Statistical Analysis

Nonparametric testing

Ana Cristina Costa

ccosta@novaims.unl.pt

Topics

- **LU8 – Nonparametric testing**
  - Introduction to nonparametric testing
  - Distribution fitting tests
  - Comparing independent samples
  - Comparing paired-samples
  - Spearman’s rank correlation test

Objectives

- **At the end of this learning unit students should be able to**
  - Distinguish parametric & nonparametric test procedures
  - Discuss the general advantages and disadvantages of nonparametric methods
  - Identify different distribution fitting tests and apply them using software
  - Discuss the general characteristics of the Shapiro-Wilk test and apply it
  - Discuss the general characteristics of the Wilcoxon Signed-Ranks test and apply it
  - Discuss the general characteristics of the Wilcoxon-Mann-Whitney test and apply it
  - Compute the Spearman's correlation coefficient and interpret it
  - Discuss the general characteristics of the Spearman's correlation test and apply it

Suggested reading

- Newbold, P., Carlson, W. L., Thorne, B. (2013). Statistics for Business and Economics. 8th Edition, Boston: Pearson, pages 602-611 (ch. 14), 622-636 (ch. 14).

- Mendenhall, W., Beaver, R. J., & Beaver, B. M. (2013). *Introduction to probability and statistics*. 14th Edition, Boston: Brooks/Cole, Cengage Learning, pages 606-615, 621-627, 637-645.

Introduction to nonparametric testing

- The nonparametric tests are applicable regardless of the distribution's form (***distribution-free***), or are applicable to a wider spectrum of distributions

- In general, the parametric tests are more powerful than the nonparametric ones
  - Have a greater ability to detect the differences that actually exist
  - But, they depend on assumptions about the population distribution

- Use nonparametric tests (NP) when
  - The form of the population's distribution is not known, or the specified family raises doubts
  - The variables are of nominal or ordinal type
  - When the data are scarce, asymmetric or strongly related, the asymptotic results may not be adequate → Use an exact NP test

Introduction to nonparametric testing

Tests for the location parameter

Sample(s)	Parametric test	Nonparametric test
One sample	Z or t for μ	▪ Test of signals ▪ Wilcoxon signed-ranks test (one sample median test) ▪ Savage's test ▪ Van der Waerden's test (*Normal scores*)
Two independent samples	Z or t for $\mu_1 - \mu_2$	▪ Wilcoxon-Mann-Whitney test (Wilcoxon rank sum test ≈ Mann-Whitney U test] ▪ Conover's test
Two paired samples	Z or t for μ_D	▪ Test of signals ▪ Wilcoxon signed-ranks test (two sample median test) ▪ McNemar's test for dichotomous variables

Introduction to nonparametric testing

Tests for the location parameter

Sample(s)	Parametric test	Nonparametric test
Three or more independent samples	ANOVA (F test)	▪ Kruskal-Wallis test
Three or more dependent samples	Repeated-measures ANOVA	▪ Friedman's test ▪ Bowker's test for nominal variables (extension of the McNemar's test)

- For a discussion on nonparametric tests for the two-way ANOVA, see

Luepsen, H. (2018). Comparison of nonparametric analysis of variance methods: A vote for van der Waerden. *Communications in Statistics-Simulation and Computation*, *47*(9), 2547-2576. https://doi.org/10.1080/03610918.2017.1353613

The McNemar's test is a special case of the Cochran-Mantel-Haenszel statistic (and the test of Signals)

Introduction to nonparametric testing

- **Tests of scale, association and homogeneity**

Application	Sample(s)	Parametric test	Nonparametric test
Scale / variance	Two or more independent samples	▪ F for the ratio of two variances ▪ Bartlett's test ▪ Levene's test	▪ Siegel-Tukey test ▪ Ansari-Bradley test ▪ Klotz's test ▪ Mood's test ▪ Conover's test
Association	Two paired samples	▪ Pearson's correlation test ▪ Simple linear regression (F test)	▪ Spearman's rank correlation test ▪ Kendall's tau-b ▪ χ^2 Independence test
Homogeneity	One sample		▪ χ^2 Homogeneity test ▪ Mann-Kendall trend test (Kendall's tau-b statistic) ▪ Wald-Wolfowitz one-sample runs test ▪ Von Neumann's ratio test

Introduction to nonparametric testing

Other tests

Application	Sample(s)	Nonparametric test
Goodness of fit (adjustment)	One sample	▪ χ^2 Adjustment test ▪ Kolmogorov-Smirnov test ▪ Lilliefors' test ▪ Shapiro-Wilk test ▪ Anderson-Darling test ▪ Cramer-von Mises test
Verify if two or more groups of observations have identical distributions	Two independent samples	▪ Kuiper's test ▪ Location tests
	Two or more independent samples	▪ Kolmogorov-Smirnov test ▪ Cramer-von Mises test
Verify if two or more groups of observations are independent	Two or more samples	▪ Fisher's exact test

Distribution fitting tests

Adjustment tests

- **AIM**: To test the goodness of fit, i.e. to know if a set of observations supports the hypothesis of being a random sample from a population with a given distribution

- **Hypotheses**

  H_0: The sample comes from a specified distribution

  H_1: The sample does **not** come from a specified distribution

  - Simple hypothesis → Completely specified distribution
    - H_0: The sample comes from a Normal population, $\mu=0$ and $\sigma=1$
  - Composite hypothesis → One or more parameters are unknown
    - H_0: The sample comes from a Normal population, μ and σ unknown

