Parallel propogation

Consider $k_\perp = 0$ [or $\cos\theta = 1$]

Simplest to return to original dispersion equations

$$\omega^2 \xi_x = V_A^2 k_\parallel^2 \xi_x$$

and

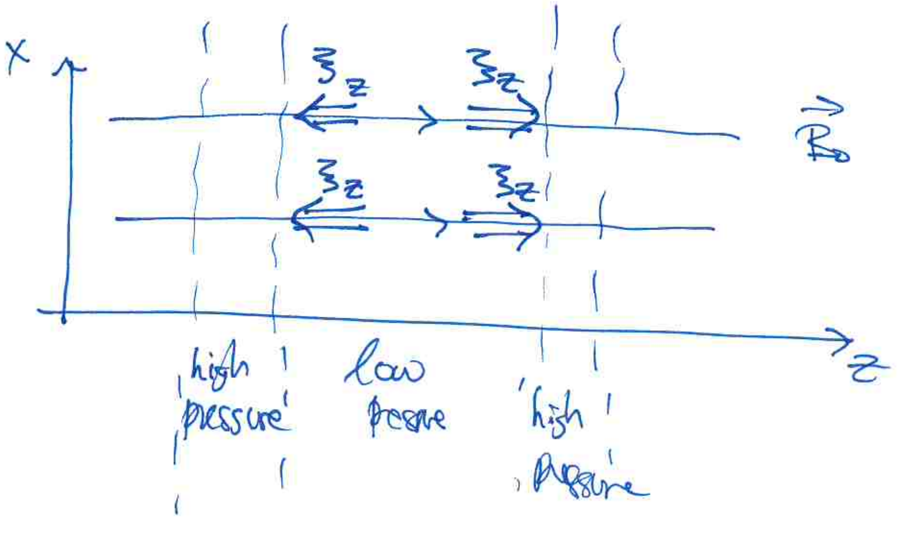
$$\omega^2 \xi_z = c_s^2 k_\parallel^2 \xi_z$$

completely seperate again, and we have one Alfvén wave [along $\hat{x}$] and one parallel propagating sound wave.

Alfvén: $\vec{\xi} = \xi_x(t)\hat{x}$, $\delta\rho = 0$, $\delta p = 0$, $\delta B = 0$, $\delta\hat{b} = ik_\parallel \xi_x \hat{x}$

Sound: $\vec{\xi} = \xi_z(t)\hat{z}$, $\frac{\delta\rho}{\rho_0} = -ik_\parallel \xi_z$, $\frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0}$, $\delta B = 0$, $\delta\hat{b} = \vec{0}$

Sound wave along $\vec{B_0}$



Perpendicular propagation

Now consider $k_{\parallel} = 0$ $[\cos\theta = 0]$.

