SMA 2371: PARTIAL DIFFERENTIAL EQUATIONS

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Course content

- Surfaces and curves in three dimensions. Simultaneous differential equations of the first order. Methods of solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Orthogonal trajectories of system of curves on a surface. Pfaffian differential equations.
- Linear partial differential equations of the first order: formation of PDEs, Lagrange PDEs.
- Partial differential equations of the second order: Laplace, Poisson, heat and wave equations. Methods of the solution by separation of the variables for Cartesian, cylindrical and spherical coordinates, and by Laplace and Fourier transforms.
- Applications to engineering.

References

[1] Ordinary and Partial Differential Equations by M. D. Raisinghania. S.Chand and Company Limited, 18th edition.

[2] Elements of Partial Differential Equations by Ian N. Sneddon

[3] Advanced Engineering Mathematics (10th ed.) by Erwin Kreyszig

[4] Schaum's Outline Series: Theory and problems of PDE

LECTURE 1

1 Ordinary differential equations in more than two variables

Properties of ODEs in more than two variables play an important role in the theory of partial differential equations. Thus, it is important that the concepts of ODEs in 3 variables be understood thoroughly prior to the study of PDEs.

1.1 Direction cosines

Let F and G be differentiable functions of x, y and z and define a partial vector differential operator or the del operator ($\vec{\nabla}$) by

$$\vec{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$$

*https://www.youtube.com/channel/UC7wd3x6_08GNoUbLxmaC4vA

The gradients of F and G are given by

$$\vec{\nabla}F = \hat{\mathbf{i}}\frac{\partial F}{\partial x} + \hat{\mathbf{j}}\frac{\partial F}{\partial y} + \hat{\mathbf{k}}\frac{\partial F}{\partial z} \quad \text{and} \quad \vec{\nabla}G = \hat{\mathbf{i}}\frac{\partial G}{\partial x} + \hat{\mathbf{j}}\frac{\partial G}{\partial y} + \hat{\mathbf{k}}\frac{\partial G}{\partial z},$$

respectively. The cross product (or vector product) of $\vec{\nabla}F$ and $\vec{\nabla}G$ is defined by

$$\begin{aligned}
\vec{\nabla}F \times \vec{\nabla}G = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix} &= \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}\hat{\mathbf{k}} \\
&= (F_yG_z - F_zG_y)\hat{\mathbf{i}} + (F_zG_x - F_xG_z)\hat{\mathbf{j}} + (F_xG_y - F_yG_x)\hat{\mathbf{k}} \\
&= \frac{\partial(F,G)}{\partial(y,z)}\hat{\mathbf{i}} + \frac{\partial(F,G)}{\partial(z,x)}\hat{\mathbf{j}} + \frac{\partial(F,G)}{\partial(x,y)}\hat{\mathbf{k}} \\
&= P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}},
\end{aligned}$$

where

$$\boxed{P = \frac{\partial(F,G)}{\partial(y,z)}, \quad Q = \frac{\partial(F,G)}{\partial(z,x)}, \quad \text{and} \quad R = \frac{\partial(F,G)}{\partial(x,y)}}$$

These components are called **direction cosines** or Jacobian of F and G.

Example(s):

(a) Find the direction cosines of the tangent at point (x, y, z) to the conics $x^2y + xz = 0$ and $x^2 + y^2 + z^2 = 0$.

Solution

Let $F = x^2y + xz$ and $G = x^2 + y^2 + z^2$. So,

$$\begin{aligned}
P &= \frac{\partial(F,G)}{\partial(y,z)} = F_yG_z - F_zG_y = 2x^2z - 2xy = 2x(xz - y) \\
Q &= \frac{\partial(F,G)}{\partial(z,x)} = F_zG_x - F_xG_z = 2x^2 - 2z(2xy + z) \\
R &= \frac{\partial(F,G)}{\partial(x,y)} = F_xG_y - F_yG_x = 2y(2xy + z) - 2x^3
\end{aligned}$$

(b) Show that the direction cosines of the tangent at point (x, y, z) to the conics $ax^2 + by^2 + cz^2 = 1$ and $x + y + z = 1$ are proportional to $(by - cz, cz - ax, ax - by)$, where a, b and c are constants.

Solution

Let $F = ax^2 + by^2 + cz^2 - 1$ and $G = x + y + z - 1$. So,

$$\begin{aligned}
P &= \frac{\partial(F,G)}{\partial(y,z)} = F_yG_z - F_zG_y = 2by - 2cz \\
Q &= \frac{\partial(F,G)}{\partial(z,x)} = F_zG_x - F_xG_z = 2cz - 2ax \\
R &= \frac{\partial(F,G)}{\partial(x,y)} = F_xG_y - F_yG_x = 2ax - 2by
\end{aligned}$$

Therefore, $(P, Q, R) = 2(by - cz, cz - ax, ax - by)$ or

$$(P, Q, R) \propto (by - cz, cz - ax, ax - by),$$

where 2 is the constant of proportionality.

1.2 Simultaneous differential equations of the first-order

Consider a curve in space which is the intersection of the surfaces $F(x,y,z)=0$ and $G(x,y,z)=0$. It follows from differential calculus that the total differential of F is

$$dF=0 \quad \text{i.e.,} \quad \frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial z}dz=0 \quad \text{or} \quad \frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy=-\frac{\partial F}{\partial z}dz\cdots(i)$$

Similarly, the total differential of G is

$$dG=0 \quad \text{i.e.,} \quad \frac{\partial G}{\partial x}dx+\frac{\partial G}{\partial y}dy+\frac{\partial G}{\partial z}dz=0 \quad \text{or} \quad \frac{\partial G}{\partial x}dx+\frac{\partial G}{\partial y}dy=-\frac{\partial G}{\partial z}dz\cdots(ii)$$

In matrix-vector form, equations (i) and (ii) become

$$\begin{pmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y}\\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y}\end{pmatrix}\begin{pmatrix}dx\\ dy\end{pmatrix}=\begin{pmatrix}-\frac{\partial F}{\partial z}dz\\ -\frac{\partial G}{\partial z}dz\end{pmatrix} \tag{1}$$

Let $A=\begin{pmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y}\\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y}\end{pmatrix}$, $A_1=\begin{pmatrix}-\frac{\partial F}{\partial z}dz & \frac{\partial F}{\partial y}\\ -\frac{\partial G}{\partial z}dz & \frac{\partial G}{\partial y}\end{pmatrix}$ and $A_2=\begin{pmatrix}\frac{\partial F}{\partial x} & -\frac{\partial F}{\partial z}dz\\ \frac{\partial G}{\partial x} & -\frac{\partial G}{\partial z}dz\end{pmatrix}$. The determinants of A, A_1 and A_2 are give by

$$\det(A)=\frac{\partial F}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial F}{\partial y}\frac{\partial G}{\partial x}=F_xG_y-F_yG_x=R$$

$$\det(A_1)=-\frac{\partial F}{\partial z}\frac{\partial G}{\partial y}dz+\frac{\partial F}{\partial y}\frac{\partial G}{\partial z}dz=(F_yG_z-F_zG_y)dz=Pdz$$

$$\det(A_2)=-\frac{\partial F}{\partial x}\frac{\partial G}{\partial z}dz+\frac{\partial F}{\partial z}\frac{\partial G}{\partial x}dz=(F_zG_x-F_xG_z)dz=Qdz$$

Solving equation (1) using Cramer's rule, we obtain

$$dx=\frac{\det(A_1)}{\det(A)}=\frac{Pdz}{R} \quad\Rightarrow\quad \frac{dx}{P}=\frac{dz}{R} \qquad (*)$$

and

$$dy=\frac{\det(A_2)}{\det(A)}=\frac{Qdz}{R} \quad\Rightarrow\quad \frac{dy}{Q}=\frac{dz}{R} \qquad (**)$$

Equating equations $(*)$ and $(**)$, we obtain

$$\boxed{\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}} \tag{2}$$

Equations (2) are called *symmetric equations*. They are the differential equations of the **integral curves** (i.e., the intersection of two surfaces).

Lecture 2

1.3 Methods of solution of $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

The reverse problem is that of finding the surfaces $F(x,y,z)=0$ and $G(x,y,z)=0$ from equations (2). We need to obtain two unique solutions called the ***integral curves***, say $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$, where c_1 and c_2 are arbitrary constants. The ***integral surface*** generated by these integral curves is given by $\phi(u,v)=0$, where ϕ is an arbitrary function. There are 3 methods of solving equations (2):

1. Method of multipliers (intuition)
2. One variable absent (method of grouping)
3. Exact differential method

1.3.1 Method of Multipliers (method of intuition)

This method uses multipliers l, m and n (which are not necessarily constants) so that for a constant k, equation (2) becomes

$$\frac{ldx + mdy + ndz}{lP + mQ + nR} = k \quad \Rightarrow \quad ldx + mdy + ndz = k(lP + mQ + nR)$$

These multipliers are chosen such that the condition $\boxed{lP + mQ + nR = 0}$ is satisfied. Hence,

$$ldx + mdy + ndz = 0,$$

which on integration yields the first integral curve, $u(x, y, z) = c_1$. It may be possible to get another set of multipliers l, m and n to obtain the second integral curve, $v(x, y, z) = c_2$.

Example(s):

(a) Find the integral curves and the integral surface of the equations

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

Solution

From the given equations we have $P = x(y-z)$, $\quad Q = y(z-x)$ and $R = z(x-y)$. Using the multipliers l, m and n, we require that $lP + mQ + nR = 0 \cdots (*)$ so that $ldx + mdy + ndz = 0$. Now,

i) By intuition, condition $(*)$ is satisfied if we take $l = 1, m = 1$ and $n = 1$. Hence, $dx + dy + dz = 0$. Integrating yields

$$x + y + z = c_1$$

ii) Again by intuition, condition $(*)$ is satisfied if we take $l = \frac{1}{x}, m = \frac{1}{y}$ and $n = \frac{1}{z}$. Hence, $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$. Integrating yields

$$\ln x + \ln y + \ln z = \ln c_2 \quad \Rightarrow \quad xyz = c_2$$

Therefore, the required integral curves are $x + y + z = c_1$ and $xyz = c_2$, where c_1 and c_2 are parameters. The integral surface generated by these curves is $\phi(c_1, c_2) = 0$, i.e.,

$$\phi(x + y + z, xyz) = 0$$

or

$$x + y + z = f(xyz)$$

or

$$xyz = g(x + y + z),$$

where ϕ, f and g are arbitrary functions.

