4. Conservation of energy

Recap: we have

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

$$\rho\left[\frac{\partial}{\partial t} + (\vec{u}\cdot\vec{\nabla})\right]\vec{u} = -\vec{\nabla}P + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B})$$

which is 7 equations for 8 unknowns $(\rho, P, \vec{u}, \vec{B})$.

To close the system we need an equation for the pressure P. This is conservation of energy.

We will begin by writing down the total energy density of the plasma, and then consider the rate of change of some components of this energy individually. This is because (e.g.) kinetic energy conservation is intimately linked to momentum conservation.

Energy density

$$\mathcal{E} = \frac{1}{2}\rho v^2 + \rho e + \frac{B^2}{2\mu_0} + \frac{1}{2}\varepsilon_0 E^2$$

internal energy per unit mass

1. $\Rightarrow$ $E \sim vB \Rightarrow \dfrac{\frac{1}{2}\varepsilon_0 E^2}{\frac{1}{2\mu_0}B^2} \sim \varepsilon_0\mu_0 v^2 \sim O\left(\dfrac{v^2}{c^2}\right)$

<u>ignore</u>.

2. $\Rightarrow$ assume perfect gas

$$PV = NkT \Rightarrow \frac{P}{\rho} = \frac{kT}{m}$$

Recall some classic thermodynamics

heat capacity @ constant volume per mole $\equiv C_V = \dfrac{R}{\gamma-1}$

$$\gamma \equiv C_P / C_V$$

heat capacity @ constant P

Internal energy of one mole of gas

$$U = C_V T = \frac{RT}{\gamma-1} = \frac{N_A kT}{\gamma-1}$$

definition of e

$$\Rightarrow e \equiv \frac{U}{mN_A} = \frac{kT}{m(\gamma-1)} = \frac{P}{\rho(\gamma-1)}$$

$$\therefore \quad \mathcal{E} = \frac{1}{2}\rho v^2 + \frac{P}{\gamma - 1} + \frac{B^2}{2\mu_0}.$$

and so, obviously

$$\underbrace{\frac{d\mathcal{E}}{dt}}_{\text{rate of change of total energy}} = \underbrace{\frac{d}{dt}}_{\text{rate of change of kinetic energy}}\left(\frac{1}{2}\rho v^2\right) + \underbrace{\frac{d}{dt}}_{\text{rate of change of thermal energy}}\left(\frac{P}{\gamma-1}\right) + \underbrace{\frac{d}{dt}}_{\text{rate of change of magnetic energy.}}\left(\frac{B^2}{2\mu_0}\right)$$

Start with rate of change of kinetic energy.

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \frac{v^2}{2}\frac{\partial \rho}{\partial t} + \rho \vec{v} \cdot \underbrace{\frac{\partial \vec{v}}{\partial t}}_{\text{momentum conservation}}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \frac{v^2}{2}\frac{\partial \rho}{\partial t} + \rho \vec{v} \cdot \left[-(\vec{v}\cdot\vec{\nabla})\vec{v} - \frac{1}{\rho}\vec{\nabla}P + \frac{1}{\rho}\vec{f} \right]$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \frac{v^2}{2}\frac{\partial \rho}{\partial t} \cancel{+} - \rho\vec{v}\cdot\left[\vec{v}\cdot\vec{\nabla}\right]\vec{v} - \vec{v}\cdot\vec{\nabla}P + \vec{v}\cdot\vec{f}$$

let's massage those terms:

$$-\rho\vec{v}\cdot[\vec{v}\cdot\vec{\nabla}]\vec{v} = -\frac{1}{2}\rho[\vec{v}\cdot\vec{\nabla}]v^2$$

$$= -\frac{1}{2}\rho\vec{v}\cdot(\vec{\nabla}v^2)$$

$$= -\frac{1}{2}\rho\left[\vec{\nabla}\cdot(v^2\vec{v}) - v^2\vec{\nabla}\cdot\vec{v}\right]$$

