MATH 300 MULTIVARIABLE CALCULUS
Complementary Notes

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*Additional topics that not covered by the lecture can be also found in this book.

1. Notations

The collection of all *real numbers* is denoted by $\mathbb{R}$. Thus $\mathbb{R}$ includes the integers

$$\ldots, -2, -1, 0, 1, 2, 3 \ldots,$$

the *rational numbers*, p/q, where p and q are integers $(q \neq 0)$,

and the *irrational numbers*, like $\sqrt{2}, \pi, e$, etc.

Members of $\mathbb{R}$ may be visualized as points on the real-number line as shown in Figure 1.

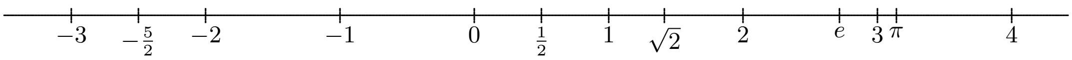


Figure 1 The Number Line

We write $a \in \mathbb{R}$ to mean a is a member of the set $\mathbb{R}$. In other words, a is a real number.

Given two real numbers a and b with $a < b$,

the *closed interval* $[a, b]$ consists of all x such that $a \leq x \leq b$,

and the *open interval* (a, b) consists of all x such that $a < x < b$.

Similarly, we may form the half-open intervals $[a, b)$ and $(a, b]$.

The *absolute value* of a number $a \in \mathbb{R}$ is written as $|a|$ and is defined as

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

For example, $|2| = 2, |-2| = 2$. Some properties of $|x|$ are summarized as follows:

1. $|-x| = |x|$ for all $x \in \mathbb{R}$.

2. $-|x| \leq x \leq |x|$, for all $x \in \mathbb{R}$.

3. For a fixed $r > 0$, $|x| < r$ if and only if $x \in (-r, r)$.

4. $\sqrt{x^2} = |x|$, $x \in \mathbb{R}$.

5. (*Triangle Inequality*) $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

A *function* $f : A \longrightarrow B$ is a rule that assigns to each $a \in A$ one specific member $f(a)$ of B.

The fact that the function f sends a to $f(a)$ is denoted symbolically by $a \mapsto f(a)$.

For example, $f(x) = x^2/(1 - x)$ assigns the number $x^2/(1 - x)$ to each $x \neq 1$ in $\mathbb{R}$. We can specify a function f by giving the rule for $f(x)$.

The set A is called the *domain* of f and B is the *codomain* of f. The *range* of f is the subset of B consisting of all the values of f. That is, the range of $f = \{f(x) \in B \mid x \in A\}$.

Given $f : A \longrightarrow \mathbb{R}$, it means that f assigns a value $f(x)$ in $\mathbb{R}$ to each $x \in A$. Such a function is called a *real-valued function*.

For a real-valued function $f : A \longrightarrow \mathbb{R}$ defined on a subset A of $\mathbb{R}$, the *graph* of f consists of all the points $(x, f(x))$ in the xy-plane.

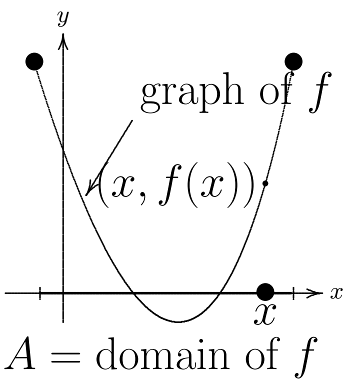


Figure 2 The graph of f

Week 1:

2. Vectors in $\mathbb{R}^3$

2.1. The Euclidean 3-space. The Euclidean 3-space denoted by $\mathbb{R}^3$ is the set

$$\{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

To specify the location of a point in $\mathbb{R}^3$ geometrically, we use a right-handed rectangular coordinate system, in which three mutually perpendicular coordinate axes meet at the origin. It is common to use the x and y axes to represent the horizontal coordinate plane and the z-axis for the vertical height.

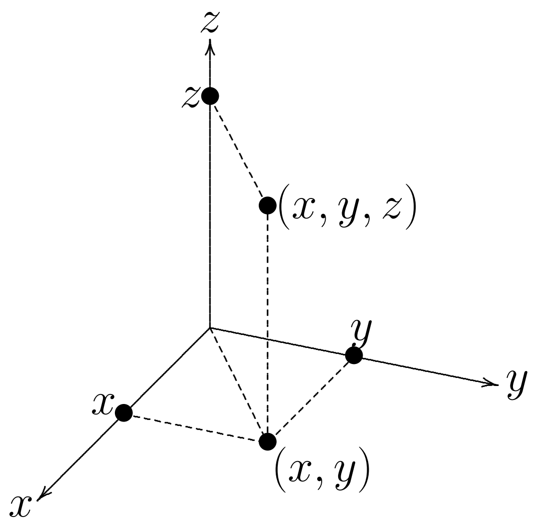


Figure 3 A right-handed coordinate system

We usually denote a point P with coordinates (x, y, z) by $P(x, y, z)$.

The distance $|P_1P_2|$ between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

An equation in x, y, z describes a surface in $\mathbb{R}^3$.

EXAMPLE 2.1. (a) $z = 3$ is the equation of a horizontal plane at level 3 above the xy-plane. (b) $y = 2$ is the equation of a vertical plane parallel to the xz-coordinate plane. Every point of this plane has y coordinate equal to 2. (c) Similarly $x = 2$ is the equation of a vertical plane parallel to the yz-coordinate plane.

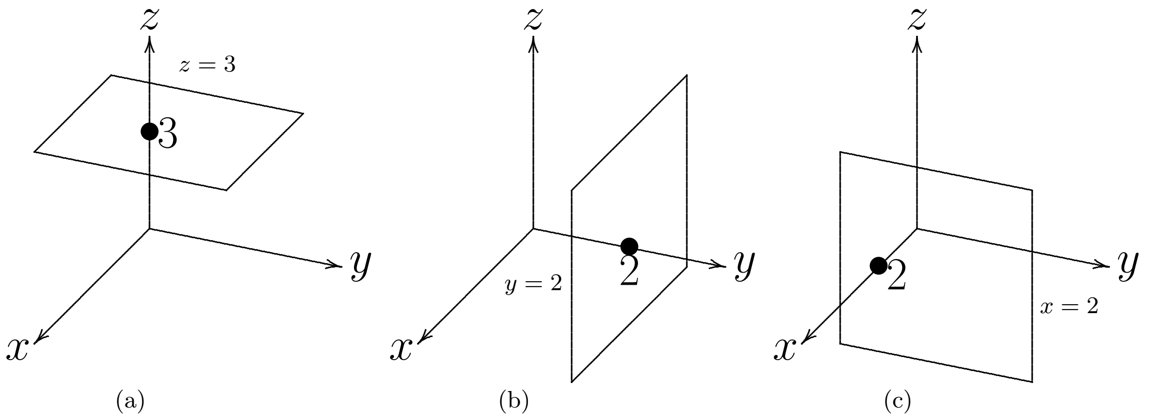


Figure 4

EXAMPLE 2.2. An equation of a sphere with centre $O(a, b, c)$ and radius r is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

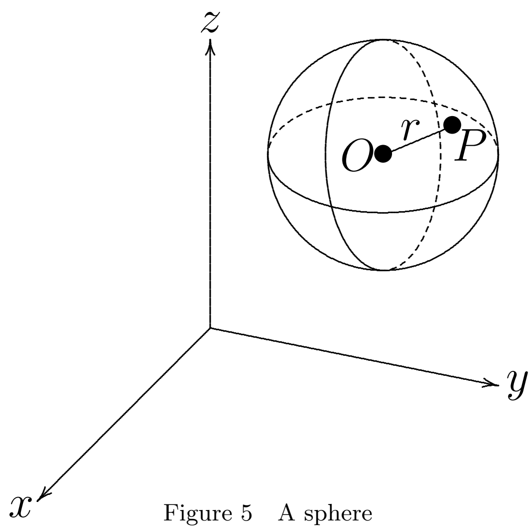


Figure 5 A sphere

EXERCISE 2.3. Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere. Describe its intersection with the plane $z = 1$.

Solution. Using the method of completing square, the given equation can be written as $(x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 8$. Hence, it is the equation of a sphere centred at $(-2, 3, -1)$ with radius $\sqrt{8}$.

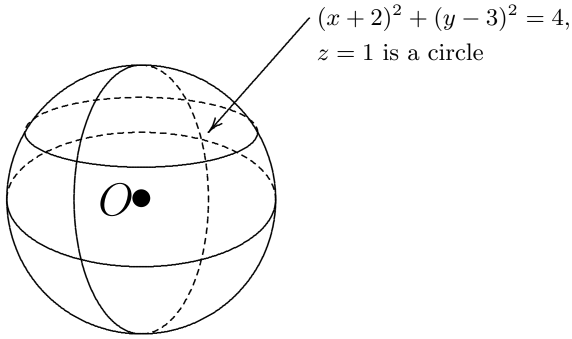


Figure 6 A circle lying on a sphere

To find the intersection with the plane $z = 1$, set $z = 1$ in the above equation. We obtain $(x + 2)^2 + (y - 3)^2 = 4$. Therefore, it is a circle lying on the horizontal plane $z = 1$ with centre at $(-2, 3, 1)$ and radius 2.

2.2. Vectors. A 3-dimensional vector is an ordered triple

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

of real numbers. a_1, a_2, a_3 are called the components of $\vec{a}$.

A vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ can be represented by an arrow from any point $P(x, y, z)$ to the point $Q(x + a_1, y + a_2, z + a_3)$ in $\mathbb{R}^3$.

In this case, we say that the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ has representation $\overrightarrow{PQ}$.

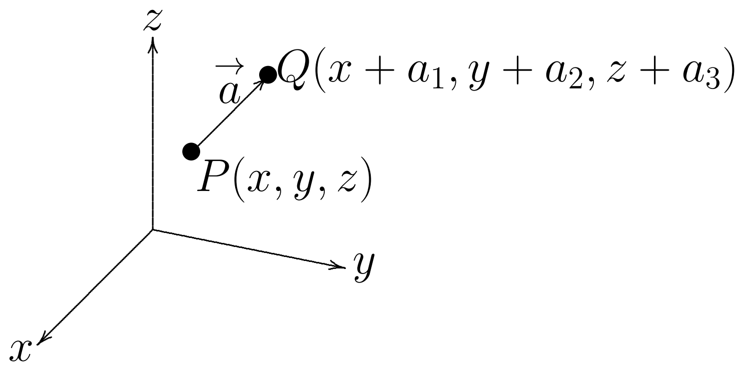


Figure 7 Vector

Instead of using an arrow on top of PQ or a, we shall suppress the arrow but write $\mathbf{PQ}$ or $\mathbf{a}$ in bold to denote the vector $\vec{a} = \overrightarrow{PQ}$.

If P is the origin O, $\mathbf{a}$ is called the position vector of the point Q.

The position vectors of $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively.

In other word, $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$ and $\mathbf{k} = \langle 0, 0, 1 \rangle$. $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are called the standard basis vectors.

Therefore, if $Q = (x, y, z)$, then $\mathbf{q} = \langle x, y, z \rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Suppose $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$.

The *magnitude* of a vector $\mathbf{PQ}$ is defined to be

$$|\mathbf{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

A vector $\mathbf{PQ}$ also has a direction determined by the orientation that the arrow is pointing.

The zero vector $\langle 0, 0, 0 \rangle$ is denoted by $\mathbf{0}$. Clearly $|\mathbf{0}| = 0$.

We say that two vectors are *equal* if and only if they have the same *direction* and the same *magnitude*.

This condition may be expressed algebraically by saying that if $\mathbf{v}_1 = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{v}_2 = \langle x_2, y_2, z_2 \rangle$, then $\mathbf{v}_1 = \mathbf{v}_2$ if and only if $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$.

If $\mathbf{v}_1 = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{v}_2 = \langle x_2, y_2, z_2 \rangle$, then define the sum $\mathbf{v}_1 + \mathbf{v}_2$ to be the vector $\langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$.

If λ is any real number and $\mathbf{v} = \langle x, y, z \rangle$, then define the scalar multiple $\lambda\mathbf{v}$ to be the vector $\langle \lambda x, \lambda y, \lambda z \rangle$.

Also $-\mathbf{v}$ is defined to be $(-1)\mathbf{v}$. Clearly $-(-\mathbf{v}) = \mathbf{v}$, and $-\mathbf{v} + \mathbf{v} = \mathbf{0}$. Also $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$.

Addition of vectors can also be described by the parallelogram law: The sum $\mathbf{v_1} + \mathbf{v_2}$ is represented by the position vector which is the diagonal of the parallelogram determined by $\mathbf{v_1}$ and $\mathbf{v_2}$.

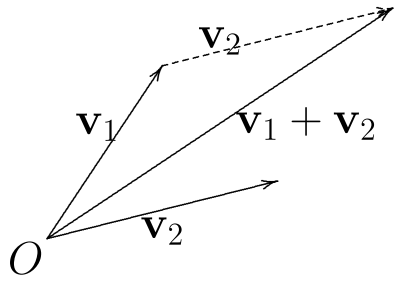


Figure 8 Vector Addition

It is straightforward to check that the set of all position vectors in $\mathbb{R}^3$ forms a vector space over $\mathbb{R}$.

Proposition 2.4. *Properties of vectors*

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$.
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$.
4. $\mathbf{a} + -\mathbf{a} = \mathbf{0}$.
5. $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{b} + \alpha\mathbf{a}$.
6. $\alpha\mathbf{a} = \mathbf{a}\alpha$.
7. $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$.
8. $(\alpha\beta)\mathbf{a} = \alpha(\beta\mathbf{a})$.
9. $1\mathbf{a} = \mathbf{a}$.
10. $|\alpha\mathbf{a}| = |\alpha||\mathbf{a}|$.

Proof: Exercise.

DEFINITION 2.5. *A unit vector is a vector whose length is 1.*

For any nonzero vector $\mathbf{a}$, $\frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$ is a unit vector that has the same direction as $\mathbf{a}$.

EXAMPLE 2.6. Find the unit vector in the direction of the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

Solution. $|2\mathbf{i} - \mathbf{j} - 2\mathbf{k}| = \left(2^2 + (-1)^2 + (-2)^2\right)^{\frac{1}{2}} = \sqrt{9} = 3$. Therefore the required unit vector is $\frac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$.

2.3. The Dot Product.

DEFINITION 2.7. *Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$. The dot product or scalar product of $\mathbf{a}$ and $\mathbf{b}$ is the number*

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

.

EXAMPLE 2.8. Let $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle -1, 0, -1\rangle$. Find $\mathbf{a}\cdot\mathbf{b}$.

Solution.

$\mathbf{a} \cdot \mathbf{b} = (1)(-1) + (2)(0) + (3)(-1) = -4$.

Clearly, we have

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0 \text{ and } \mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1.$$

PROPOSITION 2.9. *Properties of the Dot Product*
1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.

2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.

4. $(\alpha \mathbf{a}) \cdot \mathbf{b} = \alpha(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\alpha \mathbf{b})$.

5. $\mathbf{0} \cdot \mathbf{a} = 0$.

Proof. Let's prove 1. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$. Then $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$.

THEOREM 2.10. *If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, $0 \le \theta \le \pi$.*

Proof. Let $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$, where O is the origin and $\theta = \angle AOB$.

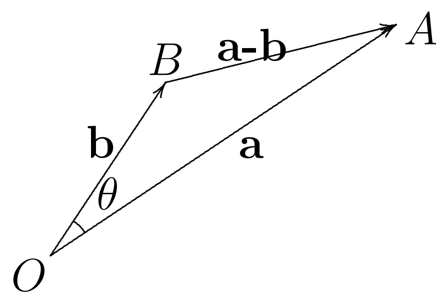


Figure 9 Angle between two vectors

Applying cosine rule to $\triangle OAB$, we have

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos\theta.$$

As $|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2$, it follows that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ or

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be orthogonal or perpendicular if the angle between them is $90°$. In other words,

$$\mathbf{a} \text{ and } \mathbf{b} \text{ are orthogonal} \iff \mathbf{a}\cdot\mathbf{b} = 0.$$

EXAMPLE 2.11. $2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ is orthogonal to $5\mathbf{i}-4\mathbf{j}+2\mathbf{k}$ because $(2\mathbf{i}+2\mathbf{j}-\mathbf{k})\cdot(5\mathbf{i}-4\mathbf{j}+2\mathbf{k}) = (2)(5)+(2)(-4)+(-1)(2) = 0.$

Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle \neq \mathbf{0}$. The angles α, β, γ in $[0,\pi]$ that $\mathbf{a}$ makes with the x, y, z axes respectively are called the direction angles of $\mathbf{a}$.

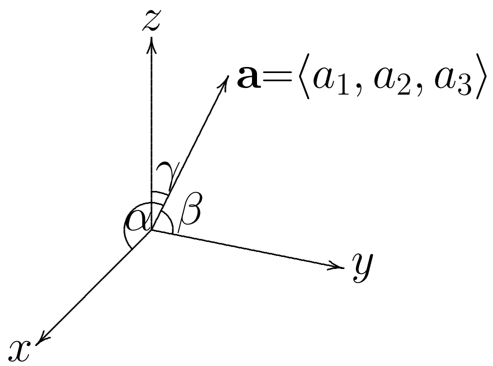


Figure 10 Direction Angles

The cosines of these angles, $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of $\mathbf{a}$.

We may express a vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ in terms of its magnitude and the direction cosines.

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{\langle a_1, a_2, a_3\rangle\cdot\langle 1,0,0\rangle}{|\langle a_1, a_2, a_3\rangle||\langle 1,0,0\rangle|} = \frac{a_1}{|\mathbf{a}|}.$$

Similarly,

$$\cos\beta = \frac{a_2}{|\mathbf{a}|} \text{ and } \cos\gamma = \frac{a_3}{|\mathbf{a}|}.$$

Thus,

$$\mathbf{a} = |\mathbf{a}|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle.$$

Next, we shall discuss the projection of a vector along another vector. Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $\mathbb{R}^3$. Let's represent $\mathbf{a}$ as $\mathbf{PQ}$ and $\mathbf{b}$ as $\mathbf{PR}$.

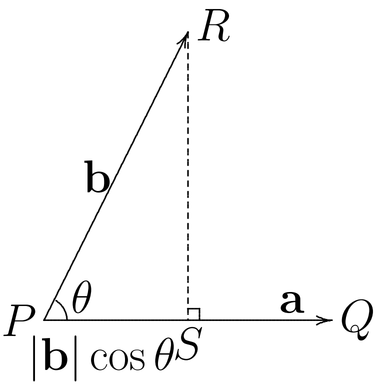


Figure 11 Vector Projection

Then

$$|\mathbf{b}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}.$$

Definition 2.12.

1. The *scalar projection of* $\mathbf{b}$ *onto* $\mathbf{a}$ *is* $|\mathbf{b}|\cos\theta = \frac{\mathbf{a}}{|\mathbf{a}|}\mathbf{b}$.

2. The *vector projection of* $\mathbf{b}$ *onto* $\mathbf{a}$ *is* $\left(\frac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$.

Note that the scalar projection is negative if $\theta > 90°$.

Moreover, in figure 11. $\mathbf{SR} = \mathbf{PR} - \mathbf{PS} = \mathbf{b} - \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$. Thus the distance from R to the line PQ is given by

$$|\mathbf{RS}| = \left|\mathbf{b} - \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}\right|.$$

EXAMPLE 2.13. Find the scalar and vector projection of $\mathbf{b} = \langle 1, 1, 2\rangle$ onto $\mathbf{a} = \langle -2, 3, 1\rangle$.

Solution. $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$.

Thus the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by

$$\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{1}{\sqrt{14}}((-2)(1) + (3)(1) + (1)(2)) = 3/\sqrt{14}.$$

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by

$$\frac{3}{\sqrt{14}}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\mathbf{a} = \langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\rangle.$$

2.4. The Cross Product.

DEFINITION 2.14. *If* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ *and* $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$*, then the cross product or vector product of* $\mathbf{a}$ *and* $\mathbf{b}$ *is*

$$
\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.
\end{aligned}
$$

EXAMPLE 2.4.1. Let $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$. Find $\mathbf{a} \times \mathbf{b}$.

Solution.

$$
\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} \\
&= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k} \\
&= -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}.
\end{aligned}
$$

Clearly, we have

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.

THEOREM 2.15. *Let* $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ *and* $\mathbf{c} = \langle c_1, c_2, c_3\rangle$. *Then*

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Proof. Exercise.

COROLLARY 2.16. $\mathbf{b} \times \mathbf{c}$ *is perpendicular to both* $\mathbf{b}$ *and* $\mathbf{c}$.

Proof.

$$\mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

$$\mathbf{c} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

THEOREM 2.17. *If* θ *is the angle between* $\mathbf{a}$ *and* $\mathbf{b}$, $0 \leq \theta \leq \pi$, *then* $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$.

Proof. First we need the following identity

$$(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

which can be easily verified by direct simplification of both sides.

Using this identity, we have

$$\begin{aligned}
|\mathbf{a} \times \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2 \cos^2\theta \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2\theta.
\end{aligned}$$

Since $0 \le \theta \le \pi$, $\sin\theta \ge 0$, we have $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$.
It follows from this result that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ is the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

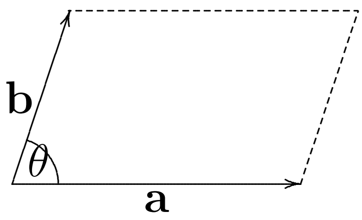


Figure 12 Area=$|\mathbf{a}||\mathbf{b}|\sin\theta$

$\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ with magnitude $|\mathbf{a}||\mathbf{b}|\sin\theta$, where $0 \le \theta \le \pi$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

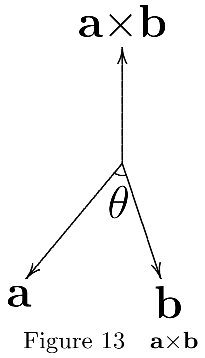


Figure 13 $\mathbf{a}\times\mathbf{b}$

There are two possible choices of such a vector. It is the one determined by the right-hand rule: $\mathbf{a} \times \mathbf{b}$ is directed so that a right-hand rotation about $\mathbf{a} \times \mathbf{b}$ through an angle θ will carry $\mathbf{a}$ to the direction of $\mathbf{b}$.

COROLLARY 2.18. $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel if and only if* $\mathbf{a}\times\mathbf{b} = \mathbf{0}$.

PROPOSITION 2.19. *Properties of the Cross Product*

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(\alpha\mathbf{a}) \times \mathbf{b} = \alpha(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\alpha\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Proof. Exercise.

The relation $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ can be proved by direct expansion in component form. Alternatively, it can be deduced by the property of the determinant: *If two rows of a determinant are switched, the determinant changes sign.* Therefore,

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.
\end{aligned}$$

In fact, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is the *algebraic or sign* volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

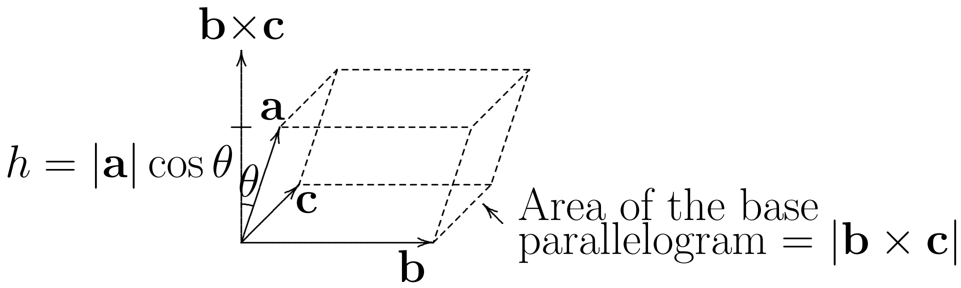


Figure 14 Volume $= |\mathbf{a}||\mathbf{b}\times\mathbf{c}||\cos\theta| = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$.

COROLLARY 2.20. *The vectors* $\mathbf{a}$,$\mathbf{b}$,$\mathbf{c}$ *are coplanar (i.e. they all lie on a plane) if and only if* $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$.

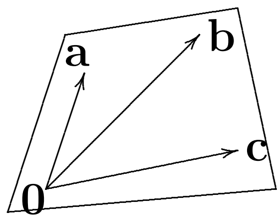


Figure 15 $\mathbf{a}$,$\mathbf{b}$,$\mathbf{c}$ are coplanar

EXAMPLE 2.21. Show that the vectors $\mathbf{a} = \langle 1, 4, -7\rangle$, $\mathbf{b} = \langle 2, -1, 4\rangle$, $\mathbf{c} = \langle 0, -9, 18\rangle$ are coplanar.

Solution. As $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 0$, it follows from 2.20 that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar.

2.5. Lines and Planes. Let L be a line passing through a point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c\rangle$.

Then any point P on L has position vector $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ for some $t \in \mathbb{R}$.

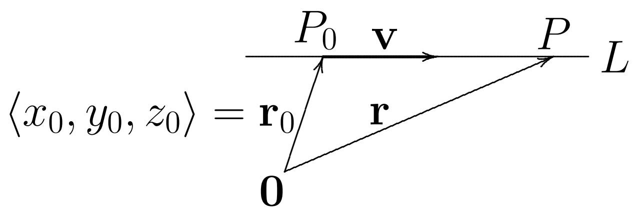


Figure 16 Vector equation of a line

Vector equation of a line: $\boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}}$

Write $\mathbf{r} = \langle x, y, z\rangle$. Then $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ is equivalent to

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

.

Parametric equations of a line: $\boxed{x = x_0 + at, y = y_0 + bt, z = z_0 + ct}$

Eliminating t, we obtain

Symmetric equations of a line: $\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$

The numbers a, b, c are called the direction numbers of the straight line. If a, b or c is zero, we may still write the symmetric equation of the line.

For example, if $a = 0$, we shall write the symmetric equations as

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

which is a line lying on the plane $x = x_0$.

EXAMPLE 2.22. Show that the lines

$$\begin{gathered} L_1 : x = 1 + t, y = -2 + 3t, z = 4 - t, \\ L_2 : x = 2s, y = 3 + s, z = -3 + 4s, \end{gathered}$$

are *skew*, i.e. they do not intersect and are not parallel. Hence they do not lie in the same plane.

Solution. L_1 and L_2 are not parallel because the corresponding vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel.

The lines L_1 and L_2 intersect if and only if the system

$$\begin{cases} 1 + t & = & 2s \\ -2 + 3t & = & 3 + s \\ 4 - t & = & -3 + 4s \end{cases}$$

has a (unique) solution in s and t.

The first two equations give $t = 11/5, s = 8/5$. But these values of t and s do not satisfy the last equation.

Thus, L_1 and L_2 do not intersect.

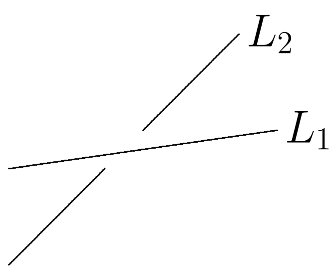


Figure 17 Skew Lines

Consider a plane in $\mathbb{R}^3$ passing through a point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n}$.

Let $P(x, y, z)$ be a point on the plane. Let $\mathbf{r}$ and $\mathbf{r}_0$ be the position vectors of P and P_0 respectively.

