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SERIES (UNIT 13)

13.2 The definition of series

1. Definition

Let $(a_n)_{n=1}^{\infty}$ 是一个 sequence. Define the nth partial sum of (a_n) to be

$$S_n = \sum_{k=1}^{n} a_k.$$

We say that the infinite series $\sum_{n=1}^{\infty} a_n$ converges if the sequence $(S_n)_{n=1}^{\infty}$ converges as a sequence, and we set

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n.$$

We say that the infinite series diverges otherwise.

2. Series vs sequence

- A series is an infinite sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

- A sequence is an infinite list:

$$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \dots$$

13,3 A telescopic series

1. Example

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2 + n}$$

$$S_1 = \frac{1}{1^2 + 1} = \frac{1}{2}$$

$$S_2 = \frac{1}{2^2 + 2} = \frac{2}{3}$$

$$S_3 = \frac{1}{3^2+3} = \frac{3}{4}$$

Conjecture: $\forall k \geq 1, S_k = \frac{k}{k+1}$

Pf: By induction

Alternative:

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2+n} = \sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$= \left[1 - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \cdots + \left[\frac{1}{k} - \frac{1}{k+1}\right] = \frac{k}{k+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

13.4 example of divergent series from the definition

1. Example

$$S = \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots = \infty$$

$$S = \lim_{k\to\infty} S_k$$

$$S_k = \sum_{n=1}^{k} 1 = k$$

$$S = \lim_{k\to\infty} k = \infty$$

$\sum_{n=1}^{\infty} 1$ is divergent

2. Example

$$S = \sum_{n=1}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - \cdots$$

$$S = \lim_{k\to\infty} S_k$$

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$$S_k = \sum_{n=1}^{k} (-1)^n$$

$$S_k = \begin{cases} 0 & \text{if k odd} \\ 1 & \text{if k even} \end{cases}$$

$\sum_{n=1}^{\infty} (-1)^n$ is divergent

13.5 Geometric series

1. Definition

Geometric Series: A geometric series is a series defined by a sequence (a_n) where $a_n = ra_{n-1}$ for some $r \in \mathbb{R}$. For example,

$$a_0 = 1,\ a_1 = \frac{1}{2},\ a_2 = \frac{1}{4},\ a_3 = \frac{1}{8},\ a_4 = \frac{1}{16}, \ldots$$

satisfies the relation $a_n = \frac{1}{2}a_{n-1}$. We can write such series as

$$\sum_{k=0}^{\infty} r^k.$$

2. Theorem

For any $|r| < 1$ we have $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$, and the series diverges otherwise.

Proof. The partial sums are given by $S_n = 1 + r + r^2 + \cdots + r^n$ and so

$$\begin{aligned} (1-r)s_n &= 1 + r + r^2 + r^3 + r^4 + \cdots + r^n \\ &\qquad - r - r^2 - r^3 - r^4 + \cdots - r^n - r^{n+1} \\ &= 1 - r^{n+1} \end{aligned}$$

so that $s_n = \frac{1 - r^{n+1}}{1-r}$. Now in the limit as $n \to \infty$ we have that $r^{n+1} \xrightarrow{n\to\infty} 0$, so

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \frac{1 - r^{n-1}}{1-r} = \frac{1}{1-r}.$$

13.6 series are linear

1. Theorem

Let $\sum_{k=1}^{\infty} a_n$ and $\sum_{k=1}^{\infty} b_n$ 是 convergent series.