$$= \frac{1}{2}\rho v^2(\vec{\nabla}\cdot\vec{v}) - \frac{1}{2}\rho\,\vec{\nabla}\cdot(v^2\vec{v})$$

and

$$= \frac{1}{2}\rho v^2(\vec{\nabla}\cdot\vec{v}) - \frac{1}{2}\vec{\nabla}\cdot(\rho v^2\vec{v}) + \frac{1}{2}v^2(\vec{v}\cdot\vec{\nabla})\rho$$

$$-\vec{v}\cdot\vec{\nabla}P = -\vec{\nabla}\cdot(\vec{v}P) + P(\vec{\nabla}\cdot\vec{v})$$

and

$$\vec{v}\cdot\vec{f} = \vec{v}\cdot(\vec{J}\times\vec{B}) \overset{\text{scalar triple product}}{=} -\vec{J}\cdot(\vec{v}\times\vec{B})$$

$$= \vec{J}\cdot\vec{E}$$

$$(\text{from } \vec{E} = -\vec{v}\times\vec{B})$$

$$\therefore\ \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \frac{v^2}{2}\frac{\partial\rho}{\partial t} + \frac{1}{2}\rho v^2(\vec{\nabla}\cdot\vec{v}) - \frac{1}{2}\vec{\nabla}\cdot(\rho v^2\vec{v})$$

$$+ \frac{1}{2}v^2(\vec{v}\cdot\vec{\nabla})\rho - \vec{\nabla}\cdot(\vec{v}P)$$

$$+ P(\vec{\nabla}\cdot\vec{v}) + \vec{J}\cdot\vec{E}$$

$$= -\vec{\nabla}\cdot\left(\frac{1}{2}\rho v^2\vec{v} + P\vec{v}\right) + P(\vec{\nabla}\cdot\vec{v}) + \vec{J}\cdot\vec{E}$$

$$+ \frac{1}{2}v^2\underbrace{\left[\frac{\partial\rho}{\partial t} + \rho(\vec{\nabla}\cdot\vec{v}) + (\vec{v}\cdot\vec{\nabla})\rho\right]}_{=0 \text{ by mass}}$$

and therefore

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\vec{\nabla}\cdot\left(\frac{1}{2}\rho v^2\vec{v} + P\vec{v}\right) + P(\vec{\nabla}\cdot\vec{v}) + \vec{j}\cdot\vec{E}$$

Let's integrate this over the volume

$$\frac{d}{dt}\iiint_V \frac{1}{2}\rho v^2\, dV = -\iiint_V \vec{\nabla}\cdot\left(\frac{1}{2}\rho v^2\vec{v} + P\vec{v}\right) dV + \iiint_V P(\vec{\nabla}\cdot\vec{v})\, dV + \iiint_V \vec{j}\cdot\vec{E}\, dV$$

$\therefore$

$$\frac{d}{dt}\iiint_V \frac{1}{2}\rho v^2\, dV = \overbrace{-\oint_{\partial V}\frac{1}{2}\rho v^2\,\vec{v}\cdot d\vec{S}}^{\text{Kinetic energy flux out of volume}} \overbrace{-\oint P\, d\vec{S}\cdot\vec{v}}^{\text{work done by pressure forces}}$$

$$+\underbrace{\iiint_V P(\vec{\nabla}\cdot\vec{v})\, dV}_{\text{Compressional heating}} + \underbrace{\iiint_V \vec{j}\cdot\vec{E}\, dV}_{\text{energy exchange with electromagnetic fields.}}$$

