Goal: Understand where atmospheric LFV (low frequency variability) comes from.

* mathematical mechanisms
* physical mechanisms.

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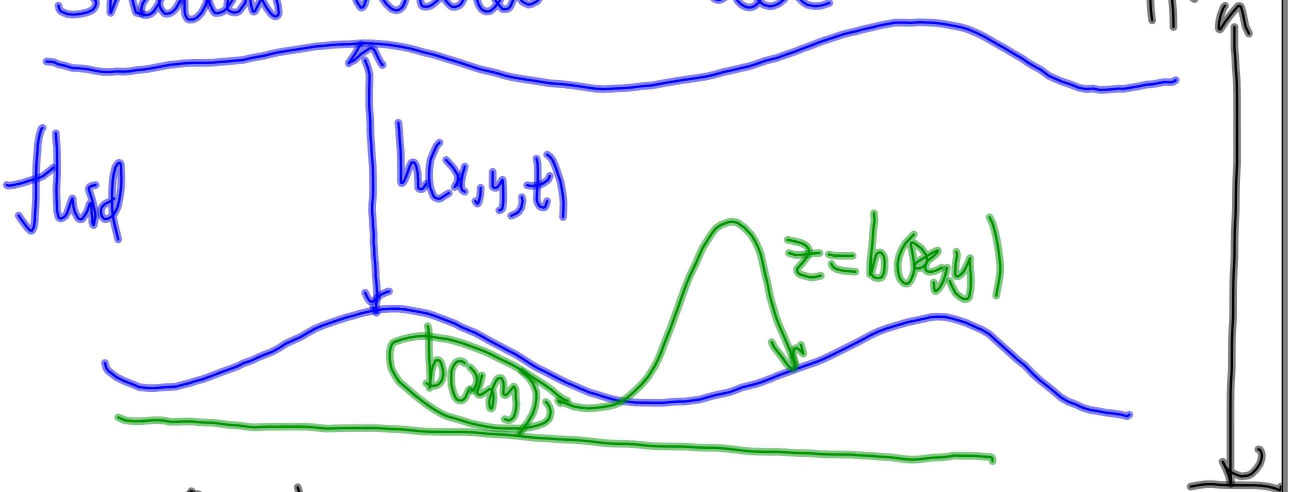
Barotropic Flow

Shallow Water Model.



Barotropic $\Rightarrow$ $w=0$.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial}{\partial x}(h+b)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + f(u) = -g\frac{\partial}{\partial y}(h+b)$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

Potential vorticity $\omega = \dfrac{\zeta + f}{h}$

$\zeta = v_x - u_y$

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Key fact: PV is conserved along trajectories, i.e.

$$\boxed{\frac{d}{dt}\left(\frac{\zeta+f}{h}\right)=0}$$

Ingredients of proof of PV conservation:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0.$$

$h \approx H \Rightarrow$ (approx)

$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

$$\left|\frac{\partial h}{\partial t}\right| \ll H \Rightarrow \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}.$$

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$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \exists\, \psi(x,y,t) \ni$$

$$u = -\psi_y$$

$$v = \psi_x$$

Note: $\zeta = v_x - u_y = \psi_{xx} + \psi_{yy} = \nabla^2 \psi$

$$(= \Delta \psi)$$

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Next step: calculate equ. for $\zeta = v_x - u_y$

① $\frac{\partial}{\partial y} \rightarrow \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial}{\partial x}(h+b)$

② $\frac{\partial}{\partial x} \rightarrow \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + f(u) = -g\frac{\partial}{\partial y}(h+b)$

② − ① $\Rightarrow$

$$\frac{\partial}{\partial t}\zeta + \zeta\frac{\partial u}{\partial x} + \zeta\frac{\partial v}{\partial y} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}$$

$$+ u f_x + v f_y + f(u_x + v_y) = 0$$

$$\frac{d\zeta}{dt} + \underbrace{u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}}_{J(\psi,\zeta)} + \underbrace{uf_x + vf_y}_{J(\psi,f)}$$

$$+ f(u_x + v_y) = 0$$

$$J(\psi, \zeta)$$

$$\| $$

$$\frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x}$$

$$J(\psi, f)$$

$$\frac{d\zeta}{dt} + J(\psi, \zeta) + J(\psi, f) + f(u_x + v_y) = 0$$

From eqn. for h:

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + h(u_x + v_y) = 0.$$

$$\frac{dh}{dt} + h(u_x + v_y) = 0.$$

$$\Rightarrow u_x + v_y = -\frac{1}{h}\frac{dh}{dt}$$

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$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta + f) + \zeta(u_x + v_y) - \frac{1}{h}\frac{dh}{dt} f = 0.$$

$$\Rightarrow \frac{d(\zeta+f)}{dt} + \zeta(u_x+v_y) - \frac{1}{h}\frac{dh}{dt} f = 0.$$

$$\frac{d}{dt}(\zeta+f) + (\zeta+f)\left(-\frac{1}{h}\frac{dh}{dt}\right) = 0,$$

$$\boxed{\frac{d}{dt}\left(\frac{\zeta+f}{h}\right) = 0.}$$

can add viscosity other forces

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Idea: in models, use PV eqn.

$$\frac{d}{dt}\left(\frac{\zeta+f}{h}\right)=0.$$

or $$\frac{\partial}{\partial t}\left(\frac{\zeta+f}{h}\right)+J\left(\psi,\frac{\zeta+f}{h}\right)=0.$$

$$\zeta=\nabla^2\psi.$$

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Coriolis force:

$2\Omega \times U$ in com. of mass

$|\Omega|(\cos\varphi\, j + \sin\varphi\, k)$

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$$\Omega \times U = \begin{vmatrix} i & j & k \\ 0 & \cos\varphi & \sin\varphi \\ u & v & w \end{vmatrix}$$

$$= |\Omega| \Big[i\big(\underbrace{\cancel{w\cos\varphi}}_{0 \text{ small } w \approx 0} - u\sin\varphi\big) + j\, u\sin\varphi + k\, u\cos\varphi \Big]$$

u mom eqn.

$f = 2|\Omega|\sin\varphi$

v mom eqn

$f = 2|\Omega|\sin\varphi$

$f = 2|\Omega|\sin\varphi$.

expand in Taylor series:

$$f = f_0 + \beta_0 y$$ ← first order correction

$f_0 = 2|\Omega|\sin\varphi_0$

Called the β-plane.

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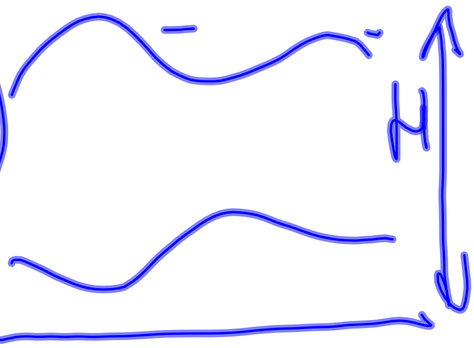
$$\frac{\partial}{\partial t}\left(\frac{\zeta + f}{h}\right) + J\left(\psi, \frac{\zeta + f}{h}\right) = 0$$

$$f = f_0 + \beta_0 y$$

$$h = H - b(x,y) + \eta(x,y,t)$$



assume $|\eta| \ll H$

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PV eqn. becomes:

$$\frac{\partial}{\partial t}\left(\zeta + f_0 + \beta_0 y\right) + J\left(\psi, \zeta + f + \frac{b}{H}(\zeta + f)\right) = 0.$$

$\frac{\zeta}{h} \to \frac{\zeta}{H}$ renamed as ζ, also absorbed H into f_0 & β_0.