EQUATIONS OF GFD

Intro:

Atmosphere & Ocean are governed by the same basic set of equations.

Two key elements in derivation:

1. Newton's laws of motion,
2. The change of frame of reference.

Equations govern motion of a fluid particle at point $(x, y, z) \in \mathbb{R}^3$ at time t.

$$(x(t), y(t), z(t))$$

Variables:

We obtain equations for the fluid particle velocities:

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt}$$

Key physical quantities:

Density: $\rho = \rho(x, y, z, t)$

Pressure: $p = p(x, y, z, t)$

Temperature: $T = T(x, y, z, t)$

One more key quantity in each of ocean and atmosphere respectively.

Salinity: $S = S(x, y, z, t)$

(salt content in ocean)

Humidity $q = q(x, y, z, t)$

(water vapor content in atmosphere)

<u>Frames of Reference:</u>

① Lagrangian vs. Eulerian

Lagrangian → follow fluid particle

Eulerian → fixed position, watch fluid particles go by

② Rotating vs. Inertial

Rotating → rotating with Earth

Inertial → fixed frame, watch Earth rotate

Our natural frame of reference:

Eulerian & Rotating

Frame of reference for laws of physics:

Lagrangian & Inertial

Transformations:

① Lagrangian → Eulerian

Consider a scalar quantity

$$f = f(x, y, z, t)$$

Evaluate f at a fluid particle:

$$f(x(t), y(t), z(t), t)$$

$\left(\frac{df}{dt}\right)_L$ = derivative of f w.r.t. time in Lagrangian frame

$\left(\frac{\partial f}{\partial t}\right)_E$ = derivative of f w.r.t. time in Eulerian (fixed) frame.

$$\left(\frac{df}{dt}\right)_L = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \left(\frac{\partial f}{\partial t}\right)_E$$

i.e.,

$$\left(\frac{df}{dt}\right)_L = U \cdot \nabla f + \left(\frac{\partial f}{\partial t}\right)_E$$

when $U = (u, v, w)$.

(2) Inertial $\rightarrow$ Rotating

Ω = vector / axis of rotation

$|\Omega|$ = speed of rotation.

Earth $\Omega = |\Omega|(0,0,1)$ and

$|\Omega| = 7.29 \times 10^{-5}\ s^{-1}$

$F = F(x,y,z,t)$ is vector-valued (in $\mathbb{R}^3$)

e.g. $F = (u,v,w)$

$$\left(\frac{dF}{dt}\right)_I = \left(\frac{dF}{dt}\right)_R + \Omega \times F$$

Newton's Second Law

$$F = ma$$ (conservation of momentum).

$$\rho \left(\frac{dU_I}{dt}\right)_{L,I} = \underbrace{-F_{gravity} - F_{pressure} + F_{friction}}_{\text{sum of all forces on fluid patch}}$$

ρ: density (mass/unit volume)

U_I = velocity of patch in inertial frame.

As mentioned above, law is formulated in Lagrangian, inertial frame, need to change to Eulerian, rotating frame.

Note: signs of forces on RHS by convention.

The 3 forces on RHS are:

$F_{gravity}$ = force due to gravity

$F_{pressure}$ = pressure force of surrounding fluid

$F_{friction}$ = forces due to friction effects

The first two force are self-explanatory. The third is more complicated and the following comments will help understand how it comes up in practice:

(1) Internal friction forces in fluids such as water or air are negligible. I.e., the molecular viscosity of these fluids is too small to be considered.

(2) Friction forces on boundaries include:

(a) wind force on surface,

(b) friction on sea-floor.

(3) A non-trivial viscosity term is often (usually?) included to parametrize the effects of sub-grid scale turbulence. In this case, it is often called "eddy viscosity". This also helps to regularize the equations and thus ease the numerical calculations.

Derivation of Equations

The first step is to transform $\left(\frac{dU}{dt}\right)_{L,R}$ by going from rotating $\rightarrow$ inertial $R \rightarrow I$.

If $r = (x, y, z)$, then

$$U = \left(\frac{dr}{dt}\right)_R = \left(\frac{dr}{dt}\right)_I + \Omega \times r$$

note: since the total derivative $\frac{d}{dt}$ already indicates a Lagrangian frame the subscript L is dropped in this calculation.

This can be rewritten as:

$$U_I = U_R + \Omega \times r$$

and so.

$$\left(\frac{dU_I}{dt}\right)_I = \left(\frac{dU_I}{dt}\right)_R + \Omega \times U_I$$

$$= \left(\frac{dU_R}{dt}\right)_R + 2\Omega \times U_R + \Omega \times \Omega \times r$$

So, Newton's 2^{nd} Law becomes:

$$\rho\left\{\left(\frac{dU_R}{dt}\right)_R + 2\Omega \times U_R + \Omega \times \Omega \times r\right\} =$$

$$= -F_{\text{gravity}} - F_{\text{pressure}} + F_{\text{friction}}$$

Now drop the "R" and convert to Eulerian co-ordinates $\Rightarrow$

$$\rho\left\{\frac{\partial U}{\partial t} + U \cdot \nabla U + 2\Omega \times U + \Omega \times \Omega \times r\right\}$$

$$= \text{sum of forces.}$$

$2\Omega \times U \Longleftrightarrow$ Coriolis force

$\Omega \times \Omega \times r \Longleftrightarrow$ centrifugal force

The force due to gravity points toward the centre of the Earth. Since Earth is actually an oblate spheroid, this is not down. It turns out, to good approximation $F_{gravity}$ can be written as:

$$F_{gravity} = \rho g \vec{k} - \Omega \times \Omega \times r$$

In other words the difference between gravity pointing down & towards centre of Earth is (almost) exactly the centrifugal force. This is intuitively not surprising (why?). Then

$$\frac{\partial U}{\partial t} + U \cdot \nabla U + 2\Omega \times U = -g\vec{k} - \frac{\nabla p}{\rho} + F$$

where $F = \frac{1}{\rho} F_{friction}$

Advection term $U \cdot \nabla U$ is nonlinear & what makes fluids difficult

Coriolis term $2\Omega \times U$ is what distinguishes GFD from FD

The Coriolis term works as follows.