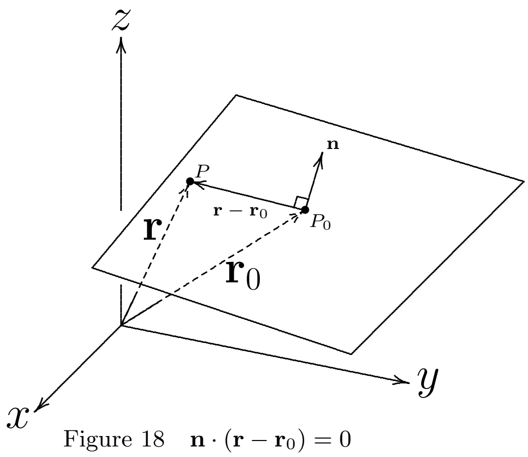


Figure 18 $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$

Then a vector equation of the plane is given by

$$\boxed{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0}$$

If $\mathbf{n} = \langle a, b, c \rangle$, then the above vector equation can be written as

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

In general, a linear equation in x, y, z, i.e. $ax + by + cz + d = 0$ is an equation of a plane in $\mathbb{R}^3$.

EXAMPLE 2.23. Find an equation of the plane passing through the points $P(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$.

Solution. $\overrightarrow{PQ}= \langle 3-1,-1-3,6-2\rangle = \langle 2,-4,4\rangle$.
$\overrightarrow{PR}= \langle 4,-1,-2\rangle$.

Thus a normal vector $\mathbf{n}$ to the plane is given by

$$\overrightarrow{PQ} \times \overrightarrow{PR}= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \langle 12, 20, 14\rangle.$$

Therefore, an equation of the plane is given by

$$\langle x-1, y-3, z-2\rangle \cdot \langle 12, 20, 14\rangle = 0.$$

That is

$$6x + 10y + 7z = 50.$$

EXERCISE 2.24. (a) Find the angle θ, $(0 \le \theta \le 90°)$ between the planes $x+y+z=1$ and $x-2y+3z=1$.

(b) Find the symmetric equations for the line of intersection of the planes in (a).

PROPOSITION 2.25. *The distance from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is*

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Proof. Pick a point $P_0(x_0, y_0, z_0)$ on the plane. Let $\mathbf{b} = \overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$.

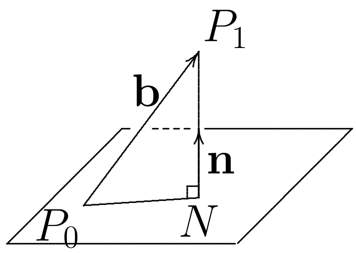


Figure 19 Distance from a point to a plane

Then

$$\begin{aligned} |\overrightarrow{NP_1}| &= |\text{projection of } \mathbf{b} \text{ along } \mathbf{n}| \\ &= \frac{|\mathbf{n}\cdot\mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|(ax_1+by_1+cz_1)-(ax_0+by_0+cz_0)|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}. \end{aligned}$$

EXAMPLE 2.26. Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

Solution. The planes are parallel because their normal vectors

$\langle 10, 2, -2\rangle$ and $\langle 5, 1, -1\rangle$ are parallel.

Pick any point on the plane $10x + 2y - 2z = 5$. For example, $(1/2, 0, 0)$ is a point on $10x + 2y - 2z = 5$.

Then the distance between the two planes is

$$\frac{|5(1/2) + 0(1) + 0(-1) - 1|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{\sqrt{3}}{6}.$$

EXAMPLE 2.27. Find the distance between the skew lines:

$$L_1 : x = 1 + t, y = -2 + 3t, z = 4 - t$$

$$L_2 : x = 2s, y = 3 + s, z = -3 + 4s$$

Solution. As L_1 and L_2 are skew, they are contained in two parallel planes respectively. A normal to these two parallel planes is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = \langle 13, -6, -5\rangle.$$

Let $s = 0$ in L_2. We get the point $(0, 3, -3)$ on L_2.

Therefore, an equation of the plane containing L_2 is

$$\langle x - 0, y - 3, z - (-3)\rangle \cdot \langle 13, -6, -5\rangle = 0.$$

That is $13x - 6y - 5z + 3 = 0$.

Let $t = 0$ in L_1. We get the point $(1, -2, 4)$ on L_1.

Thus, the distance between L_1 and L_2 is given by

$$\frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}}.$$

EXERCISE 2.28. Find the equation of the straight line passing through the point $P_0(1, 5, -1)$ and perpendicular to the lines

$$L_1 : x = 5 + t, y = -1 - t, z = 2t$$

and

$$L_2 : x = 11t, y = 7t, z = -2t$$

.

$[\frac{x-1}{2} = \frac{y-5}{-4} = \frac{z+1}{-3}.]$

3. Vector Functions

Definition 3.1. *A vector function* $\mathbf{r}(t)$ *is a function whose domain is a set of real numbers and whose range is a set of vectors.*

In other words,

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

f, g, h are called the component functions of $\mathbf{r}$.

Example 3.2. Consider the vector function

$$\mathbf{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle.$$

For each of the component functions to be defined, we must have $3 - t > 0$ and $t \geq 0$. Thus the domain of $\mathbf{r}$ is $[0, 3)$. The image of $\mathbf{r}$ traces out a curve in $\mathbb{R}^3$.

In general if $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a vector function, then $x = f(t), y = g(t), z = h(t)$ give the parametric equations of a curve in $\mathbb{R}^3$.

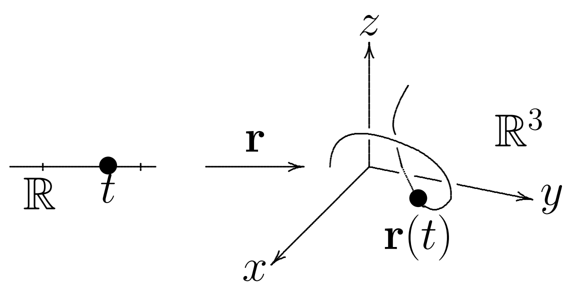


Figure 33 A vector function

EXAMPLE 3.3. The vector function $\mathbf{r}(t) = \langle 1+t, 2+5t, -1+6t\rangle$ defines a curve which is a straight line in $\mathbb{R}^3$.

EXAMPLE 3.4. Sketch the curve whose vector equation is $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$.

Solution. The parametric equations of the curve are

$$x = \cos t, y = \sin t, z = t.$$

Consider a point $P(x, y, z)$ on this curve. Since the x, y and z coordinates of P satisfy the relation $x^2 + y^2 = 1$, it lies on the cylinder $x^2 + y^2 = 1$.

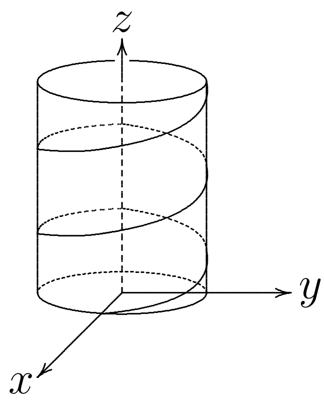


Figure 34 A helix

Moreover, P lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2 + y^2 = 1$.

Since $z = t$, the curve spirals upward around the cylinder as t increases. The curve is a Helix.

Example 3.5. *Find the vector function that represents the curve of intersection C of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.*

Solution. Since C lies on the cylinder which projects onto the circle $x^2+y^2 = 1$ on the xy-plane, we can write $x = \cos t$, $y = \sin t$ with $0 \leq t \leq 2\pi$.

Since C also lies on the plane, its x, y, z coordinates should satisfy the equation of the plane.

Thus, $z = 2 - y = 2 - \sin t$. Consequently, the vector equation of C is $\mathbf{r}(t) = \langle \cos t, \sin t, 2 - \sin t \rangle$.

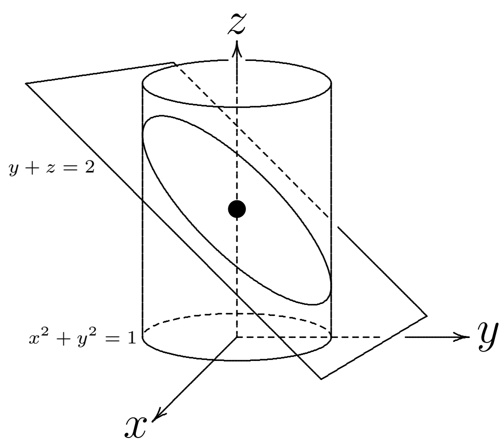


Figure 35 An ellipse

The curve C is an ellipse with centre $(0, 0, 2)$ and it inclines at an angle $45°$ to the horizontal plane.

Let $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$. The limit of $\mathbf{r}(t)$ as t tends to a is defined by:

$$\boxed{\lim_{t\to a} \mathbf{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle.}$$

EXAMPLE 3.6. Let $\mathbf{r}(t) = \langle 1 + t^3, te^{-t}, \frac{\sin t}{t}\rangle$. Find $\lim_{t\to 0} \mathbf{r}(t)$.

Solution. $\displaystyle\lim_{t\to 0} \mathbf{r}(t) = \langle \lim_{t\to 0} 1 + t^3, \lim_{t\to 0} te^{-t}, \lim_{t\to 0} \frac{\sin t}{t}\rangle = \langle 1, 0, 1\rangle.$

DEFINITION 3.7. *A vector function* $\mathbf{r}(t)$ *is continuous at* $t = a$ *if* $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.

That is $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is continuous at a if and only if $f(t), g(t), h(t)$ are continuous at a.

3.1. Derivative of a vector function. Given a vector function $\mathbf{r}(t)$.

Its *derivative* is defined by:

$$\boxed{\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}}$$

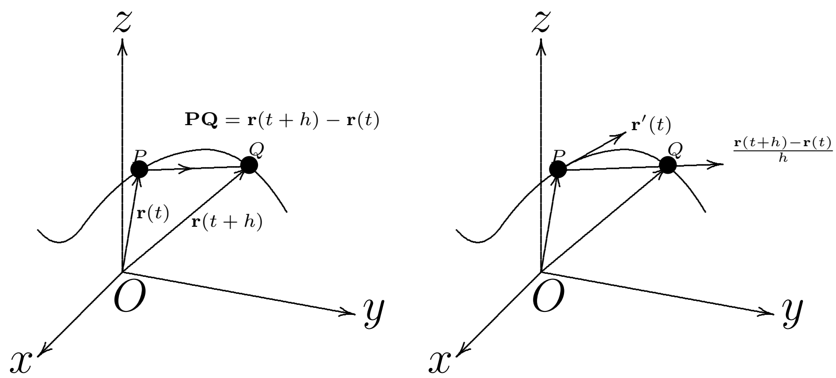


Figure 36 Derivative of a vector function

If $\mathbf{r}'(t)$ exists and is nonzero, we call it a tangent vector to the curve defined by $\mathbf{r}(t)$ at the point P. See figure 36.

In this case, $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ is called the unit tangent vector.

THEOREM 3.8. *Let* $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, *where* f, g, h *are differentiable functions of* t. *Then* $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{i} + h'(t)\mathbf{k}$.

EXAMPLE 3.9. Let $\mathbf{r}(t) = \langle 1 + t^3, 2t, 1 \rangle$. Find the unit tangent vector to the curve defined by $\mathbf{r}(t)$ at the point where $t = 0$.

Solution. First we have $\mathbf{r}'(t) = \langle 3t^2, 2, 0 \rangle$. Thus, $\mathbf{r}'(0) = \langle 0, 2, 0 \rangle = 2\mathbf{j}$. Therefore, $\mathbf{T}(0) = \frac{2\mathbf{j}}{2} = \mathbf{j}$.

EXAMPLE 3.10. Find parametric equations for the tangent line ℓ to the helix with parametric equations $x = 2\cos t, y = \sin t, z = t$ at $t = \frac{\pi}{2}$.

Solution.

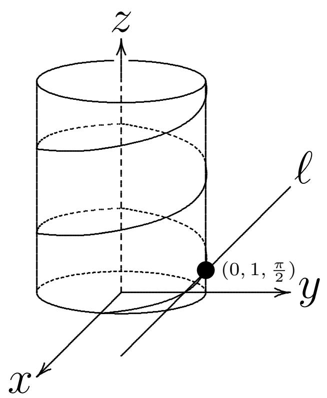


Figure 37 The tangent to the helix

The vector equation of the helix is $\mathbf{r}(t) = \langle 2\cos t, \sin t, t \rangle$. Thus, $\mathbf{r}'(t) = \langle -2\sin t, \cos t, 1 \rangle$ and $\mathbf{r}'(\frac{\pi}{2}) = \langle -2, 0, 1 \rangle$ is a tangent vector to the helix at $t = \frac{\pi}{2}$.

Therefore, the parametric equations of the tangent line ℓ are given by: $x = 0 + (-2)t, y = 1 + (0)t, z = \frac{\pi}{2} + (1)t$.

That is $x = -2t, y = 1, z = \frac{\pi}{2} + t$.

Given a vector function $\mathbf{r}(t)$, we may compute successively $\mathbf{r}'(t)$, $\mathbf{r}''(t)$, $\mathbf{r}'''(t)$ etc, provided they exist.

THEOREM 3.11. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be differentiable vector functions of* t*,* c *a scalar and* f *a real-valued function. Then we have the followings:*

1. $\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$.
2. $\frac{d}{dt}(c\mathbf{u}(t)) = c\mathbf{u}'(t)$.
3. $\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{v}'(t)$.
4. $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$.
5. $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.
6. *(Chain Rule)* $\frac{d}{dt}(\mathbf{u}(f(t))) = f'(t)\mathbf{u}'(f(t))$.

EXERCISE 3.12. Suppose $|\mathbf{r}(t)| = c$, where c is a positive constant. Show that $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$ for all t.

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a continuous vector function.

The *definite integral* of $\mathbf{r}(t)$ from $t = 1$ to $t = b$ is defined as:

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}.$$

EXAMPLE 3.13. Let $\mathbf{r}(t) = 2\cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$. Find $\displaystyle\int_0^{\frac{\pi}{2}} \mathbf{r}(t)dt$.

Solution.

$$\int_0^{\frac{\pi}{2}} \mathbf{r}(t)dt = [2\sin t]_0^{\frac{\pi}{2}}\,\mathbf{i} - [\cos t]_0^{\frac{\pi}{2}}\,\mathbf{j} + \left[t^2\right]_0^{\frac{\pi}{2}}\,\mathbf{k} = 2\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\mathbf{k}.$$

4. Functions of several variables

4.1. Functions of 2 variables.

DEFINITION 4.1. *A function f of 2 variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. Here D is called the domain of f. The set of values that f takes on is called the range of f. That is Range of $f = \{f(x, y) \mid (x, y) \in D\}$.*

We usually write $z = f(x, y)$ to indicate that z is a function of x and y. Moreover, x, y are called the independent variables and z is called the dependent variable.

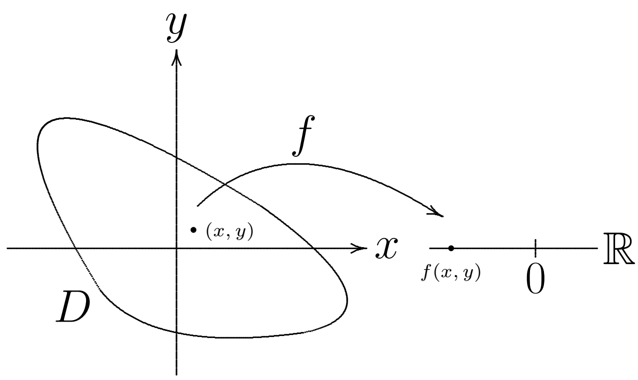


Figure 38 $\quad f : D \longrightarrow \mathbb{R}$

EXAMPLE 4.2. Find the domain of $f(x, y) = x \ln(y^2 - x)$.

Solution.

The expression $x \ln(y^2 - x)$ is defined only when $y^2 - x > 0$. That is $y^2 > x$. The curve $y^2 = x$ separates the plane into two regions, one satisfying the inequality $y^2 > x$, the other satisfying $y^2 < x$.

To find out which region is determined by the inequality $y^2 > x$. Pick any point in one of the regions and test whether it satisfies the inequality. If it does, then by 'connectivity', that whole region is the one satisfying $y^2 > x$, otherwise, it must be the other region. For example, pick the point $(3, 2)$. Since $2^2 > 3$, the region satisfying $y^2 > x$ is the one containing $(3, 2)$.

Thus, domain of f is $\{(x, y) \in \mathbb{R}^2 \mid y^2 > x\}$.

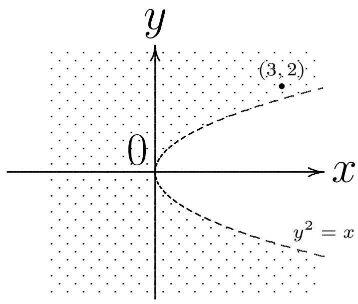


Figure 39 Domain of $x \ln(y^2 - x)$

EXAMPLE 4.3. Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution.

The domain of g is $\{(x, y) \in \mathbb{R}^2 \mid 9 - x^2 - y^2 \geq 0\} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 3^2\}$ which is a circular disk of radius 3. Since $0 \leq g(x, y) = \sqrt{9 - x^2 - y^2} \leq 3$, the range of g lies in [0,3]. Clearly every number in [0,3] can be expressed as $g(x, y)$ for certain (x, y).

Therefore the range of g is the interval [0,3].

DEFINITION 4.4. *Let f be a function of 2 variables with domain D. The graph of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$.*

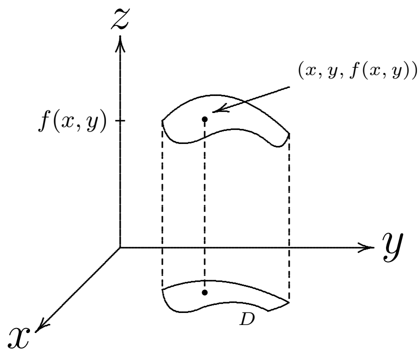


Figure 40 The graph of f

In general, the graph of $f(x, y)$ is a *surface* in $\mathbb{R}^3$.

EXAMPLE 4.5. The graph of $f(x, y) = 6 - 3x - 2y$ is a plane. See Figure 41.

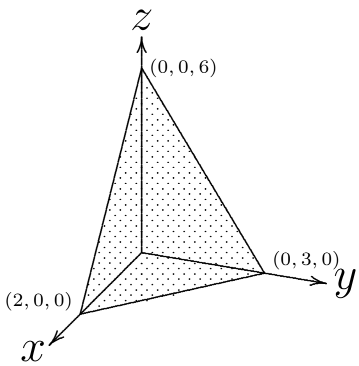


Figure 41 $z = 6 - 3x - 2y$

EXAMPLE 4.6. The graph of $h(x, y) = 4x^2 + y^2$ is an elliptic paraboloid.

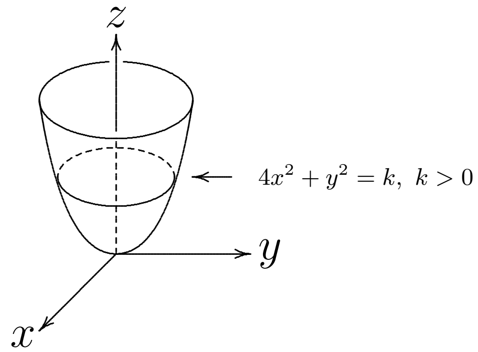


Figure 42 $z = 4x^2 + y^2$

The domain of h is $\mathbb{R}^2$. Since $4x^2 + y^2 \geq 0$, the range of h is $[0,\infty)$. Each horizontal trace is an ellipse with equation given by $4x^2 + y^2 = k$, where $k > 0$.

4.2. Level Curves.

DEFINITION 4.7. *The level curves of a function of 2 variables are the curves in the xy-plane with equation $f(x, y) = K$, where K is a constant. (K is in the range of f)*

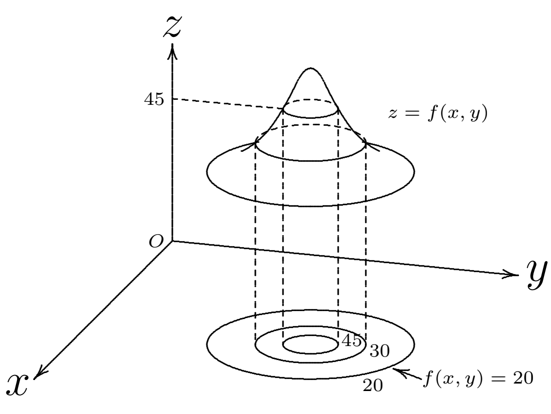


Figure 43 Level curves

EXAMPLE 4.8. Sketch the level curves of $f(x,y) = 6 - 3x - 2y$ for $K = -6, 0, 6, 12$.

Solution. The level curves are $6 - 3x - 2y = K$ which are straight lines.

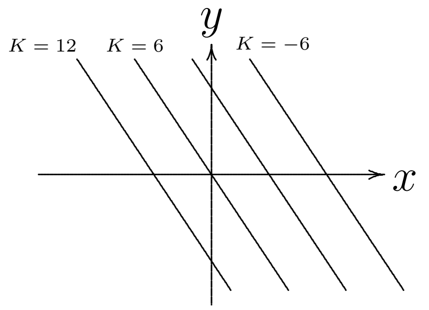


Figure 44 Level curves of $f(x,y) = 6 - 3x - 2y$

EXAMPLE 4.9. Sketch some level curves of $h(x,y) = 4x^2 + y^2$.

Solution.

If $K < 0$, then $4x^2 + y^2 = K$ has no solution in (x,y). Therefore, there are no level curves for $K < 0$.

If $K = 0$, then $4x^2 + y^2 = 0$ has only one solution $(0,0)$. Thus, the level curve consists of one single point at $(0,0)$.

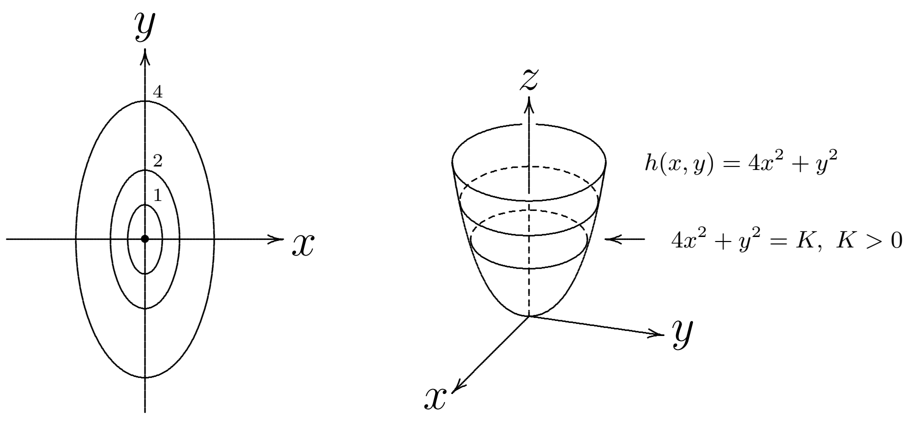


Figure 45 Level curves of $f(x,y) = 4x^2 + y^2$

If $K > 0$, the, $4x^2 + y^2 = K$ is an ellipse. We may write this equation in the standard form:

$$\frac{x^2}{(\frac{\sqrt{K}}{2})^2} + \frac{y^2}{(\sqrt{K})^2} = 1.$$

Thus, a larger K gives rise to an ellipse with longer major and minor axes.

4.3. Functions of 3 or more variables (general case).

Let $f : D \subseteq \mathbb{R}^3 \longrightarrow R$ be a function of three variables. We can describe f by examining the level surfaces of f.

These are surfaces in $\mathbb{R}^3$ given by the equations $f(x, y, z) = K$, where $K \in \mathbb{R}$.

Example 4.10. Let $f(x, y, z) = x^2+y^2+z^2$. The level surfaces of f are concentric spheres with equations of the form $x^2+y^2+z^2 = K$ for $K > 0$.

If $K = 0$, then the level surface reduces to a point at the origin of $\mathbb{R}^3$.

For $K < 0$, there is no level surface for f.

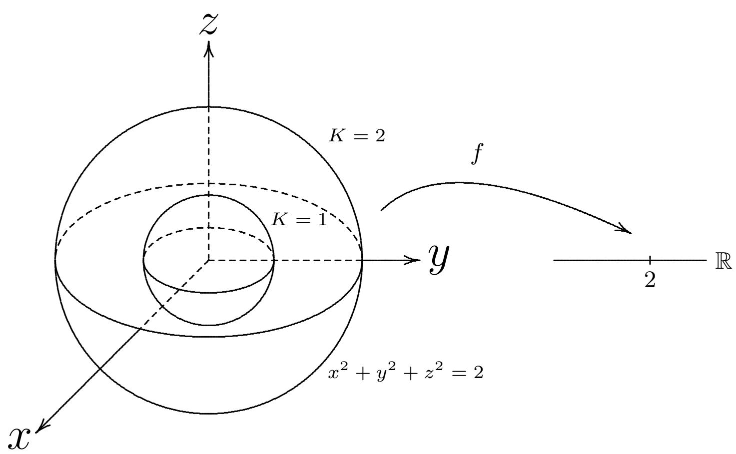


Figure 47 Level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$

5. Limits and Continuity

DEFINITION 5.1. *Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). We say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write*

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for any positive number ϵ, there is a corresponding positive number δ such that $(x, y) \in D$ and

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Longrightarrow |f(x, y) - L| < \epsilon.$$

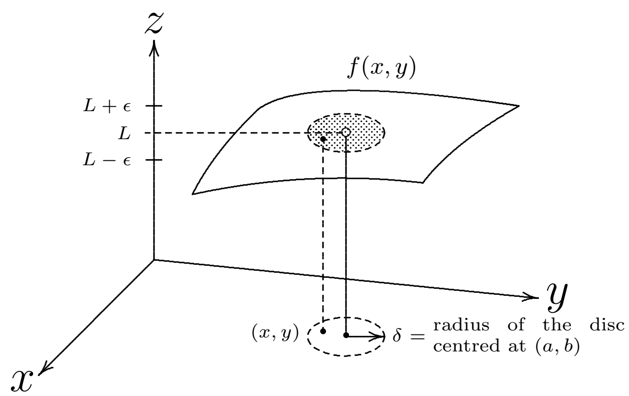


Figure 48 $\lim_{(x,y)\to(a,b)} f(x, y) = L$

Note that f is not required to be defined at (a, b).

The idea is that as (x, y) approaches (a, b), $f(x, y)$ approaches L. In other words, $f(x, y)$ can be made as close to the number L as we wish by requiring (x, y) sufficiently close to (a, b). This is the meaning of the above definition.

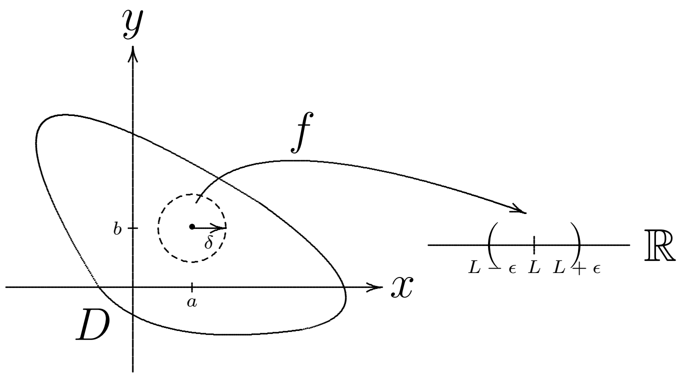


Figure 49 $\lim_{(x,y)\to(a,b)} f(x,y) = L$

The implication in definition 7.1 says that all points (x, y) which are inside the disk centered at (a, b) with radius δ are mapped by f into the interval $(L - \epsilon, L + \epsilon)$. See Figure 49.

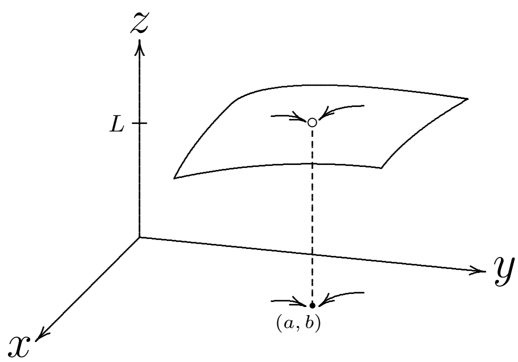


Figure 50 $f(x, y)$ approaches L along different paths

It can be proved from the definition that if $\lim_{(x,y)\to(a,b)} f(x, y) = L$ exists, then

(i) its value L is unique, and

(ii) L is independent of the choice of any path approaching (a, b).

