MHD - MMathPhys

1. Validity of the MHD model.

Magnetohydrodynamics (MHD) is the study of "large" and "slow" plasmas. It is a fluid (not kinetic) theory.

4 states of matter: solid $\rightarrow$ liquid $\rightarrow$ gas $\rightarrow$ plasma

$\Downarrow$

ionized state.

This means that constituent particles will respond dynamically to imposed $\vec{E}$ & $\vec{B}$ fields. Opens up a much wider range of behaviour than seen in classical fluid dynamics.

Plasma's make up $\sim 99\%$ of Universe, and MHD theory widely applied across astrophysics.

e.g. the Sun, stellar winds, accretion discs and jets, Planetary atmospheres.

Also, e.g. Nuclear Fusion reactors, etc.

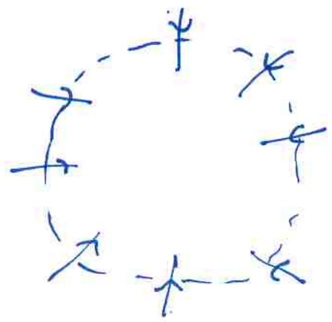
MHD describes "large" and "slow" plasmas.
We now make this precise.

Timescale

Imagine we have n_0/m^3 electrons and n_0/m^3 ions.
These are at rest ~~at~~ equillibrium.

Perturb electrons $\Rightarrow$ what happens? (In a gas, sound waves).

In plasma:



spherical perturbation inwards induces E-field

$$E = \frac{\frac{4}{3}\pi r^3 (n_e e)}{4\pi\varepsilon_0 r^2} = \frac{n_e e}{3\varepsilon_0} r$$

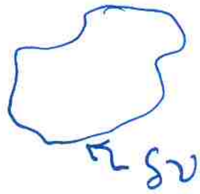
S.H.O!

typical parameters: $n_0 \sim 10^{17}/m^3$, $r \sim 1cm$; perturb $\sim 1\% \Rightarrow n_e \sim 10^{15}/m^3$.

Gives $E = 6\times10^4\ V/m \Rightarrow$ acceleration $a = eE/m_e \simeq 10^{16}\ m/s^2$.

$\therefore$ Rapid response to perturbation.

More precise: mass conservation for electron perturbation



$$\frac{d}{dt}\iiint_{\mathcal{V}} n_e\, dV = -\oint_{\delta\mathcal{V}} n_e \vec{u}\cdot d\vec{S}$$

$$= -\iiint_{\mathcal{V}} \vec{\nabla}\cdot(n_e\vec{u})\, dV$$

arbitrary volume

$$\Rightarrow \frac{\partial n_e}{\partial t} + \vec{\nabla}\cdot(n_e\vec{u}) = 0.$$

linearise: $n_e \rightarrow n_0 + n_e'$, $\vec{u} \rightarrow \vec{0} + \vec{u}'$

$$\frac{\partial}{\partial t}(\underbrace{n_0}_{\text{constant}} + n_e') + \vec{\nabla}\cdot(\underbrace{(n_0 + n_e')\vec{u}'}_{\text{second order}}) \cong \frac{\partial n_e'}{\partial t} + n_0 \vec{\nabla}\cdot\vec{u}' = 0.$$

Take Newton's second law

$$m_e \frac{d\vec{u}'}{dt} = -e\vec{E}' \qquad \text{(for each electron).}$$

take divergence

Gauss' law

$$m_e \frac{d}{dt}(\vec{\nabla}.\vec{u}') = -e\vec{\nabla}.\vec{E}' = -\frac{e}{\varepsilon_0}\rho_c = \frac{n_e' e^2}{\varepsilon_0}$$

definition of charge density

Then substitute from mass conservation

$$m_e \frac{d}{dt}\left(-\frac{dn_e'}{dt}\frac{1}{n_0}\right) = -\frac{m_e}{n_0}\frac{d^2 n_e'}{dt^2} = \frac{n_e' e^2}{\varepsilon_0}$$

Simple harmonic motion with frequency

$$\omega_{pl} = \sqrt{\frac{n_0 e^2}{m_e \varepsilon_0}}$$

Oscillations restore neutrality on timescales $\sim 1/\omega_{pl}$.

Therefore, we can treat fluid as neutral on times long compared to $t_{pl} \sim 1/\omega_{pl}$.

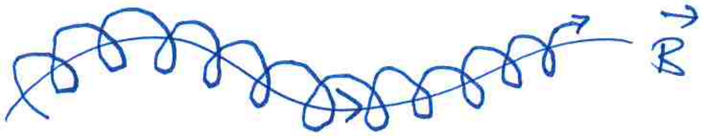
MHD only describes phenomena on timescales $t \gg t_{pl}$.

Second timescale

Particle of mass m and charge q also subject to force

$$\vec{f} = q\vec{v} \times \vec{B} .$$

Well known result that this causes uniform rotation perpendicular to field, and linear motion along field $\Rightarrow$ helix.



$$f = q v_\perp B = m v_\perp \omega \Rightarrow \omega_c = \frac{|q| B}{m}$$

known as cyclotron frequency.

Again, MHD describes systems on timescales

$$t \gg 1/\omega_c .$$

This gives us our first length scale, the radius of the circle followed by the particle

$$r_L = \frac{v_\perp}{\omega_c} \quad \text{"larmar radius".}$$

MHD works on $\quad r \gg r_L$.

Note, both particle <u>mass</u> dependent. Very different scales for electrons and ions.

Length scales

We showed that perturbing n_e locally causes an Electric field to be induced, over what length scale is this field felt?

To answer this we need more information about the equillibrium. Assume thermodynamic equillibrium so electrons obey Maxwell - Boltzmann statistics

$$n_0 = A \exp\left(eV_0/kT\right) \quad [\text{energy } U = -eV_0].$$

Perturbed

$$n_0 \to n_0 + n_e' = A\exp\left(\frac{e(V_0+V')}{kT}\right) = \underbrace{A\exp\left(\frac{eV_0}{kT}\right)}_{n_0}\exp\left(\frac{eV'}{kT}\right)$$

$$\Rightarrow n_e' = n_0\left[\exp\left(\frac{eV'}{kT}\right) - 1\right] \simeq n_0 \frac{eV'}{kT}$$

for hot plasmas.

Again, Gauss' law

$$\vec{\nabla}\cdot\vec{E}' = \frac{\rho_c}{\varepsilon_0} = -\frac{n_e' e}{\varepsilon_0}$$

with $\vec{E}' = -\vec{\nabla}V' \Rightarrow \nabla^2 V' = \frac{n_e' e}{\varepsilon_0} = \frac{n_0 e^2}{\varepsilon_0 kT} V'$

$\therefore$ V' "shielded" over a length $\lambda_D = \sqrt{\frac{\varepsilon_0 kT}{n_0 e^2}}$ known as the Debeye length.

[spherical: $\nabla^2 V' = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rV') = \frac{n_0 e^2}{\varepsilon_0 kT}V' \Rightarrow V \propto \frac{1}{r}e^{-r/\lambda_D}$].

Finally, electromagnetic waves only penetrate a certain length scale into a plasma.

Let's try and push an electro magnetic wave into the plasma. EM waves are transverse

$$\vec{E} = E_y(x,t)\,\hat{y} \; ; \quad \vec{B} = B_z(x,t)\,\hat{z}$$

wave $\propto e^{i(kx-\omega t)}$

Lenz's law : $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\frac{\partial E_y}{\partial x}\hat{z} = -\frac{\partial B_z}{\partial t}\hat{z} \quad — (1)$$

Also

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \Rightarrow -\frac{\partial B_z}{\partial x}\hat{y} = -\mu_0 n_0 e \vec{v}' \quad — (2)$$

Newton's second law $m_e \frac{d\vec{v}'}{dt} = -eE_y\hat{y} \quad — (3)$

~~[illegible]~~ Differentiate (1) w.r.t. x, use (2)

$$-\frac{\partial}{\partial t}\frac{\partial B_z}{\partial x} = -\mu_0 n_0 e \frac{dv'}{dt} = \frac{\partial^2 E_y}{\partial x^2}$$

use (3)

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{\mu_0 n_0 e^2}{m_e} E_y = \mu_0 \varepsilon_0 \cdot \left(\frac{n_0 e^2}{\varepsilon_0 m_e}\right) E_y$$

$$= \frac{c^2}{\omega_{pl}^2} E_y$$

Decay length $\Rightarrow d_e = \frac{c}{\omega_{pl}}$

Rules of the game:

- normal fluid rules
  - ↳ Plasma strongly collissional, $t \gg t_{col}$
    $L \gg \lambda_{mfp}$
    (means particles follow ~ Maxwellian).

but also

- length scale $L \gg$ larmor radius

and

- timescale $t \gg 1/$cyclotron frequency

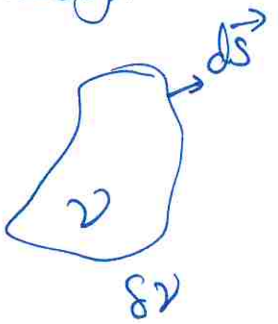
- timescale $\gg 1/$plasma frequency

- Non-relativistic (more next time).

2. Dynamical fluid evolution

- **Mass conservation**

We saw yesterday that mass conservation is constructed through



$$\frac{d}{dt}\iiint_V \rho \, dV = -\overbrace{\oint \rho \vec{u} \cdot d\vec{S}}^{\text{flux out}}$$

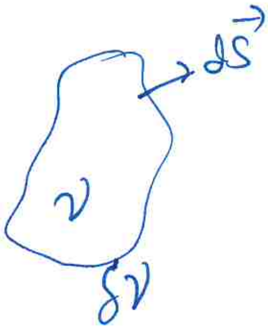
divergence theorem

$$= -\iiint_V \vec{\nabla} \cdot (\rho \vec{u}) \, dV$$

fixed arbitrary volume

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0}.$$

- **Momentum conservation**



$$\frac{d}{dt}\iiint_V \rho \vec{u} \, dV = -\overbrace{\oint_{\delta V} (\rho \vec{u}) \, \vec{u} \cdot d\vec{S}}^{\text{momentum flux out}} \overbrace{- \oint_{\delta V} P \, d\vec{S}}^{\text{pressure forces}} + \overbrace{\iiint_V \vec{f} \, dV}^{\text{force per unit volume}}$$

Divergence theorum

$$\frac{d}{dt}\iiint \rho \vec{u} \, dV = -\iiint_V \vec{\nabla} \cdot (\rho \vec{u} \vec{u}) \, dV - \iiint_V \vec{\nabla} P \, dV + \iiint_V \vec{f} \, dV$$

Arbitrary and fixed volume, and therefore

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \vec{\nabla}\cdot(\rho \vec{u}\vec{u}) = -\vec{\nabla} P + \vec{f}.$$

Let's massage this left hand side

$$\frac{\partial}{\partial t}(\rho \vec{u}) = \rho \frac{\partial \vec{u}}{\partial t} + \vec{u}\frac{\partial \rho}{\partial t}$$

$$\vec{\nabla}\cdot(\rho \vec{u}\vec{u}) = \vec{u}\left[\vec{\nabla}\cdot(\rho \vec{u})\right] + (\rho \vec{u}\cdot\vec{\nabla})\vec{u}$$

Adding

Mass conservation again

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \vec{\nabla}\cdot(\rho \vec{u}\vec{u}) = \rho \frac{\partial \vec{u}}{\partial t} + \vec{u}\left[\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho \vec{u})\right] + (\rho \vec{u}\cdot\vec{\nabla})\vec{u}$$

$\therefore$ we have

$$\rho \underbrace{\left[\frac{\partial}{\partial t} + (\vec{u}\cdot\vec{\nabla})\right]}_{\text{Convective (or Lagrangian) derivative}} \vec{u} = -\vec{\nabla} P + \vec{f}$$

[rate of change moving with flow]

What is a lagrangian derivative?

Consider sitting with a fluid element as it moves. It will trace out a path in space and time described by coordinates $(t, \vec{x}(t))$.

We can ask, how does a quantity ϕ vary as the fluid element moves?

In time δt, $\phi(\vec{x}, t)$ becomes $\phi(\vec{x} + \vec{v}\delta t, t+\delta t)$, which is

velocity of fluid element.

$$\phi(\vec{x} + \vec{v}\delta t, t+\delta t) \simeq \phi(\vec{x}, t) + \left.\frac{\partial \phi}{\partial t}\right|_{\vec{x},t} \delta t + (\delta t \vec{v} \cdot \vec{\nabla})\phi\Big|_{\vec{x},t} + \dots$$

$$\therefore \quad \frac{D\phi}{Dt} \equiv \lim_{\delta t \to 0} \frac{\phi(\vec{x} + \vec{v}\delta t, t+\delta t) - \phi(\vec{x}, t)}{\delta t}$$

$$= \frac{\partial \phi}{\partial t} + \vec{v} \cdot \vec{\nabla} \phi$$

So, we see that

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\vec{\nabla} P + \frac{\vec{f}}{\rho}$$

Note, we have neglected viscosity in this formalism. This is always a good approximation astrophysically.

What is $\vec{f}$?

General astrophysical setting

$$\vec{f} = \underbrace{-\rho \vec{\nabla} \Phi}_{\text{Gravity}} + \underbrace{\vec{f}_L}_{\text{Lorentz force per volume.}}$$

For this course we shall neglect gravity, except for occassional specific problem.

We shall focus on magnetic forces.

We assume there is a frame where charges are non-relativistic. We know (from yesterday) that charges move rapidly $(t \sim 1/\omega_{pl})$ to screen electric fields.

Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho_c \quad \text{— (M1)} \qquad [\rho_c = \text{charge density}]$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{— (M2)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{— (M3)}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \quad \text{— (M4)} \qquad [\vec{J} = \text{current}].$$

In the M.H.D limit: yesterday we argued that electrons rapidly move to screen ~~away~~ electric fields (on timescales $t \sim 1/\omega_{pl}$). Move to rest frame of ions (moving with $\vec{v}_i$):

$$\vec{B}_i = \vec{B} - \frac{1}{c^2} \vec{v}_i \times \vec{E}$$
$$\vec{E}_i = \vec{E} + \vec{v}_i \times \vec{B}$$

Ion's are <u>non-relativistic</u>

$$\vec{B}_i \simeq \vec{B}$$
$$\vec{E}_i = \vec{E} + \vec{v} \times \vec{B}$$

But $\vec{E}_i$ screened by electrons, so lab-frame

$$\vec{E}_i \simeq \vec{0} \Rightarrow \boxed{\vec{E} = -\vec{v} \times \vec{B}.}$$

and therefore $E \sim vB$ (scaling).

Current – magnetic field relationship:

M4 states

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

consider ration of the two electromagnetic terms

$$\frac{\left| \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \right|}{\left| \vec{\nabla} \times \vec{B} \right|} \sim \frac{\varepsilon_0 \mu_0 E / \tau}{B / \ell} \sim \frac{E}{B} \cdot \frac{\ell}{\tau} \cdot \frac{1}{c^2}$$

where ℓ, τ are length and timescales of process. But, $\ell/\tau \sim v$, $E \sim vB$

$$\therefore \quad \frac{\left| \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \right|}{\left| \vec{\nabla} \times \vec{B} \right|} \sim \frac{v^2}{c^2} \ll 1 .$$

$\therefore$ Current entirely specified by $\vec{B}$

$$\boxed{\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} .}$$

Ignoring the charge density

$$\vec{\nabla}\cdot\vec{E} = \rho_c/\varepsilon_0 \Rightarrow \rho_c \sim \frac{\varepsilon_0 E}{\ell} \sim \frac{\varepsilon_0 UB}{L}$$

$$\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} \Rightarrow |\vec{J}| \sim \frac{B}{\mu_0 L}$$

$$\Rightarrow \frac{|\rho_c \vec{v}|}{|\vec{J}|} \sim \frac{\varepsilon_0 v^2 B/L}{B/\mu_0 L} \sim \frac{v^2}{c^2} \ll 1.$$

So we can neglect terms of order $\rho_c \vec{v}$ compared to $\vec{J}$.

We are now in position to write $\vec{f}$.

Force per unit volume in EM

$$\vec{f} = \rho_c \vec{E} + \vec{J}\times\vec{B}$$

$$\simeq -\rho_c \vec{v}\times\vec{B} + \vec{J}\times\vec{B} \quad [\text{from } \vec{E} \simeq -\vec{v}\times\vec{B}]$$

$$\simeq \vec{J}\times\vec{B} \quad [\text{from } \rho_c\vec{v} \sim \left(\tfrac{v^2}{c^2}\right)\vec{J}]$$

$$= \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} \quad [\text{from } \vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}].$$

$$
\begin{aligned}
\therefore \quad \vec{f} &= \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} \\
\Rightarrow \quad f_i &= \frac{\varepsilon_{ijk}}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right)_j B_k \\
&= \frac{\varepsilon_{ijk}}{\mu_0}\,\varepsilon_{jlm}(\partial_l B_m) B_k \\
&= -\frac{\varepsilon_{jik}}{\mu_0}\,\varepsilon_{jlm}(\partial_l B_m) B_k \\
&= -\frac{1}{\mu_0}\left[\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}\right](\partial_l B_m) B_k \\
&= -\frac{1}{\mu_0}(\partial_i B_m) B_m + \frac{1}{\mu_0} B_k \partial_k B_i \\
&= -\frac{1}{2\mu_0}\partial_i(B_m B_m) + \frac{1}{\mu_0} B_k \partial_k B_i \\
\therefore \quad \vec{f} &= -\vec{\nabla}\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B}
\end{aligned}
$$

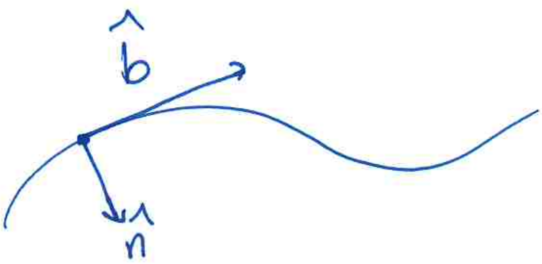
$\therefore$ we return to momentum conservation

$$\rho\left[\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right]\vec{u} = -\vec{\nabla}P - \vec{\nabla}\left(\frac{1}{2\mu_0}B^2\right) + \frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B}$$

$$= -\underbrace{\vec{\nabla}\left(P + \frac{B^2}{2\mu_0}\right)}_{\text{magnetic pressure force}} + \underbrace{\frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B}}_{\text{magnetic tension.}}$$

Let's interpret these terms:

Let's define $\hat{b}$ the unit vector tangent to the field line $\vec{B} = B\hat{b}$



then $\frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B} = \frac{B}{\mu_0}(\hat{b}\cdot\vec{\nabla})(B\hat{b})$

$$= \frac{B^2}{\mu_0}(\hat{b}\cdot\vec{\nabla})\hat{b} + \frac{\hat{b}B}{\mu_0}(\hat{b}\cdot\vec{\nabla})B$$

Call $\frac{\partial}{\partial s} \equiv \hat{b}\cdot\vec{\nabla}$ = the derivative along the field line.

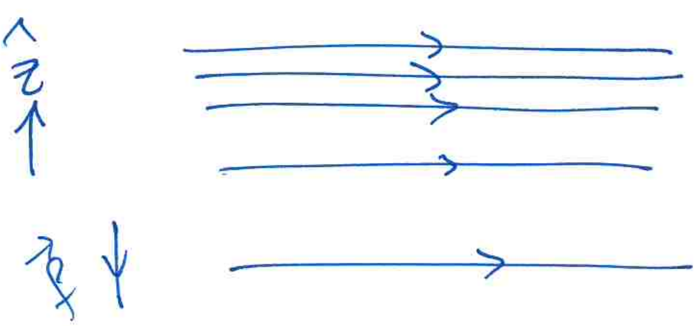
$$\therefore \quad \frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B} = \underbrace{\frac{B^2}{\mu_0}\frac{\partial \hat{b}}{\partial s}} + \hat{b}\frac{\partial}{\partial s}\left(\frac{1}{2\mu_0}B^2\right)$$

This force is <u>zero</u> if the field line is straight. This force acts to restore bent field lines.

Then adding back in the pressure term

$$-\vec{\nabla}\left(\frac{B^2}{2\mu_0}\right) + \hat{b}\frac{\partial}{\partial s}\left(\frac{1}{2\mu_0}B^2\right) = -\vec{\nabla}_\perp\left(\frac{B^2}{2\mu_0}\right)$$

we find that the magnetic pressure acts <u>perpendicular</u> to the field lines; they resist compression or rarefaction.



B^2 increases with z

$$\Rightarrow \frac{\partial}{\partial z}\left(B^2/2\mu_0\right) > 0$$

means $-\nabla_\perp(B^2/2\mu_0) < 0$

and force is along $-z$.

3. Evolution of the magnetic field

The Lorentz force [in ideal MHD $\vec{f} = \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}$] tell's the fluid how to respond to $\vec{B}$.
We now study how $\vec{B}$ responds to the fluid.

Induction equation

Recall the field transformation when we go into the rest frame of the ions

$$\vec{E}' = \vec{E} + \vec{v}\times\vec{B} .$$

Plasma (assumed to be) a perfect conductor, so

$$\vec{E}' \simeq \vec{0} \Rightarrow \vec{E} = -\vec{v}\times\vec{B} .$$

But Maxwell's 3rd equation states that

$$\frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$$

and so

$$\boxed{\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times(\vec{v}\times\vec{B}) .}$$

Quick side note:

by definition $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0 \quad \forall \vec{A}$

and so the induction equation implies

$$\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot\left[\vec{\nabla}\times(\vec{v}\times\vec{B})\right] = 0$$

$\Rightarrow \vec{\nabla}\cdot\vec{B}$ conserved $(=0)$; a good consistency check.

The induction equation $\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B})$ implies so-called "flux freezing". Flux freezing can be analysed in a number of ways, first consider the "lindquist theorem".

