Recursion

Section 3.5

The Recursion Pattern

- Recursion: when a method calls itself
- Classic example: the factorial function:

  $$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$$

- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & \text{else} \end{cases}$$

- As a C++method:

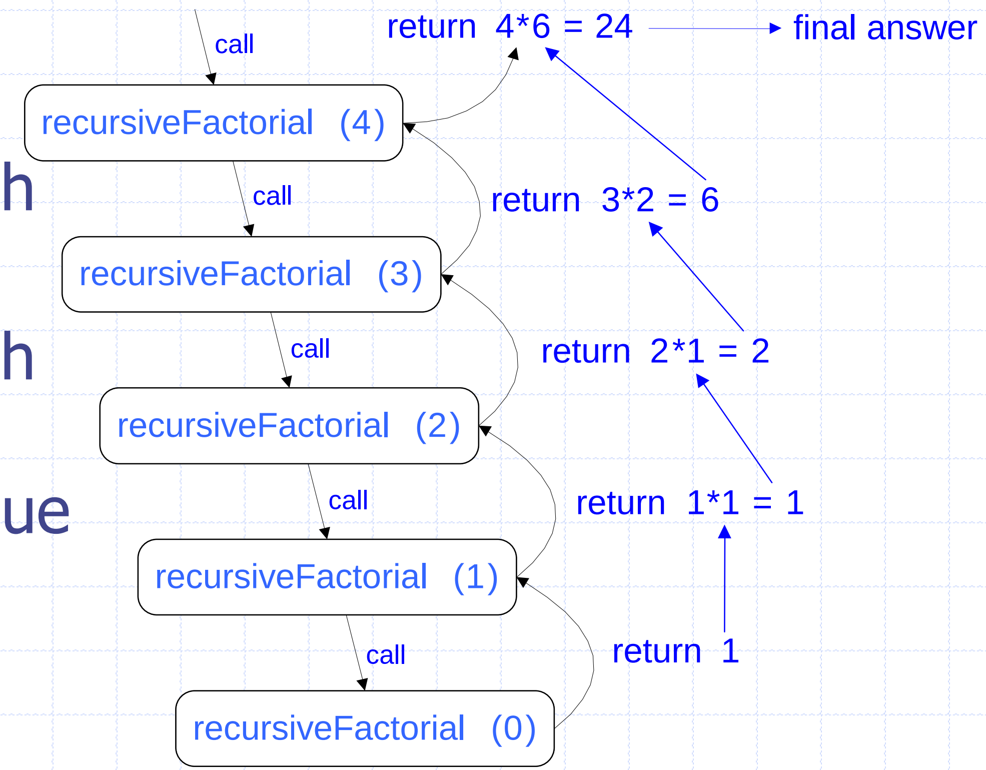
```
// recursive factorial function
int recursiveFactorial(int n) {
    if (n == 0) return 1;      // basis case
    else return n * recursiveFactorial(n- 1);  // recursive case
}
```

Content of a Recursive Method

- Base case(s)
  - Values of the input variables for which we perform no recursive calls are called base cases (there should be at least one base case).
  - Every possible chain of recursive calls must eventually reach a base case.
- Recursive calls
  - Calls to the current method.
  - Each recursive call should be defined so that it makes progress towards a base case.

Visualizing Recursion

- Recursion trace
  - A box for each recursive call
  - An arrow from each caller to callee
  - An arrow from each callee to caller showing return value
- Example



Example: English Ruler

- Print the ticks and numbers like an English ruler:

```
---- 0          ----- 0         --- 0
-               -               -
--              --              --
-               -               -
---             ---             --- 1
-               -               -
--              --              --
-               -               -
---- 1          ----            --- 2
-               -               -
--              --              --
-               -               -
---             ---             --- 3
-               -
--              --
-               -
---- 2          ----- 1
```