EXAMPLE 5.2. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

Solution. Let $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$. First let's approach $(0,0)$ along the x-axis.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2-0^2}{x^2+0^2} = \lim_{x\to 0} 1 = 1.$$

Next let's approach $(0,0)$ along the y-axis.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=0}} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0^2-y^2}{0^2+y^2} = \lim_{y\to 0} -1 = -1.$$

Since f has two different limits along 2 different paths, the given limit does not exist.

EXAMPLE 5.3. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

Solution. Let $f(x,y) = \dfrac{xy}{x^2+y^2}$. First let's approach $(0,0)$ along the x-axis.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x\cdot 0}{x^2+0^2} = 0.$$

Next let's approach $(0,0)$ along the y-axis.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=0}} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0\cdot y}{0^2+y^2} = 0.$$

At this point, we cannot conclude anything as the limit may exist or may not exist. Now let's approach $(0,0)$ along the path $y = x$.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=y}} f(x,y) = \lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}.$$

Since f has two different limits along 2 different paths, the given limit does not exist.

EXAMPLE 5.4. Let $f(x,y) = \dfrac{xy^2}{x^2+y^4}$. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Solution. Let's approach $(0,0)$ along the line $y = mx$, where m is any real number.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x,y) = \lim_{x\to 0} \frac{x\cdot(mx)^2}{x^2+(mx)^2} = \lim_{x\to 0} \frac{x^3m^2}{x^2(1+m^2x^2)} = \lim_{x\to 0} \frac{xm^2}{1+m^2x^2} = 0.$$

Thus, the limit as (x,y) approaches to the origin along any straight line is zero. However, we still cannot conclude anything as the limit may exist or may not exist. Now let's approach $(0,0)$ along the curve $y^2 = x$.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y^2=x}} f(x,y) = \lim_{y\to 0} \frac{y^2\cdot y^2}{y^4+y^4} = \tfrac{1}{2}.$$

Since f has two different limits along 2 different paths, the given limit does not exist.

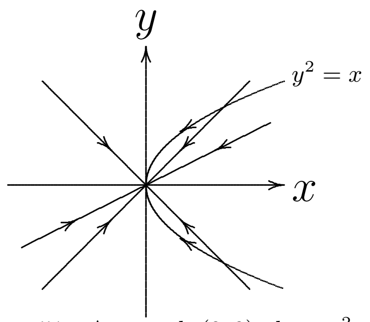


Figure 51 Approach $(0,0)$ along $y^2 = x$

Example 5.5. Prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$

Solution. Let ϵ a positive number. We wish to find a positive number δ such that

$$0 < \sqrt{x^2+y^2} < \delta \Longrightarrow \left|\frac{3x^2y}{x^2+y^2} - 0\right| < \epsilon.$$

In order to obtain the δ that enables the above implication to hold we begin by estimating the expression $\left|\dfrac{3x^2y}{x^2+y^2} - 0\right|$.
Note that

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| = 3\frac{x^2}{x^2+y^2}|y| \le 3|y| \le 3\sqrt{x^2+y^2}.$$

Thus, if we choose $\delta = \epsilon/3$, then

$$0 < \sqrt{x^2+y^2} < \delta \Longrightarrow \left|\frac{3x^2y}{x^2+y^2} - 0\right| \le 3\sqrt{x^2+y^2} < 3\delta = \epsilon.$$

By the definition of limit, we have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$

REMARK 5.6. We remark that the usual limit theorems hold for limits of functions of two variables.

For example

$$\lim_{(x,y)\to(a,b)}(f(x,y)+g(x,y)) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y).$$

DEFINITION 5.7. *A function f of two variables is said to be continuous at (a,b) if*

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

f is said to be continuous on $D \subseteq \mathbb{R}^2$ if f is continuous at each point (a,b) in D.

EXAMPLE 5.8. Every polynomial in x, y is continuous on $\mathbb{R}^2$.

Each rational function is continuous in its domain. For instance, the rational function $f(x,y) = \frac{x^2+x^3y}{x+y}$ is continuous on

$$D = \{(x,y) \in \mathbb{R}^2 \mid x+y \neq 0\}.$$

REMARK 5.9. One may compute limits using polar coordinates. This is especially convenient for limits at the origin and for those expressions that are independent of θ. More precisely, one can prove that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0^+} f(r\cos\theta, r\sin\theta).$$

EXAMPLE 5.10. Find $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

Solution. We shall change to polar coordinates.

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) &= \lim_{r\to 0^+} r^2 \ln\left(r^2\right) \\
&= \lim_{r\to 0^+} \frac{2\ln r}{r^{-2}} \\
&= \lim_{r\to 0^+} \frac{2(1/r)}{(-2)(1/r^3)} \qquad \text{using L'Hôpital's rule} \\
&= \lim_{r\to 0^+} -r^2 = 0.
\end{aligned}$$

REMARK 5.11. For functions of three or more variables, there are similar definition of limits and continuity. More precisely, for functions of three variables, these are stated as follows:

DEFINITION 5.12. $\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$ *if for any* $\epsilon > 0$, *there is a corresponding* $\delta > 0$ *such that* $(x,y,z) \in D$ *and*

$$0 < \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} < \delta \Longrightarrow |f(x,y,z)-L| < \epsilon.$$

DEFINITION 5.13. *A function f is called continuous at (a,b,c) if*

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c).$$

6. Partial Derivatives

DEFINITION 6.1. *Let f be a function of two variables. The partial derivative of f with respect to x at (a, b) is*

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

The partial derivative of f with respect to y at (a, b) is

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.$$

There are different notations for the partial derivative of a function. If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x},$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y}.$$

In other words, in order to find f_x, we may simply regard y as constant and differentiate $f(x, y)$ with respect to x.

Similarly, to find f_y, one can simply regard x as constant and differentiate $f(x, y)$ with respect to y.

That is $f_x(a, b) = \dfrac{d}{dx} f(x, b)|_{x=a}$ and $f_y(a, b) = \dfrac{d}{dy} f(a, y)|_{y=b}$.

Example 6.2. Let $f(x,y) = x^3 + x^2y^3 - 2y^2$.

Then $f_x = 3x^2 + 2xy^3$ and $f_y = 3x^2y^2 - 4y$.

Thus for example, $f_x(1,1) = 5$ and $f_y(1,1) = -1$.

Geometrically, $f_x(a,b)$ measures the rate of change of f in the direction of $\mathbf{i}$ at the point (a,b).

If we consider the line $y = b$ on the xy-plane parallel to the x-axis and passing through the point (a,b), the image of this line under f is a curve C_1 on the surface $z = f(x,y)$.

Then $f_x(a,b)$ is just the gradient of the tangent line to C_1 at (a,b).

Similarly, $f_y(a,b)$ is just the derivative at (a,b) of the curve C_2 traced out as the image of the line $x = a$ under f.

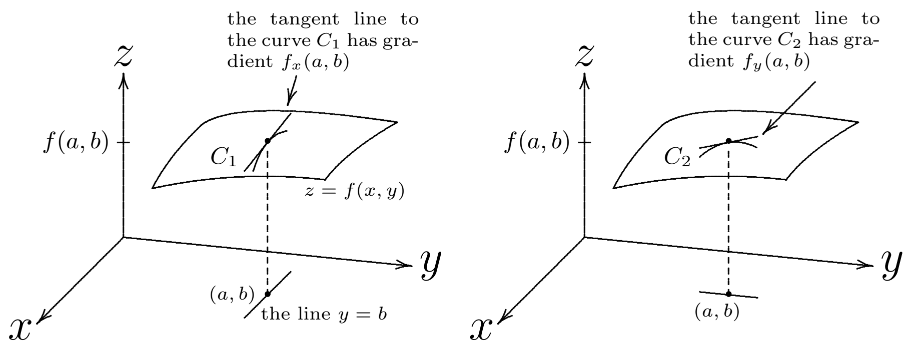


Figure 52 Partial derivatives

EXAMPLE 6.3. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

Solution. Take partial derivative with respect to x on both sides:

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6y(z + x\frac{\partial z}{\partial x}) = 0.$$

Solving for $\frac{\partial z}{\partial x}$, we have

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}.$$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$

For functions of more than two variables, as such $w = f(x, y, z)$, we can similarly define

$$f_x, f_y, f_z, \ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \ \text{or} \ \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}.$$

As in the case of function of one variable, we may also define higher order partial derivatives of a function of several variables.

Let f be a function of x and y. Then f_x and f_y are also functions of x and y.

Thus we may consider $(f_x)_x$, $(f_x)_y$, $(f_y)_x$ and $(f_y)_y$.

For convenience, we shall simply denote them by f_{xx}, f_{xy}, f_{yx} and f_{yy} respectively. These are the second order partial derivatives of f.

There are other notations for the higher order partial derivatives.

Suppose $z = f(x, y)$. Then we also write:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

EXAMPLE 6.4. Let $f(x, y) = x^3 + x^2y^3 - 2y^2$. Find f_{xx}, f_{xy}, f_{yx} and f_{yy}.

Solution. First $f_x = 3x^2 + 2xy^3$ and $f_y = 3x^2y^2 - 4y$. Thus, $f_{xx} = 6x + 2y^3$, $f_{yy} = 6x^2y - 4$, $f_{xy} = (f_x)_y = 6xy^2$ and $f_{yx} = (f_y)_x = 6xy^2$.

THEOREM 6.5. (Clairaut's Theorem) *Let f be defined on an open disk containing the point (a, b). If f_{xy} and f_{yx} are continuous at (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$.*

EXAMPLE 6.6. Let

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

On can show that

$$f_x(x,y) = \begin{cases} \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases},$$

and

$$f_y(x,y) = \begin{cases} \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

But $f_{xy}(0, 0) = -1$, while $f_{yx}(0, 0) = 1$.

EXERCISE 6.7. Let $f(x, y, z) = \sin(3x + yz)$. Find f_{xxyz} and f_{xzyx}. Show that $f_{xxyy} = f_{xyyx}$.

6.1. Tangent Plane. Let f be a function of two variables.

The graph of f is a surface in $\mathbb{R}^3$ with equation $z = f(x, y)$.

Let $P(x_0, y_0, z_0)$ be a point on this surface. Thus, $z_0 = f(x_0, y_0)$.

Assuming a tangent plane to the surface exists, we shall find its equation.

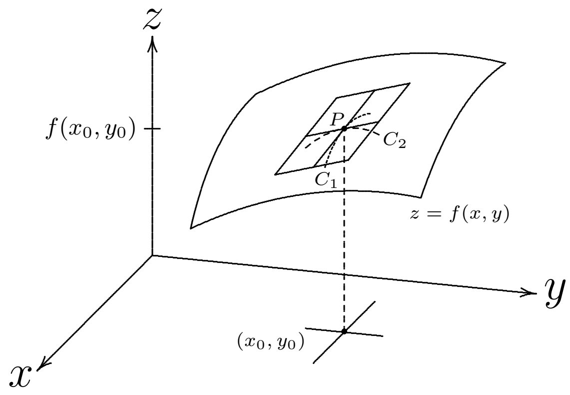


Figure 53 The tangent plane

Recall that the equation of a plane passing through $P(x_0, y_0, z_0)$ is of the form $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$.

Assuming the plane is not vertical, we have C is not zero. Thus we may write the equation of the plane as

$$z - z_0 = a(x - x_0) + b(y - y_0).$$

The tangent line to C_1 at P is obtained by taking $y = y_0$ in the above equation.

That is $z - z_0 = a(x - x_0)$. Since $f_x(x_0, y_0)$ is the gradient of the tangent line C_1 at P, we have $a = f_x(x_0, y_0)$.

Similarly, $b = f_y(x_0, y_0)$.

Consequently, the equation of the tangent plane to the surface $z = f(x, y)$ at P is

$$\boxed{z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

EXAMPLE 6.8. Find the equation of the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution. Let $f(x, y) = 2x^2 + y^2$. Then $f_x(x, y) = 4x$ and $f_y(x, y) = 2y$ so that $f_x(1, 1) = 4$ and $f_y(1, 1) = 2$.
Hence, the equation of the tangent plane at $(1, 1, 3)$ is given by

$$z = 3 + 4(x - 1) + 2(y - 1).$$

That is $z = 4x + 2y - 3$.

6.2. Linear Approximation. Since the tangent plane to the surface $z = f(x, y)$ at P is very close to the surface at least when it is near P, we may use the function defining the tangent plane as a linear approximation to f.

Recall that the equation of the tangent plane to the graph of $f(x, y)$ at $P(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Definition 6.9. *The linear function L whose graph is this tangent plane is given by*

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

L is called the linearization of f at (a, b).

The approximation

$$f(x, y) \approx L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linear approximation or tangent plane approximation of f at (a, b).

Example 6.10. Let $f(x, y) = xe^{xy}$. Find the linearization of f at $(1, 0)$. Use it to approximate $f(1.1, -0.1)$.

Solution. First we have $f_x(x, y) = e^{xy} + xye^{xy}$ and $f_y(x, y) = x^2e^{xy}$.

Thus $f_x(1, 0) = 1$ and $f_y(1, 0) = 1$.

Then,

$$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) = x + y.$$

The corresponding linear approximation is $xe^{xy} \approx x + y$.

Therefore, $f(1.1, -0.1) \approx 1.1 + (-0.1) = 1$.

The actual value of $f(1.1, -0.1)$ is 0.98542 round up to 5 decimal places.

6.3. The differential. Let $z = f(x, y)$. As in the case of functions of one variable, we take the differentials dx and dy to be independent variables.

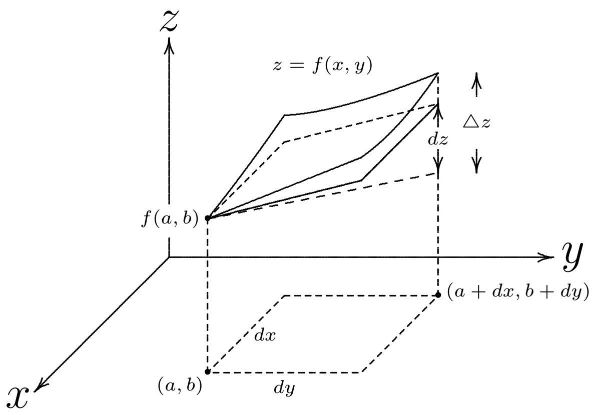


Figure 54 The differential

Definition 6.11. *The differential dz, or the total differential, is defined to be*

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

Consider the differential of f at the point (a, b).

The tangent plane approximation of f at (a, b) implies that for a small change dx of a and a small change dy of b, the actual change $\triangle z$ of z is approximately equal to dz.

In other words,

$$\triangle z \approx dz = f_x(a, b)dx + f_y(a, b)dy.$$

EXAMPLE 6.12. Let $f(x,y) = x^2 + 3xy - y^2$. Find dz. If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of $\triangle z$ and dz.

Solution. $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x+3y)dx + (3x-2y)dy$.

At the point $(2,3)$, $dz = ((2)(2)+3(3))dx + ((3)(2)-(2)(3))dy$.

That is $dz = 13dx$.

Now we take $dx = 2.05 - 2 = 0.05$ and $dy = 2.96 - 3 = -0.04$.

Thus, $dz = 13(0.05) = 0.65$.

For the actual change in z, we have $\triangle z = f(2.05, 2.96) - f(2,3) = 0.6449$.

DEFINITION 6.13. *Let $z = f(x,y)$. f is said to be differentiable at (a,b) if*

$$\triangle z = f_x(a,b)\triangle x + f_y(a,b)\triangle y + \epsilon_1 \triangle x + \epsilon_2 \triangle y,$$

where $\lim\limits_{(\triangle x, \triangle y)\to(0,0)} \epsilon_1 = 0$ *and* $\lim\limits_{(\triangle x, \triangle y)\to(0,0)} \epsilon_2 = 0$.

EXAMPLE 6.14. Prove that $f(x,y) = xy$ is differentiable at (a,b).

Solution. First $f_x(x,y) = y$ and $f_y(x,y) = x$. At the point (a,b), we have $f_x(a,b) = b$ and $f_y(a,b) = a$.

$$\begin{aligned}\triangle z &= f(a+\triangle x, b+\triangle y) - f(a,b)\\ &= (a+\triangle x)(b+\triangle y) - ab\\ &= b\triangle x + a\triangle y + \triangle x\triangle y\\ &= f_x(a,b)\triangle x + f_y(a,b)\triangle y + \triangle x\triangle y.\end{aligned}$$

Here $\epsilon_1 = 0$ and $\epsilon_2 = \triangle x$.

Clearly,

$$\lim_{(\triangle x,\triangle y)\to(0,0)} \epsilon_1 = 0 \quad\text{and}\quad \lim_{(\triangle x,\triangle y)\to(0,0)} \epsilon_2 = \lim_{(\triangle x,\triangle y)\to(0,0)} \triangle x = 0.$$

Thus, $f(x,y) = xy$ is differentiable at (a,b).

Note that from the definition of differentiability, if $f(x,y)$ is differentiable at (a,b), then $f_x(a,b)$ and $f_y(a,b)$ exist. However the converse is not necessarily true.

Exercise 6.15. Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. Show that $f_x(0,0)$ and $f_y(0,0)$ exist but f is not differentiable at $(0,0)$.

Exercise 6.16. Prove that if $f(x,y)$ is differentiable at (a,b), then $f(x,y)$ is continuous at (a,b).

Theorem 6.17. *Suppose $f_x(x,y)$ and $f_y(x,y)$ exist in an open disk containing (a,b) and are continuous at (a,b). Then f is differentiable at (a,b).*

THEOREM 6.18. (The chain rule, case 1) *Suppose $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$, $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

EXAMPLE 6.19. Let $z = x^2y + 3xy^4$, where $x = \sin 2t$, $y = \cos t$. Find $\dfrac{dz}{dt}$.

Solution.
$\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (2xy+3y^4)(2\cos 2t)+(x^2+12xy^3)(-\sin t)$.

THEOREM 6.20. (The chain rule, case 2) *Suppose $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$, $y = h(s, t)$ are both differentiable functions of s and t. Then z is a differentiable function of s and t and*

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

EXAMPLE 6.21. Let $z = e^x \sin y$, where $x = st^2$, $y = s^2t$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

Solution.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)t^2 + (e^x \cos y)(2st).$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2).$$

EXERCISE 6.22. Suppose $z = f(x, y)$ has continuous 2nd order partial derivatives and $x = r^2 + s^2,\ y = 2rs$. Find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial^2 z}{\partial r^2}$.

6.4. Implicit Differentiation. Suppose $F(x, y) = 0$ defines y implicitly as a function of x. That is $y = f(x)$.

Then $F(x, f(x)) = 0$. Now we use the chain rule(case 1) to differentiate F with respect to x.

Thus

$$F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = 0.$$

Therefore,

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}\ .}$$

EXAMPLE 6.23. Find $\dfrac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

Solution. Let $F(x, y) = x^3 + y^3 - 6xy$. The given equation is simply $F(x, y) = 0$. Therefore,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x}.$$

Next, suppose z is given implicitly as a function of x and y by an equation $F(x, y, z) = 0$.

In other words, one may solve z locally in terms of x and y in the equation $F(x, , y, z) = 0$ to obtain $z = f(x, y)$. Then $F(x, y, f(x, y)) = 0$.

We wish to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in terms of F_x, F_y and F_z. To do so, we use the chain rule to differentiate the equation $F(x, y, z) = 0$ keeping in mind that z is regarded as a function of x and y.

We thus obtain:

$$F_x \frac{\partial x}{\partial x} + F_y \frac{\partial y}{\partial x} + F_z \frac{\partial z}{\partial x} = 0.$$

Note that $\dfrac{\partial x}{\partial x} = 1$ and $\dfrac{\partial y}{\partial x} = 0$. Thus $F_x + F_z \dfrac{\partial z}{\partial x} = 0$.

Hence,

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}\ .}$$

Similarly,

$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}\ .}$$

EXERCISE 6.24. Three ants A, B and C crawl along the positive x, y and z axes respectively. A and B are crawling at a constant speed of $1\,\mathrm{cm/s}$, C is crawling at a constant speed of $3\,\mathrm{cm/s}$ and they are all traveling away from the origin. Find the rate of change of the area of triangle ABC when A is $2\,\mathrm{cm}$ away from the origin while B and C are $1\,\mathrm{cm}$ away from the origin. [The area of the triangle $A(x,0,0)B(0,y,0)C(0,0,z)$ is given by $\frac{1}{2}\sqrt{x^2y^2+y^2z^2+z^2x^2}$.]

[Answer : $4\,\mathrm{cm}^2/\mathrm{s}$.]

6.5. Directional Derivative & The Gradient Vector.

DEFINITION 6.25. *Let f be a function of x and y. Then the directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is*

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

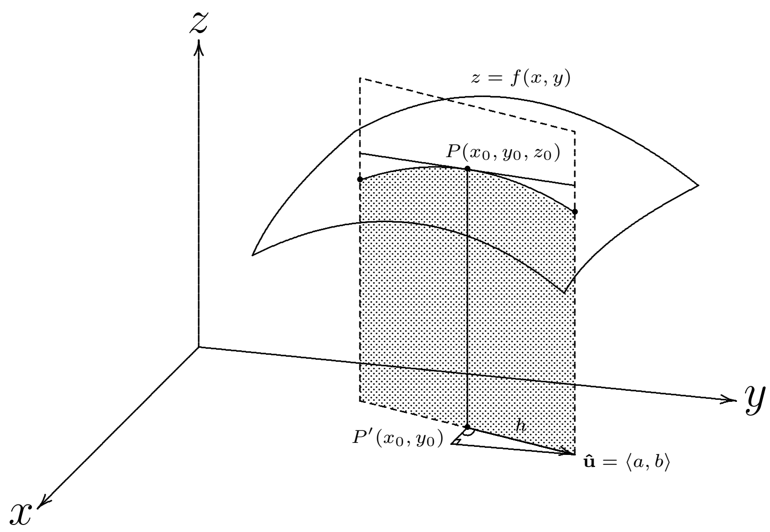


Figure 52 Directional Derivative

Note that $D_{\mathbf{i}} f(x_0, y_0) = f_x(x_0, y_0)$ and $D_{\mathbf{j}} f(x_0, y_0) = f_y(x_0, y_0)$, where $\mathbf{i}$ and $\mathbf{j}$ are the standard basis vectors in $\mathbb{R}^2$.

THEOREM 6.26. *Let f be a differentiable function of x and y. Then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and*

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = \langle f_x(x, y), f_y(x, y)\rangle \cdot \mathbf{u}.$$

Proof. Consider $g(h) = f(x_0 + ha, y_0 + hb)$.

Clearly $g'(0) = D_{\mathbf{u}} f(x_0, y_0)$.

By the chain rule, $g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}$.

At $h = 0$, we have $g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$.

Hence,

$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$.

EXAMPLE 6.27. Let $f(x, y) = x^3 - 3xy + 4y^2$. Find $D_{\mathbf{u}} f(1, 2)$, where $\mathbf{u}$ is the unit vector making an angle of $\frac{\pi}{6}$ with the positive x-axis.

Solution.

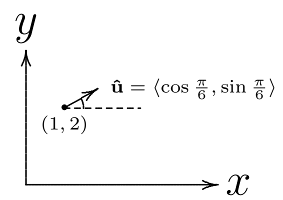


Figure 53 Directional Derivative of f

First, $f_x = 3x^2 - 3y$, $f_y = -3x + 8y$.

Thus $f_x(1, 2) = -3$ and $f_y(1, 2) = 13$.

Therefore, $D_{\mathbf{u}} f(1, 2) = \langle -3, 13 \rangle \cdot \langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \rangle = (13 - 3\sqrt{3})/2$.

DEFINITION 6.28. *Let f be a differentiable function of x and y. The gradient of f is the vector function*

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Thus we have the following formula for the directional derivative in terms of the gradient of f.

$$\boxed{D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}, \qquad \text{where } \mathbf{u} \text{ is a unit vector.}}$$

EXAMPLE 6.29. Let $f(x,y) = x^2y^3 - 4y$. Find the directional derivative of f at $(2,-1)$ in the direction $3\mathbf{i} + 4\mathbf{j}$.

Solution. The unit vector along $3\mathbf{i} + 4\mathbf{j}$ is $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$.

The gradient of f is $\nabla f = \langle 2xy^3, 3x^2y^2 - 4\rangle$.

Thus $\nabla f(2,-1) = \langle -4, 8\rangle$.

Consequently, $D_{\mathbf{u}}f(2,-1) = \nabla f(2,-1)\cdot\mathbf{u} = \langle -4, 8\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = 4$.

DEFINITION 6.30. *Let f be a function of x, y and z. The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ in $\mathbb{R}^3$ is*

$$D_{\mathbf{u}}f(x_0,y_0,z_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0,y_0,z_0)}{h}$$

if this limit exists.

Similarly, the gradient of a differentiable function f is defined to be

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

The formula $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$ is also valid for any function f of more than 2 variables.

EXERCISE 6.31. Let $f(x, y, z) = xyz^2$. Let $\mathbf{u}$ be the unit vector $\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{i}$. Find $D_{\mathbf{u}} f(1, 1, 1)$.

THEOREM 6.32. *Let f be a differentiable function of 2 or 3 variables. Let P be a point in the domain of f. The maximum value of $D_{\mathbf{u}} f(P)$ is $|\nabla f(P)|$ and it occurs where $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(P)$.*

Proof. First $D_{\mathbf{u}} f(P) = \nabla f(P) \cdot \mathbf{u} = |\nabla f(P)||\mathbf{u}| \cos \theta = |\nabla f(P)| \cos \theta$, where θ is the angle between $\nabla f(P)$ and $\mathbf{u}$.

Therefore, $D_{\mathbf{u}} f(P)$ attains its maximum value $|\nabla f(P)|$ when $\theta = 0$, i.e. when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(P)$.

EXERCISE 6.33. Let $f(x, y) = xe^y$, $P = (2, 0)$ and $Q = (\frac{1}{2}, 2)$.
(a) Find the rate of change of f at P in the direction $\overrightarrow{PQ}$. In other words, find $D_{\mathbf{u}} f(P)$.
(b) In which direction does f have the maximum rate of change? and what is this maximum rate of change?

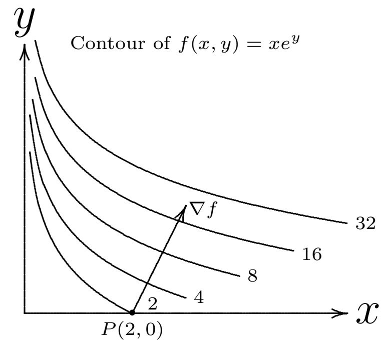


Figure 54 ∇f is the direction of steepest ascend

Theorem 8.32 implies that f has the maximal rate of increase at a point P when P is moving along the direction of the gradient.

In other words, $\nabla f(P)$ is along the direction of steepest ascend. See figure 54 for the example in the above exercise.

6.6. Tangent Planes to Level surfaces. Let S be a surface with equation $F(x, y, z) = k$, where k is a constant. That is S is a level surface of F.

Let $P(x_0, y_0, z_0)$ be a point in S.

Let's find the equation of the tangent plane to S at P.

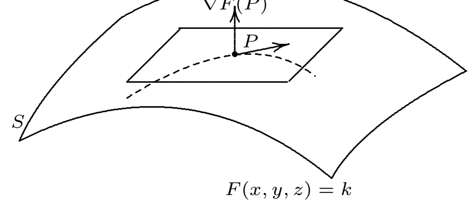


Figure 55 Tangent plane

Take any curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ on the surface S such that $\mathbf{r}(0) = (x_0, y_0, z_0)$. Its tangent vector $\mathbf{r}'(0)$ shall lie on the tangent plane to S at P.

