Obviously, if relaxation occurred with no constraints, the solution would be $\vec{B} = \vec{0}$.

However, there <u>are</u> constraints, and they turn out to be topological. This follows from the fact that ideal MHD accepts a topological conserved magnetic quantity, named helicity.

Helicity

Consider the quantity

$$H \equiv \iiint_{V} \vec{A} \cdot \vec{B} \, dV$$

where

$\vec{B} = \vec{\nabla} \times \vec{A}$, and $\vec{A}$ is the vector potential.

Let's prove some properties of H.

1. Helicity is well defined.

Not obvious, as we can always shift

$$\vec{A} \to \vec{A} + \vec{\nabla}\chi$$ and not change $\vec{B}$.

(a gauge transformation).

If we gauge transform we get

$$H \to H + \iiint_V \vec{B} \cdot \vec{\nabla}\chi \, dV$$

by parts

$$= H + \iiint_V \left(\vec{\nabla} \cdot (\vec{B}\chi) - \chi \overbrace{\cancel{\vec{\nabla} \cdot \vec{B}}}^{0} \right) dV$$

divergence theorem

$$= H + \oiint_{\delta V} \chi \vec{B} \cdot d\vec{S}$$

So if our volume encases the entire field then $\vec{B} \cdot d\vec{S} = 0$ and H is unchanged.

2. Helicity is conserved

Induction equation

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B})$$

$$\therefore \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = \vec{\nabla} \times (\vec{u} \times \vec{B})$$

"uncurl"

$$\frac{\partial A}{\partial t} = \vec{u} \times \vec{B} + \vec{\nabla} \chi$$

dot with $\vec{B}$

$$\vec{B} \cdot \frac{\partial \vec{A}}{\partial t} = \frac{\partial}{\partial t}(\vec{A} \cdot \vec{B}) - \vec{A} \cdot \frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \frac{\partial}{\partial t}(\vec{A} \cdot \vec{B}) = \vec{B} \cdot \left[\vec{u} \times \vec{B} + \vec{\nabla} \chi\right] - \vec{A} \cdot \left[\vec{\nabla} \times (\vec{u} \times \vec{B})\right]$$

term by term

$\vec{B} \cdot (\vec{u} \times \vec{B}) = 0$ (symmetry)

$$\vec{B} \cdot \vec{\nabla} \chi = \vec{\nabla} \cdot (\vec{B} \chi) - \chi \underbrace{\vec{\nabla} \cdot \vec{B}}_{0} = \vec{\nabla} \cdot (\vec{B} \chi)$$

$$\vec{A}\cdot\left[\vec{\nabla}\times(\vec{u}\times\vec{B})\right] = A_i\, \epsilon_{ijk}\, \partial_j\, \epsilon_{klm}\, u_l B_m$$

or

$$A_i\, \epsilon_{ijk}\, \partial_j B_k \overset{\text{by parts}}{=} \epsilon_{ijk}\, \partial_j (A_i B_k) - \epsilon_{ijk}\, B_k\, \partial_j A_i$$

$$= -\overset{\text{one swap}}{\epsilon_{jik}}\, \partial_j (A_i B_k) + \overset{\text{three swap}}{\epsilon_{kji}}\, B_k\, \partial_j A_i$$

$$= -\vec{\nabla}\cdot\left(\vec{A}\times(\vec{u}\times\vec{B})\right) + (\vec{u}\times\vec{B})\cdot(\vec{\nabla}\times\vec{A})$$

but $\vec{\nabla}\times\vec{A} = \vec{B}$

So $(\vec{u}\times\vec{B})\cdot\vec{B} = 0$

and we are left with

$$= -\vec{\nabla}\cdot\left[\vec{A}\times(\vec{u}\times\vec{B})\right]$$

but

$$\left(\vec{A}\times(\vec{u}\times\vec{B})\right)_i = \epsilon_{ijk} A_j\, \epsilon_{klm}\, u_l B_m$$

$$= \left[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right] A_j u_l B_m$$

$$= u_i (A_j B_j) - B_i (A_j u_j)$$

$$\therefore \quad \vec{A}\cdot\left[\vec{\nabla}\times(\vec{u}\times\vec{B})\right] = -\vec{\nabla}\cdot\left(\vec{u}\,(\vec{A}\cdot\vec{B}) - \vec{B}\,(\vec{A}\cdot\vec{u})\right)$$

and so

$$\frac{\partial}{\partial t}(\vec{A}\cdot\vec{B}) = \vec{\nabla}\cdot\left[\vec{B}\chi + \vec{B}(\vec{A}\cdot\vec{u}) - \vec{u}(\vec{A}\cdot\vec{B})\right]$$