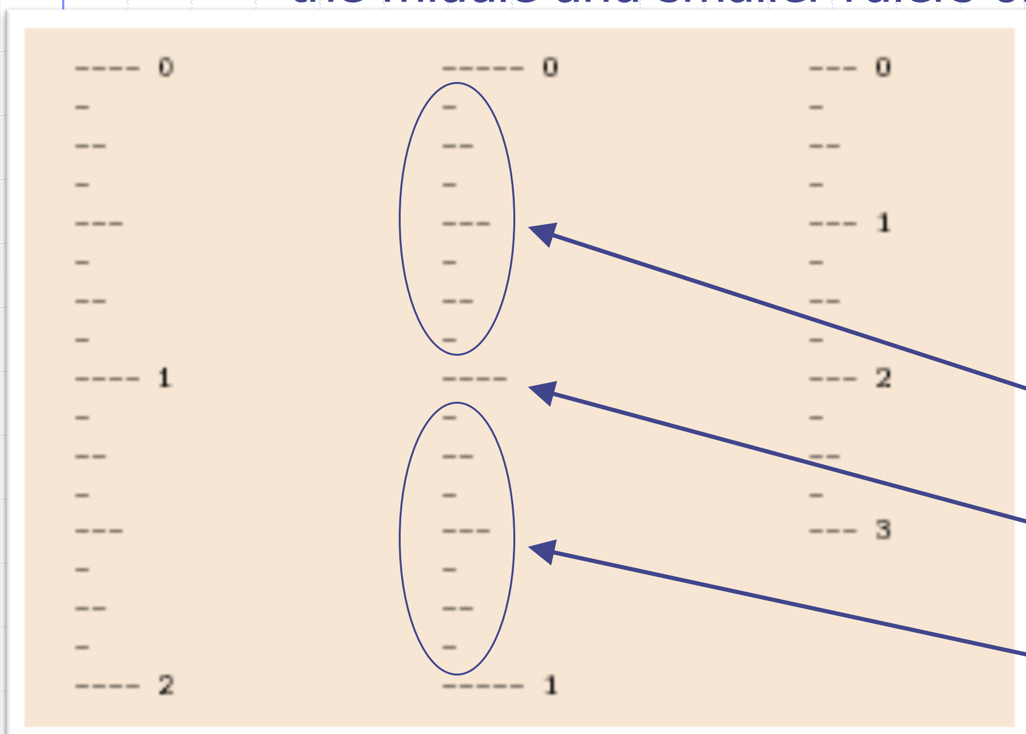
Slide by Matt Stallmann included with permission.

Using Recursion

drawTicks(length)

Input: length of a ‘tick’

Output: ruler with tick of the given length in the middle and smaller rulers on either side



drawTicks(length)

if(length > 0) then

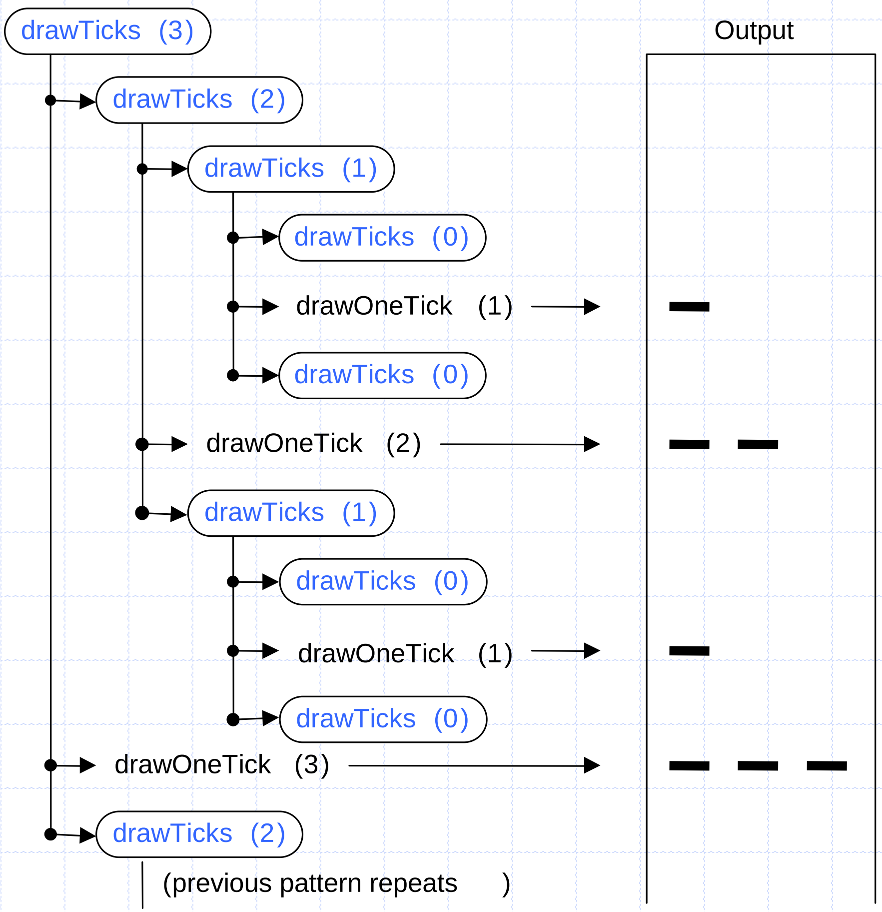
drawTicks(length – 1)

