



# MACHINE LEARNING 2018

## Midterm Exam

December 17, 2018

- This exam is due at 2 PM, January 5, 2019.
- This exam covers all the materials up to the Lecture 12 - Neural Networks.
- Please submit the exam via Google Form (Link will be sent out shortly).  
Code for programming problems should be submitted as .py (Python) or

.ipynb (Jupyter Notebook) files. Please clear all the output of Jupyter Notebooks before submission.

- You can discuss exam problems with the instructor, TAs, classmates, or others, but the work you submit must be your own work.
- You may write your answers in Vietnamese or English or a mix of both languages.
- You may consult textbooks and print and online materials.
- Please show all of your work. Answers without appropriate justification will receive very little credit. For programming questions, please submit all the code.

**Problem 1.** (*5 points*)

A symmetric  $n \times n$  real matrix  $M$  is said to be positive definite if we have

$$\mathbf{z}^T M \mathbf{z} > 0 \quad \forall \mathbf{z} \in \mathbb{R}^n \quad (1)$$

where we use  $\mathbb{R}^n$  to denote the set of  $n$ -dimensional real-valued vectors ( $\mathbf{z}$ 's dimension is  $n \times 1$ ). Recall that an eigenvalue  $\lambda_i$  of matrix  $M$  is a scalar that satisfies

$$M \mathbf{u}_i = \lambda_i \mathbf{u}_i \quad (2)$$

where  $\mathbf{u}_i$  is an eigenvector of  $M$ . Show that a necessary and sufficient condition for  $M$  to be positive definite is that all of the eigenvalues  $\lambda_i$  of  $M$  are positive.

**Problem 2.** (20 points) (Adapted from Cover & Thomas and Yurdakul)

If  $p(x)$  and  $q(x)$  are two probability mass functions (pmf's), the relative entropy or Kullback-Leibler divergence (or KL divergence), between  $p(x)$  and  $q(x)$ , is defined to be

$$D(p||q) = \sum_{x \in X} p(x) \log \frac{p(x)}{q(x)} = E_p \log \frac{p(x)}{q(x)} \quad (3)$$

where we can have different bases for the  $\log$  function (the most popular bases being  $e$  and 2), and use the convention that  $0 \ln_q^0 = 0$  and  $p \ln_q^p = \infty$ .  $X$  is the set of all possible values of  $x$ . The KL divergence is not considered to be a true distance between distributions as it is not symmetric and does not satisfy the triangle inequality.

(a) Prove that the KL divergence defined in Equation (3) is always non-negative and is zero if and only if  $p = q$ .

Let  $X = \{0, 1\}$  and consider two Bernoulli distributions  $p$  and  $q$  on  $X$ . Let  $p(0) = 1 - r$ ,  $p(1) = r$ , and let  $q(0) = 1 - s$ ,  $q(1) = s$ .

(b) Derive  $D(p||q)$  and  $D(q||p)$  as functions of  $r$  and  $s$ .

(c) Verify that if  $r=s$  then  $D(p||q) = D(q||p) = 0$ .

(d) If  $r = 1/2$  and  $s = 1/4$ , calculate  $D(p||q)$  and  $D(q||p)$  using  $\log_2$  in Equation (3).

In practice, when we have two populations  $\hat{p}$  and  $\hat{q}$ , we normally group values of  $x$  into bins and write Equation (3) as

$$D(\hat{p}||\hat{q}) = \sum_{i=1}^B \hat{p}_i(x) \log \frac{\hat{p}_i(x)}{\hat{q}_i(x)} \quad (4)$$

where  $B$  is the number of bins, and  $\hat{p}_i$  and  $\hat{q}_i$  are proportions of populations  $\hat{p}$  and  $\hat{q}$ , respectively, in bin  $i$ . The *Population Stability Index (PSI)* is then defined as

$$PSI(\hat{p}, \hat{q}) = D(\hat{p}||\hat{q}) + D(\hat{q}||\hat{p}) \quad (5)$$

PSI is widely used to measure the difference between

- feature distributions of the training samples and samples being used for the model (the current samples) in a Machine Learning model;
- feature distributions between different points in time;
- outcome distributions.

