Rogue Waves in the Nonlinear Schrödinger Equation

Math 395: Term Project

by Andy Reagan and Eric Roma

May 11, 2012

Rogue waves are now a current phenomenon in nonlinear wave theory that occur in both water and optical waves. Their general solutions in the nonlinear Schrödinger equation can be derived using the bilinear method, and then simplified to algebraic expressions using Gram determinants and Schur polynomials where the N-th order solutions have $N-1$ free parameters. Applying numerical methods yield rich and symmetrical 3-dimensional graphs, and contour plots, whose amplitudes can be maximized by varying the free parameters.

1 Introduction

Once considered to be a mythical occurrence, rogue waves are now a studied phenomenon in nonlinear wave theory. The structure and nature of these waves is imperative in our understanding of how and why they occur. Their occurrence in both water and optical waves, although neither periodic nor obvious, has an affect on our everyday lives. We live in a society that is heavily dependent on the transfer of both goods and data and any disturbance in either of the two is always an area in need of research.

In oceanography, the pragmatic approach is to define a rogue wave whenever

$$H/H_s > 2 \quad \text{or} \quad \omega_c/H_s > 1.25 \tag{1.1}$$

where H is the wave height (distance from trough to crest), ω_c is the crest height (distance from mean sea level to crest), and H_s is the significant wave height, here defined as four times the standard deviation of the surface elevation (Dysthe *et al.* 2008) They are known to cause extensive damage, and are even life-threatening, when they come into contact with ocean liners and passenger ships in the open waters. Between 1964 and 1994, it is estimated that more than 22 super-carriers have been lost at sea as a direct result of rogue waves (Kharif & Pelinovsky 2003).

More recently, there has been the discovery of a similar wave phenomenon observed in a system based on probabilistic supercontinuum generation in highly nonlinear microstructures optical fibre, i.e. optical rogue waves, but a critical challenge in observing optical rogue waves is a lack of a reliable instrument that can accurately measure real-time occurrences (Solli *et al.* 2007). A growing consensus is that both oceanic and optical rogue waves appear as a result of modulation instability of monochromatic nonlinear waves (Ohta & Yang 2012). As a result, we must turn to mathematically studying these waves using nonlinear partial differential equations (PDE). Thus we turn our attention to an integrable model, the nonlinear Schrödinger (NLS) (Zakharov & Shabat 1972).

In theory, soliton wave solutions approach zero as time goes to $\pm\infty$ whereas rogue wave solutions approach a nonzero constant as time goes to $\pm\infty$, with the first analytical solution obtained by Peregrine (1983). Ohta and Yang (2012) derived general high-order waves in the NLS equation using the bilinear method in the soliton theory, and then further simplified there results to algebraic expressions using Gram determinants and elementary Schur polynomials. The importance of both are the symmetries that they exhibit. For those not familiar with Gram determinants or Schur polynomials the authors will take the time to define both.

Definition 1.1. Let V be an inner product space over a field k with $< \cdot, \cdot >$ the inner product on V, where the inner product shall mean a symmetric bilinear form on V. Let $x_1, x_2, \ldots, x_n$ be arbitrary vectors in V. Set $r_{ij} =< x_i, x_j >$. The Gram determinant of $x_1, x_2, \ldots, x_n$ is determined to be the determinant of the symmetric matrix

$$\begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix} \tag{1.2}$$

where we denote this determinant by $\mathrm{Gram}[x_1, x_2, \ldots, x_n]$.