draw tick of the given length

drawTicks(length – 1)

Recursive Drawing Method

- The drawing method is based on the following recursive definition
- An interval with a central tick length L $\geq$1 consists of:
  - An interval with a central tick length L–1
  - An single tick of length L
  - An interval with a central tick length L–1



C++ Implementation (1)

```
// draw ruler
void drawRuler(int nInches, int majorLength) {
  drawOneTick(majorLength, 0);                 // draw tick 0 and its label
  for (int i = 1; i <= nInches; i++){
      drawTicks(majorLength- 1);               // draw ticks for this inch
      drawOneTick(majorLength, i);             // draw tick i and its label
  }
}

// draw ticks of given length
void drawTicks(int tickLength) {
  if (tickLength > 0) {                        // stop when length drops to 0
      drawTicks(tickLength- 1);                // recursively draw left ticks
      drawOneTick(tickLength);                 // draw center tick
      drawTicks(tickLength- 1);                // recursively draw right ticks
  }
}
```

C++ Implementation (2)

```
// draw a tick with no label
void drawOneTick(int tickLength) {
    drawOneTick(tickLength, - 1);
}

// draw one tick
void drawOneTick(int tickLength, int tickLabel) {
    for (int i = 0; i < tickLength; i++)
        cout << "-";
    if (tickLabel >= 0) cout << " " << tickLabel;
    cout << "\n";
}
```

Recursion Examples

Example 3.2 in the text: Programming languages are often defined in a recursive way. We can define an argument list in C++ as follows:

argument-list: ε
nonempty-argument-list

nonempty-argument-list: *argument*
nonempty-argument-list , *argument*

That is, an argument list consists of either (i) the empty string, (ii) an argument, or (iii) a nonempty argument list followed by a comma and an argument.

foo(); bar(14); bletch(23.1, ‘a’, 14);

Example of Linear Recursion

Algorithm LinearSum(*A, n*):

Input:

A integer array *A* and an integer *n* = 1, such that *A* has at least *n* elements

Output:

The sum of the first *n* integers in *A*

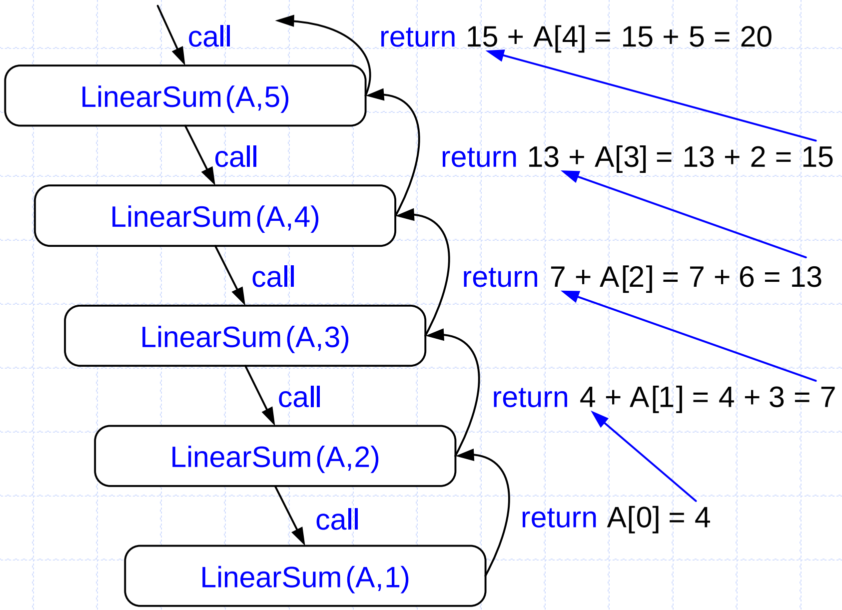
if *n* = 1 **then**

return *A*[0]

else

return LinearSum(*A, n* - 1) + *A*[*n* - 1]

