3. Evolution of the magnetic field

The Lorentz force [in ideal MHD $\vec{f} = \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}$] tell's the fluid how to respond to $\vec{B}$.

We now study how $\vec{B}$ responds to the fluid.

Induction equation

Recall the field transformation when we go into the rest frame of the ions

$$\vec{E}' = \vec{E} + \vec{v}\times\vec{B}.$$

Plasma (assumed to be) a perfect conductor, so

$$\vec{E}' \simeq \vec{0} \Rightarrow \vec{E} = -\vec{v}\times\vec{B}.$$

But Maxwell's 3rd equation states that

$$\frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$$

and so

$$\boxed{\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times(\vec{v}\times\vec{B}).}$$

Quick side note:

by definition $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0 \quad \forall \vec{A}$

and so the induction equation implies

$$\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot\left[\vec{\nabla}\times(\vec{v}\times\vec{B})\right] = 0$$

$\Rightarrow \vec{\nabla}\cdot\vec{B}$ conserved $(=0)$; a good consistency check.

The induction equation $\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B})$ implies so-called "flux freezing". Flux freezing can be analysed in a number of ways, first consider the "lindquist theorem".

Lindquist theorem

$$\vec{\nabla}\times(\vec{u}\times\vec{B}) = \varepsilon_{ijk}\partial_j(\vec{u}\times\vec{B})_k = \varepsilon_{ijk}\partial_j\varepsilon_{klm}u_l B_m$$

$$= [\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}]\partial_j(u_l B_m)$$

$$= \partial_j(u_i B_j) - \partial_j(u_j B_i)$$

$$= u_i(\partial_j B_j) + (B_j\partial_j)u_i - B_i(\partial_j u_j) - (u_j\partial_j)B_i$$

$$= \vec{u}\underbrace{(\vec{\nabla}\cdot\vec{B})}_{0} + (\vec{B}\cdot\vec{\nabla})\vec{u} - \vec{B}(\vec{\nabla}\cdot\vec{u}) - (\vec{u}\cdot\vec{\nabla})\vec{B}$$

$$= (\vec{B}\cdot\vec{\nabla})\vec{u} - (\vec{u}\cdot\vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla}\cdot\vec{u})$$

$$\therefore \quad \frac{\partial \vec{B}}{\partial t} + \underbrace{(\vec{u}\cdot\vec{\nabla})\vec{B}}_{\text{Advection}} = \underbrace{(\vec{B}\cdot\vec{\nabla})\vec{u}}_{\text{stretching}} - \underbrace{\vec{B}(\vec{\nabla}\cdot\vec{u})}_{\text{compression}}$$

Recall mass conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + (\vec{u}\cdot\vec{\nabla})\rho = -\rho(\vec{\nabla}\cdot\vec{u})$$

$$\Rightarrow \vec{\nabla}\cdot\vec{u} = -\frac{1}{\rho}\frac{D\rho}{Dt}$$ "compression of flow".

Substitute back into induction equation

$$\frac{D\vec{B}}{Dt} = (\vec{B}\cdot\vec{\nabla})\vec{u} + \frac{\vec{B}}{\rho}\frac{D\rho}{Dt}$$

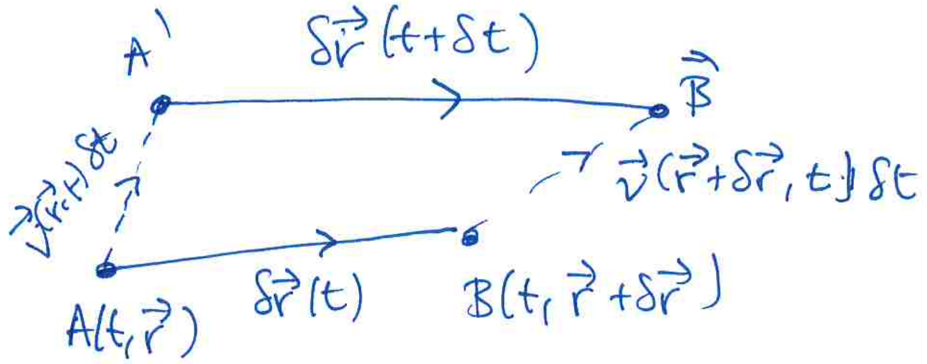
$$\therefore \quad \frac{1}{\rho}\frac{D\vec{B}}{Dt} - \frac{\vec{B}}{\rho^2}\frac{D\rho}{Dt} = \left(\frac{\vec{B}}{\rho}\cdot\vec{\nabla}\right)\vec{u}$$

