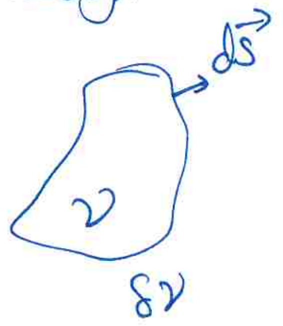
2. Dynamical fluid evolution

- **Mass conservation**

We saw yesterday that mass conservation is constructed through



$$\frac{d}{dt}\iiint_{\mathcal{V}} \rho \, d\mathcal{V} = \underbrace{-\oint \rho\vec{u}\cdot d\vec{S}}_{\text{flux out}}$$

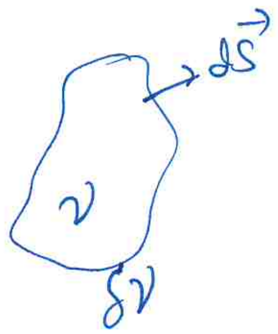
divergence theorem

$$= -\iiint_{\mathcal{V}} \vec{\nabla}\cdot(\rho\vec{u})\, d\mathcal{V}$$

fixed arbitrary volume

$$\boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0}.$$

- **Momentum conservation**



$$\frac{d}{dt}\iiint_{\mathcal{V}} \rho\vec{u}\, d\mathcal{V} = \underbrace{-\oint_{\delta\mathcal{V}} (\rho\vec{u})\,\vec{u}\cdot d\vec{S}}_{\text{momentum flux out}} \underbrace{-\oint_{\delta\mathcal{V}} P\, d\vec{S}}_{\text{pressure forces}} + \iiint_{\mathcal{V}} \underbrace{\vec{f}}_{\text{force per unit volume}} d\mathcal{V}$$

Divergence theorum

$$\frac{d}{dt}\iiint \rho\vec{u}\, d\mathcal{V} = -\iiint_{\mathcal{V}} \vec{\nabla}\cdot(\rho\vec{u}\,\vec{u})\, d\mathcal{V} - \iiint_{\mathcal{V}} \vec{\nabla}P\, d\mathcal{V} + \iiint_{\mathcal{V}} \vec{f}\, d\mathcal{V}$$

Arbitrary and fixed volume, and therefore

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \vec{\nabla}\cdot(\rho\vec{u}\vec{u}) = -\vec{\nabla}P + \vec{f}.$$

Let's massage this left hand side

$$\frac{\partial}{\partial t}(\rho\vec{u}) = \rho\frac{\partial\vec{u}}{\partial t} + \vec{u}\frac{\partial\rho}{\partial t}$$

$$\vec{\nabla}\cdot(\rho\vec{u}\vec{u}) = \vec{u}\left[\vec{\nabla}\cdot(\rho\vec{u})\right] + (\rho\vec{u}\cdot\vec{\nabla})\vec{u}$$

Adding

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \vec{\nabla}\cdot(\rho\vec{u}\vec{u}) = \rho\frac{\partial\vec{u}}{\partial t} + \vec{u}\overbrace{\left[\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u})\right]}^{\text{Mass conservation again}} + (\rho\vec{u}\cdot\vec{\nabla})\vec{u}$$

$\therefore$ we have

$$\rho\underbrace{\left[\frac{\partial}{\partial t} + (\vec{u}\cdot\vec{\nabla})\right]}_{\text{Convective (or Lagrangian) derivative}}\vec{u} = -\vec{\nabla}P + \vec{f}$$

[rate of change moving with flow]

What is a lagrangian derivative?

Consider sitting with a fluid element as it moves. It will trace out a path in space and time described by coordinates $(t, \vec{x}(t))$.

We can ask, how does a quantity ϕ vary as the fluid element moves?

In time δt, $\phi(\vec{x}, t)$ becomes $\phi(\vec{x} + \vec{v}\delta t, t + \delta t)$, which is

velocity of fluid element.

$$\phi(\vec{x} + \vec{v}\delta t, t+\delta t) \simeq \phi(\vec{x}, t) + \left.\frac{\partial \phi}{\partial t}\right|_{\vec{x},t} \delta t + \left.(\delta t \vec{v} \cdot \vec{\nabla})\phi\right|_{\vec{x},t} + \cdots$$

$$\therefore \quad \frac{D\phi}{Dt} \equiv \lim_{\delta t \to 0} \frac{\phi(\vec{x} + \vec{v}\delta t, t+\delta t) - \phi(\vec{x}, t)}{\delta t}$$

$$= \frac{\partial \phi}{\partial t} + \vec{v} \cdot \vec{\nabla}\phi$$

So, we see that

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\vec{\nabla}P + \frac{\vec{f}}{\rho}$$

Note, we have neglected viscosity in this formalism. This is always a good approximation astrophysically.

