Statistical Analysis

Point estimation

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Topics

- **LU4 – Point estimation**
  - Notation and concepts
  - Unbiasedness
  - Efficiency
  - Consistency

Objectives

- **At the end of this learning unit students should be able to**
  - Understand the properties of estimators
  - Investigate the bias of an estimator
  - Investigate the efficiency of an estimator
  - Investigate the consistency of an estimator

Suggested reading

- Newbold, P., Carlson, W. L., Thorne, B. (2013). Statistics for Business and Economics. 8th Edition, Boston: Pearson, pages 284-290 (ch. 7).
- Mendenhall, W., Beaver, R. J., & Beaver, B. M. (2013). *Introduction to Probability and Statistics*. 14th Edition, Boston: Brooks/Cole, Cengage Learning, pages 281-286.
- Pedrosa, A. C. e Gama, S. M. A. (2004). *Introdução Computacional à Probabilidade e Estatística*. Porto Editora, pages 387-398.

- **Resources on the Internet**
  - Peralta, I. M. & Portugués, E. G. (2021) . "Chapter 3. Point estimation", in A First Course on Statistical Inference, last updated: 2021-02-04, v0.9.1.
  - Hossein Pishro-Nik (2014) "8.2 Point Estimation". In *Introduction to Probability, Statistics, and Random Processes*, available at https://www.probabilitycourse.com, Kappa Research LLC.

Notation and concepts

- **Notation**
  - θ → parameter of the population
  - $X_1, X_2, \ldots, X_n$ → random sample (iid random variables)
  - $\widehat{\Theta} = g(X_1, X_2, \ldots, X_n)$ → estimator of θ

- **Issues addressed**
  - In case of more than one estimator of θ, how can we decide which one is better than another?
  - What are the desirable properties of an estimator?

Notation and concepts

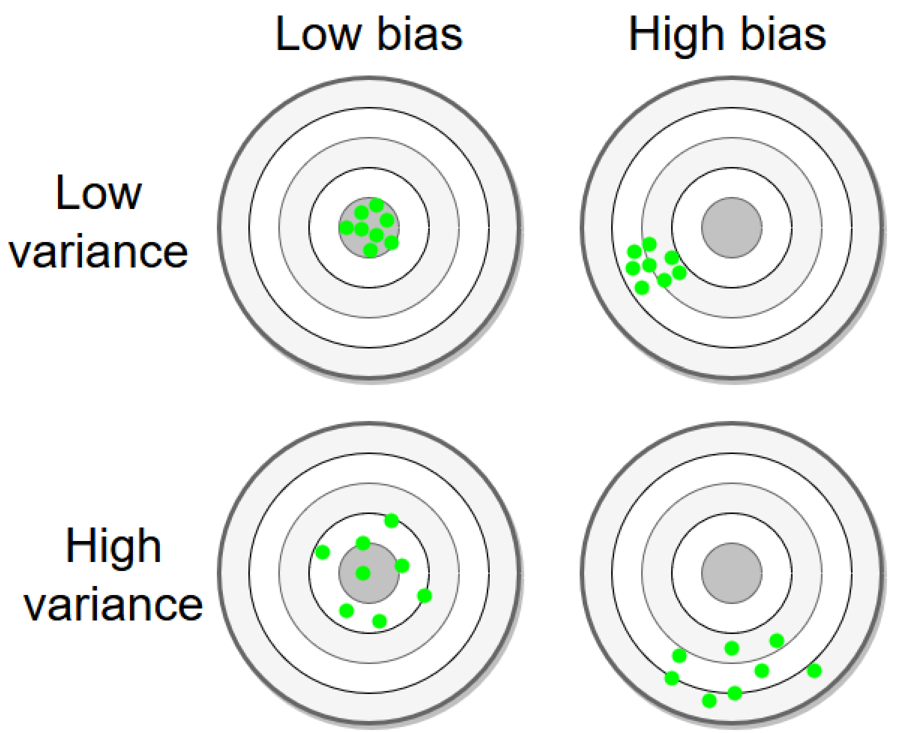
- **Concepts**
  - A **point estimator**, or simply estimator, of a parameter θ of a population is a statistic $\hat{\Theta}$ used to estimate the value of θ
  - A **point estimate**, or simply estimate, of a parameter θ of a population is the value $\hat{\theta}$ of a statistic $\hat{\Theta}$
  - **Desirable properties of estimators**
    - Sufficiency
    - Unbiasedness
    - Efficiency
    - Consistency