In practice, there is a general rule of thumb: if PSI between the training samples and current samples is

- less than 10%, the model is considered appropriate;
- between 10% and 25%, we have to investigate the current samples to see why the PSI is so high;
- beyond 25%, we should retrain the model or develop a new model using more recent samples.

(e) Prove that

$$PSI(\hat{p}, \hat{q}) = \sum_{i=1}^B [\hat{p}_i(x) - \hat{q}_i(x)] [\log(\hat{p}_i(x)) - \log(\hat{q}_i(x))] \quad (6)$$

**Problem 3.** (*5 points*) (Adapted from Bishop) It was mentioned in class that we can use '*tanh*' as an activation function for neural networks. It was also mentioned that

$$\tanh(a) = 2\sigma(2a) - 1 \quad (7)$$

Using this equation, show that a general linear combination of logistic sigmoid functions of the form

$$y(x, \mathbf{w}) = w_0 + \sum_{j=1}^M w_j \sigma\left(\frac{x - \mu_j}{s}\right) \quad (8)$$

is equivalent to a linear combination of '*tanh*' functions of the form

$$y(x, \mathbf{u}) = u_0 + \sum_{j=1}^M u_j \tanh\left(\frac{x - \mu_j}{2s}\right) \quad (9)$$

Find expressions to relate the new parameters  $\{u_0, u_1, \dots, u_M\}$  to the original parameters  $\{w_0, w_1, \dots, w_M\}$ .

**Problem 4.** (10 points) (Please look under Section 'Exam', [https://piazza.com/vef\\_academy/winter2018/ml101/resources](https://piazza.com/vef_academy/winter2018/ml101/resources) for the dataset 'uber\_usage.csv').

In a survey of Uber customers, a marketing agency interviewed 400 customers that had had at least 2 Uber rides per month in the past 12 months. As can be seen in the file 'uber\_usage.csv', they obtained the age (column 'Age') and the annual salary (in USD, column 'AnnualSalary') for each customer (column 'CustomerID'). Based on how often each customer had used Uber, they also labeled the customer as a regular rider (value 1 in column 'RegularRider') or non-regular rider (value 0 in column 'RegularRider'). We want to apply Logistic Regression to classify customers into two classes: Regular Riders and Non-regular riders based on the age and the annual salary.

(a) Split the dataset to training set and test set using the ratio training set : test set = 7 : 3. The utility *model\_selection.train\_test\_split* can be used to split the dataset.

(b) Use scikit-learn's *StandardScaler*, fit and transform each feature (age, annual salary) into a standard normal distribution. Then use the same parameters of the distributions to transform the test set.

(c) Use scikit-learn's Logistic Regression model *linear\_model.LogisticRegression* to train a model on the training set and apply it to the test set.

(d) Calculate the TP rate, FP rate, FN rate, Precision, Recall, F1 score, and AUC on the training set and test set. Compare the AUC on the training set and test set.

(e) Visualize the feature values and labels of the training set using a scatter plot. Visualize the feature values and decisions on another scatter plot.

**Problem 5.** (*10 points*) Using the same dataset in Problem 4, also split the dataset with the same ratio and standardize the feature values with StandardScaler.

(a) Use scikit-learn's SVM model *sklearn.svm.SVC* to train a model on the training set and apply it to the test set.