Lindquist theorem

$$\vec{\nabla}\times(\vec{u}\times\vec{B}) = \varepsilon_{ijk}\,\partial_j\,(\vec{u}\times\vec{B})_k = \varepsilon_{ijk}\partial_j\,\varepsilon_{klm}\,u_\ell B_m$$

$$= \left[\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}\right]\partial_j(u_\ell B_m)$$

$$= \partial_j(u_i B_j) - \partial_j(u_j B_i)$$

$$= u_i(\partial_j B_j) + (B_j\partial_j)u_i - B_i(\partial_j u_j) - (u_j\partial_j)B_i$$

$$= \vec{u}\underbrace{(\vec{\nabla}\cdot\vec{B})}_{0} + (\vec{B}\cdot\vec{\nabla})\vec{u} - \vec{B}(\vec{\nabla}\cdot\vec{u}) - (\vec{u}\cdot\vec{\nabla})\vec{B}$$

$$= (\vec{B}\cdot\vec{\nabla})\vec{u} - (\vec{u}\cdot\vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla}\cdot\vec{u})$$

$$\therefore \quad \frac{\partial \vec{B}}{\partial t} + \underbrace{(\vec{u}\cdot\vec{\nabla})\vec{B}}_{\text{Advection}} = \underbrace{(\vec{B}\cdot\vec{\nabla})\vec{u}}_{\text{stretching}} - \underbrace{\vec{B}(\vec{\nabla}\cdot\vec{u})}_{\text{compression}}$$

Recall mass conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + (\vec{u}\cdot\vec{\nabla})\rho = -\rho(\vec{\nabla}\cdot\vec{u})$$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{u} = -\frac{1}{\rho}\frac{D\rho}{Dt}$$

"compression of flow".

Substitute back into induction equation

$$\frac{D\vec{B}}{Dt} = (\vec{B}\cdot\vec{\nabla})\vec{u} + \frac{\vec{B}}{\rho}\frac{D\rho}{Dt}$$

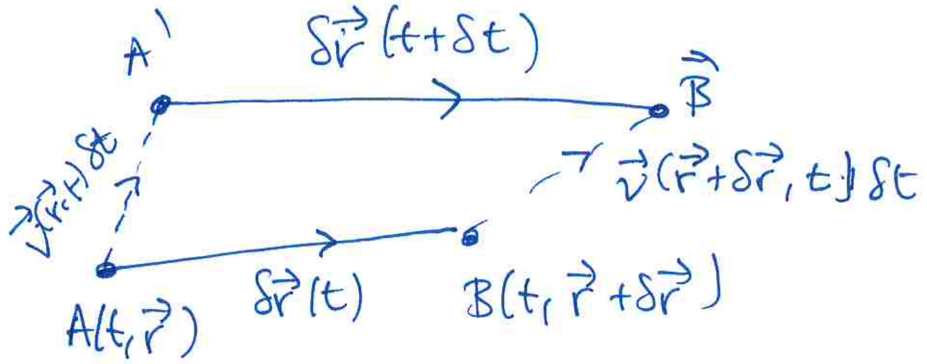
$$\therefore \quad \frac{1}{\rho}\frac{D\vec{B}}{Dt} - \frac{\vec{B}}{\rho^2}\frac{D\rho}{Dt} = \left(\frac{\vec{B}}{\rho}\cdot\vec{\nabla}\right)\vec{u}$$

$$\frac{D}{Dt}\left(\frac{\vec{B}}{\rho}\right) = \left[\left(\frac{\vec{B}}{\rho}\right)\cdot\vec{\nabla}\right]\vec{u}$$

"Lindquist identity".

Interpretation

Consider two fluid elements seperated by $\delta\vec{r}(t)$.



[i.e., during δt
$A \to A'$
$B \to B'$]

As the flow evolves $\delta\vec{r}$ will change.

Trivial vector identity

$$\vec{A'B'} \equiv \delta\vec{r}(t+\delta t) = -\vec{AA'} + \vec{AB} + \vec{BB'}$$

$$= \vec{AB} + (\vec{BB'} - \vec{AA'})$$

$$= \delta\vec{r}(t) + \vec{v}(\vec{r}+\delta\vec{r},t)\delta t - \vec{v}(\vec{r},t)\delta t$$

$$\simeq \delta\vec{r}(t) + (\delta\vec{r}\cdot\vec{\nabla})\vec{v}\,\delta t + \dots$$

$$\therefore \frac{\cancel{D}}{\cancel{D}t}\delta\vec{r} \equiv \frac{\delta\vec{r}(t+\delta t)-\delta\vec{r}(t)}{\delta t} = (\delta\vec{r}\cdot\vec{\nabla})\vec{v}$$

which is the change of line element moving with the flow.

$$\therefore \frac{D}{Dt}\left(\frac{\vec{B}}{\rho}\right) = \left(\frac{\vec{B}}{\rho}\cdot\vec{\nabla}\right)\vec{u}$$

$$\frac{D}{Dt}\delta\vec{r} = \left(\delta\vec{r}\cdot\vec{\nabla}\right)\vec{u}$$

and $\frac{\vec{B}}{\rho}$ and $\delta\vec{r}$ satisfy the same equation.

Let's assume that at time 't' $\frac{\vec{B}}{\rho}$ and $\delta\vec{r}$ are parrallel : $\frac{\vec{B}}{\rho} = \alpha\,\delta\vec{r}$

Since
$$\frac{\vec{B}}{\rho}(t+\delta t) = \frac{\vec{B}}{\rho}(t) + \frac{D}{Dt}\left(\frac{\vec{B}}{\rho}\right)\delta t$$
$$= \frac{\vec{B}}{\rho}(t) + \left(\frac{\vec{B}}{\rho}\cdot\vec{\nabla}\right)\vec{u}\,\delta t$$
$$= \alpha\,\delta r(t) + \alpha\left(\delta\vec{r}\cdot\vec{\nabla}\right)\vec{u}\,\delta t$$
$$= \alpha\,\delta\vec{r}\,(t+\delta t)$$

$\therefore$ $\frac{\vec{B}}{\rho}(t+\delta t)$ is parallel to $\delta\vec{r}\,(t+\delta t)$.

In other words, magnetic field lines move with the flow: if fluid elements start on the same field line they stay on the same field line as they move with the flow.

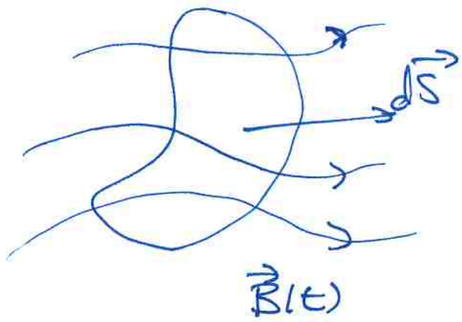
<u>But</u> we discussed last week that field lines actively resist bending (through the force $\frac{B^2}{\mu_0}\frac{\partial \hat{b}}{\partial s}$) and compression (through the force $-\vec{\nabla}_\perp(B^2/2\mu_0)$).

Following fluid elements as they move around invariably ends up with either field bending or compressing $\Rightarrow$ complex phenomena.

Alfvén's theorem

Begin by defining the magnetic flux through a surface $S(t)$ by

$$\Phi(t) = \iint_{S(t)} \vec{B}(t) \cdot d\vec{S}$$



Consider now a later time

$t \Rightarrow t + \delta t$.

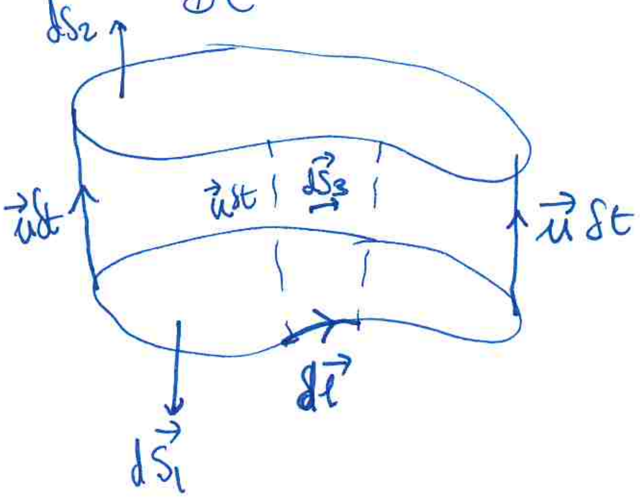
then

$$\Phi(t+\delta t) = \iint_{S(t+\delta t)} \vec{B}(t+\delta t) \cdot d\vec{S}$$

and we define the ~~ma~~ lagrangian derivative of Φ by

$$\frac{D\Phi}{Dt} \equiv \lim_{\delta t \to 0} \left[\frac{\Phi(t+\delta t) - \Phi(t)}{\delta t} \right].$$

To compute $\dfrac{D\Phi}{Dt}$, consider this volume



This describes the volume swept out by a loop $S_1 \to S_2$ over a time δt as it is advected by flow $\vec{u}$.

Trivially

$$\iiint_{\mathcal{V}} \vec{\nabla}\cdot\vec{B}\, d\mathcal{V} = 0 \quad \text{as} \quad \vec{\nabla}\cdot\vec{B} = 0$$

but $\iiint_{\mathcal{V}} \vec{\nabla}\cdot\vec{B}\, d\mathcal{V} = \oint_{\partial\mathcal{V}} \vec{B}\cdot d\vec{S}$ by divergence theorem.

Then

$$\oint_{\partial\mathcal{V}} \vec{B}\cdot d\vec{S} = -\iint_{S_1} \vec{B}(t+\delta t)\cdot d\vec{S}_1 + \iint_{S_2} \vec{B}(t+\delta t)\cdot d\vec{S}_2$$

$$+ \iint_{S_3} \vec{B}(t+\delta t)\cdot d\vec{S}_3 = 0$$

$$\therefore \underbrace{\iint_{S_2} \vec{B}(t+\delta t)\cdot d\vec{S}_2}_{\Phi(t+\delta t)} = \iint_{\underbrace{S_1}_{S(t)}} \vec{B}(t+\delta t)\cdot d\vec{S}_1 - \iint_{S_3} \vec{B}(t+\delta t)\cdot d\vec{S}_3$$

$$\Rightarrow \Phi(t+\delta t) = \iint_{S(t)} \left(\vec{B}(t) + \delta t \frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{S}_1 - \iint_{S_3} \vec{B}(t+\delta t)\cdot d\vec{S}_3$$

$\Phi(t)$

$$= \Phi(t) + \delta t \iint_{S(t)} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}_1 - \iint_{S_3} \vec{B}(t+\delta t)\cdot d\vec{S}_3$$

but $d\vec{S}_3 = d\vec{l} \times \vec{u}\,\delta t$

and

$$\iint_{S_3} \vec{B}(t+\delta t) . d\vec{S}_3 = \iint_{S_3} \vec{B}(t+\delta t) . (d\vec{l} \times \vec{u}\,\delta t)$$

and so to linear order in δt

$$\iint_{S_3} \vec{B}(t+\delta t) . d\vec{S}_3 = \iint_{S_3} \vec{B}(t) . (d\vec{l} \times \vec{u}\,\delta t)$$

(properties of scalar triple product)

$$= \iint_{S_3} (\vec{u} \times \vec{B}) . d\vec{l}\ \delta t$$

(stokes theorem)

$$= \iint_{S_1} \vec{\nabla} \times (\vec{u} \times \vec{B}) . d\vec{S}_1\, \delta t$$

$\therefore$ combining our results

$$\Phi(t+\delta t) = \Phi(t) + \delta t \iint_{S_1} \frac{\partial \vec{B}}{\partial t} . d\vec{S}_1 - \delta t \iint_{S_1} \vec{\nabla} \times (\vec{u} \times \vec{B}) . d\vec{S}_1$$

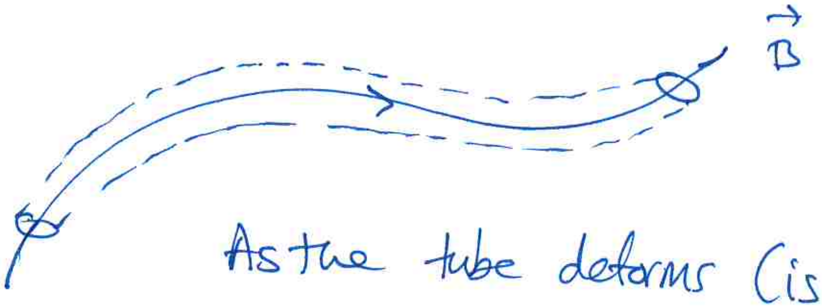
$$= \Phi(t) + \delta t \iint_{S_1} \underbrace{\left[\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times (\vec{u} \times \vec{B}) \right]}_{\text{the induction equation}} . d\vec{S}$$

$$= \Phi(t)$$

$$\Rightarrow \quad \frac{D\Phi}{Dt} = 0. \Rightarrow \Phi \text{ conserved.}$$

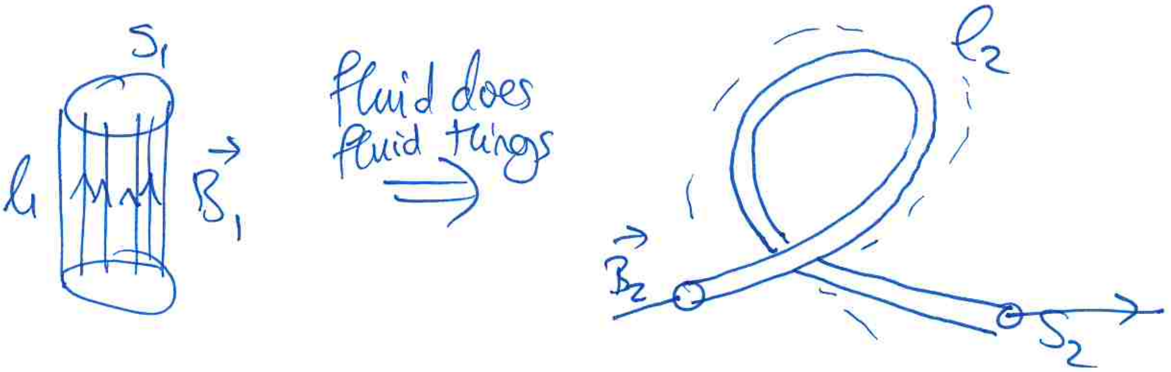
Alfvén's theorem has an interesting property: it implies that field lines are "frozen into" the flow.

To see this, take a field line and encase it in a "flux tube"



As the tube deforms (is advected with the flow) the field line must stay within it — as the flux through the two ends must stay constant, and the flux through the sides must remain zero.

This can then lead to field amplification by fluid motion



by conservation of flux

$$B_1 S_1 = B_2 S_2$$

By conservation of mass

$$\rho_1 S_1 l_1 = \rho_2 S_2 l_2$$

and therefore

$$\frac{B_1}{\rho_1 l_1} = \frac{B_2}{\rho_2 l_2} \Rightarrow \frac{B_2}{B_1} = \frac{\rho_2 l_2}{\rho_1 l_1}.$$

In an incompressible fluid $\rho_1 = \rho_2$, and the field is amplified by a factor l_2/l_1.

In a compressible fluid we could get even more amplification if $\rho_2 > \rho_1$.

Ultimate fate of the field

Recall

$$\frac{\partial \vec{B}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{B} = (\vec{B}\cdot\vec{\nabla})\vec{u} - \vec{B}(\vec{\nabla}\cdot\vec{u})$$

the question: "are there fluid flows that lead to sustained amplification of $\vec{B}$?" is the (famous) MHD dynamo problem.

Interestingly, in two dimensions we have a "no dynamo theorem" [due to Zeldovich et al. 1984].

We can write a two-dimensional B-field as

$$\vec{B} = \vec{\nabla}\times(A\hat{z}) \; ; \; A = A(x,y).$$

then

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B}) \Rightarrow \frac{\partial}{\partial t}(\vec{\nabla}\times A\hat{z}) = \vec{\nabla}\times(\vec{u}\times\vec{\nabla}\times(A\hat{z}))$$

"uncurl"

$$\frac{\partial A}{\partial t}\hat{z} = \vec{u}\times(\vec{\nabla}\times A\hat{z})$$

$$= \varepsilon_{ijk}u_j\varepsilon_{klm}\partial_l A_m$$

$$= [\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}]u_j\partial_l A_m$$

$$= \underbrace{u_j\partial_i A_j}_{\text{zero as } u^z=0} - \underbrace{u_j\partial_j A_i}_{(\vec{u}\cdot\vec{\nabla})A\hat{z}}$$

$$\Rightarrow \quad \frac{\partial A}{\partial t} + (\vec{u}\cdot\vec{\nabla})A = 0 \Rightarrow \text{no growth.}$$

(decays if non-ideal effects included).

In 3 dimensions we can however have field growth.

Simple example: consider a rotating flow

$\vec{u} = u_\phi(r)\,\hat{\phi}$ with an initial field

Initially: $\vec{B} = B_r\,\hat{r}$. [simple model of accretion flow].

Assume incompressible [$\vec{\nabla}\cdot\vec{u} = 0$], then

$$\frac{\partial \vec{B}}{\partial t} = -(\vec{u}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{u}$$

Move to ϕ component [assume $\frac{\partial}{\partial \phi} = 0$ by symmetry].

$$\frac{\partial B_\phi}{\partial t} = \underbrace{-\frac{u_\phi B_r}{r}}_{\text{advection}} + \underbrace{B_r \frac{\partial u_\phi}{\partial r}}_{\text{stretching}}$$

$$\Rightarrow \quad \frac{dB_\phi}{dt} = \left(\frac{\partial \vec{B}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{B}\right)_\phi = B_r \frac{\partial u_\phi}{\partial r}$$

Stretching of B_r produces B_ϕ.

Eventually $\frac{B^2}{2\mu_0} \sim \frac{1}{2}\rho u^2$ and lorentz force modifies flow, will not grow indefinitely in this simple manner.

4. Conservation of energy

Recap: we have

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

$$\rho\left[\frac{\partial}{\partial t} + (\vec{u}\cdot\vec{\nabla})\right]\vec{u} = -\vec{\nabla}P + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}$$

$$\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B})$$

which is 7 equations for 8 unknowns $(\rho, P, \vec{u}, \vec{B})$.

To close the system we need an equation for the pressure P. This is conservation of energy.

We will begin by writing down the total energy density of the plasma, and then consider the rate of change of some components of this energy individually. This is because (e.g.) kinetic energy conservation is intimately linked to momentum conservation.

Energy density

$$\mathcal{E} = \frac{1}{2}\rho v^2 + \rho e + \frac{B^2}{2\mu_0} + \frac{1}{2}\varepsilon_0 E^2$$

internal energy per unit mass

1. ⇒ $E \sim vB \Rightarrow \dfrac{\frac{1}{2}\varepsilon_0 E^2}{\frac{1}{2\mu_0}B^2} \sim \varepsilon_0 \mu_0 v^2 \sim O\left(\dfrac{v^2}{c^2}\right)$

ignore.

2. ⇒ assume perfect gas

$$PV = NkT \Rightarrow \frac{P}{\rho} = \frac{kT}{m}$$

Recall some classic thermodynamics

heat capacity @ constant volume per mole $\equiv C_V = \dfrac{R}{\gamma - 1}$

$\gamma \equiv C_P / C_V$

heat capacity @ constant P

Internal energy of one mole of gas

$$U = C_V T = \frac{RT}{\gamma - 1} = \frac{N_A kT}{\gamma - 1}$$

definition of e

$$\Rightarrow e \equiv \frac{U}{m N_A} = \frac{kT}{m(\gamma - 1)} = \frac{P}{\rho(\gamma - 1)}$$

$$\therefore \quad \mathcal{E} = \frac{1}{2}\rho v^2 + \frac{P}{\gamma-1} + \frac{B^2}{2\mu_0}.$$

and so, obviously

$$\underbrace{\frac{d\mathcal{E}}{dt}}_{\substack{\text{rate of}\\ \text{change of total}\\ \text{energy}}} = \underbrace{\frac{d}{dt}\left(\frac{1}{2}\rho v^2\right)}_{\substack{\text{rate of change}\\ \text{of kinetic}\\ \text{energy}}} + \underbrace{\frac{d}{dt}\left(\frac{P}{\gamma-1}\right)}_{\substack{\text{rate of change}\\ \text{of thermal}\\ \text{energy}}} + \underbrace{\frac{d}{dt}\left(\frac{B^2}{2\mu_0}\right)}_{\substack{\text{rate of}\\ \text{change of}\\ \text{magnetic}\\ \text{energy.}}}$$

Start with rate of change of kinetic energy.