Notation and concepts

- **Concepts**
  - **Sufficiency** → when the estimator takes all the relevant information about the population parameter from the sample
  - **Unbiasedness** → in medium terms, the estimator reaches the actual value of the parameter
  - **Efficiency** → the estimator is more efficient (i.e., the estimates are more *accurate*) the smaller the variability of its sampling distribution
  - **Consistency** → for *large* samples, the estimator should be *approximately* equal to the parameter

Notation and concepts

- **Unbiasedness *versus* efficiency**



Notation and concepts

- **Methods to derive estimators**
  - **Method of moments** → estimators are obtained by replacing the expressions of the sample moments in the expressions that represent the corresponding moments in the population
  - **Method of least squares** → commonly used within the linear regression
  - **Method of maximum likelihood** → it is probably the most important method. Generally, the maximum likelihood estimators enjoy desirable properties of a good estimator: usually, they are the most efficient and consistent. Although sometimes biased, they are frequently asymptotically unbiased.

Notation and concepts

- **Theorems**
  - If X is a population with **Normal distribution** of mean μ and variance σ^2, and $X_1, X_2, \ldots, X_n$ is a random sample from that population, then

$$\frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma} \right)^2 \sim \chi^2_{(n-1)}$$

  - $\bar{X}$ and S^2 **are independent random variables**

Unbiasedness

- **$\widehat{\theta}$ is an unbiased estimator if**

$$\mathrm{E}(\hat{\theta}) = \theta$$

Otherwise, the estimator is said to be **biased**, and its **bias** is given by

$$bias(\hat{\theta}) = E(\hat{\theta}) - \theta$$

- **$\widehat{\theta}$ is an asymptotically unbiased estimator of θ if**

$$\lim_{n\to+\infty} E(\hat{\theta}) = \theta \quad \text{or} \quad \lim_{n\to+\infty} bias(\hat{\theta}) = 0$$

Unbiasedness

- **Example 1**

a) Let $X_1, X_2, \ldots, X_n$ be an iid random sample from a population with mean μ. Show that $\overline{X}$ is an unbiased estimator of μ.

b) Let $X_1, X_2, \ldots, X_n$ be an iid random sample from a population $N(\mu, \sigma)$. Show that S^2 is an unbiased estimator of σ^2.

c) Show that $M_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is an asymptotically unbiased estimator of σ^2 and derive its bias.

- $M_2 = \frac{n-1}{n}S^2$, therefore

$$E(M_2) = \frac{n-1}{n}E(S^2) = \frac{n-1}{n}\sigma^2 \xrightarrow[n\to+\infty]{} \sigma^2$$

- $\text{bias}(M_2) = \frac{-\sigma^2}{n}$

Efficiency

- **The efficiency of an estimator is measured by its mean squared error**

- **Mean squared error (MSE)**
  - **Definition**: the mean squared error of an estimator $\hat{\theta}$ is

$$MSE(\hat{\theta}) = E\left[(\hat{\theta} - \theta)^2\right]$$

  - **Property**:

$$MSE(\hat{\theta}) = V(\hat{\theta}) + \left[bias(\hat{\theta})\right]^2$$

Efficiency

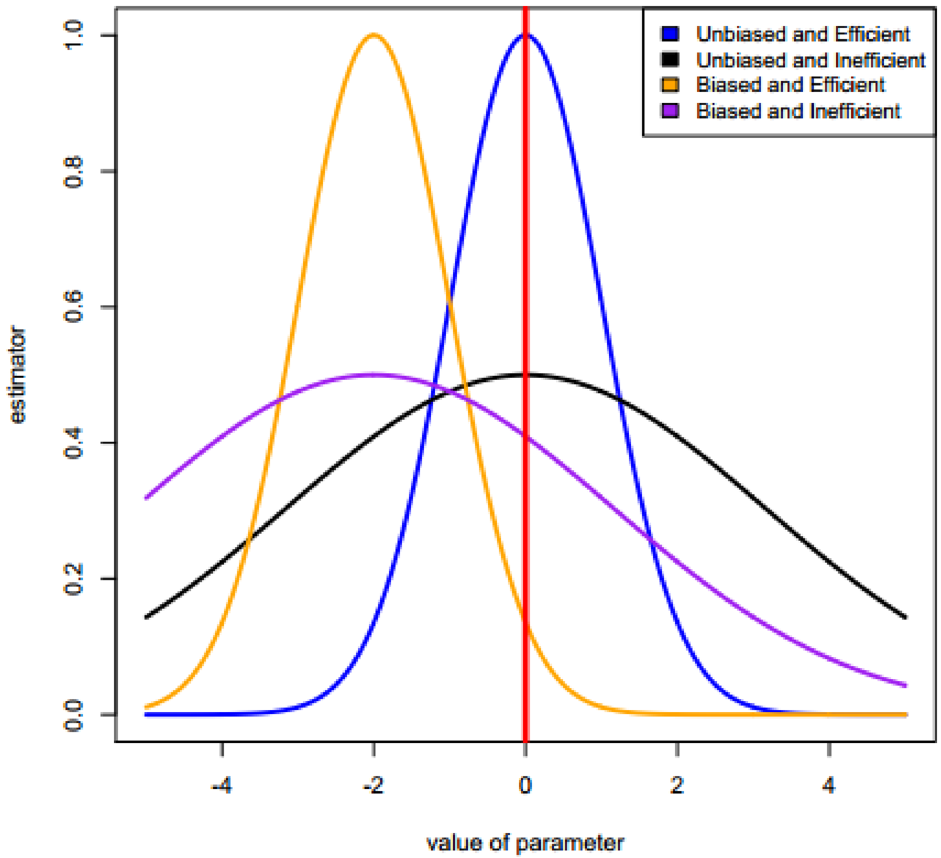
- **Relative efficiency**
  - Given two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ of a parameter θ, the **relative efficiency of $\hat{\boldsymbol{\theta}}_1$ to $\hat{\boldsymbol{\theta}}_2$** is given by

$$eff\left(\hat{\theta}_1, \hat{\theta}_2\right) = \frac{MSE\left(\hat{\theta}_1\right)}{MSE\left(\hat{\theta}_2\right)}$$

    - $\hat{\theta}_1$ is more efficient than $\hat{\theta}_2$ when $eff\left(\hat{\theta}_1, \hat{\theta}_2\right) < 1$
    - **Example**
      - If $eff\left(\hat{\theta}_1, \hat{\theta}_2\right) = 1.6$, than the variability associated to $\hat{\theta}_1$ is 1.6 higher than the variability associated to $\hat{\theta}_2$, thus $\hat{\theta}_2$ is more efficient than $\hat{\theta}_1$

