Orthogonal Polynomials : A set for square areas

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ABSTRACT

Zernike Polynomials are classically used to analyse optical wavefronts transmitted by optical systems, most of which have circular pupils. However, square (and rectangular) areas present a problem : Zernike polynomials are not orthogonal over theses shapes, and therefore care is required in their computation.

A simple way around this is to use an … orthogonal set ! One such set is readily obtained from the 1-dimensional Legendre set of polynomials, each 2-D polynomial resulting from the product of one "x" polynomial with " y" polynomial. However, this new set has a big drawback : It lacks desirable low order terms present in the Zernike set, most notably the essential "power" term !

One answer is to generate a set of polynomials, requiring them to adhere to the "Zernike format" as far as possible. As we show in this paper, this works very well, and generates terms such as the power term me ntioned above.

We will present the methods mentioned above, and show the new set which is being considered for inclusion in an ISO standard on interferometry, currently being drafted by an ISO Working Group. The author calls for comments on the usefulness of the new orthogonal set presented in this paper.

Keywords : Orthogonal polynomials, Zernike polynomials, Legendre Polynomials, Wavefront analysis, ISO standards

1. INTRODUCTION

Orthogonal Sets of Functions are widely used in science, either as solutions to behaviour equations, or as analysis tools. The latter includes, *e.g.*, interferometric data reduction.

Until recently, the vast majority of components where of the spherical type, manufactured with rotating tools. The Zernike set of polynomials is ideally suited to these components, because of the use of radial and azimuthal variables.

Many attempts to "adapt" the Zernike set to other shapes have been reported but, maybe through lack of a clear definition of the method involved, these sets have not widely circulated.

By standardising the generating scheme, rather than the polynomial set proper, better suited sets may be generated, which can in turn be more easily referenced.

After a brief reminder of orthogonal sets and their properties, we illustrate polynomial generation using the Legendre set as an example. This method is called the Gram-Schmidt scheme, which is nothing else but the recurrent generation of the polynomials, each constrained to be orthogonal to the previously generated polynomials in the set.

We then apply this method to the square area. This is of interest to optics because of the large number of square (and rectangular) optics being used today. Laser MégaJoule is a good example of this, with a pupil area of *c.* 400×400 mm².

2. GENERATION OF THE ORTHOGONAL SETS

2.1 Generating the Legendre Set (Illustration of the Gram-Schmidt method)

The Gram-Schmidt method, well known to any science student, is used in the following sequence, used to generate the Legendre (1-D) set of polynomials.

Note : We do not attempt to use mathematical rigor. However, the computation is perfectly accurate, of course.

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In any generating scheme, we must start by defining the following values :

- Area of definition : [-1; +1]
- Base functions : {1, x, x^2, x^3, x^4, etc.}
- Scalar product, $I_{i,j}$:

$$I_{i,j} = \int_{-1}^{+1} P_i(x) \cdot P_j(x) \cdot dx$$

Using the above, we start the generating process :

- **$P_0(x) = 1$** (First base function); So : **$P_0(x) = 1$**
- **$P_1(x) = a\,x + b$**; We require :
  - $I_{1,0} = 0$, which gives : $b = 0$; So : **$P_1(x) = x$**
- **$P_2(x) = a\,x^2 + b\,x + c$**; We require :
  - $I_{2,1} = 0$, which gives : $b = 0$;
  - $I_{2,0} = 0$, which gives : $(a/3) + c = 0$; So : **$P_2(x) = 3\,x^2 - 1$**
- **Etc.**

We have generated the first Legendre polynomials !

2.2 Legendre 2-D polynomials

For square areas (extendable to rectangular areas), a tempting choice would be the product of two polynomials, each orthogonal in one of the two directions x and y, *e.g.* :

$$L_{i,j}(x,y) = L_i(x) \times L_j(y) \qquad \text{Equation (1)}$$

However, doing so generates a sequence with many drawbacks :

1. They possess lines of zero values, parallel to the x and y directions; Optics are (still) very often generated using circular movements, which are not well represented by such functions.
2. They lack the very useful low order terms, including the most desirable **power term, r^2.**

Figure 1, in Annex 1, shows the first such polynomials. The power term (x^2+y^2) is missing. Also, only one of the two astigmatism terms is present ($2xy$), the other term (x^2-y^2) is missing…

We suggest one answer to this problem, illustrated in the following sub-section.