1. The sum of the series 是 convergent and

$$\sum_{k=1}^{\infty}(a_n+b_n)=\sum_{k=1}^{\infty}a_n+\sum_{k=1}^{\infty}b_n.$$

2. For any $\alpha>0,$ we have

$$\sum_{k=1}^{\infty}(\alpha a_n)=\alpha\sum_{k=0}^{\infty}a_n.$$

2. Proof

Proof. Let $(s_n)\to L$ be the partial sums of $\sum_{k=1}^{\infty} a_n$ and $(t_n)\to M$ be the partial sums for $\sum_{k=1}^{\infty} b_n$. The partial sum of the sum is given by

$$u_n=\sum_{k=1}^{\infty}(a_n+b_n)=\sum_{k=1}a_n+\sum_{k=1}b_n=s_n+t_n.$$

By Theorem 9.6, we have $(u_n)=(s_n+t_n)\to L+M$, and so

$$\sum_{k=1}^{\infty}(a_n+b_n)=\sum_{k=1}^{\infty}a_n+\sum_{k=1}^{\infty}b_n.$$

The proof for (2) is similar. □

13.7 The tail of a series

1. Theorem

$$\sum_{n=0}^{\infty} a_n \text{ is convergent } \Leftrightarrow \sum_{n=1}^{\infty} a_n \text{ is convergent}$$

13.8 A necessary condition for convergence of series

1. Theorem

If the series $\sum_{k=1}^{\infty} a^k$ is convergent, then $\lim_{n\to\infty} a_n = 0$

How do we use this in practice?

- 如果 $\lim_{n\to\infty} a_n = 0$, 那么 the series $\sum_{n=1}^{\infty} a^n$ may be convergent or divergent.
- 如果 $\lim_{n\to\infty} a_n \neq 0$, 那么 the series $\sum_{n=1}^{\infty} a^n$ is divergent.

2. Example

- Is $\sum_{n=0}^{\infty} \frac{\arctan n}{1+e^{-n}}$ convergent?

  $\lim_{n\to\infty} \frac{\arctan n}{1+e^{-n}} = \frac{\pi}{2} \neq 0.$ so $\sum_{n=0}^{\infty} \frac{\arctan n}{1+e^{-n}}$ is divergent.

- Are $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent?

  $\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^2}$, so 我们得不到任何结果.

3. Proof

Proof. Let (S_n) be the partial sums, so that $(S_n) \to L$ for some limit L. Since $S_n = S_{n-1} + a_n$ we have $a_n = S_n - S_{n-1}$, so that $(a_n) \to L - L = 0$. $\square$

13.9 Positive series

1. Definition

We call a series $\sum_{n=0}^{\infty} a_n$ positive when $\forall n \in \mathbb{N}$, $a_n > 0$.

Then A may be $\begin{cases} \text{convergent} \\ \text{divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ \text{"oscillating"} \end{cases} \end{cases}$

A positive series may be $\begin{cases} \text{convergent (a number)} \\ \text{divergent to } \infty \end{cases}$

2. proof that this are the only two options

- In general, $\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} S_k$, where $S_k = \sum_{n=0}^{k} a_n$.
- Assume the series $\sum_{n=0}^{\infty} a_n$ is positive.

  Then the sequence $\{S_n\}_{n=0}^{\infty}$ is increasing: $S_{n+1} - S_n = a_{n+1}$
- If $\{S_k\}_{n=0}^{\infty}$ is bounded above, then it is convergent. (MCT)
- if $\{S_k\}_{n=0}^{\infty}$ is unbounded above, then $\lim_{k\to\infty} S_k = \infty$

13.10 The integral test

1. Theorem [integral test]

If f is a continuous, non-negative, decreasing function on $[1,\infty)$ then

$\sum_{k=1}^{\infty} f(k)$ converges $\Leftrightarrow$ $\int_1^{\infty} f(x)dx$ converges.

Notation:

$$\int_a^{\infty} f(x)dx \sim \sum_n^{\infty} f(n)$$

13.11 the examples of the integral test

1. Example 1 "p-series"

- for which values of p>0 is the series$\sum_{n=1}^{\infty}\frac{1}{n^p}$ convergent?
- Let $f(x)=\frac{1}{x^p}$. For $x \geq 1$, f 是 continuous, positive, and decreasing.
- By integral test, $\sum_{n=1}^{\infty}\frac{1}{n^p} \sim \int_1^{\infty}\frac{1}{x^p}dx$.
- We know $\int_1^{\infty}\frac{1}{x^p}dx$ is convergent iff p>1.
- Thus, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent iff p>1.

