Ana Cristina Costa (ccosta@novaims.unl.pt)

Solved exercises about the Normal distribution

1. The random variable Z follows a Standard Normal distribution. Find the value of z_1 in the following expressions:

a) $P(Z < z_1) = 0.70$ [Exact solution: 0.5244]

b) $P(Z < z_1) = 0.25$ [Exact solution: –0.6745]

c) $P(Z > z_1) = 0.20$ [Exact solution: 0.8416]

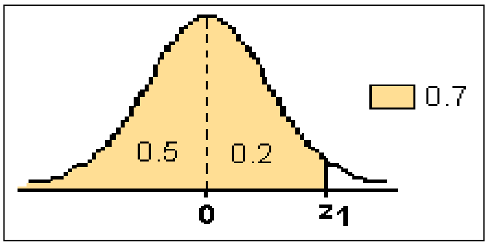
d) $P(Z > z_1) = 0.84$ [Exact solution: –0.9945]

Solving a)

Given $Z \sim N(0, 1)$ we want to determine the value of z_1 that corresponds to the 70th percentile (more formally named quantile of probability 0.70): $P(Z < z_1) = 0.70$.

First of all, we have to determine if z_1 is negative or positive.

The N(0, 1) distribution is symmetrical in zero, and therefore $P(Z<0) = P(Z>0) = 0.50$. Since $P(Z < z_1) = 0.70 > 0.50$ we can conclude that z_1 is a positive value. Note that z_1 cannot be a negative value, because in this case we would have $P(Z<z_1) < 0.50$ (see the figure below).



Now, we just need to find the value of 0.7, or the closest value to 0.7, in the Standard Normal table (see the figure below):

$$P(Z < 0.52) = 0.6985 \approx 0.7 \quad \Leftrightarrow \quad z_1 \approx 0.52$$

x	0.00	0.01	0.02	0.03	0.04	0.05
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023

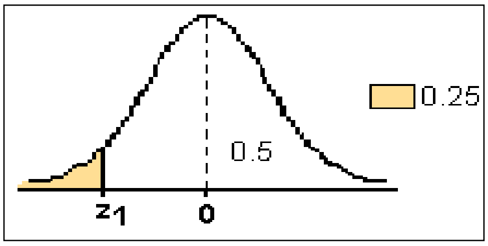
The closest value to 0.7 is the highlighted value

Solving b)

Given Z~N(0, 1) we want to determine the value of z_1 so that $P(Z < z_1) = 0.25$.

First of all, we have to determine if z_1 is negative or positive.

Using the same rationale of the previous paragraph, we conclude that z_1 is negative, because $P(Z < z_1) = 0.25 < \mathbf{0.50}$:



Given that **z_1 is negative**, we need to determine the distribution function at the respective positive value, so that we can use the Standard Normal table. Hence, using the property of symmetry of the Standard Normal distribution (see figures below):

