The magneto-thermal instability (MTI)

Discovered by Steven Balbus in 2000, 2001.
(a man of many instabilities).

Numerically explored by Parrish & Jim Stone 2007.

Extended by Quataert 2008.

[Clearly to be a named chair in Oxford, Princeton or IAS you need to study this instability].

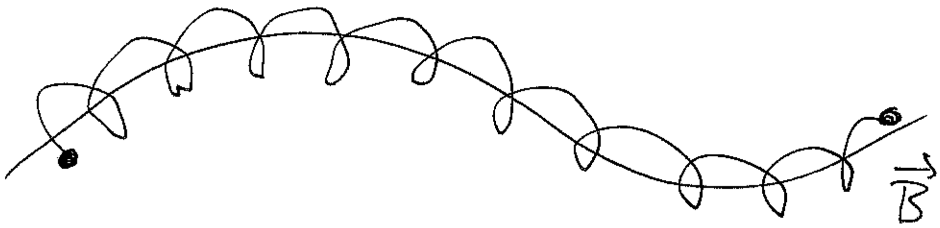
Still being actively studied now Perrone & Latter 2020, 2022 (Cambridge).

Current ~~Astrophysical~~ motivation is in understanding Galaxy clusters, one of the most famous problems with which is the "cooling flow problem" ⇒ in brief, Galaxy clusters are hotter than they should be given the naive cooling time and their age.

To model the MTI we will need to add one non-ideal ~~peice~~ piece of Physics, the flow of heat. Described by vector $\vec{Q}$, and modifies entropy equation

$$\frac{P}{\gamma-1}\frac{D}{Dt}\ln\left(P\rho^{-\gamma}\right) = -\vec{\nabla}\cdot\vec{Q}$$

Galaxy clusters are full of dilute plasmas, meaning that the larmor radius is much smaller than the mean free path.



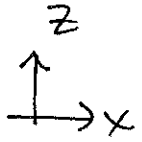
Particle undergoes many revolutions between collissions $\Rightarrow$ heat only conducted along field lines.

Normal heat conduction $\vec{Q} = -\kappa \vec{\nabla} T$, but now modify to

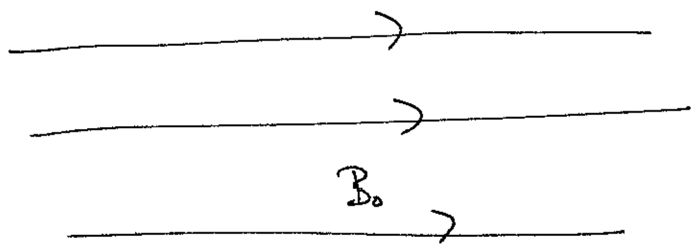
$$\vec{Q} = -\chi \hat{b} (\hat{b} \cdot \vec{\nabla}) T - \Psi \vec{\nabla} T$$

I will carry both χ & Ψ through to "tag" the magnetic field, which will help with interpretation.

Set up simple equillibrium



$\downarrow \vec{g} = -g\hat{z}$



$\vec{B_0} = B_0\hat{x}$

$P = P_0(z)$

$\rho = \rho_0(z)$

$\vec{v}_0 = \vec{0}$.

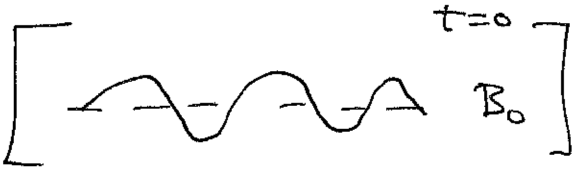
Equillibrium described by momentum equation

$$\vec{0} = -\vec{\nabla} P_0 + \rho_0\vec{g} + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B_0})\times\vec{B_0}$$

$$\Rightarrow \quad \vec{g} = +\frac{1}{\rho_0}\frac{\partial P_0}{\partial z}\hat{z} \qquad \left[\text{So } \frac{\partial P_0}{\partial z} < 0\right].$$