Distribution fitting tests

- **Chi-Square test**
  - Generally used in the analysis of grouped or classified variables (i.e. variables that divide the observations into two or more categories)
  - More generic and with fewer restrictions, but should be used with higher n because the distribution is established in asymptotic terms

- **Kolmogorov-Smirnov test**
  - Only suitable for completely specified distributions
  - Can be applied to data that are at least in an ordinal scale
  - Has advantages over exact tests based on the Chi-Square goodness-of-fit statistic, which depend on an adequate sample size and proper interval assignments for the approximations to be valid

Distribution fitting tests

- **Lilliefors' test**
  - Extension of the Kolmogorov-Smirnov test for cases where the parameters of the distribution are not known and have to be estimated
- **Anderson-Darling test**
  - Modification of the Kolmogorov-Smirnov test for cases where the parameters of the distribution are not known and have to be estimated
- **Shapiro-Wilk test**
  - Normality test (unknown parameters)
  - Usually, it presents a higher power when compared with other tests, including the Chi-Square test and the Lilliefors' test
  - With small samples, it shows better performance than the Kolmogorov-Smirnov test
  - This test is best suited to samples of less than 5000 observations

B. W. Yap & C. H. Sim (2011) Comparisons of various types of normality tests, *Journal of Statistical Computation and Simulation*, 81:12, 2141-2155

Distribution fitting tests

- **Chi-Square test (discreet case)**
  - Random experience with ***k* possible results** that constitute a division of the space of results
    - Set of events: $\{A_1, A_2, \ldots, A_k\}$
    - $p_i = P(A_i)$, $i=1, 2, \ldots, k$; $p_i > 0$; $\sum p_i = 1$
  - Test if the p_i take specified values π_i, such that $\sum \pi_i = 1$
  - **Hypotheses:** case in which the distribution is completely specified
    - H_0: $p_1=\pi_1$, $p_2=\pi_2$, ... $p_k=\pi_k$
    - H_1: $\exists i$: $p_i \neq \pi_i$

Distribution fitting tests

- **Chi-Square test (discreet case)**
  - $\boldsymbol{n_i}$ → number of times that the event A_i occurs in $\boldsymbol{n}$ independent random experiences (i=1, 2, .. ,k)
    - If H_0 is true, $\frac{n_i}{n} \xrightarrow{P} \pi_i$
    - The maximum likelihood estimator of p_i is $\hat{p}_i = {}^{n_i}/_{n}$
    - If H_0 is true, $\boldsymbol{n_i}$ are the **observed values** and $\boldsymbol{n\pi_i}$ are the **expected values**
  - **Pearson's statistic**

$$Q = \sum_{i=1}^{k} \frac{(n_i - n\pi_i)^2}{n\pi_i} = \sum_{i=1}^{k} \frac{n_i^2}{n\pi_i} - n \overset{a}{\sim} \chi^2_{(k-1)}$$

  - **Test's applicability conditions**
    - Not more than 20% of classes with $\boldsymbol{n\pi_i < 5}$
    - All classes with $\boldsymbol{n\pi_i \geq 1}$

Distribution fitting tests

- **Chi-Square test (continuous case)**
  - Partition of the support of the X variable in *k* intervals of the type
    - $\mathbf{A_1}=]-\infty, a_1]$, $\mathbf{A_2}=]a_1, a_2]$, …, $\mathbf{A_{k-1}}=]a_{k-2}, a_{k-1}]$, $\mathbf{A_k}=]a_{k-1}, \infty[$
    - $\mathbf{H_0}$: The probability density function of X is $f(x|\theta)$
      - The distribution is completely specified
  - **Test statistic** (same of the discreet case)

$$Q=\sum_{i=1}^{k}\frac{(O_i-E_i)^2}{E_i} \overset{a}{\sim} \chi^2_{(k-1)}$$

    - $\mathbf{O_i = n_i}$ → number of realizations of X belonging to the interval A_i
    - $\mathbf{E_i = n \times P(X \in A_i)}$ → estimated absolute frequency for the interval A_i, under H_0

Distribution fitting tests

- **Chi-Square test (unknown parameters)**
  - **Null hypothesis**
    - **H_0:** The probability [density] function of X is *f(x)*
  - **Test statistic**

$$Q = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \overset{a}{\sim} \chi^2_{(k-p-1)}$$

    - **p** → number of parameters that need to be estimated using sample data
    - **$O_i = n_i$** → observed absolute frequency in class A_i
    - **$E_i = n \times P(X \in A_i) = n\pi_i$** → estimated absolute frequency in class A_i, under H_0
  - **Decision**
    - Reject H_0 if $Q_{Obs} > \chi^2_{(k-p-1;\, 1-\alpha)}$
    - *p-value* = $P(\chi^2_{(k-p-1)} \geq Q_{Obs})$

Distribution fitting tests

- **Example 1**
  - A brand of mobile phones is sold in five colours: green (A_1), brown (A_2), red (A_3), blue (A_4) and white (A_5). In a market study to estimate the popularity of the various colours a random sample of 300 recent sales was analysed and it was obtained:

Colour	A_1	A_2	A_3	A_4	A_5	***Total***
Observed frequency	88	65	52	40	55	***300***

  - It is intended to test the hypothesis that consumers do not manifest tendency to prefer any of the colours:

$$H_0 : p_1 = p_2 = p_3 = p_4 = p_5 = 0.2$$

Distribution fitting tests

- **Example 1**
  - **Test statistic**

Class	Observed Frequencies (n_i)	Expected Frequencies ($n\pi_i = 300 * 0.2$)	$\frac{(n_i - n\pi_i)^2}{n\pi_i}$
A_1	88	60	13.07
A_2	65	60	0.42
A_3	52	60	1.07
A_4	40	60	6.67
A_5	55	60	0.42
Total	300	300	$\mathbf{Q_{Obs} = 21.63}$

  - **Decision**: $Q_{Obs} > \chi^2_{(4;\, 0.95)} = 9.4877 \Rightarrow$ Reject H_0 at the 5% level
  - ***p-value*** = 0.0002

Distribution fitting tests

- **Example 2**
  - A study on the life time (in days) of a sample of 1000 electronic components provided the following results. The manufacturer claims that the life time of these components has Exponential distribution with mean value equal to 200. Test this claim based on the obtained data.

