NOTES AND CORRESPONDENCE

Long-Term Variations of Daily Insolation and Quaternary Climatic Changes

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ABSTRACT

The first part of this note provides all trigonometrical formulas which allow the direct spectral analysis and the computation of those long-term variations of the earth's orbital elements which are of primary interest for the computation of the insolation. The elements are the eccentricity, the longitude of the perihelion, the precessional parameter and the obliquity. This new formulary is much more simple to use than the ones previously designed and still provides excellent accuracy, mainly because it takes into account the influence of the most important higher order terms in the series expansions. The second part is devoted to the computation of the daily insolation both for calendar and solar dates.

1. Long-term variations of the earth's orbital elements

The energy available at any given latitude ϕ on the earth (on the assumption of a perfectly transparent atmosphere) is a single-valued function (Berger, 1975a) of the solar constant S_0, the semi-major axis a of the ecliptic, its eccentricity e, its obliquity ϵ and the longitude of the perihelion $\tilde{\omega}$ measured from the moving vernal equinox (Fig. 1). To determine the time variation of such an insolation during the Quaternary ice age for example, thus requires the long-term variations of these orbital elements of the earth. As S_0 has been taken as 1353 W m^{-2} (Thekaekara, 1975) and as a has no purely secular part when the perturbations of the second order are included (Brouwer and Clemence, 1961), only the long-term variations of e, ϵ and $\tilde{\omega}$ must be determined.

From a numerical analysis of the astronomical solutions used to compute the elements of the earth's orbit over periods of time of the order of 10^6 years or more, I have shown (Berger, 1977a) that an improved, significantly different solution results if more terms are kept in the series expansions. Such long-term variations for e, ϵ and $\tilde{\omega}$ have been graphically reproduced (Berger, 1976a), the computations having been made through the classical and not easily manageable astronomical formulas.

In this note, I would like to provide the reader with trigonometric expansions of the classical astro-insolation parameters ϵ, $e\sin\tilde{\omega}$ and e:

$$\epsilon=\epsilon^*+\sum A_i\cos(f_it+\delta_i), \tag{1}$$

$$\begin{aligned} e\sin\tilde{\omega}&=\sum P_i\sin(\alpha_it+\zeta_i),\\ e\cos\tilde{\omega}&=\sum P_i\cos(\alpha_it+\zeta_i),\end{aligned} \tag{2}$$

$$e=e_0+\sum E_i\cos(\lambda_it+\phi_i), \tag{3}$$

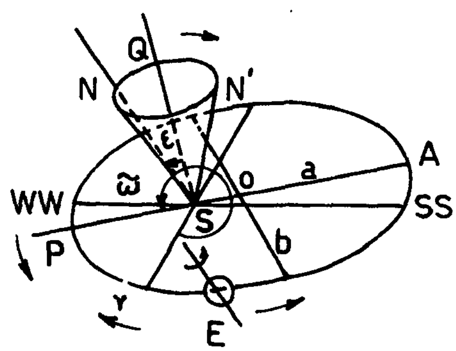

FIG. 1. Elements of the earth's orbit. The orbit of the earth E around the sun is represented by the ellipse PγEA, P being the perihelion and A the aphelion. Its eccentricity e is given by $(a^2-b^2)^{\frac{1}{2}}/a$, a being the semi-major axis and b the semi-minor axis. WW and SS are, respectively, the winter and the summer solstice and γ is the vernal equinox; WW, SS and γ are located where they are today. SQ is perpendicular to the ecliptic and the obliquity ϵ is the inclination of the equator upon the ecliptic, i.e., ϵ is equal to the angle between the earth's axis of rotation SN and SQ. $\tilde{\omega}$ is the longitude of the perihelion relatively to the moving vernal equinox and is equal to $\pi+\psi$. The annual general precession in longitude ψ describes the absolute motion of γ along the earth's orbit relative to the fixed stars. π, the longitude of the perihelion, is measured from the reference vernal equinox and describes the absolute motion of the perihelion relatively to the fixed stars. For any numerical value of $\tilde{\omega}$, 180° is subtracted for a practical purpose because observations are made from the earth, and the sun is considered as revolving around the earth.

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where for convenience, $t=0$ will refer to 1950.0 AD and t will be negative for the past.

