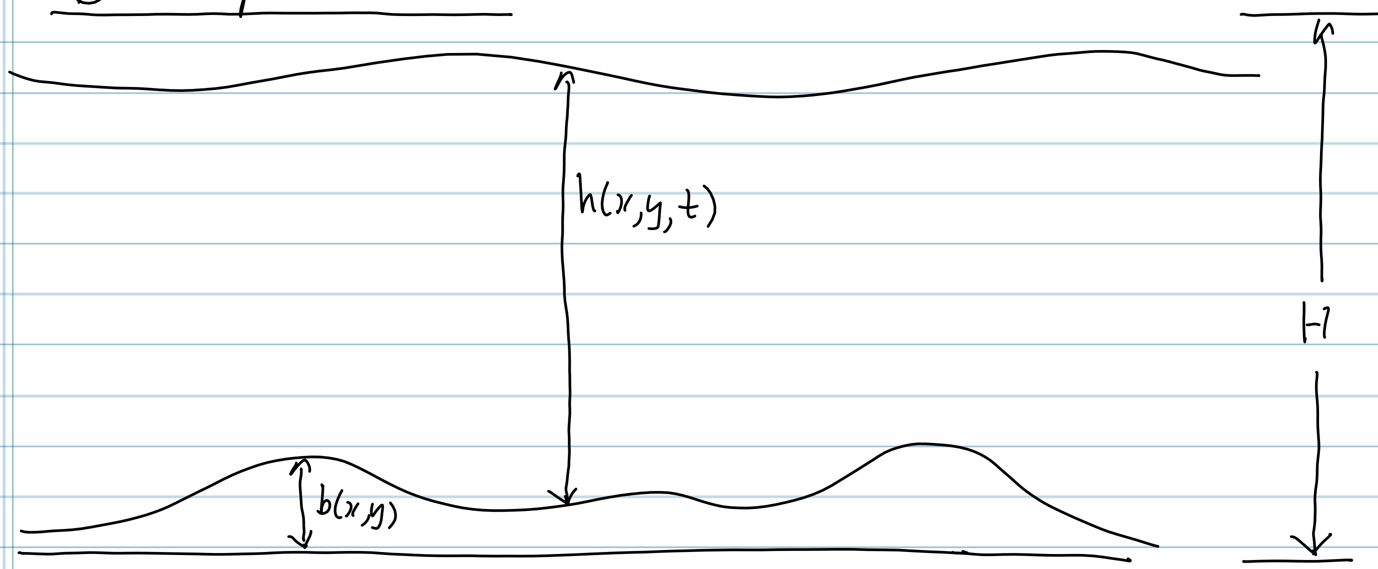
Barotropic Flow



Barotropic : $w = 0$ (no vertical velocity)

Shallow-water model :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial}{\partial x}(h+b)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial}{\partial y}(h+b)$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

Potential Vorticity:

$\omega = \dfrac{\zeta + f}{h}$ is conserved along trajectories

$\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

ζ = local fluid) vorticity
f = ambient planetary vorticity.

In continuity eqn. since ~~$h \ll H$~~ $h \simeq H$

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$d\left|\frac{\partial h}{\partial t}\right| \ll H \quad \Rightarrow \quad u_x + v_y = 0$$

$\Rightarrow \exists$ streamfunctn $\psi \ni$

$$u = -\psi_y$$

$$v = \psi_x$$

$$\Rightarrow \zeta = \psi_{xx} + \psi_{yy} = \nabla^2\psi \quad (= \Delta\psi)$$

Calculate equation for vorticity in SW model:

$$\frac{\partial}{\partial y} \rightarrow \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial}{\partial x} \rightarrow \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y}$$

$$(1) \quad \frac{\partial}{\partial t}u_y + u_y\frac{\partial u}{\partial x} + u\frac{\partial u_y}{\partial x} + v_y\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} - f_y v - f v_y = -g\frac{\partial^2 h}{\partial x \partial y}$$

$$(2) \quad \frac{\partial}{\partial t}v_x + u_x\frac{\partial v}{\partial x} + u\frac{\partial v_x}{\partial x} + v_x\frac{\partial v}{\partial y} + v\frac{\partial^2 v_y}{\partial x \partial y} + f_x u + f u_x = -g\frac{\partial^2 h}{\partial x \partial y}$$

$$(2)-(1) \Rightarrow \frac{\partial}{\partial t}(v_x - u_y) + (v_x - u_y)\frac{\partial u}{\partial x} + (v_x - u_y)\frac{\partial v}{\partial y} + u\frac{\partial}{\partial x}(v_x - u_y) + v\frac{\partial}{\partial y}(u_x - v_y)$$