$$\frac{D}{Dt}\left(\frac{\vec{B}}{\rho}\right) = \left[\left(\frac{\vec{B}}{\rho}\right)\cdot\vec{\nabla}\right]\vec{u}$$

"Lindquist identity".

Interpretation

Consider two fluid elements seperated by $\delta\vec{r}(t)$.



[i.e., during δt
$A \to A'$
$B \to B'$]

As the flow evolves $\delta\vec{r}$ will change.

Trivial vector identity

$$\vec{A'B'} \equiv \delta\vec{r}(t+\delta t) = -\vec{AA'} + \vec{AB} + \vec{BB'}$$

$$= \vec{AB} + (\vec{BB'} - \vec{AA'})$$

$$= \delta\vec{r}(t) + \vec{v}(\vec{r}+\delta\vec{r},t)\delta t - \vec{v}(\vec{r},t)\delta t$$

$$\simeq \delta\vec{r}(t) + (\delta\vec{r}\cdot\vec{\nabla})\vec{v}\,\delta t + \ldots$$

$$\therefore \frac{D}{Dt}\delta\vec{r} \equiv \frac{\delta\vec{r}(t+\delta t) - \delta\vec{r}(t)}{\delta t} = (\delta\vec{r}\cdot\vec{\nabla})\vec{v}$$

which is the change of line element moving with the flow.

$$\therefore \frac{D}{Dt}\left(\frac{\vec{B}}{\rho}\right) = \left(\frac{\vec{B}}{\rho}\cdot\vec{\nabla}\right)\vec{u}$$

$$\frac{D}{Dt}\delta\vec{r} = (\delta\vec{r}\cdot\vec{\nabla})\vec{u}$$

and $\dfrac{\vec{B}}{\rho}$ and $\delta\vec{r}$ satisfy the same equation.

Let's assume that at time 't' $\dfrac{\vec{B}}{\rho}$ and $\delta\vec{r}$ are parrallel : $\dfrac{\vec{B}}{\rho} = \alpha\,\delta\vec{r}$

Since

$$\begin{aligned}\frac{\vec{B}}{\rho}(t+\delta t) &= \frac{\vec{B}}{\rho}(t) + \frac{D}{Dt}\left(\frac{\vec{B}}{\rho}\right)\delta t \\ &= \frac{\vec{B}}{\rho}(t) + \left(\frac{\vec{B}}{\rho}\cdot\vec{\nabla}\right)\vec{u}\,\delta t \\ &= \alpha\,\delta r(t) + \alpha(\delta\vec{r}\cdot\vec{\nabla})\vec{u}\,\delta t \\ &= \alpha\,\delta\vec{r}\,(t+\delta t)\end{aligned}$$

$\therefore$ $\dfrac{\vec{B}}{\rho}(t+\delta t)$ is parallel to $\delta\vec{r}\,(t+\delta t)$.

In other words, magnetic field lines move with the flow: if fluid elements start on the same field line they stay on the same field line as they move with the flow.

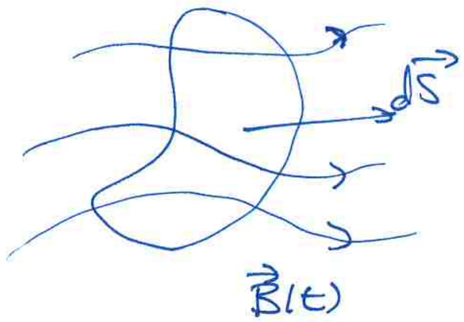
<u>But</u> we discussed last week that field lines actively resist bending (through the force $\frac{B^2}{\mu_0}\frac{\partial \hat{b}}{\partial s}$) and compression (through the force $-\vec{\nabla}_\perp(B^2/2\mu_0)$).