Starting with the classical system of the long-term planetary motion for the eccentricity e, the longitude of the perihelion based on the fixed equinox π, the inclination to the ecliptic i and the longitude of the ascending node Ω (Bretagnon, 1974)

$$e \frac{\sin}{\cos} \pi = \sum_{j=1}^{19} M_j \frac{\sin}{\cos}(g_j t+\beta_j), \tag{4}$$

$$\sin i \frac{\sin}{\cos} \Omega = \sum_{i=1}^{15} N_i \frac{\sin}{\cos}(s_i t+\delta_i), \tag{5}$$

TABLE 1. Obliquity relative to the mean ecliptic of date. Amplitude, mean rate, phase and period are listed for each of the 47 first terms of the series expansion of ϵ.

	Obliquity relative to mean ecliptic of date			
Term	Obliquity (″)	Mean rate (″ year^{-1})	Phase (°)	Period (years)
1	−2462.22	31.609970	251.90	41000
2	−857.32	32.620499	280.83	39730
3	−629.32	24.172195	128.30	53615
4	−414.28	31.983780	292.72	40521
5	−311.76	44.828339	15.37	28910
6	308.94	30.973251	263.79	41843
7	−162.55	43.668243	308.42	29678
8	−116.11	32.246689	240.00	40190
9	101.12	30.599442	222.97	42354
10	−67.69	42.681320	268.78	30365
11	24.91	43.836456	316.79	29564
12	22.58	47.439438	319.60	27319
13	−21.16	63.219955	143.80	20500
14	−15.65	64.230484	172.73	20177
15	15.39	1.010530	28.93	1282495
16	14.67	7.437771	123.59	174246
17	−11.73	55.782181	20.20	23233
18	10.27	0.373813	40.82	3466974
19	6.49	13.218362	123.47	98045
20	5.85	62.583237	155.69	20708
21	−5.49	63.593765	184.62	20379
22	−5.43	76.438309	267.27	16955
23	5.16	45.815262	55.01	28288
24	5.08	8.448301	152.52	153404
25	−4.07	56.792709	49.13	22820
26	3.72	49.747849	204.66	26051
27	3.40	12.058272	56.52	107478
28	−2.83	75.278214	200.32	17216
29	−2.66	65.241013	201.66	19865
30	−2.57	64.604294	213.55	20061
31	−2.47	1.647247	17.03	786767
32	2.46	7.811584	164.41	165907
33	2.25	12.207832	94.54	106161
34	−2.08	63.856659	131.91	20295
35	−1.97	56.155991	61.03	23079
36	−1.88	77.448837	296.20	16734
37	−1.85	6.801054	135.48	190559
38	1.82	62.209412	114.87	20833
39	1.76	20.656128	247.06	62742
40	−1.54	48.344406	256.61	26808
41	1.47	55.145462	32.10	23501
42	−1.46	69.000534	143.68	18782
43	1.42	11.071350	16.87	117059
44	−1.18	74.291306	160.68	17445
45	1.18	11.047742	27.59	117309
46	−1.13	0.636717	348.10	2035441
47	1.09	12.844549	82.64	100899

TABLE 2. Precessional parameter. Amplitude, mean rate, phase and period are listed for each of the first 46 terms of the series expansion of $e\sin\tilde{\omega}$.

	Eccentricity and longitude of moving perihelion			
Term	Amplitude	Mean rate (″ year^{-1})	Phase (°)	Period (years)
1	0.0186080	54.646484	32.01	23716
2	0.0162752	57.785370	197.18	22428
3	−0.0130066	68.296539	311.69	18976
4	0.0098883	67.659821	323.59	19155
5	−0.0033670	67.286011	282.76	19261
6	0.0033308	55.638351	90.58	23293
7	−0.0023540	68.670349	352.52	18873
8	0.0014002	76.656036	131.83	16907
9	0.0010070	56.798447	157.53	22818
10	0.0008570	66.649292	294.66	19445
11	0.0006499	53.504456	118.25	24222
12	0.0005990	67.023102	335.48	19337
13	0.0003780	68.933258	299.80	18801
14	−0.0003370	56.630219	149.16	22885
15	0.0003334	86.256454	283.91	15025
16	0.0003334	23.036499	320.11	56259
17	0.0002916	89.395340	89.08	14497
18	0.0002916	26.175385	125.27	49512
19	0.0002760	69.307068	340.62	18699
20	−0.0002330	99.906509	203.60	12972
21	−0.0002330	36.686569	239.79	35326
22	0.0001820	67.864838	155.48	19097
23	0.0001772	99.269791	215.49	13055
24	0.0001772	36.049850	251.68	35950
25	−0.0001740	56.625275	130.23	22887
26	−0.0001240	68.856720	214.05	18822
27	0.0001153	87.266983	312.84	14851
28	0.0001153	22.025970	291.17	58840
29	0.0001008	90.405869	118.01	14335
30	0.0001008	25.164856	96.34	51500
31	0.0000912	78.818680	160.31	16443
32	0.0000912	30.474274	83.70	42528
33	−0.0000806	100.917038	232.53	12842
34	−0.0000806	35.676025	210.86	36327
35	0.0000798	81.957565	325.48	15813
36	0.0000798	33.613159	248.87	38556
37	−0.0000638	92.468735	80.00	14016
38	−0.0000638	44.124329	3.39	29372
39	0.0000612	100.280319	244.42	12924
40	0.0000612	35.039322	222.75	36987
41	−0.0000603	98.895981	174.67	13105
42	−0.0000603	35.676025	210.86	36327
43	0.0000597	87.248322	342.48	14854
44	0.0000597	24.028381	18.68	53936
45	0.0000559	86.630264	324.73	14960
46	0.0000559	22.662689	279.28	57187