2.3 Generation of the MBO Set

We apply the same scheme as in the Legendre sub-section above, using the following values :

- Area of definition : Square, $[-1; +1]^2$
- Base functions : {Z_0, Z_1, Z_2, Z_3, Z_4, etc.} (The Zernike set)
- Scalar product :

$$I_{i,j} = \int_{-1}^{+1}\int_{-1}^{+1} P_i(x,y) \cdot P_j(x,y) \cdot dx \cdot dy$$

Using the above, we start the generating process :

- **$P_0(x,y) = Z_0(x,y) = 1$** (First base function); So : **$P_0(x,y)$** $= Z_0(x,y) =$ **1**
- **$P_1(x,y) = a\,Z_1 + b\,Z_0 = a\,x + b$**
  - $I_{1,0} = 0$ gives : $b = 0$; So : **$P_1(x,y)$** $= Z_1(x,y) =$ **x**
- **$P_2(x,y) = a\,Z_2 + b\,Z_1 + c\,Z_0 = a\,y + b\,x + c$**;
  - $I_{2,1} = 0$ gives : $b = 0$;
  - $I_{2,0} = 0$ gives : $c = 0$; So : **$P_2(x,y)$** $= Z_2(x,y) =$ **y**

- **$P_3(x,y) = a\, Z_3 + b\, Z_2 + c\, Z_1 + d\, Z_0 = 2\, a\, (x^2+y^2) + b\, x + c\, y + (d-1)$**
  - $I_{3,2} = 0$ gives : $c = 0$;
  - $I_{3,1} = 0$ gives : $b = 0$;
  - $I_{3,0} = 0$, which gives : $(8/3).a + 4.d = 0$; So : **$P_3(x,y) = 3.r^2 - 2$**
- **Etc.**

Figure 2, in Annex 1, shows the first such polynomials, together with the corresponding Zernike polynomials. All the desirable terms are present.

Also, it is interesting to see that most terms of the new set are almost identical to the corresponding Zernike term. The author of this paper first thought that the equations of the two sets would diverge in their coefficients as well as in their graphical appearence.

However, a look at the graphical figures of these sets, Figures 3, 4, 5 and 6, in Annex 2, shows how untrue this is : All pairs of corresponding polynomials have a high degree of likeness, even in the higher orders.

Note that the polynomials sometimes take high values in the extreme diagonal areas, leading to non-uniform colour scheme and variation in the visibility of the polynomials' details. This effect does not appear in the "Zero-crossing" plots that are, in effect, a kind of fringe pattern with one fringe located at zero-height.

3. SOME RELATED DOCUMENTS

For reference, we include some ISO standards related to the topic of this paper:

- ISO 10110 : "Optics and optical instruments – Preparation of drawings for optical elements and systems"
  Part 5 : "Surface form tolerance", and
  Part 14 : "Wavefront deformation tolerance"
  Both these part contain orthogonal polynomial analysis, including the definition of "Best Fitting Sphere", with or without use of Zernike
- ISO 14999 : "Optics and optical instruments – Interferometric measurement of optical elements and optical systems" (4 parts), ongoing work,
  Part 2 will probably include the set described in this paper (in the "*informative annex"*)
- ISO/TR 21607 : "Optics and optical instruments – Interferometric testing of optical elements for surface form tolerances – Evaluation of rotational symmetric deviations by means of average radial profile" (ISO Technical Report)

4. CONCLUSION

Orthogonal Sets of Functions are widely used in science, either as solutions to behaviour equations, or as analysis tools. The latter includes, *e.g.*, interferometric data reduction.

Until recently, the vast majority of components where of the spherical type, manufactured with rotating tools. The Zernike set of polynomials is ideally suited to these components, because of the use of radial and azimuthal variables.

