9. Instabilities

We wish to ask the following sensible question:
if we have found an equillibrium which is static $[\vec{u} = \vec{0}]$, but is rather general otherwise $P = P_0(\vec{r})$, $\rho = \rho_0(\vec{r})$, $\vec{B} = \vec{B}_0(\vec{r})$, is this ~~perturbation~~ equillibrium stable to small perturbations?

There are two general approaches to this problem:

1.) Solve a specific problem with a linearised normal mode analysis. In other words, write down a specific equillibrium, kick it $[P_0 \Rightarrow P_0 + \delta P$, etc.$]$, and look for modal solutions $\propto e^{-i\omega t + i\vec{k}\cdot\vec{r}}$. If $\omega \in \mathbb{C}$, then there may be exponentially growing modes $\Rightarrow$ instability. However, this may be complex for a given $P_0, \rho_0, \vec{B}_0$ etc.

2.) There is a general procedure, "the energy principle", which can tell you whether an equillibrium is stable, without giving you a huge amount of physical information about the instability (or lack thereof).

We will begin with this general overview approach, which is a powerful technique, before showing a particular example of (1). from Astrophysics.

The energy principle

We shall prove, but this may be intuitively obvious, that a way of tackling the instability problem is the following: compute the change in potential energy of the fluid resulting from a perturbation. If there is a way in which a perturbation can lower the potential energy of ~~ex~~ the fluid, then this perturbation leads to an instability.

Let's begin.

The total energy in MHD is (see lecture 4)

$$\mathcal{E} = \iiint_V \left(\frac{1}{2}\rho u^2 + \frac{B^2}{2\mu_0} + \frac{P}{\gamma - 1} \right) dV$$

we will define

$$\mathcal{E} \equiv \iiint_V \frac{1}{2}\rho u^2 \, dV + W.$$

As we saw in the lectures on waves (5+6), all perturbations of an MHD system can be expressed in terms of small displacements $\vec{\xi}$, where our perturbed velocity $\delta\vec{u} = \frac{\partial \vec{\xi}}{\partial t}$.

Then

(no kinetic energy in equilibrium)

$$\mathcal{E} \rightarrow \mathcal{E} + \delta\mathcal{E} = W_0 + \iiint \frac{1}{2}\rho_0 \left|\frac{\partial \vec{\xi}}{\partial t}\right|^2 dV + \delta W_1[\vec{\xi}] + \delta W_2[\vec{\xi},\vec{\xi}] + \dots$$

where we have expanded to quadratic order, W_0 is our equillibrium potential, and we have split our perturbed potentials into linear $\delta W_1[\vec{\xi}]$ and quadratic $\delta W_2[\vec{\xi},\vec{\xi}]$ orders.

Energy reduced if $\delta W < 0 \Rightarrow$ care about sign of δW_1, δW_2.

Energy must be globally conserved to <u>all</u> orders [push volume integral to infinity].

This means we can be clever and work out δW_1 & δW_2

$$\frac{d\mathcal{E}}{dt} = \frac{d\delta\mathcal{E}}{dt} = \iiint \rho_0 \frac{\partial^2\vec{\xi}}{\partial t^2}\cdot\frac{\partial\vec{\xi}}{\partial t} + \delta W_1\left[\frac{\partial\vec{\xi}}{\partial t}\right]$$

$$+ \delta W_2\left[\frac{\partial\vec{\xi}}{\partial t}, \vec{\xi}\right] + \delta W_2\left[\vec{\xi}, \frac{\partial\vec{\xi}}{\partial t}\right] + \dots = 0.$$

This must be true <u>at all times</u>. Including at $t=0$, where $\vec{\xi}$ & $\frac{\partial\vec{\xi}}{\partial t}$ are independent. (These perturbations can be chosen independently as MHD equations are second order in time for $\vec{\xi}$). Define the "force operator"

$$\vec{F}[\vec{\xi}] = \rho_0 \frac{\partial^2\vec{\xi}}{\partial t^2},$$

then, at $t=0$, calling $\vec{\eta} \equiv \frac{\partial\vec{\xi}}{\partial t}$

$$\iiint_V \vec{\eta}\cdot\vec{F}(\vec{\xi})\,dV + \delta W_1[\vec{\xi}] + \delta W_2[\vec{\eta},\vec{\xi}] + \delta W_2[\vec{\xi},\vec{\eta}] + \dots = 0.$$

Linear order: $\delta W_1[\vec{\eta}] = 0 \Rightarrow$ no linear energy perturbations.