TABLE 3. Eccentricity. Amplitude, mean rate, phase and period are listed for each of the 42 first terms of the series expansion of e.

Expansion of eccentricity amplitude larger than 0.0004

Term	Amplitude	Mean rate (″ year^{-1})	Phase (°)	Period (years)
1	0.01102940	3.138886	165.16	412885
2	−0.00873296	13.650058	279.68	94945
3	−0.00749255	10.511172	114.51	123297
4	0.00672394	13.013341	291.57	99590
5	0.00581229	9.874455	126.41	131248
6	−0.00470066	0.636717	348.10	2305441
7	−0.00254464	12.639528	250.75	102535
8	0.00231485	0.991874	58.57	1306618
9	−0.00221955	9.500642	85.58	136412
10	0.00201868	2.147012	106.59	603630
11	−0.00172371	0.373813	40.82	3466974
12	−0.00166112	12.658184	221.11	102384
13	0.00145096	1.010530	28.93	1282495
14	0.00131342	12.021467	233.00	107807
15	0.00101442	0.373813	40.82	3466974
16	−0.00088343	14.023871	320.50	92414
17	−0.00083395	6.277772	330.33	206443
18	0.00079475	6.277772	330.33	206443
19	0.00067546	27.300110	199.37	47472
20	−0.00066447	10.884985	155.34	119063
21	0.00062591	21.022339	229.03	61649
22	0.00059751	22.009552	99.82	58884
23	−0.00053262	27.300110	199.37	47472
24	−0.00052983	5.641055	342.22	229744
25	−0.00052983	6.914489	318.44	187433
26	0.00052836	12.002811	262.64	107975
27	0.00051457	16.788940	84.85	77194
28	−0.00050748	11.647654	192.18	111267
29	−0.00049048	24.535049	75.02	52822
30	0.00048888	18.870667	294.65	68678
31	0.00046278	26.026688	223.15	49795
32	0.00046212	8.863925	97.48	146211
33	0.00046046	17.162750	125.67	75512
34	0.00042941	2.151964	125.52	602241
35	0.00042342	37.174576	325.78	34863
36	0.00041713	19.748917	252.82	65624
37	−0.00040745	21.022339	229.03	61649
38	−0.00040569	3.512699	205.99	368947
39	−0.00040569	1.765073	124.34	468704
40	−0.00040385	29.802292	16.43	43487
41	0.00040274	7.746099	350.17	167310
42	0.00040068	1.142024	273.75	1134827

and using the method developed by Sharaf and Budnikova (1967), the amplitudes A_i, P_i, E_i, the mean rates f_i, α_i, λ_i and the phases δ_i, ζ_i, ϕ_i occurring in Eqs. (1)–(3) have been obtained and the main terms reproduced in Tables 1, 2 and 3. The constants of integration ϵ^* and e_0 deduced from the initial conditions are

$$\epsilon^* = 23°\ 320\ 556, \quad e_0 = 0.028\ 707.$$

These trigonometrical series not only give an easy way to compute the astro-insolation parameters but also provide their spectra (Berger, 1977b) as needed in the validation process of the astronomical theory (Hays *et al.*, 1976).