Again returning to original equations we find

$$\omega^2 \xi_x = c_s^2 k_\perp^2 \xi_x + v_A^2 k_\perp^2 \xi_x$$

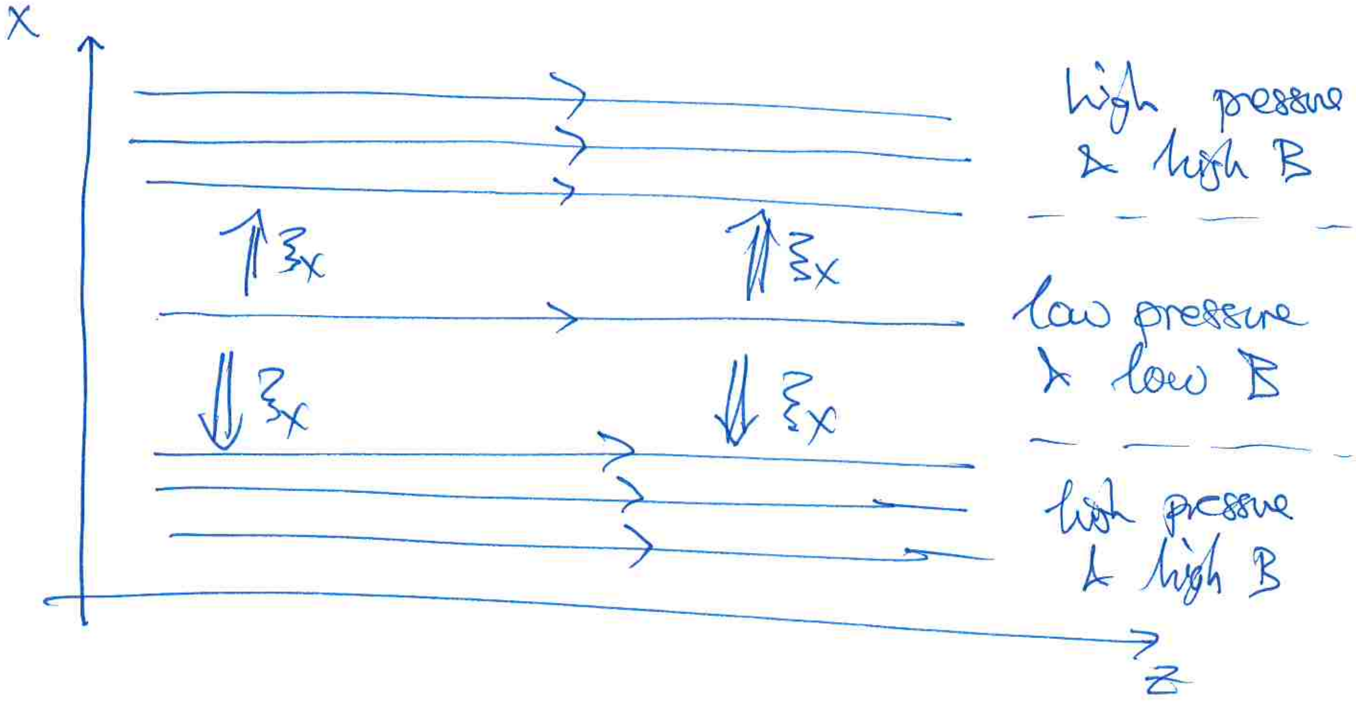
$$\Rightarrow \quad \omega^2 = k_\perp^2 \left(c_s^2 + v_A^2 \right)$$

What is this perturbation response?

Well, $\vec{\xi} = \xi_x(t)\,\hat{x}$, $\quad \dfrac{\delta\rho}{\rho_0} = -i k_\perp \xi_x(t)$

$$\frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0}, \quad \frac{\delta B}{B_0} = -i k_\perp \xi_x(t)$$

$$\delta\vec{b} = \vec{0}.$$



This perpendicular response is called a <u>sound wave</u> [no field bending].

But both thermal & magnetic pressures play a restoring role.

Note that the magnetic & thermal pressure responses are in phase.

There is <u>no</u> bending of fields.

Magnetosonic waves

The x-z plane contains a closed form 2D system

$$\omega^2 \xi_x = c_s^2 k_\perp (k_\perp \xi_x + k_\parallel \xi_z) + V_A^2 k^2 \xi_x$$

$$\omega^2 \xi_z = c_s^2 k_\parallel (k_\perp \xi_x + k_\parallel \xi_z)$$

or $$\begin{pmatrix} \omega^2 - c_s^2 k_\perp^2 - V_A^2 k^2 & -c_s^2 k_\perp k_\parallel \\ -c_s^2 k_\parallel k_\perp & \omega^2 - c_s^2 k_\parallel^2 \end{pmatrix} \begin{pmatrix} \xi_x \\ \xi_z \end{pmatrix} = \vec{0}.$$

this equation is of the form

$$\underline{\underline{M}} \vec{v} = \vec{0}.$$

If $\underline{\underline{M}}$ had an inverse, then $\underline{\underline{M}}^{-1} \underline{\underline{M}} \vec{v} = \underline{\underline{M}}^{-1} \vec{0}$

$$\Rightarrow \vec{v} = \underline{\underline{M}}^{-1} \vec{0}$$

but $\underline{\underline{M}}^{-1} \vec{0} = \vec{0} \;\; \forall \underline{\underline{M}}^{-1}$.

$\therefore$ $\underline{\underline{M}}$ has no inverse.

$\det(\underline{\underline{M}}) = 0 \Rightarrow$ <u>dispersion relationship</u>.

$$(\omega^2 - c_s^2 k_\perp^2 - V_A^2 k^2)(\omega^2 - c_s^2 k_\parallel^2) - c_s^4 k_\parallel^2 k_\perp^2 = 0$$

expanding (and canceling)

$$\omega^4 - \omega^2 (c_s^2 k^2 + V_A^2 k^2) + V_A^2 c_s^2 k^2 k_\parallel^2 = 0$$

with solutions

$$\omega^2 = \frac{1}{2} k^2 \left[c_s^2 + v_A^2 \pm \sqrt{(c_s^2+v_A^2)^2 - 4c_s^2 v_A^2 \cos^2\theta} \right]$$

where $\cos^2\theta \equiv \dfrac{k_{\parallel}^2}{k^2}$

There are two "+" solutions called "fast magnetosonic" waves, and two "−" solutions called "slow magnetosonic" waves.

