) = \nabla f_2(\theta_0) \cdot (\theta - \theta_0)$. Similarly, for all θ with $f_0(\theta) = 0$ and $f_1(\theta) \neq 0$ and $f_2(\theta) \neq 0$, we have

$$f_2(\theta) = f_1(\theta) \frac{\nabla f_2(\theta_0) \cdot (\theta - \theta_0)}{\nabla f_1(\theta_0) \cdot (\theta - \theta_0)}$$

and, therefore, $\tilde{f}_2(\theta) = \nabla f_2(\theta_0) \cdot (\theta - \theta_0)$. It follows that \tilde{f}_2 is affine.

Case 2: Suppose $\nabla f_0(\theta_0) = 0$.

Since we assumed that no type is totally indifferent, there is $a' \in A$ with $f(a', \theta_0) \neq 0$.²⁰ By choosing a different representative if necessary, we can assume without loss of generality that $f(a', \cdot)$ is constant in a neighborhood of θ_0 (while maintaining differentiability of $f(a, \cdot)$ for all $a \in A$). Because every neighborhood of θ_0 contains θ'_0 with $\nabla f_0(\theta'_0) \neq 0$ (otherwise f_0 would be identically zero in an neighborhood of θ_0), there are $\theta'_0 \in \Theta$ and $\lambda_0, \lambda_1 \in \mathbb{R}$ such that $f_0(\theta'_0) + \lambda_0 f(a', \theta'_0) = 0$, $f_1(\theta'_0) + \lambda_1 f(a', \theta'_0) = 0$, and $\nabla f_0(\theta'_0) + \lambda_0 \nabla f(a', \theta'_0) \neq 0$. The arguments for Case 1 then establish that f has an affine representative. Q.E.D.

Proof of Proposition 5. If there is $\alpha \in \mathbb{R}^n \setminus \{0\}$ such that $f(a, \cdot)$ is DSC in direction α for all $a \in A$, then preferences are one-dimensional (see Lemma 4 in the Supplementary Appendix). So suppose there are $a_0, a_1 \in A$ such that $f(a_0, \cdot)$ and $f(a_1, \cdot)$ are not DSC in a common direction, and define $f_i(\theta) := f(a_i, \theta)$ for $i = 0, 1$. Using Lemma 1, for $i = 0, 1$ the zero set of f_i is a hyperplane.²¹ Because $f_0(\cdot)$ and $f_1(\cdot)$ are not DSC in a common direction, the hyperplanes on which each function vanishes intersect. Hence, since $\Theta = \mathbb{R}^n$, there is θ_0 with $f_0(\theta_0) = f_1(\theta_0) = 0$. By assumption, there is $a_2 \in A$ such that $f(a_2, \theta_0) \neq 0$ and it follows from Lemma 3 that f has an affine representative. Q.E.D.

²⁰ Under the weaker assumption stated in footnote 15, if there is no such action (i.e., θ_0 is totally indifferent), then there are actions a'_0 and a''_0 such that $\nabla[u(a'_0, \theta_0) - u(a''_0, \theta_0)] \neq 0$. Without loss, we may assume that a'_0 is the action whose utility we normalized to zero for all types at the outset of this proof. Hence, $\nabla f(a'_0, \theta_0) \neq 0$. We can then apply Case 1 using $f(a'_0, \cdot)$ in place of $f_0(\cdot)$, and one of $f_0(\cdot)$ or $f_1(\cdot)$ in place of $f_1(\cdot)$, because at least one of $f_0(\cdot)$ and $f_1(\cdot)$ is not DSC in a common direction with $f(a'_0, \cdot)$.

²¹ To elaborate: it must be that for $i = 0, 1$, f_i is neither strictly positive nor strictly negative; otherwise, the two functions would be DSC in a common direction. By continuity, each f_i vanishes on a nonempty set. By Lemma 1, the zero set of each f_i is contained in a hyperplane; and by DSC, the set cannot be a strict (and nonempty) subset of a hyperplane.