Second order: $\iiint_V \vec{\eta} \cdot \vec{F}[\vec{\xi}]\, dV = -\delta W_2[\vec{\eta}, \vec{\xi}] - \delta W_2[\vec{\xi}, \vec{\eta}]$

Can also set $\vec{\eta} = \vec{\xi}$ [at $t=0$], this trick leads to

$$\boxed{\delta W_2[\vec{\xi}, \vec{\xi}] = -\frac{1}{2} \iiint_V \vec{\xi} \cdot \vec{F}[\vec{\xi}]\, dV}$$

It will transpire that δW_2 can have either sign, and therefore teaches us about instabilities.

Energy principle states

$$\boxed{\delta W_2[\vec{\xi}, \vec{\xi}] > 0 \quad \forall \vec{\xi} \Longleftrightarrow \text{stable equillibrium}}$$

So we need to analyse the properties of the "force operator" $\vec{F}[\vec{\xi}]$.

Obviously, we will derive the functional form of $\vec{F}$ from the linearised MHD equations, but we can actually do nearly all of our work in generality.

Property 1: $\vec{F}[\vec{\xi}]$ has simple eigenmodes $\vec{\xi}_n$.

By definition

$$\vec{F}[\vec{\xi}] = \rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2}$$

we see that for $\vec{\xi}_n = \xi_n(\vec{r}) e^{-i\omega_n t}$

we have

$$\vec{F}[\vec{\xi}_n] = -\rho_0 \omega_n^2 \vec{\xi}_n$$

so $\frac{1}{\rho_0}\vec{F}[\vec{\xi}]$ has eigenmodes $\propto e^{-i\omega_n t}$. $\square$.

Property 2. $\vec{F}[\vec{\xi}]$ is hermitian (or self adjoint)

We derived

$$\iiint_V \vec{\eta}\cdot\vec{F}[\vec{\xi}]\,dV = -\delta W_2[\vec{\eta},\vec{\xi}] - \delta W_2[\vec{\xi},\vec{\eta}]$$

R.H.S is symmetric in $\vec{\eta} \leftrightarrow \vec{\xi}$, so

$$\iiint_V \vec{\eta}\cdot\vec{F}[\vec{\xi}]\,dV = \iiint_V \vec{\xi}\cdot\vec{F}[\vec{\eta}]\,dV \quad \square.$$

Property 3 : eigenvalues ω_n^2 are real.

As $\vec{F}[\vec{\xi}_n] = -\rho_0 \omega_n^2 \vec{\xi}_n$

and $\vec{F}$ ~~hermitian~~ has no complex coefficients then

$$\vec{F}[\vec{\xi}_n^*] = -\rho_0 (\omega_n^2)^* \vec{\xi}_n^*$$

Compute the difference

hermitian!