Since both sound and Alfvén speeds are involved, clearly their ratio will be the key parameter for determining the physical regime we are in.

Convention dictates this dimensionless parameter is given by

$$\beta \equiv \frac{p_0}{B_0^2/2\mu_0} = \frac{2}{\gamma}\frac{c_s^2}{v_A^2}$$

and is known as the "plasma beta" parameter.

"Typical astrophysics" $\beta \sim 100$

"Typical fusion plasma" $\beta \sim 1/100$

Anisotropic perturbations

Consider the limit $k_{\parallel} \ll k_{\perp}$.
This will be much more interesting than $k_{\parallel} = 0$, which threw too much away.

The limit $k_{\parallel} \ll k_{\perp}$ will be especially relevant for strong magnetic fields, as those excitations which are realistic tend to propogate along the field if they can (bending a field line gives a curvature responce).

Factorising our magneto-sonic dispersion relationship

$$\omega^2 = \frac{1}{2} k^2 (c_s^2 + v_A^2)\left[1 \pm \sqrt{1 - \frac{4 c_s^2 v_A^2}{(c_s^2 + v_A^2)^2} \frac{k_{\parallel}^2}{k_{\parallel}^2 + k_{\perp}^2}}\right]$$

taylor expanding

$$\omega^2 \approx \frac{1}{2} k^2 (c_s^2 + v_A^2)\left[1 \pm 1 \mp \frac{2 c_s^2 v_A^2}{(c_s^2 + v_A^2)^2} \frac{k_{\parallel}^2}{k^2} + \dots\right]$$

where all upper/lower signs should be taken consistently.

upper sign:

$$\omega^2 = k^2 (c_s^2 + v_A^2) \quad \text{the "fast wave".}$$

Just the boosted sound wave from before, $k_{\parallel}/k_{\perp}$ corrections not interesting.

lower sign:

$$\omega^2 = k_\parallel^2 \frac{c_s^2 V_A^2}{(c_s^2 + V_A^2)}$$

This is the "slow wave", and is more interesting.

From our 'z' component of the linearized equations we have

$$\omega^2 \xi_z = c_s^2 k_\parallel k_\perp \xi_x + c_s^2 k_\parallel^2 \xi_z$$

$$\Rightarrow \xi_z \underbrace{(\omega^2 - c_s^2 k_\parallel^2)}_{\frac{k_\parallel^2 c_s^2 V_A^2}{c_s^2 + V_A^2} - c_s^4 k_\parallel^2 \;\sim\; -\frac{c_s^4 k_\parallel^2}{c_s^2 + V_A^2}} = c_s^2 k_\parallel k_\perp \xi_x$$

$$\Rightarrow \frac{\xi_x}{\xi_z} = -\frac{k_\parallel}{k_\perp} \frac{c_s^2}{c_s^2 + V_A^2} \ll 1.$$

Displacements are <u>mostly parallel</u>.

perturbations of the other fields

$$\frac{\delta\rho}{\rho_0} = -\vec{\nabla}\cdot\vec{\xi} = -i\vec{k}\cdot\vec{\xi} = -i(k_\perp \xi_x + k_\parallel \xi_z)$$

$$= -i k_\parallel \xi_z \left[\frac{k_\perp}{k_\parallel}\frac{\xi_x}{\xi_z} + 1\right]$$

$$= -i k_\parallel \xi_z \frac{v_A^2}{c_s^2 + v_A^2}$$

trivially $\quad \dfrac{\delta p}{p_0} = \gamma \dfrac{\delta\rho}{\rho_0}$

$$\delta\hat{b} = \nabla_\parallel \vec{\xi}_\perp = i k_\parallel \xi_x \hat{x} = -i \frac{k_\parallel}{k_\perp}\frac{c_s^2}{c_s^2+v_A^2} k_\parallel \xi_z \hat{x}$$

$$\frac{\delta B}{B_0} = -\vec{\nabla}_\perp\cdot\vec{\xi}_\perp = -i k_\perp \xi_x = i \frac{c_s^2}{c_s^2+v_A^2} k_\parallel \xi_z$$

Therefore to lowest order in $k_\parallel/k_\perp$ we get no field direction bending, but have pressure perturbations in anti-phase to magnetic field strength perturbations.