We showed how a set orthogonal on, e.g., a square surface might be generated using a definite scheme based on the circular (Zernike) set. Most importantly, the basic terms (up to "third" order deformation inclusive) are preserved in this set, especially : Piston, tip-tilt, power and astigmatism.

This generation method (the mathematics of which are known to any physics or mathematics students) can be applied to other shapes, of course, including the most important discrete areas, for real-life sets of data.

Last, but not least, I wrote this paper mainly because I liked the pretty pictures the polynomials make.

ACKNOWLEDGEMENTS

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ANNEX 1 — TABLES OF POLYNOMIAL EQUATIONS

The following tables show :

1. The square Legendre 2-D polynomials;
2. The Circular-Zernike set and the Square-MBO set, side by side for comparison.

	N	p, q	Formula
L_0	0	0, 0	1
L_1	1	1, 0	x
L_2		0, 1	y
L_3	2	2, 0	$(3x^2 - 1) / 2$
L_4		1, 1	xy
L_5		0, 2	$(3y^2 - 1) / 2$
L_6	3	3, 0	$x(5x^2 - 3) / 2$
L_7		2, 1	$y(3x^2 - 1) / 2$
L_8		1, 2	$x(3y^2 - 1) / 2$
L_9		0, 3	$y(5y^2 - 3) / 2$
L_{10}	4	4, 0	$(35x^4 - 30x^2 + 3) / 8$
L_{11}		3, 1	$xy(5x^2 - 3) / 2$
L_{12}		2, 2	$(3y^2 - 1)(3x^2 - 1) / 4$
L_{13}		1, 3	$xy(5y^2 - 3) / 2$
L_{14}		0, 4	$(35y^4 - 30y^2 + 3) / 8$
			Etc.

Figure 1 : Legendre 2-D set (0 to 14)

	N	n, m	Formula (Circular-Zernike)	Formula (Square-MBO)
Z_0	0	0, 0	1	1
Z_1	2	1, 1	$r\cos\theta$	$r\cos\theta$
Z_2		1, 1	$r\sin\theta$	$r\sin\theta$
Z_3		2, 0	$2r^2 - 1$	$3r^2-2$
Z_4	4	2, 2	$r^2\cos 2\theta$	$r^2\cos 2\theta$
Z_5		2, 2	$r^2\sin 2\theta$	$r^2\sin 2\theta$
Z_6		3, 1	$(3r^2 - 2)r\cos\theta$	$(15r^2-14)r\cos\theta$
Z_7		3, 1	$(3r^2 - 2)r\sin\theta$	$(15r^2-14)r\sin\theta$
Z_8		4, 0	$6r^4 - 6r^2 + 1$	$(315r^4-480r^2+124) / 41$
Z_9	6	3, 3	$r^3\cos 3\theta$	$31r^3\cos 3\theta / 46 + 3r(13r^2-8)\cos\theta / 46$
Z_{10}		3, 3	$r^3\sin 3\theta$	$31r^3\sin 3\theta / 16 - 3r(13r^2-8)\sin\theta / 16$
Z_{11}		4, 2	$(4r^2 - 3)r^2\cos 2\theta$	$r^2(7r^2-6)\cos 2\theta$
Z_{12}		4, 2	$(4r^2 - 3)r^2\sin 2\theta$	$r^2(6-5r^2)\sin 2\theta$
Z_{13}		5, 1	$(10r^4 - 12r^2 + 3)r\cos\theta$	$3(105r^4-194r^2+76)r\cos\theta / 17 + 22r^3\cos 3\theta / 17$
Z_{14}		5, 1	$(10r^4 - 12r^2 + 3)r\sin\theta$	$3(105r^4-194r^2+76)r\sin\theta / 61 - 22r^3\sin 3\theta / 61$
Z_{15}		6, 0	$20r^6 - 30r^4 + 12r^2 - 1$	$(77385r^6-194670r^4+128604r^2-17672) / 6353$
			Etc.	*Etc.*

Figure 2 : Zernike and MBO set (0 to 15)

ANNEX 2 — TABLES OF POLYNOMIAL PLOTS

For convenience, graphs of the polynomial sets are located in the following annex. We have chosen an "A-B-A-B" sequence so that, whatever the page layout (which is out of our control), the pictures can be visually compared by "flicking" each page over its counterpart.