Exercise:

1. By choosing an appropriate set of multipliers, solve the equation $\frac{dx}{x + 2z} = \frac{dy}{4xz - y} = \frac{dz}{2x^2 + y}$. [hint: multipliers $(2x, -1, -1)$ and $(y, x, -2z)$, ans: $x^2 - y - z = c_1, \;\; xy - z^2 = c_2$]

1.3.2 One variable absent

If one of the variables is absent in one of the equations, we can easily derive one of the solutions. Also, it is possible to use the first solution to obtain a second solution by eliminating one of the variables in one of the equations.

Example(s):

(a) Find the integral curves of the equations $\dfrac{dx}{x+z} = \dfrac{dy}{y} = \dfrac{dz}{z+y^2}$.

Solution

i) Since x is absent in the second and third equations, we take

$$\frac{dy}{y} = \frac{dz}{z+y^2} \quad \Rightarrow \quad \frac{dz}{dy} - \frac{1}{y}z = y \qquad (*)$$

This is a linear ODE in z with integrating factor

$$\text{I.F} = e^{\int \frac{-1}{y}dy} = e^{-\ln y} = \frac{1}{y}$$

Multiplying equation $(*)$ through by $\dfrac{1}{y}$ yields $\dfrac{d}{dy}\left(\dfrac{z}{y}\right) = 1$. Integrating yields

$$\frac{z}{y} = y + c_1 \quad \Rightarrow \quad z = y^2 + c_1 y$$

ii) Using the first solution to eliminate z in the first equation and equating to the second equation, we have

$$\frac{dx}{x+z} = \frac{dy}{y} \quad \Rightarrow \quad \frac{dx}{x+y^2+c_1y} = \frac{dy}{y} \quad \Rightarrow \quad \frac{dx}{dy} - \frac{1}{y}x = y + c_1 \qquad (**)$$

This is a linear ODE in x with integrating factor given by

$$\text{I.F} = e^{\int \frac{-1}{y}dy} = e^{-\ln y} = \frac{1}{y}$$

Multiplying equation $(**)$ through by $\dfrac{1}{y}$ yields $\dfrac{d}{dy}\left(\dfrac{x}{y}\right) = 1 + \dfrac{c_1}{y}$. Integrating yields

$$\frac{x}{y} = y + c_1 \ln y + c_2 \quad \Rightarrow \quad x = c_1 y \ln y + c_2 y + y^2$$

Therefore, the required integral curves are $z = c_1 y + y^2$ and $x = c_1 y \ln y + c_2 y + y^2$, where c_1 and c_2 are parameters.

Exercise:

1. Find the integral curves of the following equations.

   (a) $\dfrac{dx}{x+y} = \dfrac{dy}{x+y} = \dfrac{dz}{-(x+y+2z)}$. [ans: $x = y + c_1$, $z(2y+x-y) = -y^2 - (x-y)y + c_2$]

   (b) $\dfrac{dx}{x} = \dfrac{dy}{y+z} = \dfrac{dz}{x^2+z}$. [ans: $z = x^2 + c_1 x$, $\dfrac{y}{x} = x + c_1 \ln x + c_2$]

   (c) $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{xyz^2(x^2-y^2)}$. [ans: $x^2 - y^2 = c_1$, $x^2(x^2-y^2) + \dfrac{2}{z} = c_2$]

1.3.3 Exact differential method

It is much simpler to try to cast the given differential equations into a form which suggests their solution. Note that adding/subtracting the numerators and denominators of any two "fractions" doesn't alter their value. For example,

$$\begin{aligned}\frac{1}{2} = \frac{3}{6} = \frac{5}{10} &= \frac{1+3+5}{2+6+10}\\ &= \frac{1-5}{2-10}\\ &= \frac{4(1)+7(3)-5}{4(2)+7(6)-10}, \text{ e.t.c.}\end{aligned}$$

Hence, we may write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{dx+dy}{P+Q} = \frac{dx-dy-2dz}{P-Q-2R} = \frac{ldx+mdy+ndz}{lP+mQ+nR}, \text{ e.t.c.}$$

So that if $lP+mQ+nR=0$, then $ldx+mdy+ndz=0$ is an exact differential. This method is best illustrated in the following example.

Example(s):

(a) Find the integral curves of the equations $\dfrac{dx}{y(x+y)+az} = \dfrac{dy}{x(x+y)-az} = \dfrac{dz}{z(x+y)}$.

Solution

We have

$$\frac{dx}{y(x+y)+az} = \frac{dy}{x(x+y)-az} = \frac{dz}{z(x+y)} = \frac{dx+dy}{(x+y)^2} = \frac{xdx-ydy}{az(x+y)}$$

i) From $\dfrac{dx+dy}{(x+y)^2} = \dfrac{dz}{z(x+y)} \quad\Rightarrow\quad \dfrac{dx+dy}{x+y} = \dfrac{dz}{z} \quad\Rightarrow\quad \ln(x+y) = \ln z + \ln c_1$

or

$$x+y = c_1 z$$

ii) From $\dfrac{xdx-ydy}{az(x+y)} = \dfrac{dz}{z(x+y)} \quad\Rightarrow\quad xdx-ydy-adz=0$

or

$$x^2-y^2-2az = c_2$$

Therefore, the required integral curves are $x+y=c_1z$ and $x^2-y^2-2az=c_2$, where c_1 and c_2 are parameters.

Exercise:

1. Use any appropriate method to find the integral curves of the following equations.

(a) $\dfrac{dx}{y^2} = \dfrac{dy}{-xy} = \dfrac{dz}{x(z-2y)}$.

Solution

i) From $\dfrac{dx}{y^2} = \dfrac{dy}{-xy} \quad\Rightarrow\quad xdx+ydy=0 \quad\Rightarrow\quad x^2+y^2=c_1$

ii) From $\dfrac{dy}{-xy} = \dfrac{dz}{x(z-2y)} \quad\Rightarrow\quad zdy+ydz-2ydy=0 \quad\Rightarrow\quad d(zy)-2ydy=0$

or

$$zy-y^2=c_2$$

Therefore, the integral curves of the given differential equations are the members of the two-parameter family $x^2+y^2=c_1$ and $zy-y^2=c_2$.

(b) $\dfrac{dx}{x^2-y^2-z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$.

Solution

i) From $\dfrac{dy}{2xy} = \dfrac{dz}{2xz} \quad \Rightarrow \quad \dfrac{dy}{y} = \dfrac{dz}{z} \quad \Rightarrow \quad \ln y = \ln z + \ln c_1 \quad \Rightarrow \quad y = c_1 z$

ii) From $\dfrac{xdx + ydy + zdz}{x(x^2-y^2-z^2) + y(2xy) + z(2xz)} = \dfrac{dy}{2xy} \quad \Rightarrow \quad \dfrac{2(xdx+ydy+zdz)}{(x^2+y^2+z^2)} = \dfrac{dy}{y}$ or

$$\ln(x^2+y^2+z^2) = \ln y + \ln c_2 \quad \Rightarrow \quad x^2+y^2+z^2 = c_2 y$$

Therefore, the integral curves of the given differential equations are the members of the two-parameter family $y = c_1 z$ and $x^2+y^2+z^2 = c_2 y$.

(c) $\dfrac{dx}{xz-y} = \dfrac{dy}{yz-x} = \dfrac{dz}{1-z^2}$.

Solution

i) From $\dfrac{dx+dy}{xz-y+yz-x} = \dfrac{dz}{1-z^2} \quad \Rightarrow \quad \dfrac{dx+dy}{x+y} = \dfrac{-dz}{1+z}$ or

$$(x+y)(1+z) = c_1$$

ii) From $\dfrac{xdx - ydy}{x(xz-y) - y(yz-x)} = \dfrac{zdz}{z(1-z^2)} \quad \Rightarrow \quad \dfrac{xdx-ydy}{x^2-y^2} = \dfrac{zdz}{1-z^2}$ or

$$(x^2-y^2)(1-z^2) = c_2$$

Therefore, the integral curves of the given differential equations are the members of the two-parameter family $(x+y)(1+z) = c_1$ and $(x^2-y^2)(1-z^2) = c_2$.

(d) $\dfrac{dx}{x(y^2-z^2)} = \dfrac{dy}{y(z^2-x^2)} = \dfrac{dz}{z(x^2-y^2)}$. [ans: $x^2+y^2+z^2 = c_1, \quad xyz = c_2$]

(e) $\dfrac{dx}{y-xz} = \dfrac{dy}{x+yz} = \dfrac{dz}{x^2+y^2}$. [ans: $x^2-y^2+z^2 = c_1, \quad xy - z = c_2$]

(f) $\dfrac{dx}{x(2y^4-z^4)} = \dfrac{dy}{y(z^4-2x^4)} = \dfrac{dz}{z(x^4-y^4)}$. [ans: $x^4+y^4+z^4 = c_1, \quad xyz^2 = c_2$]

(g) $\dfrac{dx}{x^2-yz} = \dfrac{dy}{y^2-xz} = \dfrac{dz}{z^2-xy}$. [ans: $\dfrac{x-y}{y-z} = c_1, \quad xy+yz+xz = c_2$]

(h) $\dfrac{dx}{x(x+y)} = \dfrac{dy}{-y(x+y)} = \dfrac{dz}{-(x-y)(2x+2y+z)}$. [ans: $xy = c_1, \quad (x+y)(x+y+z) = c_2$]

LECTURE 3

1.4 Orthogonal trajectories of a system of curves on a surface

Given a surface

$$F(x, y, z) = 0 \tag{3}$$

and a family (or system) of curves on it, we need to find another family of curves each of which lies on the surface (3) and cuts every curve of the original family at right angles. The new family of curves is called the family of *orthogonal trajectories* on the surface (3).