Classes	Life time	Observed frequency
A_1	$X < 150$	543
A_2	$150 \leq X < 300$	258
A_3	$300 \leq X < 450$	120
A_4	$450 \leq X < 600$	48
A_5	$600 \leq X < 750$	20
A_6	$X \geq 750$	11

Distribution fitting tests

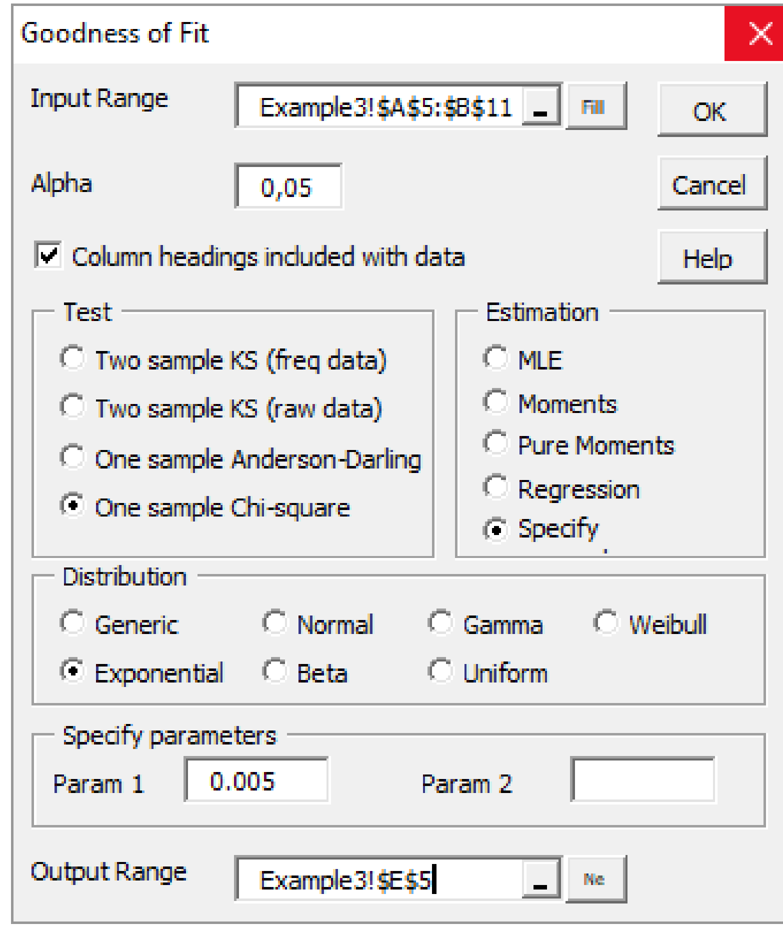
- **Example 2**
  - H_0: X~Exp(β=200) or H_0: X~Exp(λ=1/200)

Class	Observed Freq. (n_i)	Expected Freq. ($n\pi_i$)	
X < 150	543	**n×P(X< 150) = 527.63**	0.45
150 ≤ X < 300	258	249.24	0.31
300 ≤ X < 450	120	117.73	0.04
450 ≤ X < 600	48	55.61	1.04
600 ≤ X < 750	20	26.27	1.50
X ≥ 750	11	23.52	6.66
Total	**1000**	**1000**	$\mathbf{Q_{Obs} = 10.00}$

  - **Decision**: $Q_{Obs} > \chi^2_{(5;\,0.95)} = 11.0705 \Rightarrow$ Do not reject H_0 at the 5% level
  - ***p-value*** = 0.0752

Distribution fitting tests

- **Example 3**
  - **Solve Example 2 in the Real Statistics Resource Pack**
  - Organise the data as in the **Example3** sheet of the LU7_Examples Excel file
  - MISC tab + Goodness of fit



Distribution fitting tests

- ## Shapiro-Wilk test
  - **H_0:** $X \sim N(\mu, \sigma)$, in which μ and σ are unknown
  - **Test statistic:** $W = \dfrac{b^2}{(n-1)S^2}$

    b^2 → Estimator of σ^2

    $(n-1)S^2$ → Estimator of σ^2

    - Sorted sample : $x_{(1)} \le X_{(2)} \le X_{(3)} \le \ldots \le X_{(N)}$
    - The $\boldsymbol{a_i}$ coefficients and $\boldsymbol{W_{Crit}}$ values are tabulated
    - $$b = \begin{cases} \sum_{i=1}^{n/2} a_i\left(x_{(n-i+1)} - x_{(i)}\right), & \text{if } n \text{ is even} \\ \sum_{i=1}^{(n+1)/2} a_i\left(x_{(n-i+1)} - x_{(i)}\right), & \text{with } a_{[n/2]+1} = 0 \text{ if } n \text{ is odd} \end{cases}$$