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \frac{v^2}{2}\frac{\partial \rho}{\partial t} + \rho\vec{v}\cdot \underbrace{\frac{\partial \vec{v}}{\partial t}}_{\substack{\text{momentum}\\ \text{conservation}}}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \frac{v^2}{2}\frac{\partial \rho}{\partial t} + \rho\vec{v}\cdot\left[-(\vec{v}\cdot\vec{\nabla})\vec{v} - \frac{1}{\rho}\vec{\nabla}P + \frac{1}{\rho}\vec{f}\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \frac{v^2}{2}\frac{\partial \rho}{\partial t} - \rho\vec{v}\cdot\left[\vec{v}\cdot\vec{\nabla}\right]\vec{v} - \vec{v}\cdot\vec{\nabla}P + \vec{v}\cdot\vec{f}$$

let's massage those terms:

$$-\rho\vec{v}\cdot[\vec{v}\cdot\vec{\nabla}]\vec{v} = -\frac{1}{2}\rho[\vec{v}\cdot\vec{\nabla}]v^2$$

$$= -\frac{1}{2}\rho\vec{v}\cdot(\vec{\nabla}v^2)$$

$$= -\frac{1}{2}\rho\left[\vec{\nabla}\cdot(v^2\vec{v}) - v^2\vec{\nabla}\cdot\vec{v}\right]$$

$$= \frac{1}{2}\rho v^2(\vec{\nabla}\cdot\vec{v}) - \frac{1}{2}\rho\,\vec{\nabla}\cdot(v^2\vec{v})$$

and $$= \frac{1}{2}\rho v^2(\vec{\nabla}\cdot\vec{v}) - \frac{1}{2}\vec{\nabla}\cdot(\rho v^2\vec{v}) + \frac{1}{2}v^2(\vec{v}\cdot\vec{\nabla})\rho$$

$$-\vec{v}\cdot\vec{\nabla}P = -\vec{\nabla}\cdot(\vec{v}P) + P(\vec{\nabla}\cdot\vec{v})$$

and

$$\vec{v}\cdot\vec{f} = \vec{v}\cdot(\vec{J}\times\vec{B}) \overset{\text{scalar triple product}}{=} -\vec{J}\cdot(\vec{v}\times\vec{B})$$

$$= \vec{J}\cdot\vec{E}$$

(from $\vec{E} = -\vec{v}\times\vec{B}$)

$$\therefore\ \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \frac{v^2}{2}\frac{\partial\rho}{\partial t} + \frac{1}{2}\rho v^2(\vec{\nabla}\cdot\vec{v}) - \frac{1}{2}\vec{\nabla}\cdot(\rho v^2\vec{v})$$

$$+ \frac{1}{2}v^2(\vec{v}\cdot\vec{\nabla})\rho - \vec{\nabla}\cdot(\vec{v}P)$$

$$+ P(\vec{\nabla}\cdot\vec{v}) + \vec{J}\cdot\vec{E}$$

$$= -\vec{\nabla}\cdot\left(\frac{1}{2}\rho v^2\vec{v} + P\vec{v}\right) + P(\vec{\nabla}\cdot\vec{v}) + \vec{J}\cdot\vec{E}$$

$$+ \frac{1}{2}v^2\underbrace{\left[\frac{\partial\rho}{\partial t} + \rho(\vec{\nabla}\cdot\vec{v}) + (\vec{v}\cdot\vec{\nabla})\rho\right]}_{=0 \text{ by mass}}$$

and therefore

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\vec{\nabla}\cdot\left(\frac{1}{2}\rho v^2\vec{v} + P\vec{v}\right) + P(\vec{\nabla}\cdot\vec{v}) + \vec{J}\cdot\vec{E}$$

Let's integrate this over the volume

$$\frac{d}{dt}\iiint_{\mathcal{V}} \frac{1}{2}\rho v^2\, d\mathcal{V} = -\iiint_{\mathcal{V}} \vec{\nabla}\cdot\left(\frac{1}{2}\rho v^2\vec{v} + P\vec{v}\right) d\mathcal{V}$$

$$+ \iiint_{\mathcal{V}} P(\vec{\nabla}\cdot\vec{v})\, d\mathcal{V} + \iiint_{\mathcal{V}} \vec{J}\cdot\vec{E}\, d\mathcal{V}$$

$$\frac{d}{dt}\iiint_{\mathcal{V}} \frac{1}{2}\rho v^2\, d\mathcal{V} = \overbrace{-\oint_{\partial\mathcal{V}} \frac{1}{2}\rho v^2\,\vec{v}\cdot d\vec{S}}^{\text{Kinetic energy flux out of volume}} \overbrace{- \oint P\, d\vec{S}\cdot\vec{v}}^{\text{work done by pressure forces}}$$

$$+ \underbrace{\iiint_{\mathcal{V}} P(\vec{\nabla}\cdot\vec{v})\, d\mathcal{V}}_{\text{Compressional heating}} + \underbrace{\iiint_{\mathcal{V}} \vec{J}\cdot\vec{E}\, d\mathcal{V}}_{\text{energy exchange with electromagnetic fields.}}$$

Magnetic energy

We have

$$\frac{\partial \vec{B}}{\partial t} = - \vec{\nabla} \times \vec{E} \qquad (M3)$$

$$\therefore \quad \frac{\vec{B}}{\mu_0} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2\mu_0} B^2\right) = -\frac{1}{\mu_0} \vec{B} \cdot (\vec{\nabla} \times \vec{E})$$

but

$$\vec{B} \cdot (\vec{\nabla} \times \vec{E}) = B_i \, \varepsilon_{ijk} \, \partial_j E_k$$

$$= \varepsilon_{ijk} \, \partial_j (B_i E_k) - \varepsilon_{ijk} E_k \, \partial_j B_i$$

$$= \underset{\text{(one swap)}}{- \varepsilon_{jik} \, \partial_j (B_i E_k)} + \underset{\text{(three swaps)}}{\varepsilon_{kji} E_k \, \partial_j B_i}$$

$$= - \vec{\nabla} \cdot (\vec{B} \times \vec{E}) + \vec{E} \cdot (\vec{\nabla} \times \vec{B})$$

$$\therefore \quad \frac{\partial}{\partial t}\left(\frac{1}{2\mu_0} B^2\right) = \vec{\nabla} \cdot \underbrace{\left(\frac{\vec{B} \times \vec{E}}{\mu_0}\right)}_{-\vec{P}} - \vec{E} \cdot \underbrace{\left(\frac{1}{\mu_0} \vec{\nabla} \times \vec{B}\right)}_{\vec{J}}$$

$$\vec{P} \equiv \text{Poynting vector} \equiv \frac{\vec{E} \times \vec{B}}{\mu_0}$$

$$\therefore \quad \frac{\partial}{\partial t}\left(\frac{1}{2\mu_0} B^2\right) = - \vec{\nabla} \cdot \vec{P} - \vec{J} \cdot \vec{E}$$

Integrate over volume

$$\frac{d}{dt}\iiint_V \frac{B^2}{2\mu_0}\, dV = -\iiint_V \vec{\nabla}\cdot\vec{P}\, dV - \iiint_V \vec{J}\cdot\vec{E}\, dV$$

divergence theorem

$$\frac{d}{dt}\iiint_V \frac{B^2}{2\mu_0}\, dV = \underbrace{-\oint_{\delta V} \vec{P}\cdot d\vec{S}}_{\text{magnetic energy transported through the boundary by Poynting flux}} \underbrace{- \iiint_V \vec{J}\cdot\vec{E}\, dV}_{\text{Note opposite sign to kinetic energy eq}^n\text{! Energy exchange with the flow.}}$$

magnetic energy transported through the boundary by Poynting flux

Note opposite sign to kinetic energy eq^n! Energy exchange with the flow.

Thermal energy: we know kinetic energy and electromagnetic fields communicate through $\vec{J}\cdot\vec{E}$. How does thermal energy communicate?

Well, recall kinetic energy conservation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\vec{\nabla}\cdot\left(\frac{1}{2}\rho v^2 \vec{v} + P\vec{v}\right)$$
$$+ P(\vec{\nabla}\cdot\vec{v}) + \vec{J}\cdot\vec{E}$$

as all of magnetic energy communicated through $\vec{J}\cdot\vec{E}$ with kinetic energy, then thermal energy can only talk to ρv^2 (via pressure forces).

$\therefore$ Conserve kinetic + thermal energy

$$\frac{d}{dt}\iiint_V \left(\frac{1}{2}\rho v^2 + \rho e\right) dV = -\oint_{\delta V} \left(\frac{1}{2}\rho v^2 + \rho e\right)\vec{u}\cdot d\vec{S}$$
$$- \underbrace{\oint_{\delta V} P\, d\vec{S}\cdot\vec{u}}_{\text{work done by pressure forces.}}$$

Divergence theorem

$$\frac{d}{dt}\iiint_V \left(\frac{1}{2}\rho v^2 + \rho e\right) dV = -\iiint_V \vec{\nabla}\cdot\left(\left[\frac{1}{2}\rho v^2 + \rho e\right]\vec{u}\right) dV$$
$$- \iiint \vec{\nabla}\cdot(P\vec{u})\, dV$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho e\right) = -\vec{\nabla}\cdot\left(\left[\frac{1}{2}\rho v^2 + \rho e\right]\vec{u}\right) - \vec{\nabla}\cdot\left(P\vec{u}\right)$$

We can then subtract off the kinetic energy equation (in the absence of $\vec{J}\cdot\vec{E}$, as that communicates only with EM fields), which leaves

$$\frac{\partial}{\partial t}(\rho e) = -\vec{\nabla}\cdot\left[(\rho e)\vec{u}\right] - P(\vec{\nabla}\cdot\vec{u})$$

we see that, unsurprisingly, the kinetic and thermal energies communicate through compressional heating.

Then we sum and get the total energy equation:

$$\frac{\partial}{\partial t}\left[\rho e + \frac{1}{2}\rho v^2 + \frac{B^2}{2\mu_0}\right] = -\vec{\nabla}\cdot\left[\left(\rho e + \frac{1}{2}\rho v^2 + P\right)\vec{u} + \vec{P}\right]$$

As this is a pure divergence, integrating over a volume which extends to infinity $\Rightarrow$ total energy conserved.

Final manipulations of internal energy

$$\frac{\partial}{\partial t}(\rho e) = -\vec{\nabla}\cdot[(\rho e)\vec{u}] - P(\vec{\nabla}\cdot\vec{u})$$

Recall that for ideal gas $e = \dfrac{P}{\rho(\gamma-1)}$

$$\therefore \frac{1}{\gamma-1}\frac{\partial P}{\partial t} = -\vec{\nabla}\cdot\left(\frac{P\vec{u}}{\gamma-1}\right) - P(\vec{\nabla}\cdot\vec{u})$$

$$= -\frac{1}{\gamma-1}(\vec{u}\cdot\vec{\nabla})P - \frac{\gamma}{\gamma-1}P(\vec{\nabla}\cdot\vec{u})$$

$$\Rightarrow \frac{1}{\gamma-1}\left[\frac{\partial}{\partial t} + (\vec{u}\cdot\vec{\nabla})\right]P \equiv \frac{1}{\gamma-1}\frac{DP}{Dt} = -\frac{\gamma}{\gamma-1}P(\vec{\nabla}\cdot\vec{u}]$$

Recall mass conservation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = \left[\frac{\partial}{\partial t} + (\vec{u}\cdot\vec{\nabla})\right]\rho = -\rho(\vec{\nabla}\cdot\vec{u})$$

$$\Rightarrow (\vec{\nabla}\cdot\vec{u}) = -\frac{1}{\rho}\frac{D\rho}{Dt}$$

combining

$$\frac{1}{P}\frac{DP}{Dt} = \frac{\gamma}{\rho}\frac{D\rho}{Dt}$$

$$\Rightarrow \frac{D}{Dt}\ln P - \gamma\frac{D}{Dt}\ln\rho = 0$$

$$\Rightarrow \frac{D}{Dt}\ln\left(P/\rho^{\gamma}\right) = 0$$

which means (ideal) MHD conserves the quantity $P\rho^{-\gamma}$.

This is the entropy of the fluid.

Fundamental thermodynamic relationship

$$de = Tds - pdV \quad [\text{per unit mass so } V = 1/\rho]$$

$$d\left(\frac{P}{\rho(\gamma-1)}\right) = Tds - p\, d(1/\rho)$$

$$\frac{1}{\rho(\gamma-1)}dP - \frac{P}{\rho^2(\gamma-1)}d\rho = TdS + \frac{P}{\rho^2}d\rho$$

$$TdS = \frac{1}{\gamma-1}\left[\frac{dP}{\rho} - \frac{\gamma P d\rho}{\rho^2}\right]$$

$$= \frac{P}{\rho(\gamma-1)}\left[\frac{dP}{P} - \gamma\frac{d\rho}{\rho}\right]$$

ideal gas $\quad P = \frac{k_B}{m}\rho T$

$$TdS = \frac{k_B T}{m(\gamma-1)}\left[\frac{dP}{P} - \gamma\frac{d\rho}{\rho}\right]$$

$$\Rightarrow \quad S = \frac{k_B}{m(\gamma-1)}\ln\left[P\rho^{-\gamma}\right].$$

The enthalpy equation

Define $h \equiv e + P/\rho + \frac{1}{2}v^2$

Conservation of energy then

$$\frac{\partial}{\partial t}\left(\rho e + \frac{1}{2}\rho v^2 + \frac{B^2}{2\mu_0}\right) = -\vec{\nabla}\cdot\left(\rho h \vec{v} + \vec{\mathcal{P}}\right)$$

Add $\frac{\partial p}{\partial t}$ to both sides

$$\frac{\partial}{\partial t}\left(\rho h + \frac{B^2}{2\mu_0}\right) = -\vec{\nabla}\cdot(\rho h \vec{v}) - \vec{\nabla}\cdot\vec{\mathcal{P}} + \frac{\partial p}{\partial t}$$

zero by mass conservation

expand

$$\rho\frac{\partial h}{\partial t} + h\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial t}\left(B^2/2\mu_0\right) = -h\,\vec{\nabla}\cdot(\rho\vec{v}) + (\rho\vec{v}\cdot\vec{\nabla})h - \vec{\nabla}\cdot\vec{\mathcal{P}} + \frac{\partial p}{\partial t}$$

$$\boxed{\rho\left[\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right]h + \frac{\partial}{\partial t}\left(B^2/2\mu_0\right) = -\vec{\nabla}\cdot\vec{\mathcal{P}} + \frac{\partial p}{\partial t}}$$

In steady flow $[\partial/\partial t = 0]$ we find

$$\frac{dh}{dt} = -\frac{1}{\rho}\vec{\nabla}\cdot\vec{\mathcal{P}}$$

this is M.H.D version of Bernoulli's theorem:

[i.e. $h = e + P/\rho + \frac{1}{2}v^2$ constant along streamlines].

The hyperbolic structure of ideal MHD

Writing

$$\frac{\partial \rho}{\partial t} + (\vec{u}\cdot\vec{\nabla})\rho + \rho(\vec{\nabla}\cdot\vec{u}) = 0$$

$$\frac{\partial P}{\partial t} + (\vec{u}\cdot\vec{\nabla})P + \gamma p(\vec{\nabla}\cdot\vec{u}) = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} + \frac{1}{\rho}\vec{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) - \frac{1}{\rho\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B} = \vec{0}$$

$$\frac{\partial \vec{B}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{B} - (\vec{B}\cdot\vec{\nabla})\vec{u} + \vec{B}(\vec{\nabla}\cdot\vec{u}) = \vec{0}$$

We can write

$$\frac{\partial \vec{\psi}}{\partial t} + \underline{\underline{A_i}}\,\frac{\partial \vec{\psi}}{\partial x_i} = \vec{0}$$

where $\vec{\psi} = [\rho, P, u_x, u_y, u_z, B_x, B_y, B_z]$

and (e.g.)

$$\underline{\underline{A_x}} = \begin{bmatrix} u_x & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & u_x & \gamma p & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\rho & u_x & 0 & 0 & 0 & \frac{B_y}{\mu_0\rho} & \frac{B_z}{\mu_0\rho} \\ 0 & 0 & 0 & u_x & 0 & 0 & -\frac{B_z}{\mu_0\rho} & 0 \\ 0 & 0 & 0 & 0 & u_x & 0 & 0 & -\frac{B_z}{\mu_0\rho} \\ 0 & 0 & 0 & 0 & 0 & u_x & 0 & 0 \\ 0 & 0 & B_y & -B_x & 0 & 0 & u_x & 0 \\ 0 & 0 & B_z & 0 & -B_x & 0 & 0 & u_x \end{bmatrix}$$

5. M.H.D Waves

Let us begin with a recap of sound waves.

Ideal hydrodynamics is

$$\frac{\partial \rho}{\partial t} + (\vec{u}\cdot\vec{\nabla})\rho + \rho(\vec{\nabla}\cdot\vec{u}) = 0$$

$$\frac{\partial p}{\partial t} + (\vec{u}\cdot\vec{\nabla})p + \gamma p(\vec{\nabla}\cdot\vec{u}) = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} + \frac{1}{\rho}\vec{\nabla} P = \vec{0}.$$

Assume an equillibrium of $\vec{u} = \vec{0}$, $P = P_0 = cst$
$\rho = \rho_0 = cst$.

Then give the system a kick

$$\vec{u} \to \delta\vec{u}, \quad P \to P_0 + \delta P, \quad \rho \to \rho_0 + \delta\rho$$

where $\delta P/P_0 \ll 1$, $\delta\rho/\rho_0 \ll 1$.

Linearized equations

$$\frac{\partial}{\partial t}\delta\rho + \rho_0 \vec{\nabla}\cdot\delta\vec{u} = 0 \quad \text{— (1)}$$

$$\frac{\partial}{\partial t}\delta p + \gamma P_0 \vec{\nabla}\cdot\delta\vec{u} = 0 \quad \text{— (2)}$$

$$\frac{\partial}{\partial t}\delta\vec{u} + \frac{1}{\rho_0}\vec{\nabla}\delta P = 0 \quad \text{— (3)}$$

take divergence of (3)

$$\frac{\partial}{\partial t} \vec{\nabla} \cdot \delta\vec{u} + \frac{1}{\rho_0} \nabla^2 \delta P = 0$$

use (2) $\Rightarrow$ $\vec{\nabla} \cdot \delta\vec{u} = -\frac{1}{\gamma P_0} \frac{\partial}{\partial t} \delta P$

to get

$$\boxed{\frac{\partial^2 \delta P}{\partial t^2} - \frac{\gamma P_0}{\rho_0} \nabla^2 \delta P = 0}$$ wave equation.

Density equation identical, as can be seen from substituting from (1) not (2)

$$-\frac{1}{\rho_0} \frac{\partial^2}{\partial t^2} \delta\rho + \frac{1}{\rho_0} \nabla^2 \delta P = 0$$

then using entropy conservation : $P\rho^{-\gamma} = cst$

$$\delta P . \rho^{-\gamma} - \gamma \rho^{\gamma-1} P \delta\rho = 0$$

$$\Rightarrow \frac{\delta P}{P_0} - \gamma\, \delta\rho/\rho_0 = 0$$

to get

$$\boxed{\frac{\partial^2 \delta\rho}{\partial t^2} - \frac{\gamma P_0}{\rho_0} \nabla^2 \delta\rho = 0}$$

Speed of sound $c_s^2 \equiv \frac{\gamma P_0}{\rho_0}.$

MHD waves are not so simple / boring.

Our equations are

$$\frac{\partial \rho}{\partial t} + (\vec{u} \cdot \vec{\nabla})\rho + \rho(\vec{\nabla} \cdot \vec{u}) = 0$$

$$\frac{\partial p}{\partial t} + (\vec{u} \cdot \vec{\nabla}) P + \gamma p(\vec{\nabla} \cdot \vec{u}) = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} + \frac{1}{\rho}\vec{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) - \frac{1}{\rho\mu_0}(\vec{B} \cdot \vec{\nabla})\vec{B} = \vec{0}$$

$$\frac{\partial \vec{B}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{B} - (\vec{B} \cdot \vec{\nabla})\vec{u} + \vec{B}(\vec{\nabla} \cdot \vec{u}) = \vec{0}.$$

We will linearize about

$\vec{B} = B_0 \hat{z}$ (uniform)

$\rho = \rho_0 = cst$, $P = P_0 = cst$, $\vec{u} = \vec{0}$.

Again, kick the system

$\rho = \rho_0 + \delta\rho$, $P = P_0 + \delta p$, $\vec{u} = \delta\vec{u}$

$\vec{B} = B_0 \hat{z} + \delta\vec{B}$.

This time it will be prudent to write $\delta\vec{u} \equiv \frac{\partial \vec{\xi}}{\partial t}$

where $\vec{\xi}$ is a <u>displacement vector</u>.

This is used in MHD because field lines are <u>frozen into</u> the flow and dislike being <u>bent</u> or <u>compressed</u>.

In other words, magnetic fields have "memory" of where they were perturbed from.

Linearize :

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \vec{\nabla}\cdot \frac{\partial \vec{\xi}}{\partial t} = 0 \Rightarrow \boxed{\frac{\delta\rho}{\rho_0} = -\vec{\nabla}\cdot\vec{\xi}}$$

Entropy : $P\rho^{-\gamma} = cst \Rightarrow \dfrac{\delta P}{P_0} - \gamma \dfrac{\delta\rho}{\rho_0} = 0$

$$\Rightarrow \boxed{\frac{\delta P}{P_0} = -\gamma \vec{\nabla}\cdot\vec{\xi}}$$

Induction

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\vec{B} = (\vec{B}\cdot\vec{\nabla})\vec{u} - \vec{B}(\vec{\nabla}\cdot\vec{u})$$

becomes

$$\frac{\partial}{\partial t}\delta\vec{B} = \vec{B}_0\cdot\vec{\nabla}\frac{\partial\vec{\xi}}{\partial t} - \vec{B}_0\left(\vec{\nabla}\cdot\frac{\partial\vec{\xi}}{\partial t}\right)$$

$$\Rightarrow \quad \delta\vec{B} = \underbrace{(\vec{B}_0\cdot\vec{\nabla})\vec{\xi}}_{B_0 \nabla_{\parallel}\vec{\xi}} - \underbrace{\vec{B}_0(\vec{\nabla}\cdot\vec{\xi})}_{-\hat{z} B_0 \vec{\nabla}\cdot\vec{\xi}}$$

$$B_0\left(\nabla_{\parallel}\underbrace{(\xi_{\parallel}\hat{z} + \vec{\xi}_{\perp})}\right) \qquad -\hat{z}B_0\nabla_{\parallel}\xi_{\parallel} - B_0\hat{z}\vec{\nabla}_{\perp}\cdot\vec{\xi}_{\perp}$$

$$B_0\nabla_{\parallel}\vec{\xi}_{\perp} - B_0\hat{z}\vec{\nabla}_{\perp}\cdot\vec{\xi}_{\perp}$$

$$\Rightarrow \quad \frac{\delta\vec{B}}{B_0} = \nabla_{\parallel}\vec{\xi}_{\perp} - \hat{z}\vec{\nabla}_{\perp}\cdot\vec{\xi}_{\perp}$$

$$\Rightarrow \boxed{\frac{\delta B_{\parallel}}{B_0} = -\vec{\nabla}_{\perp}\cdot\vec{\xi}_{\perp}} \quad \boxed{\frac{\delta\vec{B}_{\perp}}{B_0} = \nabla_{\parallel}\vec{\xi}_{\perp}}$$

Only perpendicular perturbations modify the magnetic field. This makes sense as the fluid is carried with the flow.