A.4. Proof of Proposition 4

Let action a^* strictly dominate action a_* . We can choose a differentiable representation such that action a_* yields payoff 0 to all types and action a^* yields payoff 1 for all $\theta \in \Theta$.²²

Say that $f : \Theta \rightarrow \mathbb{R}$ is *increasing in direction* $\alpha \in \mathbb{R}^n \setminus \{0\}$ at θ if for all θ' , it holds that $\alpha \cdot (\theta' - \theta) \geq (\leq) 0 \implies f(\theta') \geq (\leq) f(\theta)$. Note that this is a strengthening of DSC.

Case 1: Suppose there are θ, a_0, a_1 such that $u(a_0, \cdot)$ and $u(a_1, \cdot)$ are not increasing in a common direction at θ .

By adding a multiple of $u(a^*, \cdot)$ if necessary, we can assume $u(a_0, \theta) = u(a_1, \theta) = 0$. Then $u(a_0, \cdot)$ and $u(a_1, \cdot)$ are each DSC but not DSC in a common direction. By Lemma 3, u has an affine representation \tilde{u} .

Case 2: Otherwise, for all θ, a_0, a_1 , the functions $u(a_0, \cdot)$ and $u(a_1, \cdot)$ are increasing in a common direction at θ .

If there is a direction $\alpha \in \mathbb{R}^n \setminus \{0\}$ such that for all $a \in A$ and $\theta \in \Theta$, $u(a, \cdot)$ is increasing in direction α at θ , then u is one-dimensional (see Lemma 4 in the Supplementary Appendix). So suppose there is $a \in A$ and $\theta_0, \theta_1 \in \Theta$ such that $u(a, \cdot)$ is increasing in direction $\alpha \in \mathbb{R}^n \setminus \{0\}$ at θ_0 but not at θ_1 . For any $a' \in A$, $u(a', \cdot)$ is increasing in direction α at θ_0 . Because θ_1 does not lie on the hyperplane in direction α through θ_0 and because of regular indifference, $u(a, \theta_1) \neq 0$ and $u(a', \theta_1) \neq 0$. Because the environment is convex, there is $a'' \in A$ such that

$$u(a'', \theta) = u(a, \theta) - [u(a', \theta) - u(a^*, \theta)u(a', \theta_0)] \frac{u(a, \theta_1)}{u(a', \theta_1) - u(a', \theta_0)}.$$

Then $u(a'', \theta) = 0$ for all θ with $(\theta - \theta_0) \cdot \alpha = 0$ and $u(a'', \theta_1) = 0$. By regular indifference, $u(a'', \theta) = 0$ for all θ ,²³ and therefore

$$u(a', \theta) = u(a, \theta)\lambda_0(a') + u(a^*, \theta)\lambda_1(a').$$

This contradicts minimal richness.

Q.E.D.

²² Indeed, the function $\frac{u(a, \theta) - u(a_*, \theta)}{u(a^*, \theta) - u(a_*, \theta)}$ is differentiable in θ because $u(a^*, \theta) - u(a_*, \theta) > 0$ for all θ .

²³ Indeed, the zero set has nonempty (relative) interior. If the linear combination is nonzero at some point, there is a line through that point that intersects the interior of the zero set; on this line, regular indifference are violated.

Appendix B: Supplementary Appendix

B.1. Omitted Proofs

Lemma 4. *Let $\alpha \in \mathbb{R}^n \setminus \{0\}$ and $f : A \times \Theta \rightarrow \mathbb{R}$. If $f(a, \cdot)$ is DSC in direction α for all $a \in A$, then f is one-dimensional.*

Proof. For any $x \in \mathbb{R}$, choose any $\theta_x \in \mathbb{R}^n$ such that $\alpha \cdot \theta_x = x$ and define, for all $a \in A$, $\tilde{f}(a, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ by $x \mapsto f(a, \theta_x)$.