2. Example 2

Is $\sum_{n=2}^{\infty}\frac{1}{n\cdot ln\, n}$ convergent?

- Let $f(x)=\frac{1}{x\ln x}$. It is positive, decreasing, continuous for $x \geq 2$.
- By integral test, $\sum_{n=2}^{\infty}\frac{1}{n\ln n} \sim \int_2^{\infty}\frac{1}{x\ln x}dx$
- $\int_2^{\infty}\frac{1}{x\ln x}dx = \lim_{b\to\infty}[\int_2^b \frac{1}{x\ln x}dx] = \lim_{b\to\infty}[\ln(\ln x)|_2^b]$
  $= \lim_{b\to\infty}[\ln(\ln b)-\ln(\ln 2)] = \infty$
- Therefore, $\sum_{n=2}^{\infty}\frac{1}{n\cdot ln\, n} = \infty$

13.12 Comparison tests for convergence of series

1. Theorem Basic Comparison Test for Series

Suppose that (a_n) and (b_n) are sequences such that $0 \le a_k \le b_k$ for sufficiently large k.

If $\sum_k b_k$ converges, then $\sum_k a_k$ converges.

If $\sum_k a_k$ diverges, then $\sum_k b_k$ diverges.

2. Theorem: Limit Comparison Test for Series

If (a_n) and (b_n) are sequences with positive terms, and $\lim_{n\to\infty} \frac{a_k}{b_k} = L$ for some $0 < L < \infty$ then

$$\sum_k a_k \text{ converges} \Leftrightarrow \sum_k b_k \text{ converges.}$$

13.13 Alternative series

1. Definition

A series $\sum_n^\infty a_n$ 是 alternating when $\forall n, a_n a_{n+1} < 0$. This means the terms “alternate” 在 positive 和 negative 之间.

2. Lemma

Let $\{c_n\}_n^\infty$ be a sequence.

- IF the sequences of even and odd terms

  $\{c_{2n}\}_n^\infty$ and $\{c_{2n+1}\}_n^\infty$

  Are convergent to the same limit
- Then the full sequence $\{c_n\}_n^\infty$ is also convergent to the same limit

3. Theorem: Alternating Series Test

If $\left(a_k\right)$ 是一个 positive decreasing sequence, the alternating series $\sum_k (-1)^k a_k$ converges if and only if $\lim_{k\to\infty} a_k = 0$.

13.14 Estimating the value of an alternating series

1. Estimate the value of

Estimate the value of

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$

With an error smaller than 0.001

By the AST, the series $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ is convergent.

- Actual value: $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = \lim_{k\to\infty} S_k$
- Estimate: $S_k = \sum_{n=1}^{k} \frac{(-1)^n}{n^4}$ for some large k?

  Which value of k to use?
- Error of the estimation: $|S - S_k|$

2. Theorem Alternating Series Test part 2

Consider a series of the form $\sum_{n=1}^{\infty} (-1)^n b_n$ or $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$, if it satisfies the alternating series test, then $|S - S_k| < b_{k+1}$ where S_k is the k-th partial sum of the series.

- Estimate: $S_k = \sum_{n=1}^{k} \frac{(-1)^n}{n^4}$
- Error of the estimation: $|S - S_k| < \frac{1}{(k+1)^4}$

  Need to choose k so that $\frac{1}{(k+1)^4} < 0.001.$

  K=5 works
- Estimate: $S_5 = -1 + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} \approx -0.94753$
- Actual value: $S \approx -0.94703$

13.15 Absolute vs conditional convergence

1. Theorem: Absolute Convergence

If $\sum_k |a_k|$ converges then $\sum a_k$ converges; that is, all absolutely convergent series are convergent.