Integrate over total volume

$$\frac{\partial}{\partial t}\iiint_V \vec{A}\cdot\vec{B}\,dV = \iiint_V \vec{\nabla}\cdot\left[\vec{B}(\chi + (\vec{A}\cdot\vec{u})) - \vec{u}(\vec{A}\cdot\vec{B})\right]dV$$

$$= \oiint_{\delta V} (\vec{B}\chi + \vec{B}(\vec{A}\cdot\vec{u}) - \vec{u}(\vec{A}\cdot\vec{B}))\cdot d\vec{S}$$

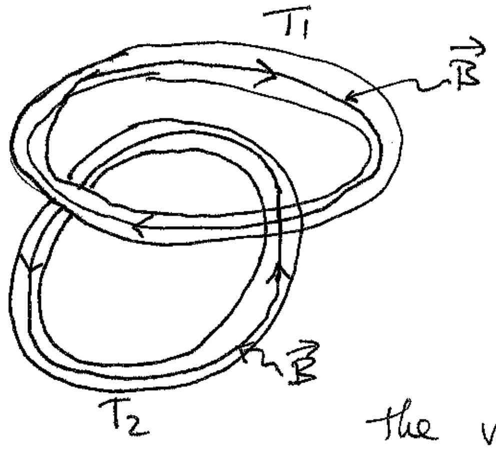
$= 0$ as nothing ($\vec{u}$ or $\vec{B}$) sticks out of the volume.

$$\therefore \boxed{\frac{dH}{dt} = 0}$$

this is the constraint subject to which energy may be minimised.

Helicity is a topological invariant

Consider two linked flux tubes [recall from Alfvén's theorem that these tubes forever enclose a field line]



Then the helicity of tube 1 is

$$H_1 = \iiint_{T_1} \vec{A} \cdot \vec{B} \, d\mathcal{V}_{T_1}$$

the volume of the tube is $\underbrace{d\vec{l}}_{\text{unit along tube } \hat{b}\, dl} . d\vec{S}$ ← surface of tube $\hat{b}\, dS$

thus

$$H_1 = \iiint_{T_1} \hat{b}\, dl \cdot \hat{b}\, dS \; \vec{A} \cdot (\hat{b} B)$$

$$= \underbrace{\iint_{T_1} \vec{B} . d\vec{S}}_{\substack{\text{flux along tube 1} \\ \equiv \Phi_1}} \int_{T_1} \vec{A} \cdot d\vec{l} = \Phi_1 \int_{T_1} \vec{A} \cdot d\vec{l}$$

by Stoke's theorem $\int_{T_1} \vec{A} \cdot d\vec{l} = \iint_{S_1} (\vec{\nabla} \times \vec{A}) . d\vec{S}$

where S_1 is the surface band by l_1

as $\vec{\nabla} \times \vec{A} = \vec{B}$ $\quad \int_{T_1} \vec{A} \cdot d\vec{l} = \underbrace{\iint_{S_1} \vec{B} \cdot d\vec{S}}_{\substack{\text{flux through middle} \\ \text{of loop 1} = \Phi_2}}$

$$\Rightarrow H_1 = \Phi_1 \Phi_2$$

By the same token, in a system of many linked tubes, the helicity of tube i is

flux of j^{th} tube

$$H_i = \Phi_i \, \Phi_{\text{through hole, } i} = \Phi_i \sum_{j \neq i} \Phi_j \underbrace{N_{ij}}$$

number of linkages between i & j

and thus the total helicity is

$$H = \sum_i \sum_{j \neq i} \Phi_i \Phi_j N_{ij}$$

Thus helicity measures the number of linkages of the flux tubes weighted by the field strength in each tube.

The physical insight is the following: if one thinks of the magnetic field as a tangled mess of field lines, while you can change this mess by moving field lines around, you cannot easily undo linkages, knots, etc.
[This is Alfvén's theorem again]. Basically any operation which would require field lines to have "ends"!

Taylor Relaxation

We can now work out which equilibrium an MHD system will relax to by minimising its energy subject to constant helicity. This is a classic lagrange multiplier problem, with "action"

$$S = \iiint_V \left(B^2 - \alpha \vec{A}\cdot\vec{B}\right) dV$$

lagrange multiplier.