We claim that $\text{sign}(f(a, \theta)) = \text{sign}(\tilde{f}(a, \alpha \cdot \theta))$ for all θ and a . Let $x = \alpha \cdot \theta$ and observe that $\tilde{f}(a, \alpha \cdot \theta) = \tilde{f}(a, \alpha \cdot \theta_x)$. Moreover, since $f(a, \cdot)$ is DSC in direction α , we obtain

$$\text{sign}(f(a, \theta)) = \text{sign}(f(a, \theta_x)) = \text{sign}(\tilde{f}(a, \alpha \cdot \theta_x)) = \text{sign}(\tilde{f}(a, \alpha \cdot \theta)). \quad \text{Q.E.D.}$$

B.2. On Tightness of Proposition 3's Assumptions

Proposition 3 has three assumptions: $\Theta = \mathbb{R}^n$; $u(a, \theta)$ is differentiable in θ ; and no type is totally indifferent (or the weaker version in footnote 15). Example 3 below shows that the first assumption cannot be dropped; Example 4 shows that the latter two cannot be jointly dropped.

Example 3. Let $\Theta = \mathbb{R}_+^2$, $X = \{x_0, x_1, x_2\}$, and $A = \Delta X$. Consider expected utility preferences over A where the vNM utility is given by

$$u(x, \theta) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{if } x = x_1 \\ \tan^{-1} \left(\frac{\theta_1}{\theta_2} \right) & \text{if } x = x_2. \end{cases}$$

So $u(x_0, \theta)$ and $u(x_1, \theta)$ are constant in θ , and, for any θ , there is a hyperplane through θ such that $u(x_2, \theta') \geq u(x_2, \theta)$ for θ' on one side of the hyperplane and $u(x_2, \theta') \leq u(x_2, \theta)$ for θ' on the other side. It follows that for any $p \in \Delta X$, $p_1 u(x_1, \cdot) + p_2 u(x_2, \cdot) + p_3 u(x_3, \cdot)$ is DSC, and so u has DSCD over lotteries. However, it can be checked that neither is u one-dimensional nor does it have an affine representation.

Example 4. Let $\Theta = \mathbb{R}^2$, $X = \{x_0, x_1, x_2\}$, and $A = \Delta X$. Consider expected utility

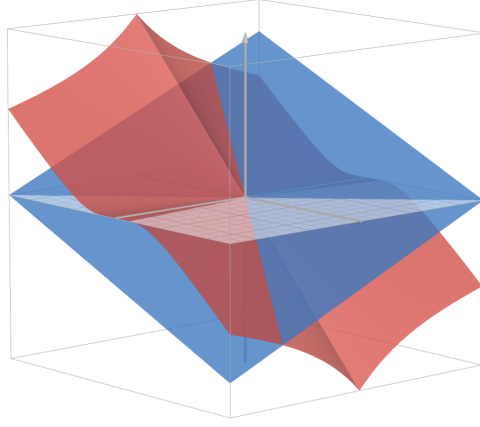


Figure 5: The utilities in [Example 4](#), with $u(x_1, \theta)$ in blue and $u(x_2, \theta)$ in red.

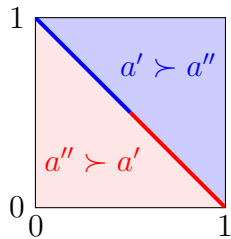
preferences over A where the vNM utility is given by

$$u(x, \theta) = \begin{cases} 0 & \text{if } x = x_0 \\ \theta_1 + \theta_2 & \text{if } x = x_1 \\ \text{sign}(\theta_2) e^{\left(\frac{-|\theta_1|}{|\theta_2|}\right)} \sqrt{\theta_1^2 + \theta_2^2} & \text{if } x = x_2. \end{cases}$$

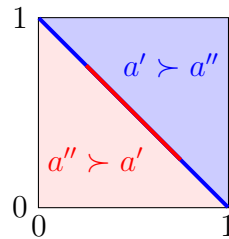
[Figure 5](#) graphs $u(x_1, \cdot)$ and $u(x_2, \cdot)$. Note that $u(x_2, \cdot)$ is not differentiable and type $\theta = 0$ is totally indifferent. It can be verified that there is DSCD over lotteries, but neither is u one-dimensional nor does it have an affine representation.