Definition 1.2. Let $S_n(\boldsymbol{x})$ be the elementary Schur polynomial defined by the generating function

$$\sum_{n=0}^{\infty} S_n(\boldsymbol{x})\lambda^n = \exp\left(\sum_{n=0}^{\infty} x_k \lambda^k\right), \tag{1.3}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots)$. Such that the first few terms are

$$S_0(\boldsymbol{x}) = 1, \quad S_1(\boldsymbol{x}) = 1, \quad S_2(\boldsymbol{x}) = \frac{1}{2}x_1^2 + x_2, \quad S_3(\boldsymbol{x}) = \frac{1}{6}x_1^3 + x_1x_2 + x_3, \ \ldots$$

The purpose of this paper will be to reproduce the main results of Ohta and Yang (2012), and extend the numerical methods to higher-order solutions. The derivations will also be based on the bilinear method in soliton theory (Hirota 2004), which will then be expanded to general rogue waves of N-th order with $N-1$ free parameters, using Gram determinants, which will be numerically simulated using a program created by the authors with Mathematica.

2 General Rogue-Wave Solutions of the NLS

Rogue waves are a special type of nonlinear wave which approaches a constant background at large distance and time (Ohta & Yang 2012). If we consider the general rogue wave solutions of the focusing NLS equation

$$iu_t = u_{xx} + 2|u|^2 u \tag{2.1}$$

we find it is invariant under scalings $x \to \alpha x$, $t \to \alpha^2 t$, $u \to u/\alpha$ for any constant $\alpha \in \mathbb{R}$, as well as the Galilean transformation $u(x,t) \to u(x - vt, t)\exp(-ivx/2 + iv^2 t/4)$ (Ohta & Yang 2012). Therefore, we will consider the rogue waves that approach a nonzero constant background at large x and t,

$$u(x,t) \to e^{-2it} \quad \text{as} \quad x,\, t \to \pm\infty.$$

If we apply the variable transformation $u \to ue^{-2it}$, then (2.1) becomes

$$\begin{aligned} i(ue^{-2it})_t &= (ue^{-2it})_{xx} + 2\left|ue^{-2it}\right|^2 \cdot ue^{-2it} \\ i\left(u_te^{-2it} - 2iue^{-2it}\right) &= e^{-2it}u_{xx} + 2|u|^2 \cdot ue^{-2it} \\ iu_te^{-2it} &= e^{-2it}u_{xx} + 2|u|^2 \cdot ue^{-2it} - 2ue^{-2it} \\ iu_t &= u_{xx} + 2u\left(|u|^2 - 1\right) \end{aligned} \tag{2.2}$$

where the boundary conditions become

$$u(x,t) \to 1 \quad \text{as} \quad x,\, t \to \pm\infty. \tag{2.3}$$

Since the Schur polynomials give the complete set of homogenous-weight algebraic solutions for the Kadomstev-Petviashvili (KP) hierarchy (Sata 1981; Jimbo & Miwa 1983), we can also use them to describe the rational solutions of rogue waves in the NLS equation (Ohta & Yang 2012).

Theorem 2.1. *The NLS equation (2.2) under the boundary conditions (2.3) has non-singular rational solutions*

$$u = \frac{\sigma_1}{\sigma_0}$$

where

$$\sigma_n = \det_{1\le i,j\le N}\left(m^{(n)}_{2i-1,j-1}\right)$$

the matrix elements in σ_m are defined by

$$m^{(n)}_{ij} = \sum_{v=0}^{\min(i,j)} \Phi^{(n)}_{iv}\Psi^{(n)}_{jv}$$

where

$$\Phi^{(n)}_{iv} = \frac{1}{2^v}\sum_{k=0}^{i-v} a_k S_{i-v-k}\left(\boldsymbol{x}^+(n) + v\boldsymbol{s}\right) \quad \textit{and} \quad \Psi^{(n)}_{jv} = \frac{1}{2^v}\sum_{l=0}^{j-v} \overline{a}_l S_{j-v-l}\left(\boldsymbol{x}^-(n) + v\boldsymbol{s}\right)$$

where a_k for $k = 0, 1, \ldots$ are complex constants, and $\boldsymbol{x}^{\pm}(n) = (x_1^{\pm}(n), \ldots)$, $\boldsymbol{s} = (s_1, lots)$ are defined by

$$x_1^{\pm}(n) = x \mp 2it \pm n - \frac{1}{2}, \quad x_k^{\pm} = \frac{x \mp 2^k it}{k!} - r_k, \quad (k \ge 2), \tag{2.4}$$

and

$$\sum_{k=1}^{\infty} r_k\lambda^k = \ln\left(\cosh\frac{\lambda}{2}\right) \quad \textit{and} \quad \sum_{k=1}^{\infty} s_k\lambda^k = \ln\left(\frac{2}{\lambda}\tanh\frac{\lambda}{2}\right).$$