2. Numerical computations of the earth's orbital elements and accuracy

In the series expansion of ϵ, from 240 terms, 104 have amplitudes larger than 0.1″, 47 larger than 1″ and 24 lead to deviations generally less than 0.002°. For $e\sin\tilde{\omega}$, among the 589 terms, 117 have amplitudes larger than 10^{-5} and 46 terms lead to a deviation less than 0.5° for $\tilde{\omega}$ and less than 0.0003 for e. As far as e is concerned, since the series expansions (2) and especially (3) are so slowly convergent, its numerical computation is recommended through (4) and Table 4, in order to save time and accuracy. For the same reasons, $\tilde{\omega}$ must be computed using the relation

$$\tilde{\omega} = \pi + \psi, \tag{6}$$

where π is given by (4) and the general precession ψ by

$$\psi = \tilde{\psi}t + \zeta + \sum F_i \sin(f_i' t + \delta_i'), \tag{7}$$

where

$$\tilde{\psi} = 50''\ 439\ 273, \quad \zeta = 3°\ 392\ 506.$$

In Eq. (7), 177 terms have an amplitude larger than 1″, but only 9 terms (Table 5) provide the required accuracy. Details about the derivation of all these equations are available in Berger (1978b) where a simple algorithm for the computation of the astro-insolation parameters and the daily insolation is also provided in Fortran.

Maximum accuracy for such a solution is needed to limit the cumulative effect in computational approximations (Berger, 1975b) and to allow input into the climatic models to be of real value. Thus the influence of the new astronomical solution on the deviations of solar radiation from their 1950 values for the classical caloric seasons of Milankovitch (1941), as recently recomputed by Vernekar (1972), has been shown in Berger (1978a) to reach as much as 10–20% in some cases. The influence on the daily insolation values will be presented in the next section.

3. *Annual variation of daily insolation*

Mainly there are two different approaches to the computation of daily insolation. Both are related to the choice of the day in the year. There will be an equinoctial type of daily insolation and a calendar day insolation.

The first is the easiest to compute. Indeed, if the insolation at equinoxes, solstices or other fixed positions of the earth relatively to the vernal equinox is considered, a constant increment of the true longitude λ must be used starting with $\lambda = 0$ at the vernal equinox. The mid-month values are thus defined by $\Delta\lambda = 30°$ and, in this case, they will be located around the 20th of each month. Because the length of the astronomical seasons is secularly variable, these mid-month values are not related to a fixed calendar date.

If the daily insolation is computed for specific calendar dates or for a whole month, the mean longitude λ_m has to be used. Because λ_m does not go to zero at the same time as λ, we employ the following strategy:

1) We let the origin of time be 21.0 March, the time of the vernal equinox ($\lambda=0$).

2) We determine λ_m at this date through the application of the following formula (Brouwer and Clemence, 1961):

$$\lambda_{m0}=\lambda-2[(\tfrac{1}{2}e+\tfrac{1}{8}e^3)(1+\beta)\sin(\lambda-\tilde{\omega}) -\tfrac{1}{4}e^2(\tfrac{1}{2}+\beta)\sin 2(\lambda-\tilde{\omega}) +\tfrac{1}{8}e^3(\tfrac{1}{3}+\beta)\sin 3(\lambda-\tilde{\omega})],$$

where

$$\beta=(1-e^2)^{\frac{1}{2}}.$$

3) For each value of λ_m obtained through an increment $\Delta\lambda_m$, i.e., $\lambda_m=\lambda_{m0}+\Delta\lambda_m$, we determine λ from

$$\lambda=\lambda_m+(2e-\tfrac{1}{4}e^3)\sin(\lambda_m-\tilde{\omega})+(5/4)e^2\sin 2(\lambda_m-\tilde{\omega}) +(13/12)e^3\sin 3(\lambda_m-\tilde{\omega}).$$

4) The daily insolation for such a calendar date is then obtained through the formulas given in the Appendix.

An important remark must be made as far as the meaning of daily mid-month and calendar date insolation is concerned. It is related to the secular drift of the dates of the solstices and the autumnal equinox relatively to the vernal equinox, i.e., to the long-term variations of the length of the astronomical seasons. Insolation values will be given here in watts per square meter but can easily be transformed to calories per square centimeter per day to allow fast comparison with previous computations (Milankovitch–Vernekar–Berger).

