5. M.H.D Waves

Let us begin with a recap of sound waves.

Ideal hydrodynamics is

$$\frac{\partial \rho}{\partial t} + (\vec{u}\cdot\vec{\nabla})\rho + \rho(\vec{\nabla}\cdot\vec{u}) = 0$$

$$\frac{\partial p}{\partial t} + (\vec{u}\cdot\vec{\nabla})p + \gamma p(\vec{\nabla}\cdot\vec{u}) = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} + \frac{1}{\rho}\vec{\nabla}p = \vec{0}.$$

Assume an equillibrium of $\vec{u} = \vec{0}$, $P = P_0 = cst$
$\rho = \rho_0 = cst$.

Then give the system a kick

$$\vec{u} \to \delta\vec{u}, \quad P \to P_0 + \delta P, \quad \rho \to \rho_0 + \delta\rho$$

where $\delta P/P_0 \ll 1$, $\delta\rho/\rho_0 \ll 1$.

Linearized equations

$$\frac{\partial}{\partial t}\delta\rho + \rho_0 \vec{\nabla}\cdot\delta\vec{u} = 0 \quad \text{— (1)}$$

$$\frac{\partial}{\partial t}\delta p + \gamma P_0 \vec{\nabla}\cdot\delta\vec{u} = 0 \quad \text{— (2)}$$

$$\frac{\partial}{\partial t}\delta\vec{u} + \frac{1}{\rho_0}\vec{\nabla}\delta P = 0 \quad \text{— (3)}$$

take divergence of (3)

$$\frac{\partial}{\partial t} \vec{\nabla} \cdot \delta \vec{u} + \frac{1}{\rho_0} \nabla^2 \delta P = 0$$

use (2) $\Rightarrow$ $\vec{\nabla} \cdot \delta \vec{u} = - \frac{1}{\gamma P_0} \frac{\partial}{\partial t} \delta P$

to get

$$\boxed{\frac{\partial^2 \delta P}{\partial t^2} - \frac{\gamma P_0}{\rho_0} \nabla^2 \delta P = 0}$$ wave equation.

Density equation identical, as can be seen from substituting from (1) not (2)

$$-\frac{1}{\rho_0} \frac{\partial^2}{\partial t^2} \delta \rho + \frac{1}{\rho_0} \nabla^2 \delta P = 0$$

then using entropy conservation : $P \rho^{-\gamma} = cst$

$$\delta P . \rho^{-\gamma} - \gamma \rho^{\gamma - 1} P \delta \rho = 0$$

$$\Rightarrow \frac{\delta P}{P_0} - \gamma \, \delta \rho / \rho_0 = 0$$

to get

$$\boxed{\frac{\partial^2 \delta \rho}{\partial t^2} - \frac{\gamma P_0}{\rho_0} \nabla^2 \delta \rho = 0}$$

Speed of sound $\quad c_s^2 \equiv \frac{\gamma P_0}{\rho_0}.$

MHD waves are not so simple / boring.

Our equations are

$$\frac{\partial \rho}{\partial t} + (\vec{u}\cdot\vec{\nabla})\rho + \rho(\vec{\nabla}\cdot\vec{u}) = 0$$

$$\frac{\partial p}{\partial t} + (\vec{u}\cdot\vec{\nabla})P + \gamma p(\vec{\nabla}\cdot\vec{u}) = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} + \frac{1}{\rho}\vec{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) - \frac{1}{\rho\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B} = \vec{0}$$

$$\frac{\partial \vec{B}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{B} - (\vec{B}\cdot\vec{\nabla})\vec{u} + \vec{B}(\vec{\nabla}\cdot\vec{u}) = \vec{0}.$$

We will linearize about

$$\vec{B} = B_0\hat{z} \quad \text{(uniform)}$$

$$\rho = \rho_0 = cst, \quad P = P_0 = cst, \quad \vec{u} = \vec{0}.$$

Again, kick the system

$$\rho = \rho_0 + \delta\rho, \quad P = P_0 + \delta p, \quad \vec{u} = \delta\vec{u}$$

$$\vec{B} = B_0\hat{z} + \delta\vec{B}.$$

This time it will be prudent to write $\delta\vec{u} \equiv \frac{\partial\vec{\xi}}{\partial t}$ where $\vec{\xi}$ is a <u>displacement vector</u>.