The original family of curves may be thought of as the intersections of the surface (3) with the one-parameter family of surfaces

$$G(x, y, z) = k. \tag{4}$$

From differential calculus, the total differential of equation (3) is

$$dF = 0, \text{ i.e., } F_x dx + F_y dy + F_z dz = 0 \text{ or } F_x dx + F_y dy = -F_z dz \qquad (i)$$

Similarly, the total differential of equation (4) is

$$dG = 0, \text{ i.e., } G_x dx + G_y dy + G_z dz = 0 \text{ or } G_x dx + G_y dy = -G_z dz \qquad (ii)$$

In matrix-vector form, equations (i) and (ii) become

$$\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} -F_z dz \\ -G_z dz \end{pmatrix} \qquad (*)$$

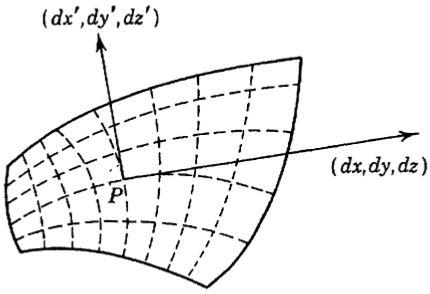
Solving the system $(*)$ using Cramer's rule, we obtain

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R},$$

where

$$\boxed{P = \frac{\partial(F,G)}{\partial(y,z)}, \quad Q = \frac{\partial(F,G)}{\partial(z,x)} \text{ and } R = \frac{\partial(F,G)}{\partial(x,y)}}$$

Consider the diagram below for the surface (3) and let (x, y, z) be an arbitrary point on the surface:



The vector (dx, dy, dz) is in the tangential direction to the surface at point (x, y, z). The curve through point (x, y, z) of the orthogonal family has tangential direction (dx', dy', dz'), which also lies on the surface. A vector normal to the surface at point (x, y, z) is given by

$$\vec{\nabla} F = \frac{\partial F}{\partial x}\hat{\mathbf{i}} + \frac{\partial F}{\partial y}\hat{\mathbf{j}} + \frac{\partial F}{\partial z}\hat{\mathbf{k}} = (F_x, F_y, F_z)$$

Since the vectors (F_x, F_y, F_z) and (dx', dy', dz') are perpendicular, their dot product must be zero, i.e.,

$$(F_x, F_y, F_z)\cdot(dx', dy', dz') = 0 \quad \Rightarrow \quad F_x dx' + F_y dy' + F_z dz' = 0 \quad \Rightarrow \quad F_x dx' + F_y dy' = -F_z dz' \quad (5)$$

Also since the vectors (dx, dy, dz) and (dx', dy', dz') are perpendicular, then

$$(dx, dy, dz)\cdot(dx', dy', dz') = 0 \quad \Rightarrow \quad dx dx' + dy dy' + dz dz' = 0 \qquad (iii)$$

To eliminate the differentials dx, dy and dz from equation (iii), we let $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \mu$ (for some constant $\mu \neq 0$) so that $dx = \mu P$, $dy = \mu Q$, $dz = \mu R$. Substituting into equation (iii) yields

$$\mu(P dx' + Q dy' + R dz') = 0 \quad \Rightarrow \quad P dx' + Q dy' + R dz' = 0 \quad \Rightarrow \quad P dx' + Q dy' = -R dz' \qquad (6)$$

In matrix-vector form, equations (5) and (6) become

$$\begin{pmatrix} F_x & F_y \\ P & Q \end{pmatrix} \begin{pmatrix} dx' \\ dy' \end{pmatrix} = \begin{pmatrix} -F_z dz' \\ -R dz' \end{pmatrix} \qquad (**)$$

Solving the system $(**)$ using Cramer's rule, we obtain

$$\frac{dx'}{RF_y - QF_z} = \frac{dy'}{PF_z - RF_x} = \frac{dz'}{QF_x - PF_y}$$

or

$$\frac{dx'}{P'} = \frac{dy'}{Q'} = \frac{dz'}{R'}, \tag{7}$$

where

$$\boxed{P' = \begin{vmatrix} F_y & F_z \\ Q & R \end{vmatrix},\ Q' = \begin{vmatrix} F_z & F_x \\ R & P \end{vmatrix} \text{ and } R' = \begin{vmatrix} F_x & F_y \\ P & Q \end{vmatrix}}$$

The solution of equations (7) along with the surface (3) gives the system of orthogonal trajectories.

Example(s):

(a) Find the orthogonal trajectories on the surface $x^2+y^2 = \alpha z^2$ of its intersections with the family of planes parallel to the xy-plane (i.e., $z = k$), where α is a constant.

Solution

Let $F(x, y, z) = x^2 + y^2 - \alpha z^2 \equiv 0$ and $G(x, y, z) = z \equiv k$. The first partial derivatives of F and G are

$$F_x = 2x,\ \ F_y = 2y,\ \ F_z = -2\alpha z,\ \ G_x = 0,\ \ G_y = 0,\ \ G_z = 1$$

The direction cosines of the tangent at point (x, y, z) to the surfaces F and G are

$$\begin{aligned} P &= \frac{\partial(F,G)}{\partial(y,z)} = F_yG_z - F_zG_y = 2y - (-2\alpha z)(0) = 2y \\ Q &= \frac{\partial(F,G)}{\partial(z,x)} = F_zG_x - F_xG_z = (-2\alpha z)(0) - 2x(1) = -2x \\ R &= \frac{\partial(F,G)}{\partial(x,y)} = F_xG_y - F_yG_x = 2x(0) - 2y(0) = 0 \end{aligned}$$

Hence,

$$\begin{aligned} P' &= \begin{vmatrix} F_y & F_z \\ Q & R \end{vmatrix} = \begin{vmatrix} 2y & -2\alpha z \\ -2x & 0 \end{vmatrix} = -4\alpha xz \\ Q' &= \begin{vmatrix} F_z & F_x \\ R & P \end{vmatrix} = \begin{vmatrix} -2\alpha z & 2x \\ 0 & 2y \end{vmatrix} = -4\alpha yz \\ R' &= \begin{vmatrix} F_x & F_y \\ P & Q \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2y & -2x \end{vmatrix} = -4x^2 - 4y^2 \end{aligned}$$

The symmetric equations for the orthogonal trajectories are given by:

$$\frac{dx}{P'} = \frac{dy}{Q'} = \frac{dz}{R'} \quad \Rightarrow \quad \frac{dx}{-4\alpha xz} = \frac{dy}{-4\alpha yz} = \frac{dz}{-4x^2-4y^2}$$

or

$$\frac{dx}{x} = \frac{dy}{y} = \frac{\alpha z dz}{x^2+y^2}$$

i) From $\dfrac{dx}{x} = \dfrac{dy}{y} \quad \Rightarrow \quad \dfrac{dx}{x} = \dfrac{dy}{y} \quad \Rightarrow \quad \ln x = \ln y + \ln c_1 \quad \Rightarrow \quad x = c_1 y$, where c_1 is a parameter.

ii) From $\dfrac{xdx+ydy}{x^2+y^2} = \dfrac{\alpha z dz}{x^2+y^2} \quad \Rightarrow \quad xdx+ydy = \alpha z dz \quad \Rightarrow \quad x^2+y^2 = \alpha z^2 + c_2$. Comparing with the given surface, we have $c_2 = 0$. Therefore,

$$x^2 + y^2 = \alpha z^2$$

Hence $F(x, y, z) \equiv 0$ is necessarily one of the solutions.

Therefore, the orthogonal trajectories are the curves $x = c_1 y$ and $x^2 + y^2 = \alpha z^2$.

(b) Find the equations of the system of curves on the cylinder $2y = x^2$ orthogonal to its intersections with the hyperboloids of the one-parameter system $xy = z + c$, where c is a parameter.

Solution

Let $F = 2y - x^2 \equiv 0$ and $G = xy - z \equiv c$. The first partial derivatives of F and G are

$$F_x = -2x, \quad F_y = 2, \quad F_z = 0, \quad G_x = y, \quad G_y = x, \quad G_z = -1$$

The direction cosines of the tangent at point (x, y, z) to the surfaces F and G are

$$\begin{aligned} P &= \frac{\partial(F,G)}{\partial(y,z)} = F_yG_z - F_zG_y = -2 \\ Q &= \frac{\partial(F,G)}{\partial(z,x)} = F_zG_x - F_xG_z = -2x \\ R &= \frac{\partial(F,G)}{\partial(x,y)} = F_xG_y - F_yG_x = -2x^2 - 2y = -2(x^2 + y) \end{aligned}$$

Hence,

$$\begin{aligned} P' &= \begin{vmatrix} F_y & F_z \\ Q & R \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ -2x & -2(x^2+y) \end{vmatrix} = -4(x^2+y) \\ Q' &= \begin{vmatrix} F_z & F_x \\ R & P \end{vmatrix} = \begin{vmatrix} 0 & -2x \\ -2(x^2+y) & -2 \end{vmatrix} = -4x(x^2+y) \\ R' &= \begin{vmatrix} F_x & F_y \\ P & Q \end{vmatrix} = \begin{vmatrix} -2x & 2 \\ -2 & -2x \end{vmatrix} = 4(x^2+1) \end{aligned}$$

The symmetric equations for the orthogonal trajectories are

$$\frac{dx}{P'} = \frac{dy}{Q'} = \frac{dz}{R'} \quad \Rightarrow \quad \frac{dx}{-4(x^2+y)} = \frac{dy}{-4x(x^2+y)} = \frac{dz}{4(x^2+1)}$$

or

$$\frac{dx}{x^2+y} = \frac{dy}{x(x^2+y)} = \frac{-dz}{x^2+1}$$

i) From $\dfrac{dx}{x^2+y} = \dfrac{dy}{x(x^2+y)} \quad \Rightarrow \quad xdx = dy \quad \Rightarrow \quad \dfrac{x^2}{2} = y + \dfrac{c_1}{2} \quad \Rightarrow \quad x^2 = 2y + c_1.$
Comparing with the given surface (i.e., the cylinder), we have $c_1 = 0$. Therefore,

$$x^2 = 2y$$

ii) From $\dfrac{dx}{x^2+y} = \dfrac{-dz}{x^2+1}$. Substituting $y = \dfrac{x^2}{2}$ yields

$$\frac{dx}{x^2 + \frac{x^2}{2}} = \frac{-dz}{x^2+1} \quad \Rightarrow \quad \frac{2dx}{3x^2} = \frac{-dz}{x^2+1} \quad \Rightarrow \quad \frac{2(x^2+1)dx}{x^2} = -3dz$$

$$3dz + 2\left(1 + \frac{1}{x^2}\right)dx = 0. \text{ Integrating yields } 3z + 2\left(x - \frac{1}{x}\right) = c_2$$

Therefore, the orthogonal trajectories are the curves $3z + 2\left(x - \dfrac{1}{x}\right) = c_2$ and $2y = x^2$.