First, a comparison is made between the 60°N daily insolation at present and 10 000 years BP[2], for $\lambda=210°$. The difference between these "mid-month" insolations amounts to 5 W m^{-2}, but $\lambda=210°$ presently refers to 24 October (109 W m^{-2}) and, at 10 000 years BP, it referred to 16 October (104 W m^{-2}). Thus, the difference reflects mainly the secular changes of both obliquity and shape of the ecliptic. Second, if a calendar date insolation is considered, the long-term variations of the length of the astronomical seasons is explicitly recognized. For the same latitude and years, if daily insolations at 16 October are compared, a difference of 29 W m^{-2} is found. This is a result of the fact that on 16 October the true longitude of the earth is presently 202° (133 W m^{-2}) and, at 10 000 years BP, 210° (104 W m^{-2}).

As far as the accuracy is concerned, three main tests were performed. One was to determine the sensitivity to the number of terms kept in the series expansions, the second was to check the improvement of this solution over previous results (Berger, 1976b, hereafter referred to as Berger 1), and the third concerned the values obtained by Vernekar (1977) from Sharaf–Budnikova earth's orbital elements. Each comparison relates to the last 10^6 years.

We will use the label Berger 2 for the mid-month daily insolation solution obtained from Eqs. (1), (4) and (7) for which 104, 19 and 177 terms, respectively, have been kept. This solution is identical to Berger 1, except for a few cases where small differences arise in the pole region at the summer solstice; there were 17 cases where the absolute value of this difference ranged between 1 and 2 W m^{-2}, a discrepancy of less than 4‰. If we remember (Bernard, 1962) that a change in the daily insolation at the solstices and the equinoxes is due

[2] BP = Before Present.

TABLE 4. Fundamental elements of the ecliptic. Amplitudes, mean rate, phase and period are listed for each term of the series expansions of eccentricity and longitude of the perihelion ($e\sin\pi$).

Eccentricity and longitude of fixed perihelion				
Term	Amplitude	Mean rate ($''$ year^{-1})	Phase (°)	Period (years)
1	0.01860798	4.2072050	28.62	308043
2	0.01627522	7.3460910	193.78	176420
3	−0.01300660	17.8572630	308.30	72576
4	0.00988829	17.2205460	320.19	75259
5	−0.00336700	16.8467330	279.37	76929
6	0.00333077	5.1990790	87.19	249275
7	−0.00235400	18.2310760	349.12	71087
8	0.00140015	26.2167580	128.44	49434
9	0.00100700	6.3591690	154.14	203800
10	0.00085700	16.2100160	291.26	79951
11	0.00064990	3.0651810	114.86	422814
12	0.00059900	16.5838290	332.09	78148
13	0.00037800	18.4939800	296.41	70077
14	−0.00033700	6.1909530	145.76	209338
15	0.00027600	18.8677930	337.23	68688
16	0.00018200	17.4255670	152.09	74373
17	−0.00017400	6.1860010	126.83	209505
18	−0.00012400	18.4174410	210.66	70368
19	0.00001250	0.6678630	72.10	1940518

TABLE 5. General precession in longitude. Amplitude, mean rate, phase and period are listed for each of the first nine terms of the series expansion of ψ.

Precession relative to mean ecliptic of date				
Term	Precession ($''$)	Mean rate ($''$ year^{-1})	Phase (°)	Period (years)
1	7391.02	31.609970	251.90	41000
2	2555.15	32.620499	280.83	39730
3	2022.76	24.172195	128.30	53615
4	−1973.65	0.636717	348.10	2035441
5	1240.23	31.983780	292.72	40521
6	953.87	3.138886	165.16	412885
7	−931.75	30.973251	263.79	41843
8	872.38	44.828339	15.37	28910
9	606.35	0.991874	58.57	1306618
10	−496.03	0.373813	40.82	3466974

to a change in the precessional parameters with, for the solstices only, a small correction related to the change in the obliquity, it can be seen that these differences are almost entirely due to differences in the obliquity values, differences which amount to a maximum of 0.07° at these dates.

If the number of terms in Berger 2 is reduced to 18, 19 and 9, respectively, in (1), (4) and (7), no significant differences arise between these daily insolations and Berger 2, this being due to the fact that the contribution of the high order terms have been taken into account in the series expansions.