	$\sum_n^{\infty} \lvert a_n \rvert$ convergent	$\sum_n^{\infty} \lvert a_n \rvert = \infty$
$\sum_n^{\infty} a_n$ convergent	ABSOLUTELY CONVERGENT	CONDITIONALLY CONVERGENT
$\sum_n^{\infty} a_n$ divergent	Impossible	DIVERGENT

2. definition

A convergent series $\sum_{n=1}^{\infty} a_n$ is

- Absolutely convergent when, in addition, $\sum_{n=1}^{\infty} |a_n|$ is convergent.
- Conditionally convergent when, in addition, $\sum_{n=1}^{\infty} |a_n| = \infty$.

Conditionally convergent series are not conmmutative

3. example

(1) Is $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ convergent?

- First, I look at $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$

  Since, it is a positive series, I can use comparison tests.
- $0 \leq \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$. We know $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

  By the BCT, $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ is also convergent.
- Thus, by the Abs.C.T. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is also convergent.

13.16 A proof of the Absolute convergence test

Proof. The crux of this proof follows from the fact that taking absolute values means that the terms of the series get larger, and hence are less likely converge. In fact, we shall use the Basic Comparison Test to show exactly this. Notice that since

$$-|a_k| \le a_k \le |a_k| \quad \text{then} \quad 0 \le a_k + |a_k| \le 2|a_k|.$$

The series formed by the terms on the right-hand-side is just $2\sum_k |a_k|$ and hence converges by assumption. By the Basic Comparison Test, it follows that the series $\sum_k (a_k + |a_k|)$ converges. Since $a_k = (a_k + |a_k|) - |a_k|$ the linearity of the series implies that

$$\sum a_k = \underbrace{\sum_k (a_k + |a_k|)}_{\text{finite}} - \underbrace{\sum |a_k|}_{\text{finite}} < \infty. \qquad \square$$

13.17 Infinite sums are not commutative

1. Theorem

- 如果 a series is absolutely convergent, 那么 it can be “reordered”. This won’t change the value of the sum.
- 如果 a series is conditionally convergent, 那么 ⋯

  ⋯ reordering it may change its value

  ⋯ it can be reordered to make it convergent to any number we want.

  ⋯ it can be reordered to make it divergent in any way we want.

2. task

Reorder $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ so the sum is 7

- Add positive terms until a partial sum goes above 7.

  (possible because the positive terms add to ∞)
- Add negative terms until a partial sum goes above 7

  (possible because the positive terms add to -∞)
- Go back to step 1.

  Since $\lim_{n\to\infty} \frac{1}{n} = 0$, each time the partial sums go above or below 7, they stay closer to 7.

  Thus, the limit of the sequence of partial sums is 7.

3. summery

Absolutely convergent series behave like finite sums

13.18 Ratio test: the theorem

1. ratio test

Let $\sum_{n=1}^{\infty} a_n$ be a series. Assume $\forall n, a_n \neq 0$. Assume the limit

$$L = \lim_{N\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Exists or is ∞

- If L<1, then the series$\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If L=1, then we draw no conclusion
- If L>1, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

2. proof

Proof. The idea of the proof is as follows: If the ratio converges to $L \neq 1$, then for sufficiently large N we can treat $a_{k+1} = La_k$ for all $k \geq N$. At this point the series effectively behaves like a geometric series $a_N \sum_i L^i$, which will converge if $L < 1$ and diverge if $L > 1$.

Assume that $L < 1$ and choose some ℓ such that $L < \ell < 1$. Since the ratio converges to L, we know there exists some $N \in \mathbb{N}$ such that if $k \geq N$ then $|a_{k+1}/a_k| < \ell$ (convince yourself of this,

using ϵ-N if necessary). In particular, this means that $|a_{k+1}| < \ell|a_k|$. In particular,

$$|a_{N+1}| < \ell|a_N|, \quad |a_{N+2}| < \ell^2|a_N|, \quad |a_{N+3}| < \ell^3|a_N|, \ldots, |a_{N+k}| < \ell^k|a_n|.$$

Using the Basic Comparison Test, we have $|a_k| \leq \ell^k|a_N|$, with the right-hand-side being a geometric series with common ratio $\ell < 1$ and hence converging. We conclude that $\sum_k |a_k|$ converges as well, so $\sum_k a_k$ converges.