To be precise, these waves are in pressure balance

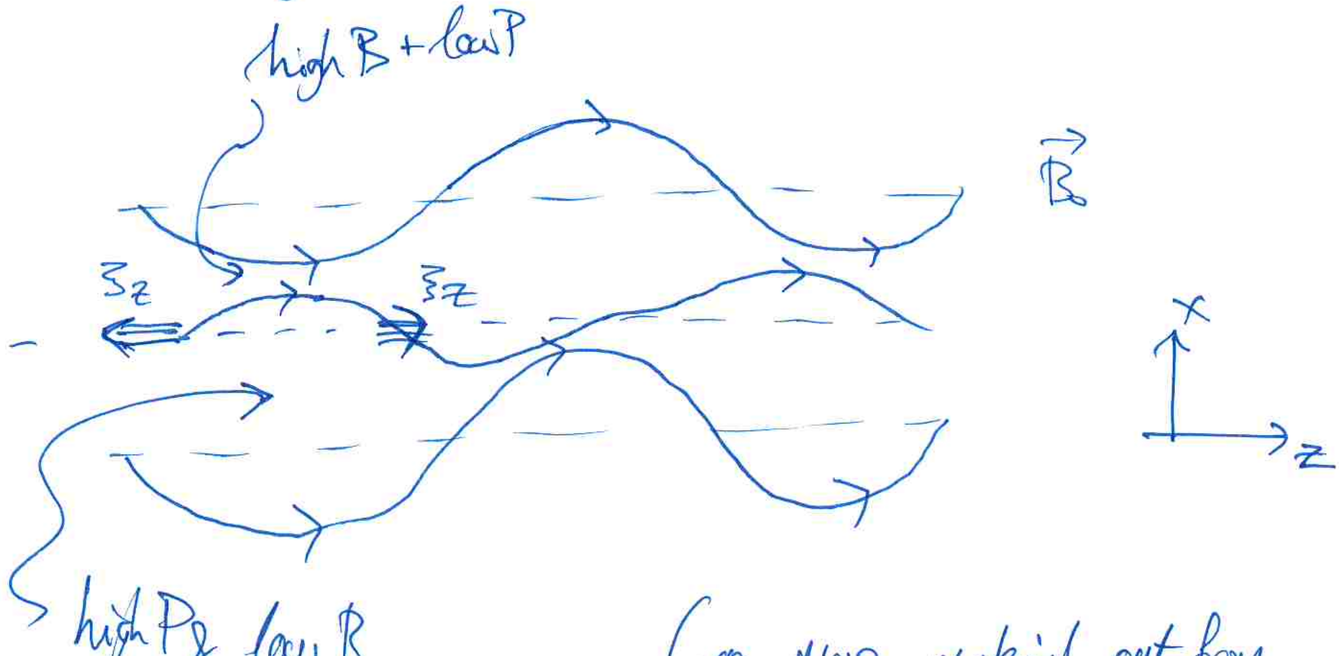
$$\delta\left(p + \frac{B^2}{2\mu_0}\right) = p_0 \frac{\delta p}{p_0} + \frac{B_0^2}{\mu_0}\frac{\delta B}{B_0}$$

$$= \frac{p_0 \gamma}{\rho_0} \delta \rho + \frac{B_0^2}{\mu_0}\frac{\delta B}{B_0}$$

$$= p_0 \gamma \left(-ik_{\parallel} \xi_z \frac{V_A^2}{c_s^2 + V_A^2}\right) + \frac{B_0^2}{\mu_0} ik_{\parallel} \xi_z \frac{c_s^2}{c_s^2 + V_A^2}$$

$$= i\rho_0 k_{\parallel} \xi_z \left[-\frac{c_s^2 V_A^2}{c_s^2 + V_A^2} + \frac{V_A^2 c_s^2}{c_s^2 + V_A^2}\right] = 0.$$

$\therefore$ where the wave causes pressure to go up, the B field goes down.



(you move material out from where the B field bunches up).

Astrophysically relevant high-β limit

Finally, let us consider $\beta \gg 1$ [$c_s^2 \gg V_A^2$].