Then $\delta S = 0$

implies

$$\delta \left[\iiint_V \left(B^2 - \alpha \vec{A}\cdot\vec{B}\right) dV\right] = 0$$

first term

$$\delta \iiint_V B^2 dV = 2\iiint_V \vec{B}\cdot\delta\vec{B}\, dV$$

$$= 2\iiint_V \vec{B}\cdot\left(\vec{\nabla}\times\delta\vec{A}\right) dV$$

$$\vec{B}\cdot\left(\vec{\nabla}\times\delta\vec{A}\right) = B_i \varepsilon_{ijk} \partial_j \delta A_k$$

$$= \varepsilon_{ijk}\partial_j\left(B_i \delta A_k\right) - \varepsilon_{ijk}\delta A_k \partial_j B_i$$

$$\overset{\text{one swap}}{=} -\varepsilon_{jik}\partial_j\left(B_i\delta A_k\right) \overset{\text{three swap}}{+} \delta A_k \varepsilon_{kji}\partial_j B_i$$

$$= -\vec{\nabla}\cdot\left(\vec{B}\times\delta\vec{A}\right) + \delta\vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)$$

$$\Rightarrow \delta\iiint_V B^2 dV = -2\iiint_V \vec{\nabla}\cdot\left(\vec{B}\times\delta\vec{A}\right) dV + 2\iiint_V \delta\vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right) dV$$

divergence Theorem

$$= -2 \oint\!\!\oint_{\delta V} (\vec{B} \times \delta\vec{A}) \cdot d\vec{S} + 2\iiint_V (\vec{\nabla} \times \vec{B}) \cdot \delta\vec{A}\, dV$$

and

$$\delta \iiint_V \vec{A} \cdot \vec{B}\, dV = \iiint_V \delta\vec{A} \cdot \vec{B}\, dV + \iiint_V \vec{A} \cdot \delta\vec{B}\, dV$$

$$= \iiint_V \delta\vec{A} \cdot \vec{B}\, dV + \iiint_V \vec{A} \cdot (\vec{\nabla} \times \delta\vec{A})\, dV$$

identical manipulation to before on second term

$$= \iiint_V \delta\vec{A} \cdot \vec{B}\, dV - \iiint_V \vec{\nabla} \cdot (\vec{A} \times \delta\vec{A})\, dV + \iiint_V \underbrace{(\vec{\nabla} \times \vec{A})}_{\vec{B}} \cdot \delta\vec{A}\, dV$$

divergence theorem + combine 1 & 3

$$= 2\iiint_V \vec{B} \cdot \delta\vec{A}\, dV - \oint\!\!\oint_{\delta V} (\vec{A} \times \delta\vec{A}) \cdot d\vec{S}$$

we must worry ~~about~~ about surface terms.

As $$\frac{\partial}{\partial t} \delta\vec{B} = \vec{\nabla} \times \left(\frac{\partial \vec{\xi}}{\partial t} \times \vec{B} \right) \qquad \left[\text{reintroducing } \delta\vec{u} = \frac{\partial \vec{\xi}}{\partial t}\right]$$

uncurling $$\delta\vec{A} = \vec{\xi} \times \vec{B}$$

Also $$\vec{A} \times \delta\vec{A} = \vec{A} \times (\vec{\xi} \times \vec{B}) = \epsilon_{ijk} A_j \epsilon_{klm} \xi_l B_m$$

$$= [\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}] A_j \xi_l B_m$$

$$= (\vec{A} \cdot \vec{B})\vec{\xi} - (\vec{A} \cdot \vec{\xi})\vec{B}$$

also

$$\vec{B} \times \delta\vec{A} = \varepsilon_{ijk} B_j \varepsilon_{klm} \xi_l B_m$$

$$\overset{\text{same argument}}{=} B^2 \vec{\xi} - (\vec{B}\cdot\vec{\xi})\vec{B}$$

and so all surface terms come with $\vec{\xi}\cdot d\vec{S}$ or $\vec{B}\cdot d\vec{S}$, which will vanish if δV encloses both the magnetic field and the plasma.

Thus, we are left with

$$\delta \iiint_V \left(B^2 - \alpha \vec{A}\cdot\vec{B}\right) dV = 2\iiint_V \left[\vec{\nabla}\times\vec{B} - \alpha\vec{B}\right]\cdot\delta\vec{A}\, dV$$

$$= 0$$

$$\therefore \quad \vec{\nabla}\times\vec{B} = \alpha\vec{B} \Rightarrow \nabla^2\vec{B} = -\alpha^2\vec{B}$$

and we have recovered the <u>linear force free field</u>.

So our system will relax to a linear force free state with system-specific boundary conditions. The boundary conditions come in through the initial helicity, and our Lagrange multiplier α is a function $\alpha = \alpha(H)$, which enforces this.