B.3. On Convex Choice, DSCD, and Strict Violation of DSCD

If $\Theta \subset \mathbb{R}$, then absent indifferences, a violation of DSCD is equivalent to a strict violation of DSCD. But not so in multiple dimensions. Consider the following examples with $\Theta = [0, 1]^2$ and $A = \{a', a''\}$:



(a) Convex choice



(b) No convex choice

Figure 6: Violations of DSCD, but not strict violations.

Both panels in the figure feature no indifferences, a violation of DSCD, but no strict

violation of DSCD. In Figure 6a, there is convex choice (so convex choice does not imply DSCD); in Figure 6b, convex choice fails (so no strict violation of DSCD does not imply convex choice).

B.4. On Grandmont's (1978) Characterization

In this subsection, we elaborate on the connection between Proposition 2 and Grandmont (1978). It is useful to begin with a strengthening of DSCD.

Definition 6. A function $f : \Theta \rightarrow \mathbb{R}$ is *strictly directionally single crossing* (strict DSC) if it is DSC in some direction α and, in addition, either $f(\cdot) = 0$ or for all $\theta, \theta' \in \Theta$,

$$\left[(\theta - \theta') \cdot \alpha > 0 \text{ and } f(\theta') = 0 \right] \implies f(\theta) \neq 0.$$

The utility function $u : A \times \Theta \rightarrow \mathbb{R}$ has *strict directionally single-crossing differences* (strict DSCD) if for all $a, a' \in A$, the difference $u(a, \theta) - u(a', \theta)$ is strict DSC.

Strict DSC can be interpreted as requiring that there is a hyperplane such that the sets $\{\theta : f(\theta) < 0\}$ and $\{\theta : f(\theta) > 0\}$ are either empty, full, or half-spaces defined by that hyperplane, and $\{\theta : f(\theta) = 0\}$ is either empty, full, or that hyperplane. So Figure 2a satisfies strict DSC whereas Figure 2b violates it.

Proposition 6. Assume Θ is open and that

$$\forall a, a' \in A, \text{ the set } \{\theta : u(a, \theta) \geq u(a', \theta)\} \text{ is closed in } \Theta. \quad (9)$$

Then u has convex choice and regular indifferences if and only if it has strict DSCD.

Hence, under Proposition 6's two assumptions, convex choice and regular indifferences are fully characterized by strict DSCD.²⁴ By comparison, Proposition 2 required neither assumption and showed that convex choice alone is "almost" characterized by just DSCD.

To tie Proposition 6 to Grandmont (1978), say that u has *convex weak choice* if for all $a, a' \in A$, the set $\{\theta : u(a, \theta) \geq u(a', \theta)\}$ is convex. In our terminology, Grandmont's condition (H.2) is the conjunction of convex weak choice, convex choice, and regular indifferences.²⁵

²⁴ Strict DSCD implies convex choice and regular indifferences without either assumption. But absent either assumption, strict DSCD can fail despite convex choice and regular indifferences. Here is an example in which Θ is not open: $\Theta = [0, 1]$, $A = \{a', a''\}$, $u(a', \cdot) = 0$, and $u(a'', \theta) = 0$ if $\theta \in \{0, 1\}$ and $u(a'', \theta) > 0$ otherwise. For an example absent (9), consider Figure 6a.

