7. MHD Equillibria & Relaxation

Last week we considered MHD in a straight field with constant density & pressure. This has some universal application, as any sufficiently zoomed-in general (static) equillibrium will look like this.

This week we shall consider what sort of large scale equillibria exist, and which of these states a MHD system will relax.

Let's take the momentum equation

$$\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla}(P) + \vec{J} \times \vec{B}$$

and look for static equillibria $\left[\vec{u} = \vec{0}, \frac{\partial}{\partial t} \equiv 0\right]$ we find

$$\boxed{-\vec{\nabla} P + \vec{J} \times \vec{B} = \vec{0}.}$$

then recalling our ideal MHD result

$$\boxed{\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B}} \quad \text{and} \quad \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

we have sufficient equations (7) for our unknowns $P, \vec{J}, \vec{B}$.

[Density drops out as nothing moves].

Two immediate consequences of static equilibria:

first:

$$\vec{B} \cdot \left[-\vec{\nabla} p + \vec{J} \times \vec{B} \right] = 0$$

but $\quad \vec{B} \cdot (\vec{J} \times \vec{B}) = - \vec{J} \cdot (\vec{B} \times \vec{B}) = 0.$

$$\therefore \quad \boxed{(\vec{B} \cdot \vec{\nabla}) P = 0}$$

pressure does not vary along field lines.

Second:

$$\vec{J} \cdot \left[-\vec{\nabla} P + \vec{J} \times \vec{B} \right] = 0$$

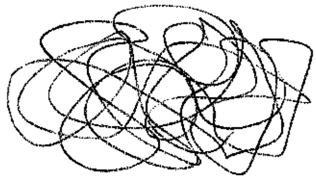
$$\Rightarrow \boxed{(\vec{J} \cdot \vec{\nabla}) P = 0}$$

Currents flow along magnetic surfaces

Consequence of

$$(\vec{B}\cdot\vec{\nabla})P = 0,$$

consider a stochastic and volume filling magnetic field.



e.g.

As on every $\vec{B}$ we must have constant pressure, then $P = \text{cst}$ everywhere.

then we have

$$\vec{J}\times\vec{B} = \vec{0}$$

from the ~~pressure~~ momentum conservation equation.

This is called a force free equillibrium.

We shall return to force-free equillibria later.

Cylindrical equillibria

The second simplest equillibrium after $\vec{B} = B_0 \hat{z}$.

Take $\frac{\partial}{\partial \theta} \equiv 0$, $\frac{\partial}{\partial z} \equiv 0$.

Properties:

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(r B_r) = 0$$

$$\Rightarrow r B_r = cst \Rightarrow B_r = 0$$

[else $B_r \to \infty$ as $r \to 0$.]

Ampere's law shows

$$\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} \Rightarrow \begin{cases} J_r = 0 \\ J_\theta = -\frac{1}{\mu_0} \frac{\partial B_z}{\partial r} \\ J_z = \frac{1}{\mu_0} \frac{1}{r} \frac{\partial}{\partial r}(r B_\theta) \end{cases}$$

So no current along radius either.

Pressure balance

$$-\vec{\nabla} p + \vec{J} \times \vec{B} = \vec{0}$$

take radial component (only non-trivial one)

$$-\frac{\partial p}{\partial r} + (\vec{J} \times \vec{B})_r = -\frac{\partial p}{\partial r} + J_\theta B_z - J_z B_\theta = 0$$

$$\Rightarrow \quad \frac{\partial p}{\partial r} = J_\theta B_z - J_z B_\theta$$

$$= \frac{1}{\mu_0}\left[-B_z \frac{\partial B_z}{\partial r} - \frac{B_\theta}{r}\frac{\partial}{\partial r}(r B_\theta)\right]$$

$$= \frac{1}{\mu_0}\left[-\frac{\partial}{\partial r}\left(\frac{B_z^2}{2}\right) - \frac{B_\theta^2}{r} - \frac{\partial}{\partial r}\left(\frac{B_\theta^2}{2}\right)\right]$$

[use $B_r = 0$]

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial r}\left(p + \frac{B^2}{2\mu_0}\right) = -\frac{B_\theta^2}{\mu_0 r}}$$

This set up therefore balances the total pressure gradient with the magnetic tension force.

A general equillibrium which satisfies this is called a "screwpinch"

Case 1: The Z-pinch

let the current flow along $\hat{z}$.

$$\Rightarrow J_\theta = 0 \Rightarrow -\frac{1}{\mu_0}\frac{\partial B_z}{\partial r} = 0 \Rightarrow B_z = 0.$$

