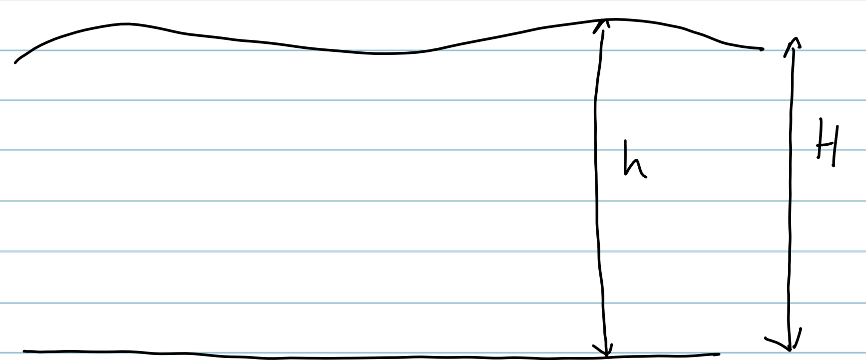
Linear Barotropic Waves

We consider here the ~~linear~~ Shallow Water Equations under these conditions.

* flat bottom
* free surface
* f-plane (constant Coriolis force)

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x} \\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y} \qquad (1) \\
&\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0
\end{aligned}
$$



$h(x,y,t) = H + \eta(x,y,t)$

In (1) have used the fact that $|\eta| << H$

Express as a "dynamical system" on function space

comprises everything (incl & RHS) in (1) except the time derivative terms.

$$
\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \tilde{f}\begin{pmatrix} u \\ v \\ \eta \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} \in X
$$

X will be some space of functions $\underset{(x,y)}{\mathbb{R}^2} \to \underset{(u,v,\eta)}{\mathbb{R}^3}$

The solution is then thought of as a trajectory in the phase space X, with specified initial values: $u(x,y,0)$, etc.

As part of X, we specify the physical domain $(x,y) \in \Omega$ and boundary conditions

We shall take periodic boundary conditions, i.e.

$$(x,y) \in [0,2\pi] \times [0,2\pi] = \Omega$$

and u, v, η are all (doubly) periodic in x & y w/ period $= 2\pi$

We then take X to consist of L^2 functions on $\mathbb{R}^2$ that are periodic in x & y as above.

A formal mathematical theory is needed to set this context properly. It leads to a view of the SW model as gives a dynamical system in function space.

In this formalism, a critical point is a $(u(x,y), v(x,y), \eta(x,y))$ so that

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv &= -g\frac{\partial \eta}{\partial x} \\
u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -g\frac{\partial \eta}{\partial y} \qquad (2) \\
H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0
\end{aligned}
$$

which is doubly periodic in x & y as above. This is a "steady state" of (1).

Particular case: rest-state

$$u = v = \eta = 0$$

From the theory of dynamical systems, we know that we can learn about small amplitude solutions of (1) by looking at the linearization of (1) at $u=0, v=0, \eta=0$.

The linearization amounts to throwing away the nonlinear terms in the u & v equations (advective terms).

This leads to the so-called "Linear Shallow Water Equations":

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \qquad (3)$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

<u>Poincaré Waves</u>

Since (3) is linear and has constant coefficients, it can be <u>solved</u> using a Fourier Transform.

The <u>outcome</u> (calculations not shown here) is that solutions are of the form:

$$\begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \\ \eta_0 \end{pmatrix} e^{i(\ell x + my - \omega t)} \qquad (4)$$

Where u_0, v_0, η_0 are constants. And ℓ, m, ω real.

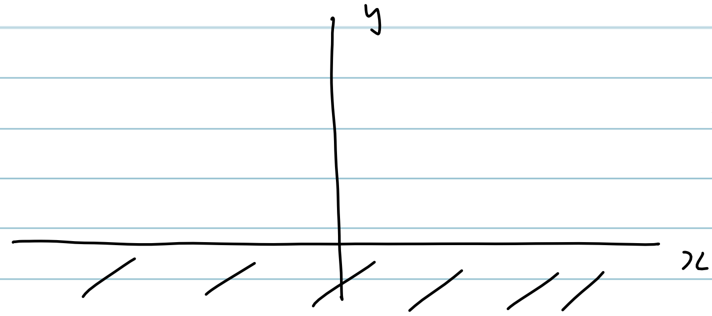
ℓ and m must be picked so that the boundary conditions are satisfied.