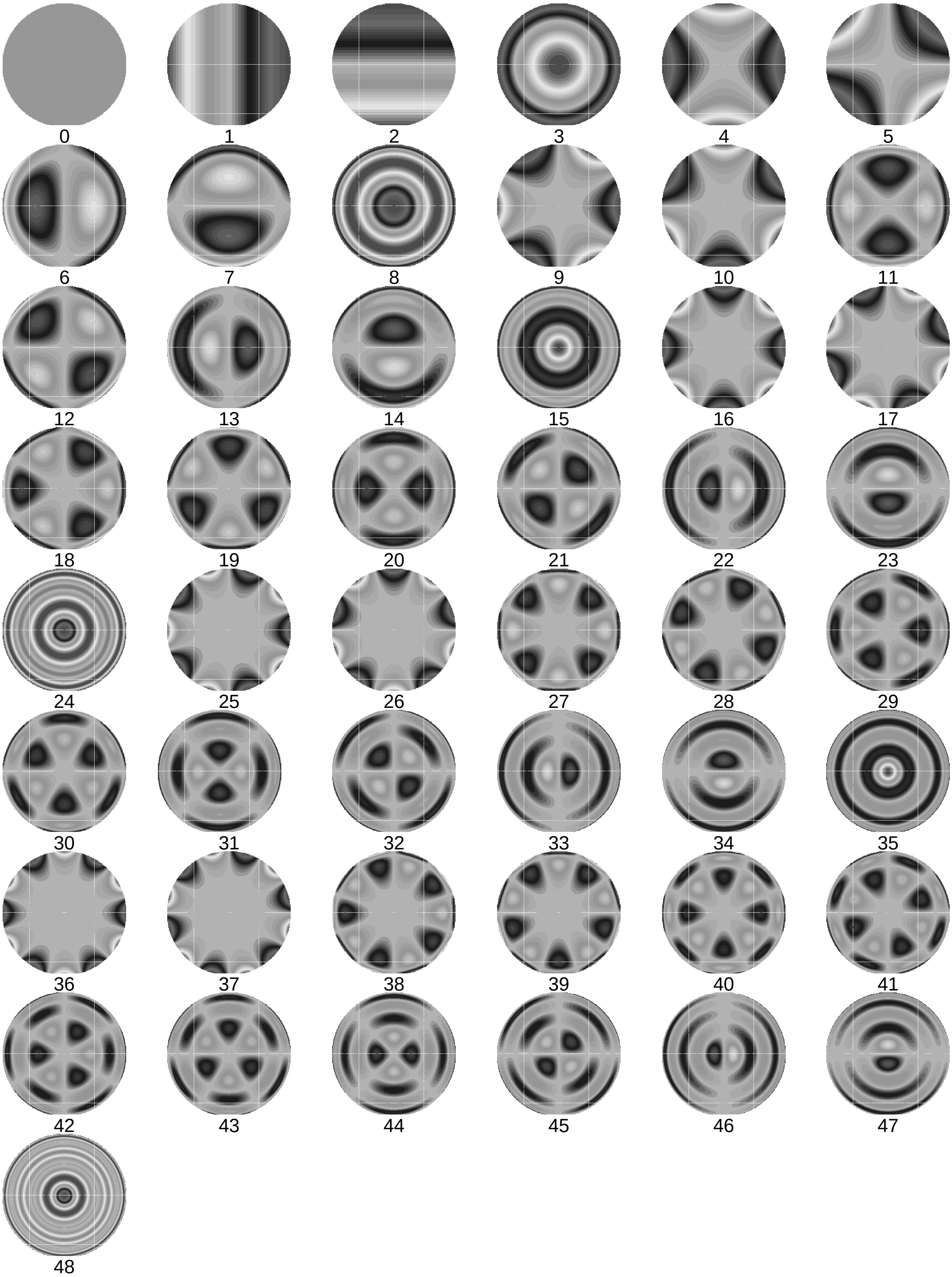

Figure 3 : Zernike set (0 to 48) — Height Plot