Finally, if we compare Berger 2 and the daily insolation obtained from the Sharaf–Budnikova formulas, differences amounting up to 3.5 W m^{-2} regularly arise for the last 100 000 years. From 100 000 to 300 000 BP, these differences increase, reach a maximum of 21 W m^{-2}, and regularly amount to between 1 and 3.6% of the daily insolation. The same is true up to 10^6 years BP, the maximum observed being 24 W m^{-2} at the South Pole, Southern Hemisphere summer (mid-December); this represents 4.5% of the daily insolation and three times the deviations from present-day values and is mainly due to the difference between corresponding ϵ values ($\Delta\epsilon=0.78°$ represents 30% of the maximum amplitude in the long-term variation of ϵ and corresponds to 16 W m^{-2}). The remaining 8 W m^{-2} comes from the difference in the precessional parameter values (0.007).

4. Application and conclusion

Mid-month and monthly mean insolation are indispensable complements to the traditional Milankovitch solar radiation values (Berger, 1978c), in simulating and explaining the long-term climatic changes that have occurred during the Quaternary ice age (Kukla, 1978). They make it possible to understand the dynamics of cooling and warming (from an insolation point of view), in particular during transitional seasons. For example, both mid-month insolation in August and the daily insolation at autumnal equinox show negative deviations around 20 000 years BP and positive deviations around 10 000 years BP, which precisely coincide, respectively, with the last glacial maximum and the beginning of the post glacial warming.

These insolation patterns have been used to tentatively forecast the insolation climate of the next 100 000 yr. Negative departures centered around 4 500 BP confirms that we are going into a glacial advance at least at a Quarternary time scale. Other significant features are the negative departures around 60 000 and 84 000 years AP[2], and the positive departures around 50 000, 71 000 and 95 000 years AP. Finally, the period extending from 20 to 40 000 AP closely resembles the situation today insofar as natural climatic changes are concerned.

[2] AP = After Present.

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APPENDIX

Daily Insolation Formulas

We define the following parameters:

- ϕ latitude
- δ declination of the sun
- ρ earth–sun distance r measured in units of the semi-major axis a
- H hour angle of the sun during the day
- v true anomaly: positional angle of the earth on its orbit, counted counterclockwise from the perihelion
- M mean anomaly: positional angle of a "mean" earth rotating around the sun with a constant angular speed equal to $2\pi/T$ and counted counterclockwise from the perihelion
- T tropical year of 365.2422 mean solar days
- λ true longitude of the earth is counted counterclockwise from the vernal equinox and is related to v through $\lambda=v+\tilde{\omega}$. As in this formula, $\tilde{\omega}$ is measured from the vernal equinox, 180° has to be added to the value numerically obtained through (6)
- λ_m mean longitude associated with the mean earth is related to M: $\lambda_m=M+\tilde{\omega}$.

Then, the classical formulas for the daily insolation W are as follows:

Latitudes where there is no sunset

$$|\phi|\geqslant\frac{\pi}{2}-|\delta|\begin{cases}\phi>0 & \text{if} \quad \delta>0\\ \phi<0 & \text{if} \quad \delta<0\end{cases}$$

$$W=\frac{86.4S_0}{\rho^2}\sin\phi\,\sin\delta. \tag{8}$$

Latitudes where there is no sunrise

$$|\phi|\geqslant\frac{\pi}{2}-|\delta|\begin{cases}\phi<0 & \text{if} \quad \delta>0\\ \phi>0 & \text{if} \quad \delta<0\end{cases}$$

$$W=0. \tag{9}$$

Latitudes where there is daily sunset and sunrise

$$-\left(\frac{\pi}{2}-|\delta|\right)<\phi<\frac{\pi}{2}-|\delta|,$$

$$W=\frac{86.4S_0}{\pi\rho^2}(H_0\sin\phi\sin\delta+\cos\phi\cos\delta\sin H_0),\quad (10)$$

where H_0 is the absolute value of the hour angle at sunrise and sunset and is given by

$$\cos H_0=-\tan\phi\tan\delta.$$

All the angles which locate the earth on its orbit are taken as being constant over the whole day. The declination is related to the true longitude of the sun by

$$\sin\delta=\sin\epsilon\sin\lambda.$$

The normalized earth's sun distance is given by

$$\rho=\frac{r}{a}=\frac{1-e^2}{1+e\cos v}.$$

In Eqs. (8), (9), (10), S_0 is expressed in W m^{-2} and the factor 86.4 provides W in kJ m^{-2} day^{-1}. If a connection with earlier computations of insolation (Milankovitch, 1941; Vernekar, 1972; Berger, 1976a, 1978a) is desired, S_0 must be replaced by 1.95 cal cm^{-2} min^{-1} and the factor 86.4 by 1440 to provide W in cal cm^{-2} day^{-1}.

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