    If n is odd: ignore the median observation

  - **Reject H_0 if $W_{Obs} < W_{Crit}$**

Distribution fitting tests

- **Example 4**
  - Check if the following (sorted) data comes from a Normal distribution

    8 9 10 10 10 12 12 16 19 24

  - Compute b:

i	$n-i+1$	a_i	X_{n-i+1}	X_i	$a_i\,(X_{n-i+1}-X_i)$
1	10	0.5739	24	8	9.1824
2	9	0.3291	19	9	3.291
3	8	0.2141	16	10	1.2846
4	7	0.1224	12	10	0.2448
5	6	0.0399	12	10	0.0798
				b =SUM =	**14.0826**

Distribution fitting tests

- **Example 4**
  - $(n–1)S^2 = (10–1)26.222 = 236$
  - $W_{Obs} = (14.0826\ )^2/236 = 0.8403$
  - $W_{(10;\ 0.05)} = 0.842$
  - **Decision**: $W_{Obs} < W_{Crit} \Rightarrow$ Reject the normality hypothesis at the 5% significance level

Comparing independent samples

- **Wilcoxon–Mann–Whitney test**
  - The data is composed by two independent random samples from two populations X and Y. Let X_1, X_2,...X_n be a random sample of size n from the X population, and Y_1, Y_2, ...,Y_m a random sample of size m from the Y population
  - The test is carried out based on the sum of the ranks, in the combined sorted sample (ranks from 1 to N=n+m), of the elements of one of the samples
    - If there are ***ties***, it is assigned to each value the average of the ranks that those tied values would have if they were different from each other
  - **Assumptions**
    - The two samples are mutually independent
    - The scale of measurement is, at least, ordinal

Comparing independent samples

Wilcoxon–Mann–Whitney test

- Test the difference between the distribution function of X, *F(x)*, and the one of Y, *G(x)*

1. Two-sided test

$$H_0: F(x) = G(x) \quad \forall x$$

$$H_1: \exists x \;\; F(x) \neq G(x)$$

2. Right-sided test

$$H_0: F(x) = G(x) \quad \forall x$$

$$H_1: \exists x \;\; F(x) > G(x)$$

The values of sample X tend to be higher than the values of sample Y

3. Left-sided test

$$H_0: F(x) = G(x) \quad \forall x$$

$$H_1: \exists x \;\; F(x) < G(x)$$

- Test the difference between the medians of the two populations

1. Two-sided test

$$H_0: \tilde{\mu}_X = \tilde{\mu}_Y$$

$$H_1: \tilde{\mu}_X \neq \tilde{\mu}_Y$$

2. Right-sided test

$$H_0: \tilde{\mu}_X \leq \tilde{\mu}_Y$$

$$H_1: \tilde{\mu}_X > \tilde{\mu}_Y$$

3. Left-sided test

$$H_0: \tilde{\mu}_X \geq \tilde{\mu}_Y$$

$$H_1: \tilde{\mu}_X < \tilde{\mu}_Y$$

Nachar, N. (2008). The Mann-Whitney U: A test for assessing whether two independent samples come from the same distribution. *Tutorials in Quantitative Methods for Psychology*, 4(1), 13-20.

Comparing independent samples

Wilcoxon–Mann–Whitney test

- **Additional assumption** when formulating the hypotheses with medians
  - The two distributions must have the same shape and variance. If they have identical forms but very distinct variances, then the probability of committing a type I error is greater than α.
  - ✓ This assumption is empirically verified based on graphics and descriptive statistics of the two samples.
- Differences in the distribution of the two groups do not invalidate the use of the test, but change the formal interpretation of its results, because the way in which statistical hypotheses are stated depends on this third assumption. In practice, the completion of the test is the same.

Comparing independent samples

- **Wilcoxon–Mann–Whitney test**
  - **Calculation of the test statistic**
    - The two samples are grouped and the origin (X or Y) of each element is identified in this new combined sample
    - This combined sample is sorted and the ranks occupied by the X and Y elements are observed
    - **W = Sum of all the ranks of the X sample**
      - If the ranks of X and Y become randomly distributed, then the final decision will be favourable to H_0
        - Under H_0, the median of the underlying population of the combined sample will be equal to the median of X and to the median of Y
        - If, for example, W takes small values then the values of the observations of the X sample are predominantly lower than those of the Y sample

Comparing independent samples

- **Wilcoxon–Mann–Whitney test**
  - **Test statistic for small samples with few ties ($n \leq 20$, $m \leq 20$)**
    - **W * = Sum of all the ranks of the X sample** (corrected for ties)
    - **W^*_{Crit} → tabulated** (exact quantile if there are no ties)
  - **Test statistic for large samples with few ties**
    - **W * = Sum of all the ranks of the X sample**
    - **N = n + m**

$$Z = \frac{W^* - \frac{m(N+1)}{2}}{\sqrt{\frac{nm(N+1)}{12}}} \overset{a}{\sim} N(0,1)$$

Comparing independent samples

- **Wilcoxon–Mann–Whitney test**
  - **Test statistic for samples with many ties**
    - **R_i → ranks assigned to the observations of the two combined samples** (corrected for ties)
    - **N = n + m**
    - The test statistic W'* is the standardized W* statistic

$$W'^* = \frac{W^* - n\frac{N+1}{2}}{\sqrt{\frac{nm}{N(N-1)}\sum_{i=1}^{N} R_i^{\,2} - \frac{nm(N+1)^2}{4(N-1)}}} \overset{a}{\sim} N(0,1)$$