Following fluid elements as they move around invariably ends up with either field bending or compressing $\Rightarrow$ complex phenomena.

Alfvén's theorem

Begin by defining the magnetic flux through a surface $S(t)$ by

$$\Phi(t) = \iint_{S(t)} \vec{B}(t) \cdot d\vec{S}$$



Consider now a later time

$t \Rightarrow t + \delta t$.

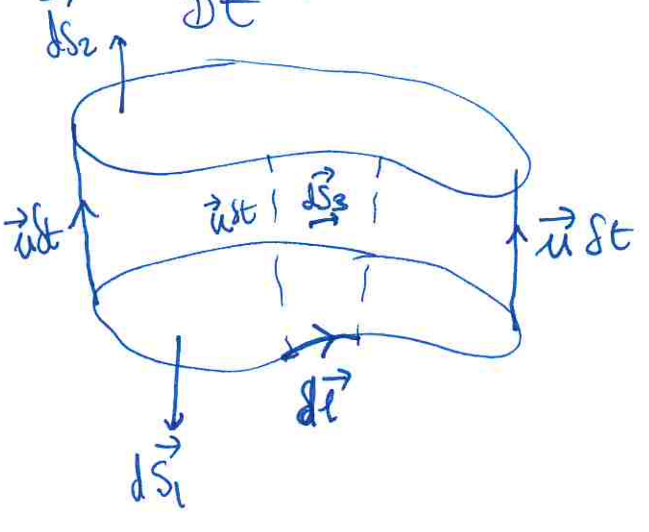
then

$$\Phi(t+\delta t) = \iint_{S(t+\delta t)} \vec{B}(t+\delta t) \cdot d\vec{S}$$

and we define the ~~ma~~ lagrangian derivative of Φ by

$$\frac{D\Phi}{Dt} \equiv \lim_{\delta t \to 0} \left[\frac{\Phi(t+\delta t) - \Phi(t)}{\delta t} \right].$$

To compute $\dfrac{D\Phi}{Dt}$, consider this volume



This describes the volume swept out by a loop $S_1 \to S_2$ over a time δt as it is advected by flow $\vec{u}$.

Trivially

$$\iiint_{\mathcal{V}} \vec{\nabla}\cdot\vec{B}\, d\mathcal{V} = 0 \quad \text{as} \quad \vec{\nabla}\cdot\vec{B} = 0$$

but $\iiint_{\mathcal{V}} \vec{\nabla}\cdot\vec{B}\, d\mathcal{V} = \oint_{\partial\mathcal{V}} \vec{B}\cdot d\vec{S}$ by divergence theorem.

Then

$$\oint_{\partial\mathcal{V}} \vec{B}\cdot d\vec{S} = -\iint_{S_1} \vec{B}(t+\delta t)\cdot d\vec{S}_1 + \iint_{S_2} \vec{B}(t+\delta t)\cdot d\vec{S}_2$$

$$+ \iint_{S_3} \vec{B}(t+\delta t)\cdot d\vec{S}_3 = 0$$

$$\therefore \quad \underbrace{\iint_{S_2} \vec{B}(t+\delta t)\cdot d\vec{S}_2}_{\Phi(t+\delta t)} = \iint_{\underbrace{S_1}_{S(t)}} \vec{B}(t+\delta t)\cdot d\vec{S}_1 - \iint_{S_3} \vec{B}(t+\delta t)\cdot d\vec{S}_3$$

$$\Rightarrow \quad \Phi(t+\delta t) = \iint_{S(t)} \left(\vec{B}(t) + \delta t\,\frac{\partial\vec{B}}{\partial t}\right)\cdot d\vec{S}_1 - \iint_{S_3} \vec{B}(t+\delta t)\cdot d\vec{S}_3$$