To elaborate, write

$$\frac{\delta \vec{B}}{B_0} = \frac{\delta(B\hat{b})}{B_0} = \delta\hat{b} + \hat{z}\,\frac{\delta B}{B_0}$$

but $\hat{b}\cdot\hat{b} = 1 \Rightarrow \delta(\hat{b}\cdot\hat{b}) = 2\hat{b}\cdot\delta\hat{b} = 0$

$\Rightarrow \delta\hat{b} \perp \hat{b}$

$\Rightarrow \delta\vec{B}_\perp = \delta\hat{b}$

$$\Rightarrow \boxed{\delta\hat{b} = \nabla_{\parallel}\vec{\xi}_\perp}$$

$$\boxed{\frac{\delta B}{B_0} = -\vec{\nabla}_\perp\cdot\vec{\xi}_\perp}$$

Let's now linearize the momentum equation

$$\underbrace{\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\right)\vec{u}}_{(1)} = \underbrace{-\vec{\nabla} P}_{(2)} \underbrace{- \vec{\nabla}\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B}}_{(3)}$$

$$(1) \Rightarrow \rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} + \text{second order.}$$

$$(2) \Rightarrow -\vec{\nabla}\delta P = P_0 \gamma \vec{\nabla}(\vec{\nabla}\cdot\vec{\xi}) \Big| \text{ [from mass + entropy].}$$

$$(3) \Rightarrow \underbrace{-\vec{\nabla}\left(\frac{B^2}{2\mu_0}\right)}_{(\alpha)} + \underbrace{\frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B}}_{(\beta)}$$

$$(\alpha) \Rightarrow B^2 = (\vec{B}_0 + \delta\vec{B})\cdot(\vec{B}_0 + \delta\vec{B}) = B_0^2 + 2\vec{B}_0\cdot\delta\vec{B} + O(\delta B^2)$$

$$\therefore -\vec{\nabla}\left(\frac{B^2}{2\mu_0}\right) \Rightarrow -\frac{B_0^2}{\mu_0}\vec{\nabla}\left(\frac{\delta B_\parallel}{B_0}\right)$$

$$(\beta) \Rightarrow \frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B} = \frac{B_0^2}{\mu_0}\nabla_\parallel\left(\delta\hat{b} + \frac{\delta B_\parallel}{B_0}\right)$$

$$\Rightarrow (\alpha)+(\beta) \Rightarrow \underbrace{-\frac{B_0^2}{\mu_0}\vec{\nabla}\left(\frac{\delta B_\parallel}{B_0}\right) + \frac{B_0^2}{\mu_0}\nabla_\parallel\left(\frac{\delta B_\parallel}{B_0}\right)}_{-\frac{B_0^2}{\mu_0}\vec{\nabla}_\perp\left(\frac{\delta B_\parallel}{B_0}\right)} + \frac{B_0^2}{\mu_0}\nabla_\parallel\delta\hat{b}$$

$$(3) \Rightarrow \; -\frac{B_0^2}{\mu_0}\vec{\nabla}_\perp\left(\frac{\delta B_\parallel}{B_0}\right) + \frac{B_0^2}{\mu_0}\nabla_\parallel \delta\hat{b}$$

using induction terms

$$\Rightarrow \; +\frac{B_0^2}{\mu_0}\vec{\nabla}_\perp(\vec{\nabla}_\perp\cdot\vec{\xi}_\perp) + \frac{B_0^2}{\mu_0}\nabla_\parallel^2\vec{\xi}_\perp$$

Assembling, we get

$$\rho_0\frac{\partial^2\vec{\xi}}{\partial t^2} = P_0\gamma\vec{\nabla}(\vec{\nabla}\cdot\vec{\xi}) + \frac{B^2}{\mu_0}\left[\vec{\nabla}_\perp(\vec{\nabla}_\perp\cdot\vec{\xi}_\perp) + \nabla_\parallel^2\vec{\xi}_\perp\right]$$

or

$$\boxed{\frac{\partial^2\vec{\xi}}{\partial t^2} = c_s^2\vec{\nabla}(\vec{\nabla}\cdot\vec{\xi}) + v_A^2\left[\vec{\nabla}_\perp(\vec{\nabla}_\perp\cdot\vec{\xi}_\perp) + \nabla_\parallel^2\vec{\xi}_\perp\right]}$$

two important speeds have emerged

$$\boxed{c_s^2 \equiv \frac{\gamma P_0}{\rho_0}} \qquad \boxed{v_A^2 \equiv \frac{B_0^2}{\mu_0\rho_0}}$$

speed of sound — Alfvén speed.

Let's seek wave-like solutions of this equation.

$$\vec{\xi} \propto \exp\left(-i\omega t + i\vec{k}\cdot\vec{r}\right)$$

giving

$$\boxed{\omega^2 \vec{\xi} = c_s^2 \vec{k}(\vec{k}\cdot\vec{\xi}) + v_A^2\left(\vec{k}_\perp(\vec{k}_\perp\cdot\vec{\xi}_\perp) + k_\parallel^2 \vec{\xi}_\perp\right)}$$

We are always allowed to pick $\vec{k} = (k_\perp, 0, k_\parallel)$

i.e. x is the direction $\perp$ and z is $\parallel$.

Then our dispersion equation becomes

$$\omega^2 \xi_x = c_s^2 k_\perp (k_\perp \xi_x + k_\parallel \xi_z) + v_A^2 \overbrace{(k_\perp^2 + k_\parallel^2)}^{k^2} \xi_x$$

$$\omega^2 \xi_y = v_A^2 k_\parallel^2 \xi_y$$

$$\omega^2 \xi_z = c_s^2 k_\parallel (k_\perp \xi_x + k_\parallel \xi_z)$$

and the other fields satisfy

$$\frac{\delta\rho}{\rho_0} = -i\vec{k}\cdot\vec{\xi} = -i(k_\perp \xi_x + k_\parallel \xi_z)$$

$$\frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0}; \quad \delta\hat{b} = i k_\parallel \vec{\xi}_\perp = i k_\parallel \begin{pmatrix} \xi_x \\ \xi_y \\ 0 \end{pmatrix}$$

$$\frac{\delta B}{B_0} = -i k_\perp \xi_x$$

Alfvén Waves

We note straight away that y-motion decouples from the system. Therefore $\vec{\xi} = (0, \xi_y, 0)$ is an eigenvector with eigenvalues given by

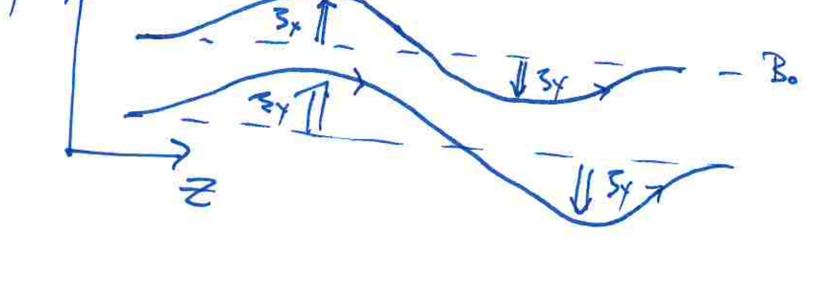
$$\omega^2 = v_A^2 k_{\parallel}^2 \Rightarrow \boxed{\omega = \pm v_A k_{\parallel}}.$$

These are Alfvén waves, they propogate parallel (and anti-parallel) to $\vec{B}_0$ [have $k_{\parallel}$].
The other fields satisfy

$$\vec{\xi} = \xi_y \hat{y}, \quad \delta\rho = 0, \quad \delta p = 0, \quad \delta B = 0$$

$$e^{i k_{\parallel}(z \pm v_A t)}, \quad \delta\hat{b} = i k_{\parallel} \xi_y \hat{y}.$$

In other words, this wave is incompressible ($\delta\rho = 0$) and involves magnetic fields acting as elastic strings, springing back against perturbing motions due to the restoring curvature force.



Note that these waves can have $k_{\perp} \neq 0$ and still be a solution.

Parallel propogation

Consider $k_\perp = 0$ [or $\cos\theta = 1$]

Simplest to return to original dispersion equations

$$\omega^2 \xi_x = V_A^2 k_\parallel^2 \xi_x$$

and

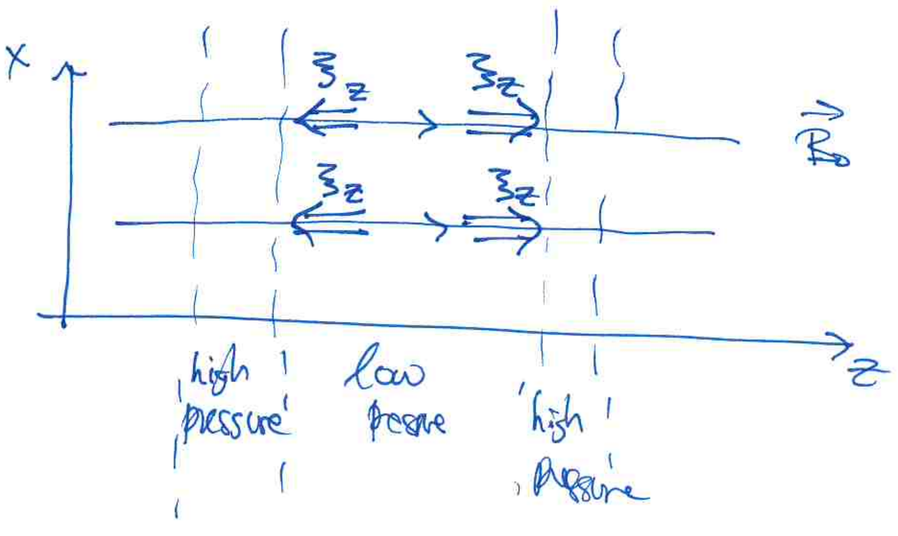
$$\omega^2 \xi_z = c_s^2 k_\parallel^2 \xi_z$$

completely seperate again, and we have one Alfvén wave [along $\hat{x}$] and one parallel propogating sound wave.

Alfvén: $\vec{\xi} = \xi_x(t)\hat{x}$, $\delta\rho = 0$, $\delta p = 0$, $\delta B = 0$, $\delta\hat{b} = ik_\parallel \xi_x \hat{x}$

Sound: $\vec{\xi} = \xi_z(t)\hat{z}$, $\frac{\delta\rho}{\rho_0} = -ik_\parallel \xi_z$, $\frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0}$, $\delta B = 0$, $\delta\hat{b} = \vec{0}$

Sound wave along $\vec{B_0}$



Perpendicular propagation

Now consider $k_{\parallel} = 0$ $[\cos\theta = 0]$.

Again returning to original equations we find

$$\omega^2 \xi_x = c_s^2 k_\perp^2 \xi_x + v_A^2 k_\perp^2 \xi_x$$

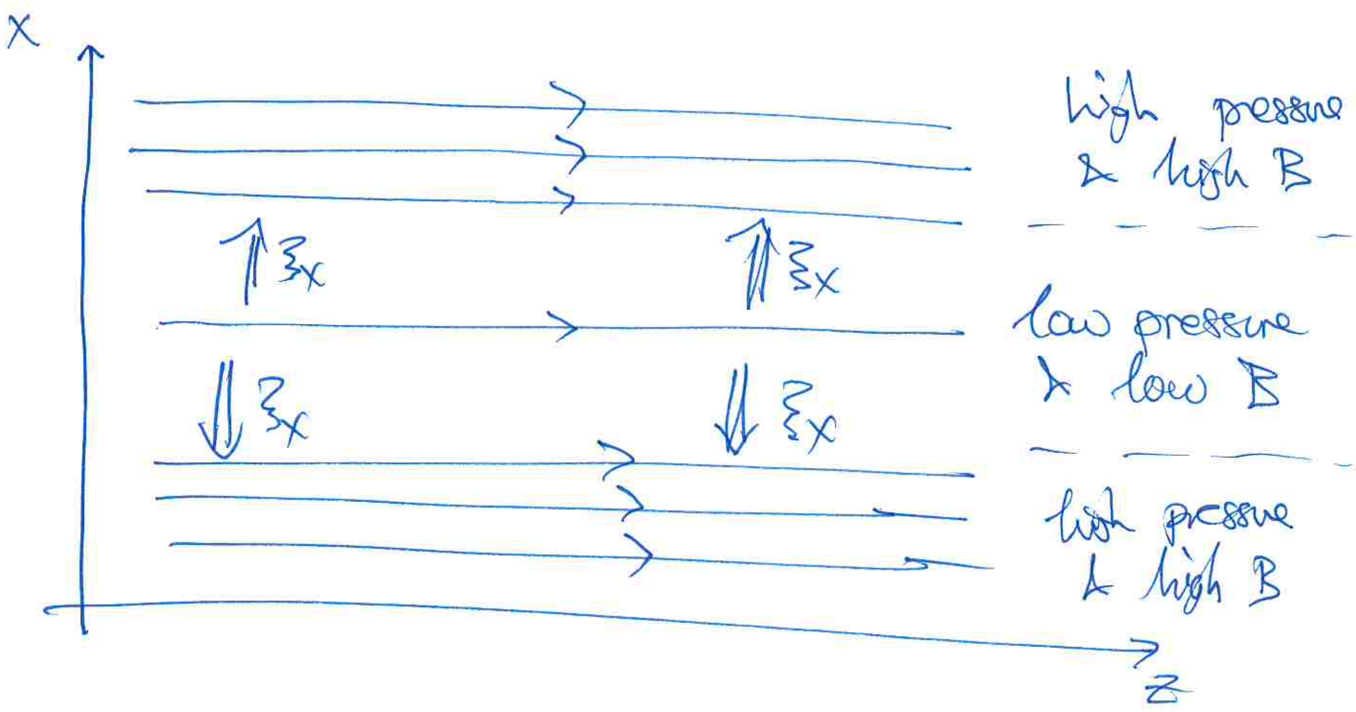
$$\Rightarrow \quad \omega^2 = k_\perp^2 \left(c_s^2 + v_A^2 \right)$$

What is this perturbation response?

Well, $\vec{\xi} = \xi_x(t)\,\hat{x}$, $\dfrac{\delta\rho}{\rho_0} = -i k_\perp \xi_x(t)$

$$\frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0}, \quad \frac{\delta B}{B_0} = -i k_\perp \xi_x(t)$$

$$\delta\hat{b} = \vec{0}.$$



This perpendicular response is called a sound wave [no field bending].

But both thermal & magnetic pressures Play a restoring role.

Note that the magnetic & thermal pressure responses are in phase.
There is no bending of fields.

Magneto sonic waves

The x-z plane contains a closed form 2D system

$$\omega^2 \xi_x = c_s^2 k_\perp (k_\perp \xi_x + k_\parallel \xi_z) + V_A^2 k^2 \xi_x$$

$$\omega^2 \xi_z = c_s^2 k_\parallel (k_\perp \xi_x + k_\parallel \xi_z)$$

or
$$\begin{pmatrix} \omega^2 - c_s^2 k_\perp^2 - V_A^2 k^2 & -c_s^2 k_\perp k_\parallel \\ -c_s^2 k_\parallel k_\perp & \omega^2 - c_s^2 k_\parallel^2 \end{pmatrix} \begin{pmatrix} \xi_x \\ \xi_z \end{pmatrix} = \vec{0}.$$

this equation is of the form

$$\underline{\underline{M}} \vec{v} = \vec{0}.$$

If $\underline{\underline{M}}$ had an inverse, then $\underline{\underline{M}}^{-1} \underline{\underline{M}} \vec{v} = \underline{\underline{M}}^{-1} \vec{0}$

$$\Rightarrow \vec{v} = \underline{\underline{M}}^{-1} \vec{0}$$

but $\underline{\underline{M}}^{-1} \vec{0} = \vec{0} \ \forall \underline{\underline{M}}^{-1}$.

$\therefore$ $\underline{\underline{M}}$ has no inverse.

$\det(\underline{\underline{M}}) = 0 \Rightarrow$ dispersion relationship.

$$(\omega^2 - c_s^2 k_\perp^2 - V_A^2 k^2)(\omega^2 - c_s^2 k_\parallel^2) - c_s^4 k_\parallel^2 k_\perp^2 = 0$$

expanding (and canceling)

$$\omega^4 - \omega^2 (c_s^2 k^2 + V_A^2 k^2) + V_A^2 c_s^2 k^2 k_\parallel^2 = 0$$

with solutions

$$\omega^2 = \frac{1}{2} k^2 \left[c_s^2 + v_A^2 \pm \sqrt{(c_s^2+v_A^2)^2 - 4 c_s^2 v_A^2 \cos^2\theta} \right]$$

where $\cos^2\theta \equiv \dfrac{k_{\parallel}^2}{k^2}$

There are two "+" solutions called "fast magnetosonic" waves, and two "−" solutions called "slow magnetosonic" waves.

Since both sound and Alfvén speeds are involved, clearly their ratio will be the key parameter for determining the physical regime we are in. Convention dictates this dimensionless parameter is given by

$$\beta \equiv \frac{p_0}{B_0^2/2\mu_0} = \frac{2}{\gamma} \frac{c_s^2}{v_A^2}$$

and is known as the "plasma beta" parameter.

"Typical astrophysics" $\beta \sim 100$

"Typical fusion plasma" $\beta \sim 1/100$

Anisotropic perturbations

Consider the limit $k_{\parallel} \ll k_{\perp}$.

This will be much more interesting then $k_{\parallel}=0$, which threw too much away.

The limit $k_{\parallel} \ll k_{\perp}$ will be especially relevant for strong magnetic fields, as those excitations which are realistic tend to propogate along the field if they can (bending a field line gives a curvature responce).

Factorising our magneto-sonic dispersion relationship

$$\omega^2 = \frac{1}{2} k^2 (c_s^2 + v_A^2)\left[1 \pm \sqrt{1 - \frac{4 c_s^2 v_A^2}{(c_s^2+v_A^2)^2} \frac{k_{\parallel}^2}{k_{\parallel}^2 + k_{\perp}^2}}\right]$$

taylor expanding

$$\omega^2 \approx \frac{1}{2} k^2 (c_s^2 + v_A^2)\left[1 \pm 1 \mp \frac{2 c_s^2 v_A^2}{(c_s^2+v_A^2)^2} \frac{k_{\parallel}^2}{k^2} + \dots\right]$$

where all upper/lower signs should be taken consistently.

upper sign:

$$\omega^2 = k^2 (c_s^2 + v_A^2) \quad \text{the "fast wave"}.$$

Just the boosted sound wave from before, $k_{\parallel}/k_{\perp}$ corrections not interesting.

lower sign:

$$\omega^2 = k_\parallel^2 \frac{c_s^2 V_A^2}{(c_s^2 + V_A^2)}$$

This is the "slow wave", and is more interesting.

From our 'z' component of the linearized equations we have

$$\omega^2 \xi_z = c_s^2 k_\parallel k_\perp \xi_x + c_s^2 k_\parallel^2 \xi_z$$

$$\Rightarrow \quad \xi_z \underbrace{(\omega^2 - c_s^2 k_\parallel^2)}_{\frac{k_\parallel^2 c_s^2 V_A^2}{c_s^2 + V_A^2} - c_s^4 k_\parallel^2 \;\simeq\; -\frac{c_s^4 k_\parallel^2}{c_s^2 + V_A^2}} = c_s^2 k_\parallel k_\perp \xi_x$$

$$\Rightarrow \quad \frac{\xi_x}{\xi_z} = -\frac{k_\parallel}{k_\perp} \frac{c_s^2}{c_s^2 + V_A^2} \ll 1.$$

Displacements are <u>mostly parallel</u>.

perturbations of the other fields

$$\frac{\delta\rho}{\rho_0} = -\vec{\nabla}\cdot\vec{\xi} = -i\,\vec{k}\cdot\vec{\xi} = -i\left(k_\perp \xi_x + k_\parallel \xi_z\right)$$

$$= -i k_\parallel \xi_z \left[\frac{k_\perp}{k_\parallel}\frac{\xi_x}{\xi_z} + 1\right]$$

$$= -i k_\parallel \xi_z \frac{v_A^2}{c_s^2 + v_A^2}$$

trivially $\quad \dfrac{\delta p}{p_0} = \gamma \dfrac{\delta\rho}{\rho_0}$

$$\delta\hat{b} = \nabla_\parallel \vec{\xi}_\perp = i k_\parallel \xi_x \hat{x} = -i \frac{k_\parallel}{k_\perp}\frac{c_s^2}{c_s^2 + v_A^2} k_\parallel \xi_z \hat{x}$$

$$\frac{\delta B}{B_0} = -\vec{\nabla}_\perp\cdot\vec{\xi}_\perp = -i k_\perp \xi_x = i\frac{c_s^2}{c_s^2 + v_A^2} k_\parallel \xi_z$$

Therefore to lowest order in $k_\parallel / k_\perp$ we get no field direction bending, but have pressure perturbations in anti-phase to magnetic field strength perturbations.