Now if we use the chain rule to differentiate $F(x(t), y(t), z(t)) = k$ with respect to t, we have

$$F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} = 0.$$

In other words, $\nabla F \cdot \mathbf{r}'(t) = 0$. At $t = 0$, we have $\nabla F(P) \cdot \mathbf{r}'(0) = 0$.

Therefore, $\nabla F(P)$ is perpendicular to the tangent plane.

$$\boxed{\text{Eqn of tangent plane:} \quad \langle x - x_0, y - y_0, z - z_0\rangle \cdot \nabla F(x_0, y_0, z_0) = 0.}$$

EXAMPLE 6.34. Find the equation of the tangent plane and the normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

Solution. Let $F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$. Then $\nabla F = \langle x/2, 2y, 2z/9\rangle$.

Thus $\nabla F(-2, 1, -3) = \langle -1, 2, -2/3\rangle$.

Therefore, the equation of the tangent plane is:

$\langle x + 2, y - 1, z + 3\rangle \cdot \langle -1, 2, -2/3\rangle = 0.$

That is $3x - 6y + 2z + 18 = 0$.

Also the equation of the normal line in symmetric form is given by

$$\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-\frac{2}{3}}.$$

In the special case in which the surface S is the graph of a function $z = f(x, y)$, S can be regarded as the level surface

$$F(x, y, z) \equiv f(x, y) - z = 0.$$

In other words, the graph of $z = f(x, y)$ is simply the level surface of $F(x, y, z)$ at level 0.

In this case, $\nabla F = \langle f_x, f_y, -1\rangle$ is a normal vector to the tangent plane of S at $(x, y, f(x, y))$.

Therefore, if we consider a point $P(x_0, y_0, f(x_0, y_0))$ on the graph of $z = f(x, y)$.

The equation of the tangent plane is given by

$$\langle x - x_0, y - y_0, z - f(x_0, y_0)\rangle \cdot \langle f_x, f_y, -1\rangle = 0.$$

That is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

This is the same formula obtained in **8.1**.

This is the end of week 2. Have a nice weekend! Remember to do your homework!

7. Maximum and Minimum Values

Definition 7.1. *$f(x,y)$ has a local maximum (minimum) at (a,b) if $f(x,y) \leq f(a,b)$ ($f(x,y) \geq f(a,b)$) for all points (x,y) in some disk with center (a,b). The number $f(a,b)$ is called a local maximum value (local minimum value).*

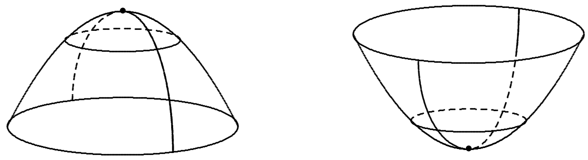


Figure 56 A local maximum A local minimum

Theorem 7.2. *If f has a local maximum or a local minimum at (a,b) and $f_x(a,b)$ and $f_y(a,b)$ exist, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$. That is $\nabla f(a,b) = \mathbf{0}$.*

Definition 7.3. *A point (a,b) is called a critical point of f if $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or if one of these partial derivatives does not exist.*

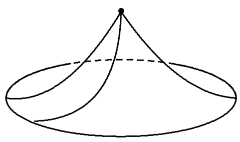


Figure 57 A critical point

Note that if f has a local minimum or a local maximum at (a,b), then (a,b) is a critical point of f.

However not all critical points of a function give rise to local maximum or local minimum.

In other words, at a critical point, a function could have a local maximum, or a local minimum or neither.

EXAMPLE 7.0.1. Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Find the local maxima and local minima of f.

Solution. First $f_x = 2x - 2$ and $f_y = 2y - 6$.

Thus, $f_x = 0$ and $f_y = 0$ if and only if $(x, y) = (1, 3)$.

Therefore f has a critical point at $(1, 3)$. So f has a possible local maximum or local minimum at $(1, 3)$.

As $f(x, y) = 4 + (x - 1)^2 + (y - 3)^2 \geq 4$, we see that f has a local (in fact absolute) minimum at $(1, 3)$.

EXAMPLE 7.0.2. Find the local extrema (i.e. local maxima or local minima) of $f(x, y) = y^2 - x^2$.

Solution. First $f_x = -2x$ and $f_y = 2y$.

Therefore, the only critical point is $(0, 0)$.

However, f has neither a maximum nor a minimum at $(0, 0)$.

To see this, consider the function f along $y = 0$, $f(x, 0) = -x^2 < 0$ for $x \neq 0$. So f has a local maximum along $y = 0$.

On the other hand, if we consider f along $x = 0$, we have $f(0, y) = y^2 > 0$ for all $y \neq 0$.

Thus f has a local minimum along $x = 0$. Therefore f has neither a maximum nor a minimum at $(0, 0)$.

Such a point is called a *saddle point*.

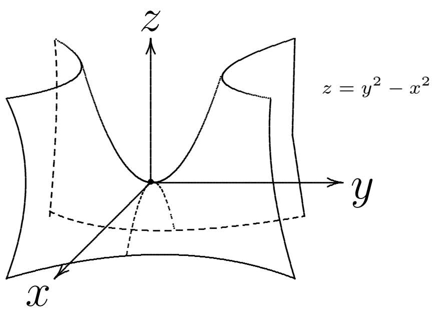


Figure 58 A saddle point

DEFINITION 7.4. *f is said to have a saddle point at (a, b) if there is a disk centered at (a, b) such that f assumes its maximum value on one diameter of the disk only at (a, b), and assume its minimum value on another diameter of the disk only at (a, b).*

In other words, f has a saddle point at (a, b) if there are some directions along which f has a local maximum at (a, b) and some directions along which f has a local minimum at (a, b).

THEOREM 7.5. *(The Second Derivative Test) Suppose f_{xx}, f_{xy}, f_{yx} and f_{yy} are continuous on a disk with centre (a, b) and suppose $f_x(a, b) = 0, f_y(a, b) = 0$. Let*

$$D = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}(a, b)^2.$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).

(c) If $D < 0$, then f has a saddle point at (a, b).

Note that if $D = 0$, then no conclusion can be drawn from it. The point can be a local maximum, a local minimum, a saddle point or neither of these.

EXAMPLE 7.0.3. Find the local maxima, local minima and saddle points (if any) of

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

Solution. First, $f_x = 4x^3 - 4y$ and $f_y = 4y^3 - 4x$.

Now we proceed to solve $4x^3 - 4y = 0$ and $4y^3 - 4x = 0$ for the critical points.

The two equations are equivalent to $y = x^3$ and $x = y^3$.

Substituting one into the other, we obtain $x^9 - x = 0$.

That is $x(x+1)(x-1)(x^2+1)(x^4+1) = 0$. Thus the real solutions are $x = 0, -1, 1$.

Therefore, the critical points are $(0, 0), (-1, -1)$ and $(1, 1)$.

To apply the second derivative test, we compute the second order partial derivatives.

$f_{xx} = 12x^2, f_{yy} = 12y^2, f_{xy} = -4.$

Thus $D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 16.$

At $(0, 0)$, $D(0, 0) = -16 < 0$. Hence, f has a saddle point at $(0, 0)$.

At $(-1,-1)$, $D(-1,-1) = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0$. Hence f has a local minimum at $(-1,-1)$.

At $(1,1)$, $D(1,1) = 128 > 0$ and $f_{xx}(1,1) = 12 > 0$. Hence f has a local minimum at $(1,1)$.

Definition 7.6. *A bounded set in $\mathbb{R}^2$ is one that is contained in some disk. A closed set in $\mathbb{R}^2$ is one that contains all its boundary points.*

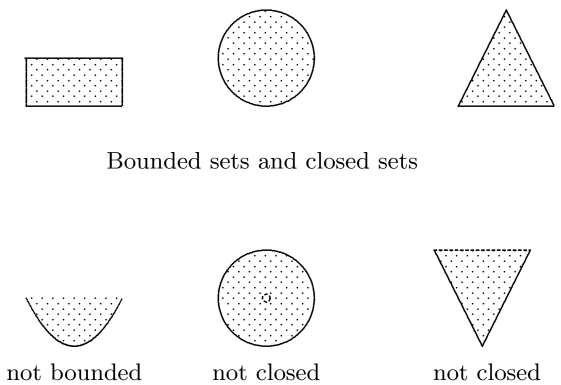


Figure 59 Sets in $\mathbb{R}^2$

Theorem 7.7. *(Extreme Value Theorem) If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.*

The following is a procedure to find the absolute maximum and the absolute minimum value of a function defined on a closed and bounded set.

1. Find the values of f at the critical points.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from 1. and 2. is the absolute maximum value and the smallest of the values from 1. and 2. is the absolute minimum value

EXAMPLE 7.8. Find the absolute maximum and minimum values of $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

Solution. First $f_x(x, y) = 2x - 2y$ and $f_y(x, y) = -2x + 2$.

Thus $f_x(x, y) = 0$ and $f_y(x, y) = 0$ if and only if $(x, y) = (1, 1)$.

That is $(1, 1)$ is the only critical point in the interior of the rectangle.

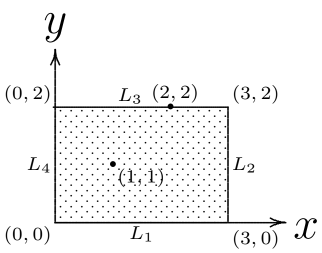


Figure 60

Along L_1: $y = 0$. That is $f(x, 0) = x^2$ for $x \in (0, 3)$ which is increasing, thus giving no critical point along L_1.

Along L_2: $x = 3$. That is $f(3, y) = 9 - 6y + 2y = 9 - 4y$ for $y \in (0, 2)$ which is decreasing, thus giving no critical point along L_2.

Along L_3: $y = 2$. That is $f(x, 2) = (x - 2)^2$ for $x \in (0, 3)$. It has a critical point (a local minimum) at $x = 2$. That is at the point $(2, 2)$.

Along L_4: $x = 0$. That is $f(0, y) = 2y$ for $y \in (0, 2)$ which is increasing, thus giving no critical point along L_4.

Now let's compute the values of f at all the critical points (including the four vertices of the rectangle).

$f(1, 1) = 1, f(2, 2) = 0, f(0, 0) = 0, f(3, 0) = 9, f(3, 2) = 1, f(0, 2) = 4.$

Thus the absolute maximum value of f is 9 and the absolute minimum value is 0.

EXERCISE 7.9. Find the maximum and minimum values of $f(x, y) = x^2 + y^2 - x - y + 1$ on the triangular region R with vertices $(0, 0), (2, 0), (0, 2)$. See Figure 61.

[Answer : Maximum value $= 3$, Minimum value $= 1/2$.]

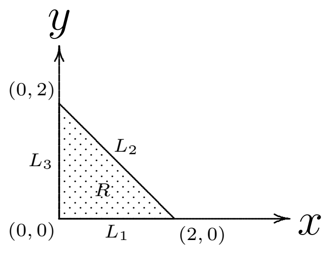


Figure 61

8. Lagrange Multipliers

In this section we consider the problem of maximizing or minimizing a function $f(x, y)$ subject to a constraint $g(x, y) = 0$.

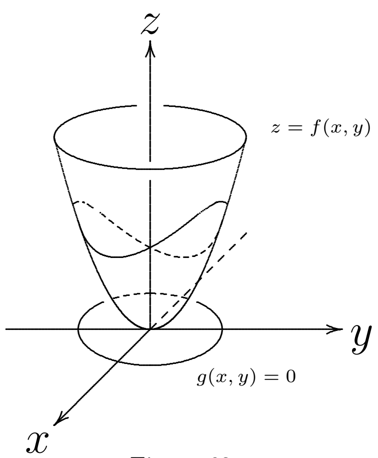


Figure 62

If we confine the point (x, y) to lie on the curve $g(x, y) = 0$ on the xy-plane, its image under f gives a curve on the graph of $z = f(x, y)$.

We are looking for the highest and lowest points of this curve.

Suppose the extreme value of $f(x, y)$ subject to the constraint $g(x, y) = 0$ is k and is attained at the point (x_0, y_0).

By examining at the contour of f, we see that at the extreme point, the curve $g(x, y) = 0$ must touch the level curve $f(x, y) = k$, because if the curve $g(x, y) = 0$ cuts across the level curve $f(x, y) = k$, one can still move the point along $g(x, y) = 0$ so as to increase or decrease the value of f.

In other words, the gradients of f and g must be parallel at the extreme point (x_0, y_0).

Consequently, we must have

$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ if $\nabla g(x_0, y_0) \neq \mathbf{0}$.

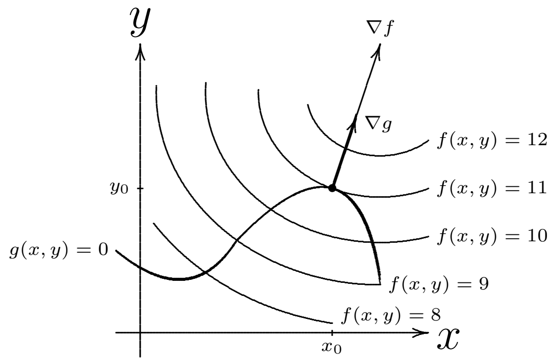


Figure 63 Lagrange Multiplier

The same principle applied to functions of three variables. Let's state the method of Lagrange Multiplier in this setting. The objective is to find the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$ (assuming that these extreme values exist). Below is an outline of the procedure.

(a) Find all x, y, z and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \qquad (*)$$

and $g(x, y, z) = 0$.

(Assuming at each of these solutions $\nabla g \neq \mathbf{0}$.)

(b) Evaluate f at all points (x, y, z) obtained in (a). The largest of these values is the absolute maximum of f; the smallest is the absolute minimum of f.

The number λ is called a *Lagrange Multiplier*.

The equation $(*)$ is equivalent to $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$.

EXAMPLE 8.1. Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$. See figure 62.

Solution: Since the circle is a closed and bounded set and f is a continuous function, there is always an absolute maximum and an absolute minimum of f over it.

They are among the extreme values of f defined over the circle.

To find them, first $\nabla f(x, y) = \langle 2x, 4y\rangle$ and $\nabla g(x, y) = \langle 2x, 2y\rangle$.

Thus $\nabla f(x, y) = \lambda \nabla g(x, y)$ is equivalent to

$$\begin{cases} 2x &= \lambda 2x, \\ 4y &= \lambda 2y. \end{cases}$$

Together with the constraint $x^2 + y^2 = 1$, we need to solve the following system of equations in x, y and λ:

$$\begin{cases} 2x(\lambda - 1) &= 0, \\ 2y(\lambda - 2) &= 0, \\ x^2 + y^2 &= 1. \end{cases}$$

The first equation gives either $x = 0$ or $\lambda = 1$.

If $x = 0$, then the constraint equation gives $y = \pm 1$. Thus we have the solutions $(0, -1), (0, 1)$.

If $\lambda = 1$, then the second equation gives $y = 0$. Thus, by the constraint equation, we have the solutions $(-1, 0), (1, 0)$.

Consequently, we have four solutions $(0, 1), (0, -1), (-1, 0), (1, 0)$.

Now $f(0,1)=2, f(0,-1)=2, f(-1,0)=1, f(1,0)=1$.

Therefore, the absolute maximum value is 2 and the absolute minimum value is 1.

EXERCISE 8.2. Find the rectangular box with the largest volume that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

[Answer: $\frac{8abc}{3^{\frac{3}{2}}}$]

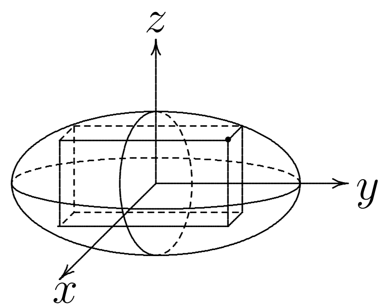


Figure 64

EXERCISE 8.3. Find the point on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(3,1,-1)$. (Consider the line passing through $(3,1,-1)$ and the centre of the sphere, it intersects the sphere diametrically at two points.)

[Answer: Min $=\sqrt{11}-2$ at $(\frac{6}{\sqrt{11}},\frac{2}{\sqrt{11}},\frac{-2}{\sqrt{11}})$, Max $=\sqrt{11}+2$ at $(\frac{-6}{\sqrt{11}},\frac{-2}{\sqrt{11}},\frac{2}{\sqrt{11}})$]

EXAMPLE 8.4. Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}.$$

Solution. First we find the critical points of f in the interior of D.

As $f_x(x, y) = 2x$ and $f_y(x, y) = 4y$, the only critical point in the interior of D is $(0, 0)$.

Next we shall find the critical points on the boundary of D, i.e. on the circle $x^2 + y^2 = 1$.

Using the method of Lagrange multipliers as in example 10.1, we obtain 4 critical points $(-1, 0), (1, 0), (0, -1), (0, 1)$.

Now, $f(0, 0) = 0, f(0, -1) = 2, f(0, -1) = 2, f(-1, 0) = 1, f(1, 0) = 1$.

Therefore, the absolute maximum value is 2 and the absolute minimum value is 0.

REMARK 8.5. *If we define a new function* $L(x, y; \lambda) = f(x, y) - \lambda g(x, y)$. *Then* $\frac{\partial L}{\partial x} = f_x - \lambda g_x$, $\frac{\partial L}{\partial y} = f_y - \lambda g_y$ *and* $\frac{\partial L}{\partial \lambda} = g$. *Therefore, the critical points of* L *correspond to the extreme points of the original problem. The* L *is a called a Lagrangian.*

The method of Lagrange Multipliers can be applied to the case of more than one constraints.

Consider the problem of maximizing or minimizing $f(x, y, z)$ subject to the constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$.

If f attains an extreme value at (x_0, y_0, z_0) , then

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0).$$

(For this linear combination to be valid, we need to assume

$\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$ and $\nabla h(x_0, y_0, z_0) \neq \mathbf{0}$ and that they are not parallel.)

Solving this vector equation and the constraint equations give all the possible extreme points. Equating components, these equations are equivalent to the following system:

$$\begin{cases} f_x(x,y,z) &= \lambda g_x(x,y,z) + \mu h_x(x,y,z) \\ f_y(x,y,z) &= \lambda g_y(x,y,z) + \mu h_y(x,y,z) \\ f_z(x,y,z) &= \lambda g_z(x,y,z) + \mu h_z(x,y,z) \\ g(x,y,z) &= 0 \\ h(x,y,z) &= 0 \end{cases}$$

In this case, we have two multipliers.

EXAMPLE 8.6. Find the maximum value of $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Solution. We wish to maximize $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z - 1$ and $h(x, y, z) = x^2 + y^2 - 1$.

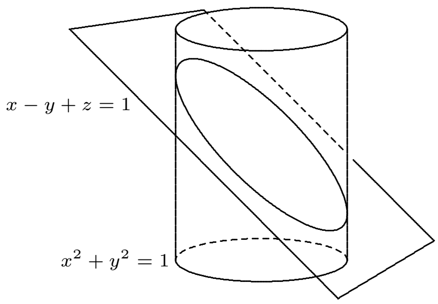


Figure 65

First we have $\nabla f = \langle 1,2,3\rangle$, $\nabla g = \langle 1,-1,1\rangle$ and $\nabla h = \langle 2x, 2y, 0\rangle$.

Thus we need to solve the system of equations:

$\nabla f = \lambda \nabla g + \mu \nabla h, \quad x-y+z=1, \quad x^2+y^2=1.$

That is

$$\begin{cases} 1 &= \lambda + 2x\mu & (1)\\ 2 &= -\lambda + 2y\mu & (2)\\ 3 &= \lambda + 0 & (3)\\ x-y+z &= 1 & (4)\\ x^2+y^2 &= 1 & (5)\end{cases}$$

From (3), $\lambda = 3$. Substituting this into (1) and (2), we get $x = -\frac{1}{\mu}$ and $y = \frac{5}{2\mu}$. Note that $\mu \neq 0$ by (2) and (3).

From (4),

we have $z = 1 - x + y = 1 + \frac{1}{\mu} + \frac{5}{2\mu} = 1 + \frac{7}{2\mu}$. (6)

Using (5), we have $(-\frac{1}{\mu})^2 + (\frac{5}{2\mu})^2 = 1$. From this, we can solve for μ, giving $\mu = \pm\frac{\sqrt{29}}{2}$. Thus $x = -\frac{2}{\sqrt{29}}$ or $x = \frac{2}{\sqrt{29}}$. The corresponding values of y are $\frac{5}{\sqrt{29}}, -\frac{5}{\sqrt{29}}$. Using (6), the corresponding values of z are $1 + \frac{7}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}}$.

Therefore, the two possible extreme values are at

$P_1 = (-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}})$ and

$P_2 = (\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}})$.

As $f(P_1) = 3 + \sqrt{29}$ and $f(P_2) = 3 - \sqrt{29}$, the maximum value is $3 + \sqrt{29}$ and the minimum value is $3 - \sqrt{29}$.

9. Multiple Integrals

9.1. Volume and Double Integrals. Let f be a function of two variables defined over a rectangle $R = [a, b] \times [c, d]$. We would like to define the double integral of f over R as the (algebraic) volume of the solid under the graph of $z = f(x, y)$ over R.

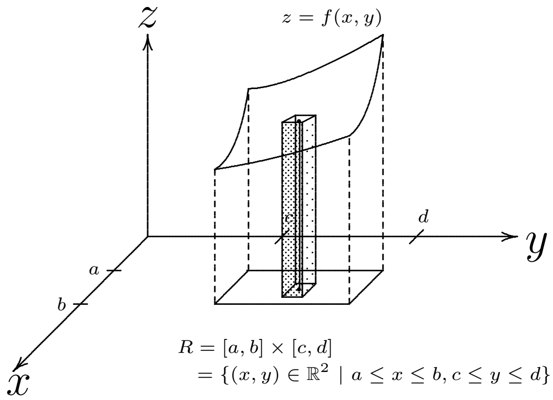


Figure 66

To do so, we first subdivide R into mn small rectangles R_{ij}, each having area $\triangle A$, where $i = 1, \cdots, m$ and $j = 1, \cdots, n$.

For each pair (i, j), pick an arbitrary point (x_{ij}^*, y_{ij}^*) inside R_{ij}.

We then use the value $f(x_{ij}^*, y_{ij}^*)$ as the height of a rectangular solid erected over R_{ij}.

Thus its volume is $f(x_{ij}^*, y_{ij}^*)\triangle A$.

The sum of the volume of all these small rectangular solids approximates the volume of the solid under the graph of $z = f(x, y)$ over R.

This sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \triangle A$$

is called a Riemann sum of f.

We define the double integral of f over R as the limit of the Riemann sum as m and n tend to infinity. In other words,

Definition 9.1. *The double integral of f over R is*

$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \triangle A$$

if this limit exists.

Theorem 9.2. *If $f(x, y)$ is continuous on R, then $\iint_R f(x, y)\, dA$ always exists.*

If $f(x, y) \geq 0$, then the volume V of the solid lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA.$$

9.2. Iterated Integrals. Let $f(x, y)$ be a function defined on $R = [a, b] \times [c, d]$.

We write $\displaystyle\int_c^d f(x, y)dy$ to mean that x is regarded as a constant and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$.

Therefore, $\int_c^d f(x,y)dy$ is a function of x and we can integrate it with respect to x from $x = a$ to $x = b$.

The resulting integral

$$\int_a^b \int_c^d f(x,y)dydx$$

is called an iterated integral. Similarly one can define the iterated integral $\int_c^d \int_a^b f(x,y)dxdy$.

Example 9.3. Evaluate the iterated integrals (a) $\int_0^3 \int_1^2 x^2y\,dydx$, (b) $\int_1^2 \int_0^3 x^2y\,dxdy$.

Solution. (a) $\int_0^3 \int_1^2 x^2y\,dydx = \int_0^3 \left[\frac{x^2y^2}{2}\right]_{y=1}^{y=2} dx = \int_0^3 \frac{3x^2}{2}\,dx$
$= \left[\frac{x^3}{2}\right]_{x=0}^{x=3} = 27/2.$

(b) $\int_1^2 \int_0^3 x^2y\,dxdy = \int_1^2 \left[\frac{x^3y}{3}\right]_{x=0}^{x=3} dy = \int_1^2 9y\,dy = \left[\frac{9y^2}{2}\right]_{y=1}^{y=2} =$
$27/2.$

Consider a positive function $f(x,y)$ defined on a rectangle $R = [a,b] \times [c,d]$.

Let V be the volume of the solid under the graph of f over R.

We may compute V by means of either one of the iterated integrals:

$$\int_a^b \int_c^d f(x,y)dydx, \quad \text{or} \quad \int_c^d \int_a^b f(x,y)dxdy.$$

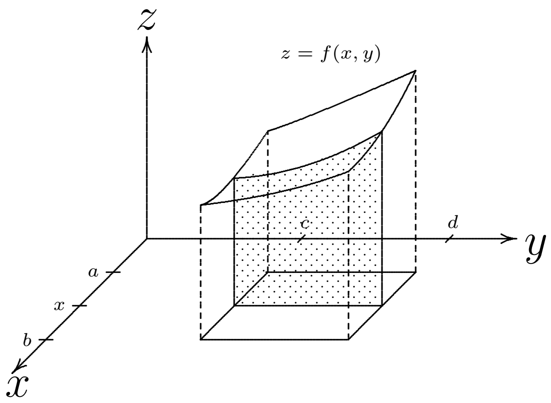


Figure 67 $\int_a^b \int_c^d f(x,y)dydx$

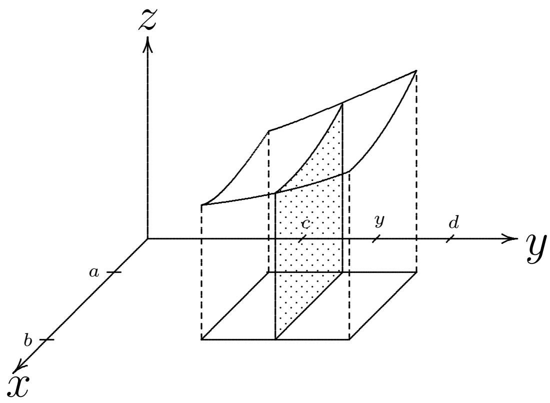


Figure 68 $\int_c^d \int_a^b f(x,y)dxdy$

THEOREM 9.4. *(Fubini's Theorem) If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$, then*

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy.$$

More generally, this is true if f is bounded on R, f is discontinuous only at a finite number of smooth curves, and the iterated integrals exist.

Furthermore, the theorem is valid for a general closed and bounded region as discussed in the subsequent sections.

EXAMPLE 9.5. Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$, $y = 2$, and the 3 coordinate planes. See figure 69.

Solution:

$$\text{Volume} = \iint_R 16 - x^2 - 2y^2 dA = \int_0^2 \int_0^2 16 - x^2 - 2y^2 dxdy = 48.$$

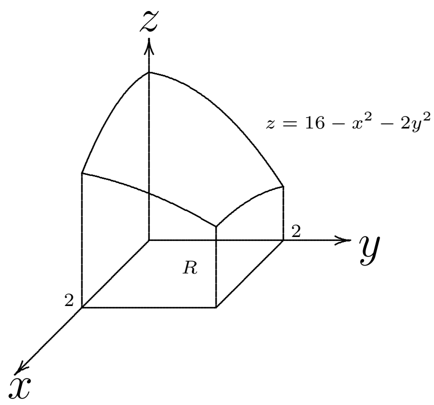


Figure 69 $\iint_R 16 - x^2 - 2y^2 dA$

EXAMPLE 9.6. Let $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$. Evaluate $\displaystyle\iint_R \sin x \cos y \, dA$.