Example recursion trace:



Reversing an Array

Algorithm ReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

```
if i < j then
    Swap A[i] and A[ j]
    ReverseArray(A, i + 1, j - 1)
return
```

Defining Arguments for Recursion

- In creating recursive methods, it is important to define the methods in ways that facilitate recursion.
- This sometimes requires we define additional paramaters that are passed to the method.
- For example, we defined the array reversal method as ReverseArray(A, i, j), not ReverseArray(A).

Computing Powers

- The power function, $p(x,n)=x^n$, can be defined recursively:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x,n-1) & \text{else} \end{cases}$$

- This leads to an power function that runs in O(n) time (for we make n recursive calls).
- We can do better than this, however.

Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } x > 0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } x > 0 \text{ is even} \end{cases}$$

- For example,

$$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$$
$$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$$
$$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$$
$$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128.$$

Recursive Squaring Method

Algorithm Power(x, n):
 Input: A number x and integer $n = 0$
 Output: The value x^n
 if $n = 0$ **then**
  return 1
 if n is odd **then**
  $y = $ Power(x, $(n - 1)/ 2$)
  return $x \cdot y \cdot y$
 else
  $y = $ Power(x, $n/ 2$)
  return $y \cdot y$

Analysis

Algorithm Power(x, n):
Input: A number x and integer $n = 0$
Output: The value x^n
if $n = 0$ **then**
  return 1
if n is odd **then**
  $y = \text{Power}(x, (n - 1)/2)$
  return $x \cdot y \cdot y$
else
  $y = \text{Power}(x, n/2)$
  return $y \cdot y$

Each time we make a recursive call we halve the value of n; hence, we make log n recursive calls. That is, this method runs in O(log n) time.

It is important that we use a variable twice here rather than calling the method twice.

Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its last step.
- The array reversal method is an example.
- Such methods can be easily converted to non-recursive methods (which saves on some resources).
- Example:

Algorithm IterativeReverseArray(A, i, j):

__Input:__ An array A and nonnegative integer indices i and j

__Output:__ The reversal of the elements in A starting at index i and ending at j

while $i < j$ **do**

Swap $A[i]$ and $A[j]$

$i = i + 1$

$j = j - 1$

return

Binary Recursion

- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.
- Example: the DrawTicks method for drawing ticks on an English ruler.

```
---- 0          ----- 0          --- 0
-               -                -
--              --               --
-               -                -
---             ---              --- 1
-               -                -
--              --               --
-               -                -
---- 1          ----             --- 2
-               -                -
--              --               --
-               -                -
---             ---              --- 3
-               -
--              --
-               -
---- 2          ----- 1
```

A Binary Recursive Method for Drawing Ticks

```
      // draw a tick with no label
public static void drawOneTick(int tickLength) { drawOneTick(tickLength, - 1); }
      // draw one tick
public static void drawOneTick(int tickLength, int tickLabel) {
   for (int i = 0; i < tickLength; i++)
      System.out.print("-");
   if (tickLabel >= 0) System.out.print(" " + tickLabel);
   System.out.print("\n");
}
public static void drawTicks(int tickLength) { // draw ticks of given length
   if (tickLength > 0) {            // stop when length drops to 0
      drawTicks(tickLength- 1);     // recursively draw left ticks
      drawOneTick(tickLength);      // draw center tick
      drawTicks(tickLength- 1);     // recursively draw right ticks
   }
}
public static void drawRuler(int nInches, int majorLength) { // draw ruler
   drawOneTick(majorLength, 0);     // draw tick 0 and its label
   for (int i = 1; i <= nInches; i++) {
      drawTicks(majorLength- 1);    // draw ticks for this inch
      drawOneTick(majorLength, i);  // draw tick i and its label
   }
}
```