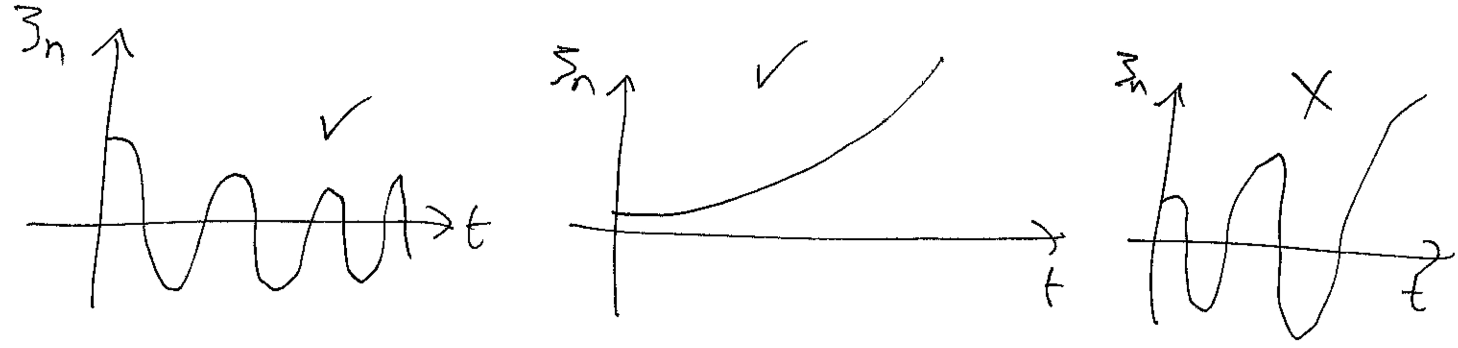
$$\iiint_V \vec{\xi}_n \cdot \vec{F}[\vec{\xi}_n^*]\, dV - \iiint_V \vec{\xi}_n^* \cdot \vec{F}[\vec{\xi}_n]\, dV = 0$$

$$= -\left[\omega_n^2 - (\omega_n^2)^*\right] \iiint \rho_0 |\vec{\xi}_n|^2 dV$$

but $\rho_0 |\vec{\xi}_n|^2 > 0 \Rightarrow \omega_n^2 = (\omega_n^2)^*$

$$\Rightarrow \boxed{\omega_n^2 \in \mathbb{R}.}$$

This means that ω_n is either pure real or pure imaginary $\Rightarrow$ perturbations oscillate or diverge, but don't oscillate with changing amplitude (in MHD).



Property 4: the eigenmodes $\vec{\xi}_n$ are orthogonal

Identical calculation as above, only with $\vec{\xi}_n$ & $\vec{\xi}_m$:

$$\vec{F}[\vec{\xi}_m] = -\rho_0 \omega_m^2 \vec{\xi}_m$$

then

$$-\omega_n^2 \iiint \rho_0 \vec{\xi}_m \cdot \vec{\xi}_n \, dV + \omega_m^2 \iiint_V \rho_0 \vec{\xi}_n \cdot \vec{\xi}_m \, dV$$

$$= \iiint_V \vec{\xi}_m \cdot \vec{F}[\vec{\xi}_n] dV - \iiint \vec{\xi}_n \cdot \vec{F}[\vec{\xi}_m] dV$$

$$= 0$$

$$\Rightarrow -(\omega_n^2 - \omega_m^2) \iiint_V \rho_0 \vec{\xi}_n \cdot \vec{\xi}_m \, dV = 0$$

$$\Rightarrow \iiint_V \rho_0 \vec{\xi}_n \cdot \vec{\xi}_m \, dV = \delta_{n,m} \iiint \rho_0 |\vec{\xi}_n|^2 dV$$

We can now prove the energy principle.

Proof of energy principle

Write

$$\vec{\xi}(\vec{r},t) = \sum_n a_n(t)\,\vec{\xi}_n(\vec{r})$$

Then

$$\delta W_2[\vec{\xi},\vec{\xi}] = -\frac{1}{2}\iiint_V \vec{\xi}\cdot\vec{F}[\vec{\xi}]\,dV$$

$$= -\frac{1}{2}\sum_n\sum_m \iiint_V a_n a_m \vec{\xi}_n\cdot\vec{F}[\vec{\xi}_m]\,dV$$

$$= +\frac{1}{2}\sum_n\sum_m a_n a_m \omega_m^2 \iiint_V \rho_0\,\vec{\xi}_n\cdot\vec{\xi}_m\,dV$$

$$= \frac{1}{2}\sum_n a_n^2\omega_n^2 \iiint \rho_0\,|\vec{\xi}_n|^2\,dV$$

~~The kinetic term in δE is~~
we can define

$$K[\vec{\xi},\vec{\xi}] \equiv \frac{1}{2}\iiint_V \rho_0\,|\vec{\xi}|^2\,dV$$

....

$$= \frac{1}{2}\sum_n a_n^2 \iiint \rho_0\,|\vec{\xi}_n|^2\,dV$$

and therefore, our smallest eigenvalue (call it ϖ_1) satisfies

$$\varpi_1^2 = \min_{\vec{\xi}} \left(\frac{\delta W_2[\vec{\xi},\vec{\xi}]}{K[\vec{\xi},\vec{\xi}]} \right)$$

as $K > 0$, if $\delta W_2 > 0$ for all possible $\vec{\xi}$, then $\varpi_1^2 > 0 \Rightarrow$ all $\varpi_n^2 > 0 \Rightarrow$ stable.