Solution:

$$\iint_R \sin x \cos y \, dA = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin x \cos y \, dydx = \int_0^{\frac{\pi}{2}} \sin x \, dx \int_0^{\frac{\pi}{2}} \cos y \, dy$$
$$= 1 \times 1 = 1.$$

In general, if $f(x, y) = g(x)h(y)$, then

$$\iint_R g(x)h(y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right),$$

where $R = [a, b] \times [c, d]$.

9.3. Doubles Integral over General Regions. Let $f(x, y)$ be a continuous function defined on a closed and bounded region D in $\mathbb{R}^2$.

The double integral $\displaystyle\iint_D f(x, y)\,dA$ can be defined similarly as the limit of a Riemann sum.

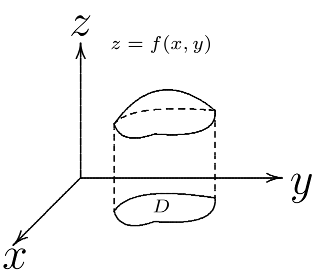
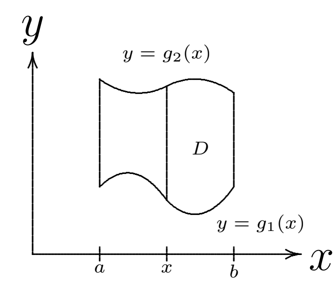


Figure 70 Double integral over a more general region

In particular, if D is one of the following two types of regions in $\mathbb{R}^2$. We may set up the corresponding iterated integral to compute it.

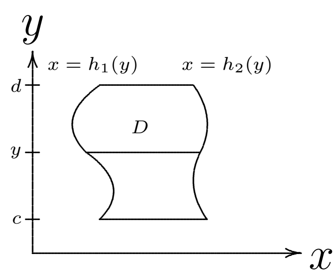
If D is the region bounded by two curves $y = g_1(x)$ and $y = g_2(x)$ from $x = a$ to $x = b$, where $g_2(x) \geq g_1(x)$ for all $x \in [a, b]$, we called it a type 1 region. In this case, the double integral of f over D can be expressed as an iterated integral as given in figure 71.

Similarly, if D is the region bounded by two curves $x = h_1(y)$ and $x = h_2(y)$ from $y = c$ to $y = d$, where $h_2(y) \geq h_1(y)$ for all $y \in [c, d]$, we called it a type 2 region. In this case, the double integral of f over D can be expressed as an iterated integral as given in figure 72.



$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dydx$$

Figure 71 Type 1 region



$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dxdy$$

Figure 72 Type 2 region

EXAMPLE 9.7. Evaluate $\displaystyle\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution. The region D is a type 1 region as shown in figure 73.

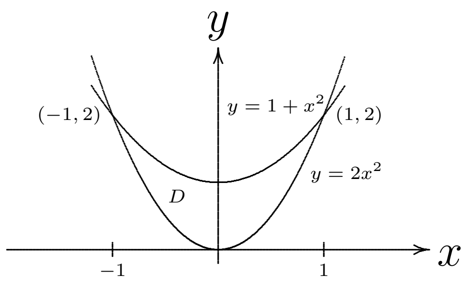


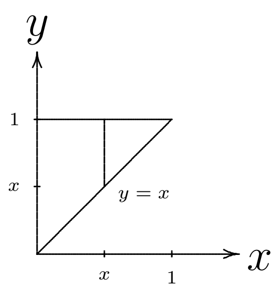
Figure 73

$$\iint_D (x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)\,dydx = \int_{-1}^{1}\left[xy+y^2\right]_{y=2x^2}^{y=1+x^2} dx$$
$$= \int_{-1}^{1}(-3x^4-x^3+2x^2+x+1)\,dx = 32/15.$$

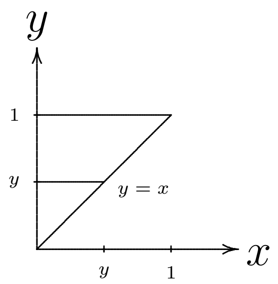
EXERCISE 9.8. Evaluate $\displaystyle\iint_D xy\,dA$, where D is the region bounded by the line $y=x-1$ and the parabola $y^2=2x+6$.[Answer: 36]

EXAMPLE 9.9. Evaluate the iterated integral $I=\displaystyle\int_0^1\int_x^1 \sin(y^2)\,dydx$ by interchanging the order of integration.

Solution. The region of integration is the triangular region bounded by the lines $y=x, x=0$ and $y=1$.



$$I=\int_0^1\int_x^1 \sin(y^2)\,dydx$$



$$I = \int_0^1 \int_0^y \sin(y^2)\, dxdy$$

Figure 74

$\displaystyle\int_0^1 \int_0^y \sin(y^2)\, dxdy = \int_0^1 \left[x \sin(y^2)\right]_{x=0}^{x=y} dy = \int_0^1 y \sin(y^2)\, dy = \left[-\frac{1}{2}\cos(y^2)\right]_0^1 = \frac{1}{2}(1 - \cos 1).$

Example 9.10. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the plane $z = 0$ and $z = y$. See figure 75.

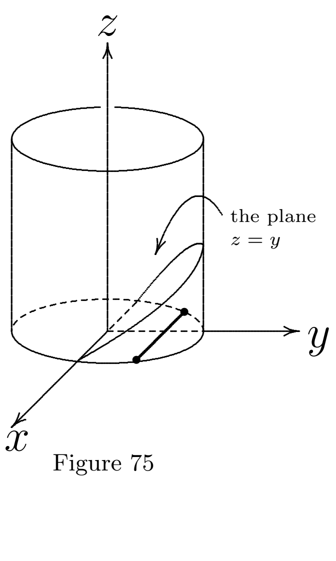


Figure 75

Solution. Since the plane $z = y$ is the top face of the solid, we may use the function defining this plane as the height function of this solid.

The function whose graph is the plane $z = y$ is simply $f(x, y) = y$.

Therefore, the volume of the solid can be computed by integrating this f over the bottom face of the solid which is the semi-circular disk $D = \{(x, y) \mid x^2 + y^2 \leq 1, y \geq 0\}$.

$$\text{Volume} = \iint_D f(x,y)\,dA = \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\,dxdy$$

$$= \int_0^1 [xy]_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}}\,dy = \int_0^1 2y\sqrt{1-y^2}\,dy = \left[-\frac{2}{3}(1-y^2)^{\frac{3}{2}}\right]_0^1 = 2/3.$$

Properties of Double Integrals

(1) $\displaystyle\iint_D (f(x,y)+g(x,y))\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA.$

(2) $\displaystyle\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$, where c is a constant.

(3) If $f(x,y) \geq g(x,y)$ for all $(x,y) \in D$, then

$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA.$$

(4) $\displaystyle\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$, where $D = D_1 \cup D_2$ and D_1, D_2 do not overlap except perhaps on their boundary.

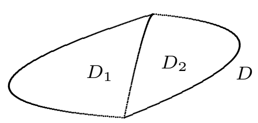


Figure 76

(5) $\displaystyle\iint_D dA \left(= \iint_D 1\, dA\right) = A(D)$, the area of D.

(6) If $m \leq f(x,y) \leq M$ for all $(x,y) \in D$,

$$\text{then } mA(D) \leq \iint_D f(x,y)\, dA \leq MA(D).$$

9.4. Double Integrals in Polar Coordinates. Consider a point (r,θ) on the plane in polar coordinates as in figure 77.

An increment dr in r and $d\theta$ in θ give rise to an area $dA = rd\theta dr$.

This is the area differential in polar coordinates.

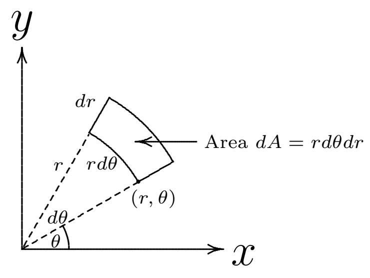


Figure 77

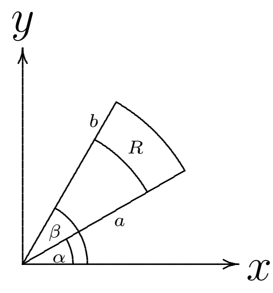


Figure 78 A polar rectangle

Let f be a continuous function defined on a polar rectangle

$$R = \{(r,\theta) \mid 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\},$$

where $0 \leq \beta - \alpha \leq 2\pi$.

The double integral of f over R can be expressed in polar coordinates as follow:

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r dr d\theta.$$

EXERCISE 9.11. Evaluate $\iint_R (3x+4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$. See figure 79.

[Answer: $15\pi/2$]

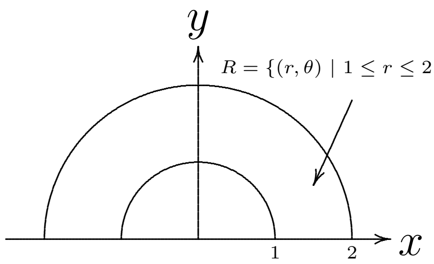


Figure 79 A semi-circular annulus

In general, if f is continuous on a polar region of the form

$$D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\},$$

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta.$$

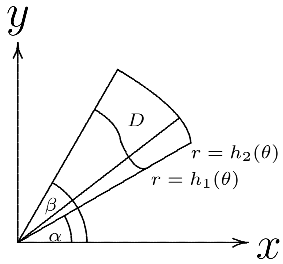


Figure 80 A general polar region

EXAMPLE 9.12. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Solution. The cylinder $x^2 + y^2 = 2x$ lies over the circular disk D which can be described in polar coordinates as

$$D = \{(r, \theta) \mid -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 2\cos\theta\}.$$

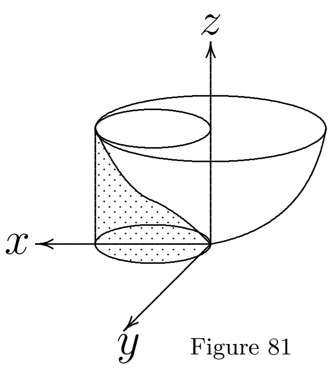


Figure 81

The height of the solid is the z-value of the paraboloid. Hence the volume V of the solid is

$$V = \iint_D (x^2 + y^2)\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 r dr d\theta = 3\pi/2.$$

EXERCISE 9.13. Show that the volume of the solid region bounded by the three cylinders $x^2 + y^2 = 1$, $y^2 + z^2 = 1$ and $x^2 + z^2 = 1$ is $16 - 8\sqrt{2}$.

10. Triple Integrals

Let $f : B \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a continuous function, where $B = [a, b] \times [c, d] \times [r, s]$ is a rectangular solid.

Divide $[a, b]$, $[c, d]$ and $[r, s]$ into l, m and n equal subintervals, respectively.

Thus B is divided into $l \times m \times n$ small rectangular solids.

Label each small rectangular solid by C_{ijk}, where $1 \le i \le l$, $1 \le j \le m$ and $1 \le k \le n$. Inside each such C_{ijk}, pick a point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$.

Denote the volume of C_{ijk} by $\triangle V$.

Then we may form the Riemann sum:

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\triangle V.$$

The triple integral of f over B is defined to

$$\iiint_B f(x, y, z)\, dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\triangle V.$$

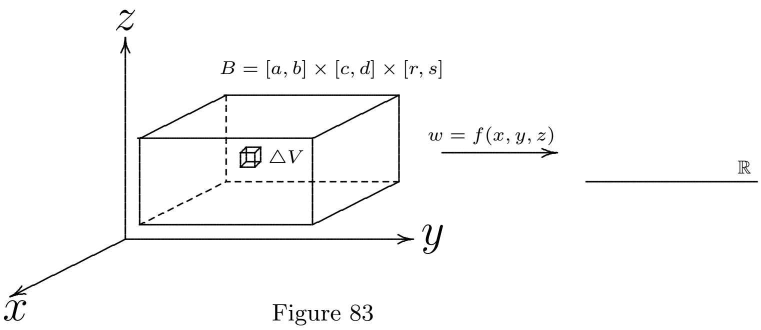


Figure 83

The limit exists if f is continuous.

The triple integral of a continuous function defined on a more general closed and bounded solid in $\mathbb{R}^3$ can be defined in a similar way.

THEOREM 10.1. *(Fubini's Theorem for triple integrals) If $f(x, y, z)$ is continuous on $B = [a, b] \times [c, d] \times [r, s]$, then*

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z) dx dy dz$$
$$= \int_r^s \int_a^b \int_c^d f(x, y, z) dy dx dz = etc.$$

(Note that there are 3!=6 such iterated integrals involved and they are all equal.) Furthermore, the theorem is valid for a general closed and bounded solid.

EXAMPLE 10.2. Evaluate $\displaystyle\iiint_B xyz^2\, dV$, where $B = [0, 1] \times [-1, 2] \times [0, 3]$.

Solution.

$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 dxdydz = 27/4.$$

10.1. Triple Integrals over a General Bounded Region. For each of the following three types of solid regions, we may write down the triple integral as an iterated integral of a double integral and a simple integral.

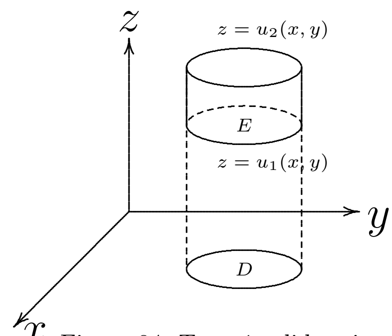


Figure 84 Type 1 solid region

$$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dA$$

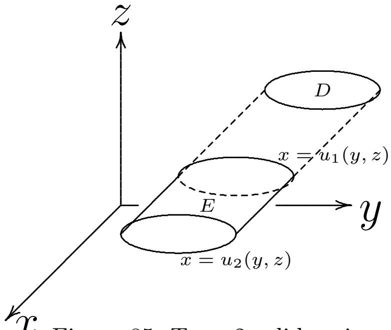


Figure 85 Type 2 solid region

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA$$

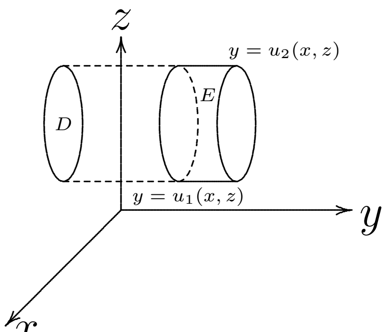


Figure 86 Type 3 solid region

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA$$

Note that if $f(x,y,z) = 1$ for all $(x,y,z) \in E$, then $\displaystyle\iiint_E 1\,dV$ is just the volume of E.

Example 10.3. Evaluate $\displaystyle\iiint_E \sqrt{x^2+z^2}\,dV$, where E is the solid region bounded by the paraboloid $y = x^2+z^2$ and the plane $y = 4$. See figure 87.

Solution. E is a type 3 solid region whose projection onto the xz-plane is

$$D = \{(x,z) \mid x^2+z^2 \le 4\} = \{(r,\theta) \mid 0 \le \theta \le 2\pi, 0 \le r \le 2\}.$$

$$\begin{aligned}
\iiint_E \sqrt{x^2+z^2}\,dV &= \iint_D \int_{x^2+z^2}^4 \sqrt{x^2+z^2}dy\,dA \\
&= \iint_D \left[y\sqrt{x^2+z^2}\right]_{x^2+z^2}^4 dA \\
&= \iint_D \sqrt{x^2+z^2}(4-x^2-z^2)\,dA \\
&= \int_0^{2\pi}\int_0^2 r(4-r^2)r\,dr d\theta \qquad \text{(polar coords)} \\
&= 128\pi/5.
\end{aligned}$$

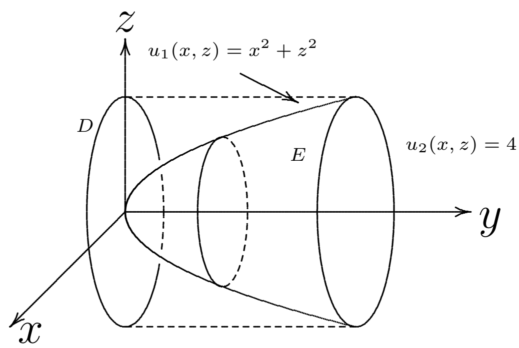


Figure 87 A type 3 solid

EXERCISE 10.4. Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x+y+z = 1$.

[Answer: 1/24]

EXERCISE 10.5. Find the volume of the solid tetrahedron bounded by the planes $x = 2y, x = 0, z = 0$ and $x + 2y + z = 2$.

[Answer: 1/3]

10.2. Triple Integrals in Cylindrical Coordinates. Consider a rectangle in cylindrical coordinates as in figure 88:

$$E = \{(r, \theta, z) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta), u_1(r, \theta) \leq z \leq u_2(r, \theta)\}.$$

The triple integral of $f(x, y, z)$ over E can be expressed as:

$$\begin{aligned}\iiint_E f(x, y, z)\, dV &= \iint_D \left[\int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, dz\right] dA \\ &= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r dz dr d\theta.\end{aligned}$$

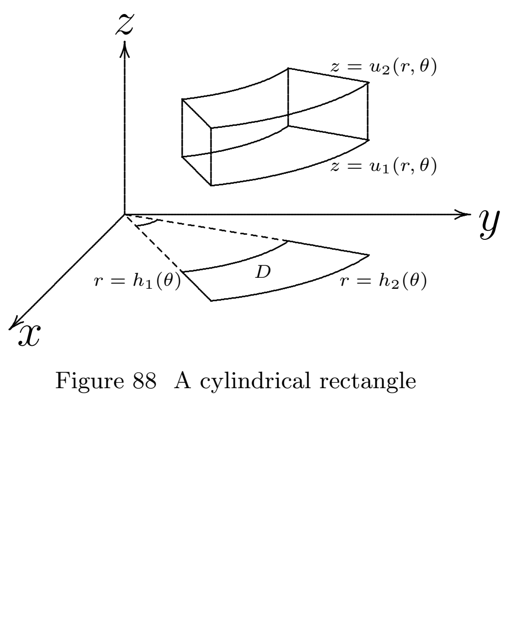


Figure 88 A cylindrical rectangle

EXAMPLE 10.6. Let E be the solid within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. Evaluate $\displaystyle\iiint_E \sqrt{x^2+y^2}\,dV$.

Solution. The solid can be described in cylindrical coordinates as:

$$E = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, 1-r^2 \le z \le 4\}.$$

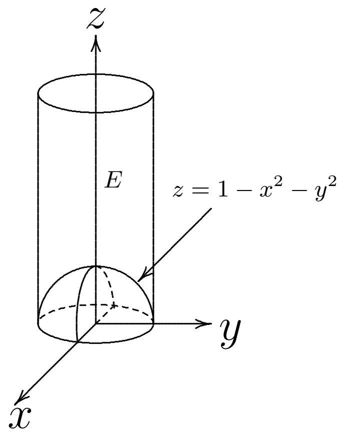


Figure 89

Thus,

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 r\,r\,dz dr d\theta = 12\pi/5.$$

EXERCISE 10.7. Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dzdydx.$

[Answer=$16\pi/5$]

10.3. Triple Integrals in Spherical Coordinates. Let us consider the volume element in spherical coordinates.

To do so, take any point $P(\rho, \theta, \phi)$. Make an increment in each of the coordinates. See figure 90.

Let's calculate the volume of the solid arising from these increments.

The projection of OP onto the xy-plane has length $\rho \sin \phi$.

Thus the thickness of this volume element is $\rho \sin \phi d\theta$.

It opens up a sector of width of $\rho d\phi$.

Thus, the volume is

$$dV = \rho^2 \sin \phi d\rho d\theta d\phi.$$

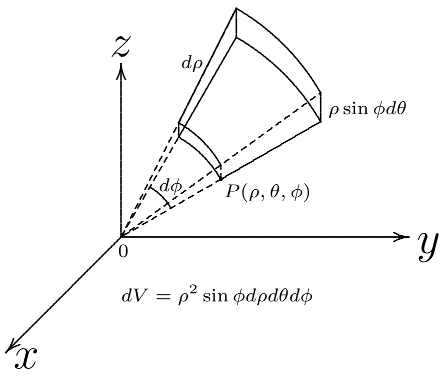


Figure 90 The volume element in spherical coordinate

Now consider a spherical rectangle

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\},$$

where $a \geq 0$, $\beta - \alpha \leq 2\pi$, $d - c \leq \pi$.

The triple integral of f over E can be expressed as follow:

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi d\rho d\theta d\phi.$$

EXAMPLE 10.8. Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dV$, where B is the unit ball $\{(x,y,z) \mid x^2+y^2+z^2 \leq 1\}$.

Solution. Using spherical coordinates, we have

$$\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dV = \int_0^\pi \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{\frac{3}{2}}} \rho^2 \sin\phi d\rho d\theta d\phi = \frac{4}{3}\pi(e-1).$$

Note that the corresponding triple integral formulated in Cartesian coordinates is very hard to evaluate.

EXERCISE 10.9. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$. See figure 91.

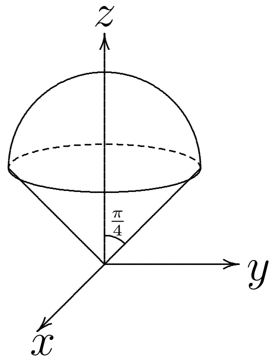


Figure 91

[Answer: $\pi/8$]

11. Change of Variables in Multiple Integrals

Let T be a transformation from the uv-plane to the xy-plane.

That is $(x, y) = T(u, v)$ or $x = x(u, v), y = y(u, v)$.

We assume that T is a C^1-transformation, i.e. both $x(u, v)$ and $y(u, v)$ have continuous partial derivatives with respect to u and v.

We also assume T is an injective function so that its inverse T^{-1} exists (from the range of T back to the domain of T).

Thus T maps a region S in the uv-plane bijectively onto a region R in the xy-plane.

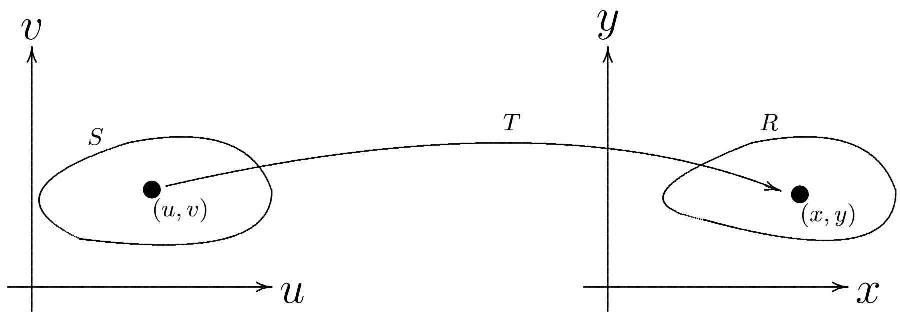


Figure 92

For example if T is the transformation to polar coordinates $T(r, \theta) = (r\cos\theta, r\sin\theta)$, then T maps a rectangle $[r_1, r_2] \times [\theta_1, \theta_2]$ in the $r\theta$-plane to a polar rectangle in the xy-plane.

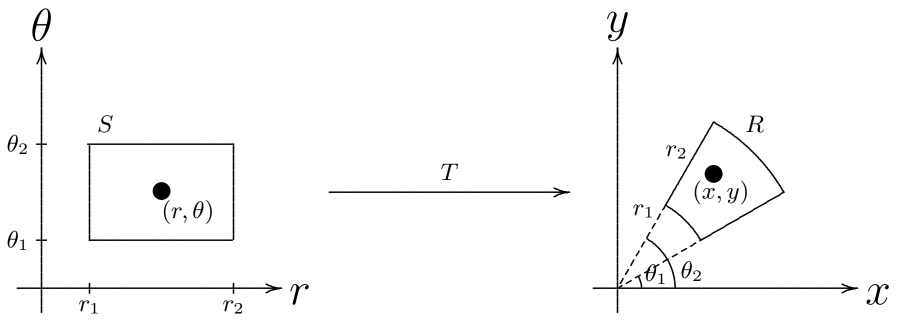


Figure 93 Polar coordinates

EXAMPLE 11.1. Consider $T(u, v) = (u^2 - v^2, 2uv)$. Find the image of the square $S = \{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$.

Solution. First let's find out the boundary of the image. Label the edges of the square S by S_1, S_2, S_3 and S_4 as shown in figure 94.

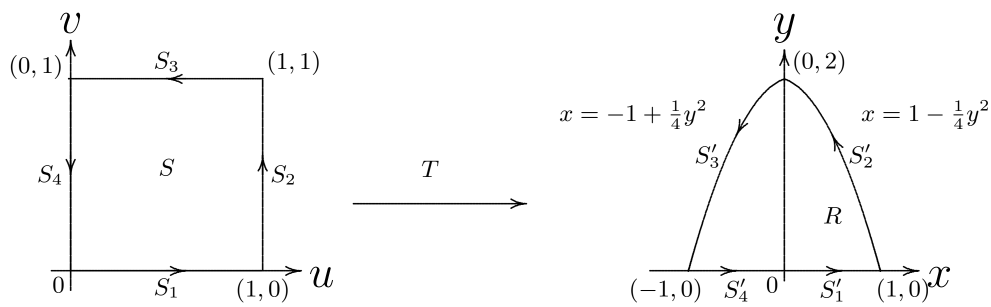


Figure 94

S_1 is described by $v = 0, 0 \le u \le 1$.

Thus the image S_1' in the xy-plane is given by $x = u^2 - 0^2 = u^2, y = 2u(0) = 0$.

That is $x = u^2$ for $0 \le u \le 1$ and $y = 0$.

Therefore, S_1' is described by $y = 0, 0 \le x \le 1$, which is just the line segment on the x-axis from $(0, 0)$ to $(1, 0)$.

Next S_2 is described by $u = 1, 0 \leq v \leq 1$.

Thus the image S_2' in the xy-plane is given by $x = 1 - v^2, y = 2v$.

Eliminating v, we obtain $x = 1 - \frac{1}{4}y^2$. As $0 \leq v \leq 1$, we have $0 \leq y \leq 2$.

Therefore, S_2' is described by $x = 1 - \frac{1}{4}y^2$ for $0 \leq y \leq 2$.

Similarly, we find out S_3' as $x = -1 + \frac{1}{4}y^2$ for y from 2 to 0 and S_4' as $y = 0$ for x from -1 to 0.

The boundary of the image of S encloses a region R.

We are going to show that T maps S bijectively onto R. We leave it the reader to verify that T is a bijective function for $u, v \geq 0$.

As we traverse the boundary of S in the counterclockwise direction, the above calculation shows that the image of the boundary of S also traverses in the counterclockwise direction.

This means that T preserves orientation. In other words, points on the left hand side of the boundary of S go under T to points on the left hand side of the boundary of R.

Therefore, T maps S onto R.

Another easy way to confirm this is to pick a point P, say $(1/2, 1/2)$ inside S and check that $T(P)$ is inside R.

Then the region S must be mapped by T into R.

Before we derive the formula for change of variables in a multiple integral, let's review the formula for functions of 1 variable.

Let the continuous function $f(x)$ be integrated over the interval $[a, b]$.

Suppose we make a substitution $x = g(u)$ so that $a = g(c)$ and $b = g(d)$. Thus we obtain:

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du.$$

Here the formula is valid provided g is differentiable and $g'(u) \neq 0$, except possibly at a finite number of points.

The function g is also required to be bijective so that g^{-1} exists.

Observe that c may not be less than d. More precisely, if $g'(u) > 0$ for all u between c and d, then g and hence g^{-1} is increasing.

Thus g^{-1} preserves orientation or ordering. This means that $c < d$ and $[c, d]$ is an interval.

On the other hand, if $g'(u) < 0$ for all u between c and d, then g and hence g^{-1} is decreasing. Thus g^{-1} reverses orientation or ordering. This means that $c > d$ and it does not make sense to write $[c, d]$ though we could still integrate from c to d.

In this case, the formula can be rewritten as:

$$\int_a^b f(x)\,dx = \int_d^c f(g(u))(-g'(u))\,du,$$

so as to keep the lower limit of integration smaller than the upper limit.