To be precise, these waves are in pressure balance

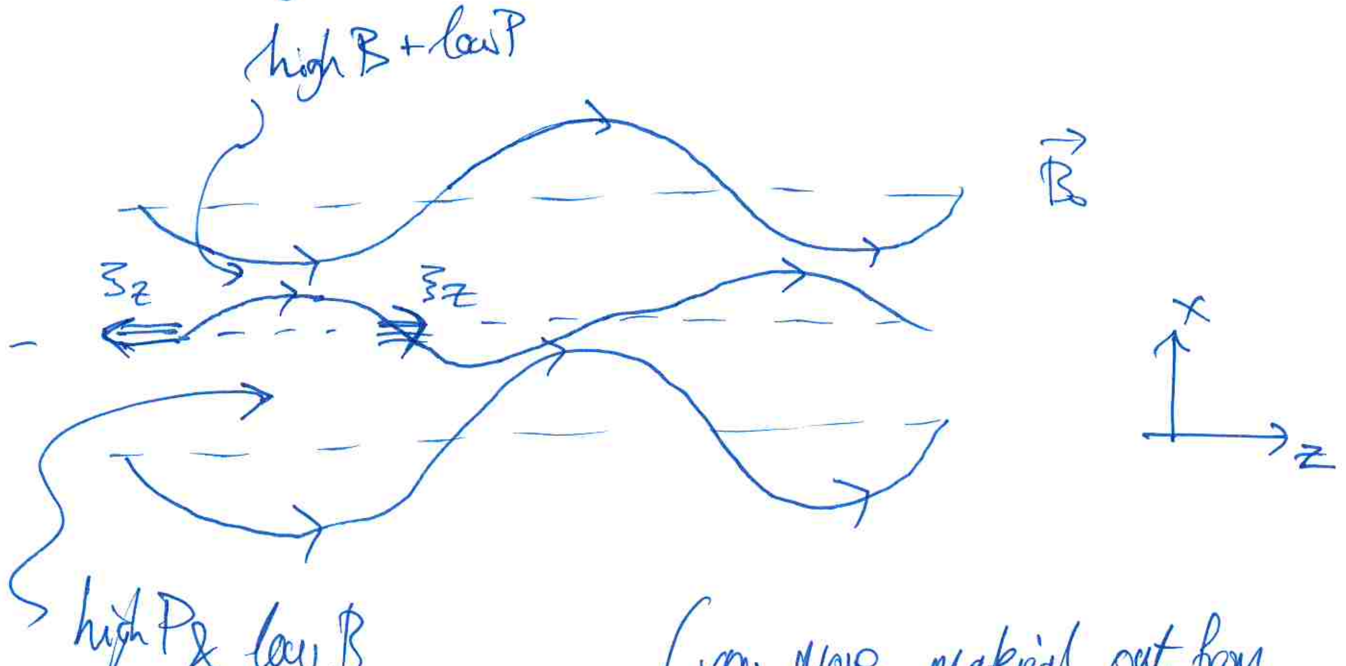
$$\delta\left(p+\frac{B^2}{2\mu_0}\right) = p_0\frac{\delta p}{p_0} + \frac{B_0^2}{\mu_0}\frac{\delta B}{B_0}$$

$$= \frac{p_0\gamma}{\rho_0}\delta\rho + \frac{B_0^2}{\mu_0}\frac{\delta B}{B_0}$$

$$= p_0\gamma\left(-ik_{\parallel}\xi_z\frac{V_A^2}{c_s^2+V_A^2}\right) + \frac{B_0^2}{\mu_0}ik_{\parallel}\xi_z\frac{c_s^2}{c_s^2+V_A^2}$$

$$= i\rho_0 k_{\parallel}\xi_z\left[-\frac{c_s^2V_A^2}{c_s^2+V_A^2} + \frac{V_A^2c_s^2}{c_s^2+V_A^2}\right] = 0.$$

$\therefore$ where the wave causes pressure to go up, the B field goes down.



(you move material out from where the B field bunches up).

Astrophysically relevant high-β limit

Finally, let us consider $\beta \gg 1$ $[c_s^2 \gg v_A^2]$.

Very relevant for astrophysics, because fields generated in [e.g.] interstellar medium topout at energetic scale that created them

$$\rho u^2 \sim \frac{B^2}{\mu_0} \Rightarrow v_A \sim u$$

and interstellar medium is sub-sonic

$$u \ll c_s \Rightarrow v_A \ll c_s.$$

Our taylor expansion of

$$\omega^2 = \frac{1}{2} k^2 (c_s^2 + v_A^2) \left[1 \pm \sqrt{1 - \frac{4 c_s^2 v_A^2}{(c_s^2+v_A^2)^2} \frac{k_\parallel^2}{k^2}}\right]$$

still valid in this limit, but now small parameter is v_A/c_s, not $k_\parallel/k_\perp$.

Fast wave becomes (upper sign in expension)

$$\omega^2 \simeq k^2 c_s^2$$

and slow wave becomes (lower sign in expension)

$$\omega^2 \approx k_\parallel^2 v_A^2.$$

This looks like an Alfvén wave, but is not. Called a pseudo-Alfvén wave as eigenvector very different.

Let's look at $\vec{\nabla}\cdot\vec{\xi}$ for this eigenvalue

$$\vec{\nabla}\cdot\vec{\xi} = k_\perp \xi_x + k_\parallel \xi_z = k_\parallel \xi_z \left[1 + \frac{k_\perp}{k_\parallel}\frac{\xi_x}{\xi_z}\right]$$

$$= -i k_\parallel \xi_z \frac{V_A{}^2}{c_s{}^2 + V_A{}^2} \rightarrow 0 .$$

To lowest order in $1/\beta$ the perturbations are incompressible.

In contrast to the anisotropic limit the parallel & perpendicular perturbations are comparable

$$\frac{\xi_x}{\xi_z} = -\frac{k_\parallel}{k_\perp}\frac{c_s{}^2}{c_s{}^2 + V_A{}^2} \simeq -\frac{k_\parallel}{k_\perp} \sim O(1)$$

in general

We have then

$$\frac{\delta\rho}{\rho_0} = -\vec{\nabla}\cdot\vec{\xi} = -i\,\frac{V_A^2}{c_s^2}\,k_{\parallel}\,\xi_z \rightarrow 0$$

$$\frac{\delta P}{P_0} = \gamma\,\delta\rho/\rho_0 \rightarrow 0.$$

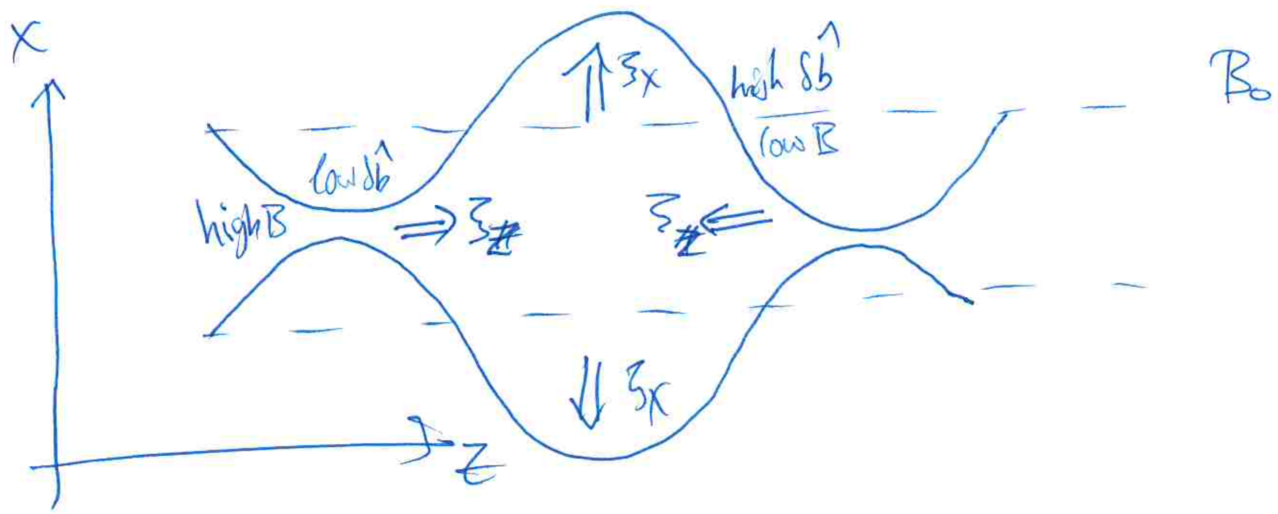
$$\delta\hat{b} = i k_{\parallel}\,\xi_x\,\hat{x} = -i\,\frac{k_{\parallel}}{k_{\perp}}\,k_{\parallel}\,\xi_z\,\hat{x}$$

$$\frac{\delta B}{B_0} = -i k_{\perp}\,\xi_x = i k_{\parallel}\,\xi_z$$

$\therefore$ δB and $\delta\hat{b}$ are in <u>counterphase</u>

as are ξ_x & ξ_z.

This looks like



This $\beta \gg 1$ wave is also in pressure balance

$$P_0 \frac{\delta P}{P_0} + \frac{B_0^2}{\mu_0} \frac{\delta B}{B_0} = \rho_0 \left[c_s^2 \frac{\delta \rho}{\rho_0} + V_A^2 \frac{\delta B}{B_0} \right]$$

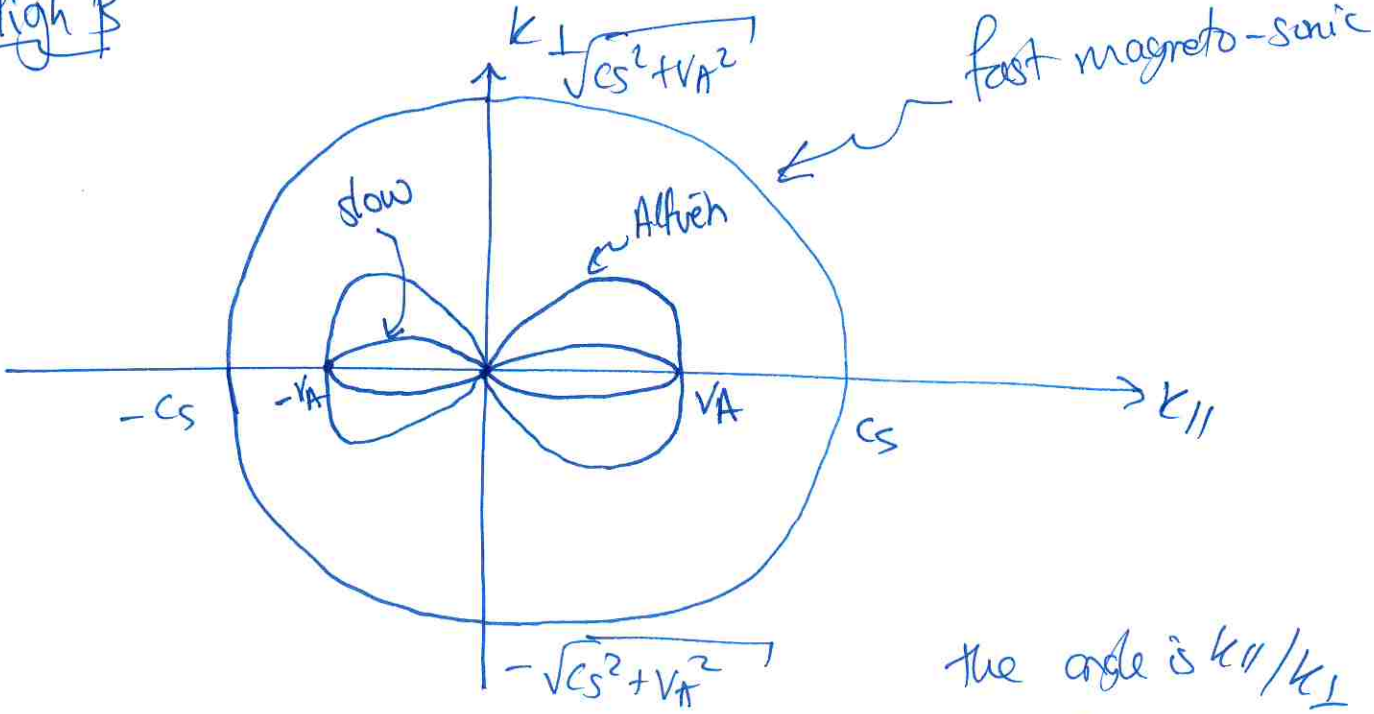
$$= -i k_{\parallel} \xi_z \left[-c_s^2 \frac{V_A^2}{c_s^2} + V_A^2 \right] = 0 .$$

Despite $\frac{\delta P}{P_0} \to 0$ background thermal energy density much bigger than magnetic energy density and so can maintain pressure balance.

Friedricks diagram

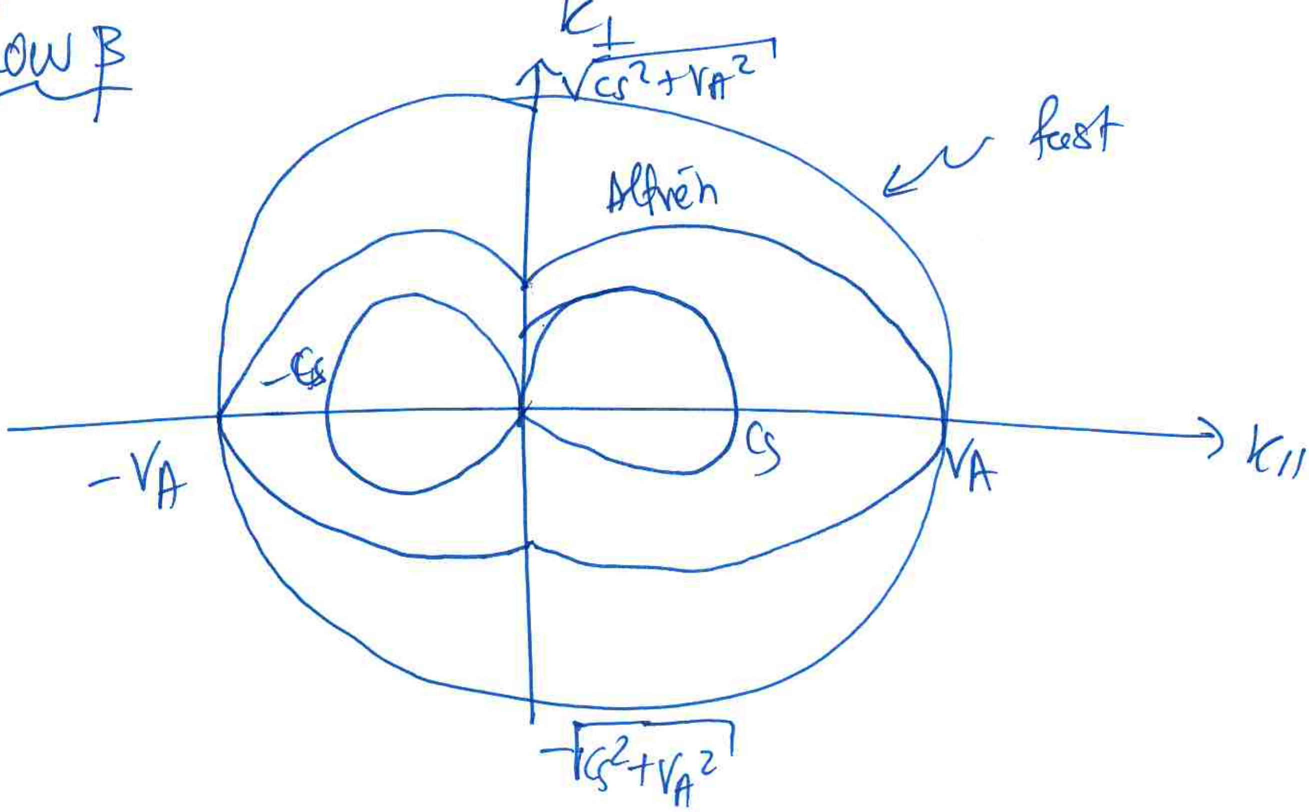
Can represent the proprties of these different waves on a so-called "Friedricks diagram".

High β



the angle is $k_{\parallel}/k_\perp$
The radius is ω/k

Low β



7. MHD Equillibria & Relaxation

Last week we considered MHD in a straight field with constant density & pressure. This has some universal application, as any sufficiently zoomed-in general (static) equillibrium will look like this.

This week we shall consider what sort of large scale equillibria exist, and which of these states a MHD system will relax.

Let's take the momentum equation

$$\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla}(P) + \vec{J} \times \vec{B}$$

and look for static equillibria $[\vec{u} = \vec{0}, \frac{\partial}{\partial t} \equiv 0]$ we find

$$\boxed{-\vec{\nabla} P + \vec{J} \times \vec{B} = \vec{0}.}$$

then recalling our ideal MHD result

$$\boxed{\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B}} \quad \text{and} \quad \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

we have sufficient equations (7) for our unknowns $P, \vec{J}, \vec{B}$.

[Density drops out as nothing moves].

Two immediate consequences of static equilibria:

first:

$$\vec{B} \cdot \left[-\vec{\nabla} p + \vec{J} \times \vec{B} \right] = 0$$

but $$\vec{B} \cdot (\vec{J} \times \vec{B}) = -\vec{J} \cdot (\vec{B} \times \vec{B}) = 0.$$

$$\therefore \quad \boxed{(\vec{B} \cdot \vec{\nabla}) P = 0}$$

pressure does not vary along field lines.

Second:

$$\vec{J} \cdot \left[-\vec{\nabla} P + \vec{J} \times \vec{B} \right] = 0$$

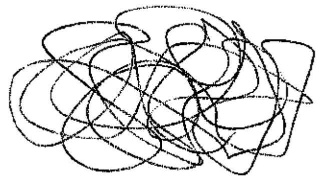
$$\Rightarrow \boxed{(\vec{J} \cdot \vec{\nabla}) P = 0}$$

Currents flow along magnetic surfaces

Consequence of

$$(\vec{B}\cdot\vec{\nabla})P = 0,$$

consider a stochastic and volume filling magnetic field.



e.g.

As on every $\vec{B}$ we must have constant pressure, then $P = \text{const}$ everywhere.

then we have

$$\vec{J} \times \vec{B} = \vec{0}$$

from the ~~pressure~~ momentum conservation equation.

This is called a force free equillibrium.

We shall return to force-free equillibria later.

Cylindrical equillibria

The second simplest equillibrium after $\vec{B} = B_0 \hat{z}$.

Take $\frac{\partial}{\partial \theta} \equiv 0$, $\frac{\partial}{\partial z} \equiv 0$.

Properties:

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \frac{1}{r} \frac{\partial}{\partial r}(r B_r) = 0$$

$$\Rightarrow r B_r = cst \Rightarrow B_r = 0$$

[else $B_r \to \infty$ as $r \to 0$.]

Ampere's law shows

$$\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} \Rightarrow \begin{cases} J_r = 0 \\ J_\theta = -\frac{1}{\mu_0} \frac{\partial B_z}{\partial r} \\ J_z = \frac{1}{\mu_0} \frac{1}{r} \frac{\partial}{\partial r}(r B_\theta) \end{cases}$$

So no current along radius either.

Pressure balance

$$-\vec{\nabla} p + \vec{J} \times \vec{B} = \vec{0}$$

take radial component (only non-trivial one)

$$-\frac{\partial p}{\partial r} + (\vec{J} \times \vec{B})_r = -\frac{\partial p}{\partial r} + J_\theta B_z - J_z B_\theta = 0$$

$$\Rightarrow \quad \frac{\partial p}{\partial r} = J_\theta B_z - J_z B_\theta$$

$$= \frac{1}{\mu_0}\left[-B_z \frac{\partial B_z}{\partial r} - \frac{B_\theta}{r}\frac{\partial}{\partial r}(r B_\theta) \right]$$

$$= \frac{1}{\mu_0}\left[-\frac{\partial}{\partial r}\left(\frac{B_z^2}{2}\right) - \frac{B_\theta^2}{r} - \frac{\partial}{\partial r}\left(\frac{B_\theta^2}{2}\right) \right]$$

[use $B_r = 0$]

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial r}\left(p + \frac{B^2}{2\mu_0}\right) = -\frac{B_\theta^2}{\mu_0 r}}$$

This set up therefore balances the total pressure gradient with the magnetic tension force.

A general equillibrium which satisfies this is called a "screwpinch"

Case 1: The Z-pinch

let the current flow along $\hat{z}$.

$$\Rightarrow J_\theta = 0 \Rightarrow -\frac{1}{\mu_0}\frac{\partial B_z}{\partial r} = 0 \Rightarrow B_z = 0.$$

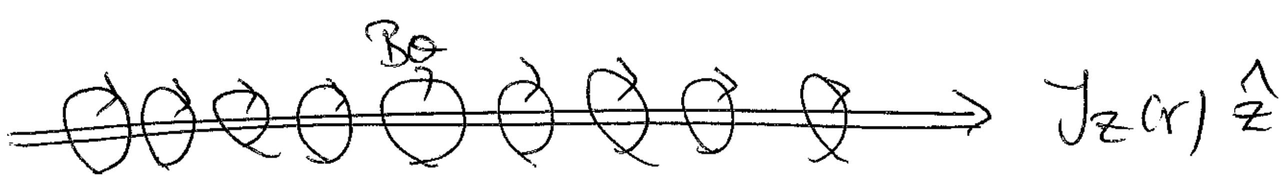
$$J_z \neq 0 \Rightarrow \frac{1}{\mu_0 r}\frac{\partial}{\partial r}(r B_\theta) = J_z(r)$$

$$\Rightarrow B_\theta = \frac{\mu_0}{r}\int^r r' J_z(r')\,dr'$$

and pressure balance is just

$$\frac{\partial p}{\partial r} = -J_z B_\theta = -\frac{\mu_0 J_z(r)}{r}\int^r r' J_z(r')\,dr'$$

what does this look like?



The loops want to contract inwards, and the pressure gradient opposses this. This means that the plasma is <u>confined</u>.

To verify this, try

$$J_z(r) = \frac{J_0}{1+(r/R)^2}$$

$$B_\theta = \frac{\mu_0}{r}\int_0^r \frac{r' J_0}{1+(r'/R)^2}\,dr = \mu_0 J_0 R^2 \frac{\ln\left(1+(r/R)^2\right)}{r}$$

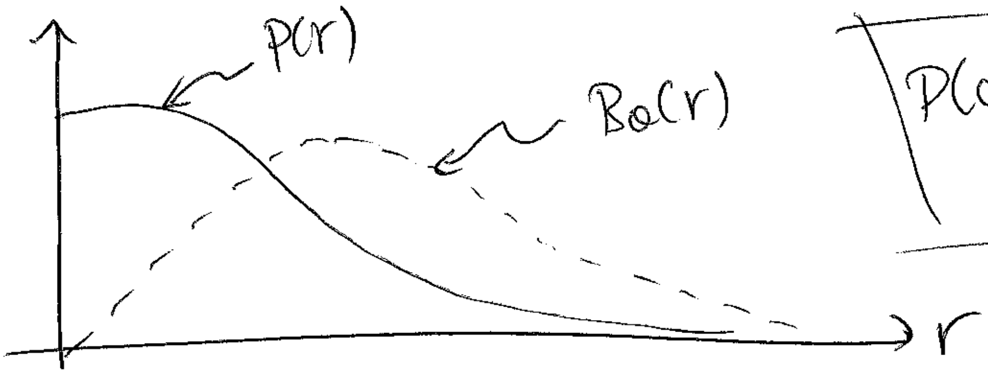
$B_\theta \to 0$ as $r \to 0$.