However, if $\delta W_2 < 0$ for any $\vec{\xi}$, then at least one of $\varpi_n^2 < 0 \Rightarrow$ an instability exists.

Our procedure is therefore clear, write down the force operator $\vec{F}[\vec{\xi}] = \rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2}$ from the linearised MHD equations, use this to compute δW_2, and this will let us analyse a wide class of instability problems.

Deriving The force operator

Mass: $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$

Entropy: $\frac{D}{Dt}(P\rho^{-\gamma}) = 0 \Rightarrow \frac{\partial p}{\partial t} + (\vec{u} \cdot \vec{\nabla}) P + \gamma P (\vec{\nabla} \cdot \vec{u}) = 0$

~~[illegible]~~

Momentum: $\rho \left[\frac{\partial}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \right] \vec{u} = -\vec{\nabla} P + \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) \times \vec{B}$.

Induction: $\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B})$.

let's linearise; with $\vec{u} = \vec{0}$, $\delta \vec{u} = \frac{\partial \vec{\xi}}{\partial t}$

Mass: $\delta \rho = -\vec{\nabla} \cdot (\rho_0 \vec{\xi})$

Pressure: $\delta p = -(\vec{\xi} \cdot \vec{\nabla}) p_0 - \gamma p_0 (\vec{\nabla} \cdot \vec{\xi})$

Induction: $\delta \vec{B} = \vec{\nabla} \times (\vec{\xi} \times \vec{B}_0)$

Momentum: $\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = -\vec{\nabla} \delta p + \frac{1}{\mu_0} (\vec{\nabla} \times \delta \vec{B}) \times \vec{B}_0$

$+ \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}_0) \times \delta \vec{B}$.

$\therefore$

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = \vec{\nabla}\left[(\vec{\xi}\cdot\vec{\nabla})p_0\right] + \gamma\vec{\nabla}\left[p_0(\vec{\nabla}\cdot\vec{\xi})\right]$$

$$+ \frac{1}{\mu_0}\left(\vec{\nabla}\times\left[\vec{\nabla}\times(\vec{\xi}\times B_0)\right]\right)\times\vec{B_0}$$

$$+ \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B_0})\times\left[\vec{\nabla}\times(\vec{\xi}\times\vec{B_0})\right].$$

and we therefore have derived our force operator.

Therefore

$$\delta W_2 = -\frac{1}{2}\iiint_V \vec{\xi}\cdot\vec{F}[\vec{\xi}]\,dV$$

$$= -\frac{1}{2}\iiint_V \vec{\xi}\cdot\left(\vec{\nabla}\left[(\vec{\xi}\cdot\vec{\nabla})p_0\right]\right)dV$$

$$-\frac{1}{2}\iiint_V \gamma\,\vec{\xi}\cdot\left(\vec{\nabla}\left[p_0(\vec{\nabla}\cdot\vec{\xi})\right]\right)dV$$

$$-\frac{1}{2}\iiint \frac{1}{\mu_0}\vec{\xi}\cdot\left[(\vec{\nabla}\times\delta\vec{B})\times\vec{B_0}\right]dV$$

$$-\frac{1}{2}\iiint \vec{\xi}\cdot(\vec{J_0}\times\delta\vec{B})\,dV$$

there are a myriad number of equivalent formulations of this integral, all based on different numbers of integrations by parts. I will derive the "textbook" version

$$I_1 = \iiint_V \vec{\xi} \cdot \left(\vec{\nabla} \left[(\vec{\xi} \cdot \vec{\nabla}) p_0 \right] \right) dV$$