Here we have $\overline{a}_l$ is the complex conjugate of a_l. Therefore, σ^n can be expressed by

$$\sigma^n = \sum_{v_1=0}^{1}\sum_{v_2=v_1+1}^{3}\sum_{v_3=v_2+1}^{5}\cdots\sum_{v_N=v_{N-1}+1}^{2N-1} \det_{1\le i,j\le N}\left(\Phi^{(n)}_{2i-1,v_j}\right)\det_{1\le i,j\le N}\left(\Psi^{(n)}_{2i-1,v_j}\right)$$

such that

$$\Phi^{(n)}_{iv} \quad \textit{and} \quad \Psi^{(n)}_{iv} \quad \textit{for} \quad i < v.$$

According to Ohta and Yang (2012), since the generators for r_k and s_k are even functions we can set all the odd terms to zero. With the code provided in the appendix you will be able to generate the even terms. Also, without loss of generality we can set

$$a_0 = 1, \quad a_2 = a_4 = \cdots = a_{\text{even}} = 0.$$

3 General Rogue-Wave Solutions and the Bilinear Method

First we derive the bilinear form of the transformed NLS from (2.1). Let $u = g/f$ for $g \in \mathbb{C}, f \in \mathbb{R}$. Then we have

$$\begin{aligned}
0 &= \left(\frac{g}{f}\right)_{xx} + 2\left(\left|\frac{g}{f}\right|^2 - 1\right)\frac{g}{f} - i\left(\frac{g}{f}\right)_t \\
&= \left(\frac{fg_x - gf_x}{f^2}\right)_x + 2\left(\frac{|g|^2 g}{f^3} - \frac{g}{f}\right) - i\left(\frac{fg_t - gf_t}{f^2}\right) \\
&= \frac{f^2(fg_{xx} - gf_{xx}) - (fg_x - gf_x)2f}{f^4} + 2\left(\frac{|g|^2 g}{f^3} - \frac{g}{f}\right) - i\left(\frac{fg_t - gf_t}{f^2}\right).
\end{aligned}$$

By multiplying through by f^3, and then grouping by g, f we have

$$\begin{aligned}
0 &= f(f_{xx} - gf_{xx}) - 2(fg_x - gf_x) + 2|g|^2 g - gf^2 - if(fg_t - gf_t) \\
&= f((fg_{xx} - gf_{xx}) - i(fg_t - gf_t)) + g\left(2|g|^2 - f^2 - 2f\frac{g_x}{g} - 2f_x\right) \\
&= f(D_x^2 - iD_t)(gf) + g((D_x^2 + 2)(f^2)) - 2|g|^2)
\end{aligned}$$

so we have the desired bilinear form

$$\left.\begin{aligned}
(D_x^2 + 2)f \cdot f &= 2|g|^2 \\
\text{and} \quad (D_x^2 - iD_t)g \cdot f &= 0,
\end{aligned}\right\} \tag{3.1}$$

where D is Hirota's bilinear differential operator such that $D_x(fg) = f_x g - g_x f$. Then for h another complex variable, we consider the $2+1$-dimensional generalization of the above bilinear form

$$\left.\begin{aligned}
(D_x D_y + 2)f \cdot f &= 2gh \\
\text{and} \quad (D_x^2 - iD_t)g \cdot f &= 0.
\end{aligned}\right\} \tag{3.2}$$

Solutions to equations (3.2) under the conditions

$$(\partial_x + \partial_y)f = Cf, \qquad \text{where} \quad f \in \mathbb{R} \quad \text{and} \quad h = \overline{g} \tag{3.3}$$

for C some constant then also satisfy the bilinear NLS equations, since then $gh = |g|^2$ and $D_x D_y(f) = D_x^2(f)$.