$$+ u f_x + v f_y + f(u_x + v_y) = 0$$

$$\cancel{\Rightarrow \frac{d\zeta}{dt} + \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{df}{dt} = 0}$$

$$\cancel{\Rightarrow \frac{d}{dt}}$$

$$\Rightarrow \frac{d\zeta}{dt} + \zeta(u_x + v_y) + \frac{d}{dt}f + f(u_x + v_y) = 0$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + h(u_x + v_y) = 0$$

$$u_x + v_y = -\frac{1}{h}\frac{dh}{dt}$$

$$\frac{d}{dt}(\zeta + f) + (\zeta + f)\left(-\frac{1}{h}\frac{dh}{dt}\right) = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0.}$$ Conservation of PV

$$\frac{d}{dt}\left(\frac{\zeta+f}{h}\right) = \frac{\partial}{\partial t}\left(\frac{\zeta+f}{h}\right) + u\frac{\partial}{\partial x}\left(\frac{\zeta+f}{h}\right) + v\frac{\partial}{\partial y}\left(\frac{\zeta+f}{h}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\zeta+f}{h}\right) - \psi_y\left(\frac{\zeta+f}{h}\right)_x + \psi_x\left(\frac{\zeta+f}{h}\right)_y = 0$$

Set $J(\rho,\eta) = \frac{\partial \rho}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial \eta}{\partial x}$ (Jacobian bracket)

$$\frac{\partial}{\partial t}\left(\frac{\zeta+f}{h}\right) + J\left(\psi, \frac{\zeta+f}{h}\right) = 0$$

Note: can add other forces

* viscosity
* Ekman damping
* viscous drag
* wind forcing

& RHS (check carries thru' for SW model)

$$\frac{\partial}{\partial t}\left(\frac{\zeta + f}{h}\right) + J\left(\psi, \frac{\zeta + f}{h}\right) = F$$

$$\zeta = \nabla^2 \psi$$

Expand f in Taylor series at particular latitude

$$f = \underbrace{f_0}_{f\text{-plane}} + \underbrace{\beta_0 y}_{\beta\text{-plane}} + \cdots$$

Next, $h = H - b(x,y) + \eta(x,y,t)$

$$\frac{\zeta + f}{h} = \frac{\zeta + f}{H - b + \eta} = \frac{\zeta/H + f/H}{1 - \frac{b}{H} + \frac{\eta}{H}} = \frac{\zeta/H + \frac{f}{H}}{1 + \frac{\eta - b}{H}}$$

$$= \left(\frac{\zeta}{H} + \frac{f}{H}\right)\left(1 - \frac{\eta - b}{H} + \cdots\right)$$

$$\approx \left(\frac{\zeta}{H} + \frac{f}{H}\right)\left(1 + \frac{\cancel{\eta - b}\ b}{H}\right)$$

$$\frac{\zeta}{H} \to \zeta$$

$$\frac{f}{H} \to f$$

PV eqn. $\Rightarrow$

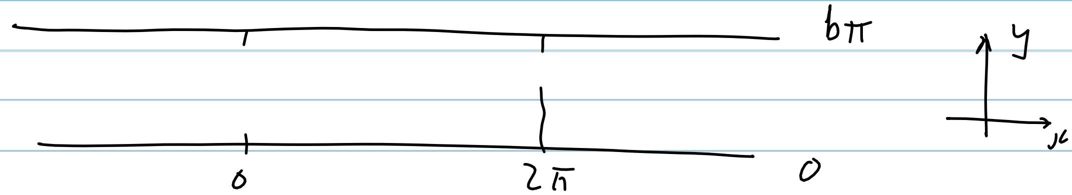
$$\frac{\partial}{\partial t}\left(\zeta + f_0 + \beta_0 y\right) + J\left(\psi, \zeta + f + \frac{b}{H}(\zeta + f)\right) = F$$

$$\frac{\partial \zeta}{\partial t} + J\left(\psi, \zeta + \beta_0 y + \frac{f_0 b}{H}\right) = F$$

$b(x,y) \to h(x,y)$ topography

$$\boxed{\frac{\partial \zeta}{\partial t} + J\left(\psi, \zeta + \beta_0 y + \frac{f_0 h}{H}\right) = F}$$

Pose on a channel:



$\cancel{\zeta(0) = \zeta(0,y,t) = \zeta(2\pi,y,t)}$ $\quad \psi(0,y,t) = \psi(2\pi,y,t)$

$$\frac{\partial \psi}{\partial x}(x,0,t) = \frac{\partial \psi}{\partial x}(x,b\pi,t) = 0$$

$$\int_0^{2\pi} \frac{\partial \psi}{\partial y}(x,0,t)\,dx = 0 \quad \& \quad \int_0^{2\pi} \frac{\partial \psi}{\partial y}(x,b\pi,t)\,dx \to$$