Magnetic energy

We have

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} \qquad (M3)$$

$$\therefore \quad \frac{\vec{B}}{\mu_0} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2\mu_0} B^2\right) = -\frac{1}{\mu_0} \vec{B} \cdot \left(\vec{\nabla} \times \vec{E}\right)$$

but $\vec{B} \cdot \left(\vec{\nabla} \times \vec{E}\right) = B_i \, \varepsilon_{ijk} \, \partial_j E_k$

$$= \varepsilon_{ijk} \, \partial_j (B_i E_k) - \varepsilon_{ijk} E_k \, \partial_j B_i$$

$$= \underset{\text{(one swap)}}{-\varepsilon_{jik} \, \partial_j (B_i E_k)} + \underset{\text{(three swaps)}}{\varepsilon_{kji} E_k \, \partial_j B_i}$$

$$= -\vec{\nabla} \cdot \left(\vec{B} \times \vec{E}\right) + \vec{E} \cdot \left(\vec{\nabla} \times \vec{B}\right)$$

$$\therefore \quad \frac{\partial}{\partial t}\left(\frac{1}{2\mu_0} B^2\right) = \vec{\nabla} \cdot \underbrace{\left(\frac{\vec{B} \times \vec{E}}{\mu_0}\right)}_{-\vec{P}} - \vec{E} \cdot \underbrace{\left(\frac{1}{\mu_0} \vec{\nabla} \times \vec{B}\right)}_{\vec{J}}$$

$\vec{P} \equiv$ Poynting vector $\equiv \dfrac{\vec{E} \times \vec{B}}{\mu_0}$

$$\therefore \quad \frac{\partial}{\partial t}\left(\frac{1}{2\mu_0} B^2\right) = -\vec{\nabla} \cdot \vec{P} - \vec{J} \cdot \vec{E}$$

Integrate over volume

$$\frac{d}{dt}\iiint_{\mathcal{V}} \frac{B^2}{2\mu_0}\, d\mathcal{V} = -\iiint_{\mathcal{V}} \vec{\nabla}\cdot\vec{P}\, d\mathcal{V} - \iiint_{\mathcal{V}} \vec{J}\cdot\vec{E}\, d\mathcal{V}$$

divergence theorem

$$\frac{d}{dt}\iiint_{\mathcal{V}} \frac{B^2}{2\mu_0}\, d\mathcal{V} = \underbrace{-\oint_{\delta\mathcal{V}} \vec{P}\cdot d\vec{S}}_{\substack{\text{magnetic energy}\\ \text{transported through}\\ \text{the boundary by}\\ \text{Poynting flux}}} \underbrace{- \iiint_{\mathcal{V}} \vec{J}\cdot\vec{E}\, d\mathcal{V}}_{\substack{\text{Note opposite sign}\\ \text{to kinetic energy}\\ \text{eq}^n\text{! Energy}\\ \text{exchange with}\\ \text{the flow.}}}$$

Thermal energy: we know kinetic energy and electromagnetic fields communicate through $\vec{J}\cdot\vec{E}$. How does thermal energy communicate?

Well, recall kinetic energy conservation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\vec{\nabla}\cdot\left(\frac{1}{2}\rho v^2\vec{v} + P\vec{v}\right)$$
$$+ P(\vec{\nabla}\cdot\vec{v}) + \vec{J}\cdot\vec{E}$$

as all of magnetic energy communicated through $\vec{J}\cdot\vec{E}$ with kinetic energy, then thermal energy can only talk to ρv^2 (via pressure forces).

$\therefore$ Conserve kinetic + thermal energy

$$\frac{d}{dt}\iiint_V \left(\frac{1}{2}\rho v^2 + \rho e\right) dV = -\oint_{\delta V}\left(\frac{1}{2}\rho v^2 + \rho e\right)\vec{u}\cdot d\vec{S}$$
$$- \underbrace{\oint_{\delta V} P\, d\vec{S}\cdot\vec{u}}_{\text{work done by pressure forces.}}$$