Perturb with the following

$$\vec{\xi} = \xi_z(t)\, e^{ikx}\,\hat{z}$$



this means

$$\vec{\nabla}\cdot\vec{\xi} = \frac{\partial \xi_z}{\partial z} = 0$$

$$\Rightarrow \quad \delta\rho = -\vec{\nabla}\cdot(\rho_0\vec{\xi}) = -(\vec{\xi}\cdot\vec{\nabla})\rho_0$$

$$= -\xi_z(t)\, e^{ikx}\frac{\partial \rho_0}{\partial z}$$

Magnetic field perturbation

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B}) \Rightarrow \delta \vec{B} = \vec{\nabla} \times (\vec{\xi} \times \vec{B}_0)$$

$$\delta \vec{B} = B_0 \xi_z(t) \vec{\nabla} \times (e^{ikx} \hat{y})$$

$$= B_0 \xi_z(t) (ik) e^{ikx} \hat{z}$$

Heat flux perturbation

$$\delta \vec{Q} = -\chi \delta\hat{b} (\hat{b} \cdot \vec{\nabla}) T - \chi \hat{b} (\delta\hat{b} \cdot \vec{\nabla}) T$$

$$- \chi \hat{b} (\hat{b} \cdot \vec{\nabla}) \delta T - \psi \vec{\nabla} \delta T$$

note that $(\hat{b} \cdot \vec{\nabla}) \delta T = 0$ as $\hat{b} = \hat{x}$, $T = T(z)$

So

$$\delta \vec{Q} = -\chi \hat{b} (\delta\hat{b} \cdot \vec{\nabla}) T - \chi \hat{b} (\hat{b} \cdot \vec{\nabla}) \delta T$$

$$- \psi \vec{\nabla} \delta T$$

Employ the "Boussinesq" approximation, where we perturb slowly $[t \gg L/c_s]$ so pressure equilibrium maintained. $\Rightarrow$ $\delta P = 0$.

$$\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla} P + \frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) \times \vec{B} + \rho \vec{g}$$

Perturb

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = \frac{1}{\mu_0}(\vec{\nabla} \times \delta\vec{B}) \times \vec{B}_0 + \frac{1}{\mu_0}\cancel{(\vec{\nabla} \times \vec{B}_0)}^{\,0} \times \delta\vec{B}$$
$$+ \delta\rho \vec{g}$$

$$\frac{\partial^2 \vec{\xi}}{\partial t^2} = \frac{1}{\mu_0 \rho}(\vec{\nabla} \times \delta\vec{B}) \times \vec{B}_0 - \frac{\delta\rho}{\rho_0} \vec{g}$$

Use $\quad \vec{g} = +\frac{1}{\rho_0} \frac{\partial P_0}{\partial z} \hat{z}$

and $\quad \delta\vec{B} = ikB_0 \vec{\xi}$

$$\frac{\partial^2 \vec{\xi}}{\partial t^2} = \frac{1}{\rho_0^2} \delta\rho \frac{\partial P_0}{\partial z} \hat{z} + \frac{ikB_0^2}{\mu_0 \rho_0} (\vec{\nabla} \times \vec{\xi}) \times \hat{x}$$

$$= \frac{1}{\rho_0^2} \delta\rho \frac{\partial P_0}{\partial z} \hat{z} - \frac{k^2 B_0^2}{\mu_0 \rho_0} \vec{\xi}$$

Finally, perturb entropy

$$\frac{P}{\gamma-1}\left[\frac{\partial}{\partial t}+(\vec{u}\cdot\vec{\nabla})\right]\ln(P\rho^{-\gamma}) = -\vec{\nabla}\cdot\vec{Q}$$

Perturb $[\delta P=0]$

$$\frac{P_0}{\gamma-1}\left[-\frac{\gamma}{\rho_0}\frac{\partial}{\partial t}\delta\rho+\frac{\partial \xi_z}{\partial t}\frac{\partial}{\partial z}\ln(P_0\rho_0^{-\gamma})\right] = -\vec{\nabla}\cdot\delta\vec{Q}$$

We now have everything in place to derive a dispersion relation.

Ideal gas

$$\frac{P}{\rho} = \frac{N k_B T}{m_p} \Rightarrow \frac{\delta T}{T} = -\frac{\delta\rho}{\rho} \quad [\delta P = 0].$$

Manipulations

Induction + Momentum

$$\frac{\partial^2 \xi_z}{\partial t^2} + k^2 V_A^2 \xi_z = \frac{1}{\rho_0^2} \delta\rho \frac{\partial P_0}{\partial z}$$

Entropy:

$$\frac{P_0}{\gamma-1}\left[-\frac{\gamma}{\rho_0}\frac{\partial}{\partial t}\delta\rho + \frac{\partial \xi_z}{\partial t}\frac{\partial}{\partial z}\ln(P\rho^{-\gamma})\right]$$

$$= +\vec{\nabla}\cdot\left[\chi\hat{b}(\hat{b}\cdot\vec{\nabla})\delta T + \chi\hat{b}(\delta\hat{b}\cdot\vec{\nabla})T + \psi\vec{\nabla}\delta T\right]$$