$B_\theta \to 0$ as $r \to \infty$

$$\frac{\partial p}{\partial r} = -J_z B_\theta = -\mu_0 J_0^2 R^2 \frac{\ln\left(1+(r/R)^2\right)}{r\left(1+r^2/R^2\right)}$$

$\frac{\partial p}{\partial r} \to 0$ at $r \to 0$ $\Rightarrow$ $p \simeq$ cst at $r=0$

$\frac{\partial p}{\partial r} \sim \frac{-1}{r^3}$ as $r \to \infty$ $\Rightarrow$ $p \sim 1/r^2$ at large r



Plasma inside B field!

It turns out the z-pinch is violently unstable. [although it is popular for laboratory experiments].

θ-pinch

Alternatively, put the magnetic field along $\hat{z}$.

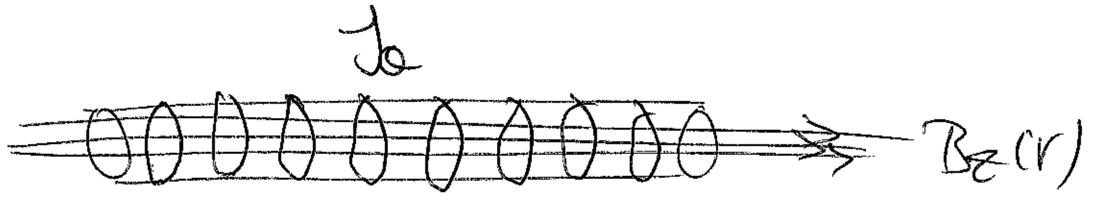
$$B_\theta = 0 \Rightarrow J_z = 0$$

$$B_z \neq 0 \Rightarrow J_\theta = -\frac{1}{\mu_0}\frac{\partial B_z}{\partial r}$$

Need $B_z = B_z(r)$.

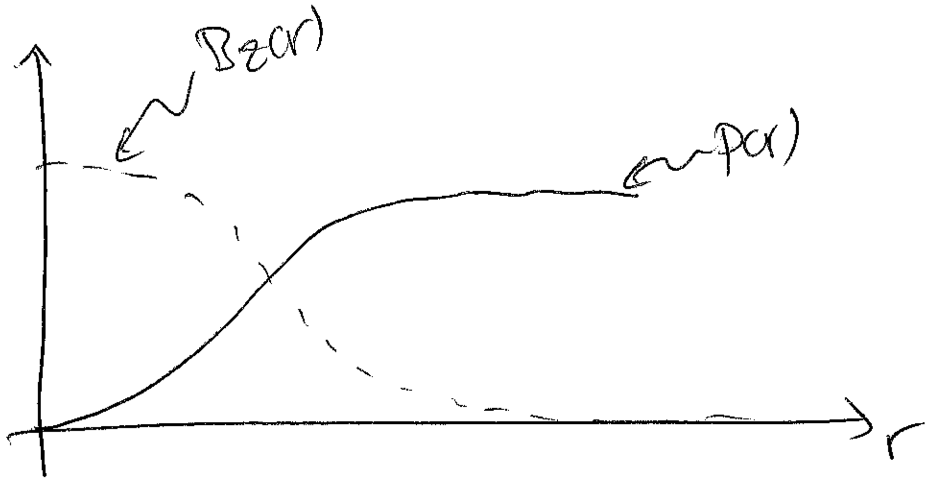
Then momentum equation becomes

$$\frac{\partial}{\partial r}\left(P + \frac{B^2}{2\mu_0}\right) = 0 \Rightarrow \text{pressure balance.}$$



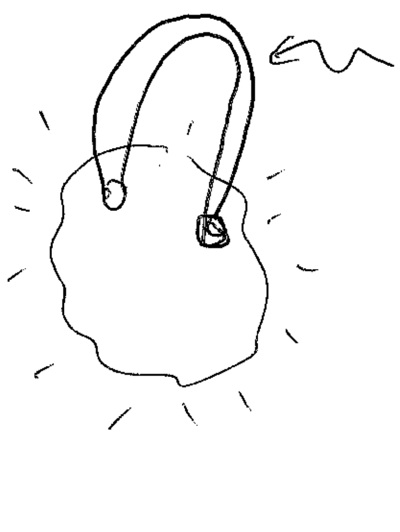
It is possible to confine <u>either</u> the plasma <u>or</u> magnetic field in this setup.

"plasma confined".



"magnetic field confined".

this second case occurs in sun spots



flux tube connects two sun spot, plasma confines B-field.

The θ-pinch is <u>stable</u>.

Force - free equillibria

Another interesting class of equillibria occour in situations where we can neglect $\vec{\nabla} p$ in the momentum equations. This can happen in two situations

1. $\vec{B}$ is stochastic & volume filling
2. $\beta \equiv P/(B^2/2\mu_0) \ll 1$ (typical fusion limit).

This is then a purely magnetic equillibrium, and

$$\vec{J} \times \vec{B} = \vec{0} \Rightarrow \vec{J} \propto \vec{B} \Rightarrow \vec{J} = \frac{1}{\mu_0}\alpha(\vec{r})\vec{B}$$

but $\vec{J} = \frac{1}{\mu_0}\vec{\nabla} \times \vec{B} \Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \alpha(\vec{r})\vec{B}}$

$\alpha(\vec{r})$ is (for the moment) an albitrary scalar field.

I have defined α to have units 1/length.

Taking divergence $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0 = \vec{\nabla} \cdot (\alpha(\vec{r})\vec{B})$

$$= (\vec{B} \cdot \vec{\nabla})\alpha + \alpha \cancelto{0}{(\vec{\nabla} \cdot \vec{B})}$$

$$\Rightarrow (\vec{B} \cdot \vec{\nabla})\alpha = 0$$

constant on magnetic surfaces.

$\alpha(\vec{r}) = \alpha_0 = \text{cst.}$

Is called "linear" force free field.

Then $\vec{\nabla} \times \vec{B} = \alpha_0 \vec{B}$

take curl

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \alpha_0 \vec{\nabla} \times \vec{B} = \alpha_0^2 \vec{B}.$$

but $\vec{\nabla} \times (\vec{\nabla} \times \vec{B})_i = \varepsilon_{ijk} \partial_j \varepsilon_{klm} \partial_l B_m$

$$= [\delta_{il} \delta_{jm} - \delta_{jl} \delta_{im}] \partial_j \partial_l B_m$$

$$= \partial_i \cancelto{0}{\partial_j B_j} - \partial_j \partial_j B_i$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\nabla^2 \vec{B} = \alpha_0^2 \vec{B}$$

$$\Rightarrow (\nabla^2 + \alpha_0^2) \vec{B} = \vec{0}.$$

the so-called Helmholtz equation.

Therefore, there is a potentially huge zoo of MHD equilibria. Some will be stable, others not, so some are interesting / others not.

The question is: to what equilibrium will a MHD system settle?

Imagine we have a plasma, and we set up some initial configuration of $\vec{B}$, say by driving a current (in a wire / the plasma). This will exert forces on the plasma, which will move & produce currents $\Rightarrow$ evolving $\vec{B}$.

In the long time limit, everything will settle down into some static equillibrium.

While we have been considering ideal MHD, with no losses (e.g., viscosity, resistivity), there will always be some losses in a real system. These losses will sap some of the energy content of the initial field. In nature we expect the final state to be a <u>minimum energy state</u>, and so we find it by minimising the magnetic energy

$$\iiint d^3\vec{r} \, \frac{B^2}{2\mu_0} \rightarrow \text{minimum.}$$

Obviously, if relaxation occurred with no constraints, the solution would be $\vec{B} = \vec{0}$.

However, there <u>are</u> constraints, and they turn out to be topological. This follows from the fact that Ideal MHD accepts a topological conserved magnetic quantity, named helicity.

<u>Helicity</u>

Consider the quantity

$$H \equiv \iiint_{V} \vec{A} \cdot \vec{B} \, dV$$

where

$\vec{B} = \vec{\nabla} \times \vec{A}$, and $\vec{A}$ is the vector potential.

Let's prove some properties of H.

1. Helicity is well defined.

Not obvious, as we can always shift

$$\vec{A} \to \vec{A} + \vec{\nabla}\chi \quad \text{and not change } \vec{B}.$$

(a gauge transformation).

If we gauge transform we get

$$H \to H + \iiint_V \vec{B} \cdot \vec{\nabla}\chi \, dV$$

by parts

$$= H + \iiint_V \left(\vec{\nabla} \cdot (\vec{B}\chi) - \chi \overbrace{\vec{\nabla} \cdot \vec{B}}^{0} \right) dV$$

divergence theorem

$$= H + \oiint_{\delta V} \chi \vec{B} \cdot d\vec{S}$$

So if our volume encases the entire field then $\vec{B} \cdot d\vec{S} = 0$ and H is unchanged.

2. Helicity is conserved

Induction equation

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B})$$

$$\therefore \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = \vec{\nabla} \times (\vec{u} \times \vec{B})$$

"uncurl"

$$\frac{\partial A}{\partial t} = \vec{u} \times \vec{B} + \vec{\nabla}\chi$$

dot with $\vec{B}$

$$\vec{B} \cdot \frac{\partial \vec{A}}{\partial t} = \frac{\partial}{\partial t}(\vec{A} \cdot \vec{B}) - \vec{A} \cdot \frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \frac{\partial}{\partial t}(\vec{A} \cdot \vec{B}) = \vec{B} \cdot \left[\vec{u} \times \vec{B} + \vec{\nabla}\chi\right] - \vec{A} \cdot \left[\vec{\nabla} \times (\vec{u} \times \vec{B})\right]$$

term by term

$$\vec{B} \cdot (\vec{u} \times \vec{B}) = 0 \quad \text{(symmetry)}$$

$$\vec{B} \cdot \vec{\nabla}\chi$$

$$= \vec{\nabla} \cdot (\vec{B}\chi) - \chi \cancelto{0}{\vec{\nabla} \cdot \vec{B}} = \vec{\nabla} \cdot (\vec{B}\chi)$$

$$\vec{A}\cdot\left[\vec{\nabla}\times(\vec{u}\times\vec{B})\right] = A_i\,\epsilon_{ijk}\,\partial_j\,\epsilon_{klm}\,u_l B_m$$

or

$$A_i\,\epsilon_{ijk}\,\partial_j B_k \overset{\text{by parts}}{=} \epsilon_{ijk}\,\partial_j(A_i B_k)$$

$$- \epsilon_{ijk}\,B_k\,\partial_j A_i$$

$$= -\underbrace{\epsilon_{jik}}_{\text{one swap}}\,\partial_j(A_i B_k) + \underbrace{\epsilon_{kji}}_{\text{three swap}}\,B_k\,\partial_j A_i$$

$$= -\vec{\nabla}\cdot\left(\vec{A}\times(\vec{u}\times\vec{B})\right) + (\vec{u}\times\vec{B})\cdot(\vec{\nabla}\times\vec{A})$$

but $\vec{\nabla}\times\vec{A} = \vec{B}$

so $(\vec{u}\times\vec{B})\cdot\vec{B} = 0$

and we are left with

$$= -\vec{\nabla}\cdot\left[\vec{A}\times(\vec{u}\times\vec{B})\right]$$

but $\vec{A}\times(\vec{u}\times\vec{B})\big)_i = \epsilon_{ijk}\,A_j\,\epsilon_{klm}\,u_l B_m$

$$= \left[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right] A_j u_l B_m$$

$$= u_i(A_j B_j) - B_i(A_j u_j)$$

$$\therefore\ \vec{A}\cdot\left[\vec{\nabla}\times(\vec{u}\times\vec{B})\right] = -\vec{\nabla}\cdot\left(\vec{u}(\vec{A}\cdot\vec{B}) - \vec{B}(\vec{A}\cdot\vec{u})\right)$$

and so

$$\frac{\partial}{\partial t}(\vec{A}\cdot\vec{B}) = \vec{\nabla}\cdot\left[\vec{B}\chi + \vec{B}(\vec{A}\cdot\vec{u}) - \vec{u}(\vec{A}\cdot\vec{B})\right]$$

Integrate over total volume

$$\frac{\partial}{\partial t}\iiint_V \vec{A}\cdot\vec{B}\,dV = \iiint_V \vec{\nabla}\cdot\left[\vec{B}(\chi + (\vec{A}\cdot\vec{u})) - \vec{u}(\vec{A}\cdot\vec{B})\right]dV$$

$$= \oiint_{\delta V}(\vec{B}\chi + \vec{B}(\vec{A}\cdot\vec{u}) - \vec{u}(\vec{A}\cdot\vec{B}))\cdot d\vec{S}$$

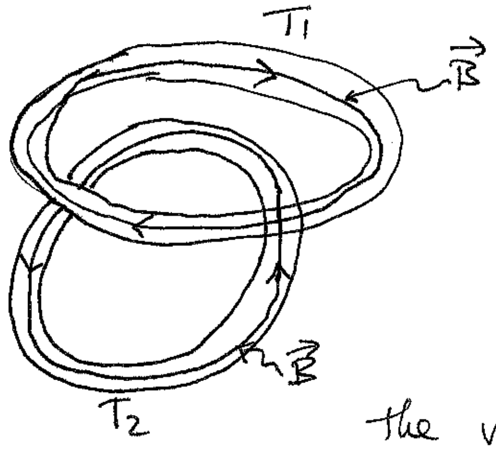
$= 0$ as nothing ($\vec{u}$ or $\vec{B}$) sticks out of the volume.

$$\therefore \quad \boxed{\frac{dH}{dt} = 0}$$

this is the constraint subject to which energy may be minimised.

Helicity is a topological invariant

Consider two linked flux tubes [recall from Alfvén's theorem that these tubes forever enclose a field line]



Then the helicity of tube 1 is

$$H_1 = \iiint_{T_1} \vec{A} \cdot \vec{B} \, d\mathcal{V}_{T_1}$$

the volume of the tube is $\underbrace{d\vec{l}}_{\text{unit along tube } \hat{b}\, dl} . \, d\vec{S}$ $\rightarrow$ surface of tube $\hat{b}\, dS$

thus

$$H_1 = \iiint_{T_1} \hat{b}\, dl \cdot \hat{b}\, dS \; \vec{A} \cdot (\hat{b} B)$$

$$= \underbrace{\iint_{T_1} \vec{B} . d\vec{S}}_{\substack{\text{flux along tube 1} \\ \equiv \Phi_1}} \int_{T_1} \vec{A} \cdot d\vec{l} = \Phi_1 \int_{T_1} \vec{A} \cdot d\vec{l}$$

by Stoke's theorem $\displaystyle\int_{T_1} \vec{A} \cdot d\vec{l} = \iint_{S_1} (\vec{\nabla} \times \vec{A}) . d\vec{S}$

where S_1 is the surface bound by l_1

as $\vec{\nabla} \times \vec{A} = \vec{B}$ $\displaystyle\int_{T_1} \vec{A} \cdot d\vec{l} = \underbrace{\iint_{S_1} \vec{B} \cdot d\vec{S}}_{\substack{\text{flux through middle} \\ \text{of loop } 1 = \Phi_2}}$

$$\Rightarrow H_1 = \Phi_1 \Phi_2$$

by the same token, in a system of many linked tubes, the helicity of tube i is

flux of j^{th} tube

$$H_i = \Phi_i \, \Phi_{\text{through hole}, i} = \Phi_i \sum_{j \neq i} \Phi_j \underbrace{N_{ij}}_{\substack{\text{number of} \\ \text{linkages between} \\ i \,\&\, j}}$$

and thus the total helicity is

$$H = \sum_i \sum_{j \neq i} \Phi_i \Phi_j N_{ij}$$

Thus helicity measures the number of linkages of the flux tubes weighted by the field strength in each tube.

The physical insight is the following: if one thinks of the magnetic field as a tangled mess of field lines, while you can change this mess by moving field lines around, you cannot easily undo linkages, knots, etc.
[This is Alfvén's theorem again]. Basically any operation which would require field lines to have "ends"!

Taylor Relaxation

We can now work out which equillibrium an MHD system will relax to by minimising its energy subject to constant helicity. This is a classic lagrange multiplier problem, with "action"

$$S = \iiint_V (B^2 - \alpha \vec{A}\cdot\vec{B})\, dV$$

α ← lagrange multiplier.

Then $\delta S = 0$

implies

$$\delta\left[\iiint_V (B^2 - \alpha\vec{A}\cdot\vec{B})\, dV\right] = 0$$

first term

$$\delta \iiint_V B^2 dV = 2\iiint_V \vec{B}\cdot\delta\vec{B}\, dV$$

$$= 2\iiint_V \vec{B}\cdot(\vec{\nabla}\times\delta\vec{A})\, dV$$

$$\vec{B}\cdot(\vec{\nabla}\times\delta\vec{A}) = B_i \varepsilon_{ijk}\partial_j \delta A_k$$

$$= \varepsilon_{ijk}\partial_j(B_i\delta A_k) - \varepsilon_{ijk}\delta A_k \partial_j B_i$$

$$\overset{\text{one swap}}{=} -\varepsilon_{jik}\partial_j(B_i\delta A_k) \overset{\text{three swap}}{+} \delta A_k \varepsilon_{kji}\partial_j B_i$$

$$= -\vec{\nabla}\cdot(\vec{B}\times\delta\vec{A}) + \delta\vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$$\Rightarrow \delta\iiint_V B^2 dV = -2\iiint_V \vec{\nabla}\cdot(\vec{B}\times\delta\vec{A})\, dV + 2\iiint_V \delta\vec{A}\cdot(\vec{\nabla}\times\vec{B})\, dV$$

divergence Theorem

$$= -2\oint\!\!\!\oint_{\delta V} (\vec{B}\times\delta\vec{A})\cdot d\vec{S} + 2\iiint_V (\vec{\nabla}\times\vec{B})\cdot\delta\vec{A}\, dV$$

and

$$\delta\iiint_V \vec{A}\cdot\vec{B}\, dV = \iiint_V \delta\vec{A}\cdot\vec{B}\, dV + \iiint_V \vec{A}\cdot\delta\vec{B}\, dV$$

$$= \iiint_V \delta\vec{A}\cdot\vec{B}\, dV + \iiint_V \vec{A}\cdot(\vec{\nabla}\times\delta\vec{A})\, dV$$

identical manipulation to before on second term

$$= \iiint_V \delta\vec{A}\cdot\vec{B}\, dV - \iiint_V \vec{\nabla}\cdot(\vec{A}\times\delta\vec{A})\, dV + \iiint_V \underbrace{(\vec{\nabla}\times\vec{A})}_{\vec{B}}\cdot\delta\vec{A}\, dV$$

divergence theorem + combine 1 & 3

$$= 2\iiint_V \vec{B}\cdot\delta\vec{A}\, dV - \oint\!\!\!\oint_{\delta V} (\vec{A}\times\delta\vec{A})\cdot d\vec{S}$$

we must worry about surface terms.

As $$\frac{\partial}{\partial t}\delta\vec{B} = \vec{\nabla}\times\left(\frac{\partial\vec{\xi}}{\partial t}\times\vec{B}\right) \qquad \left[\text{reintroducing } \delta\vec{u} = \frac{\partial\vec{\xi}}{\partial t}\right]$$

uncurling $$\delta\vec{A} = \vec{\xi}\times\vec{B}$$

Also $$\vec{A}\times\delta\vec{A} = \vec{A}\times(\vec{\xi}\times\vec{B}) = \epsilon_{ijk} A_j \epsilon_{klm} \xi_l B_m$$

$$= [\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}] A_j \xi_l B_m$$

$$= (\vec{A}\cdot\vec{B})\vec{\xi} - (\vec{A}\cdot\vec{\xi})\vec{B}$$

also

$$\vec{B} \times \delta\vec{A} = \varepsilon_{ijk} B_j \varepsilon_{klm} \xi_l B_m$$

$$\overset{\text{same argument}}{=} B^2 \vec{\xi} - (\vec{B}\cdot\vec{\xi})\vec{B}$$

and so all surface terms come with $\vec{\xi}\cdot d\vec{S}$ or $\vec{B}\cdot d\vec{S}$, which will vanish if δV encloses both the magnetic field and the plasma.

Thus, we are left with

$$\delta \iiint_V \left(B^2 - \alpha \vec{A}\cdot\vec{B}\right) dV = 2\iiint_V \left[\vec{\nabla}\times\vec{B} - \alpha\vec{B}\right]\cdot\delta\vec{A}\, dV$$

$$= 0$$

$$\therefore \quad \vec{\nabla}\times\vec{B} = \alpha\vec{B} \Rightarrow \nabla^2\vec{B} = -\alpha^2\vec{B}$$

and we have recovered the <u>linear force free field</u>.

So our system will relax to a linear force free state with system-specific boundary conditions. The boundary conditions come in through the initial helicity, and our lagrange multiplier α is a function $\alpha = \alpha(H)$, which enforces this.

This linear force free state has a particularly simple helicity, which follows from

$$\vec{\nabla}\times\vec{B} = \alpha\vec{B} = \alpha\vec{\nabla}\times\vec{A} \Rightarrow \vec{B} = \alpha\vec{A} + \vec{\nabla}\chi$$

$$H(\alpha) = \iiint_V \vec{A}\cdot\vec{B}\,dV = \frac{1}{\alpha}\iiint_V B^2 dV - \frac{1}{\alpha}\iiint \vec{B}\cdot\vec{\nabla}\chi dV$$

final term vanishes under volume integral

$$\iiint_V \vec{B}\cdot\vec{\nabla}\chi\, dV = \iiint_V \vec{\nabla}\cdot(\vec{B}\chi)\,dV - \iiint_V \chi\,\cancel{\vec{\nabla}\cdot\vec{B}}^{\,0}\, dV$$

$$= \oint\!\!\oint_{\delta V} \chi\vec{B}\cdot d\vec{S} = 0 \quad [\text{enclose field}].$$

$\therefore$ Constant 'α' set by

$$H = \frac{1}{\alpha}\iiint_V B^2 dV.$$

General procedure:

- solve $\nabla^2\vec{B} = -\alpha^2\vec{B}$, get $\vec{B}(\vec{r},\alpha)$.
- Calculate $H(\alpha) = \frac{1}{\alpha}\iiint_V B^2(\vec{r},\alpha)\,dV$
- Set $H(\alpha) = H_0$, the initial value of helicity
- Set α by inverting to $\alpha = \alpha(H_0)$
- Then relaxed state $\vec{B} = \vec{B}(\vec{r}, \alpha[H_0])$.