$\Phi(t)$

$$= \Phi(t) + \delta t \iint_{S(t)} \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}_1 - \iint_{S_3} \vec{B}(t+\delta t)\cdot d\vec{S}_3$$

but $d\vec{S}_3 = d\vec{l} \times \vec{u}\,\delta t$

and

$$\iint_{S_3} \vec{B}(t+\delta t) . d\vec{S}_3 = \iint_{S_3} \vec{B}(t+\delta t) . (d\vec{l} \times \vec{u}\,\delta t)$$

and so to linear order in δt

$$\iint_{S_3} \vec{B}(t+\delta t) . d\vec{S}_3 = \iint_{S_3} \vec{B}(t) . (d\vec{l} \times \vec{u}\,\delta t)$$

(properties of scalar triple product)

$$= \iint_{S_3} (\vec{u} \times \vec{B}) . d\vec{l}\ \delta t$$

(stokes theorem)

$$= \iint_{S_1} \vec{\nabla} \times (\vec{u} \times \vec{B}) . d\vec{S}_1\, \delta t$$

$\therefore$ combining our results

$$\Phi(t+\delta t) = \Phi(t) + \delta t \iint_{S_1} \frac{\partial \vec{B}}{\partial t} . d\vec{S}_1 - \delta t \iint_{S_1} \vec{\nabla} \times (\vec{u} \times \vec{B}) . d\vec{S}_1$$

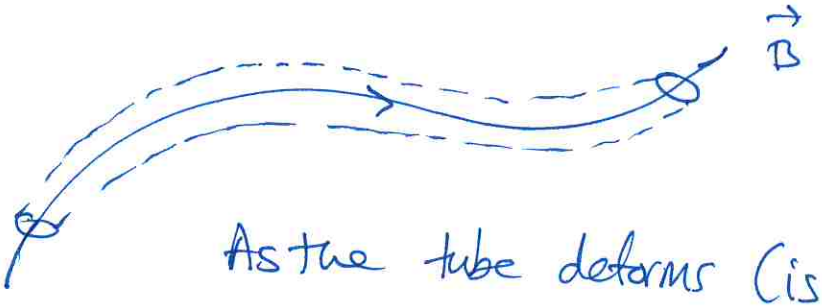
$$= \Phi(t) + \delta t \iint_{S_1} \Big[\underbrace{\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times (\vec{u} \times \vec{B})}_{\text{the induction equation}} \Big] . d\vec{S}_1$$

$$= \Phi(t)$$

$$\Rightarrow \quad \frac{D\Phi}{Dt} = 0. \Rightarrow \Phi \text{ conserved.}$$

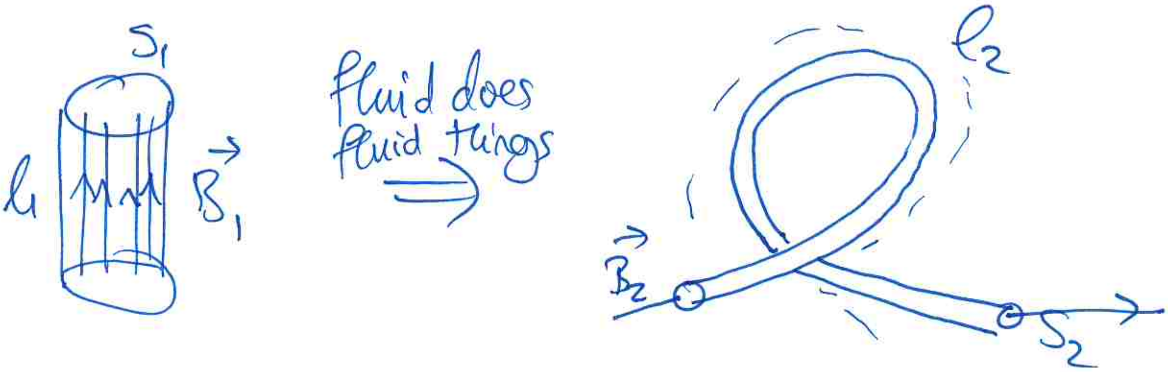
Alfvén's theorem has an interesting property: it implies that field lines are "frozen into" the flow.

