MACHINE LEARNING 2018

Homework 2's Solutions

December 12, 2018

Problem 1. *(10 points)*

Using the formula for multinomial coefficients, the number of divisions is given by

$$\binom{15}{4,5,6} = \frac{15!}{4!\ 5!\ 6!} = 630630$$

One will get full mark if he/she states that he/she applies the formula for multinomial coefficients as above.

Problem 2. *(15 points)*

(a) *(7 points)* If we draw the second marble after replacing the first marble, the sample space will be

$$S = \{(R,R),(R,G),(R,B),(G,R),(G,G),(G,B),(B,R),(B,G),(B,B)\}$$

(b) *(8 points)* If we draw the second marble without replacing the first marble, the sample space is given by

$$S = \{(R,R),(R,G),(R,B),(G,R),(G,B),(B,R),(B,G)\}$$

Note that when we do not replace the first marble, we still have (R,R) as there are two red marbles in the box.

Problem 3. *(10 points)*
(a) *(5 points)* The number of ways we can draw two cards from a deck of 52 playing cards:

$$\binom{52}{2} = 1326 \tag{1}$$

Out of these 1326 combinations, the number of ways we can draw two Jacks from the same deck:

$$\binom{4}{2} = 6 \tag{2}$$

Note that all the above combinations are equally likely (having the same probability). Thus the probability that the two cards are both Jacks is

$$P\{both\ Jacks\} = \frac{6}{1326} \approx 0.0045$$

(b) *(5 points)* The number of ways we can draw 2 cards of the same value for a given value is given in Eq (2). As there are a total of 13 different values, the number of ways we can draw 2 cards of the same value for any value is given by $13 \times 6 = 78$. Thus the probability that the two cards have the same value is

$$P\{same\ value\} = \frac{78}{1326} \approx 0.059$$

Problem 4. *(10 points)*
$P\{at\ least\ one\ 5 \mid i\} = 0$ for $i = 3, 4, 5$.
For $i = 6$, there are 5 possible outcomes $\{(1,5),(2,4),(3,3),(4,2),(5,1)\}$, out of which there are 2 outcomes with at least one 5. Thus
$P\{at\ least\ one\ 5 \mid i = 6\} = 0.4$.
Similarly,
$P\{at\ least\ one\ 5 \mid i = 7\} = 1/3$ ($\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$).
$P\{at\ least\ one\ 5 \mid i = 8\} = 0.4$ ($\{(2,6),(3,5),(4,4),(5,3),(6,2)\}$).
$P\{at\ least\ one\ 5 \mid i = 9\} = 0.5$ ($\{(3,6),(4,5),(5,4),(6,3)\}$).
$P\{at\ least\ one\ 5 \mid i = 10\} = 1/3$ ($\{(4,6),(5,5),(6,4)\}$).
$P\{at\ least\ one\ 5 \mid i = 11\} = 1$ ($\{(5,6),(6,5)\}$).
$P\{at\ least\ one\ 5 \mid i = 12\} = 0$ ($\{(6,6)\}$).

Problem 5. *(15 points)*
The pmf of a Poisson distribution with parameter λ is given as

$$P\{X = i\} = \frac{e^{-\lambda}\lambda^i}{i!}$$

Thus we have

$$\begin{aligned} \frac{P\{X = i\}}{P\{X = i-1\}} &= \frac{e^{-\lambda}\lambda^i}{i!} \Big/ \frac{e^{-\lambda}\lambda^{i-1}}{(i-1)!} \\ &= \frac{\lambda}{i} \end{aligned}$$

As i increases and $i < \lambda$, $(\lambda/i) > 1$, so $P\{X = i\}$ increases monotonically. If λ is a nonnegative integer, when $i = \lambda$, $P\{X = i\} = P\{X = i - 1\}$ and $P\{X = i\}$ will decrease monotonically when $i > \lambda$. If λ is not an integer, then $P\{X = i\}$ will reach its maximum when i is the largest integer not exceeding λ. After that, $P\{X = i\}$ will decrease monotonically as $(\lambda/i) < 1$.