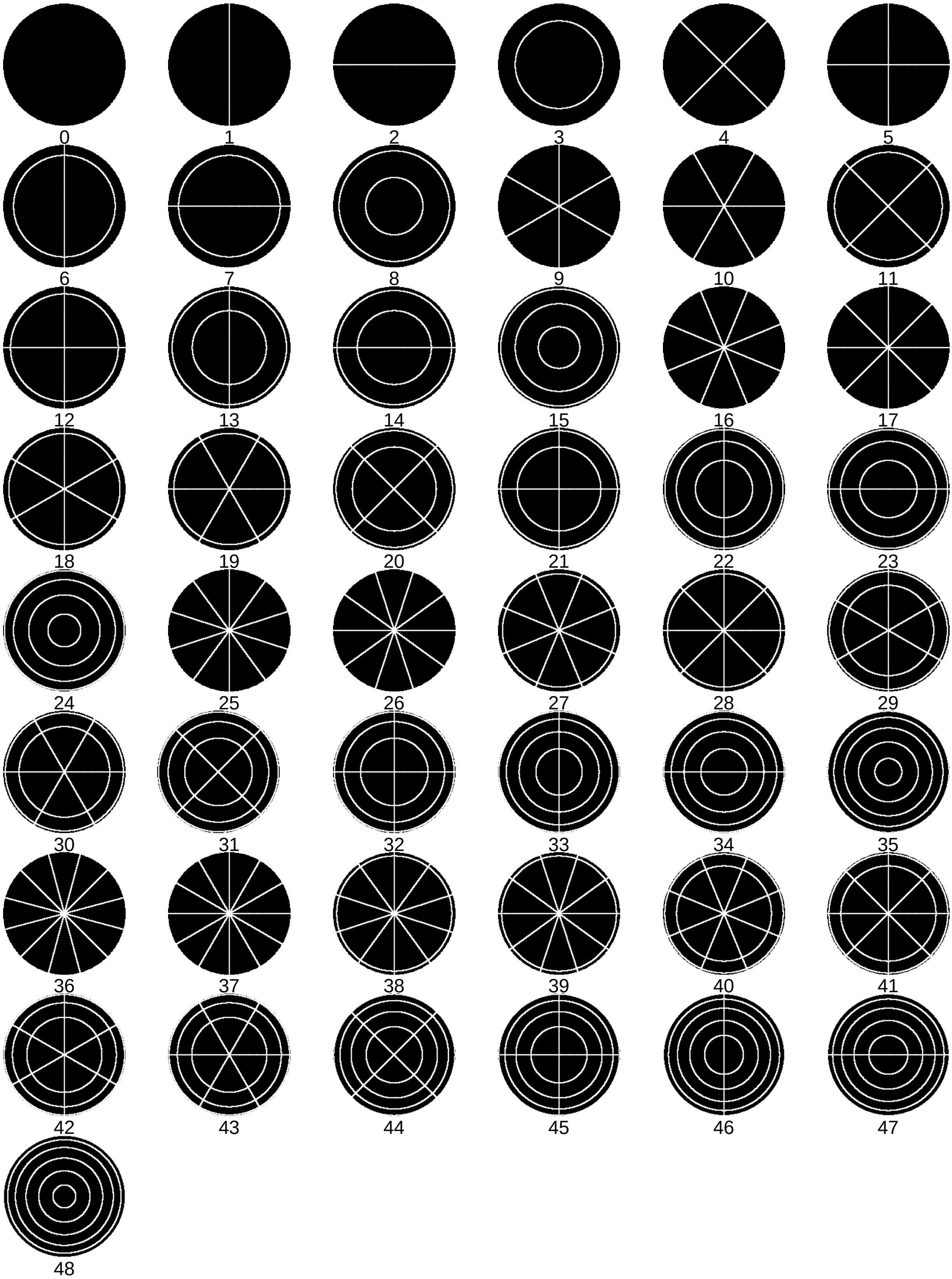

Figure 4 : Zernike set (0 to 48) — Zero-crossing Points