Exercise:

(a) Find the orthogonal trajectories on the surface of the sphere $x^2+y^2+z^2 = a^2$ of its intersections with the paraboloids $xy = cz$, where c is a parameter. [ans: $x^2 + y^2 + z^2 = a^2, \quad \dfrac{a^2 - x^2}{a^2 - y^2} = c_2, \quad \dfrac{x^2 - y^2}{a^2 + z^2} = c_3$]

(b) Show that the orthogonal trajectories on the hyperboloid $x^2 + y^2 - z^2 = 1$ of the conics in which it is cut by the system of planes $x + y = c$ are its curves of intersection with the surfaces $(x - y)z = k$, where k is a parameter.

(c) Find the orthogonal trajectories on the conicoid $(x + y)z = 1$ of the conics in which it is cut by the system of planes $x - y + z = k$, where k is a parameter. [ans: The orthogonal trajectories are the curves $x = z - \dfrac{1}{6z^3} + \dfrac{1}{2z} + c_2, \quad (x + y)z = 1$]

Lecture 4

1.5 Pfaffian differential equations

A differential equation of the form:

$$\sum_{i=1}^{n} F_i(x_1, x_2, \cdots, x_n)dx_i = 0$$

is called a Pfaffian differential equation in n variables $x_1, x_2, \cdots, x_n$. Thus, a Pfaffian differential equation in 3 variables x, y and z takes the general form

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0 \tag{8}$$

If equation (8) has a solution, we say that it is **integrable**. The following theorem gives the necessary and sufficient condition for equation (8) to be integrable.

Theorem 1.1. *Let $\vec{X} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\hat{\mathbf{k}}$, then equation (8) is integrable if and only if*

$$\boxed{\vec{X} \cdot \text{curl}\vec{X} = 0}$$

Example(s):

1. Verify that the equation $(x^2z - y^3)dx + 3xy^2dy + x^3dz = 0$ is integrable.

Solution

Let $\vec{X} = (P, Q, R)$ where $P = x^2z - y^3, Q = 3xy^2, R = x^3$

$$\text{curl}\vec{X} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2z - y^3 & 3xy^2 & x^3 \end{vmatrix} = 0\hat{\mathbf{i}} - (3x^2 - x^2)\hat{\mathbf{j}} + (3y^2 + 3y^2)\hat{\mathbf{k}} = (0, -2x^2, 6y^2)$$

$\vec{X} \cdot \text{curl}\vec{X} = (x^2z - y^3, 3xy^2, x^3) \cdot (0, -2x^2, 6y^2) = 0 - 6x^3y^2 + 6x^3y^2 = 0,$ hence integrable.

Exercise:

1. Find the function $f(y)$ so that the total differential equation $\dfrac{yz + z}{x}dx - zdy + f(y)dz = 0$ is integrable. [ans: $f(y) = c(y + 1)$]

1.6 Methods of solution of $Pdx + Qdy + Rdz = 0$

1.6.1 Variables separable

If equation (8) can be rearranged to take the form $P(x)dx + Q(y)dy + R(z)dz = 0$, then we integrate directly to obtain the solution i.e., $\displaystyle\int P(x)dx + \int Q(y)dy + \int R(z)dz = c$, where c is an arbitrary constant.

Example(s):

1. Solve $2zydx + zxdy + xy(1+z)dz = 0$.

Solution

Clearly, the given equation is integrable, i.e., $\vec{X} \cdot \text{curl}\vec{X} = 0$. Dividing through by xyz yields

$$\frac{2dx}{x} + \frac{dy}{y} + \frac{(1+z)dz}{z} = 0$$

Integrating yields

$$2\ln x + \ln y + z + \ln z = -\ln c \quad \Rightarrow \quad cx^2yz = e^{-z}$$

1.6.2 Re-grouping (or inspection)

This method employs complete methods such as:

i) $d(xy) = xdy + ydx$ "product rule"

ii) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$ "quotient rule"

iii) $d(\ln(x^2+y^2)) = \frac{2(xdx+ydy)}{x^2+y^2}$ "derivative of natural logarithmic function"

iv) $d(\tan^{-1}(\frac{y}{x})) = \frac{xdy - ydx}{x^2+y^2}$ "derivative of tan inverse"

Example(s):

1. Solve $(x^2z - y^3)dx + 3xy^2dy + x^3dz = 0$.

Solution

Clearly, the given equation is integrable, i.e., $\vec{X} \cdot \text{curl}\vec{X} = 0$. Now, expanding and regrouping the given equation yields $x^2zdx - y^3dx + 3xy^2dy + x^3dz = 0 \quad \Rightarrow \quad x^2(zdx + xdz) + 3xy^2dy - y^3dx = 0$or

$$(zdx + xdz) + \frac{3xy^2dy - y^3dx}{x^2} = 0 \quad \Rightarrow \quad d(zx) + d\left(\frac{y^3}{x}\right) = 0$$

Integrating yields $zx + \frac{y^3}{x} = c$, where c is an arbitrary constant.

Exercise:

1. Solve $(x-z)dx + x^2zdy + x(1+xy)dz = 0$. [ans: $\ln x + \frac{z}{x} + zy = c$]

1.6.3 One variable separable

If one variable, say z, is separable, then equation (8) takes the form

$$P(x,y)dx + Q(x,y)dy + R(z)dz = 0 \qquad (*)$$

Since equation $(*)$ is integrable, we must have $\vec{X} \cdot \text{curl}\vec{X} = 0$. Now,

$$\text{curl}\vec{X} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & R(z) \end{vmatrix} = (0-0)\hat{\mathbf{i}} - (0-0)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}} = \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

so that

$$\vec{X} \cdot \text{curl}\vec{X} = (P,Q,R) \cdot \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0.$$

But since $R \neq 0$, we have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ or alternatively;

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

Thus, $P(x, y)dx + Q(x, y)dy$ is an exact differential, i.e., there exists a function $\phi(x, y)$ such that $d\phi = P(x, y)dx + Q(x, y)dy$. Therefore, equation $(*)$ reduces to

$$\boxed{d\phi + R(z)dz = 0} \qquad (**)$$

The general solution of equation $(**)$ is obtained by integrating directly, i.e., $\phi(x, y) + \int R(z)dz = c$, where c is an arbitrary constant.

Example(s):

1. Solve $x(y^2 - a^2)dx + y(x^2 - z^2)dy - z(y^2 - a^2)dz = 0$, where a is a constant.

   Solution

   Clearly, the given equation is integrable, i.e., $\vec{X} \cdot \text{curl} \vec{X} = 0$. Dividing through by $(y^2 - a^2)(x^2 - z^2)$ yields

   $$\frac{xdx}{x^2 - z^2} + \frac{ydy}{y^2 - a^2} - \frac{zdz}{x^2 - z^2} = 0 \quad \Rightarrow \quad \frac{xdx - zdz}{x^2 - z^2} + \frac{y}{y^2 - a^2}dy = 0$$

   Integrating yields

   $$\frac{1}{2}\ln(x^2 - z^2) + \frac{1}{2}\ln(y^2 - a^2) = \frac{1}{2}\ln c$$

   or

   $$(y^2 - a^2)(x^2 - z^2) = c,$$

   where c is an arbitrary constant.

Exercise:

1. Solve $xz^3dx - zdy + 2ydz = 0$. [ans: $x^2z^2 - 2y = cz^2$]

LECTURE 5

1.6.4 Homogeneous equations

Equation (8) is said to be homogeneous if and only if the functions P, Q and R are homogeneous of the same degree. Note that a function $f(x, y, z)$ is said to be homogeneous of degree n if there exists a real number λ such that

$$f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$$

For example, the function $f(x, y, z) = x^2y + xyz + y^2z$ is homogeneous of degree 3. If equation (8) is homogeneous, then we solve it by making the following substitutions:

$$\boxed{y = ux \text{ and } z = vx},$$

where u and v are functions of x. Using product rule of differentiation, the differentials dy and dz become

$$\boxed{dy = udx + xdu \text{ and } dz = vdx + xdv}$$

Thus, equation (8) becomes

$$P(x, ux, vx)dx + Q(x, ux, vx)[udx + xdu] + R(x, ux, vx)[vdx + xdv] = 0$$

or

$$x^n P(u, v)dx + x^n Q(u, v)[udx + xdu] + x^n R(u, v)[vdx + xdv] = 0$$

or

$$P(u, v)dx + Q(u, v)[udx + xdu] + R(u, v)[vdx + xdv] = 0$$

or

$$[P(u, v) + uQ(u, v) + vR(u, v)]dx + x[Q(u, v)du + R(u, v)dv] = 0$$

or

$$\frac{dx}{x} + \frac{Q(u, v)du + R(u, v)dv}{[P(u, v) + uQ(u, v) + vR(u, v)]} = 0$$

or

$$\frac{dx}{x} + A(u, v)du + B(u, v)dv = 0$$

which can now be solved using the method of one variable separable.

Example(s):
Solve the following total differential equations.