Comparing independent samples

- **Example 5**
  - A Statistics course is being taught to two degree programs, A and B, having the students performed the same exam. Two mutually independent random samples of 15 students from each class were collected.
  - The exam grades from both samples are in the **Example5** sheet of the LU7_Examples Excel file
  - At the 10% significance level, can it be said that there is a significant difference between the grades of the two degree programs?
    - **Hypotheses**

$$H_0 : \tilde{\mu}_A = \tilde{\mu}_B$$

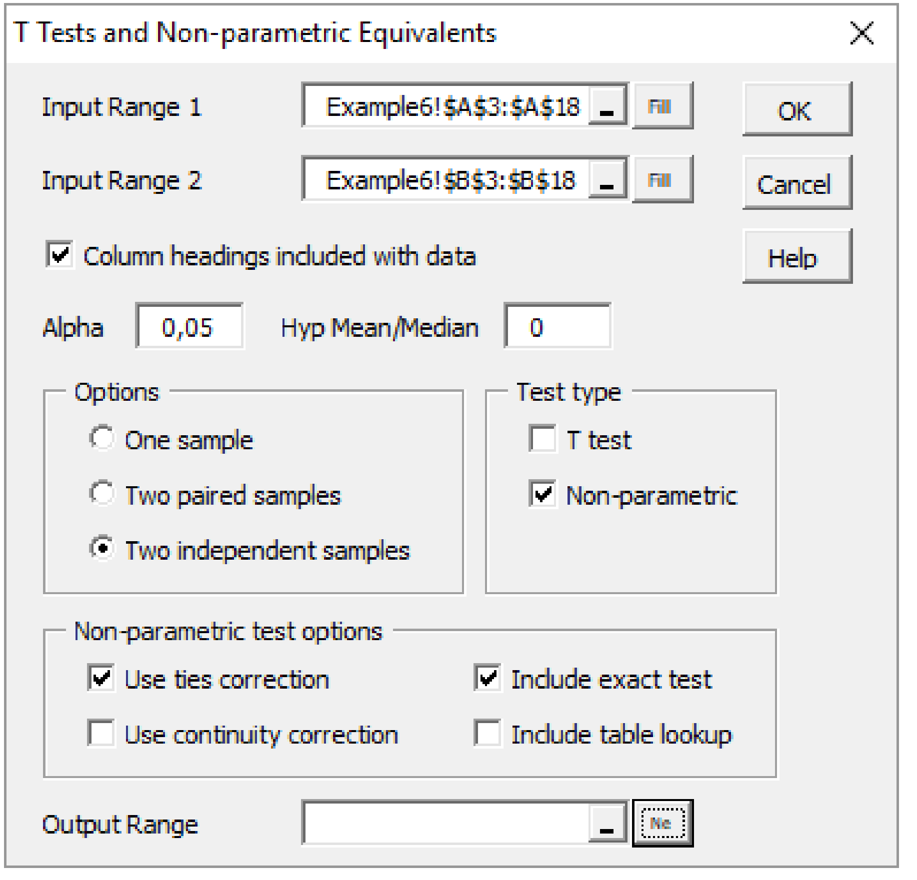
$$H_1 : \tilde{\mu}_A \neq \tilde{\mu}_B$$

Comparing independent samples

- **Example 5**
  - Determine the ranks in the combined sample using the **RANK.AVG** function
  - Sum of the ranks of class A (consider that there are few ties): $W^*_{Obs} = 277.5$
  - Inferior critical value: $\mathbf{W^*_{(15,\,15;\;\alpha/2=0.05)} = 193}$
  - Superior critical value: $\mathbf{W^*_{(15,\,15;\;1-\alpha/2=0.95)} = 15(15+15+1)-193 = 272}$
  - Reject H_0 at the 10% significance level
    - There is a significant difference between the grades of the two degree programs

Comparing independent samples

- **Example 6**
  - **Solve Example 5 in the Real Statistics Resource Pack**
  - Consider the data in the **Example6** sheet of the LU7_Examples Excel file
  - MISC tab + T Tests and Non-parametric Equivalents



Comparing paired-samples

- **Wilcoxon signed-ranks test**
  - The data consists of ***n*** pairs of random observations from a bivariate random variable (X, Y)

$$(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$$

    that constitute a random paired sample from the X and Y populations

  - The sample of the bivariate data is taken as a sample of the **population of differences D = X – Y**

Comparing paired-samples

- **Wilcoxon signed-ranks test**
  - **Assumptions**
    - The variable of interest is measured on a scale that is at least ordinal
    - The distribution of the population of differences, D, is continuous
    - The distribution of each r.v. $D_i = X_i - Y_i$ (i=1, ..., n) has to be symmetrical, all of them must have the same mean, and their scale must be continuous
  - The assumptions of this test are not tested
    - They are necessary for the calculation of the exact distribution of the test statistic (tabulated), since they guarantee that the ranks of the observations do not have ties in the sample of the differences. However, it is also possible to use an exact test based on permutation tests (available in some software). On the other hand, when there are many ties (can occur in large samples, especially for variables in Likert scales), the approximation to the Normal distribution is valid.