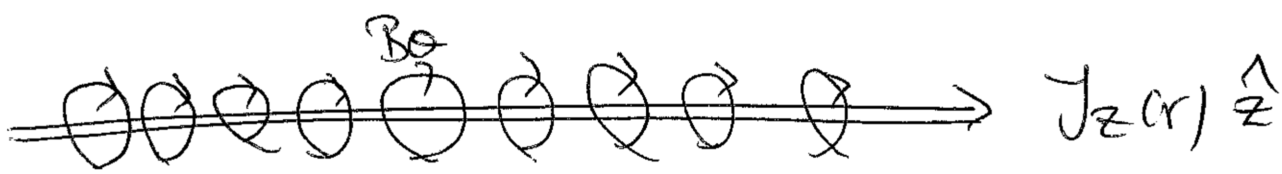
$$J_z \neq 0 \Rightarrow \frac{1}{\mu_0 r}\frac{\partial}{\partial r}(r B_\theta) = J_z(r)$$

$$\Rightarrow B_\theta = \frac{\mu_0}{r}\int^r r' J_z(r')\,dr'$$

and pressure balance is just

$$\frac{\partial p}{\partial r} = -J_z B_\theta = -\frac{\mu_0 J_z(r)}{r}\int^r r' J_z(r')\,dr'$$

what does this look like?



the loops want to contract inwards, and the pressure gradient opposses this. This means that the plasma is <u>confined</u>.

To verify this, try

$$J_z(r) = \frac{J_0}{1+(r/R)^2}$$

$$B_\theta = \frac{\mu_0}{r}\int_0^r \frac{r' J_0}{1+(r'/R)^2}\,dr = \mu_0 J_0 R^2 \frac{\ln\left(1+(r/R)^2\right)}{r}$$

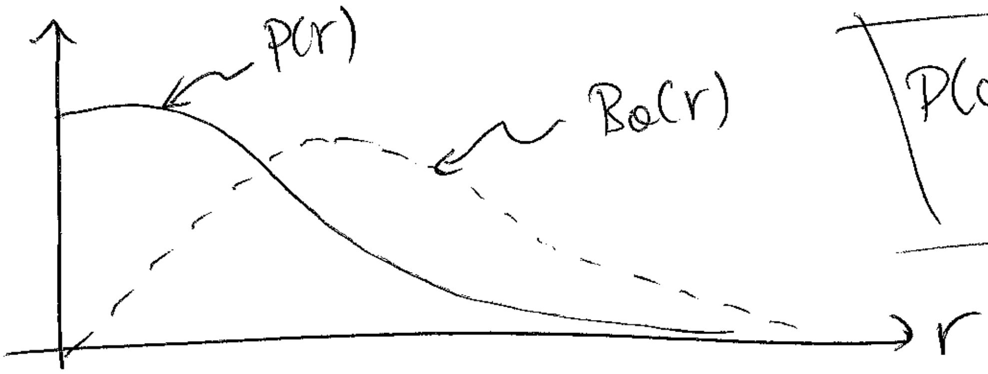
$B_\theta \to 0$ as $r \to 0$.

$B_\theta \to 0$ as $r \to \infty$

$$\frac{\partial p}{\partial r} = -J_z B_\theta = -\mu_0 J_0^2 R^2 \frac{\ln\left(1+(r/R)^2\right)}{r\left(1+r^2/R^2\right)}$$

$\frac{\partial p}{\partial r} \to 0$ at $r \to 0 \Rightarrow p \simeq cst$ at $r=0$

$\frac{\partial p}{\partial r} \sim \frac{-1}{r^3}$ as $r \to \infty \Rightarrow p \sim 1/r^2$ at large r



Plasma inside B field!

It turns out the z-pinch is violently unstable. [although it is popular for laboratory experiments].

θ-pinch

Alternatively, put the magnetic field along $\hat{z}$.

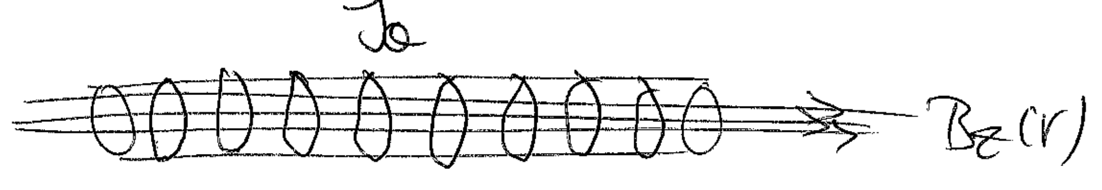
$$B_\theta = 0 \Rightarrow J_z = 0$$

$$B_z \neq 0 \Rightarrow J_\theta = -\frac{1}{\mu_0}\frac{\partial B_z}{\partial r}$$

Need $B_z = B_z(r)$.