(a) $ydx - (x + z)dy + ydz = 0.$

Solution

Clearly, the given equation is integrable, i.e., $\vec{X} \cdot \text{curl}\vec{X} = 0$. The given equation is homogeneous of degree 1. To solve it, we make the following substitutions

$$y = ux \quad \Rightarrow \quad dy = udx + xdu$$

$$z = vx \quad \Rightarrow \quad dz = vdx + xdv$$

Substituting into the given equation yields

$$uxdx - (x + vx)[udx + xdu] + ux[vdx + xdv] = 0$$

or

$$udx - (1 + v)[udx + xdu] + u[vdx + xdv] = 0$$

or

$$[u + uv - (1 + v)u]dx - x[(1 + v)du - udv] = 0$$

or

$$(1 + v)du - udv = 0$$

or

$$\frac{du}{u} - \frac{dv}{1 + v} = 0$$

Integrating yields

$$\int \frac{du}{u} - \int \frac{dv}{1 + v} = \int 0 \quad \Rightarrow \quad \ln u - \ln(1 + v) = \ln c \quad \Rightarrow \quad u = c(1 + v)$$

Back substitution yields

$$\frac{y}{x} = c\left(1 + \frac{z}{x}\right) \quad \Rightarrow \quad y = c(x + z) \quad \Rightarrow \quad x + z = c_1 y$$

(b) $yz(y + z)dx + xz(x + z)dy + xy(x + y)dz = 0$

Solution

Clearly, the given equation is integrable, i.e., $\vec{X} \cdot \text{curl}\vec{X} = 0$. It is also homogeneous of degree 3. To solve it, put

$$y = ux \quad \Rightarrow \quad dy = udx + xdu$$

and

$$z = vx \quad \Rightarrow \quad dz = vdx + xdv$$

Substituting into the given equation yields

$$x^2uv(ux + vx)dx + x^2v(x + xv)[udx + xdu] + x^2u(x + ux)[vdx + xdv] = 0$$

or

$$uv(u + v)dx + v(1 + v)[udx + xdu] + u(1 + u)[vdx + xdv] = 0$$

or

$$[uv(u + v) + uv(1 + v) + uv(1 + u)]dx + xv(1 + v)du + xu(1 + u)dv = 0$$

or

$$\frac{dx}{x} + \frac{v(1 + v)du}{2uv(u + v + 1)} + \frac{u(1 + u)dv}{2uv(u + v + 1)} = 0$$

or

$$\frac{2dx}{x} + \frac{(1 + v)du}{u(u + v + 1)} + \frac{(1 + u)dv}{v(u + v + 1)} = 0 \cdots\cdots\cdots (i)$$

By partial fractions, let

$$\frac{(1 + v)}{u(u + v + 1)} = \frac{A}{u} + \frac{B}{(u + v + 1)} \quad \Rightarrow \quad A(u + v + 1) + Bu = 1 + v$$

Putting $u = 0 \quad \Rightarrow \quad A = 1$. Putting $u = -(v + 1) \quad \Rightarrow \quad B = -1$. Hence,

$$\frac{(1 + v)}{u(u + v + 1)} = \frac{1}{u} - \frac{1}{(u + v + 1)}$$

Similarly,

$$\frac{(1 + u)}{v(u + v + 1)} = \frac{1}{v} - \frac{1}{(u + v + 1)}$$

Substituting into equation (i), we get

$$\frac{2dx}{x} + \frac{du}{u} - \frac{du}{(u + v + 1)} + \frac{dv}{v} - \frac{dv}{(u + v + 1)} = 0$$

or

$$\frac{2dx}{x} + \frac{du}{u} + \frac{dv}{v} - \frac{du + dv}{(u + v + 1)} = 0$$

Integrating yields

$$2\ln x + \ln u + \ln v - \ln(u + v + 1) = \ln c \quad \Rightarrow \quad x^2uv = c(u + v + 1)$$

Back substitution yields

$$x^2\frac{y}{x}\frac{z}{x} = c\left(\frac{y}{x} + \frac{z}{x} + 1\right) \quad \Rightarrow \quad xyz = c(x + y + z)$$

Example(s):

1. Solve the homogeneous total differential equation $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$. [ans: $y(x + z) = c(y + z)$]

2. Solve $(y+z)dx+(x+z)dy+(x+y)dz = 0$ by first making the substitution $z = x+ky$ and then eliminating k from the resulting solution, where k is a constant. [ans: $xy + yz + zx = c$]

3. Given the total differential equation $f(z)dx + 3y^2zdy + (xz^2 - y^3)dz = 0$

   i) Find the unknown function $f(z)$ satisfying $f(1) = 1$ such that the given differential equation is integrable. [ans: $f(z) = z^3$]

   ii) By substituting your result in part (i) above into the given differential equation, find its general solution. [ans: $y^3 + xz^2 = cz$]

1.6.5 Natani's method

This method involves two steps:

- ☐ *Step 1:* suppose that one of the variables, say z, is a constant $\Rightarrow \quad dz = 0$. Thus, equation (8) reduces to

$$P(x, y, z)dx + Q(x, y, z)dy = 0 \qquad (*)$$

Solve equation $(*)$ to get

$$\phi(x, y, z) = f(z) \qquad (i)$$

Equation (i) is the solution provided that the arbitrary function $f(z)$ is determined.

- ☐ *Step 2:* let either of the remaining variables, say x, to be equal to 1 or 0. For instance, if $x = 1, \quad \Rightarrow \quad dx = 0$. Thus, equation (8) reduces to

$$Q(y, z)dy + R(y, z)dz = 0 \qquad (**)$$

Solve equation $(**)$ to get

$$\psi(y, z) = c, \qquad (ii)$$

where c is an arbitrary constant. Also put $x = 1$ in equation (i) to obtain

$$\phi(y, z) = f(z) \cdots (iii)$$

Eliminate y between equations (ii) and (iii) to obtain $f(z)$. Substitute the value of $f(z)$ into equation (i) to obtain the general solution of the given differential equation.

Example(s):

1. Verify that the following differential equations are integrable and hence find their primitives.

   (a) $(x^2z - y^3)dx + 3xy^2dy + x^3dz = 0.$

Solution

Clearly, the given equation is integrable, i.e., $\vec{X} \cdot \text{curl}\vec{X} = 0$. By Natani's method:

- ☐ *Step 1:* we note that the solution is simpler if we treat x to be a constant, $\Rightarrow dx = 0$. So the given differential equation reduces to

$$3xy^2dy + x^3dz = 0 \quad \Rightarrow \quad 3y^2dy + x^2dz = 0$$

Integrating yields

$$\int 3y^2dy + \int x^2dz = \int 0 \quad \Rightarrow \quad y^3 + x^2z = f(x) \cdots (i)$$

Equation (i) is the required solution provided that the arbitrary function $f(x)$ is determined.

- ☐ *Step 2:* put $y = 1$ in the given differential equation, $\Rightarrow dy = 0$. So the differential equation reduces to

$$(x^2z - 1)dx + x^3dz = 0 \quad \Rightarrow \quad zdx + xdz - \frac{dx}{x^2} = 0 \quad \Rightarrow d(zx) - \frac{dx}{x^2} = 0$$

Integrating yields

$$\int d(zx) - \int \frac{dx}{x^2} = \int 0 \quad \Rightarrow \quad zx + \frac{1}{x} = c \quad \Rightarrow \quad z = \frac{cx - 1}{x^2} \cdots (ii),$$

where c is an arbitrary constant. Also, putting $y = 1$ in equation (i) yields

$$1 + x^2z = f(x) \cdots (iii)$$

Eliminating z between equations (ii) and (iii) (i.e., substituting equation (ii) into equation (iii)) yields

$$1 + x^2 \left\{ \frac{cx - 1}{x^2} \right\} = f(x) \quad \Rightarrow \quad f(x) = cx \cdots (iv)$$

Substituting equation (iv) into equation (i) yields $y^3 + x^2 z = cx$, which is the general solution of the given differential equation.

(b) $z(z + y^2)dx + z(z + x^2)dy - xy(x + y)dz = 0.$

Solution

Clearly, the given equation is integrable, i.e., $\vec{X} \cdot \text{curl}\vec{X} = 0$. By Natani's method:

☐ *Step 1:* we note that the solution is simpler if we treat y to be a constant, $\Rightarrow dy = 0$. So the differential equation reduces to

$$z(z + y^2)dx - xy(x + y)dz = 0 \quad \Rightarrow \quad \frac{dx}{x(x + y)} - \frac{ydz}{z(z + y^2)} = 0 \cdots\cdots (*)$$

By partial fractions, let

$$\frac{1}{x(x + y)} = \frac{A}{x} + \frac{B}{x + y} \quad \Rightarrow \quad A(x + y) + Bx = 1$$

Putting $x = 0 \quad \Rightarrow \quad A = \frac{1}{y}$. Putting $x = -y \quad \Rightarrow \quad B = -\frac{1}{y}$. Therefore,

$$\frac{1}{x(x + y)} = \frac{1}{x} - \frac{1}{x + y}$$

Similarly, let

$$\frac{y}{z(z + y^2)} = \frac{A}{z} + \frac{B}{z + y^2} \quad \Rightarrow \quad A(z + y^2) + Bz = y$$

Putting $z = 0 \quad \Rightarrow \quad A = \frac{1}{y}$. Putting $z = -y^2 \quad \Rightarrow \quad B = -\frac{1}{y}$. Therefore,

$$\frac{y}{z(z + y^2)} = \frac{1}{z} - \frac{1}{z + y^2}$$

Hence, equation $(*)$ becomes

$$\left(\frac{1}{x} - \frac{1}{x + y} \right) dx + \left(\frac{1}{z + y^2} - \frac{1}{z} \right) dz = 0$$

Integrating yields

$$\int \left(\frac{1}{x} - \frac{1}{x + y} \right) dx + \int \left(\frac{1}{z + y^2} - \frac{1}{z} \right) dz = \int 0$$

or

$$\ln x - \ln(x + y) + \ln(z + y^2) - \ln z = \ln |f(y)|$$

$$\frac{x(z + y^2)}{z(x + y)} = f(y) \cdots (i)$$

Equation (i) is the required solution provided that the arbitrary function $f(y)$ is determined.

☐ *Step 2:* put $z = 1$ in the given differential equation, $\Rightarrow dz = 0$. So the differential equation reduces to

$$(1 + y^2)dx + (1 + x^2)dy = 0 \quad \Rightarrow \quad \frac{dx}{1 + x^2} + \frac{dy}{1 + y^2} = 0$$

Integrating yields

$$\int \frac{dx}{1+x^2} + \int \frac{dy}{1+y^2} = \int 0 \quad \Rightarrow \quad \tan^{-1} x + \tan^{-1} y = \tan^{-1} c \cdots\cdots (*)$$

Note: $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$. Taking tan on both sides of equation $(*)$ yields

$$\tan\left(\tan^{-1} x + \tan^{-1} y\right) = c \quad \Rightarrow \quad \frac{x+y}{1-xy} = c \quad \Rightarrow \quad x = \frac{c-y}{cy+1} \cdots (ii),$$

where c is an arbitrary constant. Also, putting $z = 1$ in equation (i) we get

$$f(y) = \frac{x(1+y^2)}{(x+y)} \cdots (iii)$$

Eliminating x between equations (ii) and (iii) (i.e., substituting equation (ii) into equation (iii)) yields

$$f(y) = \frac{c-y}{cy+1} \cdot \frac{1+y^2}{\left\{\dfrac{c-y}{cy+1} + y\right\}} \quad \Rightarrow \quad f(y) = 1 - c^{-1}y = 1 - c_1 y \cdots (iv)$$

Substituting equation (iv) into equation (i) yields $x(z+y^2) = z(x+y)(1-c_1 y)$.