Define $\chi' \equiv \frac{(\gamma-1)\chi}{P}$, $\psi' \equiv \frac{(\gamma-1)\psi}{P}$

$$-\frac{\gamma}{\rho_0}\frac{\partial}{\partial t}\delta\rho + \frac{\partial \xi_z}{\partial t}\frac{\partial}{\partial z}\ln(P\rho^{-\gamma}) = \chi'\left[\vec{\nabla}\cdot(\hat{b}[\hat{b}\cdot\vec{\nabla}]\delta T) + \vec{\nabla}\cdot(\hat{b}[\delta\hat{b}\cdot\vec{\nabla}]T)\right] + \psi'\nabla^2\delta T$$

both $\vec{\nabla} \cdot \hat{b} = 0$ and $\vec{\nabla} \cdot \delta\hat{b} = 0$

leaving

$$-\frac{\gamma}{\rho_0}\frac{\partial}{\partial t}\delta\rho + \frac{\partial \xi_z}{\partial t}\frac{\partial}{\partial z}\ln(P\rho^{-\gamma})$$

$$= \chi' \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\delta T\right]$$

$$+ \chi' \frac{\partial}{\partial x}\left[ik\xi_z e^{ikx}\frac{\partial}{\partial z}T\right]$$

$$+ \varphi' \frac{\partial^2}{\partial x^2}\delta T$$

use $\delta T = -\frac{T_0}{\rho_0}\delta\rho$

$$-\frac{\gamma}{\rho_0}\frac{\partial}{\partial t}\delta\rho + \frac{\partial \xi_z}{\partial t}\frac{\partial}{\partial z}\ln(P_0\rho_0^{-\gamma})$$

$$= +\frac{T_0\chi'}{\rho_0}k^2\delta\rho - \chi' k^2\xi_z\frac{\partial T}{\partial z}$$

$$+ k^2\varphi'\frac{T_0}{\rho_0}\delta\rho$$

Now assume $e^{-i\omega t}$

Momentum:

$$-\omega^2 \xi_z + k^2 V_A^2 \xi_z = \frac{1}{\rho_0^2}\,\delta\rho\,\frac{\partial P_0}{\partial z}$$

Entropy:

$$i\omega\gamma\,\frac{\delta\rho}{\rho_0} \cancel{=} -i\omega\,\xi_z\,\frac{\partial}{\partial z}\ln\left(P_0\rho_0^{-\gamma}\right)$$

$$= \frac{T_0}{\rho_0}\chi' k^2\,\delta\rho - \chi' k^2\,\xi_z\,\frac{\partial T_0}{\partial z} + k^2\psi'\,\delta\rho/\rho_0$$

Use $\xi_z = \frac{1}{k^2V_A^2-\omega^2}\,\frac{\delta\rho}{\rho_0^2}\,\frac{\partial P_0}{\partial z}$

$$\Rightarrow\quad i\omega\gamma\,\frac{\delta\rho}{\rho_0} - \frac{i\omega\,\frac{\partial}{\partial z}\ln\left(P_0\rho_0^{-\gamma}\right)}{k^2V_A^2-\omega^2}\cdot\frac{\delta\rho}{\rho_0^2}\,\frac{\partial P_0}{\partial z}$$

$$= T_0\chi' k^2\,\frac{\delta\rho}{\rho_0} - \frac{\chi' k^2\,\frac{\partial T}{\partial z}}{k^2V_A^2-\omega^2}\,\frac{\delta\rho}{\rho_0^2}\,\frac{\partial P_0}{\partial z} + k^2\psi'\,T_0\,\delta\rho/\rho_0$$

Dispersion

$$i\omega\gamma\left(k^2V_A^2-\omega^2\right) - i\omega\,\frac{1}{\rho_0}\frac{\partial P_0}{\partial z}\frac{\partial}{\partial z}\ln\left(P_0\rho_0^{-\gamma}\right)$$

$$= k^2T_0(\chi'+\psi')\left(k^2V_A^2-\omega^2\right) - \chi' k^2\frac{\partial T_0}{\partial z}\frac{1}{\rho_0}\frac{\partial P_0}{\partial z}$$

$$\Rightarrow i\omega^3 - \frac{k^2T_0}{\gamma}(\chi'+\psi')\omega^2$$

$$+ i\omega\left(\frac{1}{\gamma\rho_0}\frac{\partial P_0}{\partial z}\frac{\partial}{\partial z}\ln\left(P_0\rho_0^{-\gamma}\right) - k^2V_A^2\right)$$

$$+ \frac{k^2T_0}{\gamma}\left[(\chi'+\psi')k^2V_A^2 - \frac{\chi'}{\rho_0}\frac{\partial P_0}{\partial z}\frac{\partial \ln T_0}{\partial z}\right] = 0$$