Any solution to (3) can then be expressed as an (infinite) linear combination (i.e. convergent series in L^2) of solutions of the form (4)

$\boxed{\text{In class:}}$ plug in RHS of (4) into (3) and obtain conditions on (ℓ, m, ω)

Kelvin Waves

We change the geometry now and introduce a real boundary. From the GFD viewpoint, you can see this as a "coastline".



$$\Omega = \{(x,y) \mid y \geq 0\}.$$

BC: we still assume period 2π in x
for y, need

$$v = 0 \quad \text{on} \quad y = 0.$$

~~To simplify calculations, we set $v \equiv 0$. This gives the most elementary Kelvin waves. It is not necessary, but significantly simplifies the calculation!~~

Looking for solutions in $L^2_{x\text{-per}}([0,2\pi]\times[0,\infty])$ that are periodic in x, we expect decay as $y\to+\infty$. Hence, it is natural to look for solutions of the form:

$$\begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \\ \eta_0 \end{pmatrix} \cdot e^{i(\ell x-\omega t)-my} \qquad \cancel{(5)}$$

ℓ, ω, m real & $m>0$.

Note that at $y=0$

$$v = v_0 e^{i(\ell x-\omega t)}$$

and so the BC $\Rightarrow$ $v_0 = 0$. So we need solutions of the form

$$\begin{pmatrix} u \\ \eta \end{pmatrix} = \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix} e^{i(\ell x-\omega t)-my}, \quad v\equiv 0 \qquad (5)$$

$\boxed{\text{In class:}}$ plug (5) into (3) and find conditions on solutions.

Rossby Waves

Sometimes called "planetary waves", these exist because of the change in the Coriolis effect with latitude. This is most easily expressed in the beta-effect

Recall that

$$f = 2|\Omega|\sin\theta$$

where $|\Omega|$ is the rotation speed of the Earth and θ the latitudinal angle.

In (x,y) co-ordinates, f depends only on y.

At a given latitude we can expand

$$f = f_0 + \beta y$$

On the f-plane (used for Poincaré waves) $f \equiv f_0$

On the β-plane, ~~$\beta > 0$~~ $\beta \neq 0$.

Rossby waves would be stationary on the f-plane but start to move with the beta effect.

In this case (3) becomes:

$$\frac{\partial u}{\partial t} - (f_0 + \beta y) v = -g \frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + (f_0 + \beta y) u = -g \frac{\partial \eta}{\partial y} \qquad (6)$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

The problem with (6) is that it no longer has constant coefficients. But, miraculously, the vorticity form does:

Set $\Gamma = v_x - u_y$

$\boxed{\text{In class}}$: derive vorticity equation:

$$\frac{\partial \Gamma}{\partial t} - \frac{f_0}{H} \frac{\partial \eta}{\partial t} + \beta v = 0 \qquad (7)$$

Now assume that (u, v, η) is close to geostrophic balance (QG), then:

$$u \approx \frac{g}{f_0}\frac{\partial \eta}{\partial x}$$

$$v \approx -\frac{g}{f_0}\frac{\partial \eta}{\partial y}$$

Thus $\psi = -\frac{g}{f_0}\eta$ acts as a streamfunction.

Plugging into (7) renders:

$$\frac{\partial}{\partial t}\frac{g}{f_0}\nabla^2\eta - \frac{f_0}{H}\frac{\partial \eta}{\partial t} - \beta\frac{g}{f_0}\frac{\partial \eta}{\partial y} = 0$$

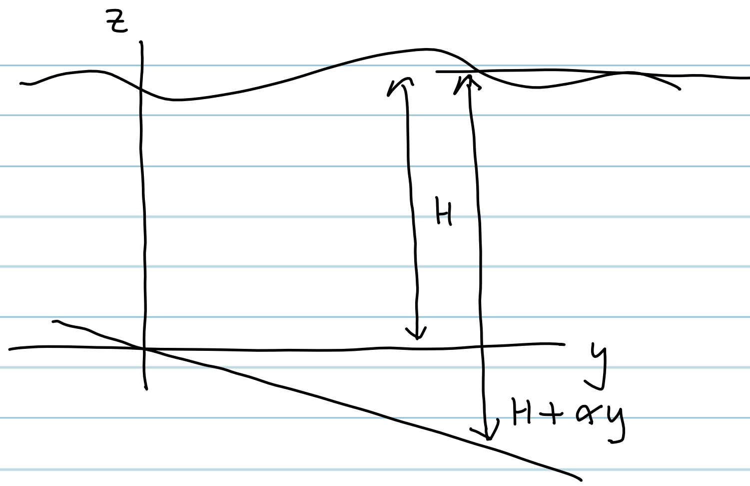
In second term, approximate f by f_0 and then:

$$\frac{\partial}{\partial t}\left(\nabla^2\eta - R^2\eta\right) - \beta\frac{\partial \eta}{\partial y} = 0 \qquad (8)$$

$\boxed{\text{In class:}}$ plug $\eta = \eta_0 e^{i(kx+my-\omega t)}$ into (8) and derive the dispersion relation.

Topographic Waves

Meanwhile, back on the f-plane, very similar waves can be formed through a variation of bottom topography.



$$h = H + \alpha y + \eta(x,y,t)$$

Look at the equation for h

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

Assume αy & η dominated by H $\Rightarrow$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \alpha v = 0$$

$\boxed{\text{In class:}}$ derive vorticity equation analogous to (8).