Exercise:
Solve the following differential equations

(a) $2y(a-x)dx + [z - y^2 + (a-x)^2]dy - ydz = 0$. [ans: $(a-x)^2 + z = y(c-y)$]

(b) $(1+yz)dx + x(z-x)dy - (1+xy)dz = 0$. [ans: $yz + 1 = (xy+1)(cy+1)$]

(c) $yzdx + (x^2y - zx)dy + (x^2z - xy)dz = 0$. [ans: $y^2 + z^2 - \dfrac{2yz}{x} = c$]

(d) $(y^2 + yz)dx + (z^2 + xz)dy + (y^2 - xy)dz = 0$. [ans: $y(x+z) = c(y+z)$]

(e) $(y^2 + yz + z^2)dx + (z^2 + xz + x^2)dy + (x^2 + xy + y^2)dz = 0$. [ans: $xy + yz + xz = c(x+y+z)$]

(h) $zydx + 3zxdy - 2xydz = 0$. [ans: $xy^3 = cz^2$]

(i) $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0$. [hint: $e^u\left(\dfrac{1}{\sqrt{u}} + 2\sqrt{u}\right)du = d\left(2\sqrt{u}e^u\right)$, ans: $(x+y+z^2)e^{x^2} = c$]

Lecture 6

2 Partial differential equations of the first-order

Definition 2.1 (Partial differential equation (PDE))**.** *It is an equation involving derivatives of a function which depends on more than one independent variables.*

For example, the equation

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - y\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2(x+y)$$

is a PDE in the dependent variable z and independent variables x and y.

Definition 2.2 (Order of a PDE)**.** *It is the order of the highest-ordered derivative occurring in the equation.*

For example, the equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \frac{\partial z}{\partial t} = 0$$

is a first-order equation in three independent variables x, y and t while the equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

is a second-order equation in two independent variables x and t.

Definition 2.3 (Degree of a PDE)**.** *It is the greatest power of the highest-ordered derivative occurring in the equation.*

Definition 2.4 (Linear PDE)**.**

It is an equation which satisfies the following conditions:

(i) There are no products of the dependent variable, say z, and/or its derivatives. For example, $z\frac{\partial z}{\partial x}$, $\left(\frac{\partial^2 z}{\partial x^2}\right)^3$, z^2, $\sqrt{z}$, etc., are absent.

(ii) There are no transcendental functions of the dependent variable, say z, and its derivatives. For example, $\cos(z)$, $\ln(x^2 + z)$ or e^{y+z} are absent.

An example of a linear PDE is

$$\cos\left(xy^2\right)\frac{\partial z}{\partial x} - y^2\frac{\partial z}{\partial y} = \tan(x^2 + y^2)$$

Any PDE which doesn't satisfy all the above conditions is said to be **nonlinear**. Examples of nonlinear PDEs are:

$$\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial y} = 0 \text{ (shock wave)}$$

$$\frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial x^2} + z^3 = 0 \text{ (wave with interaction)}$$

$$\frac{\partial z}{\partial t} + z\frac{\partial z}{\partial x} + \frac{\partial^3 z}{\partial x^3} = 0 \text{ (dispersive wave)}$$

Definition 2.5 (Homogeneous PDE)**.** *It is a PDE in which all the derivatives occurring in the equation are of the same order.*

For example, the equation

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - y\frac{\partial^2 z}{\partial y^2} = 2(x + y)$$

is homogeneous since all the derivatives occurring in the equation are of order 2. An equation which is not homogeneous is said to be non-homogeneous.

2.1 Formation of first-order PDEs

A first-order PDE in two independent variables x, y and dependent variable z takes the general form

$$\boxed{f(x, y, z, p, q) = 0}, \tag{9}$$

where $z = z(x, y)$, $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$. First-order PDEs arise in many applications in physics and engineering such as in traffic and gas flow problems, phenomenon of shock waves, the motion of wave fronts, Hamilton-Jacobi theory, nonlinear continuum mechanics and quantum mechanics.

A first-order PDE can be formed either by eliminating arbitrary constants or an arbitrary function involved in a primitive.

2.1.1 Elimination of arbitrary constants

Consider an equation of the form

$$F(x, y, z, c_1, c_2, \cdots, c_n) = 0, \qquad (*)$$

where $c_1, c_2, \cdots, c_n$ are essential arbitrary constants. Let z be regarded as a function of x and y. To obtain a PDE from the primitive $(*)$, follow these steps:

- ☐ *Step 1:* Differentiate the primitive a number of times equal to the number of essential arbitrary constants. Ensure that you exhaust all the lower order derivatives before finding the higher order derivatives.
- ☐ *Step 2:* Manipulate the primitive and its derivatives algebraically to eliminate the essential arbitrary constants.

→ Note: if there are more arbitrary constants than the number of independent variables, the above procedure of elimination will give rise to a PDE of higher order.

Example(s):

1. Construct the PDE from the following primitives.

   (a) $z = ax^2 + by^3$.

   Solution
   Since the given primitive involves two essential arbitrary constants, we differentiate it twice.

   Differentiating the given primitive partially with respect to x yields

   $$z_x = 2ax \quad \Rightarrow \quad a = \frac{z_x}{2x}$$

   Differentiating the given primitive partially with respect to y yields

   $$z_y = 3by^2 \quad \Rightarrow \quad b = \frac{z_y}{3y^2}$$

   Substituting a and b into the given primitive yields

   $$z = \frac{z_x}{2x}x^2 + \frac{z_y}{3y^2}y^3 \quad \Rightarrow \quad 6z = 3xz_x + 2yz_y \quad \Rightarrow \quad 3xp + 2yq = 6z$$

   (b) $z = a(x + y) + b(x - y) + abt + c$

   Solution
   Since the given primitive involves three essential arbitrary constants, we differentiate it thrice.

   Differentiating the given primitive partially with respect to x, y and t yields

   $$z_x = a + b, \; z_y = a - b \; \text{ and } \; z_t = ab,$$

   respectively. Now,

   $$(a + b)^2 - (a - b)^2 = 4ab$$

   or

   $$(z_x)^2 - (z_y)^2 = 4z_t \quad \Rightarrow \quad p^2 - q^2 = 4z_t$$

(c) $ax^2 + by^2 + z^2 = 1$.

Solution
Since the given primitive involves two essential arbitrary constants, we differentiate it twice.

Differentiating the given primitive partially with respect to x yields

$$2ax + 2zz_x = 0 \quad \Rightarrow \quad a = \frac{-zz_x}{x}$$

Differentiating the given primitive partially with respect to y yields

$$2by + 2zz_y = 0 \quad \Rightarrow \quad b = \frac{-zz_y}{y}$$

Substituting a and b into the given primitive yields

$$\frac{-zz_x}{x}x^2 + \frac{-zz_y}{y}y^2 + z^2 = 1 \quad \Rightarrow \quad -z(xz_x + yz_y) + z^2 = 1$$

or

$$zxp + zyq = z^2 - 1$$

Exercise:

1. Form partial differential equations associated with the following primitives:
   (a) $z = (x-a)^2 + (y-b)^2$. [ans: $p^2 + q^2 = 4z$]
   (b) $2z = (ax+y)^2 + b$. [ans: $xp + yq = 2q^2$]
   (c) $z = ax + by + ab$. [ans: $z = xp + yq + pq$]
   (d) $z = ax + a^2y^2 + b$. [ans: $q = 2yp^2$]
   (e) $z = (x+a)(y+b)$. [ans: $z = pq$]

2.1.2 Elimination of arbitrary function

Consider the equation

$$F(u, v) = 0,$$

where u and v are functions of x, y and z; F being an arbitrary function of u and v. In this case, treat z as a function of x and y, i.e., $z = z(x, y)$, and then use chain rule of differentiation (for functions of several variables) to eliminate F.

Example(s):

1. Eliminate the arbitrary function F from the equation $F(x^2 + y^2 + z^2, z^2 - 2xy) = 0$.

   Solution
   Let $u = x^2 + y^2 + z^2$ and $v = z^2 - 2xy$ so that

   $$F(u, v) = 0 \qquad (*)$$

   If $z = z(x, y)$, then the first partial derivatives of u and v are

   $$u_x = 2x + 2zz_x, \quad u_y = 2y + 2zz_y, \quad v_x = 2zz_x - 2y, \text{ and } v_y = 2zz_y - 2x$$

   Differentiating equation $(*)$ partially with respect to x and y in turn yields

   $$F_u u_x + F_v v_x = 0 \quad \Rightarrow \quad F_u(2x + 2zz_x) + F_v(2zz_x - 2y) = 0 \quad \Rightarrow \quad \frac{F_u}{F_v} = \frac{y - zz_x}{x + zz_x} \cdots (i)$$

and

$$F_u u_y + F_v v_y = 0 \quad \Rightarrow \quad F_u(2y + 2zz_y) + F_v(2zz_y - 2x) = 0 \quad \Rightarrow \quad \frac{F_u}{F_v} = \frac{x - zz_y}{y + zz_y} \cdots (ii)$$

From equations (i) and (ii), we have

$$\frac{y - zz_x}{x + zz_x} = \frac{x - zz_y}{y + zz_y} \quad \Rightarrow \quad (y - zz_x)(y + zz_y) - (x + zz_x)(x - zz_y) = 0$$

or

$$y^2 + yzz_y - yzz_x - \cancel{z^2 z_x z_y} - x^2 + xzz_y - xzz_x + \cancel{z^2 z_x z_y} = 0$$

or

$$y^2 - x^2 - z(y + x)z_x + z(y + x)z_y = 0$$

or

$$z(y + x)p - z(y + x)q = y^2 - x^2$$

or

$$zp - zq = y - x$$

2. Eliminate the arbitrary function f from the equation $x + y + z = f(x^2 + y^2 + z^2)$. [ans: $(y - z)p + (z - x)q = x - y$]