Comparing paired-samples

Wilcoxon signed-ranks test

- Test if one of the r.v. tends to take on average identical values to another r.v.; or whether it takes different values (higher values)

1. **Two-sided test**

   $H_0: E[D] = 0$

   $H_1: E[D] \neq 0$

2. **Right-sided test**

   $H_0: E[D] \leq 0$

   $H_1: E[D] > 0$

3. **Left-sided test**

   $H_0: E[D] \geq 0$

   $H_1: E[D] < 0$

- Test if the two paired samples come from populations with the same median

1. **Two-sided test**

   $H_0: \tilde{\mu}_X = \tilde{\mu}_Y$

   $H_1: \tilde{\mu}_X \neq \tilde{\mu}_Y$

2. **Right-sided test**

   $H_0: \tilde{\mu}_X \leq \tilde{\mu}_Y$

   $H_1: \tilde{\mu}_X > \tilde{\mu}_Y$

3. **Left-sided test**

   $H_0: \tilde{\mu}_X \geq \tilde{\mu}_Y$

   $H_1: \tilde{\mu}_X < \tilde{\mu}_Y$

Comparing paired-samples

- **Wilcoxon signed-ranks test**
  - **Wilcoxon test for a single sample** → Apply the *Wilcoxon signed-ranks test* to the x_i values of the sample, and consider the observations of Y equal to the value that is intended to test
  - **Calculation of the test statistic**
    - Calculate the differences $\mathbf{d_i = x_i - y_i}$ and retain the sign of the result (+ or –)
      - If $\mathbf{d_i = 0}$, eliminate this d_i and correct the sample size for ***n'***
    - Determine the ranks $|d_1|, |d_2|, \ldots, |d_{n'}|$, and associate them with the sign '+' if d_i is positive and with '–' if it is negative
      - If there are any ties, assign the average of the ranks that the values would have if they were different

Comparing paired-samples

- **Wilcoxon signed-ranks test**
  - **n'** → number of differences D_i that are not null
  - **R_i** → **signed-rank** = rank of the absolute difference $|D_i|$ affected by its sign (positive or negative; corrected for the ties)

- **Test statistic for small samples with few ties**
  - **W^+ = Sum of ranks of the $|D_i|$ with sign +**

$$W^+ = \sum_{\substack{i=1 \\ (D_i>0)}}^{n'} R_i$$

  - W_{Crit} of the inferior quantile → **W_α tabulated for n'** (exact if there are no ties)
  - W_{Crit} of the superior quantile → **$W_{1-\alpha} = n'(n'+1)/2 - w_\alpha$**

Comparing paired-samples

- **Wilcoxon signed-ranks test**
  - **Test statistic for large samples or with many ties**

$$Z = \frac{\sum_{i=1}^{n'} R_i}{\sqrt{\sum_{i=1}^{n'} R_i^2}} \overset{a}{\sim} N(0,1)$$

    - **If there are no ties**, the Z statistic simplifies to

$$Z = \frac{\sum_{i=1}^{n'} R_i}{\sqrt{n'(n'+1)(2n'+1)/6}} \overset{a}{\sim} N(0,1)$$

Comparing paired-samples

- **Example 7**
  - A random sample of 12 financial analysts of a multinational bank was collected, and they were asked to provide an estimate of the increase (%) of the quotations of two shares on the stock market
  - The estimates of the increase (%) of the quotations are in the **Example7** sheet of the LU7_Examples Excel file
  - Test the hypothesis that, for the population of analysts of the bank, there is no preference for each one of the two shares
    - Admit that the distribution of the differences of values is symmetrical
    - **Hypotheses**

$$H_0 : \tilde{\mu}_A = \tilde{\mu}_B$$

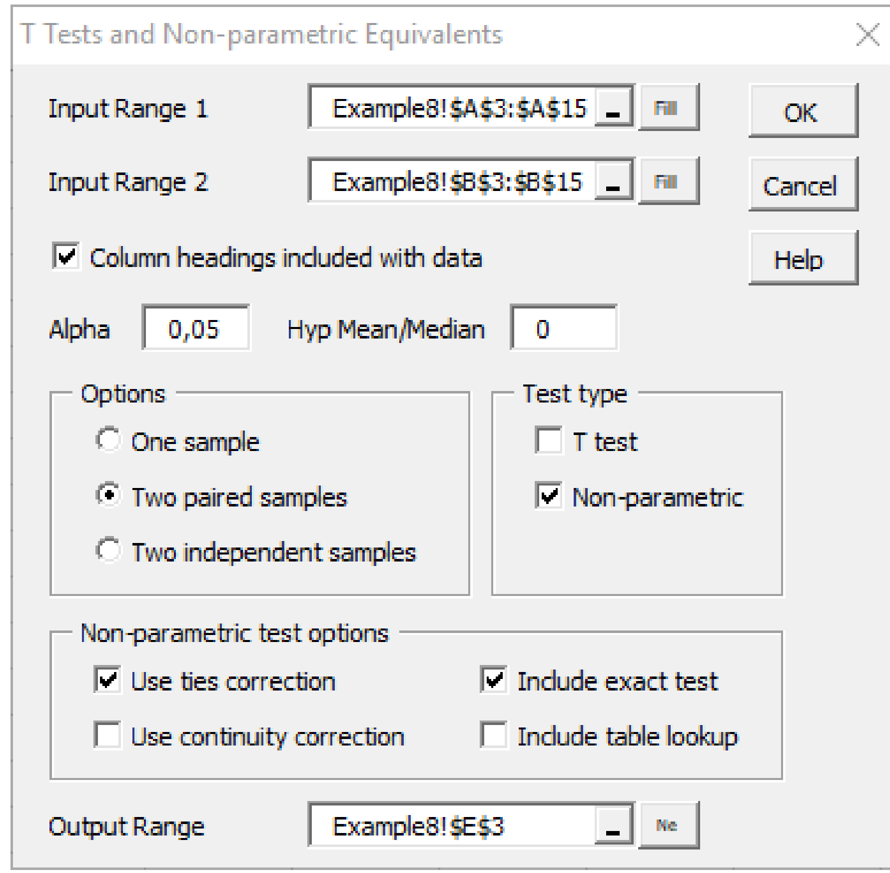
$$H_1 : \tilde{\mu}_A \neq \tilde{\mu}_B$$