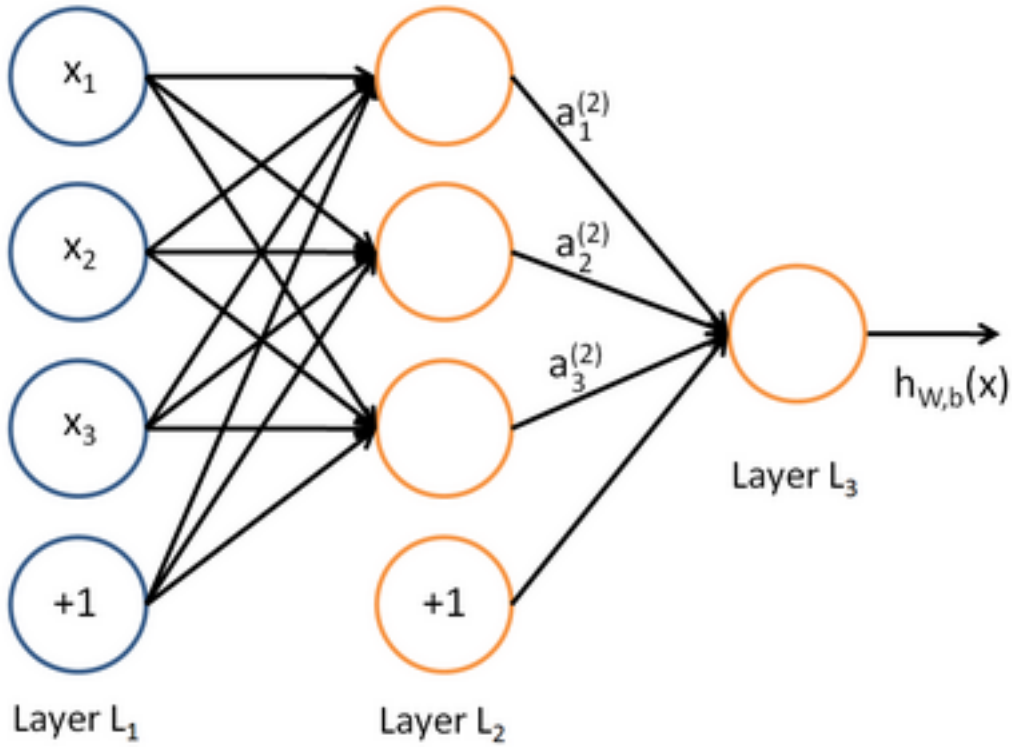
(b) Calculate the TP rate, FP rate, FN rate, Precision, Recall, F1 score, and AUC on the training set and test set. Compare the AUC on the training set and test set.

(c) Visualize the feature values and labels of the training set using a scatter plot. Also plot the line that is used to separate the two classes.

(d) Visualize the feature values and decisions on another scatter plot. Also plot the line that is used to separate the two classes.

(e) Calculate the PSI between the age distributions in the training set and test set, and the PSI between the annual salary distributions in the training set and test set.





**Problem 6.** (50 points) (a) Suppose we have a neural network in the above graph (Figure from [5]).

In this problem, we are going to use the notations from [4], where  $\mathbf{W}^{(l)}$  and  $\mathbf{b}^{(l)}$  are respectively the weight matrix and the bias vector between the  $(l-1)^{th}$  layer and  $l^{th}$  layer,  $l = 1, 2, 3$ . Suppose further that we use the sigmoid function as the activation function.

- Write the feedforward equations to calculate the output from the inputs
- Write the backpropagation equations to calculate the gradient of the loss function  $J(\mathbf{W}, \mathbf{b}, \mathbf{X}, \mathbf{Y})$  with respect to  $\mathbf{W}^{(l)}$  and  $\mathbf{b}^{(l)}$ .

(b) Using the training set and test set from <https://github.com/zalando-research/fashion-mnist>, build a neural network with 784 inputs and the biases, and 10 outputs. You have to implement the Backpropagation algorithm and Gradient Descent algorithm by yourself.

(c) Calculate the multi-class AUC for the training set and test set using the formulas given in [8].

## References

1. Thomas M. Cover and Joy A. Thomas. 2006. Elements of Information Theory (Wiley Series in Telecommunications and Signal Processing). Wiley-Interscience, New York, NY, USA.
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3. Christopher M. Bishop. 2006. Pattern Recognition and Machine Learning (Information Science and Statistics). Springer-Verlag, Berlin, Heidelberg.
4. Vu Huu Tiep – Machine Learning co ban, <https://machinelearningcoban.com/2017/02/24/mlp/>
5. Neural Networks – [http://ufldl.stanford.edu/wiki/index.php/Neural\\_Networks](http://ufldl.stanford.edu/wiki/index.php/Neural_Networks)
6. M. Pathak, "Introduction to t-SNE", <https://www.datacamp.com/community/tutorials/introduction-t-sne>
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8. Tom Fawcett. 2006. An introduction to ROC analysis. Pattern Recogn. Lett. 27, 8 (June 2006), 861-874. DOI=<http://dx.doi.org/10.1016/j.patrec.2005.10.010>