1 Primitive Equations

The main set of equations for geophysical fluids govern the variables: (u, v, w) the three velocities of a fluid particle in, respectively, the x, y and z directions, p the pressure and ρ the density:

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}, \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}, \\
0 &= -\frac{\partial p}{\partial z} - \rho g, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \\
\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} &= \kappa\frac{\partial^2 \rho}{\partial z^2}.
\end{aligned}$$

where $f = 2\Omega \sin\phi$.

2 Various Simplifications

- **Rapidly rotating**: forces on left hand side dominated by Coriolis ($Ro_T <<$ 1, $Ro << 1$);
- **Frictionless**: no friction force on right hand side, $\nu = 0$ ($Ek << 1$);
- **Homogeneous**: constant density ($\rho = 0$);
- **Barotropic**: u, v are independent of depth z;
- **Stratified**: non-homogeneous, i.e., density not constant, usually assume $\frac{\partial \rho}{\partial z} \neq 0$;
- **Layered model**: layers of constant density;
- **Reduced gravity model**: 1 layer over inert, infinitely deep layer;
- **f-plane**: f is a constant;
- **β-plane**: $f = f_0 + \beta y$.

2.1 Geostrophic flows

If the Coriolis forces dominate (the flow is *rapidly rotating*), then on the left hand side of the primitive equations all but the terms involving f are omitted.

If the flow is also frictionless and homogeneous then there is a balance between the Coriolis force and the HPG:

$$
\begin{aligned}
-fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \\
+fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y}.
\end{aligned}
$$

Note that p is independent of z from the third equation.

2.2 Barotropic flows-*shallow water model*

If we assume that the flow is homogeneous and frictionless and that the flow is barotropic, we can derive the shallow water model, where $h = h(x, y, t)$ is the height of the surface above the bottom, given by $b = b(x, y)$:

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv &= -g\frac{\partial h}{\partial x}, \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -g\frac{\partial h}{\partial y}, \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0.
\end{aligned}
$$

The pressure here is given by $p = \rho_0 g(h + b)$, called the dynamic pressure. If the bottom is not flat, then h in the first two equations needs to be replaced by $h + b$.

2.3 Stratified flow

We assume that $\frac{\partial \rho}{\partial z} \neq 0$, and then use the density as an independent variable replacing depth. The resulting equations look very much like the shallow water model, BUT $u = u(x, y, \rho, t)$, etc.

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv &= -\frac{1}{\rho_0}\frac{\partial P}{\partial x}, \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -\frac{1}{\rho_0}\frac{\partial P}{\partial y}, \\
\frac{\partial P}{\partial \rho} &= gz, \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0.
\end{aligned}
$$

The pressure here is actually the Montgomery potential: $P = p + \rho g z$ and $h = -c\frac{\partial z}{\partial \rho}$ with c being an arbitrary constant, usually taken to be $c = \Delta\rho$.

2.4 Reduced gravity model

This holds for a single, constant density layer of depth h over an inert ($p = 0$), infinitely deep layer. The equations are exactly the same as the shallow-water equations but with the gravity g replaced by the reduced gravity:

$$g' = g(\rho_2 - \rho_1)/\rho_0.$$

2.5 Quasi-geostrophic equation

When Ro_T, Ro, and Ek are simultaneuously small, the geohysical flow is near-geostrophic balance. The primitive equations can be simplified by reducing to the quasi-geostrophic (QG) equation.

$$\frac{\partial q}{\partial t} + J(\psi, q) = \nu \frac{\partial^2 \nabla^2 \psi}{\partial z^2}$$

$$q = \nabla^2 \psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) + \beta_0 y$$

Once the solution to the QG equation is obtained, the original variables can be derived as follows.

$$\begin{aligned}
u &= -\partial\psi/\partial y \\
v &= \partial\psi/\partial x \\
w &= -\frac{f_0}{N^2}\left[\frac{\partial^2\psi}{\partial t \partial z} + J\left(\psi, \frac{\partial \psi}{\partial z}\right)\right] \\
p' &= \rho_0 f_0 \psi \\
\rho' &= -\frac{\rho_0 f_0}{g}\frac{\partial \psi}{\partial z}
\end{aligned}$$

3 Special Effects

3.1 Vertical rigidity

Geostrophic flows are barotropic. In other words,

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0,$$

which implies that vertical water columns move in concert (Taylor-Proudman Theorem). On the f-plane, the flow is horizontally non-divergent:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

and hence $\frac{\partial w}{\partial z} = 0$ and the vertical velocity is independent of depth.

If the flow has a flat bottom, or a flat surface, then the flow is strictly two-dimensional. This also happens in the shallow water model if both a flat bottom and surface are assumed.

3.2 Potential vorticity

In the **shallow water model**, potential vorticity is conserved:

$$\frac{d}{dt}\left(\frac{f+\zeta}{h}\right) = 0,$$

where

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

is the relative vorticity. This also holds in stratified flow, but h is now given by $h = -c\frac{\partial z}{\partial \rho}$ and not the height.

4 Some important constants

Temporal Rossby number:

$$Ro_T = \frac{1}{\Omega T}$$

Rossby number:

$$Ro = \frac{U}{\Omega L}$$

Eckman number:

$$Ek = \frac{\nu}{\Omega H^2}$$

Rossby radius of deformation (external):

$$R = \frac{\sqrt{gH}}{f}$$

Brunt-Väisälä frequency:

$$N(z) = \sqrt{\frac{-g}{\rho_0}\frac{d\bar{\rho}}{dz}}$$

Burger's number:

$$Bu = \frac{N^2 H^2}{f_0^2 L^2}$$

Internal radius of deformation:

$$R = \frac{NH}{f_0} = \frac{\sqrt{g' H}}{f_0}$$