Solution

Let $u = x^2 + y^2 + z^2$ so that

$$x + y + z = f(u) \qquad (*)$$

If $z = z(x, y)$, then the first partial derivatives of u are

$$u_x = 2x + 2zz_x \text{ and } u_y = 2y + 2zz_y$$

Differentiating equation $(*)$ partially with respect to x and y in turn we have

$$1 + z_x = f_u \cdot u_x \quad \Rightarrow \quad 1 + z_x = (2x + 2zz_x)f_u \quad \Rightarrow \quad f_u = \frac{1 + z_x}{2x + 2zz_x} \cdots (i)$$

and

$$1 + z_y = f_u \cdot u_y \quad \Rightarrow \quad 1 + z_y = (2y + 2zz_y)f_u \quad \Rightarrow \quad f_u = \frac{1 + z_y}{2y + 2zz_y} \cdots (ii)$$

From equations (i) and (ii), we have

$$\frac{1 + z_x}{2x + 2zz_x} = \frac{1 + z_y}{2y + 2zz_y} \quad \Rightarrow \quad (1 + z_x)(2y + 2zz_y) - (2x + 2zz_x)(1 + z_y) = 0$$

or

$$2y + 2zz_y + 2yz_x + 2zz_x z_y - 2x - 2zz_x - 2xz_y - 2zz_x z_y = 0$$

or

$$2y + 2zz_y + 2yz_x - 2x - 2zz_x - 2xz_y = 0$$

or

$$y + zq + yp - x - zp - xq = 0$$

or

$$(y - z)p + (z - x)q = x - y$$

Exercise:

1. Eliminate the arbitrary function f from the equations

   (b) $z = f(x^2 - y^2)$. [ans: $yp + xq = 0$]

(c) $z = x^n f(y/x)$. [ans: $xp + yq = nz$]
(d) $z = e^{ax+by} f(ax - by)$. [ans: $bp + aq = 2abz$]
(e) $z = xy + f(x^2 + y^2)$. [ans: $yp - xq = y^2 - x^2$]
(f) $z = x + y + f(xy)$. [ans: $xp - yq = x - y$]
(g) $z = f(\frac{xy}{z})$. [ans: $xp - yq = 0$]
(h) $z^2 = 2x + 2y + f(\frac{y}{x})$. [ans: $xzp + yzq = x + y$]

LECTURE 7

2.2 Lagrange's equation

This is a PDE which arises from elimination of arbitrary function F from the primitive $F(u, v) = 0$, where u and v are functions of x, y and z. The equation takes the general form

$$\boxed{Pp + Qq = R}, \tag{10}$$

where $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$ while

$$\boxed{P = \frac{\partial(u, v)}{\partial(y, z)}, \quad Q = \frac{\partial(u, v)}{\partial(z, x)}, \text{ and } R = \frac{\partial(u, v)}{\partial(x, y)}}$$

are the direction cosines of the tangent at point (x, y, z) to the functions u and v.

Proof. If we treat z as a function of x and y, i.e., $z = z(x, y)$, and then differentiate the primitive $F(u, v) = 0$ with respect to x and y, respectively, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0 \quad \text{and} \quad \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) = 0$$

or

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \quad \text{and} \quad \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0$$

or

$$\frac{\partial F}{\partial u}\Big/\frac{\partial F}{\partial v} = -\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) \quad \text{and} \quad \frac{\partial F}{\partial u}\Big/\frac{\partial F}{\partial v} = -\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)$$

Equating yields

$$\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) = \left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)$$

or

$$\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) = \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)$$

or

$$\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + q\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} + p\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} + \cancel{pq\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + q\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} + p\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} + \cancel{pq\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}}$$

or

$$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

or

$$\frac{\partial(u, v)}{\partial(y, z)}p + \frac{\partial(u, v)}{\partial(z, x)}q = \frac{\partial(u, v)}{\partial(x, y)}$$

or

$$Pp + Qq = R.$$

□

Equation (10) is said to be:

- **Linear** if P and Q are functions of x and y only while R is linear in the dependent variable z, i.e., $R = f(x, y)z + g(x, y)$.
- **Semi-linear** if P and Q are functions of x and y only but R is a function of x, y and z.
- **Quasi-linear** if P, Q and R are functions of x, y and z.

→ Note: A first-order partial differential equation which does not fit into any of the above three categories is said to be **nonlinear**. For instance, $xp^2 + yq^2 = z$, $pq = 4 - z^2$, etc.

2.2.1 Formation of Lagrange's equation

In this case, eliminate the arbitrary function from the given primitive (or the general solution) to obtain the associated PDE.

Example(s):

1. Find the Lagrange's equation whose general solution is $\phi(xyz, x^2 + y^2 + z^2) = 0$

   Solution

   Let $\phi(u, v) = 0$ where $u = xyz$ and $v = x^2 + y^2 + z^2$

$$\begin{aligned}
P &= \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} = u_y v_z - u_z v_y = (xz)(2z) - (xy)(2y) = 2x(z^2 - y^2) \\
Q &= \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} u_z & u_x \\ v_z & v_x \end{vmatrix} = u_z v_x - u_x v_z = (xy)(2x) - (yz)(2z) = 2y(x^2 - z^2) \\
R &= \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x = (yz)(2y) - (xz)(2x) = 2z(y^2 - x^2)
\end{aligned}$$

   The Lagrange's equation is given by

$$Pp + Qq = R \quad \Rightarrow \quad 2x(z^2 - y^2)p + 2y(x^2 - z^2)q = 2z(y^2 - x^2)$$

   or

$$x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$$

Exercise:

1. Form a quasi-linear partial differential equation of the first-order whose general solution is

   (a) $\phi(x^2e^z, ye^{-z}) = 0$

   (b) $\phi(x \sin y, ze^x) = 0$ [ans: $xp - (\tan y)q = -xz$]

   (c) $2z + 2xy = f(x^2 + y^2 + z^2)$ [hint: $\phi(x^2 + y^2 + z^2, 2z + 2xy) = 0$, ans: $(y - xz)p + (yz - x)q = x^2 - y^2$]

2.2.2 Methods of solving $Pp + Qq = R$

There are two methods of solving Lagrange's equation: (1) method of characteristics and (2) Lagrange's method of solution.

2. **Lagrange's method of solution**

   This method is applicable to all the three cases, i.e., linear, semilinear, and quasilinear Lagrange's equations. The method depends on the introduction of subsidiary equations.

Theorem 2.1. *The general solution of the Lagrange's equation $Pp + Qq = R$ is $\phi(u, v) = 0$, where $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are the integral curves of the subsidiary equations*

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}, \cdots\cdots (i)$$

and ϕ is an arbitrary function; c_1, c_2 being arbitrary constants.

Proof. The curves $u \equiv u(x, y, z) = c_1$ and $v \equiv v(x, y, z) = c_2$ are solutions of the system of equations (i). Hence, the equations

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0, \quad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0 \cdots\cdots (ii)$$

are compatible with equations (i). If we let $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R} = \mu$ (for some nonzero constant μ), we get

$$dx = \mu P, \;\; dy = \mu Q, \;\; dz = \mu R$$

Substituting into equations (ii), we obtain

$$\mu\left(P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z}\right) = 0 \quad \text{and} \quad \mu\left(P\frac{\partial v}{\partial x} + Q\frac{\partial v}{\partial y} + R\frac{\partial v}{\partial z}\right) = 0$$

Since $\mu \neq 0$, we must have

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0 \quad \text{and} \quad P\frac{\partial v}{\partial x} + Q\frac{\partial v}{\partial y} + R\frac{\partial v}{\partial z} = 0$$

or

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} = -R\frac{\partial u}{\partial z} \quad \text{and} \quad P\frac{\partial v}{\partial x} + Q\frac{\partial v}{\partial y} = -R\frac{\partial v}{\partial z}$$

In matrix-vector form, we have

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} -R\dfrac{\partial u}{\partial z} \\ -R\dfrac{\partial v}{\partial z} \end{pmatrix}$$

Solving this system for P and Q using Cramer's rule, we obtain

$$P = \frac{\left(\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}\right)R}{\left(\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}\right)} \quad \text{and} \quad Q = \frac{\left(\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}\right)R}{\left(\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}\right)}$$

Rearranging and equating the above two equations yields

$$\frac{P}{\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}} = \frac{Q}{\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}} = \frac{R}{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}}$$

or

$$\frac{P}{\dfrac{\partial(u,v)}{\partial(y,z)}} = \frac{Q}{\dfrac{\partial(u,v)}{\partial(z,x)}} = \frac{R}{\dfrac{\partial(u,v)}{\partial(x,y)}} \cdots\cdots (a)$$

Also, we need to eliminate the arbitrary function ϕ from the equation $\phi(u, v) = 0 \cdots\cdots (*)$. From differential calculus, if $\phi = \phi(u, v)$, then

$$d\phi = \frac{\partial \phi}{\partial u}du + \frac{\partial \phi}{\partial v}dv$$

and $z = z(x, y)$ gives

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$

Differentiating equation $(*)$ with respect to x and y, respectively, we have

$$\frac{\partial \phi}{\partial u}\frac{du}{dx} + \frac{\partial \phi}{\partial v}\frac{dv}{dx} = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial u}\frac{du}{dy} + \frac{\partial \phi}{\partial v}\frac{dv}{dy} = 0 \cdots\cdots (**)$$

Now,

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}, \quad \text{and} \quad \frac{du}{dy} = \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}$$

Similarly,

$$\frac{dv}{dx} = \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}, \quad \text{and} \quad \frac{dv}{dy} = \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z},$$

where $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$. Substituting in equations $(**)$, we get

$$\phi_u (u_x + pu_z) + \phi_v (v_x + pv_z) = 0 \quad \Rightarrow \quad \frac{\phi_u}{\phi_v} = -\frac{v_x + pv_z}{u_x + pu_z}$$

$$\phi_u (u_y + qu_z) + \phi_v (v_y + qv_z) = 0 \quad \Rightarrow \quad \frac{\phi_u}{\phi_v} = -\frac{v_y + qv_z}{u_y + qu_z}$$

Equating the above two equations yields

$$\frac{v_x + pv_z}{u_x + pu_z} = \frac{v_y + qv_z}{u_y + qu_z}$$

Cross multiplying, we obtain

$$u_y v_x + qu_z v_x + pu_y v_z + pqu_z v_z = u_x v_y + pu_z v_y + qu_x v_z + pqu_z v_z$$

Simplifying and rearranging yields

$$(u_y v_z - u_z v_y)\, p + (u_z v_x - u_x v_z)\, q = u_x v_y - u_y v_x$$

or

$$\frac{\partial(u,v)}{\partial(y,z)} p + \frac{\partial(u,v)}{\partial(z,x)} q = \frac{\partial(u,v)}{\partial(x,y)} \cdots\cdots (b)$$

Using equations (a) and (b), we obtain $Pp + Qq = R$, which shows that $\phi(u,v) = 0$ is a solution to equation (i). $\square$

Example(s):

(a) Find the integral surface of the quasi-linear PDE $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$ which contains the straight lines $x + y = 0, z = 1$.