To see this, take a field line and encase it in a "flux tube"



As the tube deforms (is advected with the flow) the field line must stay within it — as the flux through the two ends must stay constant, and the flux through the sides must remain zero.

This can then lead to field amplification by fluid motion



by conservation of flux

$$B_1 S_1 = B_2 S_2$$

By conservation of mass

$$\rho_1 S_1 l_1 = \rho_2 S_2 l_2$$

and therefore

$$\frac{B_1}{\rho_1 l_1} = \frac{B_2}{\rho_2 l_2} \Rightarrow \frac{B_2}{B_1} = \frac{\rho_2 l_2}{\rho_1 l_1}.$$

In an incompressible fluid $\rho_1 = \rho_2$, and the field is amplified by a factor l_2/l_1.

In a compressible fluid we could get even more amplification if $\rho_2 > \rho_1$.

Ultimate fate of the field

Recall

$$\frac{\partial \vec{B}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{B} = (\vec{B}\cdot\vec{\nabla})\vec{u} - \vec{B}(\vec{\nabla}\cdot\vec{u})$$

the question: "are there fluid flows that lead to sustained amplification of $\vec{B}$?" is the (famous) MHD dynamo problem.

Interestingly, in two dimensions ~~we~~ have a "no dynamo theorem" [due to Zeldovich et al. 1984].

We can write a two-dimensional B-field as

$$\vec{B} = \vec{\nabla}\times(A\hat{z}) \quad ; \quad A = A(x,y).$$

then $\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B}) \Rightarrow \frac{\partial}{\partial t}(\vec{\nabla}\times A\hat{z}) = \vec{\nabla}\times(\vec{u}\times\vec{\nabla}\times A\hat{z}))$

"uncurl"

$$\frac{\partial A}{\partial t}\hat{z} = \vec{u}\times(\vec{\nabla}\times A\hat{z})$$

$$= \varepsilon_{ijk} u_j \varepsilon_{klm} \partial_l A_m$$

$$= [\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}] u_j \partial_l A_m$$

$$= \underbrace{u_j \partial_i A_j}_{\text{zero as } u^z = 0} - \underbrace{u_j \partial_j A_i}_{(\vec{u}\cdot\vec{\nabla})A\hat{z}}$$

$$\Rightarrow \quad \frac{\partial A}{\partial t} + (\vec{u}\cdot\vec{\nabla})A = 0 \Rightarrow \text{no growth.}$$

(decays if non-ideal effects included).

In 3 dimensions we can however have field growth.

Simple example: consider a rotating flow

$\vec{u} = u_\phi(r)\,\hat{\phi}$ with an initial field

Initially: $\vec{B} = B_r\,\hat{r}$. [simple model of accretion flow].

Assume incompressible [$\vec{\nabla}\cdot\vec{u} = 0$], then

$$\frac{\partial \vec{B}}{\partial t} = -(\vec{u}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{u}$$

Move to ϕ component [assume $\frac{\partial}{\partial \phi} = 0$ by symmetry].

$$\frac{\partial B_\phi}{\partial t} = \underbrace{-\frac{u_\phi B_r}{r}}_{\text{advection}} + \underbrace{B_r \frac{\partial u_\phi}{\partial r}}_{\text{stretching}}$$

$$\Rightarrow \quad \frac{dB_\phi}{dt} = \left(\frac{\partial \vec{B}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{B}\right)_\phi = B_r \frac{\partial u_\phi}{\partial r}$$

Stretching of B_r produces B_ϕ.

Eventually $\frac{B^2}{2\mu_0} \sim \frac{1}{2}\rho u^2$ and Lorentz force modifies flow, will not grow indefinitely in this simple manner.