$$\Omega = |\Omega| (\cos\varphi \, j + \sin\varphi \, k)$$

Setting $U = (u, v, w)$

$$\frac{\Omega \times U}{|\Omega|} = \begin{vmatrix} i & j & k \\ 0 & \cos\varphi & \sin\varphi \\ u & v & w \end{vmatrix}$$

$$= i(\cos\varphi \, w - \sin\varphi \, v) + j \sin\varphi \, u - k \cos\varphi \, u$$

Thus writing the equations out in co-ords.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - 2|\Omega|\sin\varphi \, v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F^x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + 2|\Omega|\sin\varphi \, u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F^y$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - 2|\Omega|\cos\varphi \, u = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + F^z$$

where $F = (F^x, F^y, F^z)$

Note: w term omitted in first equation due to size.

Hydrostatic Balance

The horizontal motion will dominate the vertical motion and it is usually assumed that the fluid is in hydrostatic balance which means that the first two terms on the RH dominate everything else $\Rightarrow$ 3rd eqn. becomes

$$0 = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \quad , \text{ or}$$

$$\frac{\partial p}{\partial z} = -\rho g$$

Equation Inventory

So far we have 3 equations, one for each of u, v and w (although w does not appear in the w equation anymore). BUT, there are 5 unknowns:

$$\underbrace{u \quad v \quad w}_{\text{velocities}} \qquad \underset{\text{density}}{\rho} \qquad \underset{\text{pressure}}{p}$$

The above equation, derived from $F=ma$, capture the conservation of momentum.

We need 2 more equations and they come from the 2 other basic principles of conservation:

1. Conservation of mass
2. Conservation of energy

Conservation of Mass

The equation is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho U = 0, \quad \text{or}$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot U + U \cdot \nabla \rho = 0$$

$\boxed{\text{Ocean}}$: incompressible $\Rightarrow \nabla \cdot U = 0$

$$\frac{\partial \rho}{\partial t} + U \cdot \nabla \rho = 0 \quad \Rightarrow$$

$$\frac{D}{Dt} \rho = 0$$

$\boxed{\text{Atmosphere}}$ compressible, so need full density equation.

Conservation of Energy

$$c\left(\frac{dT}{dt}\right)_{E,R} = K_T \nabla^2 T + \frac{1}{\rho}\left(\frac{dp}{dt}\right)_{E,R} + G$$

where:

$T = T(x,y,z,t)$ = temperature

K_T = thermal diffusivity $(0 < K_T << 1)$

G accounts for heat forcing due to evaporation, condensation & radiation.

If we throw away diffusivity & express the conservation of energy in terms of potential temperature, which then means the pressure source term is subsumed into the temperature term. We then set

$$c\frac{dT}{dt} = G\left(\begin{Bmatrix}\text{water vapour content} \\ \text{salinity}\end{Bmatrix}, \ldots\right)$$

atmos

ocean

Note: as indicated on RHS, water vapour in atmos. is analogous & salt content in ocean. Unfortunately, the equation for energy conservation does not help us in our quest for a complete set of equations as it has introduced another variable, namely temperature. So we still need another equation.

Equation of State

This will relate pressure and/or density to temperature (and other variables). Since the atmosphere and ocean are in different "states", this is a key point at which their equations differ.

Atmosphere $\rightarrow$ Ideal Gas Law

$$P = \rho R T$$

Ocean $\rightarrow$

$$\rho = \rho_0 \left[1 + \alpha (S - S_0) + \beta (T - T_0) \right]$$

This then gives a complete set of equations, known as the Primitive Equations of GFD.

Remarks on Balances

There are 2 fundamental balance in GFD:

1. Hydrostatic balance (already mentioned & used) This allows for special treatment of the vertical co-ordinate and the formulation of layered models

In layered models, either pressure or density is used to replace the vertical co-ordinate z. (atmos $\longleftrightarrow$ pressure, ocean $\longleftrightarrow$ density)

Note that hydrostatic balance is almost always used in GFD calculations

An interesting and important point is that the set-up for deep-water convection in the North Atlantic, which plays a key role in the overturning circulation, defies hydrostatic balance. It therefore cannot appear in calculation based on the primitive eqns. of GFD. Indeed, it is usually included in OGCMs through a parametrization as vertical diffusivity.

Geostrophic Balance

The limiting balance in GFD is quite different from classical FD as it does not involve advection at all. It is a balance between the Coriolis terms and the horizontal pressure gradient:

$$-fu = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$fv = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

where $f = 2|\Omega|\sin\varphi$.

If a flow is in a geostrophic balance, then we have immediate expressions for the horizontal velocities u and v. Of course, not all geophysical flows are in such balance, but it gives the lowest order approx.

Coriolis approximations

The angle φ in the Coriolis term is the latitude. The Coriolis term $2|\Omega|\sin\varphi$ is usually approximated in one of 2 ways:

1. F-plane. Fix $\varphi = \tilde{\varphi}$ (i.e. particular lat) assume $F = 2|\Omega|\sin\tilde{\varphi}$ is constant
2. β-plane. Include first order variation of Coriolis with latitude

$$f = 2|\Omega|\sin\varphi \cong f_0 + \beta y$$