This is used in MHD because field lines are <u>frozen into</u> the flow and dislike being <u>bent</u> or <u>compressed</u>.

In other words, magnetic fields have "memory" of where they were perturbed from.

Linearize :

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \vec{\nabla}\cdot\frac{\partial\vec{\xi}}{\partial t} = 0 \Rightarrow \boxed{\frac{\delta\rho}{\rho_0} = -\vec{\nabla}\cdot\vec{\xi}}$$

Entropy : $P\rho^{-\gamma} = cst \Rightarrow \frac{\delta P}{P_0} - \gamma\frac{\delta\rho}{\rho_0} = 0$

$$\Rightarrow \boxed{\frac{\delta P}{P_0} = -\gamma\vec{\nabla}\cdot\vec{\xi}}$$

Induction

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\vec{B} = (\vec{B}\cdot\vec{\nabla})\vec{u} - \vec{B}(\vec{\nabla}\cdot\vec{u})$$

becomes

$$\frac{\partial}{\partial t}\delta\vec{B} = \vec{B}_0\cdot\vec{\nabla}\frac{\partial\vec{\xi}}{\partial t} - \vec{B}_0\left(\vec{\nabla}\cdot\frac{\partial\vec{\xi}}{\partial t}\right)$$

$$\Rightarrow \quad \delta\vec{B} = \underbrace{(\vec{B}_0\cdot\vec{\nabla})\vec{\xi}}_{B_0\nabla_{\parallel}\vec{\xi}} - \underbrace{\vec{B}_0(\vec{\nabla}\cdot\vec{\xi})}_{\hat{z}B_0\vec{\nabla}\cdot\vec{\xi}}$$

$$B_0\nabla_{\parallel}\vec{\xi} = B_0\left(\underbrace{\nabla_{\parallel}(\xi_{\parallel}\hat{z} + \vec{\xi}_{\perp})}\right) \qquad -\hat{z}B_0\underbrace{\vec{\nabla}\cdot\vec{\xi}} = -\hat{z}B_0\nabla_{\parallel}\xi_{\parallel} - B_0\hat{z}\vec{\nabla}_{\perp}\cdot\vec{\xi}_{\perp}$$

$$B_0\nabla_{\parallel}\vec{\xi}_{\perp} - B_0\hat{z}\vec{\nabla}_{\perp}\cdot\vec{\xi}_{\perp}$$

$$\Rightarrow \quad \frac{\delta\vec{B}}{B_0} = \nabla_{\parallel}\vec{\xi}_{\perp} - \hat{z}\vec{\nabla}_{\perp}\cdot\vec{\xi}_{\perp}$$

$$\Rightarrow \boxed{\frac{\delta B_{\parallel}}{B_0} = -\vec{\nabla}_{\perp}\cdot\vec{\xi}_{\perp}} \quad \boxed{\frac{\delta\vec{B}_{\perp}}{B_0} = \nabla_{\parallel}\vec{\xi}_{\perp}}$$

Only perpendicular perturbations modify the magnetic field. This makes sense as the fluid is carried with the flow.