Precisely the same reasoning holds for (2), though this train of argument cannot be used when $L = 1$. $\square$

13.19 ratio test: example

1. example 1

$$\sum_{n}^{\infty} \frac{(-1)^n e^{2n}}{n!\,(n+3)}$$

Call $a_n = \frac{(-1)^n e^{2n}}{n!(n+3)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{e^{2(n+1)}}{(n+1)!\,(n+4)}}{\frac{e^{2n}}{n!\,(n+3)}}$$

$$= \lim_{n\to\infty}\frac{e^{2(n+1)}}{e^{2n}}\frac{n!}{(n+1)!}\frac{n+3}{n+4}$$

$$= \lim_{n\to\infty}\frac{e^2(n+3)}{(n+1)(n+4)} = 0$$

0<1 so, by the Ratio Tests the original series converges absolutely.

2. example 2

$$\sum_{n}^{\infty} \frac{n+1}{n^2+2}$$

Call $a_n = \frac{n+1}{n^2+2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{(n+1)+1}{(n+1)^2+2}}{\frac{n+1}{n^2+2}} = \lim_{n\to\infty}\frac{(n+2)(n^2+2)}{(n^2+2n+3)(n+1)} = \cdots = 1$$

Since the limit is 1m the Ratio Test is inconclusive.

Use limit-Comparison Test instead! Compare with $\sum_{n}^{\infty}\frac{1}{n}$

Test	When to Use	Conclusions
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ $if\ \lvert r\rvert < 1;\ diverges\ if\ \lvert r\rvert \geq 1.$
Necessary Condition	All series	If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.
Integral Test	• $a_n = f(n)$ • f is continuous, positive and decreasing. • $\int_1^{\infty} f(x)dx$ is easy to compute	$\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ Both converge or both diverge.
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	Converges for p > 1; diverges for $p \leq 1$.
Basic Comparison Test	$0 \leq a_n \leq b_n$	If $\sum_{n=1}^{\infty} b_n$ converges,then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges
Limit Comparison Test	$a_n, b_n > 0\ and\ \lim_{n\to\infty} \frac{a_n}{b_n} = L\ (0 < L < \infty)$	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge
Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^n b_n, b_n \geq 0$	If $b_n > 0, \forall n$ • $\{b_n\}$ is decreasing • $\lim_{n\to\infty} b_n = 0$ Then $\sum_{n=1}^{\infty} (-1)^n b_n$ is convergent.
Absolute Convergence	Series with some positive terms and some negative terms (including alternating series)	If $\sum_{n=1}^{\infty} \lvert a_n\rvert$ converges, then $\sum_{n=1}^{\infty} a_n$ converges (absolutely).
Ratio Test	Any series (especially those involving exponentials and/or factorials)	For $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = L\ (including\ L = \infty)$, • If L < 1,then $\sum_{n=1}^{\infty} a_n$ converges absolutely • If L >1, then $\sum_{n=1}^{\infty} a_n$ diverges • If L = 1, then we can draw no conclusion.

EXERCISE

1. Geometric series. You have learned that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad if \quad |x|<1$$

and the series is divergent if $|x| \geq 1$. Calculate the following infinite sums:

a) $\sum_{n=0}^{\infty} (\ln 2)^n$

b) $\sum_{n=0}^{\infty} (\ln 3)^n$

c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{e^{2n+3}}$

d) $\sum_{n=m}^{\infty} x^n$

e) $\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^5}+\frac{1}{5^7}+\frac{1}{5^8}+\frac{1}{5^{10}}+\frac{1}{5^{11}}+\dots$

f) $\frac{1}{2^{0.5}}+\frac{1}{2}+\frac{1}{2^{1.5}}-\frac{1}{2^2}+\frac{1}{2^{2.5}}+\frac{1}{2^3}+\frac{1}{2^{3.5}}-\frac{1}{2^4}+\frac{1}{2^{4.5}}+\frac{1}{2^5}+\frac{1}{2^{5.5}}-\frac{1}{2^6}+\dots$