Comparing paired-samples

- **Example 7**
  - Sum of ranks of the positive d_i values: $W_{Obs} = 3$
  - n' =11, critical value tabulated: $W_{(11;\ \alpha/2=0{,}025)} = 11$
  - $W_{Obs} < W_{Crit}$, therefore H_0 is rejected at the 5% significance level
    - There is evidence that the estimates of the increase (%) of the quotations of the two shares are significantly different

Comparing paired-samples

- **Example 8**
  - ➢ **Solve Example 7 in the Real Statistics Resource Pack**
  - Consider the data in the **Example8** sheet of the LU7_Examples Excel file
  - MISC tab + T Tests and Non-parametric Equivalents

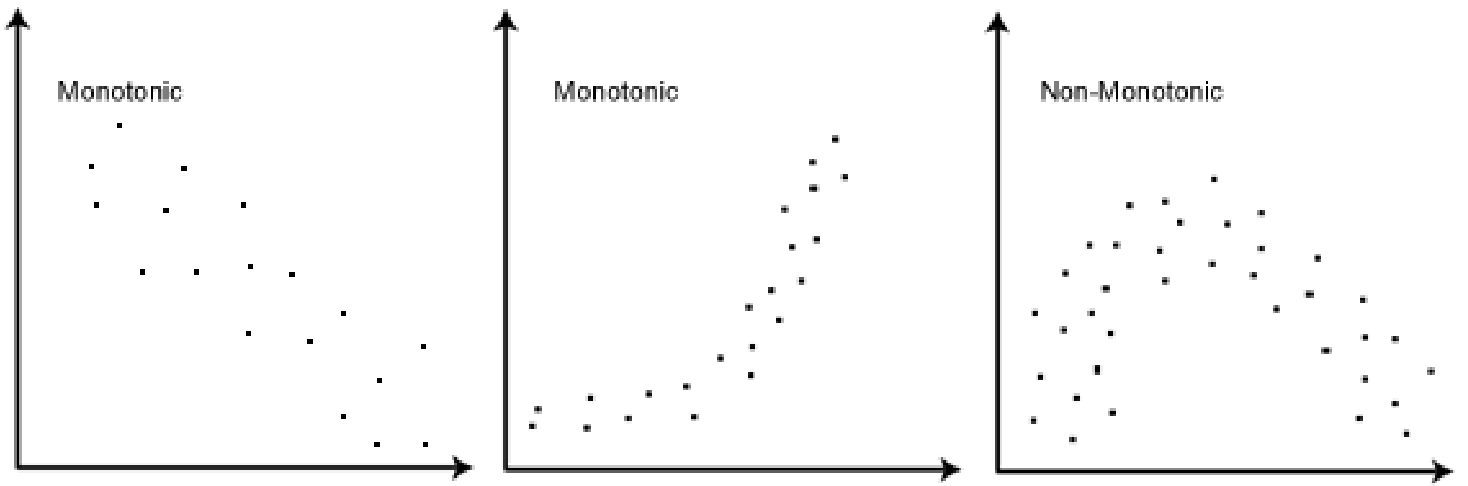


Spearman’s rank correlation test

- **Spearman’s rank correlation test**
  - Alternative to Pearson's correlation test (*Pearson's product-moment correlation*) that assumes that the random pair (X, Y) has a bivariate normal distribution
  - Spearman’s correlation coefficient is a statistical measure of the strength of a **monotonic relationship** between paired data
    - The Spearman’s rank correlation is a measure of association (not correlation)
    - Its interpretation is similar to that of Pearson's
  - **The measurement scale of the variables X and Y must be at least ordinal**
  - The Spearman’s rank correlation (also called Spearman’s rho) is the Pearson’s correlation coefficient on the ranks of the data

Spearman's rank correlation test

- **Spearman's rank correlation test**
  - **Monotonic function**
    - A monotonic function is one that either never increases or never decreases as its independent variable increases



Spearman's rank correlation test

- **Calculation of the Spearman's rank correlation coefficient**
  - Bivariate random sample : $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$
  - Sort the *n* observations of each variable, separately, and associate them with their corresponding rank
    - **$R(X_i)$** → Rank of the X_i value (corrected for ties)
    - **$R(Y_i)$** → Rank of the Y_i value (corrected for ties)
  - Apply the Pearson's linear correlation coefficient to the pairs $[R(x_i), R(y_i)]$

$$\rho = \frac{\sum_{i=1}^{n} R(X_i)R(Y_i) - n\left(\frac{n+1}{2}\right)^2}{\sqrt{\left(\sum_{i=1}^{n}[R(X_i)]^2 - n\left(\frac{n+1}{2}\right)^2\right)\left(\sum_{i=1}^{n}[R(Y_i)]^2 - n\left(\frac{n+1}{2}\right)^2\right)}}$$

Spearman’s rank correlation test

- ## Calculation of the Spearman’s rank correlation coefficient
  - **Samples without ties**

$$\rho = 1 - \frac{6\sum_{i=1}^{n}\left[R(X_i) - R(Y_i)\right]^2}{n\left(n^2 - 1\right)}$$