²⁵ The one-dimensional example in footnote 24 has convex choice and regular indifferences but not convex weak choice. Note that given convex choice and regular indifferences, convex weak choice is equivalent to convex indifference, i.e., for all a, a' , the set $\{\theta : u(a, \theta) = u(a', \theta)\}$ is convex.

Grandmont's condition (H.1) is our (9). His result can thus be stated as:

Proposition 7 (Grandmont (1978)). *Assume Θ is open. Then u satisfies (9) and has convex weak choice, convex choice, and regular indifference if and only if for all $a, a' \in A$, either*

1. $\{\theta : u(a', \theta) > u(a, \theta)\} = \Theta$ or $\{\theta : u(a', \theta) < u(a, \theta)\} = \Theta$ or $\{\theta : u(a', \theta) = u(a, \theta)\} = \Theta$;
or
2. there is $\alpha \in \mathbb{R}^n \setminus 0$ and $c \in \mathbb{R}$ such that $\alpha \cdot \theta > c$ if $u(a, \theta) > u(a', \theta)$, $\alpha \cdot \theta < c$ if $u(a, \theta) < u(a', \theta)$, and $\alpha \cdot \theta = c$ if $u(a, \theta) = u(a', \theta)$.

The “if” directions of both Proposition 6 and Proposition 7 are straightforward, so let us explain how each proposition’s “only if” can be obtained from the other. To go from Proposition 7 to Proposition 6, we can first observe that convex choice and regular indifference imply convex weak choice when Θ is open,²⁶ and then observe that the consequent of Proposition 7 implies strict DSCD. To go from Proposition 6 to Proposition 7, it suffices to observe that given (9), strict DSCD implies the consequent of Proposition 7.²⁷

B.5. On McAfee and McMillan's (1988) Single Crossing

In this subsection, we construct a one-dimensional example in which the utility function satisfies McAfee and McMillan's (1988)'s generalized single crossing (GSC) but does not have convex choice. We construct a mechanism that is locally IC but not IC, as predicted by Proposition 1.

Let $\Theta = [0, 1]$, $A = [0, 1]$, and $\kappa > 0$.²⁸ Preferences are given by

$$u(a, \theta) = \begin{cases} -\kappa a - \frac{1}{2} \left(a - \frac{1}{2}\right)^2 \int_0^\theta (2-s) ds & \text{if } a \leq \frac{1}{2} \\ -\kappa a - \frac{1}{2} \left(a - \frac{1}{2}\right)^2 \int_0^\theta (1+s) ds & \text{if } a > \frac{1}{2}. \end{cases}$$

²⁶ Suppose that, contrary to convex weak choice, $u(a, \theta_1) \geq u(a', \theta_1)$ and $u(a, \theta_3) \geq u(a', \theta_3)$ but $u(a, \theta_2) < u(a', \theta_2)$ for some $\theta_2 \in \ell(\theta_1, \theta_3)$. If Θ is open, then there are θ_0 and θ_4 such that $\theta_1, \theta_3 \in \ell(\theta_0, \theta_4)$. If $u(a, \theta) < u(a', \theta)$ for either $\theta \in \{\theta_0, \theta_4\}$, then convex choice fails. If $u(a, \theta) = u(a', \theta)$ for either $\theta \in \{\theta_0, \theta_4\}$, then regular indifference fails. But if $u(a, \theta) > u(a', \theta)$ for both $\theta \in \{\theta_0, \theta_4\}$, then convex choice fails.

²⁷ Suppose u has strict DSCD in direction $\alpha \neq 0$ and the sign of $u(a, \cdot) - u(a', \cdot)$ is not constant (otherwise, point 1 of Proposition 7 follows immediately). Then, (9) implies that there is θ' with $u(a, \theta') = u(a', \theta')$. Consider the hyperplane in the direction of α passing through θ' . Strict DSCD implies that all types on that hyperplane are indifferent between a and a' , all types to the left (i.e., types θ with $\alpha \cdot \theta < \alpha \cdot \theta'$) strictly prefer a , and all types to the right strictly prefer a' . Point 2 of Proposition 7 follows, with $c = \alpha \cdot \theta'$.