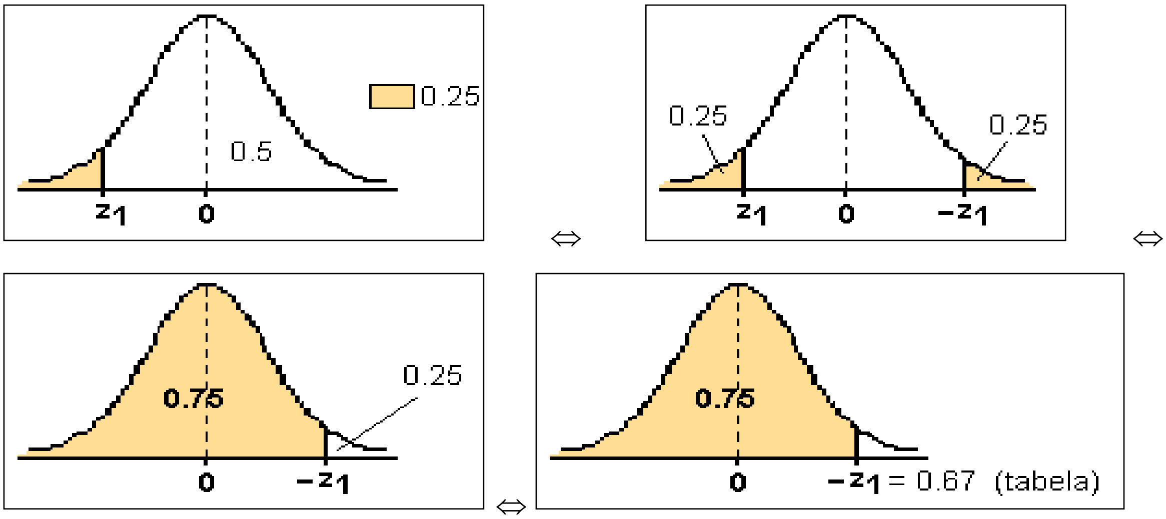
$$P(Z < z_1) = 0.25 \Leftrightarrow P(Z > -z_1) = 0.25 \Leftrightarrow$$

$$\Leftrightarrow 1 - P(Z \leq -z_1) = 0.25 \Leftrightarrow$$

$$\Leftrightarrow P(Z \leq -z_1) = 0.75$$

Given that **$-z_1$ is positive**, we now have to find the value of 0.75, or the closest value to 0.75, in the Standard Normal table:

$$P(Z < 0.67) = 0.7486 \approx 0.75 \quad \Leftrightarrow \quad -z_1 \approx 0.67 \quad \Leftrightarrow \quad z_1 \approx -0.67$$

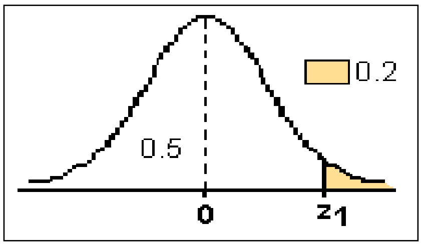


Solving c)

Given $Z \sim N(0, 1)$ we want to determine the value of z_1 so that $P(Z > z_1) = 0.20$.

First of all, we have to determine if z_1 is negative or positive.

Using the same rationale of the previous paragraphs, we conclude that z_1 is positive, because $P(Z > z_1) = 0.20 < \mathbf{0.50}$:



Given that z_1 is positive, we simply need to calculate the probability of the complementary event to use the table:

$$P(Z > z_1) = 0.2 \Leftrightarrow 1 - P(Z \leq z_1) = 0.2 \Leftrightarrow P(Z \leq z_1) = 0.8$$

We now have to find the value of 0.8, or the closest value to 0.8, in the Standard Normal table:

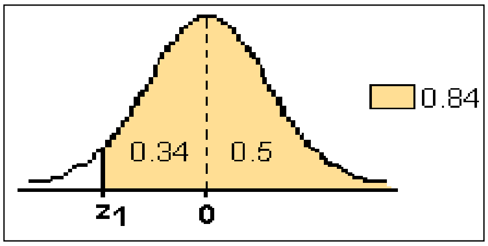
$$P(Z \leq 0.84) = 0.7995 \approx 0.8 \qquad \Leftrightarrow \qquad z_1 \approx 0.84$$

Solving d)

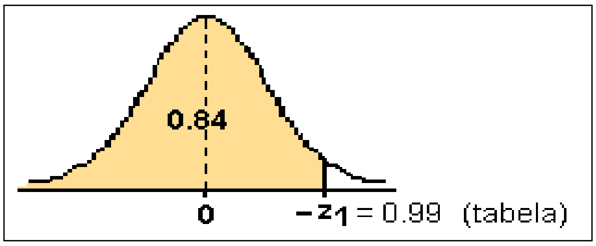
Given Z~N(0, 1) we want to determine the value of z_1 so that $P(Z > z_1) = 0.84$.