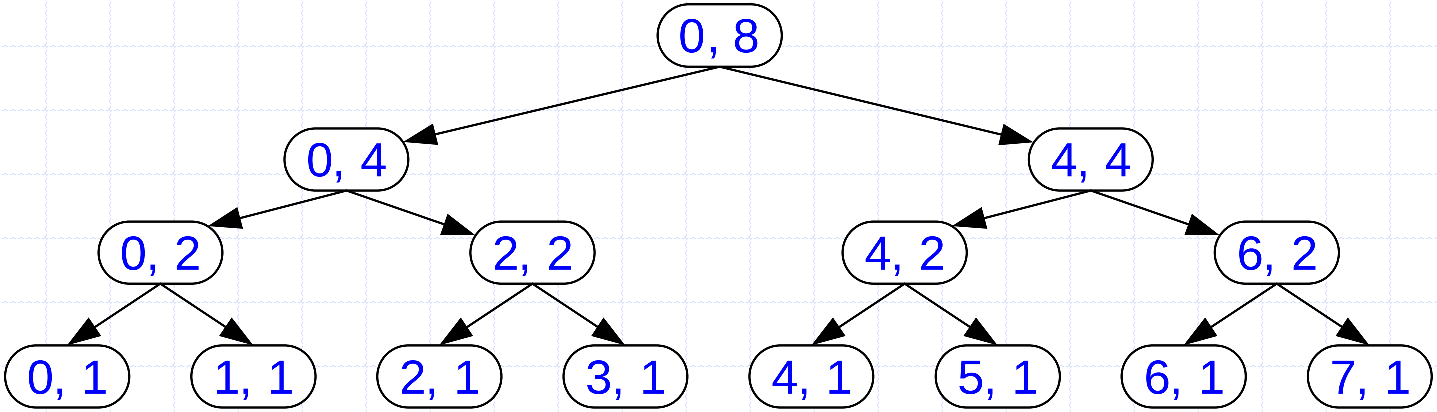
Note the two recursive calls

Another Binary Recusive Method

- Problem: add all the numbers in an integer array A:

Algorithm BinarySum(A, i, n):
  Input: An array A and integers i and n
  Output: The sum of the n integers in A starting at index i
  if $n = 1$ **then**
  return $A[i]$
  return BinarySum(A, i, $n/2$) + BinarySum(A, $i + n/2$, $n/2$)

- **Example trace:**



Computing Fibonacci Numbers

- Fibonacci numbers are defined recursively:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_i = F_{i-1} + F_{i-2} \quad \text{for } i > 1.$$

- Recursive algorithm (first attempt):

Algorithm BinaryFib(k):

Input: Nonnegative integer k

Output: The kth Fibonacci number F_k

if $k = 1$ **then**

return k

else

return BinaryFib(k - 1) + BinaryFib(k - 2)

Analysis

- Let n_k be the number of recursive calls by BinaryFib(k)
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67.$
- Note that n_k at least doubles every other time
- That is, $n_k > 2^{k/2}$. It is exponential!

A Better Fibonacci Algorithm

- Use linear recursion instead

```
Algorithm LinearFibonacci(k):
   Input: A nonnegative integer k
   Output: Pair of Fibonacci numbers (F_k , F_{k−1})
  if k = 1 then
      return (k, 0)
  else
      (i,  j)  =  LinearFibonacci(k − 1)
      return (i +j, i)
```

Algorithm LinearFibonacci(k):
Input: A nonnegative integer k
Output: Pair of Fibonacci numbers (F_k , F_{k-1})
if $k = 1$ **then**
return (k, 0)
else
(i, j) = LinearFibonacci($k - 1$)
return (i +j, i)

- LinearFibonacci makes k−1 recursive calls

Multiple Recursion

- Motivating example:
  - summation puzzles
    - *pot* + *pan* = *bib*
    - *dog* + *cat* = *pig*
    - *boy* + *girl* = *baby*
- Multiple recursion:
  - makes potentially many recursive calls
  - not just one or two

Algorithm for Multiple Recursion

Algorithm PuzzleSolve(k,S,U):
 Input: Integer k, sequence S, and set U (universe of elements to test)
 Output: Enumeration of all k-length extensions to S using elements in U without repetitions
 for all e in U **do**
  Remove e from U {e is now being used}
  Add e to the end of S
  if k = 1 **then**
   Test whether S is a configuration that solves the puzzle
   if S solves the puzzle **then**
    return "Solution found: " S
  else
   PuzzleSolve(k - 1, S,U)
  Add e back to U {e is now unused}
  Remove e from the end of S