What is $\vec{f}$?

General astrophysical setting

$$\vec{f} = \underbrace{-\rho \vec{\nabla} \Phi}_{\text{Gravity}} + \underbrace{\vec{f}_L}_{\text{Lorentz force per volume.}}$$

For this course we shall neglect gravity, except for occassional specific problem.

We shall focus on magnetic forces.

We assume there is a frame where charges are non-relativistic. We know (from yesterday) that charges move rapidly ($t \sim 1/\omega_{pl}$) to screen electric fields.

Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho_c \quad \text{— (M1)} \qquad [\rho_c = \text{charge density}]$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{— (M2)}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{— (M3)}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \quad \text{— (M4)} \qquad [\vec{J} = \text{current}].$$

In the M.H.D limit: Yesterday we argued that electrons rapidly move to screen ~~...~~ electric fields (on timescales $t \sim 1/\omega_{pl}$). Move to rest frame of ions (moving with $\vec{v}_i$):

$$\vec{B}_i = \vec{B} - \frac{1}{c^2}\vec{v}_i\times\vec{E}$$
$$\vec{E}_i = \vec{E} + \vec{v}_i\times\vec{B}$$

Ion's are <u>non-relativistic</u>

$$\vec{B}_i \simeq \vec{B}$$
$$\vec{E}_i = \vec{E} + \vec{v}\times\vec{B}$$

But $\vec{E}_i$ screened by electrons, so lab-frame

$$\vec{E}_i \simeq \vec{0} \Rightarrow \boxed{\vec{E} = -\vec{v}\times\vec{B}.}$$

and therefore $E \sim vB$ (scaling).

Current – magnetic field relationship:

M4 states

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

consider ration of the two electromagnetic terms

$$\frac{\left| \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \right|}{|\vec{\nabla} \times \vec{B}|} \sim \frac{\varepsilon_0 \mu_0 \, E/\tau}{B/\ell} \sim \frac{E}{B} \cdot \frac{\ell}{\tau} \cdot \frac{1}{c^2}$$

Where ℓ, τ are length and timescales of process. But, $\ell/\tau \sim v$, $E \sim vB$

$$\therefore \quad \frac{\left| \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \right|}{|\vec{\nabla} \times \vec{B}|} \sim \frac{v^2}{c^2} \ll 1 .$$

$\therefore$ Current entirely specified by $\vec{B}$

$$\boxed{\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} .}$$

Ignoring the charge density

$$\vec{\nabla}\cdot\vec{E} = \rho_c/\varepsilon_0 \Rightarrow \rho_c \sim \frac{\varepsilon_0 E}{\ell} \sim \frac{\varepsilon_0 UB}{L}$$

$$\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} \Rightarrow |\vec{J}| \sim \frac{B}{\mu_0 L}$$

$$\Rightarrow \frac{|\rho_c \vec{v}|}{|\vec{J}|} \sim \frac{\varepsilon_0 v^2 B/L}{B/\mu_0 L} \sim \frac{v^2}{c^2} \ll 1.$$

So we can neglect terms of order $\rho_c \vec{v}$ compared to $\vec{J}$.

We are now in position to write $\vec{f}$.