Divergence theorem

$$\frac{d}{dt}\iiint_V \left(\frac{1}{2}\rho v^2 + \rho e\right) dV = -\iiint_V \vec{\nabla}\cdot\left(\left[\frac{1}{2}\rho v^2 + \rho e\right]\vec{u}\right) dV$$
$$- \iiint \vec{\nabla}\cdot(P\vec{u})\, dV$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho e\right) = -\vec{\nabla}\cdot\left(\left[\frac{1}{2}\rho v^2 + \rho e\right]\vec{u}\right) - \vec{\nabla}\cdot(P\vec{u})$$

We can then subtract off the kinetic energy equation (in the absence of $\vec{J}\cdot\vec{E}$, as that communicates only with EM fields), which leaves

$$\frac{\partial}{\partial t}(\rho e) = -\vec{\nabla}\cdot\left[(\rho e)\vec{u}\right] - P(\vec{\nabla}\cdot\vec{u})$$

we see that, unsurprisingly, the kinetic and thermal energies communicate through compressional heating.

Then we sum and get the total energy equation:

$$\frac{\partial}{\partial t}\left[\rho e + \frac{1}{2}\rho v^2 + \frac{B^2}{2\mu_0}\right] = -\vec{\nabla}\cdot\left[\left(\rho e + \frac{1}{2}\rho v^2 + P\right)\vec{u} + \vec{P}\right]$$

As this is a pure divergence, integrating over a volume which extends to infinity $\Rightarrow$ total energy conserved.

final manipulations of internal energy

$$\frac{\partial}{\partial t}(\rho e) = -\vec{\nabla}\cdot[(\rho e)\vec{u}] - P(\vec{\nabla}\cdot\vec{u})$$

Recall that for ideal gas $e = \dfrac{P}{\rho(\gamma-1)}$

$$\therefore \frac{1}{\gamma-1}\frac{\partial P}{\partial t} = -\vec{\nabla}\cdot\left(\frac{P\vec{u}}{\gamma-1}\right) - P(\vec{\nabla}\cdot\vec{u})$$

$$= -\frac{1}{\gamma-1}(\vec{u}\cdot\vec{\nabla})P - \frac{\gamma}{\gamma-1}P(\vec{\nabla}\cdot\vec{u})$$

$$\Rightarrow \frac{1}{\gamma-1}\left[\frac{\partial}{\partial t} + (\vec{u}\cdot\vec{\nabla})\right]P \equiv \frac{1}{\gamma-1}\frac{DP}{Dt} = -\frac{\gamma}{\gamma-1}P(\vec{\nabla}\cdot\vec{u})$$

Recall mass conservation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = \left[\frac{\partial}{\partial t} + (\vec{u}\cdot\vec{\nabla})\right]\rho = -\rho(\vec{\nabla}\cdot\vec{u})$$

$$\Rightarrow (\vec{\nabla}\cdot\vec{u}) = -\frac{1}{\rho}\frac{D\rho}{Dt}$$

combining

$$\frac{1}{P}\frac{DP}{Dt} = \frac{\gamma}{\rho}\frac{D\rho}{Dt}$$

$$\Rightarrow \frac{D}{Dt}\ln P - \gamma\frac{D}{Dt}\ln\rho = 0$$

$$\Rightarrow \frac{D}{Dt}\ln\left(P/\rho^{\gamma}\right) = 0$$

which means (ideal) MHD conserves the quantity $\underline{\underline{P\rho^{-\gamma}}}$.

This is the entropy of the fluid.

Fundamental thermodynamic relationship

$$de = Tds - pdV \quad [\text{per unit mass so } V = 1/\rho]$$

$$d\left(\frac{P}{\rho(\gamma-1)}\right) = Tds - p\,d(1/\rho)$$

$$\frac{1}{\rho(\gamma-1)}dP - \frac{P}{\rho^2(\gamma-1)}d\rho = TdS + \frac{P}{\rho^2}d\rho$$

$$TdS = \frac{1}{\gamma-1}\left[\frac{dP}{\rho} - \frac{\gamma P d\rho}{\rho^2}\right]$$

$$= \frac{P}{\rho(\gamma-1)}\left[\frac{dP}{P} - \gamma\frac{d\rho}{\rho}\right]$$

ideal gas $\quad P = \frac{k_B}{m}\rho T$

$$TdS = \frac{k_B T}{m(\gamma-1)}\left[\frac{dP}{P} - \gamma\frac{d\rho}{\rho}\right]$$

$$\Rightarrow \quad S = \frac{k_B}{m(\gamma-1)}\ln\left[P\rho^{-\gamma}\right].$$