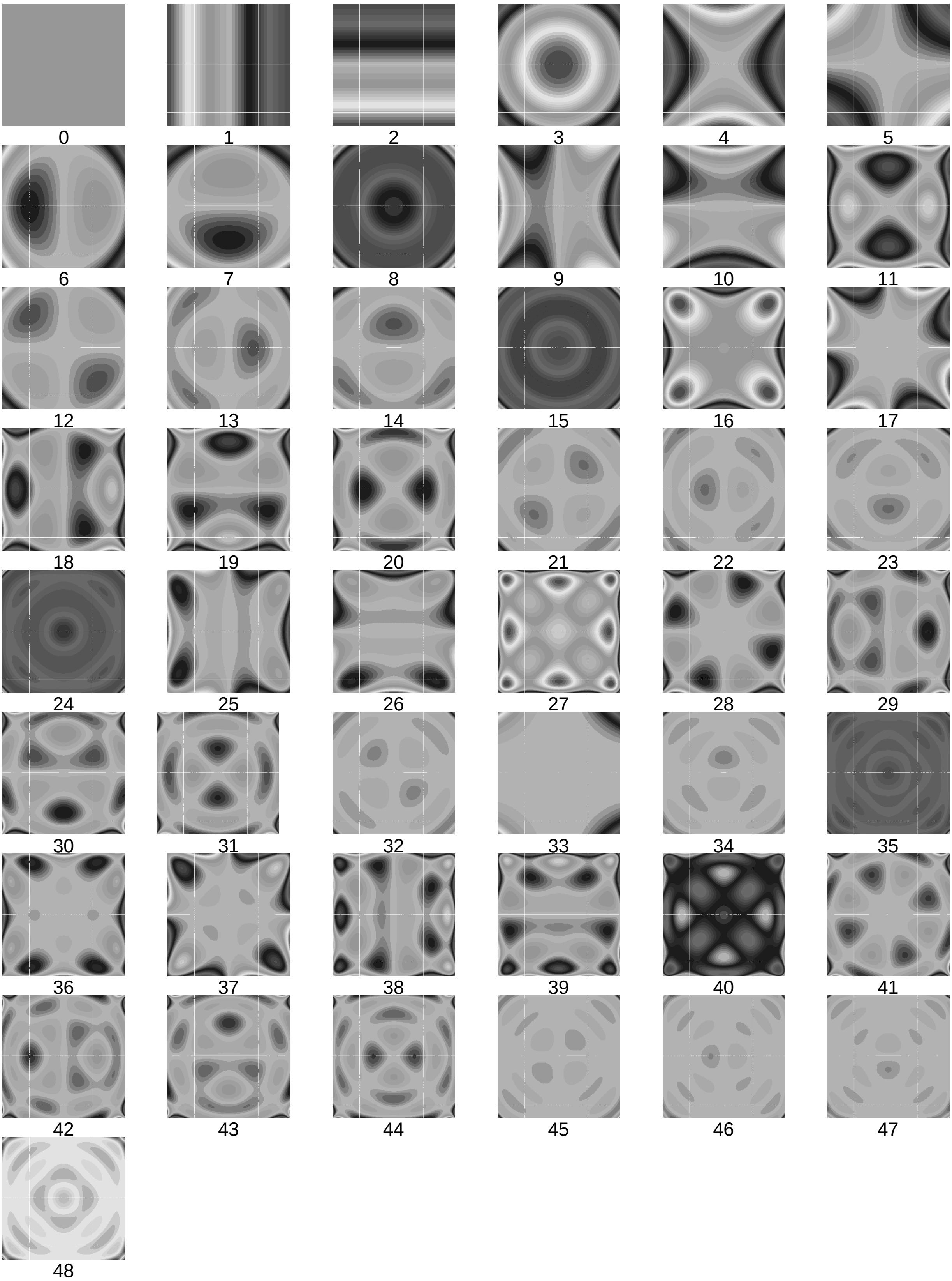

Figure 5 : MBO set (0 to 48) — Height Plot