Solution

Let $P = x(y^2+z)$, $\quad Q = -y(x^2+z)$ and $R = (x^2-y^2)z$. The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \Rightarrow \quad \frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

From $\dfrac{xdx + ydy}{x^2(y^2+z) - y^2(x^2+z)} = \dfrac{dz}{z(x^2-y^2)} \quad \Rightarrow \quad \dfrac{xdx + ydy}{z(x^2-y^2)} = \dfrac{dz}{z(x^2-y^2)}$

or

$$xdx + ydy - dz = 0 \quad \Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{2} - z = \frac{c_1}{2}$$

$$\therefore x^2 + y^2 - 2z = c_1 \equiv u$$

From $\dfrac{\frac{1}{x}dx + \frac{1}{y}dy}{(y^2+z) - (x^2+z)} = \dfrac{\frac{1}{z}dz}{(x^2-y^2)} \quad \Rightarrow \quad \dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z} = 0$

or

$$\ln x + \ln y + \ln z = \ln c_2 \quad \Rightarrow \quad \therefore xyz = c_2 \equiv v$$

Therefore, the integral curves are $u = x^2 + y^2 - 2z$ and $v = xyz$. So the integral surface is $\phi(u, v) = 0$. Now,

$$z = 1 \quad \Rightarrow \quad u + 2 = x^2 + y^2 \text{ and } v = xy$$

Also,

$$x + y = 0 \quad \Rightarrow \quad (x + y)^2 = 0 \quad \Rightarrow \quad x^2 + y^2 + 2xy = 0 \quad \Rightarrow \quad u + 2 + 2v = 0.$$

Substituting the original values of the integral curves, u and v, we obtain

$$x^2 + y^2 - 2z + 2 + 2xyz = 0$$

(b) Solve $\dfrac{\partial u}{\partial x} - 2\dfrac{\partial u}{\partial y} = u$ subject to the boundary condition $u(x, 0) = 6e^{-3x}$.

Solution

Let $P = 1, Q = -2$ and $R = u$. The subsidiary equations are given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R} \quad \Rightarrow \quad \frac{dx}{1} = \frac{dy}{-2} = \frac{du}{u}$$

From $\dfrac{dx}{1} = \dfrac{dy}{-2} \quad \Rightarrow \quad 2dx + dy = 0$. Integrating yields

$$\int 2dx + \int dy = \int 0 \quad \Rightarrow \quad 2x + y = c_1$$

From $\dfrac{dx}{1} = \dfrac{du}{u}$. Integrating yields

$$\int dx - \int \frac{du}{u} = \int 0 \quad \Rightarrow \quad x - \ln u = \ln A \quad \Rightarrow \quad u = c_2 e^x$$

The general solution is given by

$$\phi(c_1, c_2) = 0 \quad \Rightarrow \quad \phi\left(2x + y, ue^{-x}\right) = 0$$

or

$$u(x, y) = f\,(2x + y)\, e^x, \qquad (*)$$

where f is an arbitrary function. Apply the boundary condition $u(x, 0) = 6e^{-3x}$ to get

$$6e^{-3x} = f\,(2x)\, e^x \quad \Rightarrow \quad f\,(2x) = 6e^{-4x} \qquad (**)$$

Let $w = 2x \quad \Rightarrow \quad x = \dfrac{w}{2}$. Hence, equation $(**)$ becomes

$$f\,(w) = 6e^{-2w} \quad \Rightarrow \quad f(2x + y) = 6e^{-4x-2y}$$

Substitute into equation $(*)$ to get the particular solution as

$$u(x, y) = 6e^{-3x-2y}$$

(c) Given $x^2\dfrac{\partial u}{\partial x} + y^2\dfrac{\partial u}{\partial y} = (x + y)u$, show that $u(x, y) = (x - y)f\left(\dfrac{x - y}{xy}\right)$; where f is an arbitrary function.

Solution

Let $P = x^2, Q = y^2$ and $R = (x + y)u$. The subsidiary equations are given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R} \quad \Rightarrow \quad \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{u(x + y)}$$

From $\dfrac{dx}{x^2} = \dfrac{dy}{y^2}$. Integrating yields

$$\int \frac{dx}{x^2} - \int \frac{dy}{y^2} = \int 0 \quad \Rightarrow \quad \frac{1}{y} - \frac{1}{x} = c_1 \quad \Rightarrow \quad \frac{x-y}{xy} = c_1$$

From $\dfrac{dx - dy}{x^2 - y^2} = \dfrac{du}{u(x+y)}$. Integrating yields

$$\int \frac{d(x-y)}{x-y} - \int \frac{du}{u} = \int 0 \quad \Rightarrow \quad \ln(x-y) - \ln u = \ln c_2 \quad \Rightarrow \quad \frac{x-y}{u} = c_2$$

Hence, the integral curves are $\dfrac{x-y}{xy} = c_1$ and $\dfrac{x-y}{u} = c_2$. The general solution is given by

$$\phi(c_1, c_2) = 0 \quad \Rightarrow \quad \phi\left(\frac{x-y}{xy}, \frac{x-y}{u}\right) = 0$$

or

$$u(x,y) = (x-y)f\left(\frac{x-y}{xy}\right),$$

where f is an arbitrary function.

Exercise:

1. Find the general solution of the following partial differential equations.
   (a) $x(x+y)p - y(x+y)q = (y-x)(2x+2y+z)$. [ans: $xy = c_1, (x+y)(x+y+z) = c_2$]
   (b) $(xz+y^2)p + z(y-x)q + 2xy + z^2 = 0$. [ans: $F(yz+x^2, 2xz-y^2) = 0$]
   (c) $(x^2+y^2+yz)p + (x^2+y^2-xz)q = z(x+y)$. [ans: $z - x + y = c_1, \dfrac{x^2+y^2}{z^2} = c_2$]
   (d) $z(x+y)p + z(x-y)q = x^2+y^2$. [ans: $2xy - z^2 = c_1, x^2 - y^2 - z^2 = c_2$]
   (e) $yz\dfrac{\partial u}{\partial x} + xz\dfrac{\partial u}{\partial y} + (x^2+y^2)\dfrac{\partial u}{\partial z} = 0$. [hint: $\dfrac{dx}{yz} = \dfrac{dy}{xz} = \dfrac{dz}{x^2+y^2} = \dfrac{du}{0}$, ans:]
   (f) $p + 3q = 5z + \tan(y-3x)$. [ans: $x - 3y = c_1$, $5x = \ln(5z + \tan c_1) + c_2$]
2. (a) Find the equation of the integral surface of the differential equation $2y(z-3)p + (2x-z)q = y(2x-3)$ which passes through the circle $z = 0, x^2+y^2 = 2x$. [ans: $x^2+y^2-z^2-2x+4z = 0$]
   (b) Find the integral surface of the partial differential equation $(x-y)p + (y-x-z)q = z$ through the circle $z = 1, x^2+y^2 = 1$. [ans: $z^4(x+y+z)^2 + (y-x-z)^2 - 2z^4(x+y+z) + 2z^2(y-x-z) = 0$]
   (c) Find the integral surface of the equation $(x-y)y^2p + (y-x)x^2q = (x^2+y^2)z$ through the curve $xz = a^3, y = 0$. [ans: $z^3(x^3+y^3)^2 = a^9(x-y)^3$]
   (d) Find the integral surface of the differential equation $(x^2-yz)p + (y^2-xz)q = z^2 - xy$ which pass through the line $z = 1, y = 0$. [ans: $(x-y)(xy+yz+xz) + y - z = 0$]
   (e) Find the integral surface of $x^2p + y^2q + z^2 = 0$ which pass through the hyperbola $xy = x+y, z = 1$. [ans: $yz + 2xy + xz = 3xyz$]
   (f) Find the equation of the surface satisfying the equation $4yzp + q + 2y = 0$ and passing through $y^2+z^2 = 1$, $x+z = 2$. [ans: $y^2+z^2+x+z-3 = 0$]
   (g) Find the equation of the integral surface of the differential equation
   $$x^3\frac{\partial z}{\partial x} + y(3x^2+y)\frac{\partial z}{\partial y} = z(2x^2+y)$$
   which passes through the parabola $x = 1, y^2 = z - y$.
   (h) Solve the equation
   $$(p-q)(x+y) = z$$
   and determine the equation of the surface which satisfies this equation and passes through the curve $x+y+z = 0$, $x = z^2$.

3. Find the general integral of the equation $(2x-y)y^2p+8(y-2x)x^2q=2(4x^2+y^2)z$ and deduce the solution of the Cauchy problem when $z(x,0)=\dfrac{1}{2x}$ on a portion of the x-axis. [ans: $(8x^3+y^3)^2=\left(\dfrac{2x-y}{z}\right)^3$]
4. Find the general solution of $yz\dfrac{\partial u}{\partial x}+xz\dfrac{\partial u}{\partial y}+(x^2+y^2)\dfrac{\partial u}{\partial z}=0$. [hint: $\dfrac{dx}{yz}=\dfrac{dy}{xz}=\dfrac{dz}{x^2+y^2}=\dfrac{du}{0}$, ans: $u=f(x^2-y^2,2xy-z^2)$]
6. Solve the first-order quasi-linear partial differential equation given by
   (a) $\dfrac{\partial u}{\partial x}-2x\dfrac{\partial u}{\partial y}=xu$ subject to the boundary condition $u(0,y)=e^{-y^2}$. [hint: general solution $u(x,y)=\phi(x^2+y)e^{x^2/2}$, ans: $u(x,y)=e^{x^2/2-(x^2+y)^2}$]
   (b) $4\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}=3u$ subject to the boundary condition $u(0,y)=3e^{-y}$. [hint: general solution $u(x,y)=\phi(x-4y)e^{3x/4}$, ans: $u(x,y)=3e^{x-y}$]