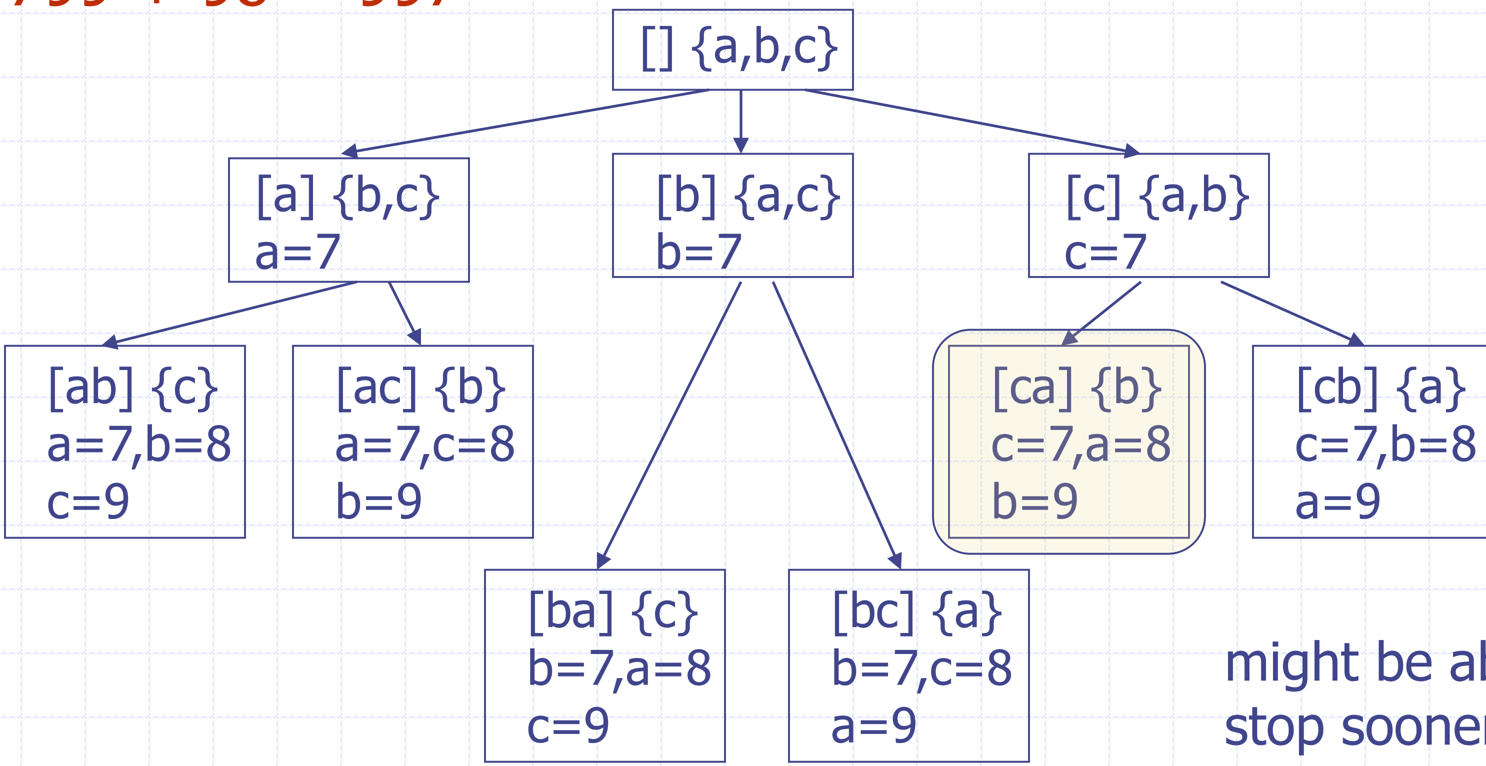
Slide by Matt Stallmann included with permission.

Example

cbb + ba = abc

799 + 98 = 997

a,b,c stand for 7,8,9; not necessarily in that order



might be able to stop sooner

Visualizing PuzzleSolve