~~Take weak field limit~~

Take zero field limit $v_A \to 0$, $\chi' \to 0$.

$$i\varpi^3 - \frac{k^2 T_0}{\gamma} \psi' \varpi^2 + \frac{i\varpi}{\gamma} \frac{1}{\rho_0} \frac{\partial P_0}{\partial z} \frac{\partial}{\partial z} \ln(P_0 \rho_0^{-\gamma}) = 0$$

$$-(i\varpi)^3 + \frac{k^2 T_0}{\gamma} \psi' (i\varpi)^2 + (i\varpi) \frac{1}{\gamma\rho_0} \frac{\partial P_0}{\partial z} \frac{\partial}{\partial z} \ln(P_0 \rho_0^{-\gamma}) = 0$$

$$i\varpi = \frac{k^2 T_0 \psi'}{\gamma} \pm \sqrt{\frac{k^4 T_0^2 \psi'^2}{\gamma^2} + \frac{1}{\gamma\rho_0} \frac{\partial P_0}{\partial z} \frac{\partial}{\partial z} \ln(P_0 \rho_0^{-\gamma})}$$

Consider long wave length modes $k \to 0$

$$\varpi = \pm i \sqrt{\frac{1}{\gamma\rho_0} \frac{\partial P_0}{\partial z} \frac{\partial}{\partial z} \ln(P_0 \rho_0^{-\gamma})}$$

if $\dfrac{\partial}{\partial z} \ln(P_0 \rho_0^{-\gamma}) < 0$

then $\varpi \in \mathbb{C}$

$\Rightarrow$ unstable.

Take weak field limit

$v_A \to 0 \quad \chi' \neq 0 . \quad \psi' \to 0 .$

$$i\varpi^3 - \frac{k^2 T_0}{\gamma}\chi'\varpi^2 + i\varpi \frac{1}{\gamma\rho_0}\frac{\partial P_0}{\partial z}\frac{\partial}{\partial z}\ln(P_0\rho_0^{-\gamma})$$

$$- \frac{k^2 T_0}{\gamma}\frac{\chi'}{\rho_0}\frac{\partial P_0}{\partial z}\frac{\partial \ln T_0}{\partial z} = 0$$

there are the same entropy gradient instabilities available for $k \to 0$, but what if $\frac{\partial s}{\partial z} > 0$?

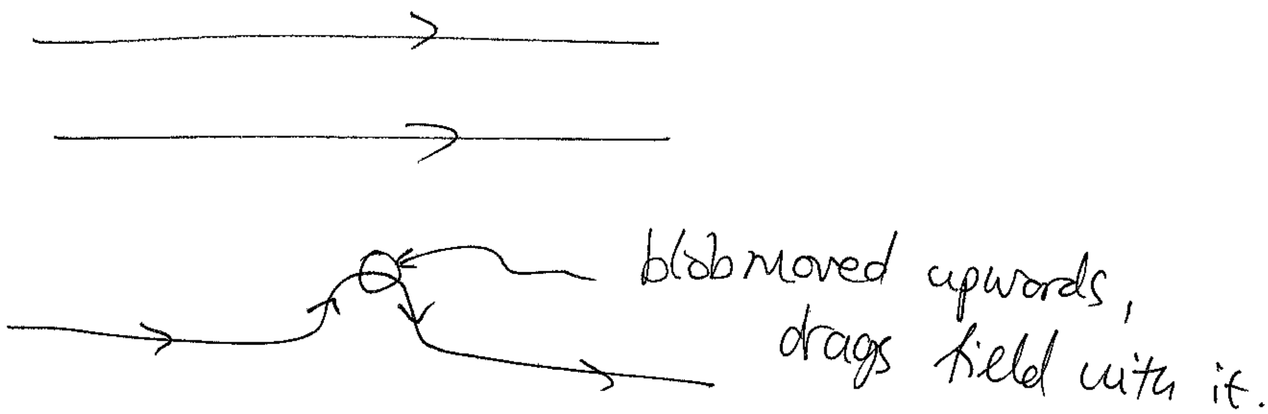
Consider small $|\varpi|$, then

$$\varpi \simeq - i\chi' k^2 \frac{\partial T}{\partial z}\left[\frac{\partial}{\partial z}\ln(P_0\rho_0^{-\gamma})\right]^{-1}$$

if $\frac{\partial T}{\partial z} < 0 \Rightarrow -i\varpi$ is real & positive.

$\Rightarrow$ instability.

What is the physics?



temperature only conducts along field, so perturbation is isothermal.

but $\frac{dT}{dz} < 0$ so gas now hotter than surroundings, so buyant.

$\Rightarrow$ keeps rising.