2. Telescopic series. Calculate the value of the following infinite sums. In all cases, you can start by finding a formula for the N-th partial sum, and then taking the limit.

   a) $\sum_{n=0}^{\infty}\left[\arctan n - \arctan(n+1)\right]$

   b) $\sum_{n=1}^{\infty}\left[\ln\frac{n}{n+1}\right]$

   c) $\sum_{n=1}^{\infty}\frac{1}{n^2+3n}$

   d) $\sum_{n=3}^{\infty}\frac{n+2}{n^3-n}$

Hint: For Question 2c, write $\frac{1}{n^2+3n}=\frac{A}{n}+\frac{B}{n+3}$. Something similar helps for 2d.

3. Infinite decimal expansions. We can interpret any finite decimal expansion as a finite sum. For example:

$$2.13096 = 2 + \frac{1}{10} + \frac{3}{10^2} + \frac{0}{10^3} + \frac{9}{10^4} + \frac{6}{10^5}$$

Similarly, we can interpret any infinite decimal expansion as an infinite series.

Interpret the following numbers as series, and add up the series to calculate their value as fractions:

a) $0.99999\cdots$
b) $0.11111\cdots$
c) $0.252525\cdots$
d) $0.3121212\cdots$

Determine which ones of the following series are absolutely convergent, conditionally convergent, or divergent.

1. $\sum_{n=1}^{\infty} \frac{e^{1-1/n}}{3+\sin n}$
2. $\sum_{n=1}^{\infty} \frac{1}{n}$
3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
4. $\sum_{n=1}^{\infty} \frac{1}{n^2}$
5. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+6}$
6. $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^5+4n+11}}$
7. $\sum_{n=1}^{\infty} (-1)^n \sin\frac{1}{n}$
8. $\sum_{n=1}^{\infty} (-1)^n n \sin\frac{1}{n}$

9. $\sum_{n=1}^{\infty}\frac{(n+3)2^n}{n!}$

10. $\sum_{n=1}^{\infty}\frac{n!(2n)!}{(3n)!}$

11. $\sum_{n=1}^{\infty}\frac{1}{n^n}$

12. $\sum_{n=1}^{\infty}\frac{2\cdot4\cdot6\cdot\ldots\cdot(2n)}{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}\cdot\frac{\pi^{n+1}}{e^{2n-1}}$

13. $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$

14. $\sum_{n=1}^{\infty}\frac{n!}{10^{4n}}$

15. $\frac{1}{2}+\frac{2}{3^2}-\frac{4}{4^3}+\frac{8}{5^4}+\frac{16}{6^5}-\frac{32}{7^6}+\ldots$

16. $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\ln n}$
17. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
18. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$
19. $\sum_{n=2}^{\infty} \frac{\ln n}{n^{1.1}}$
20. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$

1. Assume we know the following about a sequence: $\lim_{n\to\infty} a_n = 0$. With only this information, we are going to show you various series. For each one of them decide whether the series must be convergent, the series must be divergent, or it could be either. If you think the series must be convergent, prove it. If you think the series must be divergent, prove it. If you think it could be either, give one example that shows the series may be convergent and one example that shows the series may be divergent.

   (a) The series $\sum_{n=0}^{\infty} a_n$.

   Solution: EITHER.

   For example, notice that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent. However, those series start at $n = 1$, so we will modify them a bit so that they start at $n = 0$ instead. Instead of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ we could use $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ or $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$ or $\sum_{n=0}^{\infty} a_n$ where $a_n = 1/n$ when $n \geq 1$ and $a_0 = 0$, for example. We will do the first one in detail.