Very relevant for astrophysics, because fields generated in [e.g.] interstellar medium topout at energetic scale that created them

$$\rho u^2 \sim \frac{B^2}{\mu_0} \Rightarrow V_A \sim u$$

and interstellar medium is sub-sonic

$$u \ll c_s \Rightarrow V_A \ll c_s.$$

Our taylor expansion of

$$\omega^2 = \frac{1}{2} k^2 (c_s^2 + V_A^2)\left[1 \pm \sqrt{1 - \frac{4 c_s^2 V_A^2}{(c_s^2+V_A^2)^2} \frac{k_\parallel^2}{k^2}}\right]$$

still valid in this limit, but now small parameter is V_A/c_s, not $k_\parallel/k_\perp$.

Fast wave becomes (upper sign in expansion)

$$\omega^2 \simeq k^2 c_s^2$$

and slow wave becomes (lower sign in expansion)

$$\omega^2 \approx k_\parallel^2 V_A^2.$$

This looks like an Alfvén wave, but is not. Called a pseudo-Alfvén wave as eigenvector very different.

Let's look at $\vec{\nabla}\cdot\vec{\xi}$ for this eigenvalue

$$\vec{\nabla}\cdot\vec{\xi} = k_\perp \xi_x + k_\parallel \xi_z = k_\parallel \xi_z \left[1 + \frac{k_\perp}{k_\parallel}\frac{\xi_x}{\xi_z}\right]$$

$$= -i k_\parallel \xi_z \frac{V_A^2}{c_s^2 + V_A^2} \rightarrow 0.$$

To lowest order in $1/\beta$ the perturbations are incompressible.

In contrast to the anisotropic limit the parallel & perpendicular perturbations are comparable

$$\frac{\xi_x}{\xi_z} = -\frac{k_\parallel}{k_\perp}\frac{c_s^2}{c_s^2+V_A^2} \simeq -\frac{k_\parallel}{k_\perp} \sim O(1)$$

in general

We have then

$$\frac{\delta\rho}{\rho_0} = -\vec{\nabla}\cdot\vec{\xi} = -i\,\frac{V_A^2}{c_s^2}\,k_{\parallel}\,\xi_z \rightarrow 0$$

$$\frac{\delta P}{P_0} = \gamma\,\delta\rho/\rho_0 \rightarrow 0.$$

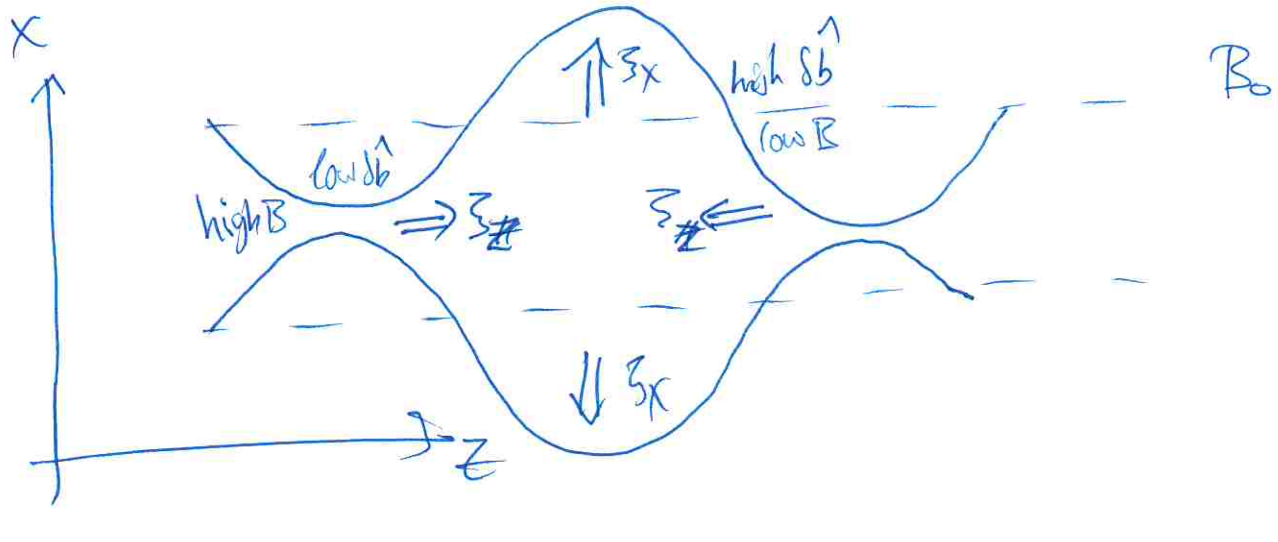
$$\delta\hat{b} = i k_{\parallel}\,\xi_x\,\hat{x} = -i\,\frac{k_{\parallel}}{k_{\perp}}\,k_{\parallel}\,\xi_z\,\hat{x}$$

$$\frac{\delta B}{B_0} = -i k_{\perp}\,\xi_x = i k_{\parallel}\,\xi_z$$

$\therefore$ δB and $\delta\hat{b}$ are in counterphase

as are ξ_x & ξ_z.