Therefore, if the interval $[c, d]$, $(c < d)$ is mapped onto the interval $[a, b]$ under $x = g(u)$, then the formula for change of variables can be stated as:

$$\int_{[a,b]} f(x)\, dx = \int_{[c,d]} f(g(u))|g'(u)|\, du.$$

It is this formula that we are going to generalize.

How does a change of variables affect a double integral?

Let T be a transformation mapping a point (u_0, v_0) to a point (x_0, y_0).

Consider a small increment du and dv at the point (u_0, v_0) along the u and v directions respectively.

These increments generate a rectangle of area $dudv$ whose image under T is a curved parallelogram in the xy-plane.

The area of this curved parallelogram up to the first order approximation is given by the area of the parallelogram generated by the two tangent vectors $\mathbf{a}du$ and $\mathbf{b}dv$ at (x_0, y_0), where $\mathbf{a}$ is the derivative of the curve $T(u, v_0)$ at (u_0, v_0), and $\mathbf{b}$ is the derivative of curve $T(u_0, v)$ at (u_0, v_0).

That is

$$\mathbf{a} = \left.\frac{dT(u, v_0)}{du}\right|_{u=u_0} = \langle \frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0)\rangle,$$

$$\mathbf{b} = \left.\frac{dT(u_0, v)}{dv}\right|_{v=v_0} = \langle \frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0) \rangle.$$

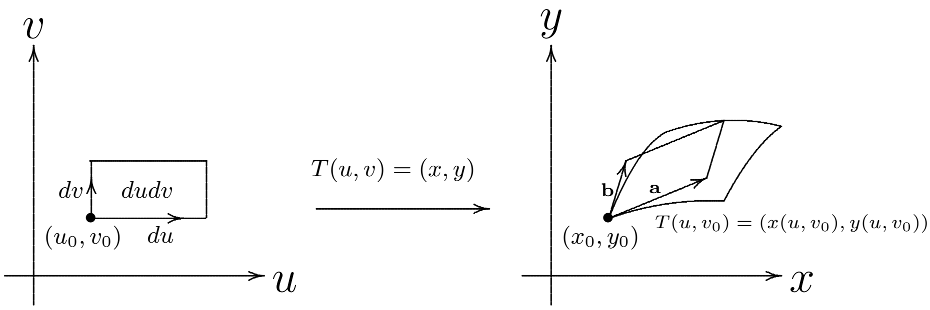


Figure 95

Therefore, the area element dA in the xy-plane is $dudv$ times the magnitude of

$$\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0 \rangle \times \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0 \rangle = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \mathbf{k}.$$

That is $dA = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| dudv.$

DEFINITION 11.2. *The Jacobian of the transformation T given by $x = x(u, v), y = y(u, v)$ is*

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Therefore,

$$dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv.$$

THEOREM 11.3. *Let $T(u, u)$ be a bijective C^1-transformation whose Jacobian is nonzero except possibly at a finite number of points. Suppose T maps a region S in the uv-plane onto a region R of the xy-plane. Suppose f is continuous on R.*

Then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv.$$

EXAMPLE 11.0.1. Find the Jacobian of the transformation from polar coordinates to Cartesian coordinates.

Solution. $x = r\cos\theta$ and $y = r\sin\theta$. Thus,

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Therefore,

$$\iint_R f(x, y)\, dA = \iint_S f(r\cos\theta, r\sin\theta) r dr d\theta.$$

EXAMPLE 11.0.2. Use the change of variables $x = u^2 - v^2, y = 2uv$ to evaluate the integral

$$\iint_R y\, dA,$$

where R is the region bounded by the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, and the x-axis.

Solution. The transformation is the one discussed in example 14.1.

First, let's compute the Jacobian of T.

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2.$$

Therefore,

$$\iint_R y\, dA = \iint_S (2uv) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv = \int_0^1 \int_0^1 (2uv) 4(u^2 + v^2) dudv = 2.$$

For the case of triple integrals, we have a completely analogous formula for change of variables.

Suppose

$T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$

is a C^1-transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ mapping a solid region S bijective onto the solid region R.

First, the Jacobian of T is defined to be

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

If $f(x, y, z)$ is a continuous function defined on R. Then

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right|$$

EXERCISE 11.4. Show that the Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$ of the transformation from spherical coordinates to Cartesian coordinates is $\rho^2 \sin\phi$.

12. Vector Fields

DEFINITION 12.1. *Let $D \subseteq \mathbb{R}^2$. A vector field on D is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two dimensional vector $\mathbf{F}(x, y)$.*

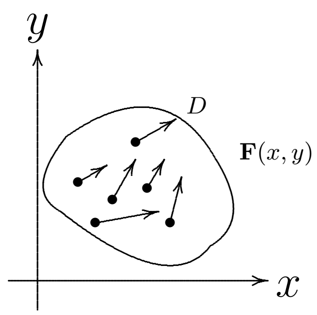


Figure 97

We may write $\mathbf{F}(x, y)$ in terms of its component functions.

That is

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y)\rangle,$$

or simply

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j}.$$

DEFINITION 12.2. *Let $E \subseteq \mathbb{R}^3$. A vector field on E is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three dimensional vector $\mathbf{F}(x, y, z)$.*

That is

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

$$= \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle.$$

EXAMPLE 12.3. A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$. Show that $\mathbf{F}(x,y)$ is always perpendicular to the position vector of the point (x,y).

Solution.

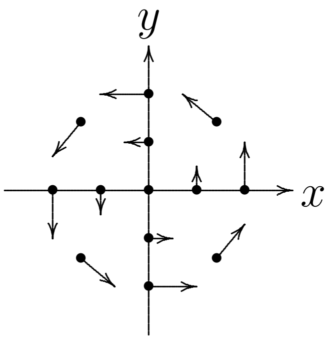


Figure 98

Figure 98 shows the vector field $\mathbf{F}$.

Note that $\langle x,y\rangle \cdot \mathbf{F}(x,y) = \langle x,y\rangle \cdot \langle -y,x\rangle = 0$.

Also $|\mathbf{F}(x,y)| = \sqrt{x^2+y^2}$.

The vector assigned by $\mathbf{F}$ to the origin is the zero vector.

DEFINITION 12.4. *A vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ defined on a domain D in $\mathbb{R}^2$ is continuous on D if $P(x,y)$ and $Q(x,y)$ are continuous functions on D.*

A vector field $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ defined on a domain D in $\mathbb{R}^3$ is continuous on D if $P(x,y,z)$, $Q(x,y,z)$ and $R(x,y,z)$ are continuous functions on D.

For example, $\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is a continuous vector field on $\mathbb{R}^2$.

12.1. Gradient Fields.

DEFINITION 12.5. *If $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a differentiable function, then ∇f is a vector field on $\mathbb{R}^2$ and it is called the gradient vector field of f.*

Similarly, if $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a differentiable function, then ∇f is a vector field on $\mathbb{R}^3$ and it is called the gradient vector field of f.

EXAMPLE 12.6. Find the gradient vector field of $f(x, y) = x^2y - y^3$.

Solution. $\nabla f(x, y) = 2xy\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$. The gradient field and the contours of f are drawn on the diagram in figure 99.

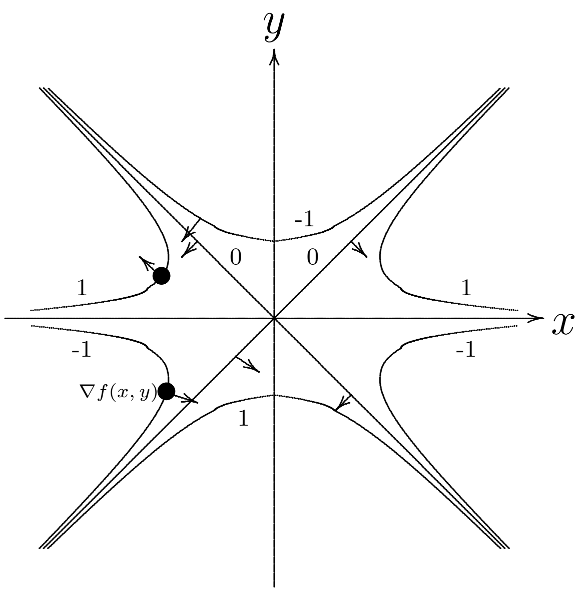


Figure 99

Notice that the gradient vectors are perpendicular to the level curves as is proved in 8.6 using the chain rule.

EXERCISE 12.7. Find the gradient vector field ∇f of $f(x,y) = \sqrt{x^2+y^2}$. Sketch ∇f.

DEFINITION 12.8. *A vector field* $\mathbf{F}$ *is called a conservative vector field if it is the gradient of some scalar function, that is there exists a differentiable function* f *such that* $\mathbf{F} = \nabla f$*. In this situation,* f *is called a potential function for* $\mathbf{F}$*.*

For example, $\mathbf{F}(x,y) = 2xy\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is conservative since it has a potential function $f(x,y) = x^2y - y^3$.

Not all vector fields are conservative, but such fields do arise frequently in physics. For instance, the gravitational field given by

$$\mathbf{F} = \frac{-mMGx}{(x^2+y^2+z^2)^{\frac{3}{2}}}\mathbf{i} + \frac{-mMGy}{(x^2+y^2+z^2)^{\frac{3}{2}}}\mathbf{j} + \frac{-mMGz}{(x^2+y^2+z^2)^{\frac{3}{2}}}\mathbf{k}$$

is conservative because it is the gradient of the gravitational potential function

$$f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}},$$

where G is the gravitational constant, M and m are the masses of two objects.

Think of the mass M at the origin that creates the field and f is the potential energy attained by the mass m situated at (x,y,z).

In later sections, we will derive conditions when a vector field is conservative.

13. Line Integrals

Consider a plane curve $C : x = x(t), y = y(t)$ or $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $a \leq t \leq b$.

We assume C is a smooth curve, meaning that $\mathbf{r}'(t) \neq \mathbf{0}$, and $\mathbf{r}'(t)$ is continuous for all t.

Let $f(x, y)$ be a continuous function defined in a domain containing C.

To define the line integral of f along C, we subdivide arc from $\mathbf{r}(a)$ to $\mathbf{r}(b)$ into n small arcs of (equal) length $\triangle s_i$, $i = 1, \cdots, n$.

Pick an arbitrary point (x_i^*, y_j^*) inside the ith small arc and form the Riemann sum $\sum_{i=1}^n f(x_i^*, y_j^*)\triangle s_i$.

The line integral of f along C is the limit of this Riemann sum.

Definition 13.1. *The integral of f along C is defined to be*

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*, y_j^*)\triangle s_i.$$

We can pull back the integral to an integral in terms of t using the parametrization $\mathbf{r}$.

Recall that the arc length differential is given by $ds = |\mathbf{r}'(t)||dt|$.

Thus,

$$\int_C f(x,y)\,ds = \int_a^b f(\mathbf{r})|\mathbf{r}'(t)|dt = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Note that since $a \le t \le b$, $|dt| = dt$.

EXAMPLE 13.2. Evaluate $\int_C (2 + x^2y)ds$, where C is the upper half of the unit circle traversed in the counterclockwise sense.

Solution. We may parametrize C by $x = \cos t, y = \sin t, t \in [0, \pi]$.

Thus

$$\int_C (2 + x^2y)ds = \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t}\,dt$$

$$= \int_0^\pi (2 + \cos^2 t \sin t)dt = \left[2t - \frac{1}{3}\cos^3 t\right]_0^\pi = 2\pi + \tfrac{2}{3}.$$

DEFINITION 13.3. *A piecewise smooth curve C is a union of a finite number of smooth curves $C_1, C_2, \cdots, C_n$, where the initial point of C_{i+1} is the terminal point of C_i, $i = 0, \cdots, n-1$. In that case, we write $C = C_1 + C_2 + \cdots + C_n$.*

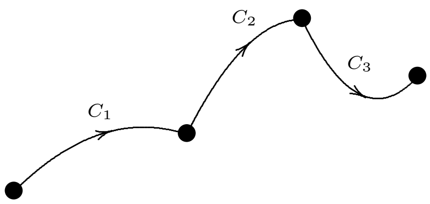


Figure 100 $C = C_1 + C_2 + C_3$

Then the line integral f along C is defined to be

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds.$$

EXERCISE 13.4. Evaluate $\displaystyle\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

[Answer: $\frac{1}{6}(5\sqrt{5} - 11)$]

Next we define two more line integrals:

DEFINITION 13.5. *Given a smooth curve C: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $a \le t \le b$.*

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt,$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))y'(t)\,dt,$$

are called the line integrals of f along C with respect to x and y.

Sometimes, we refer to the original line integral of f along C, namely,

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt,$$

as the line integral of f along C with respect to arc length.

We make the following abbreviation:

$$\int_C P(x,y)dx + Q(x,y)dy = \int_C P(x,y)dx + \int_C Q(x,y)dy.$$

EXAMPLE 13.6. Evaluate $\displaystyle\int_C y^2dx + xdy$, where

(a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$,

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Solution. (a) $C_1 : x = 5t - 5, y = 5t - 3$, $0 \le t \le 1$.

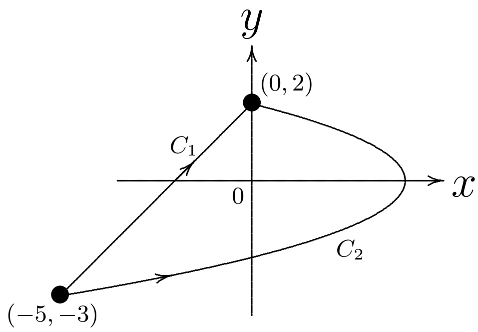


Figure 101

Thus,

$$\int_{C_1} y^2dx + xdy = \int_0^1 (5t-3)^2 5dt + \int_0^1 (5t-5)5dt = -5/6.$$

(b) $C_2 : x = 4 - t^2, y = t$, $-3 \le t \le 2$.

Thus

$$\int_{C_2} y^2dx + xdy = \int_{-3}^{2} t^2(-2t)dt + \int_{-3}^{2} (4-t^2)dt = 245/6.$$

A parametrization $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $t \in [a,b]$ determines an orientation of C. In other words, C is an oriented curve. Note that if we reverse the orientation of C, we obtain a curve with the opposite orientation of C. We denote this oriented curve by $-C$.

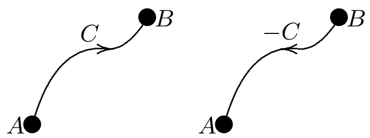


Figure 102

For example the upper semicircle C in the xy-plane centered at the origin with radius 1 joins the point $(1,0)$ to $(-1,0)$.

It has a vector equation in the form $\mathbf{r}_1(t) = \langle \cos(\pi t), \sin(\pi t)\rangle$, $t \in [0,1]$.

Then $-C$ can be parametrized by $\mathbf{r}_2(t) = \langle \cos(\pi(1-t)), \sin(\pi(1-t))\rangle$, $t \in [0,1]$ and $-C$ joins $(-1,0)$ to $(1,0)$.

Note that because the sign of $x'(t)$ and $y'(t)$ reverses in $-C$, we have

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx$$

and

$$\int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy.$$

But because the arclength differential is always positive,

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds.$$

For line integral of a function $f(x,y,z)$ along a parametrized space curve C, we have the similar definitions:

$$\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r})|\mathbf{r}'(t)|dt$$

$$= \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt,$$

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))x'(t)\,dt,$$

$$\int_C f(x,y,z)\,dy = \int_a^b f(x(t),y(t),z(t))y'(t)\,dt,$$

$$\int_C f(x,y,z)\,dz = \int_a^b f(x(t),y(t),z(t))z'(t)\,dt.$$

EXAMPLE 13.7. Evaluate $\displaystyle\int_C y\sin z\,ds$, where C is the circular helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, $t \in [0, 2\pi]$.

Solution.

$$\int_C y \sin z \, ds = \int_0^{2\pi} (\sin t)(\sin t) \sqrt{\sin^2 t + \cos^2 t + 1} \, dt$$

$$= \frac{\sqrt{2}}{2} \int_0^{2\pi} (1 - \cos(2t)) \, dt$$

$$= \frac{\sqrt{2}}{2} \left[t - \frac{1}{2} \sin(2t) \right]_0^{2\pi} = \sqrt{2}\pi.$$

14. Line Integrals of Vector Fields

DEFINITION 14.1. *Let* $\mathbf{F}$ *be a continuous vector field defined on a domain containing a smooth curve* C *given by a vector function* $\mathbf{r}(t)$, $t \in [a, b]$. *The line integral of* $\mathbf{F}$ *along the curve* C *is*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

Note that $\displaystyle\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$ as $\mathbf{r}'(t)$ changes sign in $-C$.

EXAMPLE 14.2. Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle xy, yz, zx \rangle$, and C is the curve $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $t \in [0, 1]$.

Solution. First $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$.

Thus

$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \langle t \cdot t^2, t^2 \cdot t^3, t^3 \cdot t \rangle \cdot \langle 1, 2t, 3t^2 \rangle = t^3 + 5t^6.$

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^1 t^3 + 5t^6\, dt = 27/28.$$

Let's rewrite $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ in the component form.

Suppose

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k},$$

and

$$C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \qquad t \in [a, b].$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_a^b \langle P(\mathbf{r}(t)), Q(\mathbf{r}(t)), R(\mathbf{r}(t))\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\, dt$$

$$= \int_a^b P(\mathbf{r}(t))x'(t)\, dt + \int_a^b Q(\mathbf{r}(t))y'(t)\, dt + \int_a^b R(\mathbf{r}(t))z'(t)\, dt$$

$$= \int_C Pdx + Qdy + Rdz.$$

Sometimes, it is helpful to think of $\mathbf{F}\cdot d\mathbf{r}$ as $\langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle = Pdx + Qdy + Rdz$.

15. The Fundamental Theorem for Line Integrals

Let's recall the fundamental theorem for Calculus:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

It has the following generalization in terms of line integrals:

THEOREM 15.1. *Let C be a smooth curve given by $\mathbf{r}(t)$, $t \in [a,b]$. Let f be a function of 2 or 3 variables whose gradient ∇f is continuous. Then*

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Proof. Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $t \in [a,b]$.

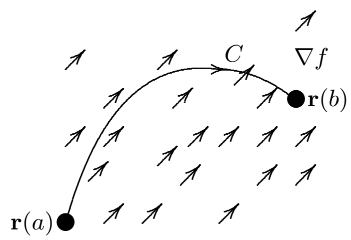


Figure 103

$$\begin{aligned}
\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)\,dt \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\,dt && \text{by Chain Rule} \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a)). && \text{by Fund. Thm. of Calc.}
\end{aligned}$$

EXAMPLE 15.2. Consider the gravitational (force) field $\mathbf{F}(\mathbf{r}) = -\dfrac{mMG}{|\mathbf{r}|^3}\mathbf{r}$, where $\mathbf{r} = \langle x, y, z\rangle$.
Recall that $\mathbf{F} = \nabla f$, where $f(x, y, z) = \dfrac{mMG}{\sqrt{x^2 + y^2 + z^2}}$.

Find the work done by the gravitational field in moving a particle of mass m from the point $(3, 4, 12)$ to the point $(1, 0, 0)$ along a piecewise smooth curve C.

Solution. $W \equiv \displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(1, 0, 0) - f(3, 4, 12) =$ $12mMG/13$.

16. Independence of Path

Let $\mathbf{F}$ be a continuous vector field with domain D.

DEFINITION 16.1. *The line integral* $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ *is independent of path in D if* $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ *for any 2 paths C_1 and C_2 in D that have the same initial and terminal points.*

DEFINITION 16.2. *A path is called closed if its terminal point coincides with its initial point.*

THEOREM 16.3. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ *is independent of path in D if and only if* $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for every closed path in D.*

Proof. To prove the necessity, let C be a closed path starting from the point A and ending at A.

Pick any point B on C other than A.

Denote the subpath along C from A to B by C_1 and the subpath along C from B to A by C_2.

Then $C = C_1 + C_2$. See figure 104.

Now both C_1 and $-C_2$ are paths from A and B.

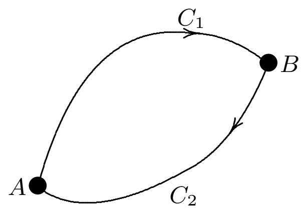


Figure 104 $\quad C = C_1 + C_2$

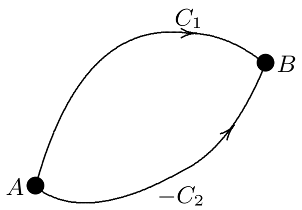


Figure 105 $\quad C = C_1 - C_2$

We have $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} - \int_{-C_2} \mathbf{F}\cdot d\mathbf{r}$ $= 0$, since both C_1 and $-C_2$ are paths from A and B and the line integral is by assumption independent of path.

To prove the sufficiency, consider 2 paths C_1 and C_2 having the same initial point A and terminal point B. See figure 105.

Then $C = C_1 - C_2$ is a closed path. Thus,

$\displaystyle 0 = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1 - C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} - \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$. Hence, $\displaystyle\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$.

Consider the following statements:

(1) $\mathbf{F}$ is conservative on D.

(2) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D.

(3) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path C in D.

By 18.3 and the fundamental theorem for line integrals, we have the following implication and equivalence: (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3).

In fact, the implication (2) $\Longrightarrow$ (1) is true with some suitable assumptions on the domain D.

Definition 16.4. *A subset D in $\mathbb{R}^2$ (or $\mathbb{R}^3$) is said to be open if for any point p in D, there is a disk (ball) with center at p that lies entirely in D. (This means D does not contain any boundary points.)*

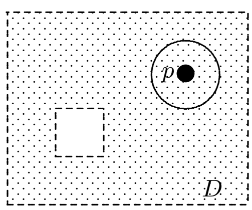


Figure 106 An open set D in $\mathbb{R}^2$

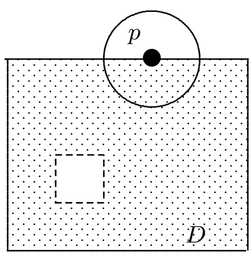


Figure 107 D is not open

Definition 16.5. *A subset D in $\mathbb{R}^2$ (or $\mathbb{R}^3$) is said to be connected if any two points in D can be joined by a path that lies in D.*

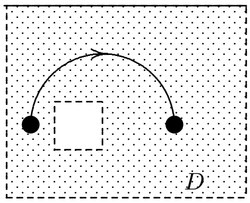


Figure 108 D is connected

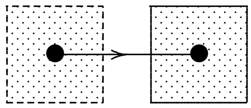


Figure 109 D is not connected

THEOREM 16.6. *Suppose* $\mathbf{F}$ *is a vector field that is continuous on an open connected region* D*. If* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *is independent of path in* D*, then* $\mathbf{F}$ *is conservative. That is there exists a function* f *such that* $\nabla f = \mathbf{F}$*.*

Proof. Let's prove the case in $\mathbb{R}^2$. The case in $\mathbb{R}^3$ is similar.

Fix a point $A(a, b)$ in D. Let

$$f(x, y) = \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r},$$

where $(x, y) \in D$ and the line integral is taken along a path C in D joining (a, b) to (x, y).

Since $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, f is well-defined. As the domain D is open, there exists a disk centered at (x, y) that lies entirely in D. Pick a point (x_1, y) in the disk with $x_1 < x$. Let C consist of any path C_1 from (a, b) to (x_1, y) followed by the horizontal line segment C_2 from (x_1, y) to (x, y). See figure 111.

Then

$$f(x,y)=\int_{C_1}\mathbf{F}\cdot d\mathbf{r}+\int_{C_2}\mathbf{F}\cdot d\mathbf{r}=\int_{(a,b)}^{(x_1,y)}\mathbf{F}\cdot d\mathbf{r}+\int_{C_2}\mathbf{F}\cdot d\mathbf{r}.$$

Thus, $\dfrac{\partial f}{\partial x}=0+\dfrac{\partial}{\partial x}\displaystyle\int_{C_2}\mathbf{F}\cdot d\mathbf{r}$ because the first line integral along C_1 is independent of x.

Let's write $\mathbf{F}=P\mathbf{i}+Q\mathbf{j}$. Then $\displaystyle\int_{C_2}\mathbf{F}\cdot d\mathbf{r}=\int_{C_2}Pdx+Qdy$. On C_2, $y=$ constant so that $\displaystyle\int_{C_2}Qdy=0$.

Hence,

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\int_{C_2}\mathbf{F}\cdot d\mathbf{r}=\frac{\partial}{\partial x}\int_{x_1}^{x}P\,dx=P(x,y).$$

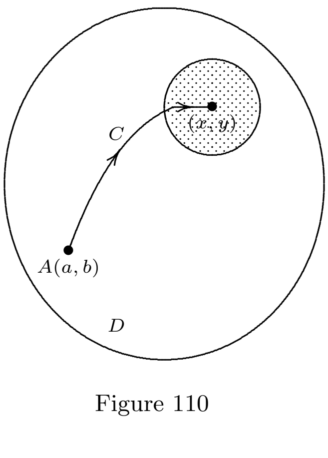


Figure 110

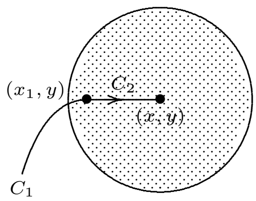


Figure 111

Similarly, by considering a path from $A(a,b)$ to a point (x,y_1) with $y_1<y$ inside the disk followed by the vertical segment from (x,y_1) to (x,y), we can prove that

$\dfrac{\partial f}{\partial y} = Q(x, y).$

Therefore,

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}j = \nabla f.$$

The openness of D is to ensure the points (x_1, y) and (x, y_1) exist corresponding to every (x, y) in D.

There is an obvious necessary condition for a vector field on $\mathbb{R}^2$ to be conservative due to Clairaut's Theorem

THEOREM 16.7. *Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field on $D \subset \mathbb{R}^2$, where P and Q have continuous partial derivatives in D. If $\mathbf{F}$ is conservative, then*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Proof. As $\mathbf{F}$ is conservative in D, there exists a differentiable function $f(x, y)$ in D such that $\nabla f = \mathbf{F}$. That is $f_x = P$ and $f_y = Q$.

By Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

The converse is true for a special kind of domain in $\mathbb{R}^2$.

DEFINITION 16.8. *A simple curve is a curve which does not intersect itself.*

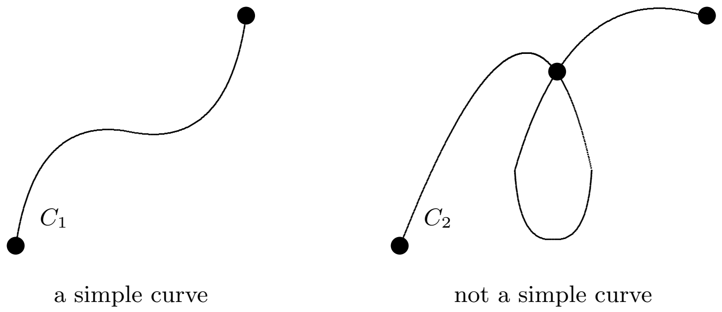


Figure 112

DEFINITION 16.9. *A simply-connected region in the plane is a connected region such that every simple closed curve in D encloses only points that are in D.*

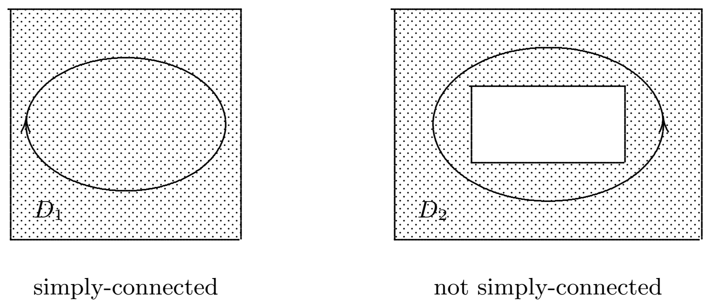


Figure 113

THEOREM 16.10. *Let $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a vector field on an open simply-connected region $D \subset \mathbb{R}^2$, where P and Q have continuous partial derivatives in D. If $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, then $\mathbf{F}$ is conservative.*

This is a consequence of Green's Theorem in the next section.