$$= \underbrace{\iiint_V \vec{\nabla} \cdot \left(\vec{\xi} \left[(\vec{\xi} \cdot \vec{\nabla}) p_0 \right] \right) dV}_{0 \text{ by divergence theorem}} - \iiint_V \left[(\vec{\xi} \cdot \vec{\nabla}) p_0 \right] \left[\vec{\nabla} \cdot \vec{\xi} \right] dV$$

$$I_2 = \iiint_V \gamma \vec{\xi} \cdot \left(\vec{\nabla} \left[p_0 (\vec{\nabla} \cdot \vec{\xi}) \right] \right) dV$$

$$= \underbrace{\iiint_V \vec{\nabla} \cdot \left[\gamma \vec{\xi} \left[p_0 (\vec{\nabla} \cdot \vec{\xi}) \right] \right]}_{0} - \iiint \gamma p_0 (\vec{\nabla} \cdot \vec{\xi})^2 dV$$

$$I_4 = \iiint_V \vec{\xi} \cdot \left(\vec{J}_0 \times \delta \vec{B} \right) dV = - \iiint_V \vec{J}_0 \cdot (\vec{\xi} \times \delta \vec{B}) dV$$

$$I_3 = \iiint_V \frac{\vec{\xi}}{\mu_0} \cdot \left[(\vec{\nabla} \times \delta \vec{B}) \times \vec{B}_0 \right] dV$$

$$= \frac{1}{\mu_0} \iiint_V (\vec{\nabla} \times \delta\vec{B}) \cdot (\vec{\xi} \times \vec{B}_0) \, dV \qquad [\text{vector triple}]$$

$$= -\frac{1}{\mu_0} \iiint_V (\delta\vec{B} \times \vec{\nabla}) \cdot (\vec{\xi} \times \vec{B}_0) \, dV \qquad [\text{by parts}]$$

$$= -\frac{1}{\mu_0} \iiint_V \delta\vec{B} \cdot \left(\vec{\nabla} \times (\vec{\xi} \times \vec{B}_0)\right) dV$$

$$= -\iiint_V \frac{|\delta\vec{B}|^2}{\mu_0} \, dV$$

and therefore

$$\delta W_2 = \frac{1}{2} \iiint_V \Big[\big((\vec{\xi} \cdot \vec{\nabla}) p_0\big) \vec{\nabla} \cdot \vec{\xi} + \overbrace{\gamma p_0 (\vec{\nabla} \cdot \vec{\xi})^2}^{\text{stabilising}}$$
$$+ \vec{J}_0 \times (\vec{\xi} \times \delta\vec{B}) + \underbrace{\frac{1}{\mu_0} |\delta\vec{B}|^2}_{\text{stabilising}} \Big] dV$$

Let's use this as the largest hammer to ever hit a small nail.

Consider adding a constant gravitational field to the momentum equation

$$\rho\left[\frac{\partial}{\partial t}+(\vec{u}\cdot\vec{\nabla})\right]\vec{u} = -\vec{\nabla}P + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} + \rho\vec{g}$$

then a perturbation gets an extra force term

$$\delta\rho\,\vec{g} = -\vec{g}\left(\vec{\nabla}\cdot(\rho_0\vec{\xi})\right)$$

and our energy integral has

$$\delta W_2 = \frac{1}{2}\iiint_V \Big[\left((\vec{\xi}\cdot\vec{\nabla})P_0\right)(\vec{\nabla}\cdot\vec{\xi}) + \gamma P_0(\vec{\nabla}\cdot\vec{\xi})^2 + (\vec{\xi}\cdot\vec{g})(\vec{\nabla}\cdot(\rho_0\vec{\xi})) + \vec{J}_0\times(\vec{\xi}\times\delta\vec{B}) + \frac{|\delta\vec{B}|^2}{\mu_0}\Big]dV$$

Let us return to hydrodynamics $\ni$ $\vec{B}_0 = \vec{J}_0 = \vec{0}$.