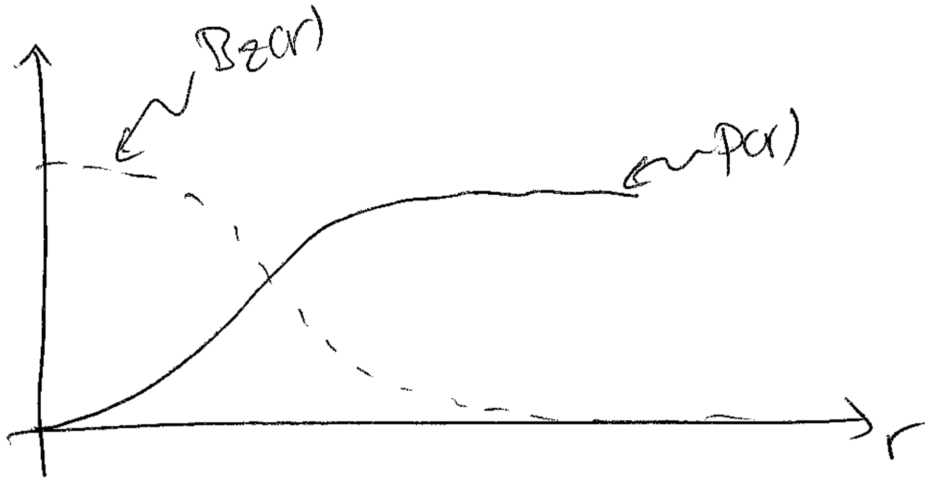
Then momentum equation becomes

$$\frac{\partial}{\partial r}\left(p + \frac{B^2}{2\mu_0}\right) = 0 \Rightarrow \text{pressure balance.}$$



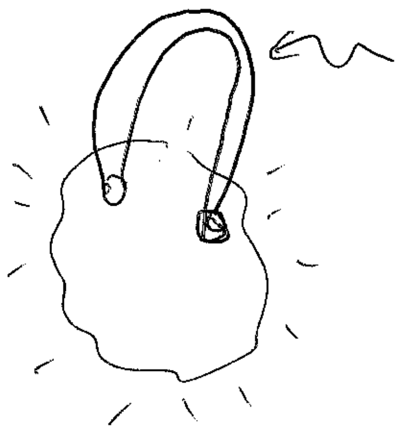
It is possible to confine either the plasma or magnetic field in this setup.

"plasma confined".



"magnetic field confined".

this second case occurs in sun spots



flux tube connects two sun spot, plasma confines B-field.

The θ-pinch is <u>stable</u>.

Force - free equillibria

Another interesting class of equillibria occour in situations where we can neglect $\vec{\nabla} p$ in the momentum equations. This can happen in two situations

1. $\vec{B}$ is stochastic & volume filling
2. $\beta \equiv P/(B^2/2\mu_0) \ll 1$ (typical fusion limit).

This is then a purely magnetic equillibrium, and

$$\vec{J} \times \vec{B} = \vec{0} \Rightarrow \vec{J} \propto \vec{B} \Rightarrow \vec{J} = \frac{1}{\mu_0}\alpha(\vec{r})\vec{B}$$

but $$\vec{J} = \frac{1}{\mu_0}\vec{\nabla} \times \vec{B} \Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \alpha(\vec{r})\vec{B}}$$

$\alpha(\vec{r})$ is (for the moment) an arbitrary scalar field.

I have defined α to have units 1/length.

Taking divergence $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 = \vec{\nabla}\cdot(\alpha(\vec{r})\vec{B})$

$$= (\vec{B}\cdot\vec{\nabla})\alpha + \alpha\,\cancelto{0}{(\vec{\nabla}\cdot\vec{B})}$$

$$\Rightarrow (\vec{B}\cdot\vec{\nabla})\alpha = 0$$ constant on magnetic surfaces.

$\alpha(\vec{r}) = \alpha_0 = cst.$

Is called "linear" force free field.

Then $\vec{\nabla} \times \vec{B} = \alpha_0 \vec{B}$

take curl

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \alpha_0 \vec{\nabla} \times \vec{B} = \alpha_0^2 \vec{B}.$$

but $\vec{\nabla} \times (\vec{\nabla} \times \vec{B})_i = \varepsilon_{ijk} \partial_j \varepsilon_{klm} \partial_l B_m$

$$= [\delta_{il} \delta_{jm} - \delta_{jl} \delta_{im}] \partial_j \partial_l B_m$$

$$= \partial_i \cancelto{0}{\partial_j B_j} - \partial_j \partial_j B_i$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\nabla^2 \vec{B} = \alpha_0^2 \vec{B}$$

$$\Rightarrow (\nabla^2 + \alpha_0^2) \vec{B} = \vec{0}.$$

the so-called Helmholtz equation.

Therefore, there is a potentially huge zoo of MHD equilibria. Some will be stable, others not, so some are interesting / others not.

The question is: to what equillibrium will a MHD system settle?

Imagine we have a plasma, and we set up some initial configuration of $\vec{B}$, say by driving a current (in a wire / the plasma). This will exert forces on the plasma, which will move & produce currents $\Rightarrow$ evolving $\vec{B}$.

In the long time limit, everything will settle down into some static equillibrium.

While we have been considering ideal MHD, with no losses (e.g., viscosity, resistivity), there will always be some losses in a real system. These losses will sap some of the energy content of the initial field. In nature we expect the final state to be a <u>minimum energy state</u>, and so we find it by minimising the magnetic energy

$$\iiint d^3\vec{r}\, \frac{B^2}{2\mu_0} \rightarrow \text{minimum.}$$