<u>LV4</u> <u>Exerc. 4</u> $X \sim N(\mu, \sigma)$

$$\tilde{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$$

a) show that $E[\tilde{\sigma}^2] = \sigma^2$

- Pag. 1 of Form: 1st theorem:

  $x_i \sim N(\mu, \sigma)$ then

  $$\sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$$

- Last page of Form:

  Mean and variance of a $\chi^2_{(n)}$:

  $E[\chi^2_{(n)}] = n$ (= degrees of freedom)

  $V[\chi^2_{(n)}] = 2n$ (= twice the deg. of. freed.)

- Using these results:

$$E\left[\sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2\right] = E\left[\chi^2_{(n)}\right] = n$$

$$\Leftrightarrow E\left[\frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right] = n$$

$$\Leftrightarrow \frac{1}{\sigma^2} E\left[\sum_{i=1}^{n} (x_i - \mu)^2\right] = n$$

$$\Leftrightarrow E\left[\sum_{i=1}^{n} (x_i - \mu)^2\right] = n\sigma^2$$

$$\Leftrightarrow \frac{1}{n} E\left[\sum_{i=1}^{n} (x_i - \mu)^2\right] = \sigma^2$$

$$\Leftrightarrow E\left[\frac{1}{n}\sum_{i=1}^{n} (x_i - \mu)^2\right] = \sigma^2 \quad \Leftrightarrow E[\tilde{\sigma}^2] = \sigma^2$$

$\therefore \tilde{\sigma}^2$ is unbiased for the estimation of σ^2.

b) Since both $\tilde{\sigma}^2$ and S^2 are unbiased, we simply need to compare their variances to determine which one is more efficient.

→ Determine $V(\tilde{\sigma}^2) = V\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\right]$

Using the theorem and results from (a):

$$V\left[\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2\right] = V\left[\chi^2_{(n)}\right] = 2n$$

$$\Leftrightarrow V\left[\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2\right] = 2n$$

$$\Leftrightarrow \frac{1}{\sigma^4} V\left[\sum_{i=1}^{n}(X_i-\mu)^2\right] = 2n$$

$$\Leftrightarrow V\left[\sum_{i=1}^{n}(X_i-\mu)^2\right] = 2n\sigma^4$$

$$\Leftrightarrow \frac{1}{n^2} V\left[\sum_{i=1}^{n}(X_i-\mu)^2\right] = \frac{2n\sigma^4}{n^2}$$

$$\Leftrightarrow V\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\right] = \frac{2\sigma^4}{n}$$

$$\Leftrightarrow V(\tilde{\sigma}^2) = \frac{2\sigma^4}{n}$$

→ In class, we showed that $V(S^2) = \frac{2\sigma^4}{n-1}$

→ Relative efficiency:

$$\text{eff}(\tilde{\sigma}^2; S^2) = \frac{V(\tilde{\sigma}^2)}{V(S^2)} = \frac{\frac{2\sigma^4}{n}}{\frac{2\sigma^4}{n-1}} = \frac{n-1}{n} < 1$$

$\therefore V(\tilde{\sigma}^2) < V(S^2)$, therefore $\tilde{\sigma}^2$ is more efficient than S^2.