$\delta\vec{B} = 0$.

$$\delta W_2 = \frac{1}{2}\iiint_V \Big[(\vec{\nabla}\cdot\vec{\xi})(\vec{\xi}\cdot\vec{\nabla})P_0 + \gamma P_0(\vec{\nabla}\cdot\vec{\xi})^2 + (\vec{\xi}\cdot\vec{g})\,\vec{\nabla}\cdot(\rho_0\vec{\xi})\Big]dV$$

Interchange instability

$$\vec{g} = -g\hat{z}, \quad \rho = \rho_0(z), \quad P = P_0(z).$$

Vertical momentum equation

$$\vec{0} = -\vec{\nabla}P + \rho\vec{g} \Rightarrow \frac{dP_0}{dz} = -\rho_0 g$$

Perturbation δW_2 satisfies

$$\delta W_2 = \frac{1}{2}\iiint\left[\xi_z \frac{dP_0}{dz}(\vec{\nabla}\cdot\vec{\xi}) + \gamma P_0(\vec{\nabla}\cdot\vec{\xi})^2 - g\xi_z\left(\frac{d\rho_0}{dz}\xi_z + \rho_0\vec{\nabla}\cdot\vec{\xi}\right)\right]dV$$

$$\left(\text{use } \rho_0 = -\frac{1}{g}\frac{dP_0}{dz}\right)$$

$$= \frac{1}{2}\iiint\left(2\xi_z\frac{dP_0}{dz}(\vec{\nabla}\cdot\vec{\xi}) + \gamma P_0(\vec{\nabla}\cdot\vec{\xi})^2 - g\xi_z^2\frac{d\rho_0}{dz}\right)dV$$

$$\therefore \quad \delta W_2 = \iiint_V f(\xi_z, \vec{\nabla}\cdot\vec{\xi})\,dV$$

Let's look for the most unstable perturbation, by minimising f. Treat ξ_z & $\vec{\nabla}\cdot\vec{\xi}$ as independent

$$\frac{\partial f}{\partial(\vec{\nabla}\cdot\vec{\xi})} = 2\xi_z\frac{dP_0}{dz} + 2\gamma P_0(\vec{\nabla}\cdot\vec{\xi}) = 0$$

$$\Rightarrow (\vec{\nabla}\cdot\vec{\xi}) = -\frac{1}{\gamma P_0}\frac{dP_0}{dz}\xi_z$$

Substituting back

$$\delta W_2 = \frac{1}{2}\iiint_V\left(-\frac{2\xi_z^2}{\gamma P_0}\left(\frac{dP_0}{dz}\right)^2 + \frac{1}{\gamma P_0}\left(\frac{dP_0}{dz}\right)^2\xi_z^2 - g\xi_z^2\frac{d\rho_0}{dz}\right)dV$$

simplifying

$$\delta W_2 = \frac{1}{2}\iiint_V \left[-\frac{1}{\gamma p_0}\left(\frac{dp_0}{dz}\right)^2 \xi_z^2 - g\xi_z^2 \frac{d\rho_0}{dz}\right] dV$$

with $\frac{dp_0}{dz} = -\rho_0 g$

$$\delta W_2 = \frac{1}{2}\iiint_V \left[\frac{\rho_0 g}{\gamma p_0}\frac{dp_0}{dz}\xi_z^2 - g\xi_z^2 \frac{d\rho_0}{dz}\right] dV$$

$$= \frac{1}{2}\iiint_V \frac{\rho_0 g \xi_z^2}{\gamma}\left[\frac{1}{p_0}\frac{dp_0}{dz} - \frac{\gamma}{\rho_0}\frac{d\rho_0}{dz}\right] dV$$

$$= \frac{1}{2}\iiint_V \frac{\rho_0 g \xi_z^2}{\gamma}\left[\frac{d}{dz}\ln\left(p_0\rho_0^{-\gamma}\right)\right] dV$$

$\therefore$ unstable iff. $\frac{d}{dz}\ln\left(p_0\rho_0^{-\gamma}\right) < 0$

$\Rightarrow$ entropy decreases upwards.

Highlights the pro's & con's of this method.

- Pro: easy calculation, no underlying solutions of the hydro required
- Con: no insight about <u>why</u> this is unstable.