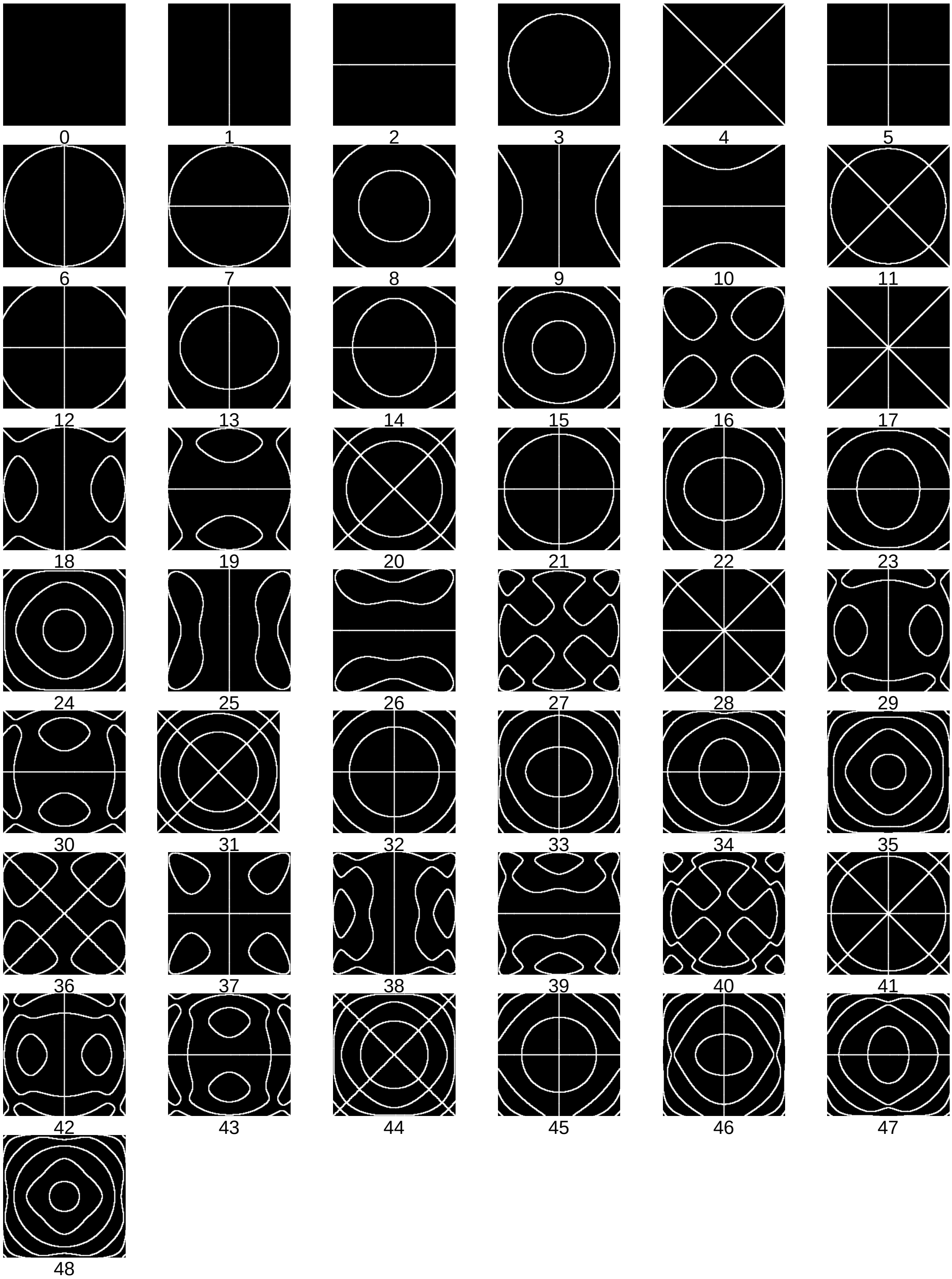

Figure 6 : MBO set (0 to 48) — Zero-crossing Points