    - When there are few ties, this formulation can also be used, but the tabulated critical values are approximations

Spearman's rank correlation test

- ## Hypotheses

H_0: X_i and Y_i are mutually independent $\forall i$

1. **Two-sided test**

   H_1: Or ***(a)*** there is a tendency for higher values of X to be associated to high values of Y, or ***(b)*** there is a tendency for low values of X to be associated to high values of Y

2. **Right-sided test**

   H_1: There is a tendency for higher values of X to be associated to high values of Y

3. **Left-sided test**

   H_1: There is a tendency for low values of X to be associated to high values of Y

Spearman’s rank correlation test

- **Test statistic**
  - Spearman’s correlation coefficient (ρ)

- **Sampling distribution of ρ**

1. For $n \leq 30$ and when there are not (or there are few) ties, assuming that H_0 is true, the higher quantiles from the exact distribution of the Spearman's coefficient are tabulated
   - The lower quantiles are symmetric to the higher quantiles: $\mathbf{W_{\alpha} = -W_{1-\alpha}}$
2. For $n>30$ or when there are many ties, the approximation to the N(0,1) is used: $w_{\alpha} \approx \frac{z_{\alpha}}{\sqrt{n-1}}$

- ***p-value* of the two-sided test**: $p - value \approx 2P\left(Z > |\rho_{obs}|\sqrt{n-1}\right)$

Spearman's rank correlation test

- **Example 9**
  - The following data are in the **Example9** sheet of the LU7_Examples Excel file. Test if there is an association between the two variables.

X	Y
550	80
620	60
580	10
580	20
540	30

- **Hypotheses**

H_0: X and Y are independent

H_1: Either there is a tendency for higher values of X to be associated to high values of Y, or there is a tendency for low values of X to be associated to high values of Y

Spearman's rank correlation test

- **Example 9**
  - $\rho_{obs} = -0.125$
  - $W_{(5;\ 0,025)} = 1$
  - $W_{(5;\ 0,975)} = -1$
  - *p-value* $= 2\,[1 - P(Z \leq |-0.125| *2)] = 0.8026$
    - H_0 is not rejected at usual significance levels
    - There is no evidence of a significant association between X and Y

Spearman's rank correlation test

- **Example 10**
  - Ten students were disposed according to their progress in laboratory work (X) and the level of theoretical knowledge (Y)
  - The data are in the **Example10** sheet of the LU7_Examples Excel file
  - Test if there is an association between the two variables
    - **Hypotheses**

      H_0: X and Y are independent

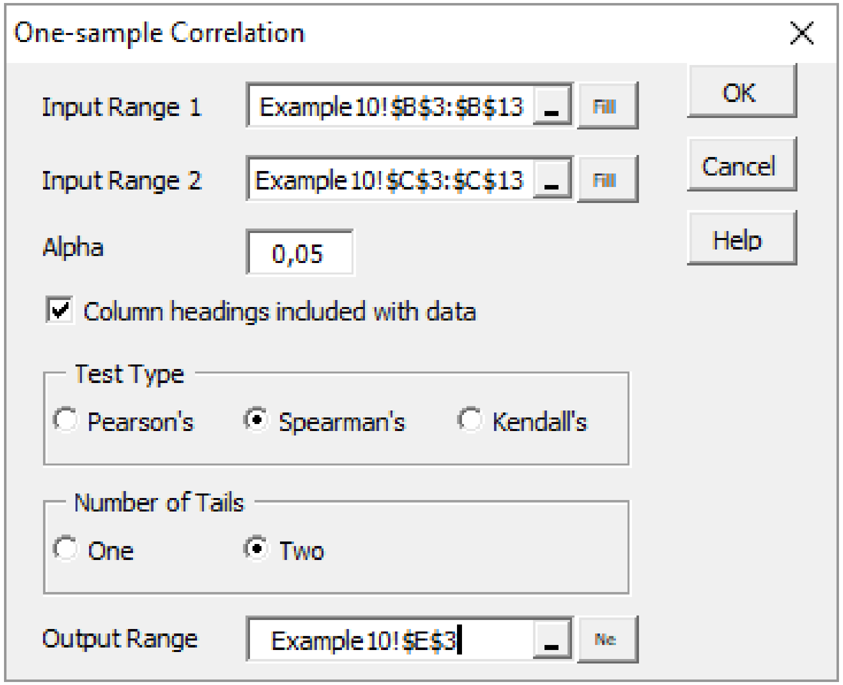
      H_1: Either there is a tendency for higher values of X to be associated to high values of Y, or there is a tendency for low values of X to be associated to high values of Y

Spearman’s rank correlation test

- **Example 10**
  - $\rho_{obs} = 0.8545$
  - $W_{(10;\ 0{,}025\ )} = 0.648$
  - $W_{(10;\ 0{,}975\ )} = -0.648$
    - H_0 is rejected at the 5% significance level
    - There is evidence of a significant positive association between X and Y
  - *p-value* $= 2\ [1 - P(Z \leq 0.8545 * 3)] = 0.0104$

Spearman’s rank correlation test

- **Example 11**
  - **Solve Example 10 in the Real Statistics Resource Pack**
  - Consider the data in the **Example11** sheet of the LU7_Examples Excel file
  - CORR tab + Correlation Tests



Nonparametric testing

Do the homework*!*

Ana Cristina Costa, ccosta@novaims.unl.pt