Next, we verify Lemma 3.1 of Ohta and Yang (2012) to begin constructing Gram determinant solutions to the $2+1$-dimensional bilinear system.

3.1 Gram determinant solutions to $2+1$-dimensional bilinear equations

Lemma 3.1. *Let $M_{ij}^{(n)}, \phi_i^{(n)}, \psi_j^{(n)}$ be functions of x_1, x_2, x_{-1} satisfying the following differential and difference relations,*

$$\left.\begin{aligned} \partial_{x_1} m_{ij}^{(n)} &= \phi_i^{(n)}\psi_j^{(n)}, \\ \partial_{x_2} m_{ij}^{(n)} &= \phi_i^{(n+1)}\psi_j^{(n)} + \phi_i^{(n)}\psi^{(n-1)}, \\ \partial_{x_{-1}} m_{ij}^{(n)} &= -\phi_i^{(n-1)}\psi_j^{(n+1)}, \\ m_{ij}^{(n+1)} &= m_{ij}^{(n)} + \phi_i^{(n)}\psi_j^{(n+1)}, \\ \text{and} \qquad \partial_{x_k}\phi_i^{(n)} &= \phi_i^{(n+k)}, \ \ \partial_{x_k}\psi_j^{(n)} = -\psi_j^{(n-k)}, \ (k=1,2,-1). \end{aligned}\right\}$$

Then the determinant,

$$\tau_n = \det_{1\le i,j\le N}\left(m_{ij}^{(n)}\right),$$

satisfies the bilinear equations,

$$\left.\begin{aligned} (D_{x_1}D_{x_{-1}} - 2)\tau_n\cdot\tau_n &= -2\tau_{n+1}\tau_{n-1} \\ \text{and} \qquad (D_{x_1}^2 - D_{x_2})\tau_{n+1}\cdot\tau_n &= 0. \end{aligned}\right\}$$

Proof. We then verify

$$\partial_{x_1}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n)} \\ -\psi_j^{(n)} & 0 \end{vmatrix},$$

$$\partial_{x_1}^2\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n+1)} \\ -\phi_j^{(n)} & 0 \end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n)} \\ \phi_j^{(n-1)} & 0 \end{vmatrix},$$

$$\partial_{x_2}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n+1)} \\ -\phi_j^{(n)} & 0 \end{vmatrix} - \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n)} \\ \phi_j^{(n-1)} & 0 \end{vmatrix},$$

$$\partial_{x_{-1}} = \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n-1)} \\ -\psi_j^{(n+1)} & 0 \end{vmatrix},$$

$$(\partial_{x_1}\partial_{x_{-1}} - 1)\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n-1)} & \phi_i^{(n)} \\ \psi_j^{(n+1)} & 0 & -1 \\ -\psi_j^{(n)} & -1 & 0 \end{vmatrix},$$

$$\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n)} \\ -\psi_j^{(n+1)} & 1 \end{vmatrix},$$

$$\tau_{n-1} = \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n-1)} \\ \psi_j^{(n)} & 1 \end{vmatrix},$$

$$\partial_{x_1}\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n+1)} \\ -\psi_j^{(n+1)} & 0 \end{vmatrix},$$

$$\partial_{x_1}^2\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n+2)} \\ -\psi_j^{(n+1)} & 0 \end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n)} & \phi_i^{(n+1)} \\ -\psi_j^{(n)} & 0 & 0 \\ -\psi_j^{(n+1)} & 1 & 0 \end{vmatrix},$$

$$\text{and} \quad \partial_{x_2}\tau_{n+1} = \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n+1)} \\ -\psi_j^{(n+1)} & 0 \end{vmatrix} - \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n)} & \phi_i^{(n+1)} \\ -\psi_j^{(n)} & 0 & 0 \\ -\psi_j^{(n+1)} & 1 & 0 \end{vmatrix}.$$