This linear force free state has a particularly simple helicity, which follows from

$$\vec{\nabla}\times\vec{B} = \alpha\vec{B} = \alpha\vec{\nabla}\times\vec{A} \Rightarrow \vec{B} = \alpha\vec{A} + \vec{\nabla}\chi$$

$$H(\alpha) = \iiint_{\mathcal{V}} \vec{A}\cdot\vec{B}\,d\mathcal{V} = \frac{1}{\alpha}\iiint_{\mathcal{V}} B^2 d\mathcal{V} - \frac{1}{\alpha}\iiint \vec{B}\cdot\vec{\nabla}\chi d\mathcal{V}$$

final term vanishes under volume integral

$$\iiint_{\mathcal{V}} \vec{B}\cdot\vec{\nabla}\chi\,d\mathcal{V} = \iiint_{\mathcal{V}} \vec{\nabla}\cdot(\vec{B}\chi)\,d\mathcal{V} - \iiint_{\mathcal{V}} \chi\,\cancel{\vec{\nabla}\cdot\vec{B}}^{\,0}\,d\mathcal{V}$$

$$= \oiint_{\delta\mathcal{V}} \chi\vec{B}\cdot d\vec{S} = 0 \quad [\text{enclose field}].$$

$\therefore$ Constant 'α' set by

$$H = \frac{1}{\alpha}\iiint_{\mathcal{V}} B^2 d\mathcal{V}.$$

General procedure:

- solve $\nabla^2\vec{B} = -\alpha^2\vec{B}$, get $\vec{B}(\vec{r},\alpha)$.
- calculate $H(\alpha) = \frac{1}{\alpha}\iiint_{\mathcal{V}} B^2(\vec{r},\alpha)\,d\mathcal{V}$
- set $H(\alpha) = H_0$, the initial value of helicity
- set α by inverting to $\alpha = \alpha(H_0)$
- Then relaxed state $\vec{B} = \vec{B}(\vec{r}, \alpha[H_0])$.

Example

Let us consider the case of cylindrical & axial symmetry once again. We know $B_r = 0$, and $B_\theta, B_z \neq 0$.

z-component of $\nabla^2 \vec{B} = \alpha^2 \vec{B} \Rightarrow \dfrac{\partial^2 B_z}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial B_z}{\partial r} + \alpha^2 B_z = 0.$

This is a Bessel equation, with solution

$$B_z(r) = B_0 J_0(\alpha r)$$

Bessel function of order 0.

This solution satisfies the b.c. $B_z(0) = B_0$, $B_z(r \to \infty) \to 0$.

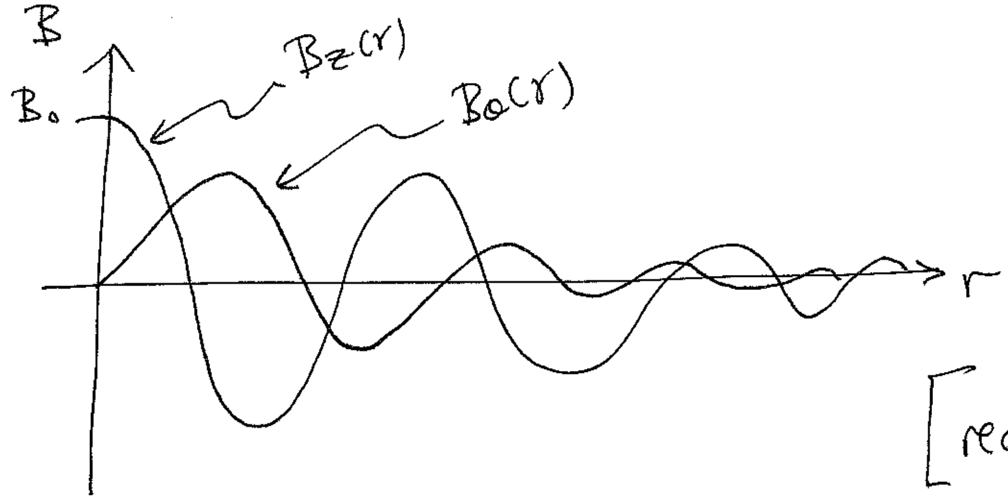
We can use

$$\vec{\nabla} \times \vec{B} = \alpha \vec{B} \Rightarrow \alpha B_\theta = (\vec{\nabla} \times \vec{B})_\theta = -\frac{\partial B_z}{\partial r}$$

using Bessel identities

$$B_\theta(r) = B_0 J_1(\alpha r)$$

Bessel function of order 1.



[recall $J(x) \sim \frac{1}{\sqrt{x}} \sin(x)$ for large x.]

which is an interesting twisted field geometry which can maintain itself in equilibrium.

Enforce helicity constraint

$$H = \frac{1}{\alpha} \iiint_V B^2 \, dV$$

Assume volume is of a cylinder of length L and radius R

$$H = \frac{1}{\alpha} \cdot B_0^2 \cdot 2\pi \cdot L \cdot \int_0^R r\left[J_0^2(\alpha r) + J_1^2(\alpha r)\right] dr$$

......

$$H = \frac{B_0^2 \, \pi R^2 L}{\alpha}\left[J_0^2(\alpha R) + 2J_1^2(\alpha R) + J_2^2(\alpha R) - \frac{2}{\alpha R} J_1(\alpha R) J_2(\alpha R)\right]$$

picking H_0 and inverting for $\alpha = \alpha(H_0)$ gives us our final field state.