²⁸ McAfee and McMillan (1988) allow for transfers. We implicitly set transfers to zero to simplify.

Generalized single crossing. To verify that u satisfies [McAfee and McMillan's GSC](#), we must show that for all a, θ, θ' there is $\lambda > 0$ such that

$$u_a(a, \theta) - u_a(a, \theta') = \lambda u_{a,\theta}(a, \theta')(\theta - \theta'). \quad (10)$$

Using subscripts on u to denote partial derivatives,

$$u_a(a, \theta) = \begin{cases} -\kappa - (a - \frac{1}{2}) \int_0^\theta (2 - s) \, ds & \text{if } a \leq \frac{1}{2} \\ -\kappa - (a - \frac{1}{2}) \int_0^\theta (1 + s) \, ds & \text{if } a > \frac{1}{2}, \end{cases}$$

and

$$u_{a,\theta}(a, \theta) = \begin{cases} -(a - \frac{1}{2})(2 - \theta) & \text{if } a \leq \frac{1}{2} \\ -(a - \frac{1}{2})(1 + \theta) & \text{if } a > \frac{1}{2}. \end{cases}$$

If $u_{a\theta}(a, \theta) > 0$ then $a < \frac{1}{2}$, hence

$$u_a(a, \theta) - u_a(a, \theta') = -\left(a - \frac{1}{2}\right) \int_{\theta'}^\theta (2 - s) \, ds$$

is strictly positive (negative) if and only if $\theta' < (>)\theta$. Similarly, if $u_{a\theta}(a, \theta) < 0$ then $a > \frac{1}{2}$, hence

$$u_a(a, \theta) - u_a(a, \theta') = -\left(a - \frac{1}{2}\right) \int_{\theta'}^\theta (1 + s) \, ds$$

is strictly positive (negative) if and only if $\theta' < (>)\theta$. Lastly, if $u_{a\theta}(a, \theta) = 0$ then $a = \frac{1}{2}$ and $u_a(a, \theta) - u_a(a, \theta') = 0$. Therefore, for all a, θ, θ' there is $\lambda > 0$ satisfying [Equation 10](#).

No convex choice and the insufficiency of local IC. Let $\kappa > 0$ be small and compare $a = 0$ with $a = 1$:

$$\begin{aligned} u(0, \theta) &= -\frac{1}{2} \left(\frac{1}{2}\right)^2 \int_0^\theta (2 - s) \, ds, \\ u(1, \theta) &= -\kappa - \frac{1}{2} \left(\frac{1}{2}\right)^2 \int_0^\theta (1 + s) \, ds. \end{aligned}$$

Hence,

$$\begin{aligned}
u(1, \theta) - u(0, 0) &= \int_0^\theta [(2 - s) - (1 + s)] ds - 8\kappa \\
&= \int_0^\theta (1 - 2s) ds - 8\kappa \\
&= \theta - \theta^2 - 8\kappa,
\end{aligned}$$

with roots

$$\underline{\theta} = \frac{1 - \sqrt{1 - 32\kappa}}{2} \text{ and } \bar{\theta} = \frac{1 + \sqrt{1 - 32\kappa}}{2}.$$

These two types are indifferent between the actions $a = 0$ and $a = 1$, types in $(\underline{\theta}, \bar{\theta})$ strictly prefer $a = 1$, and types outside $[\underline{\theta}, \bar{\theta}]$ strictly prefer $a = 0$. So convex choice—and hence also DSCD—fails. Moreover, the following mechanism is locally IC but not IC:

$$m(\theta) = \begin{cases} 0 & \text{if } \theta \leq \underline{\theta} \\ 1 & \text{otherwise.} \end{cases}$$

B.6. Connected Choice

For this subsection, we replace the maintained assumption that the type space Θ is convex with the weaker assumption that it is connected.

