Proof that eigenvalues in left half plane implies stability:

$$\dot{x} = f(x) \qquad f(\bar{x}) = 0$$

<u>Step 1</u> set $y = x - \bar{x}$

$$\dot{y} = f(y + \bar{x})$$

Write $f(y+\bar{x}) = Df(\bar{x})y + g(y)$ (i.e. expanding in Taylor series)

and $g(y)$ is "higher order". This is interpreted as follows for our purposes: given $\delta > 0$ there is an ε-ball in y so that

$$|g(y)| \leq \delta |y| \quad (1)$$

Also set $A = Df(\bar{x})$

<u>Step 2</u> Assume that for all eigenvalues λ of A we have:

$$\mathrm{Re}\,\lambda < \alpha < 0$$

then there is a basis on $\mathbb{R}^n$ so that

$$\langle Ay, y\rangle \leq \alpha \langle y, y\rangle \quad (2)$$

Note: (1) $\langle y, y\rangle = y_1^2 + y_2^2 + \dots + y_n^2$ i.e. the "dot" product

(2) estimate (2) involves non-trivial linear algebra including the Jordan canonical form.

Step 3 Calculate

$$\frac{d}{dt}|y|^2 = \frac{d}{dt}\langle y,y\rangle = 2\left\langle \frac{dy}{dt}, y\right\rangle$$

$$= 2\left\{\langle Ay, y\rangle + \langle g(y), y\rangle\right\}$$

using estimates (1) and (2)

$$\frac{d}{dt}\langle y,y\rangle \le 2\left\{\alpha\langle y,y\rangle + \delta\langle y,y\rangle\right\}$$

$$\Rightarrow \frac{d}{dt}|y|^2 \le 2(\alpha+\delta)|y|^2 \quad (3)$$

Step 4 Let us recall the goal: to show that if $\alpha < 0$ (i.e. $\mathrm{Re}\,\lambda < 0$ for all eigenvalues of A) then we can find a ~~ball~~ neighborhood around $\bar{x}$, new co-ords → (say $|x-\bar{x}| \le \varepsilon$) so that if x_0 is in this ~~ball~~ nbhd then $x(t) \to \bar{x}$ as $t \to +\infty$ and $x(t)$ stays in the ~~ball~~ nbh.

(note: changed ball → nbhd = neighborhood as we will find a ball in the co-ordinates for which (2) holds and this is just a nbhd. in the original co-ordinates).

Since $y = x - \bar{x}$, we can use (3) and need to show that (3) $\Rightarrow$ if $|y_0| < \varepsilon$ (for some $\varepsilon > 0$) then

$$|y(t)| \to 0 \text{ as } t \to +\infty$$

$$\& \quad |y(t)| < \varepsilon \quad \forall\, t \ge 0.$$

Set $z = |y|^2$ (3) says

$$\frac{dz}{dt} \leq 2(\alpha+\delta) z$$

$$\Rightarrow \quad \frac{1}{z}\frac{dz}{dt} \leq 2(\alpha+\delta) \quad (4)$$

and this holds for some $\varepsilon > 0$ & $0 \leq z < \varepsilon^2$.

Integrating (4) $\Rightarrow$

$$\int_0^t \frac{1}{z}\frac{dz}{dt}\,dt \leq \int_0^t 2(\alpha+\delta)\,dt$$

$$\ln \frac{z(t)}{z(0)} \leq 2(\alpha+\delta)t$$

$$\Rightarrow \quad z(t) \leq z(0)\, e^{2(\alpha+\delta)t}$$

$$\Rightarrow \quad |y(t)|^2 \leq |y_0|\, e^{2(\alpha+\delta)t}$$

taking square roots of both sides $\Rightarrow$

$$|y(t)| \leq |y_0|\, e^{(\alpha+\delta)t}$$

Since $\alpha < 0$, choose $\varepsilon > 0$ so that $\alpha + \delta < 0$ and we see that if $|y_0| < \varepsilon$ then $|y(t)| < \varepsilon$ and

$$|y(t)| \to 0 \quad \text{as} \quad t \to +\infty$$

as desired.