First of all, we have to determine if z_1 is negative or positive.

Using the same rationale of the previous paragraphs, we conclude that z_1 is negative, because $P(Z > z_1) = 0.84 > 0.50$:



Given that **z_1 is negative**, we need to determine the distribution function at the respective positive value, so that we can use the Standard Normal table. Hence, using the property of symmetry of the Standard Normal distribution:



$P(Z > z_1) = 0.84 \Leftrightarrow P(Z < -z_1) = 0.84$

Given that **$-z_1$ is positive**, we now have to find the value of 0.84, or the closest value to 0.84, in the Standard Normal table:

$P(Z < 0.99) = 0.8389 \approx 0.84 \quad \Leftrightarrow \quad -z_1 \approx 0.99 \quad \Leftrightarrow \quad z_1 \approx -0.99$

2. Let X~N(μ, σ).

a) Knowing that $P(X < 24) = 0.01$ and $P(X > 67) = 0.45$, find the parameters of the probability distribution of X. [Solution: μ=64.73, σ=17.48]

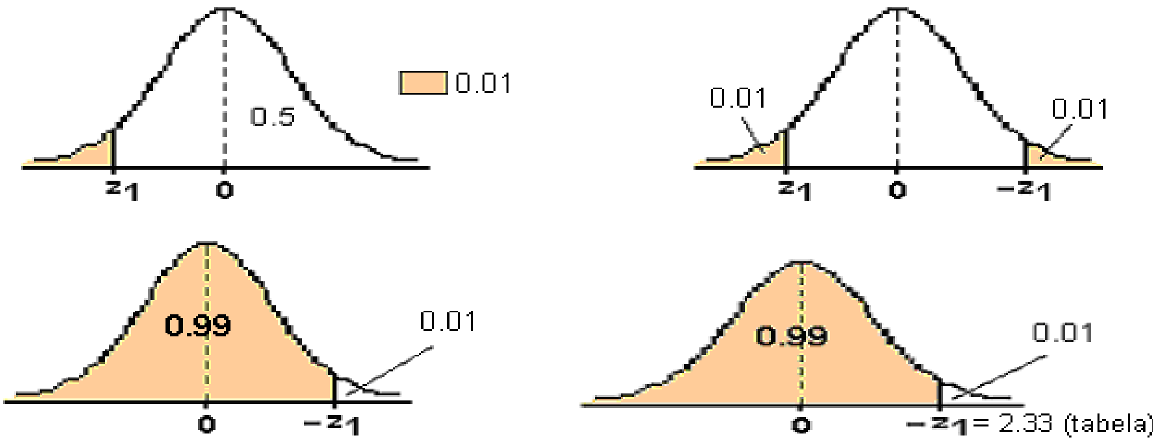
b) If $\mu = -3$ and $\sigma^2 = 4$, compute $P[-4 < X < -2]$. [Solution: 0.383]

Solving a)

$$P(X<24)=0.01 \Leftrightarrow P\left[\frac{X-\mu}{\sigma}<\frac{24-\mu}{\sigma}\right]=0.01 \Leftrightarrow P\left[Z<\frac{24-\mu}{\sigma}\right]=0.01\,,\ \ Z\sim N(0{,}1)$$

Let $z_1 = (24-\mu)/\sigma$. In the table of the N(0,1), we need to find the value of z_1 so that $P(Z < z_1)=0.01$.

Similarly to the previous exercise, the figures below help us to conclude that **z_1 is a negative value**, and that we must find the value of 0.99, or the closest value to 0.99, in the Standard Normal table:



Therefore, $z_1 = (24-\mu)/\sigma = -2.33$.