EXAMPLE 16.11. Determine whether the vector field $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is conservative.

Solution. As

$$\frac{\partial(x^2 - 3y^2)}{\partial x} = 2x = \frac{\partial(3 + 2xy)}{\partial x},$$

and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, $\mathbf{F}$ is conservative by the Theorem 18.10.

EXAMPLE 16.12. Let $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$. Find a function f such that $\nabla f = \mathbf{F}$.

Also evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by $\mathbf{r}(t) = e^t \sin t\mathbf{i} + \cos t\mathbf{j}, \quad t \in [0, \pi]$.

Solution. As $\nabla f = \mathbf{F}$, we have $f_x(x, y) = 3 + 2xy$.

Integrating with respect to x, we get $f(x, y) = 3x + x^2y + g(y)$, where $g(y)$ is an integration constant, but it could be a function of y.

Thus $f_y(x, y) = x^2 + g'(y)$ so that $x^2 + g'(y) = x^2 - 3y^2$. That is $g'(y) = -3y^2$.

Integrating $g'(y)$ with respect to y, we obtain $g(y) = -y^3 + K$, where K is a constant.

Consequently, $f(x, y) = 3x + x^2y - y^3 + K$.

Since $\mathbf{F}$ is conservative, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path. In fact, f is a potential function for $\mathbf{F}$.

Thus by the fundamental theorem for line integrals, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0,-1) - f(0,1) = 2.$$

EXERCISE 16.13. If $\mathbf{F}(x,y,z) = y^2\mathbf{i} + (2xy + e^{3z})\mathbf{j} + 3ye^{3z}\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

[Answer: $f(x,y,z) = xy^2 + ye^{3z} + C$]

17. Green's Theorem

Green's Theorem gives the relationship between a line integral along a simple closed curve C on the plane and the double integral over the plane region D that C bounds.

By the Jordan curve theorem, every simple closed curve C on the plane bounds a region D. The *positive orientation* of C refers to the orientation of C such that as one traverses along C in the direction of this orientation, the region D that it bounds is always on the left hand side.

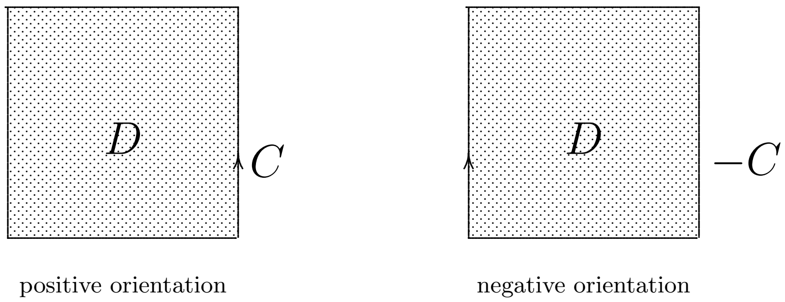


Figure 114

For example, if D is a circular region on the plane, then the boundary C of D oriented in the counterclockwise sense is the positive orientation. We use the notation ∂D to denote the boundary of D with the positive orientation.

THEOREM 17.1. *(Green's Theorem) Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open simply connected region that contains D, then*

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

The line integral $\displaystyle\int_C Pdx + Qdy$, where C is positively oriented has other notations such as

$$\oint_C Pdx + Qdy, \quad \text{or} \quad \oint_{\partial D} Pdx + Qdy.$$

They all indicate that the line integral is calculated using the positive orientation of C.

Proof. We shall first verify that Green's Theorem is true for D being a region which is both of type I and Type II. (See figure 71 for type I and type II regions.)

The general case can be proved by cutting the region into a finite number of regions of both type I and type II and applying the result to each of them.

Let's consider a type I region. The proof for type II region is similar.

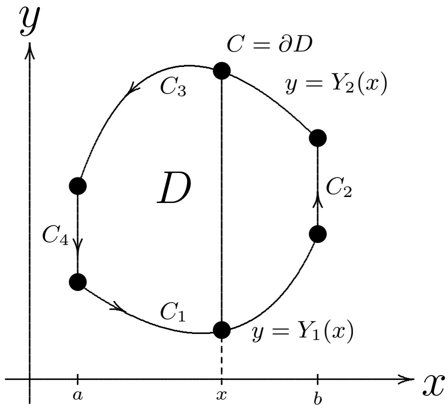


Figure 115

In this case, the lower and upper boundaries of D consist of simple smooth curves C_1 and C_3 respectively, and the left and right boundaries are vertical lines $C_4 : x = a$ and $C_2 : x = b$. See figure 115.

Let C_1 and C_3 be parametrized as the graphs of $y = Y_1(x)$ and $y = Y_2(x)$ respectively for $x \in [a, b]$.

Here we assume $Y_2(x) > Y_1(x)$ for all $x \in (a, b)$ so that C_3 is higher than C_1.

Thus C_1 is given the orientation which goes from left to right, whereas C_3 is given the orientation which goes from right to left so that $C = C_1 + C_2 + C_3 + C_4$. Then

$$\iint_D \frac{\partial P}{\partial y}\, dxdy = \int_a^b \left[\int_{Y_1(x)}^{Y_2(x)} \frac{\partial P}{\partial y}\, dy \right] dx$$

$$= \int_a^b \left[P(x, y) \right]_{y=Y_1(x)}^{y=Y_2(x)} dx$$

$$
\begin{aligned}
&= \int_a^b P(x, Y_2) - P(x, Y_1)\, dx \\
&= - \int_a^b P(x, Y_1)\, dx - \int_b^a P(x, Y_2)\, dx \\
&= - \int_{C_1} P\, dx - \int_{C_3} P\, dx \\
&= - \int_{C_1} P\, dx - \int_{C_2} P\, dx - \int_{C_3} P\, dx - \int_{C_4} P\, dx \\
&= - \int_C P\, dx.
\end{aligned}
$$

Note that $\displaystyle\int_{C_2} P\, dx$ and $\displaystyle\int_{C_4} P\, dx$ are in fact zero because C_2 and C_4 are vertical segments: $x = b$ and $x = a$, so that $dx = 0$.

Similarly, using the fact that D is also a type II region,

$$
\iint_D \frac{\partial Q}{\partial x}\, dxdy = \int_C Q\, dy.
$$

Therefore, $\displaystyle\oint_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$

Now to extend this result to the general case, first consider a region D which is a union of two regions D_1 and D_2 meeting along a curve L in their common boundaries. See figure 116.

Thus $\partial D_1 = L_1 + L$, $\partial D_2 = -L + L_2$ and $\partial D = L_1 + L_2$.

Suppose Green's Theorem holds for the regions D_1 and D_2.

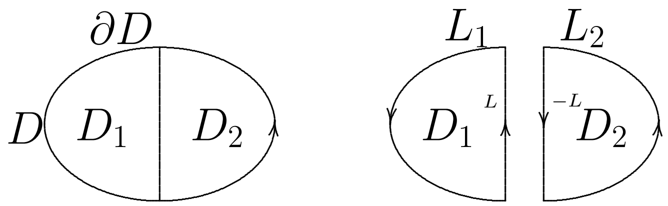


Figure 116

Then, suppressing the terms involving P, Q etc, the following calculation shows that Green's theorem holds for the region D.

$$\int_{\partial D} = \int_{L_1+L_2} = \int_{L_1} + \int_{L_2} = \int_{L_1} + \int_{L} + \int_{-L} + \int_{L_2}$$

$$= \int_{L_1+L} + \int_{-L+L_2} = \int_{\partial D_1} + \int_{\partial D_2}$$

$$= \iint_{D_1} + \iint_{D_2} = \iint_{D}.$$

Now any simple closed curve in the plane bounds a region which can be cut into regions both of type I and type II. See figure 117. Thus, by the above consideration, Green's Theorem is valid for any simple closed curve in the plane.

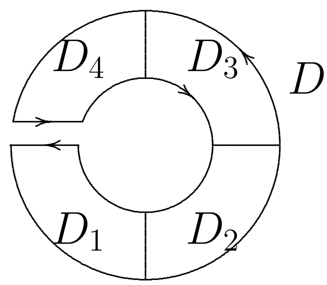


Figure 117

Lastly, the proof of Theorem 19.10 follows from Green's Theorem and Theorem 19.6.

EXAMPLE 17.2. Evaluate $\displaystyle\int_C x^4dx + xydy$, where C is the triangular curve consisting of the oriented line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$ and from $(0,1)$ to $(0,0)$.

Solution. The functions $P(x,y) = x^4$ and $Q(x,y) = xy$ have continuous partial derivatives on the whole of $\mathbb{R}^2$, which is open and simply connected.

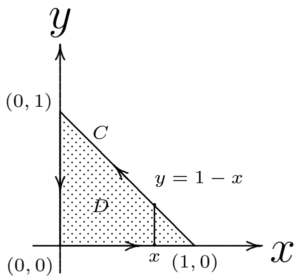


Figure 118

By Green's Theorem, $$\int_C x^4dx + xydy = \iint_D \left[\frac{\partial(xy)}{\partial x} - \frac{\partial x^4}{\partial y}\right] dA$$
$$= \iint_D y\, dydx$$
$$= \int_0^1 \int_0^{1-x} y\, dydx$$
$$= \tfrac{1}{6}.$$

17.1. Application of Green's Theorem to Find Area.

Recall that the area of a region D in $\mathbb{R}^2$ is $\displaystyle\iint_D 1\, dA$.

Therefore, if we choose $P(x,y)$ and $Q(x,y)$ such that $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$, then by Green's Theorem we have

$$\text{Area of } D = \iint_D 1\, dA = \oint_{\partial D} Pdx + Qdy.$$

There are various choices of P and Q that satisfy this requirement. For example:

(1) $P(x,y) = 0, Q(x,y) = x$.
(2) $P(x,y) = -y, Q(x,y) = 0$.
(3) $P(x,y) = -y/2, Q(x,y) = x/2$.

Therefore,

$$\text{Area of } D = \oint_{\partial D} x\, dy = -\oint_{\partial D} y\, dx = \frac{1}{2}\oint_{\partial D} xdy - ydx.$$

EXAMPLE 17.3. Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution. Let the parametric equations for the ellipse be

$x = a\cos t, y = b\sin t$, for $t \in [0, 2\pi]$.

Then,

$$\text{Area } = \frac{1}{2}\oint_{\partial D} xdy - ydx = \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t) - (b\sin t)(-a\sin t)\, dt$$
$$= \frac{1}{2}\int_0^{2\pi} ab\, dt = \pi ab.$$

EXERCISE 17.4. Evaluate by Green's Theorem

$$\oint_C e^{-x}\sin y\, dx + e^{-x}\cos y\, dy,$$

where C is the rectangle with vertices at $(0,0)$, $(\pi,0)$, $(\pi,\pi/2)$, $(0,\pi/2)$.

[Answer: $2(e^{-\pi} - 1)$]

EXERCISE 17.5. Let $\mathbf{F}(x,y) = \dfrac{-y}{\sqrt{x^2+y^2}}\mathbf{i} + \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{j}$ be defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. Show that $\mathbf{F}$ is not conservative. Show that $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = 2\pi a$, where C is a circle centred at the origin with radius a traversed in the counterclockwise direction.

18. The Curl and Divergence of a Vector Field

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field on $\mathbb{R}^3$. The curl of $\mathbf{F}$ is defined by

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

The curl of a vector field $\mathbf{F}$ is a vector field which measures the rotational effect of $\mathbf{F}$. The geometric meaning of $\operatorname{curl}\mathbf{F}$ can be seen after we learn Stokes' Theorem.
At this point, let's introduce the del operator ∇. We let

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

We regard ∇ as a 3-dimensional vector consisting of the operators of partial differentiations with respect to x, y, z. We can multiply ∇ by a scalar function (on the right), take the dot product with a function, or the cross product with a vector field. For example, we may regard the gradient of a function f as being the scalar multiplication of ∇ and f.

That is

$$\text{grad } f = \nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}.$$

The curl of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ can be regarded as the cross product between ∇ and $\mathbf{F}$.

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

EXAMPLE 18.1. Let $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$. Find curl $\mathbf{F}$.

Solution.

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = -(2y + xy)\mathbf{i} + x\mathbf{j} + yz\mathbf{k}.$$

THEOREM 18.2. *If $f(x, y, z)$ has continuous 2nd order partial derivatives, then* curl $(\nabla f) = \mathbf{0}$.

Proof.

$$\text{curl } \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k}$$

$= \mathbf{0}$. by Clairaut's Theorem

COROLLARY 18.3. *If* $\mathbf{F}$ *is conservative (i.e.* $\mathbf{F} = \nabla f$*), then* curl $\mathbf{F} = \mathbf{0}$.

REMARK 18.4. If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field on $\mathbb{R}^2$, we may regard $\mathbf{F}$ as the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + 0\mathbf{k}$ in $\mathbb{R}^3$ with zero $\mathbf{k}$ component.

Then $\text{curl } \mathbf{F} = \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k}$.

Thus in this case, if $\mathbf{F}$ is conservative, then $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ which is 19.7.

For example $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ is not conservative because $\text{curl } \mathbf{F} = -(2y + xy)\mathbf{i} + x\mathbf{j} + yz\mathbf{k} \neq \mathbf{0}$.

Using Stokes' Theorem in the next section, one can prove the following:

THEOREM 18.5. *Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ whose component functions have continuous partial derivatives. If* $\text{curl } \mathbf{F} = \mathbf{0}$*, then $\mathbf{F}$ is conservative.*

EXERCISE 18.6. Show that the vector field $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ is conservative. Find a function f such that $\nabla f = \mathbf{F}$.

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field on $\mathbb{R}^3$. The divergence of $\mathbf{F}$ is defined by

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) \cdot (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) .$$

$$= \nabla \cdot \mathbf{F}.$$

EXAMPLE 18.7. Let $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$. Find div $\mathbf{F}$.

Solution. div $\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(xz) + \dfrac{\partial}{\partial y}(xyz) + \dfrac{\partial}{\partial z}(-y^2) = z + xz.$

THEOREM 18.8. *Let* $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. *Suppose* P, Q, R *have continuous 2nd order partial derivatives. Then* div curl $\mathbf{F} = 0$.

Proof.
div curl $\mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)$

$$= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y}$$

$$= 0.$$

because the terms cancel in pairs by Clairaut's Theorem.

For a velocity vector field $\mathbf{F}$, div $\mathbf{F}$ measures the amount of flow radiating at a point. If the flow is uniform and without compression or expansion, then div $\mathbf{F} = 0$. Thus, if div $\mathbf{F} = 0$, we say that $\mathbf{F}$ is *incompressible*. Whereas curl $\mathbf{F}$ measures the rotational effect of the vector field $\mathbf{F}$. Therefore, if curl $\mathbf{F} = \mathbf{0}$, then we say that $\mathbf{F}$ is *irrotational*.

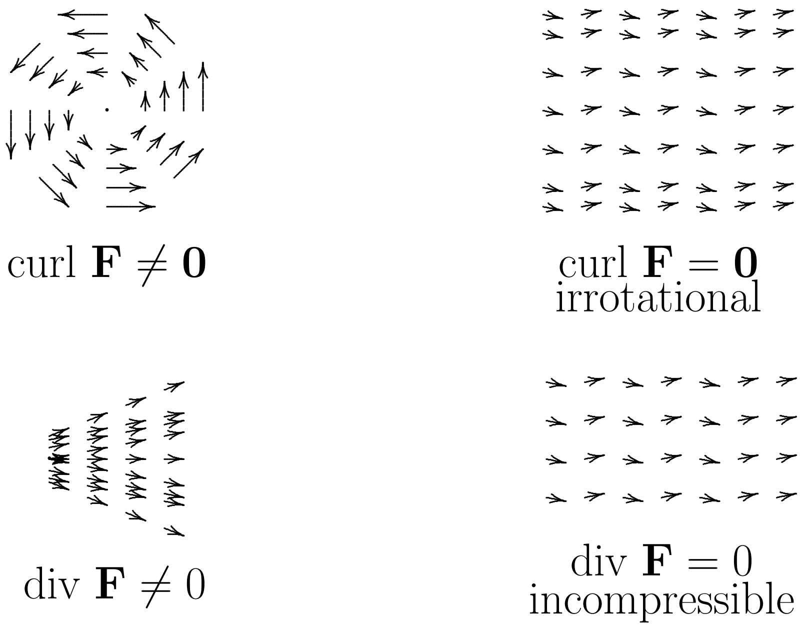


Figure 121

Another differential operator occurs when we compute the divergence of a gradient vector field ∇f. If f is a function of three variables, we have

$$\text{div } (\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

We abbreviate this expression as $\nabla^2 f$. The operator $\nabla^2 = \nabla \cdot \nabla$ is called the *Laplace operator* because of its relation to Laplace's equation:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

We can also apply the Laplace operator ∇^2 to a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ in terms of its components:

$$\nabla^2 \mathbf{F} = \nabla^2 P\mathbf{i} + \nabla^2 Q\mathbf{j} + \nabla^2 R\mathbf{k}.$$

EXERCISE 18.9. Let $r = \sqrt{x^2 + y^2 + z^2}$. Find $\nabla^2(r^3)$.

EXERCISE 18.10. Prove that $\text{div}(f\mathbf{F}) = f\text{div}\mathbf{F} + \mathbf{F} \cdot \nabla f$.

18.1. Green's Theorem in Vector Forms. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field. Green's Theorem says that

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Regard $\mathbf{F}$ as a vector field in $\mathbb{R}^3$. That is $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + 0\mathbf{k}$.

Then

curl $\mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$, so that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (\text{curl } \mathbf{F}) \cdot \mathbf{k}$.

Therefore, we may state Green's Theorem in the following form:

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA.$$

To get a better meaning of this equation, let ∂D be parametrized by the vector equation

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle \quad \text{for } t \in [a, b].$$

We assume the parametrization gives the positive orientation of ∂D. Then the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \langle \frac{x'(t)}{|\mathbf{r}'(t)|}, \frac{y'(t)}{|\mathbf{r}'(t)|} \rangle.$$

Thus

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)|\, dt$$

$$= \int_a^b (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{T}(t))\ |\mathbf{r}'(t)|\, dt = \int_{\partial D} \mathbf{F} \cdot \mathbf{T}\, ds,$$

where $ds = |\mathbf{r}'(t)|\, dt$ is the arc length differential. Then, we can also state Green's Theorem in the following form:

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dA.$$

In this form, the equation expresses the line integral of the tangential component of $\mathbf{F}$ along ∂D as the double integral of the vertical component of curl $\mathbf{F}$ over the region D. This is a special case of Stoke's Theorem in which D is not necessarily a planar region but is a surface in $\mathbb{R}^3$ with a boundary curve ∂D.

We could also derive a formula involving the normal component of $\mathbf{F}$ along ∂D. In that way, Greens' theorem will be stated in terms of the divergence of the vector field $\mathbf{F}$. Using the above parametrization of C, one can easily verify (by taking dot product with $\mathbf{T}$) that the *outward* unit normal vector to ∂D is given by

$$\mathbf{n}(t) = \langle \frac{y'(t)}{|\mathbf{r}'(t)|}, -\frac{x'(t)}{|\mathbf{r}'(t)|} \rangle.$$

(It is the outward pointing normal because C is given the positive orientation.) Now we consider the line integral of the normal component of $\mathbf{F}$ along ∂D.

$$\begin{aligned}
\int_{\partial D} \mathbf{F}\cdot\mathbf{n}\,ds &= \int_a^b (\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{n}(t))\,|\mathbf{r}'(t)|\,dt \\
&= \int_a^b \left[\frac{P(x(t),y(t))y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t),y(t))x'(t)}{|\mathbf{r}'(t)|}\right]|\mathbf{r}'(t)|\,dt \\
&= \int_a^b P(x(t),y(t))y'(t)dt - Q(x(t),y(t))x'(t)dt \\
&= \int_C Pdy - Qdx \\
&= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\,dA \qquad \text{by Green's Theorem} \\
&= \iint_D \mathrm{div}\mathbf{F}\,dA.
\end{aligned}$$

Therefore,

$$\int_{\partial D} \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \mathrm{div}\mathbf{F}\,dA.$$

This version says that the line integral of the normal component of $\mathbf{F}$ along ∂D is equal to the double integral of the divergence of $\mathbf{F}$ over the region D. This result can be generalized to the case of closed surface enclosing a solid region in $\mathbb{R}^3$ which is the content of the Divergence Theorem.

In the next two exercises, we assume D satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous. The first exercise is a consequence of 21.10.

EXERCISE 18.11. Prove that

$$\iint_D f\nabla^2 g\, dA = \int_{\partial D} f(\nabla g)\cdot \mathbf{n}\, ds - \iint_D \nabla f\cdot \nabla g\, dA.$$

EXERCISE 18.12. Prove that

$$\iint_D (f\nabla^2 g - g\nabla^2 f)\, dA = \int_{\partial D} (f(\nabla g) - g(\nabla f))\cdot \mathbf{n}\, ds.$$

19. Parametric Surfaces and their Areas

DEFINITION 19.1. *Let* $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ *be a vector-valued function defined on a region* D *in the* uv*-plane. Then*

$$S = \{(x,y,z) \mid x = x(u,v), y = y(u,v), z = z(u,v),\ (u,v) \in D\}$$

is called a parametric surface.

The equations: $x = x(u,v), y = y(u,v), z = z(u,v)$ *are called the parametric equations of* S.

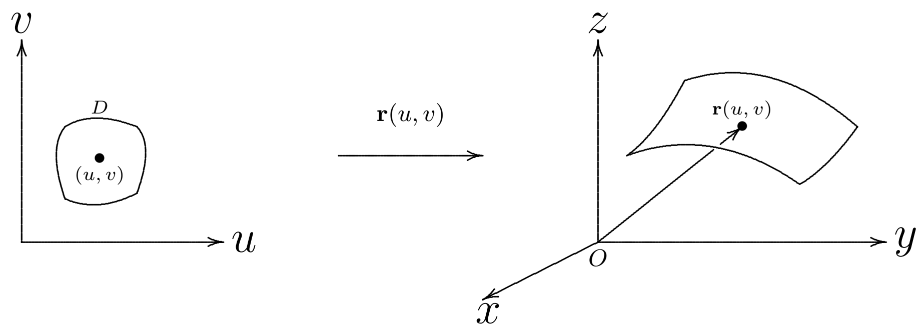


Figure 122

EXAMPLE 19.2. Identity the surface with vector equation $\mathbf{r}(u,v) = \langle 2\cos u, v, 2\sin u\rangle$.

Solution. The point $(x,y,z) = (2\cos u, v, 2\sin u)$ lies on this surface if and only if $x^2 + z^2 = 4\cos^2 u + 4\sin^2 u = 4$.

Therefore, the surface is the cylinder $x^2 + z^2 = 4$. The domain of $\mathbf{r}$ can be taken as the infinite strip $D = \{(u,v) \mid 0 \le u \le 2\pi, -\infty < v < \infty\}$. The function $\mathbf{r}$ simply identifies the two vertical sides of this strip to form the cylinder. Here we omit the line $u = 2\pi$ so that $\mathbf{r}$ is injective.

We could take the domain of $\mathbf{r}$ to be the whole xy-plane. In that case, $\mathbf{r}$ takes the whole xy-plane and wraps it up around the cylinder infinitely many times.

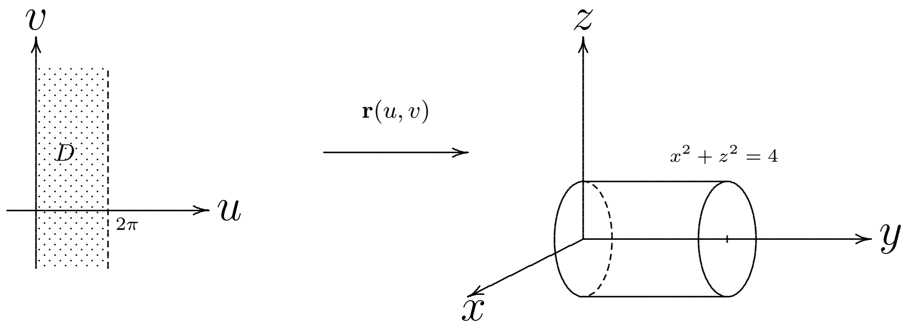


Figure 123

EXAMPLE 19.3. Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and contains two non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$.

Solution. Let O be the origin. For any point P on the plane, its position vector $\mathbf{r}$ can be expressed as

$$\mathbf{r} = \overrightarrow{OP_0} + \overrightarrow{P_0P} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b},$$

for some numbers u and v.

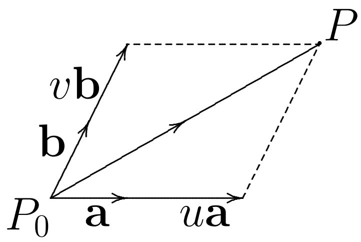


Figure 124

Therefore, $\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$ is the vector equation of the plane. If we let $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$, $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then the parametric equations of the plane are : $x = x_0 + ua_1 + vb_1, y = y_0 + ua_2 + vb_2, z = z_0 + ua_3 + vb_3$. Here u and v are the parameters.

EXAMPLE 19.4. Find a parametric representation of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution. We use the angles ϕ and θ in spherical coordinates. For a point on the sphere, $\rho = a$. Thus

$$x = a \sin\phi \cos\theta, \quad y = a \sin\phi \sin\theta, \quad z = a \cos\phi.$$

That is

$$\mathbf{r}(\phi, \theta) = \langle a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi \rangle.$$

EXAMPLE 19.5. Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

Solution. We simply use x and y as the parameters. Thus, $\mathbf{r}(x, y) = \langle x, y, x^2 + 2y^2 \rangle$.

In general, if the surface S is the graph of a function $z = f(x, y)$, then a natural parametric representation of S is

$$\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle.$$

EXAMPLE 19.6. Find a parametric representation of the cone $z = 2\sqrt{x^2 + y^2}$.

Solution. Since the cone is the graph of the function $z = 2\sqrt{x^2 + y^2}$, we can simply take

$$\mathbf{r}(x, y) = \langle x, y, 2\sqrt{x^2 + y^2} \rangle.$$

Alternatively, we can consider cylindrical coordinates. The equation of the cone $z = 2\sqrt{x^2+y^2}$ in cylindrical coordinates is $z = 2r$. Thus if we use polar coordinates (r, θ) of the xy-plane, we can write $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 2r\rangle$.

19.1. Tangent Planes. Let S be a parametric surface defined by

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle.$$

We shall find the equation of the tangent plane to S at a point P_0 with position vector $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$.

Consider the horizontal line $v = v_0$ in the uv-plane and within the domain of $\mathbf{r}$, its image under $\mathbf{r}$ is a curve C_1 on S passing through the point P_0.