   (i) Letting $a_n = \frac{1}{1+n^2}$, we claim that $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ is convergent. To see this, note that $f(x) = \frac{1}{1+x^2}$ is positive, continuous, and decreasing on $[0, \infty)$, and that

   $$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{c\to\infty} \int_0^{c} \frac{1}{1+x^2}\,dx = \lim_{c\to\infty} [\arctan x]_0^c = \frac{\pi}{2}.$$

   By the Integral Test, it follows that $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ is convergent.

   (ii) Letting $a_n = \frac{1}{1+\sqrt{n}}$, we claim that $\sum_{n=0}^{\infty} \frac{1}{1+\sqrt{n}}$ is divergent. To see this, we note that

   $$\lim_{n\to\infty} \frac{\left(\frac{1}{1+\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1,$$

and we recall that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent. By the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ must also be divergent. It follows that $\sum_{n=0}^{\infty} \frac{1}{1+\sqrt{n}}$ is divergent, as claimed.

(b) The series $\sum_{n=0}^{\infty} a_n^2$.

Solution: EITHER.

(i) Let $a_n = \frac{1}{\sqrt{1+n^2}}$. We have $\sum_{n=0}^{\infty} a_n^2 = \sum_{n=0}^{\infty} \frac{1}{1+n^2}$, which was shown to be convergent in the solution to (a).

(ii) Let $a_n = \frac{1}{\sqrt{1+\sqrt{n}}}$. We have $\sum_{n=0}^{\infty} a_n^2 = \sum_{n=0}^{\infty} \frac{1}{1+\sqrt{n}}$, which was shown to be divergent in the solution to (a).

(c) The series $\sum_{n=0}^{\infty} \sqrt{|a_n|}$.

Solution: EITHER.

(i) Let $a_n = \left(\frac{1}{1+n^2}\right)^2$. We have $\sum_{n=0}^{\infty} \sqrt{|a_n|} = \sum_{n=0}^{\infty} \frac{1}{1+n^2}$, which was shown to be convergent in the solution to (a).

(ii) Let $a_n = \left(\frac{1}{1+\sqrt{n}}\right)^2$. We have $\sum_{n=0}^{\infty} \sqrt{|a_n|} = \sum_{n=0}^{\infty} \frac{1}{1+\sqrt{n}}$, which was shown to be divergent in the solution to (a).

(d) The series $\sum_{n=1}^{\infty} \frac{a_n}{n}$.

Solution: EITHER.

(i) Let $a_n = \frac{1}{n}$. We have $\sum_{n=1}^{\infty} \frac{a_n}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent series.

(ii) Letting $a_n = \frac{1}{1+\ln(n)}$, we claim that the series $\sum_{n=1}^{\infty} \frac{a_n}{n} = \sum_{n=1}^{\infty} \frac{1}{n(1+\ln(n))}$ is divergent. To see this, note that the function $f(x) = \frac{1}{x(1+\ln x)}$ is positive, continuous, and decreasing on $[1,\infty)$, and that

$$\begin{aligned}\int_1^\infty \frac{1}{x(1+\ln x)}\,dx &= \lim_{c\to\infty}\int_1^c \frac{1}{x(1+\ln x)}\,dx\\ &= \lim_{c\to\infty}\left[\ln|1+\ln x|\right]_1^c\\ &= \infty.\end{aligned}$$

(Note: In the calculation above, the antiderivative $\ln|1+\ln x|$ was obtained after the substitution $u = 1+\ln x$.)

By the Integral Test, it follows that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(1+\ln(n))}$ is divergent.

2. Read the Limit Comparison Test for series (Theorem 12.3.7 on the book) and its proof.

   (a) Assume that in the statement of the LCT we accept the case $L = 0$. Show with an example that the theorem is no longer true.

   Solution: Let LCT2 be the version of the LCT in which the case $L = 0$ is accepted. Now, note that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ are series of positive terms and

$$\lim_{n\to\infty}\frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{1}{n} = 0.$$

   Hence, we have satisfied the hypotheses of LCT2. However, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent while $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, so that the conclusion of LCT2 fails to hold. In other words, LCT2 is not true.

