MHD - MMathPhys

1. Validity of the MHD model.

Magnetohydrodynamics (MHD) is the study of "large" and "slow" plasmas. It is a fluid (not kinetic) theory.

4 states of matter: solid → liquid → gas → plasma
⇓
ionized state.

This means that constituent particles will respond dynamically to imposed $\vec{E}$ & $\vec{B}$ fields. Opens up a much wider range of behaviour than seen in classical fluid dynamics.

Plasma's make up ~99% of Universe, and MHD theory widely applied across astrophysics.

e.g. the Sun, stellar winds, accretion discs and jets, Planetary atmospheres.

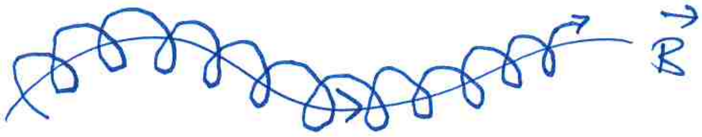
Also, e.g. Nuclear Fusion reactors, etc.

Second timescale

Particle of mass m and charge q also subject to force

$$\vec{f} = q\vec{v} \times \vec{B} .$$

Well known result that this causes uniform rotation perpendicular to field, and linear motion along field $\Rightarrow$ helix.



$$f = q v_\perp B = m v_\perp \omega \Rightarrow \omega_c = \frac{|q| B}{m}$$

known as cyclotron frequency.

Again, MHD describes systems on timescales

$$t \gg 1/\omega_c .$$

This gives us our first length scale, the radius of the circle followed by the particle

$$r_L = \frac{v_\perp}{\omega_c} \quad \text{"larmor radius".}$$

MHD works on $r \gg r_L$.

Note, both particle <u>mass</u> dependent. Very different scales for electrons and ions.

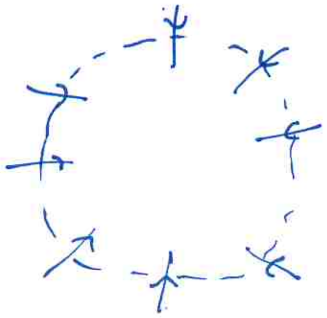
MHD describes "large" and "slow" plasmas.
We now make this precise.

Timescale

Imagine we have n_0/m^3 electrons and n_0/m^3 ions.
These are at rest ~~at~~ equillibrium.

Perturb electrons $\Rightarrow$ what happens? (In a gas, sound waves).

In plasma :



spherical perturbation inwards induces E-field

$$E = \frac{\frac{4}{3}\pi r^3 (n_e e)}{4\pi\varepsilon_0 r^2} = \frac{n_e e}{3\varepsilon_0} r$$

S.H.O!

typical parameters : $n_0 \sim 10^{17}/m^3$, $r \sim 1cm$; perturb $\sim 1\%$ $\Rightarrow n_e \sim 10^{15}/m^3$.

Gives $E = 6\times10^4\ V/m$ $\Rightarrow$ acceleration $a = eE/m_e \simeq 10^{16}\ m/s^2$.

$\therefore$ Rapid response to perturbation.

More precise : mass conservation for electron perturbation

δv

$$\frac{d}{dt}\iiint_v n_e\, dV = -\oint_{\delta v} n_e \vec{u}\cdot d\vec{S}$$

$$= -\iiint_v \vec{\nabla}\cdot(n_e\vec{u})\, dV$$

arbitrary volume

$$\Rightarrow \frac{\partial n_e}{\partial t} + \vec{\nabla}\cdot(n_e\vec{u}) = 0.$$

linearise : $n_e \to n_0 + n_e'$, $\vec{u} \to \vec{0} + \vec{u}'$

$$\frac{\partial}{\partial t}(\underbrace{n_0}_{\text{constant}} + n_e') + \vec{\nabla}\cdot(\underbrace{(n_0 + n_e')\vec{u}'}_{\text{second order}}) \simeq \frac{\partial n_e'}{\partial t} + n_0 \vec{\nabla}\cdot\vec{u}' = 0.$$

Take Newton's second law

$$m_e \frac{d\vec{u}'}{dt} = -e\vec{E}' \qquad \text{(for each electron).}$$

take divergence

Gauss' law

$$m_e \frac{d}{dt}(\vec{\nabla}\cdot\vec{u}') = -e\vec{\nabla}\cdot\vec{E}' = -\frac{e}{\varepsilon_0}\rho_c = \frac{n_e' e^2}{\varepsilon_0}$$

definition of charge density

Then substitute from mass conservation

$$m_e \frac{d}{dt}\left(-\frac{dn_e'}{dt}\frac{1}{n_0}\right) = -\frac{m_e}{n_0}\frac{d^2 n_e'}{dt^2} = \frac{n_e' e^2}{\varepsilon_0}$$

Simple harmonic motion with frequency

$$\omega_{pl} = \sqrt{\frac{n_0 e^2}{m_e \varepsilon_0}}$$

Oscillations restore neutrality on timescales $\sim 1/\omega_{pl}$.