This looks like



This $\beta \gg 1$ wave is also in pressure balance

$$p_0 \frac{\delta p}{p_0} + \frac{B_0^2}{\mu_0} \frac{\delta B}{B_0} = \rho_0 \left[c_s^2 \frac{\delta \rho}{\rho_0} + v_A^2 \frac{\delta B}{B_0} \right]$$

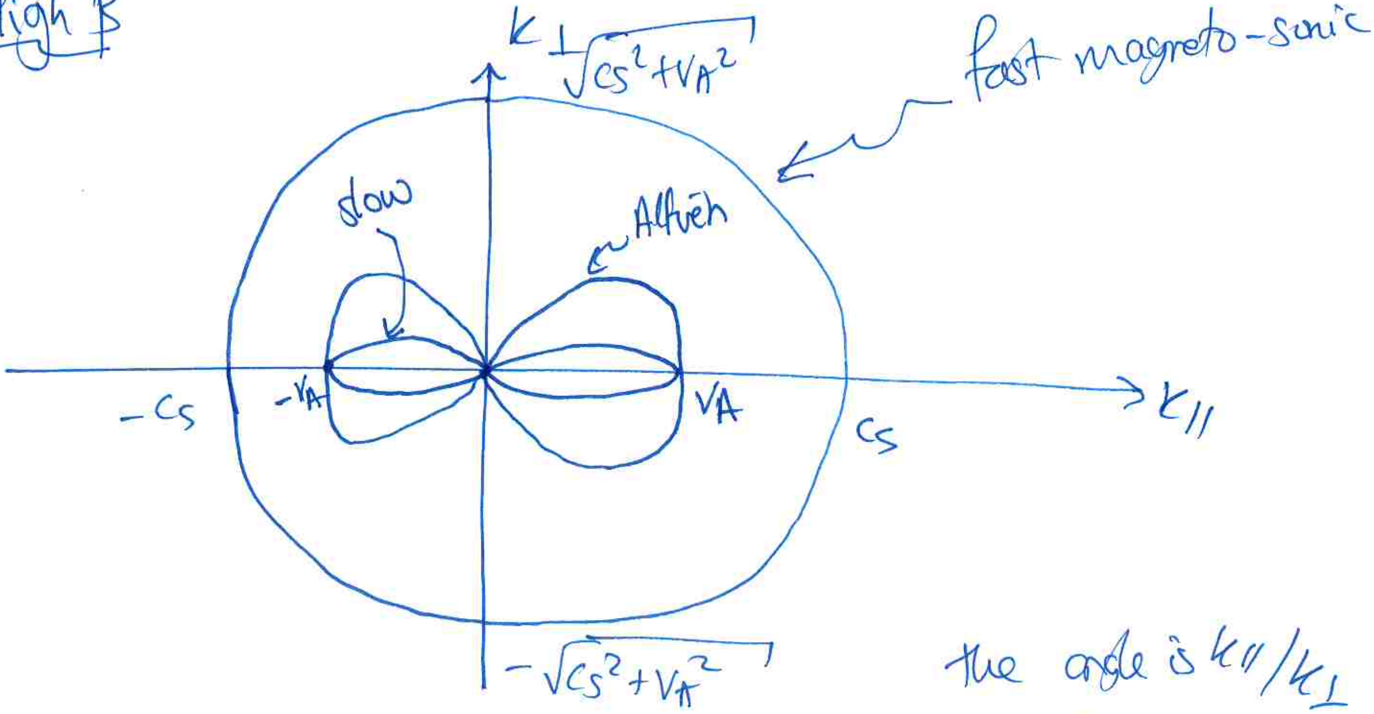
$$= -i k_{\parallel} \xi_z \left[-c_s^2 \frac{v_A^2}{c_s^2} + v_A^2 \right] = 0 .$$

Despite $\frac{\delta p}{p_0} \to 0$ backgrond thermal energy density much bigger then magnetic energy density and so can maintain pressure balance.

Friedricks diagram

Can represent the properties of these different waves on a so-called "Friedricks diagram".

High β



the angle is $k_{\parallel}/k_\perp$

The radius is ω/k

Low β