   (b) While the conclusion of the theorem is no longer true, one of the two directions in the "iff" statement is true. Which one is it? Prove it.

   Solution: We claim that the direction $\sum b_n$ converges $\Longrightarrow \sum a_n$ converges is still true. In other words, we will prove the following:

   Let $\sum a_n$ and $\sum b_n$ be series of positive terms such that $\frac{a_n}{b_n} \to L \geq 0$. Then $\sum b_n$ converges $\Longrightarrow \sum a_n$ converges.

To begin the proof, we assume that $\sum b_n$ converges. Now, applying the formal definition of $\frac{a_n}{b_n} \to L$ with $\varepsilon = 1$, there exists $K \in \mathbb{N}$ such that

$$\begin{aligned} n > K &\implies |\frac{a_n}{b_n} - L| < 1 \\ &\implies \frac{a_n}{b_n} < L + 1 \\ &\implies a_n < (L+1)b_n. \end{aligned}$$

Hence, $0 < a_n < (L+1)b_n$ for all $n > K$. Also, since $\sum b_n$ converges, $\sum (L+1)b_n$ converges. We may therefore apply the BCT to conclude that $\sum a_n$ converges.

3. **Infinite decimal expansions.** We can interpret any finite decimal expansion as a finite sum. For example:

$$2.13096 = 2 + \frac{1}{10} + \frac{3}{10^2} + \frac{0}{10^3} + \frac{9}{10^4} + \frac{6}{10^5}$$

Similarly, we can interpret any infinite decimal expansion as an infinite series. For example:

$$\pi = 3.141592... = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \dots$$

Interpret the following numbers as a series, then add up the series to calculate its exact value as a rational number.

(a) $0.11111\dots$

Solution: We have

$$\begin{aligned} 0.11111\dots &= \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \dots \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n \\ &= \left[\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n\right] - 1 \\ &= \frac{1}{1 - \frac{1}{10}} - 1 \\ &= \frac{1}{9}. \end{aligned}$$

(b) $0.9999\ldots$

Solution: We have

$$\begin{aligned}
0.9999\ldots &= \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots \\
&= \sum_{n=1}^{\infty} 9\left(\frac{1}{10}\right)^n \\
&= \left[\sum_{n=0}^{\infty} 9\left(\frac{1}{10}\right)^n\right] - 9 \\
&= 9\left[\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n\right] - 9 \\
&= 9\left(\frac{1}{1-\frac{1}{10}}\right) - 9 \\
&= 1.
\end{aligned}$$

(c) $0.25252525\ldots$

Solution: We have

$$\begin{aligned}
0.25252525 &= \frac{25}{100} + \frac{25}{100^2} + \frac{25}{100^3} + \ldots \\
&= \sum_{n=1}^{\infty} 25\left(\frac{1}{100}\right)^n \\
&= \left[\sum_{n=0}^{\infty} 25\left(\frac{1}{100}\right)^n\right] - 25 \\
&= 25\left[\sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n\right] - 25 \\
&= 25\left(\frac{1}{1-\frac{1}{100}}\right) - 25 \\
&= \frac{25}{99}.
\end{aligned}$$

(d) $0.3121212\ldots$

Solution: We have

$$
\begin{aligned}
0.3121212 &= 0.3 + \frac{12}{10^3} + \frac{12}{10^5} + \frac{12}{10^7} + \dots \\
&= 0.3 + \frac{1.2}{100} + \frac{1.2}{100^2} + \frac{1.2}{100^3} + \dots \\
&= 0.3 + \sum_{n=1}^{\infty} 1.2 \left(\frac{1}{100}\right)^n \\
&= 0.3 + \left[\sum_{n=0}^{\infty} 1.2 \left(\frac{1}{100}\right)^n\right] - 1.2 \\
&= -0.9 + 1.2 \sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n \\
&= -0.9 + 1.2 \left(\frac{1}{1 - \frac{1}{100}}\right) \\
&= \frac{103}{330}.
\end{aligned}
$$