Therefore, we can treat fluid as neutral on times long compared to $t_{pl} \sim 1/\omega_{pl}$.

MHD only describes phenomena on timescales $t \gg t_{pl}$.

Length scales

We showed that perturbing n_e locally causes an Electric field to be induced, over what length scale is this field felt?

To answer this we need more information about the equillibrium. Assume thermodynamic equillibrium so electrons obey Maxwell – Boltzmann statistics

$$n_0 = A \exp\left(eV_0/kT \right) \qquad [\text{energy } U = -eV_0].$$

Peturbed

$$n_0 \to n_0 + n_e' = A \exp\left(\frac{e(V_0+V')}{kT} \right) = \underbrace{A \exp\left(\frac{eV_0}{kT}\right)}_{n_0} \exp\left(\frac{eV'}{kT}\right)$$

$$\Rightarrow n_e' = n_0 \left[\exp\left(\frac{eV'}{kT}\right) - 1 \right] \simeq n_0 \frac{eV'}{kT}$$

for hot plasmas.

Again, Gauss' law

$$\vec{\nabla} \cdot \vec{E}' = \frac{\rho_c}{\varepsilon_0} = -\frac{n_e' e}{\varepsilon_0}$$

with $\vec{E}' = -\vec{\nabla} V' \Rightarrow \nabla^2 V' = \frac{n_e' e}{\varepsilon_0} = \frac{n_0 e^2}{\varepsilon_0 kT} V'$

$\therefore$ V' "shielded" over a length $\lambda_D = \sqrt{\frac{\varepsilon_0 kT}{n_0 e^2}}$ known as the Debeye length.

$\left[\text{spherical: } \nabla^2 V' = \frac{1}{r} \frac{\partial^2}{\partial r^2}(rV') = \frac{n_0 e^2}{\varepsilon_0 kT} V' \Rightarrow V \propto \frac{1}{r} e^{-r/\lambda_D} \right].$

Finally, electromagnetic waves only penetrate a certain length scale into a plasma.

Let's try and push an electro magnetic wave into the plasma. EM waves are transverse

$$\vec{E} = E_y(x,t)\,\hat{y}\,; \quad \vec{B} = B_z(x,t)\,\hat{z}$$

wave $\propto e^{i(kx-\omega t)}$

Lenz's law: $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\frac{\partial E_y}{\partial x}\hat{z} = -\frac{\partial B_z}{\partial t}\hat{z} \quad \text{— (1)}$$

Also

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \Rightarrow -\frac{\partial B_z}{\partial x}\hat{y} = -\mu_0 n_0 e\vec{v}' \quad \text{— (2)}$$

Newton's second law $m_e\frac{d\vec{v}'}{dt} = -eE_y\hat{y}$ — (3)

~~Calculate~~ Differentiate (1) w.r.t. x, use (2)

$$-\frac{\partial}{\partial t}\frac{\partial B_z}{\partial x} = -\mu_0 n_0 e\frac{dv'}{dt} = \frac{\partial^2 E_y}{\partial x^2}$$

use (3)

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{\mu_0 n_0 e^2}{m_e}E_y = \mu_0\varepsilon_0\cdot\left(\frac{n_0 e^2}{\varepsilon_0 m_e}\right)E_y$$

$$= \frac{c^2}{\omega_{pl}^2}E_y$$

Decay length $\Rightarrow d_e = \frac{c}{\omega_{pl}}$

Rules of the game:

- normal fluid rules

  ↳ Plasma strongly collisional, $t \gg t_{col}$

  $L \gg \lambda_{mfp}$

  (means particles follow ~ Maxwellian).

but also

- length scale $L \gg$ Larmor radius

and

- timescale $t \gg 1/$cyclotron frequency

- timescale $\gg 1/$plasma frequency

- Non-relativistic (more next time).