So we have

$$(\partial_{x_1}\partial_{x_{-1}} - 1)\tau_n \times \tau_n = \partial_{x_1}\tau_n \times \partial_{x_{-1}}\tau_n - (-\tau_{n-1})(-\tau_{n+1}),$$
$$\tfrac{1}{2}(\partial_{x_1}^2 - \partial_{x_2})\tau_{n+1} \times \tau_n = \partial_{x_1}\tau_{n+1} \times \partial_{x_1}\tau_n - \tau_{n+1}\tfrac{1}{2}(\partial_{x_1}^2 + \partial_{x_2})\tau_n,$$

which is the bilinear form of the $2+1$-dimensional NLS, as desired. □

Since we write

$$m_{ij}^{(n)} = \int^{x_1} \phi_i^{(n)}\psi_j^{(n)} dx_1$$

the determinant τ_n is the Gram determinant solution. By defining

$$f = \tau_0, \quad g = \tau_1, \quad h = \tau_{-1}, \tag{3.11}$$

these are the Gram determinant solutions for the $2+1$-dimensional bilinear system where

$$x_1 = x,\ x_2 = -it, \text{ and } x_{-1} = -y. \tag{3.12}$$

3.2 Algebraic solutions satisfying reduction condition

Here, we verify that by choosing the matrix elements appropriately in τ_n, we have solutions that also satisfy the reduction condition.

Lemma 3.2. *Define matrix elements*

$$m_{ij}^{(n)} = A_i B_j m^{(n)}\Big|_{p=1,q=1}$$

and

$$m^{(n)} = \frac{1}{p+q}\left(-\frac{p}{q}\right)^n e^{\xi+\eta}, \quad \xi = px_1 + p^2x_2, \quad \eta = qx_1 - q^2x_2,$$

where A_i, B_j are differential operators with respect to p, q defined as

$$\begin{aligned} A_0 &= a_0, \\ A_1 &= a_0 p\partial_p + a_1, \\ A_2 &= \frac{a_0}{2}(p\partial_p)^2 + a_1 p\partial_p + a_2, \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} B_0 &= b_0, \\ B_1 &= b_0 q\partial_q + b_1, \\ B_2 &= \frac{b_0}{2}(q\partial_q)^2 + b_1 q\partial_q + b_2, \\ &\vdots \end{aligned}$$

where a_k, b_l are constants. Then the determinant

$$\tau_n = \det_{1\le i,j\le N}(m^{(n)}_{2i-1,2j-1}) = \begin{vmatrix} m^{(n)}_{1,1} & m^{(n)}_{1,3} & \cdots & m^{(n)}_{1,2N-1} \\ m^{(n)}_{3,1} & m^{(n)}_{3,3} & \cdots & m^{(n)}_{3,2N-1} \\ \vdots & \vdots & & \vdots \\ m^{(n)}_{2N-1,1} & m^{(n)}_{2N-1,3} & \cdots & m^{(n)}_{2N-1,2N-1} \end{vmatrix}$$

satisfies the NLS bilinear equations.

The proof to Lemma 3.2 can be found in Ohta and Yang (2012). Already verifying the proof for Lemma 3.1 we will leave out the proof of 3.2 in order to avoid more redundancy.

3.3 Complex Conjugacy and Simplification of Rogue-Wave Solutions

If we combine the complex conjugate conditions of (3.3) with (3.11) we have that

$$\tau_0 \in \mathbb{R} \quad \text{and} \quad \tau_{-1} = \overline{\tau}_1 \in \mathbb{C}. \tag{3.31}$$

Thus combining (3.31) with (3.12), we can satisfy the above condition by taking the parameters a_k and b_k of Lemma 3.2 to be complex conjugates of one another such that $b_k = \overline{a}_k$. With this condition in mind we know that the rational solution from Theorem 2.1 is non-singular since $u = g/f = \tau_1/\tau_0$ where $f = \tau_0$ is the determinant of a positive definite matrix and therefore $f > 0$.