Definition 7. u has *connected choice* if for all $B \subset A$ and $a \in B$,

$$\left\{ \theta : \{a\} = \arg \max_{a' \in B} u(a', \theta) \right\} \text{ is connected.} \quad (11)$$

Definition 8. u has *thin indifference*s if for all $B \subset A$ and $a \in B$,

$$\{\theta : u(a, \theta) \geq u(b, \theta) \forall b \in B\} \subset \text{cl}\{\theta : u(a, \theta) > u(b, \theta) \forall b \in B\}. \quad (12)$$

Thin indifference can hold without **regular indifference**s: consider $\Theta = [0, 1]$, $A = \{a', a''\}$, and $u(a, \theta) = \mathbb{1}\{a = a'\}$ for $\theta \notin \{1/4, 1/2\}$ and $u(a', \theta) = u(a'', \theta)$ for $\theta \in \{1/4, 1/2\}$. Regular indifference can also hold without thin indifference: simply consider a case of total indifference. However, if for every pair of actions there is some type that strictly prefers one and some type that strictly prefers the other, then regular indifference implies thin indifference.

Proposition 8. *If u has connected choice and thin indifferences, then every locally IC mechanism with finite range is IC.*

The idea of the proof below is as follows. Suppose m is locally IC, but it is not optimal for some type θ to be truthful. Let θ' be an optimal report for type θ and suppose, for simplicity, that it is optimal for type θ' to be truthful. Let Θ_b be the types for which $m(\theta')$ is most-preferred in the range of m , let $\Theta_1 \subset \Theta_b$ be the types that get $m(\theta')$ under truthtelling, and let $\Theta_2 := \Theta_b \setminus \Theta_1$. Since Θ_b is connected by assumption, either the closure of Θ_1 intersects Θ_2 or the closure of Θ_2 intersects Θ_1 . In either case, local IC is violated, a contradiction.

Proof of Proposition 8. Fix a locally IC mechanism m with finite range and suppose there is a type θ_0 for which it is not optimal to be truthful. Let $a := m(\theta_0)$ and let $b \neq a$ be one of type θ_0 's most-preferred alternatives in $m(\Theta)$. Without loss of generality, type θ_0 strictly prefers b to any other alternative.²⁹

Define $B := \{\theta : m(\theta) = b\}$. Let A' be a maximal subset of $m(\Theta)$ with the property that $a, b \in m(\Theta_b^{A'})$, where

$$\Theta_b^{A'} := \left\{ \theta : \{b\} = \arg \max_{a' \in A'} u(a', \theta) \right\}.$$

Such a maximal subset exists because $a, b \in m(\Theta_b^{\{b\}}) = m(\Theta)$ and $m(\Theta)$ is finite. Also note that $\Theta_b^{A'} \not\subset B$ because $\theta_0 \in \Theta_b^{A'}$ and $m(\theta_0) = a$. Since u has connected choice, $\Theta_b^{A'}$ is connected; therefore, $\Theta_b^{A'} \cap B$ and $\Theta_b^{A'} \setminus B$ are not separated.³⁰ Hence, either (i) there is $\theta \in B \cap \Theta_b^{A'}$ with $m(N_\theta \cap \Theta_b^{A'}) \not\subset \{b\}$ or (ii) there is $\theta \in \Theta_b^{A'} \setminus B$ with $b \in m(N_\theta \cap \Theta_b^{A'})$. In case (i), there is $\theta' \in N_\theta \cap \Theta_b^{A'}$ with $m(\theta') \neq b$; local IC implies $u(m(\theta'), \theta') \geq u(b, \theta')$ and therefore $m(\theta') \notin A'$. In case (ii), there is $\theta' \in N_\theta \cap \Theta_b^{A'}$ with $m(\theta') = b$; local IC implies $u(m(\theta), \theta) \geq u(b, \theta)$ and therefore $m(\theta) \notin A'$. Hence, in either case, there is $\theta_1 \in \Theta_b^{A'}$ and $c \in m(\Theta) \setminus A'$ such that $m(\theta_1) = b$ and $u(b, \theta_1) \geq u(c, \theta_1)$. Since