To elaborate, write

$$\frac{\delta \vec{B}}{B_0} = \frac{\delta(B\hat{b})}{B_0} = \delta\hat{b} + \hat{z}\,\frac{\delta B}{B_0}$$

but $\hat{b}\cdot\hat{b} = 1 \Rightarrow \delta(\hat{b}\cdot\hat{b}) = 2\hat{b}\cdot\delta\hat{b} = 0$

$$\Rightarrow \delta\hat{b} \perp \hat{b}$$

$$\Rightarrow \delta\vec{B}_\perp = \delta\hat{b}$$

$$\Rightarrow \boxed{\delta\hat{b} = \nabla_\parallel \vec{\xi}_\perp}$$

$$\boxed{\frac{\delta B}{B_0} = -\vec{\nabla}_\perp \cdot \vec{\xi}_\perp}$$

Let's now linearize the momentum equation

$$\underbrace{\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\right)\vec{u}}_{(1)} = \underbrace{-\vec{\nabla}P}_{(2)} \underbrace{- \vec{\nabla}\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B}}_{(3)}$$

$$(1) \Rightarrow \rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} + \text{second order.}$$

$$(2) \Rightarrow -\vec{\nabla}\delta P = P_0 \gamma \vec{\nabla}(\vec{\nabla}\cdot\vec{\xi}) \Big| \text{ [from mass + entropy].}$$

$$(3) \Rightarrow \underbrace{-\vec{\nabla}\left(\frac{B^2}{2\mu_0}\right)}_{(\alpha)} + \underbrace{\frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B}}_{(\beta)}$$

$$(\alpha) \Rightarrow B^2 = (\vec{B}_0 + \delta\vec{B})\cdot(\vec{B}_0 + \delta\vec{B}) = B_0^2 + 2\vec{B}_0\cdot\delta\vec{B} + O(\delta B^2)$$

$$\therefore -\vec{\nabla}\left(\frac{B^2}{2\mu_0}\right) \Rightarrow -\frac{B_0^2}{\mu_0}\vec{\nabla}\left(\frac{\delta B_\parallel}{B_0}\right)$$

$$(\beta) \Rightarrow \frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B} = \frac{B_0^2}{\mu_0}\nabla_\parallel\left(\delta\hat{b} + \frac{\delta B_\parallel}{B_0}\right)$$

$$\Rightarrow (\alpha)+(\beta) \Rightarrow \underbrace{-\frac{B_0^2}{\mu_0}\vec{\nabla}\left(\frac{\delta B_\parallel}{B_0}\right) + \frac{B_0^2}{\mu_0}\nabla_\parallel\left(\frac{\delta B_\parallel}{B_0}\right)}_{-\frac{B_0^2}{\mu_0}\vec{\nabla}_\perp\left(\frac{\delta B_\parallel}{B_0}\right)} + \frac{B_0^2}{\mu_0}\nabla_\parallel \delta\hat{b}$$

(3) $\Rightarrow$ $$-\frac{B_0^2}{\mu_0}\vec{\nabla}_\perp\left(\frac{\delta B_\parallel}{B_0}\right)+\frac{B_0^2}{\mu_0}\nabla_\parallel \delta\hat{b}$$

using induction terms

$$\Rightarrow +\frac{B_0^2}{\mu_0}\vec{\nabla}_\perp(\vec{\nabla}_\perp\cdot\vec{\xi}_\perp)+\frac{B_0^2}{\mu_0}\nabla_\parallel^2\vec{\xi}_\perp$$

Assembling, we get

$$\rho_0\frac{\partial^2\vec{\xi}}{\partial t^2}=P_0\gamma\vec{\nabla}(\vec{\nabla}\cdot\vec{\xi})+\frac{B^2}{\mu_0}\left[\vec{\nabla}_\perp(\vec{\nabla}_\perp\cdot\vec{\xi}_\perp)+\nabla_\parallel^2\vec{\xi}_\perp\right]$$

or $$\boxed{\frac{\partial^2\vec{\xi}}{\partial t^2}=c_s^2\vec{\nabla}(\vec{\nabla}\cdot\vec{\xi})+v_A^2\left[\vec{\nabla}_\perp(\vec{\nabla}_\perp\cdot\vec{\xi}_\perp)+\nabla_\parallel^2\vec{\xi}_\perp\right]}$$

two important speeds have emerged

$$\boxed{c_s^2\equiv\frac{\gamma P_0}{\rho_0}}\qquad\boxed{v_A^2\equiv\frac{B_0^2}{\mu_0\rho_0}}$$

speed of sound

Alfvén speed.