Example

Let us consider the case of cylindrical & axial symmetry once again. We know $B_r = 0$, and $B_\theta, B_z \neq 0$.

z-component of $\nabla^2 \vec{B} = \alpha^2 \vec{B} \Rightarrow \dfrac{\partial^2 B_z}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial B_z}{\partial r} + \alpha^2 B_z = 0.$

This is a Bessel equation, with solution

$$B_z(r) = B_0 \underbrace{J_0}_{\text{Bessel function of order 0.}}(\alpha r)$$

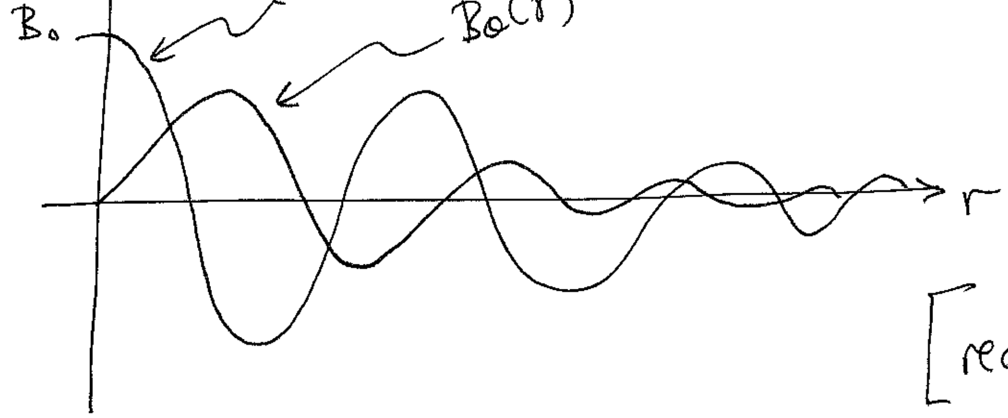
This solution satisfies the b.c. $B_z(0) = B_0$, $B_z(r\to\infty) \to 0$.

We can use

$$\vec{\nabla}\times\vec{B} = \alpha\vec{B} \Rightarrow \alpha B_\theta = (\vec{\nabla}\times\vec{B})_\theta = -\frac{\partial B_z}{\partial r}$$

using Bessel identities

$$B_\theta(r) = B_0 \underbrace{J_1}_{\text{Bessel function of order 1.}}(\alpha r)$$



[recall $J(x) \sim \frac{1}{\sqrt{x}}\sin(x)$ for large x.]

which is an interesting twisted field geometry which can maintain itself in equilibrium.

Enforce helicity constraint

$$H = \frac{1}{\alpha} \iiint_V B^2 \, dV$$

Assume volume is of a cylinder of length L and radius R

$$H = \frac{1}{\alpha} \cdot B_0^2 \cdot 2\pi \cdot L \cdot \int_0^R r\left[J_0^2(\alpha r) + J_1^2(\alpha r)\right] dr$$

......

$$H = \frac{B_0^2 \pi R^2 L}{\alpha} \left[J_0^2(\alpha R) + 2J_1^2(\alpha R) + J_2^2(\alpha R) - \frac{2}{\alpha R} J_1(\alpha R) J_2(\alpha R) \right]$$

picking H_0 and inverting for $\alpha = \alpha(H_0)$ gives us our final field state.

9. Instabilities

We wish to ask the following sensible question:
if we have found an equillibrium which is static $[\vec{u}=\vec{0}]$, but is rather general otherwise $p = p_0(\vec{r})$, $\rho = \rho_0(\vec{r})$, $\vec{B} = \vec{B}_0(\vec{r})$, is this ~~perturbation~~ equillibrium stable to small perturbations?

There are two general approaches to this problem:

1.) Solve a specific problem with a linearised normal mode analysis. In other words, write down a specific equillibrium, kick it [$p_0 \Rightarrow p_0 + \delta p$, etc.], and look for modal solutions $\propto e^{-i\omega t + i\vec{k}\cdot\vec{r}}$. If $\omega \in \mathbb{C}$, then there may be exponentially growing modes $\Rightarrow$ instability. However, this may be complex for a given $p_0, \rho_0, \vec{B}_0$ etc.

2.) There is a general procedure, "the energy principle", which can tell you whether an equillibrium is stable, without giving you a huge amount of physical information about the instability (or lack thereof).

We will begin with this general overview approach, which is a powerful technique, before showing a particular example of (1). from Astrophysics.

The energy principle

We shall prove, but this may be intuitively obvious, that a way of tackling the instability problem is the following: compute the change in potential energy of the fluid resulting from a perturbation. If there is a way in which a perturbation can lower the potential energy of ~~ex~~ the fluid, then this perturbation leads to an instability.

Let's begin.

The total energy in MHD is (see lecture 4)

$$\mathcal{E} = \iiint_{V} \left(\frac{1}{2}\rho u^2 + \frac{B^2}{2\mu_0} + \frac{P}{\gamma - 1} \right) dV$$

we will define

$$\mathcal{E} \equiv \iiint_{V} \frac{1}{2}\rho u^2 \, dV + W.$$

As we saw in the lectures on waves (5+6), all perturbations of an MHD system can be expressed in terms of small displacements $\vec{\xi}$, where our perturbed velocity $\delta \vec{u} = \frac{\partial \vec{\xi}}{\partial t}$.

Then

(no kinetic energy in equilibrium)

$$\mathcal{E} \rightarrow \mathcal{E} + \delta\mathcal{E} = W_0 + \iiint \frac{1}{2}\rho_0 \left| \frac{\partial \vec{\xi}}{\partial t} \right|^2 dV + \delta W_1[\vec{\xi}] + \delta W_2[\vec{\xi}, \vec{\xi}] + \dots$$

where we have expanded to quadratic order, W_0 is our equillibrium potential, and we have split our perturbed potentials into linear $\delta W_1[\vec{\xi}]$ and quadratic $\delta W_2[\vec{\xi},\vec{\xi}]$ orders.

Energy reduced if $\delta W < 0 \Rightarrow$ care about sign of δW_1, δW_2.

Energy must be globally conserved to <u>all</u> orders [push volume integral to infinity].

This means we can be clever and work out δW_1 & δW_2

$$\frac{d\mathcal{E}}{dt} = \frac{d\,\delta\mathcal{E}}{dt} = \iiint \rho_0 \frac{\partial^2\vec{\xi}}{\partial t^2}\cdot\frac{\partial\vec{\xi}}{\partial t} + \delta W_1\left[\frac{\partial\vec{\xi}}{\partial t}\right]$$

$$+ \delta W_2\left[\frac{\partial\vec{\xi}}{\partial t},\vec{\xi}\right] + \delta W_2\left[\vec{\xi},\frac{\partial\vec{\xi}}{\partial t}\right] + \dots = 0.$$

This must be true <u>at all times</u>. Including at $t=0$, where $\vec{\xi}$ & $\frac{\partial\vec{\xi}}{\partial t}$ are independent. (These perturbations can be chosen independently as MHD equations are second order in time for $\vec{\xi}$). Define the "force operator"

$$\vec{F}[\vec{\xi}] = \rho_0 \frac{\partial^2\vec{\xi}}{\partial t^2},$$

then, at $t=0$, calling $\vec{\eta} \equiv \frac{\partial\vec{\xi}}{\partial t}$

$$\iiint_V \vec{\eta}\cdot\vec{F}(\vec{\xi})\,dV + \delta W_1[\vec{\xi}] + \delta W_2[\vec{\eta},\vec{\xi}] + \delta W_2[\vec{\xi},\vec{\eta}] + \dots = 0.$$

Linear order: $\delta W_1[\vec{\eta}] = 0 \Rightarrow$ no linear energy perturbations.

Second order: $$\iiint_V \vec{\eta} \cdot \vec{F}[\vec{\xi}]\, dV = -\delta W_2[\vec{\eta}, \vec{\xi}] - \delta W_2[\vec{\xi}, \vec{\eta}]$$

Can also set $\vec{\eta} = \vec{\xi}$ [at $t=0$], this trick leads to

$$\boxed{\delta W_2[\vec{\xi}, \vec{\xi}] = -\frac{1}{2} \iiint_V \vec{\xi} \cdot \vec{F}[\vec{\xi}]\, dV}$$

It will transpire that δW_2 can have either sign, and therefore teaches us about instabilities.

Energy principle states

$$\boxed{\delta W_2[\vec{\xi}, \vec{\xi}] > 0 \quad \forall \vec{\xi} \iff \text{stable equillibrium}}$$

So we need to analyse the properties of the "force operator"

$$\vec{F}[\vec{\xi}].$$

Obviously, we will derive the functional form of $\vec{F}$ from the linearised MHD equations, but we can actually do nearly all of our work in generality.

Property 1: $\vec{F}[\vec{\xi}]$ has simple eigenmodes $\vec{\xi}_n$.

By definition

$$\vec{F}[\vec{\xi}] = \rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2}$$

we see that for $\vec{\xi}_n = \xi_n(\vec{r})\, e^{-i\omega_n t}$

we have

$$\vec{F}[\vec{\xi}_n] = -\rho_0 \omega_n^2 \vec{\xi}_n$$

so $\frac{1}{\rho_0}\vec{F}[\vec{\xi}]$ has eigenmodes $\propto e^{-i\omega_n t}$. $\square$.

Property 2. $\vec{F}[\vec{\xi}]$ is hermitian (or self adjoint)

We derived

$$\iiint_V \vec{\eta}\cdot\vec{F}[\vec{\xi}]\, dV = -\delta W_2[\vec{\eta},\vec{\xi}] - \delta W_2[\vec{\xi},\vec{\eta}]$$

R.H.S is symmetric in $\vec{\eta} \leftrightarrow \vec{\xi}$, so

$$\iiint_V \vec{\eta}\cdot\vec{F}[\vec{\xi}]\, dV = \iiint_V \vec{\xi}\cdot\vec{F}[\vec{\eta}]\, dV \quad \square.$$

Property 3 : eigenvalues ω_n^2 are real.

As $\vec{F}[\vec{\xi}_n] = -\rho_0 \omega_n^2 \vec{\xi}_n$

and $\vec{F}$ ~~hermitian~~ has no complex coefficients then

$$\vec{F}[\vec{\xi}_n^*] = -\rho_0 (\omega_n^2)^* \vec{\xi}_n^*$$

Compute the difference

hermitian!

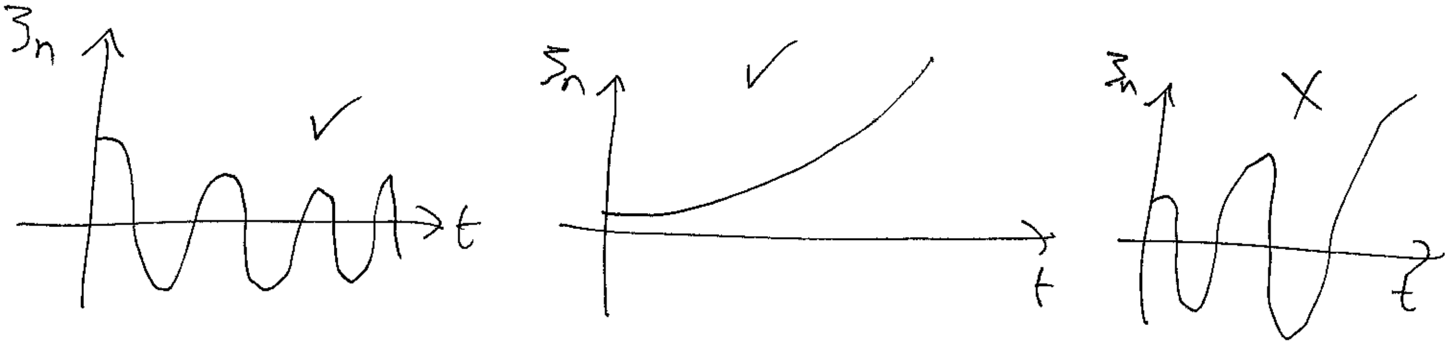
$$\iiint_V \vec{\xi}_n \cdot \vec{F}[\vec{\xi}_n^*]\, dV - \iiint_V \vec{\xi}_n^* \cdot \vec{F}[\vec{\xi}_n]\, dV = 0$$

$$= -\left[\omega_n^2 - (\omega_n^2)^*\right] \iiint \rho_0 |\vec{\xi}_n|^2 dV$$

but $\rho_0 |\vec{\xi}_n|^2 > 0 \Rightarrow \omega_n^2 = (\omega_n^2)^*$

$$\Rightarrow \boxed{\omega_n^2 \in \mathbb{R}.}$$

This means that ω_n is either pure real or pure imaginary $\Rightarrow$ perturbations oscillate or diverge, but don't oscillate with changing amplitude (in MHD).



Property 4: the eigenmodes $\vec{\xi}_n$ are orthogonal

Identical calculation as above, only with $\vec{\xi}_n$ & $\vec{\xi}_m$:

$$\vec{F}[\vec{\xi}_m] = -\rho_0 \omega_m^2 \vec{\xi}_m$$

then

$$-\omega_n^2 \iiint \rho_0 \vec{\xi}_m \cdot \vec{\xi}_n \, dV + \omega_m^2 \iiint_V \rho_0 \vec{\xi}_n \cdot \vec{\xi}_m \, dV$$

$$= \iiint_V \vec{\xi}_m \cdot \vec{F}[\vec{\xi}_n] \, dV - \iiint \vec{\xi}_n \cdot \vec{F}[\vec{\xi}_m] \, dV$$

$$= 0$$

$$\Rightarrow -(\omega_n^2 - \omega_m^2) \iiint_V \rho_0 \vec{\xi}_n \cdot \vec{\xi}_m \, dV = 0$$

$$\Rightarrow \iiint_V \rho_0 \vec{\xi}_n \cdot \vec{\xi}_m \, dV = \delta_{n,m} \iiint \rho_0 |\vec{\xi}_n|^2 \, dV$$

We can now prove the energy principle.

Proof of energy principle

Write

$$\vec{\xi}(\vec{r},t) = \sum_n a_n(t)\,\vec{\xi}_n(\vec{r})$$

Then

$$\delta W_2[\vec{\xi},\vec{\xi}] = -\frac{1}{2}\iiint_V \vec{\xi}\cdot\vec{F}[\vec{\xi}]\,dV$$

$$= -\frac{1}{2}\sum_n\sum_m \iiint_V a_n a_m \vec{\xi}_n\cdot\vec{F}[\vec{\xi}_m]\,dV$$

$$= +\frac{1}{2}\sum_n\sum_m a_n a_m \omega_m^2 \iiint_V \rho_0\,\vec{\xi}_n\cdot\vec{\xi}_m\,dV$$

$$= \frac{1}{2}\sum_n a_n^2\omega_n^2 \iiint \rho_0\,|\vec{\xi}_n|^2\,dV$$

~~the kinetic term is~~ we can define

$$K[\vec{\xi},\vec{\xi}] \equiv \frac{1}{2}\iiint_V \rho_0\,|\vec{\xi}|^2\,dV$$

....

$$= \frac{1}{2}\sum_n a_n^2 \iiint \rho_0\,|\vec{\xi}_n|^2\,dV$$

and therefore, our smallest eigenvalue (call it ϖ_1) satisfies

$$\varpi_1^2 = \min_{\vec{\xi}} \left(\frac{\delta W_2[\vec{\xi}, \vec{\xi}]}{K[\vec{\xi}, \vec{\xi}]} \right)$$

as $K > 0$, if $\delta W_2 > 0$ for all possible $\vec{\xi}$, then $\varpi_1^2 > 0 \Rightarrow$ all $\varpi_n^2 > 0 \Rightarrow$ stable.

However, if $\delta W_2 < 0$ for any $\vec{\xi}$, then at least one of $\varpi_n^2 < 0 \Rightarrow$ an instability exists.

Our procedure is therefore clear, write down the force operator $\vec{F}[\vec{\xi}] = \rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2}$ from the linearised MHD equations, use this to compute δW_2, and this will let us analyse a wide class of instability problems.

Deriving the force operator

Mass: $$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

Entropy: $$\frac{D}{Dt}(p\rho^{-\gamma}) = 0 \Rightarrow \frac{\partial p}{\partial t} + (\vec{u}\cdot\vec{\nabla})p + \gamma p(\vec{\nabla}\cdot\vec{u}) = 0$$

Momentum: $$\rho\left[\frac{\partial}{\partial t} + (\vec{u}\cdot\vec{\nabla})\right]\vec{u} = -\vec{\nabla}p + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}.$$

Induction: $$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B}).$$

let's linearise; with $\vec{u} = \vec{0}$, $\delta\vec{u} = \frac{\partial\vec{\xi}}{\partial t}$

Mass: $$\delta\rho = -\vec{\nabla}\cdot(\rho_0\vec{\xi})$$

Pressure: $$\delta p = -(\vec{\xi}\cdot\vec{\nabla})p_0 - \gamma p_0(\vec{\nabla}\cdot\vec{\xi})$$

Induction: $$\delta\vec{B} = \vec{\nabla}\times(\vec{\xi}\times\vec{B}_0)$$

Momentum: $$\rho_0\frac{\partial^2\vec{\xi}}{\partial t^2} = -\vec{\nabla}\delta p + \frac{1}{\mu_0}(\vec{\nabla}\times\delta\vec{B})\times\vec{B}_0 + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}_0)\times\delta\vec{B}.$$

$\therefore$

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = \vec{\nabla}\left[(\vec{\xi}\cdot\vec{\nabla})p_0\right] + \gamma\vec{\nabla}\left[p_0(\vec{\nabla}\cdot\vec{\xi})\right]$$

$$+ \frac{1}{\mu_0}\left(\vec{\nabla}\times\left[\vec{\nabla}\times(\vec{\xi}\times\vec{B_0})\right]\right)\times\vec{B_0}$$

$$+ \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B_0})\times\left[\vec{\nabla}\times(\vec{\xi}\times\vec{B_0})\right].$$

and we therefore have derived our force operator.

Therefore

$$\delta W_2 = -\frac{1}{2}\iiint_V \vec{\xi}\cdot\vec{F}[\vec{\xi}]\,dV$$

$$= -\frac{1}{2}\iiint_V \vec{\xi}\cdot\left(\vec{\nabla}\left[(\vec{\xi}\cdot\vec{\nabla})p_0\right]\right)dV$$

$$-\frac{1}{2}\iiint_V \gamma\,\vec{\xi}\cdot\left(\vec{\nabla}\left[p_0(\vec{\nabla}\cdot\vec{\xi})\right]\right)dV$$

$$-\frac{1}{2}\iiint \frac{1}{\mu_0}\vec{\xi}\cdot\left[(\vec{\nabla}\times\delta\vec{B})\times\vec{B_0}\right]dV$$

$$-\frac{1}{2}\iiint \vec{\xi}\cdot(\vec{J_0}\times\delta\vec{B})\,dV$$

there are a myriad number of equivalent formulations of this integral, all based on different numbers of integrations by parts. I will derive the "textbook" version

$$I_1 = \iiint_V \vec{\xi} \cdot \left(\vec{\nabla} \left[(\vec{\xi} \cdot \vec{\nabla}) p_0 \right] \right) dV$$

$$= \underbrace{\iiint_V \vec{\nabla} \cdot \left(\vec{\xi} \left[(\vec{\xi} \cdot \vec{\nabla}) p_0 \right] \right) dV}_{0 \text{ by divergence theorem}} - \iiint_V \left[(\vec{\xi} \cdot \vec{\nabla}) p_0 \right] \left[\vec{\nabla} \cdot \vec{\xi} \right] dV$$

$$I_2 = \iiint_V \gamma \vec{\xi} \cdot \left(\vec{\nabla} \left[p_0 (\vec{\nabla} \cdot \vec{\xi}) \right] \right) dV$$

$$= \underbrace{\iiint_V \vec{\nabla} \cdot \left[\gamma \vec{\xi} \left[p_0 (\vec{\nabla} \cdot \vec{\xi}) \right] \right]}_{0} - \iiint \gamma p_0 (\vec{\nabla} \cdot \vec{\xi})^2 \, dV$$

$$I_4 = \iiint_V \vec{\xi} \cdot \left(\vec{J}_0 \times \delta \vec{B} \right) dV = -\iiint_V \vec{J}_0 \cdot (\vec{\xi} \times \delta \vec{B}) \, dV$$

$$I_3 = \iiint_V \frac{\vec{\xi}}{\mu_0} \cdot \left[(\vec{\nabla} \times \delta \vec{B}) \times \vec{B}_0 \right] dV$$

$$= \frac{1}{\mu_0} \iiint_V (\vec{\nabla} \times \delta\vec{B}) \cdot (\vec{\xi} \times \vec{B}_0)\, dV \qquad \text{[vector triple]}$$

$$= -\frac{1}{\mu_0} \iiint_V (\delta\vec{B} \times \vec{\nabla}) \cdot (\vec{\xi} \times \vec{B}_0)\, dV \qquad \text{[by parts]}$$

$$= -\frac{1}{\mu_0} \iiint_V \delta\vec{B} \cdot \left(\vec{\nabla} \times (\vec{\xi} \times \vec{B}_0)\right) dV$$

$$= -\iiint_V \frac{|\delta\vec{B}|^2}{\mu_0}\, dV$$

and therefore

$$\delta W_2 = \frac{1}{2} \iiint_V \Big[\left((\vec{\xi} \cdot \vec{\nabla}) p_0\right) \vec{\nabla} \cdot \vec{\xi} + \overbrace{\gamma p_0 (\vec{\nabla} \cdot \vec{\xi})^2}^{\text{stabilising}}$$

$$+ \vec{J}_0 \times (\vec{\xi} \times \delta\vec{B}) + \underbrace{\frac{1}{\mu_0} |\delta\vec{B}|^2}_{\text{stabilising}} \Big] dV$$

Let's use this as the largest hammer to ever hit a small nail.

