Maasch - Saltzman Model

$$\dot{X} = -X - Y$$
$$\dot{Y} = -pZ + rY - sZ^2 - Z^2 Y$$
$$\dot{Z} = -q(X + Z)$$

p, q, r, s parameters

$X \leftrightarrow$ ice mass
$Y \leftrightarrow$ atmospheric CO_2
$Z \leftrightarrow$ NADW

Fixed points:

$$X = -Y, \quad X = -Z \quad \Rightarrow \quad Y = Z.$$

and so, for 2^{nd} eq.

$$-pZ + rZ - sZ^2 - Z^3 = 0$$
$$-Z\left(Z^2 + sZ + p - r\right) = 0$$

$$Z = 0 \quad \text{or} \quad Z^2 + SZ + p - r = 0$$
$$Z = -\frac{s \pm \sqrt{s^2 - 4(p-r)}}{2}$$

f.p.'s at $(-a, a, a)$ for

(1) $a = 0$

(2) $a = -\frac{s}{2} + \frac{\sqrt{s^2 - 4(p-r)}}{2}$

(3) $a = -\frac{s}{2} - \frac{\sqrt{s^2 - 4(p-r)}}{2}$

} if <u>real</u>

Linearize eqs. @ $(-a, a, a)$

$$f(X, Y, Z) = \begin{pmatrix} -X - Y \\ -pZ + rY - sZ^2 - Z^2 Y \\ -q(X+Z) \end{pmatrix}$$

Linearization = Jacobian

$$Df(-a, a, a) = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1\ \cancel{2}a^2 & -p - 2sa - 2a^2 \\ -q & 0 & -q \end{bmatrix}$$

Characteristic polynomial:

$$(-1-\lambda)\left((r-a^2-\lambda)(-q-\lambda) + 1 \cdot q(-p-2sa-2a^2)\right) = 0$$

$$(\lambda+q)(1+\lambda)(r-a^2-\lambda) - q(p+2sa+2a^2) = 0$$

<u>ex.</u> $a = 0$

$$(\lambda+q)(1+\lambda)(r-\lambda) - qp = 0$$

Does a Hopf bifurcation occur at $(0,0,0)$?

A necessary condition is to find parameter values at which the Jacobian has an imaginary eigenvalue:

Set $\lambda = i\gamma$

$$(i\gamma + q)(1+i\gamma)(r-i\gamma) - qp = 0$$

$$(-\gamma^2 + i\gamma(q+1) + q)(r - i\gamma) - qp = 0$$

Equate real & imaginary parts:

Re: $-\gamma^2 r + \gamma^2(q+1) + rq - pq = 0.$

$$\gamma^2(q+1-r) = q(p-r)$$

$$\gamma^2 = \frac{q(p-r)}{q+1-r}$$

Im: $-\gamma^3 + \gamma(q+1)r - \gamma q = 0.$

$$\gamma(-\gamma^2 + (q+1)r - q) = 0.$$

$\gamma = 0$ (not a Hopf!) or

$$\gamma^2 = r(q+1) - q.$$

Need: $r(q+1) - q = \dfrac{q(p-r)}{q+1-r}$

$$\Rightarrow \quad (q+1-r)(r(q+1)-q) = q(p-r)$$

Note: quadratic in q!

$$(q+1-r)(q(r-1)+r)+q(r-p)=0.$$

$$q^2(r-1) - q\left[(r-1)^2 - r + p - r\right] + r(1-r) = 0.$$

$4ac = -4r(1-r)^2 < 0$ if $r>0$

$\Rightarrow$ potential Hopf!