Efficiency

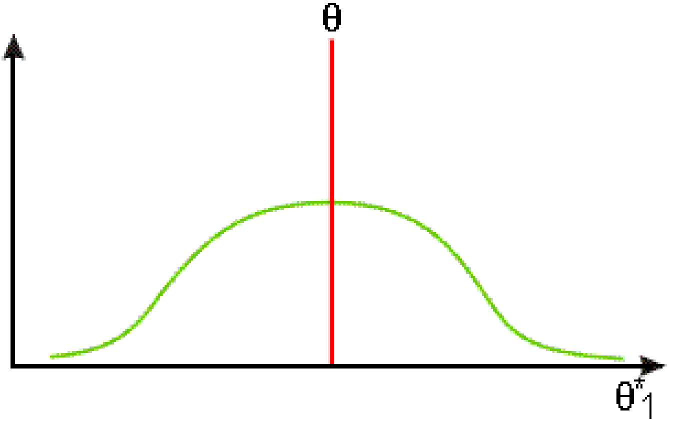
- **Efficiency and bias**

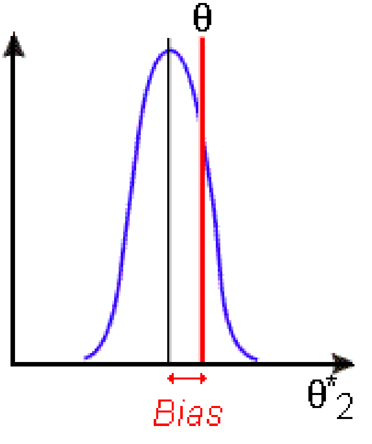


Efficiency

- **Efficiency and bias trade-off**
  - In practice, a biased estimator can be a better estimator than an unbiased one when its bias is small and its efficiency is higher
    - $\theta^*_1 \rightarrow$ unbiased and inefficient
    - $\theta^*_2 \rightarrow$ biased and efficient

Sampling distribution of θ^*_1 and θ^*_2:





Efficiency

- **Example 2**
  - Let $X_1, X_2, \ldots, X_n$ be an iid random sample from a population $N(\mu, \sigma)$. Consider the two estimators of σ^2: S^2 and $M_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

a) Show that $V(S^2) = \frac{2\sigma^4}{n-1}$ and $V(M_2) = \frac{n-1}{n^2}2\sigma^4$

b) Considering that *n* is large, determine the relative efficiency of S^2 to M_2.

   ✓ For large samples, M_2 is more efficient than S^2

Consistency

- $\hat{\theta}$ is a **consistent** estimator of θ if and only if $\hat{\theta}$ converges in probability to θ:

$$\lim_{n \to +\infty} P\left(|\hat{\theta} - \theta| \leq \varepsilon\right) = 1 \quad \text{for all } \varepsilon>0$$

  - An estimator is consistent if increasing the sample size implies an increase in the probability of the estimated value to be in a neighbourhood of the true value of the parameter

Consistency

- **An estimator $\widehat{\boldsymbol{\theta}}$ is consistent in mean square error if**

$$\lim_{n \to +\infty} MSE\left(\hat{\theta}\right) = 0$$

- **Hence, an <u>unbiased</u> estimator $\widehat{\boldsymbol{\theta}}$ is consistent if**

$$\lim_{n \to +\infty} V\left(\hat{\theta}\right) = 0$$

Consistency

- **Properties**
  - If $\widehat{\Theta}$ and $\widehat{\Theta}'$ are consistent estimators of θ and θ', respectively, then
    - $\widehat{\Theta} + \widehat{\Theta}'$ is a consistent estimator of $\theta+\theta'$
    - $\widehat{\Theta} \times \widehat{\Theta}'$ is a consistent estimator of $\theta\times\theta'$
    - $\widehat{\Theta}/\widehat{\Theta}'$ is a consistent estimator of θ/θ', with $\theta'\neq0$
    - If $g(\cdot)$ is a real continuous function in θ, then $g(\widehat{\Theta})$ is a consistent estimator of $g(\theta)$

Consistency

- **Example 3**
  - Let X_1, X_2,…,X_n be a random sample from a population with Poisson(λ) distribution. Show that the sample mean is a consistent estimator of λ.
    - ✓ $E(\bar{X}) = \lambda$
    - ✓ $V(\bar{X}) = {}^{\lambda}/_{n}$
    - ✓ $\lim_{n\to+\infty} V(\bar{X}) = \lim_{n\to+\infty} {}^{\lambda}/_{n} = 0$

Point estimation

Do the homework!

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