Problem 6. *(10 points)*
$Var(aX+b) = Var(aX) + Var(b)$ (a constant is independent of another random variable / constant).

$$\begin{aligned} Var(aX+b) &= Var(aX) \quad \text{(Note that Var(b)=0)} \\ &= a^2 Var(X) \\ &= a^2\sigma^2 \end{aligned}$$

Thus

$$SD(aX+b) \;=\; |a|\sigma$$

2 points should be deducted if there are no absolute sign for a.

Problem 7. *(30 points)*
(a) *(10 points)* Recall that in the jointly continuous case, x and y are independent iff (if and only if):

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \forall x, y \tag{3}$$

Here $f_{X,Y}(x,y)$ does not factor into a function of x and a function of y, so X and Y are not independent.
We can also calculate $f_X(x)$ (as in Part (b)) and $f_Y(y)$ to disprove Equation (3).

(b) *(10 points)*

$$\begin{aligned} f_X(x) &= \int_0^1 f(x,y)dy \\ &= \int_0^1 (x+y)dy \\ &= \left(xy + \frac{y^2}{2}\right)\Big|_0^1 \\ &= x + \frac{1}{2}, \quad 0 < x < 1 \end{aligned}$$

(c) *(10 points)* We only taking the integration in the area $(0 < x < 1,\ 0 < y <$

$1 - x$) (as $X + Y < 1$).

$$\begin{aligned} P\{X+Y<1\} &= \int_0^1 \left(\int_0^{1-x} f(x,y)dy \right) dx \\ &= \int_0^1 \left(xy + \frac{y^2}{2} \right) \Big|_0^{1-x} dx \\ &= \int_0^1 \left(x(1-x) + \frac{(1-x)^2}{2} \right) dx \\ &= \int_0^1 \left(\frac{-x^2+1}{2} \right) dx \\ &= \frac{1}{2} \left(\frac{-x^3}{3} + x \right) \Big|_0^1 \\ &= \frac{1}{3} \end{aligned}$$

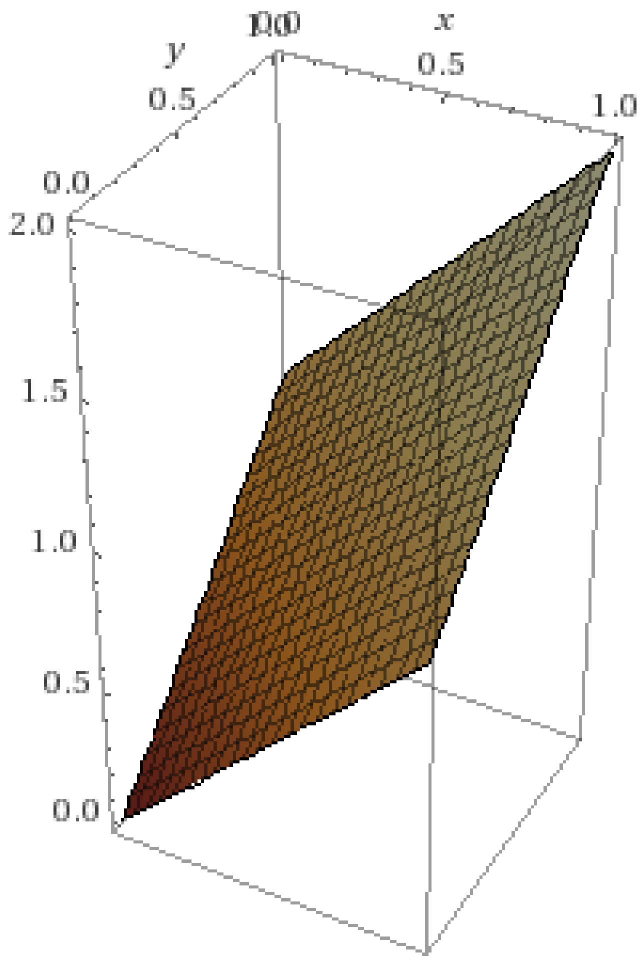

Figure 1: Graph of $f(x, y)$ (Problem 7) (Using Wolfram Alpha).