$$b \in \arg \max_{a' \in A' \cup m(N_{\theta_1}) \cup \{c\}} u(a', \theta_1)$$

and u has thin indifferences, there is a type $\theta_2 \in N_{\theta_1}$ with

$$\{b\} = \arg \max_{a' \in A' \cup m(N_{\theta_1}) \cup \{c\}} u(a', \theta_2).$$

²⁹ If not, by thin indifferences, every neighborhood of θ_0 contains a type with b as the uniquely most-preferred alternative; such a type in N_{θ_0} does not get b as otherwise local IC would be violated and we can apply our arguments for this type.

³⁰ Two sets are separated if the closure of each set is disjoint from the other set.

Hence, $m(\theta_2) = b$ and the set

$$\Theta_b^{A' \cup \{c\}} := \left\{ \theta : \{b\} = \arg \max_{a' \in A' \cup \{c\}} u(a', \theta) \right\}$$

contains θ_0 and θ_2 and therefore satisfies $a, b \in m\left(\Theta_b^{A' \cup \{c\}}\right)$. Hence, A' was not maximal, a contradiction. Q.E.D.

Note that [Proposition 8](#) also holds under the weaker assumption of connected choice in finite choice problems, i.e., the analog of [Definition 7](#) restricted to finite choice sets B . The next result shows that this assumption is in fact necessary.

Proposition 9. *If u violates connected choice on a finite set,³¹ then there is a mechanism that is locally IC but not IC.*

Proof. Suppose there is a finite subset $B \subset A$ and $a \in B$ such that

$$\Theta_a := \left\{ \theta : \{a\} = \arg \max_{a' \in B} u(a', \theta) \right\}$$

is not connected. Then Θ_a can be partitioned into two nonempty sets Θ_1 and Θ_2 that are separated. Note that $|B| \geq 2$, since Θ is connected. Consider any mechanism m such that $m(\theta) = a$ for all $\theta \in \Theta_1$ and $m(\theta) \in \arg \max_{a' \in B \setminus \{a\}} u(a', \theta)$ for all $\theta \in \Theta \setminus \Theta_1$. Mechanism m is locally IC but not IC. Q.E.D.

We can also show that the assumption of thin indifferences (at least on all finite sets) is necessary in [Proposition 8](#) subject to the following assumption:

$$\text{for all } a \in A \text{ there is } \theta_a \text{ with } \{a\} = \arg \max_{a' \in A} u(a', \theta_a). \quad (13)$$

Proposition 10. *Assume (13). If u violates thin indifferences on a finite set,³² then there is a mechanism that is locally IC but not IC.*

Proof. Suppose u violates thin indifferences on a finite set $B \subset A$. That is, there are $a \in B$, $\theta_0 \in \Theta$, and a neighborhood of θ_0 , call it N_{θ_0} , such that $u(a, \theta_0) \geq u(b, \theta_0)$ for all $b \in B$ and, for all $\theta' \in N_{\theta_0}$, there is $b \in B$ with $u(b, \theta') \geq u(a, \theta')$. By (13), there is θ_a satisfying $\{a\} = \arg \max_{b \in B} u(b, \theta_a)$. Consider any mechanism m such that $m(\theta_0) = a$ and $m(\theta) \in \arg \max_{b \in B \setminus \{a\}} u(b, \theta)$ for all $\theta \neq \theta_0$. Mechanism m is locally IC but not IC. Q.E.D.

³¹ I.e., there is a finite set $B \subset A$ and $a \in B$ such that (11) fails.

³² I.e., there is a finite set $B \subset A$ and $a \in B$ such that (12) fails.