Consider adding a constant gravitational field to the momentum equation

$$\rho\left[\frac{\partial}{\partial t}+(\vec{u}\cdot\vec{\nabla})\right]\vec{u} = -\vec{\nabla}P+\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}+\rho\vec{g}$$

then a perturbation gets an extra force term

$$\delta\rho\,\vec{g} = -\vec{g}\left(\vec{\nabla}\cdot(\rho_0\vec{\xi})\right)$$

and our energy integral has

$$\delta W_2 = \frac{1}{2}\iiint_V \Big[\big((\vec{\xi}\cdot\vec{\nabla})P_0\big)(\vec{\nabla}\cdot\vec{\xi})+\gamma P_0(\vec{\nabla}\cdot\vec{\xi})^2 + (\vec{\xi}\cdot\vec{g})(\vec{\nabla}\cdot(\rho_0\vec{\xi}))+\vec{J}_0\times(\vec{\xi}\times\delta\vec{B})+\frac{|\delta\vec{B}|^2}{\mu_0}\Big]dV$$

Let us return to hydrodynamics $\ni$ $\vec{B}_0=\vec{J}_0=\vec{0}$.

$\delta\vec{B}=0$.

$$\delta W_2 = \frac{1}{2}\iiint_V\Big[(\vec{\nabla}\cdot\vec{\xi})(\vec{\xi}\cdot\vec{\nabla})P_0+\gamma P_0(\vec{\nabla}\cdot\vec{\xi})^2+(\vec{\xi}\cdot\vec{g})\,\vec{\nabla}\cdot(\rho_0\vec{\xi})\Big]dV$$

Interchange instability

$$\vec{g} = -g\hat{z}, \quad \rho = \rho_0(z), \quad P = P_0(z).$$

Vertical momentum equation

$$\vec{0} = -\vec{\nabla}P + \rho\vec{g} \Rightarrow \frac{dP_0}{dz} = -\rho_0 g$$

Perturbation δW_2 satisfies

$$\delta W_2 = \frac{1}{2}\iiint\left[\xi_z \frac{dP_0}{dz}(\vec{\nabla}\cdot\vec{\xi}) + \gamma P_0(\vec{\nabla}\cdot\vec{\xi})^2 - g\xi_z\left(\frac{d\rho_0}{dz}\xi_z + \rho_0\vec{\nabla}\cdot\vec{\xi}\right)\right]dv$$

(use $\rho_0 = -\frac{1}{g}\frac{dP_0}{dz}$)

$$= \frac{1}{2}\iiint\left(2\xi_z\frac{dP_0}{dz}(\vec{\nabla}\cdot\vec{\xi}) + \gamma p_0(\vec{\nabla}\cdot\vec{\xi})^2 - g\xi_z^2\frac{d\rho_0}{dz}\right)dV$$

$$\therefore \quad \delta W_2 = \iiint_V f(\xi_z, \vec{\nabla}\cdot\vec{\xi})\,dv$$

Let's look for the most unstable perturbation, by minimising f. Treat ξ_z & $\vec{\nabla}\cdot\vec{\xi}$ as independent

$$\frac{\partial f}{\partial(\vec{\nabla}\cdot\vec{\xi})} = 2\xi_z\frac{dP_0}{dz} + 2\gamma p_0(\vec{\nabla}\cdot\vec{\xi}) = 0$$

$$\Rightarrow (\vec{\nabla}\cdot\vec{\xi}) = -\frac{1}{\gamma p_0}\frac{dP_0}{dz}\xi_z$$

Substituting back

$$\delta W_2 = \frac{1}{2}\iiint_V\left(-\frac{2\xi_z^2}{\gamma p_0}\left(\frac{dP_0}{dz}\right)^2 + \frac{1}{\gamma p_0}\left(\frac{dP_0}{dz}\right)^2\xi_z^2 - g\xi_z^2\frac{d\rho_0}{dz}\right)dV$$

simplifying

$$\delta W_2 = \frac{1}{2}\iiint_V \left[-\frac{1}{\gamma p_0}\left(\frac{dp_0}{dz}\right)^2 \xi_z^2 - g\xi_z^2 \frac{d\rho_0}{dz}\right] dV$$

with $\frac{dp_0}{dz} = -\rho_0 g$

$$\delta W_2 = \frac{1}{2}\iiint_V \left[\frac{\rho_0 g}{\gamma p_0}\frac{dp_0}{dz}\xi_z^2 - g\xi_z^2 \frac{d\rho_0}{dz}\right] dV$$

$$= \frac{1}{2}\iiint_V \frac{\rho_0 g \xi_z^2}{\gamma}\left[\frac{1}{p_0}\frac{dp_0}{dz} - \frac{\gamma}{\rho_0}\frac{d\rho_0}{dz}\right] dV$$

$$= \frac{1}{2}\iiint_V \frac{\rho_0 g \xi_z^2}{\gamma}\left[\frac{d}{dz}\ln\left(p_0\rho_0^{-\gamma}\right)\right] dV$$

$\therefore$ unstable iff. $\frac{d}{dz}\ln\left(p_0\rho_0^{-\gamma}\right) < 0$

$\Rightarrow$ entropy decreases upwards.

Highlights the pro's & con's of this method.

- Pro: easy calculation, no underlying solutions of the hydro required
- Con: no insight about <u>why</u> this is unstable.

The magneto-thermal instability (MTI)

Discovered by Steven Balbus in 2000, 2001.
(a man of many instabilities).

Numerically explored by Parrish & Jim Stone 2007.

Extended by Quataert 2008.

[Clearly to be a named chair in Oxford, Princeton or IAS you need to study this instability].

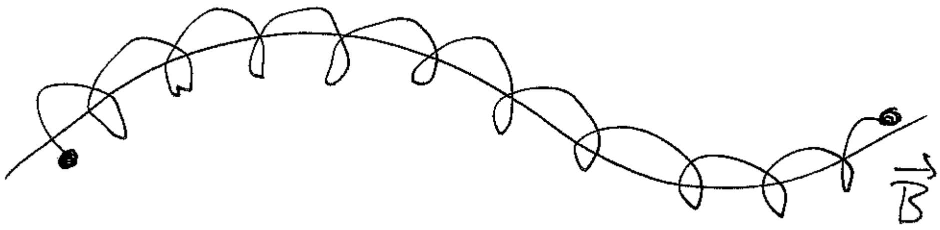
Still being actively studied now Perrone & Latter 2020, 2022 (Cambridge).

Current Astrophysical motivation is in understanding Galaxy clusters, one of the most famous problems with which is the "cooling flow problem" $\Rightarrow$ in brief, Galaxy clusters are hotter than they should be given the naïve cooling time and their age.

To model the MTI we will need to add one non-ideal ~~peice~~ piece of physics, the flow of heat. Described by vector $\vec{Q}$, and modifies entropy equation

$$\frac{P}{\gamma-1}\frac{D}{Dt}\ln\left(P\rho^{-\gamma}\right) = -\vec{\nabla}\cdot\vec{Q}$$

Galaxy clusters are full of dilute plasmas, meaning that the larmor radius is much smaller than the mean free path.



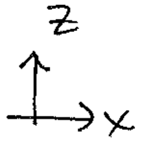
Particle undergoes many revolutions between collissions $\Rightarrow$ heat only conducted along field lines.

Normal heat conduction $\vec{Q} = -\kappa \vec{\nabla} T$, but now modify to

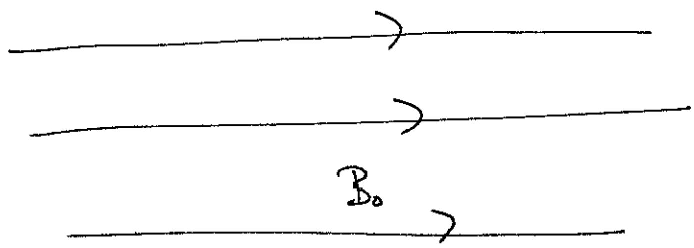
$$\vec{Q} = -\chi \hat{b}(\hat{b} \cdot \vec{\nabla})T - \psi \vec{\nabla} T$$

I will carry both χ & ψ through to "tag" the magnetic field, which will help with interpretation.

Set up simple equillibrium



$\downarrow \vec{g} = -g\hat{z}$



$\vec{B_0} = B_0\hat{x}$

$P = P_0(z)$

$\rho = \rho_0(z)$

$\vec{v}_0 = \vec{0}.$

Equillibrium described by Momentum equation

$$\vec{0} = -\vec{\nabla} P_0 + \rho_0\vec{g} + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B_0})\times\vec{B_0}$$

$$\Rightarrow \quad \vec{g} = +\frac{1}{\rho_0}\frac{\partial P_0}{\partial z}\hat{z} \qquad \left[\text{So } \frac{\partial P_0}{\partial z} < 0\right].$$

Perturb with the following

$$\vec{\xi} = \xi_z(t)\,e^{ikx}\,\hat{z}$$

[]

this means

$$\vec{\nabla}\cdot\vec{\xi} = \frac{\partial \xi_z}{\partial z} = 0$$

$$\Rightarrow \quad \delta\rho = -\vec{\nabla}\cdot(\rho_0\vec{\xi}) = -(\vec{\xi}\cdot\vec{\nabla})\rho_0$$

$$= -\xi_z(t)\,e^{ikx}\,\frac{\partial \rho_0}{\partial z}$$

Magnetic field perturbation

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B}) \Rightarrow \delta\vec{B} = \vec{\nabla}\times(\vec{\xi}\times\vec{B}_0)$$

$$\delta\vec{B} = B_0\,\xi_z(t)\,\vec{\nabla}\times\left(e^{ikx}\,\hat{y}\right)$$

$$= B_0\,\xi_z(t)\,(ik)\,e^{ikx}\,\hat{z}$$

Heat flux perturbation

$$\delta\vec{Q} = -\chi\,\delta\hat{b}\,(\hat{b}\cdot\vec{\nabla})T - \chi\,\hat{b}\,(\delta\hat{b}\cdot\vec{\nabla})T$$

$$-\chi\,\hat{b}\,(\hat{b}\cdot\vec{\nabla})\,\delta T - \psi\,\vec{\nabla}\,\delta T$$

note that $(\hat{b}\cdot\vec{\nabla})\,\delta T = 0$ as $\hat{b} = \hat{x}$, $T = T(z)$

so

$$\delta\vec{Q} = -\chi\,\hat{b}\,(\delta\hat{b}\cdot\vec{\nabla})T - \chi\,\hat{b}\,(\hat{b}\cdot\vec{\nabla})\,\delta T$$

$$-\psi\,\vec{\nabla}\,\delta T$$

Employ the "Boussinesq" approximation, where we perturb slowly $[t \gg l/c_s]$ so pressure equilibrium maintained. $\Rightarrow$ $\delta P = 0$.

$$\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla} P + \frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) \times \vec{B} + \rho \vec{g}$$

Perturb

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = \frac{1}{\mu_0}(\vec{\nabla} \times \delta\vec{B}) \times \vec{B_0} + \frac{1}{\mu_0}\cancel{(\vec{\nabla} \times \vec{B_0})}^{0} \times \delta\vec{B}$$

$$+ \delta\rho \vec{g}$$

$$\frac{\partial^2 \vec{\xi}}{\partial t^2} = \frac{1}{\mu_0 \rho}(\vec{\nabla} \times \delta\vec{B}) \times \vec{B_0} - \frac{\delta\rho}{\rho_0}\vec{g}$$

Use $\vec{g} = +\frac{1}{\rho_0}\frac{\partial P_0}{\partial z}\hat{z}$

and $\delta\vec{B} = ikB_0\vec{\xi}$

$$\frac{\partial^2 \vec{\xi}}{\partial t^2} = \frac{1}{\rho_0^2}\delta\rho\frac{\partial P_0}{\partial z}\hat{z} + \frac{ikB_0^2}{\mu_0\rho_0}(\vec{\nabla} \times \vec{\xi}) \times \hat{x}$$

$$= \frac{1}{\rho_0^2}\delta\rho\frac{\partial P_0}{\partial z}\hat{z} - \frac{k^2 B_0^2}{\mu_0\rho_0}\vec{\xi}$$

Finally, perturb entropy

$$\frac{P}{\gamma-1}\left[\frac{\partial}{\partial t}+(\vec{u}\cdot\vec{\nabla})\right]\ln(P\rho^{-\gamma}) = -\vec{\nabla}\cdot\vec{Q}$$

Perturb [$\delta P=0$]

$$\frac{P_0}{\gamma-1}\left[-\frac{\gamma}{\rho_0}\frac{\partial}{\partial t}\delta\rho + \frac{\partial \xi_z}{\partial t}\frac{\partial}{\partial z}\ln(P_0\rho_0^{-\gamma})\right] = -\vec{\nabla}\cdot\delta\vec{Q}$$

We now have everything in place to derive a dispersion relation.

Ideal gas

$$\frac{P}{\rho} = \frac{N k_B T}{m_p} \Rightarrow \frac{\delta T}{T} = -\frac{\delta\rho}{\rho} \quad [\delta P = 0].$$

<u>Manipulations</u>

Induction + Momentum

$$\frac{\partial^2 \xi_z}{\partial t^2} + k^2 V_A^2 \xi_z = \frac{1}{\rho_0^2}\,\delta\rho\,\frac{\partial P_0}{\partial z}$$

Entropy:

$$\frac{P_0}{\gamma-1}\left[-\frac{\gamma}{\rho_0}\frac{\partial}{\partial t}\delta\rho + \frac{\partial \xi_z}{\partial t}\frac{\partial}{\partial z}\ln(P\rho^{-\gamma})\right]$$

$$= +\vec{\nabla}\cdot\left[\chi\hat{b}(\hat{b}\cdot\vec{\nabla})\delta T + \chi\hat{b}(\delta\hat{b}\cdot\vec{\nabla})T + \psi\vec{\nabla}\delta T\right]$$

Define $\chi' \equiv \frac{(\gamma-1)\chi}{P}$, $\psi' \equiv \frac{(\gamma-1)\psi}{P}$

$$-\frac{\gamma}{\rho_0}\frac{\partial}{\partial t}\delta\rho + \frac{\partial \xi_z}{\partial t}\frac{\partial}{\partial z}\ln(P\rho^{-\gamma}) = \chi'\left[\vec{\nabla}\cdot(\hat{b}[\hat{b}\cdot\vec{\nabla}]\delta T) + \vec{\nabla}\cdot(\hat{b}[\delta\hat{b}\cdot\vec{\nabla}]T)\right] + \psi'\nabla^2\delta T$$

both $\vec{\nabla}\cdot\hat{b} = 0$ and $\vec{\nabla}\cdot\delta\hat{b} = 0$

leaving

$$-\frac{\gamma}{\rho_0}\frac{\partial}{\partial t}\delta\rho + \frac{\partial \xi_z}{\partial t}\frac{\partial}{\partial z}\ln(P_0\rho_0^{-\gamma})$$

$$= \chi' \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\delta T\right]$$

$$+ \chi' \frac{\partial}{\partial x}\left[ik\xi_z e^{ikx}\frac{\partial}{\partial z}T\right]$$

$$+ \chi' \frac{\partial^2}{\partial x^2}\delta T$$

use $\delta T = -\frac{T_0}{\rho_0}\delta\rho$

$$-\frac{\gamma}{\rho_0}\frac{\partial}{\partial t}\delta\rho + \frac{\partial \xi_z}{\partial t}\frac{\partial}{\partial z}\ln(P_0\rho_0^{-\gamma})$$

$$= +\frac{T_0\chi'}{\rho_0}k^2\delta\rho - \chi' k^2\xi_z\frac{\partial T}{\partial z}$$

$$+ k^2\chi'\frac{T_0}{\rho_0}\delta\rho$$

Now assume $e^{-i\omega t}$

Momentum:

$$-\omega^2 \xi_z + k^2 V_A^2 \xi_z = \frac{1}{\rho_0^2}\,\delta\rho\,\frac{\partial P_0}{\partial z}$$

Entropy:

$$i\omega\gamma\frac{\delta\rho}{\rho_0} \cancel{=} -i\omega\xi_z\frac{\partial}{\partial z}\ln(P_0\rho_0^{-\gamma})$$

$$= \frac{T_0}{\rho_0}\chi' k^2\delta\rho - \chi' k^2 \xi_z\frac{\partial T_0}{\partial z}$$

$$+ k^2\psi'\,\delta\rho/\rho_0$$

Use $\xi_z = \dfrac{1}{k^2V_A^2-\omega^2}\,\dfrac{\delta\rho}{\rho_0^2}\,\dfrac{\partial P_0}{\partial z}$

$$\Rightarrow\quad i\omega\gamma\frac{\delta\rho}{\rho_0} - \frac{i\omega\frac{\partial}{\partial z}\ln(P_0\rho_0^{-\gamma})}{k^2V_A^2-\omega^2}\cdot\frac{\delta\rho}{\rho_0^2}\frac{\partial P_0}{\partial z}$$

$$= T_0\chi' k^2\frac{\delta\rho}{\rho_0} - \frac{\chi' k^2\frac{\partial T}{\partial z}}{k^2V_A^2-\omega^2}\,\frac{\delta\rho}{\rho_0^2}\frac{\partial P_0}{\partial z}$$

$$+ k^2\psi' T_0\,\delta\rho/\rho_0$$

Dispersión

$$i\omega\gamma\left(k^2 V_A^2-\omega^2\right)-i\omega\frac{1}{\rho_0}\frac{\partial P_0}{\partial z}\frac{\partial}{\partial z}\ln\left(P_0\rho_0^{-\gamma}\right)$$

$$=k^2 T_0\left(\chi'+\psi'\right)\left(k^2 V_A^2-\omega^2\right)-\chi' k^2\frac{\partial T_0}{\partial z}\frac{1}{\rho_0}\frac{\partial P_0}{\partial z}$$

$$\Rightarrow i\omega^3-\frac{k^2 T_0}{\gamma}\left(\chi'+\psi'\right)\omega^2$$

$$+i\omega\left(\frac{1}{\gamma\rho_0}\frac{\partial P_0}{\partial z}\frac{\partial}{\partial z}\ln\left(P_0\rho_0^{-\gamma}\right)-k^2 V_A^2\right)$$

$$+\frac{k^2 T_0}{\gamma}\left[\left(\chi'+\psi'\right)k^2 V_A^2-\frac{\chi'}{\rho_0}\frac{\partial P_0}{\partial z}\frac{\partial \ln T_0}{\partial z}\right]=0$$

~~Take weak field limit~~

Take zero field limit $v_A \to 0$, $\chi' \to 0$.

$$i\varpi^3 - \frac{k^2 T_0}{\gamma} \psi' \varpi^2 + \frac{i\varpi}{\gamma} \frac{1}{\rho_0} \frac{\partial P_0}{\partial z} \frac{\partial}{\partial z} \ln(P_0 \rho_0^{-\gamma}) = 0$$

$$-(i\varpi)^3 + \frac{k^2 T_0}{\gamma} \psi' (i\varpi)^2 + (i\varpi) \frac{1}{\gamma \rho_0} \frac{\partial P_0}{\partial z} \frac{\partial}{\partial z} \ln(P_0 \rho_0^{-\gamma}) = 0$$

$$i\varpi = \frac{k^2 T_0 \psi'}{\gamma} \pm \sqrt{\frac{k^4 T_0^2 \psi'^2}{\gamma^2} + \frac{1}{\gamma \rho_0} \frac{\partial P_0}{\partial z} \frac{\partial}{\partial z} \ln(P_0 \rho_0^{-\gamma})}$$

Consider long wavelength modes $k \to 0$

$$\varpi = \pm i \sqrt{\frac{1}{\gamma \rho_0} \frac{\partial P_0}{\partial z} \frac{\partial}{\partial z} \ln(P_0 \rho_0^{-\gamma})}$$

if $\frac{\partial}{\partial z} \ln(P_0 \rho_0^{-\gamma}) < 0$

then $\varpi \in \mathbb{C}$

$\Rightarrow$ unstable.

Take weak field limit

$$v_A \to 0 \quad \chi' \neq 0. \quad \psi' \to 0.$$

$$i\varpi^3 - \frac{k^2 T_0}{\gamma}\chi'\varpi^2 + i\varpi\frac{1}{\gamma\rho_0}\frac{\partial P_0}{\partial z}\frac{\partial}{\partial z}\ln(P_0\rho_0^{-\gamma})$$

$$- \frac{k^2 T_0}{\gamma}\frac{\chi'}{\rho_0}\frac{\partial P_0}{\partial z}\frac{\partial \ln T_0}{\partial z} = 0$$

there are the same entropy gradient instabilities available for $k \to 0$, but what if $\frac{\partial s}{\partial z} > 0$?

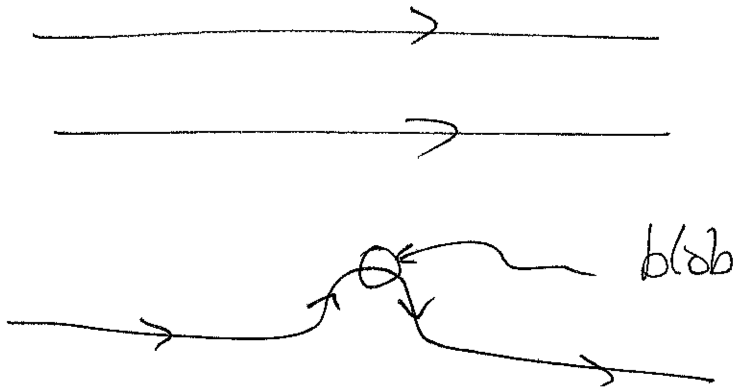
Consider small $|\varpi|$, then

$$\varpi \simeq -i\chi' k^2 \frac{\partial T}{\partial z}\left[\frac{\partial}{\partial z}\ln(P_0\rho_0^{-\gamma})\right]^{-1}$$

if $\frac{\partial T}{\partial z} < 0 \Rightarrow -i\varpi$ is real & positive.

$\Rightarrow$ instability.

What is the physics?



blob moved upwards, drags field with it.

temperature only conducts along field, so pertubation is isothermal.

but $\frac{dT}{dz} < 0$ so gas now hotter than surroundings, so buyant.

$\Rightarrow$ keeps rising.