The enthalpy equation

Define $h \equiv e + P/\rho + \frac{1}{2}v^2$

Conservation of energy then

$$\frac{\partial}{\partial t}\left(\rho e + \frac{1}{2}\rho v^2 + \frac{B^2}{2\mu_0}\right) = -\vec{\nabla}\cdot\left(\rho h \vec{v} + \vec{P}\right)$$

Add $\frac{\partial P}{\partial t}$ to both sides

$$\frac{\partial}{\partial t}\left(\rho h + \frac{B^2}{2\mu_0}\right) = -\vec{\nabla}\cdot(\rho h \vec{v}) - \vec{\nabla}\cdot\vec{P} + \frac{\partial P}{\partial t}$$

expand

zero by mass conservation

$$\rho\frac{\partial h}{\partial t} + h\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial t}\left(B^2/2\mu_0\right) = -h\vec{\nabla}\cdot(\rho\vec{v}) + (\rho\vec{v}\cdot\vec{\nabla})h - \vec{\nabla}\cdot\vec{P} + \frac{\partial P}{\partial t}$$

$$\boxed{\rho\left[\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right]h + \frac{\partial}{\partial t}\left(B^2/2\mu_0\right) = -\vec{\nabla}\cdot\vec{P} + \frac{\partial P}{\partial t}}$$

In steady flow $[\partial/\partial t = 0]$ we find

$$\frac{dh}{dt} = -\frac{1}{\rho}\vec{\nabla}\cdot\vec{P}$$

this is M.H.D version of Bernoulli's theorem:

[i.e. $h = e + P/\rho + \frac{1}{2}v^2$ constant along streamlines].

The hyperbolic structure of ideal MHD

Writing

$$\frac{\partial \rho}{\partial t} + (\vec{u}\cdot\vec{\nabla})\rho + \rho(\vec{\nabla}\cdot\vec{u}) = 0$$

$$\frac{\partial P}{\partial t} + (\vec{u}\cdot\vec{\nabla})P + \gamma p(\vec{\nabla}\cdot\vec{u}) = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} + \frac{1}{\rho}\vec{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) - \frac{1}{\rho\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B} = \vec{0}$$

$$\frac{\partial \vec{B}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{B} - (\vec{B}\cdot\vec{\nabla})\vec{u} + \vec{B}(\vec{\nabla}\cdot\vec{u}) = \vec{0}$$

We can write

$$\frac{\partial \vec{\psi}}{\partial t} + \underline{\underline{A_i}}\,\frac{\partial \vec{\psi}}{\partial x_i} = \vec{0}$$

where $\vec{\psi} = [\rho, P, u_x, u_y, u_z, B_x, B_y, B_z]$

and (e.g.)

$$\underline{\underline{A_x}} = \begin{bmatrix} u_x & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & u_x & \gamma p & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\rho & u_x & 0 & 0 & 0 & \frac{B_y}{\mu_0\rho} & \frac{B_z}{\mu_0\rho} \\ 0 & 0 & 0 & u_x & 0 & 0 & -\frac{B_z}{\mu_0\rho} & 0 \\ 0 & 0 & 0 & 0 & u_x & 0 & 0 & -\frac{B_z}{\mu_0\rho} \\ 0 & 0 & 0 & 0 & 0 & u_x & 0 & 0 \\ 0 & 0 & B_y & -B_x & 0 & 0 & u_x & 0 \\ 0 & 0 & B_z & 0 & -B_x & 0 & 0 & u_x \end{bmatrix}$$