Force per unit volume in EM

$$\vec{f} = \rho_c \vec{E} + \vec{J}\times\vec{B}$$

$$\simeq -\rho_c \vec{v}\times\vec{B} + \vec{J}\times\vec{B} \quad [\text{from } \vec{E} \simeq -\vec{v}\times\vec{B}]$$

$$\simeq \vec{J}\times\vec{B} \quad [\text{from } \rho_c\vec{v} \sim \left(\tfrac{v^2}{c^2}\right)\vec{J}]$$

$$= \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} \quad [\text{from } \vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}].$$

$\therefore$

$$\vec{f} = \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}$$

$$\Rightarrow \quad f_i = \frac{\varepsilon_{ijk}}{\mu_0}(\vec{\nabla}\times\vec{B})_j B_k$$

$$= \frac{\varepsilon_{ijk}}{\mu_0}\varepsilon_{jlm}(\partial_l B_m) B_k$$

$$= -\frac{\varepsilon_{jik}}{\mu_0}\varepsilon_{jlm}(\partial_l B_m) B_k$$

$$= -\frac{1}{\mu_0}\left[\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}\right](\partial_l B_m) B_k$$

$$= -\frac{1}{\mu_0}(\partial_i B_m) B_m + \frac{1}{\mu_0} B_k \partial_k B_i$$

$$= -\frac{1}{2\mu_0}\partial_i(B_m B_m) + \frac{1}{\mu_0} B_k \partial_k B_i$$

$\therefore$

$$\vec{f} = -\vec{\nabla}\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B}$$

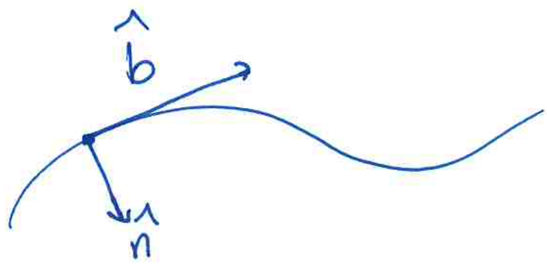
$\therefore$ we return to momentum conservation

$$\rho\left[\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right]\vec{u} = -\vec{\nabla}P - \vec{\nabla}\left(\frac{1}{2\mu_0}B^2\right) + \frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B}$$

$$= -\vec{\nabla}\underbrace{\left(P + \frac{B^2}{2\mu_0}\right)}_{\text{magnetic pressure force}} + \underbrace{\frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B}}_{\text{magnetic tension.}}$$

Let's interpret these terms:

Let's define $\hat{b}$ the unit vector tangent to the field line $\vec{B} = B\hat{b}$



then $\frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B} = \frac{B}{\mu_0}(\hat{b}\cdot\vec{\nabla})(B\hat{b})$

$$= \frac{B^2}{\mu_0}(\hat{b}\cdot\vec{\nabla})\hat{b} + \frac{\hat{b}B}{\mu_0}(\hat{b}\cdot\vec{\nabla})B$$

Call $\frac{\partial}{\partial s} \equiv \hat{b}\cdot\vec{\nabla}$ = the derivative along the field line.

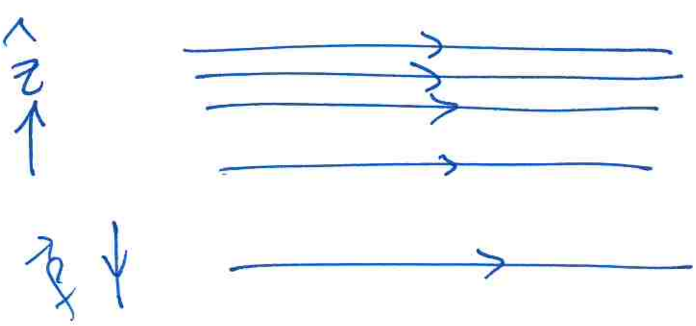
$$\therefore \quad \frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B} = \underbrace{\frac{B^2}{\mu_0}\frac{\partial \hat{b}}{\partial s}} + \hat{b}\frac{\partial}{\partial s}\left(\frac{1}{2\mu_0}B^2\right)$$

This force is <u>zero</u> if the field line is straight. This force acts to restore bent field lines.

Then adding back in the pressure term

$$-\vec{\nabla}\left(\frac{B^2}{2\mu_0}\right) + \hat{b}\frac{\partial}{\partial s}\left(\frac{1}{2\mu_0}B^2\right) = -\vec{\nabla}_\perp\left(\frac{B^2}{2\mu_0}\right)$$

we find that the magnetic pressure acts <u>perpendicular</u> to the field lines; they resist compression or rarefaction.



B^2 increases with z

$$\Rightarrow \frac{\partial}{\partial z}\left(B^2/2\mu_0\right) > 0$$

means $-\nabla_\perp(B^2/2\mu_0) < 0$

and force is along $-z$.