Let's seek wave-like solutions of this equation.

$$\vec{\xi} \propto \exp(-i\omega t + i\vec{k}\cdot\vec{r})$$

giving

$$\boxed{\omega^2 \vec{\xi} = c_s^2 \vec{k}(\vec{k}\cdot\vec{\xi}) + v_A^2\left(\vec{k}_\perp(\vec{k}_\perp\cdot\vec{\xi}_\perp) + k_\parallel^2 \vec{\xi}_\perp\right)}$$

We are always allowed to pick $\vec{k} = (k_\perp, 0, k_\parallel)$

i.e. x is the direction $\perp$ and z is $\parallel$.

Then our dispersion equation becomes

$$\omega^2 \xi_x = c_s^2 k_\perp (k_\perp \xi_x + k_\parallel \xi_z) + v_A^2 \overbrace{(k_\perp^2 + k_\parallel^2)}^{k^2} \xi_x$$

$$\omega^2 \xi_y = v_A^2 k_\parallel^2 \xi_y$$

$$\omega^2 \xi_z = c_s^2 k_\parallel (k_\perp \xi_x + k_\parallel \xi_z)$$

and the other fields satisfy

$$\frac{\delta\rho}{\rho_0} = -i\vec{k}\cdot\vec{\xi} = -i(k_\perp \xi_x + k_\parallel \xi_z)$$

$$\frac{\delta p}{p_0} = \gamma \frac{\delta \rho}{\rho_0} \; ; \quad \delta\hat{b} = ik_\parallel \vec{\xi}_\perp = ik_\parallel \begin{pmatrix} \xi_x \\ \xi_y \\ 0 \end{pmatrix}$$

$$\frac{\delta B}{B_0} = -ik_\perp \xi_x$$

Alfvén Waves

We note straight away that y-motion decouples from the system. Therefore $\vec{\xi} = (0, \xi_y, 0)$ is an eigenvector with eigenvalues given by

$$\omega^2 = v_A^2 k_{\parallel}^2 \Rightarrow \boxed{\omega = \pm v_A k_{\parallel}}.$$

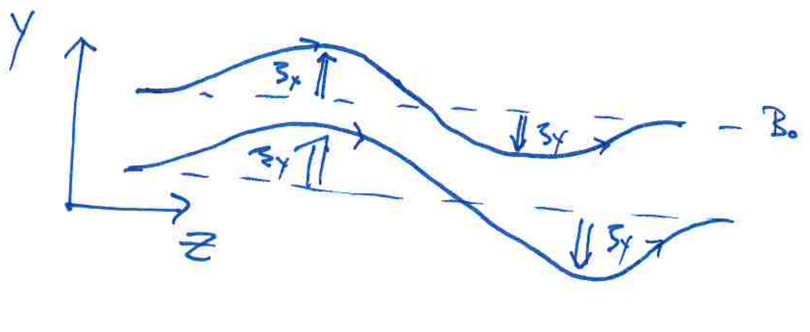
These are Alfvén waves, they propogate parallel (and anti-parallel) to $\vec{B}_0$ [have $k_{\parallel}$].

The other fields satisfy

$$\vec{\xi} = \xi_y \hat{y}, \quad \delta\rho = 0, \quad \delta p = 0, \quad \delta B = 0$$

$$e^{i k_{\parallel}(z \pm v_A t)}, \quad \delta\hat{b} = i k_{\parallel} \xi_y \hat{y}.$$

In other words, this wave is incompressible ($\delta\rho = 0$) and involves magnetic fields acting as elastic strings, springing back against perturbing motions due to the restoring curvature force.



Note that these waves can have $k_\perp \neq 0$ and still be a solution.