Considering the second probability given in the exercise question, and proceeding likewise, we conclude:

$$P(X>67)=0.45 \Leftrightarrow P\left[\frac{X-\mu}{\sigma}>\frac{67-\mu}{\sigma}\right]=0.45 \Leftrightarrow P\left[Z>\frac{67-\mu}{\sigma}\right]=0.45\,,\ \ Z\sim N(0{,}1)$$

Let $z_2 = (67-\mu)/\sigma$. In the table of the N(0,1), we need to find the value of z_2 so that: $P(Z > z_2)=0.45$.

Similarly to the previous case, we conclude that **z_2 is a positive value**, and that we must find the value of 0.55, or the closest value to 0.55, in the Standard Normal table.

Therefore, $z_2 = (67-\mu)/\sigma = 0.13$.

Solving the system of two equations for μ and σ, we can now obtain the solution:

$$\begin{cases} \dfrac{24-\mu}{\sigma} = -2.33 \\ \dfrac{67-\mu}{\sigma} = 0.13 \end{cases} \Leftrightarrow \ldots \Leftrightarrow \begin{cases} \mu \approx 64.73 \\ \sigma \approx 17.48 \end{cases}$$

Solving b)

$$P(-4 < X < -2) = P\left(\frac{-4+3}{2} < \frac{X+3}{2} < \frac{-2+3}{2}\right) = P(-0.5 < Z < 0.5)\ , \quad \text{sendo } Z \sim N(0,1)$$

$$= P(Z < 0.5) - P(Z \leq -0.5) = \qquad \text{(pela simetria)}$$

$$= P(Z < 0.5) - P(Z \geq 0.5) = \qquad \text{(passando ao complementar)}$$

$$= P(Z < 0.5) - [1 - P(Z < 0.5)] = \qquad \text{(pela tabela)}$$

$$= 0.6915 - 1 + 0.6915 = 0.383$$

3. Suppose that a factory produces three types of electrical materials A, B, and C, whose sales prices are 120€, 62,50€ and 128€, respectively. The number of units sold weekly of each type of material can be considered independent, and each of them with a Normal distribution of mean value 190, 221, 225, and variance 2304, 4096, 3600, respectively. Calculate the probability that, within a week, the total sales value of all units exceeds 70 000€. [Solution: 0.33]

Solving

$X_1 -$ nr. of units sold of A $\qquad X_1 \sim N(190, \sqrt{2304})$

$X_2 -$ nr. of units sold of B $\qquad X_2 \sim N(221, \sqrt{4096})$

$X_3 -$ nr. of units sold of C $\qquad X_3 \sim N(225, \sqrt{3600})$

X_1, X_2 and X_3 are independent.

We want to find $P(120X_1 + 62.5X_2 + 128X_3 > 70000) = ?$

Let's consider the random variable $Y = 120X_1 + 62.5X_2 + 128X_3$. Given that Y is a linear combination of independent normal variables, we can conclude by a theorem that $Y \sim N(\mu, \sigma)$. Using the properties of the mean and variance:

$$\mu = E(120X_1 + 62.5X_2 + 128X_3) = 120E(X_1) + 62.5(X_2) + 128E(X_3) =$$

$$= 120(190) + 62.5(221) + 128(225) = 65412.5$$

$$\sigma^2 = V(120X_1 + 62.5X_2 + 128X_3) \underset{\substack{\downarrow \\ \text{independentes}}}{=} (120)^2 V(X_1) + (62.5)^2 V(X_2) + (128)^2 V(X_3) =$$

$$= 108160000 \Rightarrow \sigma = 10400$$

Hence, we found that Y~N(65412.5 , 10400). We can now compute the probability:

$$P(Y > 70000) = P\left[\frac{Y - 65412.5}{10400} > \frac{70000 - 65412.5}{10400}\right] \underset{\substack{\downarrow \\ Z\sim N(0,1)}}{=} P(Z > 0.44) =$$

$$= 1 - P(Z \leq 0.44) \underset{\substack{\downarrow \\ \text{tabela}}}{=} 1 - 0.67 = 0.33$$