This curve C_1 has a vector equation

$$\mathbf{r}(u, v_0) = \langle x(u, v_0), y(u, v_0), z(u, v_0)\rangle.$$

The tangent vector to C_1 at P_0 is given by $\frac{d}{du}\mathbf{r}(u, v_0)\,|_{u=u_0}$, which is simply

$$\mathbf{r}_u \equiv \langle \frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0)\rangle.$$

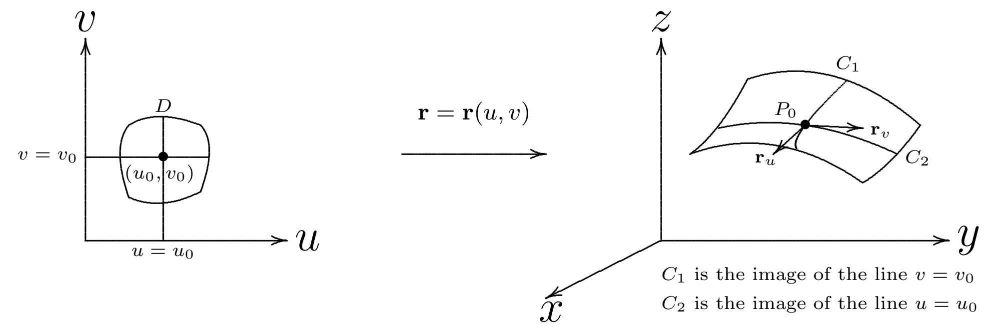


Figure 125

Similarly, the image of the vertical line $u = u_0$ under $\mathbf{r}$ is a curve C_2 whose tangent vector at P_0 is given by

$$\mathbf{r}_v \equiv \langle \frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial v}(u_0, v_0) \rangle.$$

Both vectors $\mathbf{r}_u$ and $\mathbf{r}_u$ lie in the tangent plane to S at P_0.

Therefore, the cross product $\mathbf{r}_u \times \mathbf{r}_v$, assuming it is nonzero, provides a normal vector to the tangent plane to S at P_0.

Therefore, $(\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 0$ is the equation of the tangent plane. At this point, let's make a definition.

Definition 19.7. *The surface S is said to be smooth if $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ for all points $(x, y) \in D$.*

Thus a smooth surface always has a tangent plane at each of its points. Basically, a smooth surface is one which has no corners and no breaks.

Example 19.8. Find the equation of the tangent plane to the surface with parametric equations $x = u^2, y = v^2, z = u + 2v$ at the point $(1, 1, 3)$.

Solution. The vector equation of the surface is

$$\mathbf{r}(u, v) = \langle u^2, v^2, u + 2v \rangle.$$

Therefore, $\mathbf{r}_u = \langle 2u, 0, 1 \rangle$ and $\mathbf{r}_v = \langle 0, 2v, 2 \rangle$.

Thus, a normal vector to the tangent plane is $\mathbf{r}_u \times \mathbf{r}_v = \langle -2v, -4u, 4uv \rangle$.

At the point $(1, 1, 3)$, we have $(u, v) = (1, 1)$.

Then, the normal vector at $(u, v) = (1, 1)$ is $\langle -2, -4, 4 \rangle$.

Therefore, the equation of the tangent plane to the surface at $(1, 1, 3)$ is $(\langle x, y, z\rangle - \langle 1, 1, 3\rangle) \cdot \langle -2, -4, 4\rangle = 0$, or equivalently, $x + 2y - 2z + 3 = 0$.

19.2. Surface Area. If a smooth parametric surface is given by

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle, \quad (u, v) \in D,$$

and $\mathbf{r}$ is injective except possibly on the boundary of D, then the surface area of S over D is defined to be

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA.$$

EXAMPLE 19.9. Find the surface area of the sphere $x^2+y^2+z^2 = a^2$.

Solution. A parametric representation of the sphere is given by

$$\mathbf{r}(\phi, \theta) = \langle a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi\rangle,$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. Thus,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \langle a^2 \sin^2\phi \cos\theta, a^2 \sin^2\phi \sin\theta, a^2 \sin\phi \cos\phi\rangle$$

Therefore, $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin\phi$. Hence,

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \int_0^{2\pi} \int_0^{\pi} a^2 \sin\phi \, d\phi d\theta = 4\pi a^2.$$

19.3. Surface Area of the Graph of a Function. Let S be a surface which is the graph of a function $f(x, y)$ defined on a domain $D \subset \mathbb{R}^2$.

Then a parametric representation of S is $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$.

Thus, $\mathbf{r}_x = \langle 1, 0, f_x\rangle$ and $\mathbf{r}_y = \langle 0, 1, f_y\rangle$ so that $\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x, -f_y, 1\rangle$, and $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + f_x^2 + f_y^2}$.

Therefore, the surface area of S over D is given by

$$A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA.$$

This is the same formula derived in section 12.

19.4. Surface Integrals. Let S be a parametric surface with vector equation $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, where $(u, v) \in D$. Let $f(x, y, z)$ be a continuous function defined on S.

Definition 19.10. *The surface integral of f over S is*

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))|\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

If S is the graph of $z = g(x, y)$, then

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y))\sqrt{1 + g_x^2 + g_y^2}\, dA.$$

EXAMPLE 19.11. Evaluate $\iint_S x^2\,dS$, where S is the unit sphere $x^2+y^2+z^2=1$.

Solution. A parametric representation of the unit sphere is given by

$$\mathbf{r}(\phi,\theta)=\langle\sin\phi\cos\theta,\sin\phi\sin\theta,\cos\phi\rangle,$$

where $0\le\phi\le\pi$ and $0\le\theta\le 2\pi$. From example 22.9, we have $|\mathbf{r}_\phi\times\mathbf{r}_\theta|=\sin\phi$.

Therefore,

$$\begin{aligned}\iint_S x^2\,dS &= \iint_D(\sin\phi\cos\theta)^2|\mathbf{r}_\phi\times\mathbf{r}_\theta|\,dA\\ &= \int_0^{2\pi}\int_0^{\pi}\sin^3\phi\cos^2\theta\,d\phi d\theta\\ &= \int_0^{\pi}\sin^3\phi\,d\phi\int_0^{2\pi}\cos^2\theta\,d\theta\\ &= 4\pi/3.\end{aligned}$$

EXAMPLE 19.12. Evaluate $\displaystyle\iint_S z\,dS$, where S is the surface whose side face S_1 is part of the cylinder $x^2+y^2=1$ bounded by the bottom face S_2 which is the xy-plane and the top face S_3 which is part of the plane $z=x+1$ above the xy-plane. See figure 126.

Solution. The surface integral is the sum of three surface integrals:

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS.$$

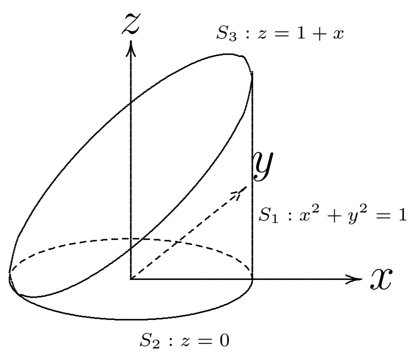


Figure 126

Let's first calculate the surface integral over S_1.

The surface S_1 is a cylinder. By example 22.2, it has a parametric representation $\mathbf{r}(\phi, z) = \langle \cos\theta, \sin\theta, z\rangle$, where $0 \le \theta \le 2\pi, 0 \le z \le 1 + x = 1 + \cos\theta$.
Thus, $\mathbf{r}_\theta \times \mathbf{r}_z = \langle \cos\theta, \sin\theta, 0\rangle$ and $|\mathbf{r}_\theta \times \mathbf{r}_z| = 1$.

Therefore,

$$\iint_{S_1} z\,dS = \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dzd\theta = \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2\,d\theta = \frac{3\pi}{2}.$$

On S_2, we have $z = 0$. Thus the integrand of $\displaystyle\iint_{S_2} z\,dS$ is zero so that the integral has value zero. Therefore,

$$\iint_{S_2} z\,dS = 0.$$

The surface S_3 is the graph of the function $z = 1 + x$.

Therefore, using polar coordinates, we have

$$\iint_{S_3} z\,dS = \iint_D (1+x)\sqrt{1 + z_x^2 + z_y^2}\,dA = \iint_D (1+x)\sqrt{2}\,dA$$

$$= \int_0^{2\pi}\int_0^1 (1 + r\cos\theta)\sqrt{2}\,r dr d\theta = \sqrt{2}\pi.$$

Consequently, $\displaystyle\iint_S z\,dS = \frac{3\pi}{2} + \sqrt{2}\pi.$

Exercise 19.13. Evaluate $\displaystyle\iint_S z^2\,dS$, where S is the portion of the cone $z = \sqrt{x^2 + y^2}$ for which $1 \le x^2 + y^2 \le 4$.

[Answer: $15\pi\sqrt{2}/2$]

20. Oriented Surfaces

A surface S is said to be orientable if it is two-sided, otherwise it is non-orientable.

For example, a sphere is orientable because it has an inside and an outside.

Whereas the Möbius strip in figure 127 is non-orientable. When one walks on one side along the center curve of the Möbius strip, one arrives after one turn to the same position on the opposite side.

This means the Möbius strip is only a one-side surface and is non-orientable.

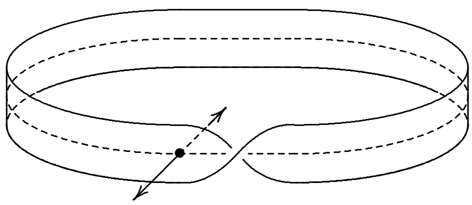


Figure 127 A Möbius band

If S is orientable, then it is possible to choose a unit normal vector $\mathbf{n}$ at every point S so that $\mathbf{n}$ varies continuously over S.

In that case, S is called an oriented surface and the choice of $\mathbf{n}$ is called an orientation of S.

There are only two orientations of an orientable surface S, namely one for each side of the surface S which corresponds to the choice where all the unit normal vectors point away from that side of the surface.

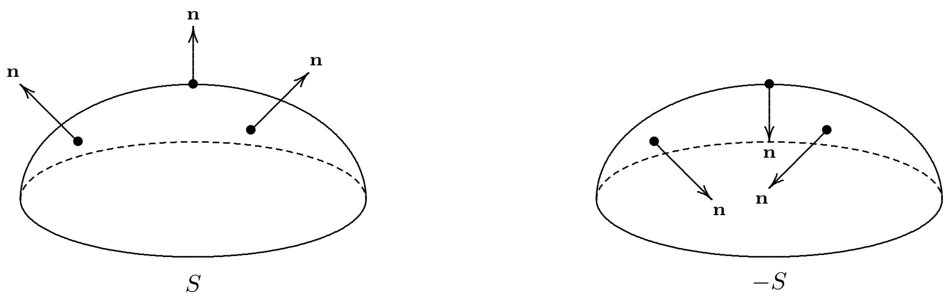


Figure 128 There are two possible orientations of S.

If S is the graph of $z = g(x, y)$, then

$$\mathbf{n} = \frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{g_x^2 + g_y^2 + 1}}$$

is the upward orientation of S because the $\mathbf{k}$-component is positive.

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r} = \mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

The opposite orientation is denoted by $-\mathbf{n}$ and the corresponding oriented surface is denoted by $-S$.

As an example, consider the unit sphere. It has a parametric representation given by

$$\mathbf{r}(\phi, \theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle,$$

where

$$0 \leq \phi \leq \pi \qquad 0 \leq \theta \leq 2\pi.$$

From example 22.9, we have

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\rangle$$

and

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi.$$

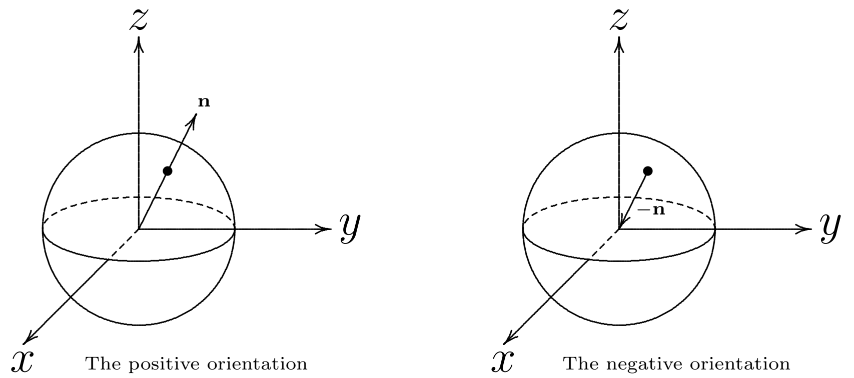


Figure 129 The sphere

Therefore,

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle = \mathbf{r}(\phi,\theta),$$

which is the outward pointing normal.

For a closed surface (i.e. a surface which is the boundary of a solid region E), the convention is that the positive orientation is the one for which the normal vector points outward from E, and the inward-pointing normal gives the negative orientation.

21. Surface Integrals of Vector Fields

Let $\mathbf{F}$ be a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$. The surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS.$$

This integral is also called the flux of $\mathbf{F}$ over S.

If S is the graph of a function $z = g(x, y)$ over a region D in the xy-plane, and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_S \mathbf{F}\cdot\mathbf{n}\, dS \\ &= \iint_D \langle P, Q, R\rangle \cdot \frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{g_x^2 + g_y^2 + 1}}\sqrt{g_x^2 + g_y^2 + 1}\, dA \\ &= \iint_D (-Pg_x - Qg_y + R)\, dA.\end{aligned}$$

EXAMPLE 21.1. Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle y, x, z\rangle$, and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$. Here S is given the positive orientation with respect to E.

Solution. Let S_1 be the paraboloid above the xy-plane and S_2 the unit disk D on the xy-plane. Then S is the union of the surfaces S_1 and S_2. The surface integral over S is the sum of the surface integrals over S_1 and S_2.

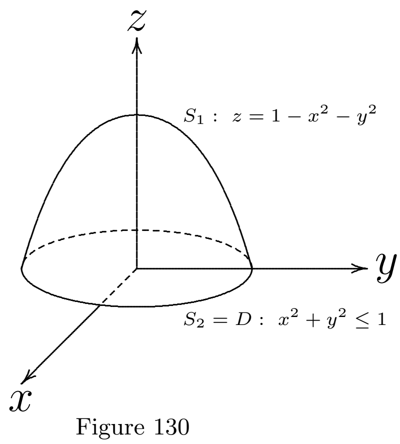


Figure 130

First let's compute the surface integral over S_1.

The surface S_1 is the graph of the function $g(x, y) = 1 - x^2 - y^2$ over the disk $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$. We have $g_x = -2x$ and $g_y = -2y$.

Therefore,

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_D (-Pg_x - Qg_y + R)\, dA$$

$$= \iint_D [(-y)(-2x) - x(-2y) + (1 - x^2 - y^2)]\, dA$$

$$= \iint_D (1 + 4xy - x^2 - y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos\theta \sin\theta - r^2)\, r dr d\theta = \frac{\pi}{2}.$$

The disk S_2 is oriented downward, so its unit normal vector is $-\mathbf{k}$.

Then, since $z = 0$ on S_2, we have

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot (-\mathbf{k})\, dS = \iint_{S_2} -z\, dS = 0.$$

Therefore, $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2}$.

If S is a parametric surface defined by a vector function $\mathbf{r} = \mathbf{r}(u, v) : D \longrightarrow \mathbb{R}^3$, then

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS \\
&= \iint_D \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v|\, dA \\
&= \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.
\end{aligned}$$

Therefore,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

EXAMPLE 21.2. Let $\mathbf{F}(x, y, z) = \langle z, y, x \rangle$. Evaluate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$, oriented with the outward pointing normal.

Solution. A parametric representation of the unit sphere is given by

$$\mathbf{r}(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle,$$

where
$0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. We have

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \rangle,$$

and

$$\mathbf{F}(\mathbf{r}(\phi,\theta)) = \langle \cos\phi, \sin\phi\sin\theta, \sin\phi\cos\theta \rangle.$$

Thus, $\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta$.

Therefore,

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi} (2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)\,d\phi d\theta \\ &= 2\int_0^{\pi} \sin^2\phi\cos\phi\,d\phi \int_0^{2\pi} \cos\theta\,d\theta + \int_0^{\pi} \sin^3\phi\,d\phi \int_0^{2\pi} \sin^2\theta\,d\theta \\ &= 0 + 4\pi/3.\end{aligned}$$

22. Stokes' Theorem

THEOREM 22.1. *(Stokes' Theorem) Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}.$$

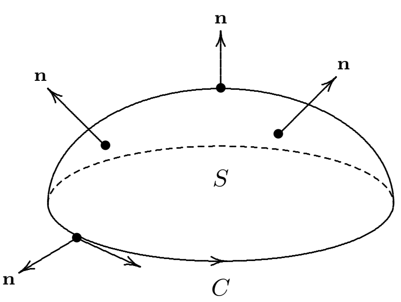


Figure 131 Stokes' Theorem $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}$

Proof of a special case of Stokes' Theorem.

Assume S is the graph of $z = g(x, y)$, over a region D which is of type I and II in the xy-plane, and g has continuous 2nd order partial derivatives.

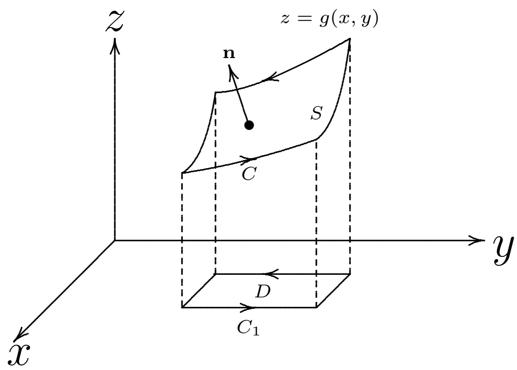


Figure 132

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then

$$\iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S} = \iint_D -(R_y - Q_z)g_x - (P_z - R_x)g_y + (Q_x - P_y)\, dA.$$

Let $x = x(t)$, $y = y(t)$, $t \in [a, b]$ be a parametric representation of the boundary curve C_1 of D.

Then $x = x(t), y = y(t), z = g(x(t), y(t))$ is a parametric representation of C. Therefore

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right] dt \\
&= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R(g_x\frac{dx}{dt} + g_y\frac{dy}{dt}) \right] dt \\
&= \int_a^b \left[(P + Rg_x)\frac{dx}{dt} + (Q + Rg_y)\frac{dy}{dt} \right] dt
\end{aligned}$$

$$= \int_{C_1} (P + Rg_x)dx + (Q + Rg_y)dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}(Q + Rg_y) - \frac{\partial}{\partial y}(P + Rg_x) \right] dA \quad \text{by Green's Thm on } D$$

$$= \iint_D \begin{bmatrix} Q_x + Q_z \frac{\partial z}{\partial x} + (R_x + R_z \frac{\partial z}{\partial x})g_y + Rg_{yx} \\ -P_y - P_z \frac{\partial z}{\partial y} - (R_y + R_z \frac{\partial z}{\partial y})g_x - Rg_{xy} \end{bmatrix} dA$$

$$= \iint_D -(R_y - Q_z)g_x - (P_z - R_x)g_y + (Q_x - P_y)\, dA$$

$$= \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}.$$

Note that $Q(x, y, z) = Q(x, y, g(x, y))$ is a function of x and y in D.

By chain rule, we have $\dfrac{\partial Q}{\partial x} = Q_x \dfrac{\partial x}{\partial x} + Q_y \dfrac{\partial y}{\partial x} + Q_z \dfrac{\partial z}{\partial x}$.

That is $\dfrac{\partial Q}{\partial x} = Q_x + Q_z \dfrac{\partial z}{\partial x} = Q_x + Q_z g_x$.

Similarly, we have $\dfrac{\partial R}{\partial x} = R_x + R_z g_x$.

Example 22.2. Evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$, where C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$, and C is oriented in the counterclockwise sense when viewed from above.

Solution. Let S be the surface enclosed by C on the plane $y + z = 2$. S is the graph of $z = g(x, y) = 2 - y$ over the disk $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

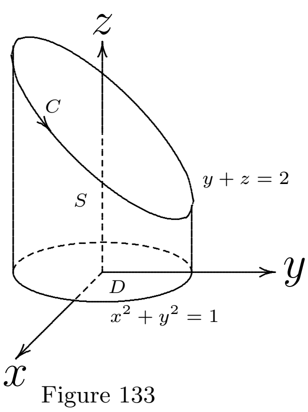


Figure 133

Also curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y)\mathbf{k}$.

By Stokes' Theorem,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S} = \iint_D (1 + 2y)\, dA = \int_0^{2\pi}\int_0^1 (1 + 2r\sin\theta) r\,dr\,d\theta$$

$$= \int_0^{2\pi} (\frac{1}{2} + \frac{2}{3}\sin\theta)\, d\theta = \pi.$$

EXAMPLE 22.3. Use Stokes' Theorem to compute $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cylinder $x^2 + y^2 = 1$ and above the xy-plane. S as part of the sphere is given the outward orientation.

Solution. The cylinder $x^2+y^2 = 1$ intersects the upper hemisphere $z = \sqrt{4 - x^2 + y^2}$ in a curve C at height $z = \sqrt{3}$.

The curve C has a vector equation given by $\mathbf{r}(t) = \langle \cos t, \sin t, \sqrt{3}\rangle$ and $\mathbf{r}'(t) = \langle -\sin t, \cos t, 0\rangle$. Also $\mathbf{F}(\mathbf{r}(t)) = \langle \sqrt{3} \sin t, \sqrt{3} \cos t, \cos t \sin t\rangle$.

By Stokes' Theorem,

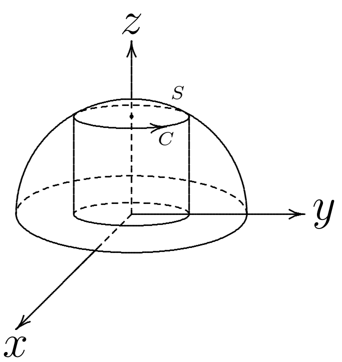


Figure 134

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} (-\sqrt{3} \sin^2 t + \sqrt{3} \cos^2 t)\, dt = 0.$$

Note that one can compute a surface integral simply by knowing the values of $\mathbf{F}$ on the boundary curve C.

This means if we have another oriented surface with the same boundary curve C, then we get exactly the same value for the surface integral.

If S and S' are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' theorem, then

$$\iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S'} (\text{curl } \mathbf{F}) \cdot d\mathbf{S}.$$

This fact is useful when it is hard to integrate over one surface but easy to integrate over the other.

Corollary 22.4. *If* curl $\mathbf{F} = \mathbf{0}$ *on all of* $\mathbb{R}^3$*, then* $\mathbf{F}$ *is conservative.*

Proof. By Stokes' Theorem,

$\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0$ for all simple closed curve C in $\mathbb{R}^3$.

By cutting any closed curve into a finite number of simple closed curves, the line integral is zero for any closed curve. Thus $\mathbf{F}$ is conservative by 19.6.

23. The Divergence Theorem

THEOREM 23.1. *(The Divergence Theorem or Gauss' Theorem) Let E be a solid region which is both of type I, II and III, and let S be the boundary of E, given with the positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region containing E. Then*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F}\, dV.$$

Proof. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then div $\mathbf{F} = P_x + Q_y + R_z$.

Thus,

$$\iiint_E \text{div } \mathbf{F}\, dV = \iiint_E P_x\, dV + \iiint_E Q_x\, dV + \iiint_E R_x\, dV.$$

Let $\mathbf{n}$ be the unit outward normal of S. Then

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \mathbf{n}\, dS \\ &= \iint_S P\mathbf{i} \cdot \mathbf{n}\, dS + \iint_S Q\mathbf{j} \cdot \mathbf{n}\, dS + \iint_S R\mathbf{k} \cdot \mathbf{n}\, dS.\end{aligned}$$

We shall show

$$\iint_S P\mathbf{i} \cdot \mathbf{n}\, dS = \iiint_E P_x\, dV, \quad \iint_S Q\mathbf{j} \cdot \mathbf{n}\, dS = \iiint_E Q_y\, dV,$$

and $\displaystyle\iint_S R\mathbf{k} \cdot \mathbf{n}\, dS = \iiint_E R_z\, dV.$

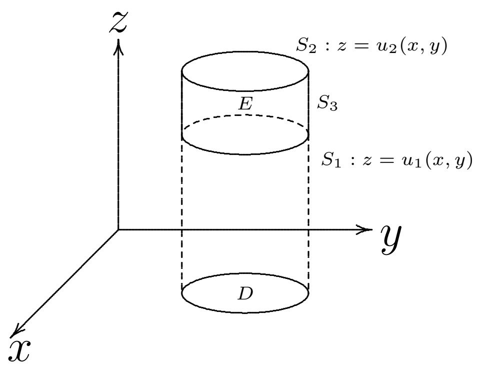


Figure 136 Type 1 solid region

To prove the third equation, we use the fact that E is a type I solid region:

$$E = \{(x, y, z) \mid u_1(x, y) \leq z \leq u_2(x, y), (x, y) \in D\},$$

where D is the projection of E onto the xy-plane.

The boundary of E consists of S_1, S_2, and S_3.

On S_3, $\mathbf{n}$ is perpendicular to $\mathbf{k}$. Thus $\displaystyle\iint_{S_3} R\mathbf{k} \cdot \mathbf{n}\, dS = 0.$

The surface S_2 is given as the graph of $z = u_2(x, y)$ with $(x, y) \in D$.

On S_2, the outward normal is given by

$$\mathbf{n} = \langle -(u_2)_x, -(u_2)_y, 1\rangle / ((u_2)_x^2 + (u_2)_y^2 + 1)^{\frac{1}{2}}.$$

Thus,

$$\iint_{S_2} R\mathbf{k} \cdot \mathbf{n}\, dS = \iint_D R(x, y, u_2(x, y))\, dA.$$

The surface S_1 is given as the graph of $z = u_1(x, y)$ with $(x, y) \in D$.

On S_1, the outward normal is given by

$$\mathbf{n} = \langle (u_1)_x, (u_1)_y, -1 \rangle / ((u_1)_x^2 + (u_1)_y^2 + 1)^{\frac{1}{2}}.$$

Thus,

$$\iint_{S_1} R\mathbf{k} \cdot \mathbf{n} \, dS = -\iint_D R(x, y, u_1(x, y)) \, dA.$$

Therefore,

$$\iint_S R\mathbf{k} \cdot \mathbf{n} \, dS = \iint_D R(x, y, u_2(x, y)) - R(x, y, u_1(x, y)) \, dA$$

$$= \iint_D [R(x, y, z)]_{z=u_1(x,y)}^{z=u_2(x,y)} \, dA = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} \frac{\partial R}{\partial z} dz \, dA$$

$$= \iiint_E R_z \, dV.$$

Similarly, using the fact that E is a type II and III solid region, one can prove the second and first equation. Combining the three equations, we get the Divergence Theorem.

EXAMPLE 23.2. Let $\mathbf{F}(x,y,z) = z\mathbf{i}+y\mathbf{j}+x\mathbf{k}$. Evaluate $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S}$, where S is the unit sphere $x^2+y^2+z^2=1$ given with the outward orientation.

Solution.
By the Divergence Theorem, $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E 1\,dV$ = volume of the unit ball $= 4\pi/3$.

EXAMPLE 23.3. Evaluate $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = xy\mathbf{i} + (y^2 + e^{xz^2})\mathbf{j} + \sin(xy)\mathbf{k}$ and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0, y = 0$ and $y + z = 2$. S is given the outward orientation.

Solution. The solid region E can be described as

$$E = \{(x,y,z) \mid -1 \le x \le 1, 0 \le z \le 1 - x^2, 0 \le y \le 2 - z\}.$$

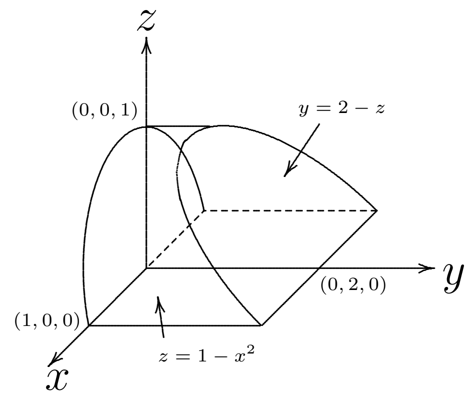


Figure 137

By the Divergence Theorem, we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 3y \, dV =$$

$$3 \int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} y \, dydzdx = \frac{184}{35}.$$

Bibliography

[1] Stewart,J., *Calculus*, Brooks/Cole., 5th edition, 2003