We can then simplify the rogue-wave solutions such that if we take the generator $\mathcal{G}$ of the differential operators $(p\partial_p)^k(q\partial_q)^l$ defined by

$$\mathcal{G} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\kappa^k}{k!}\frac{\lambda^l}{l!}(p\partial_p)^k(q\partial_q)^l = \exp(\kappa_p\partial_p + \lambda q\partial q) = \exp(\kappa\partial_{\ln p} + \lambda\partial_{\ln q}) \tag{3.32}$$

then for any function $F(p,q)$, we have

$$\mathcal{G}F(p,q) = F(e^{\kappa}p, e^{\lambda}q) \tag{3.33}$$

such that expanding the right-hand side of (3.33) into a Taylor series of (κ, λ) around the point $(0,0)$, rewriting the exponent in terms of x_k^+ and x_l^- as defined in (2.4), calculating the matrix element of the Gram determinant, σ_n, and applying the Laplace expansion to the determinant we are left with rational solutions of the form of Theorem 2.1 (Ohta & Yang 2012).

4 Numerical Simulations of Rogue Waves Solutions

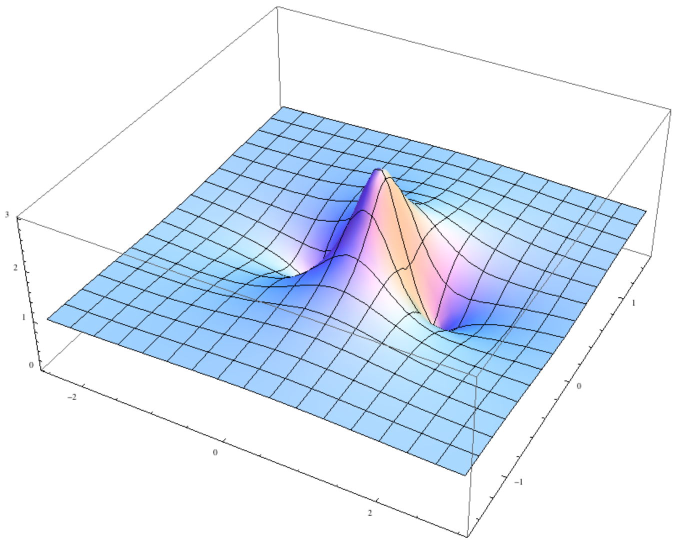

Figure 1: Mathematica plot of 1st order rogue wave solution.

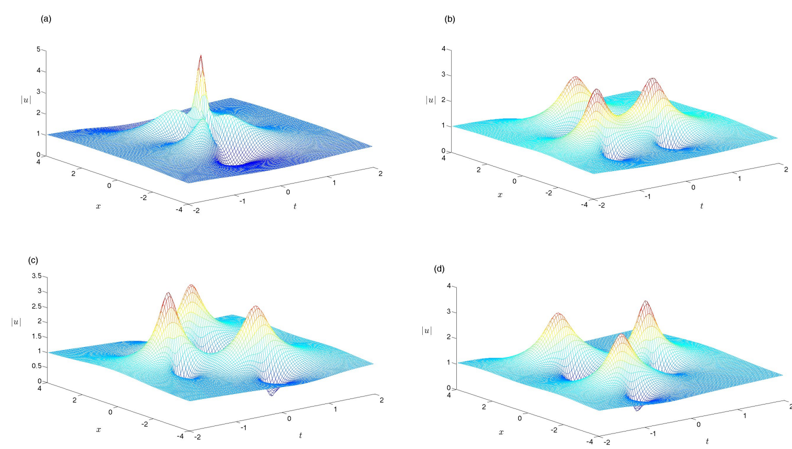

Figure 2: 2nd order wave solutions plotted in MATLAB for the following values of a_3: (a) -1/12, (b) 5/3, (c) $-5i/2$, and (d) $5i/2$.

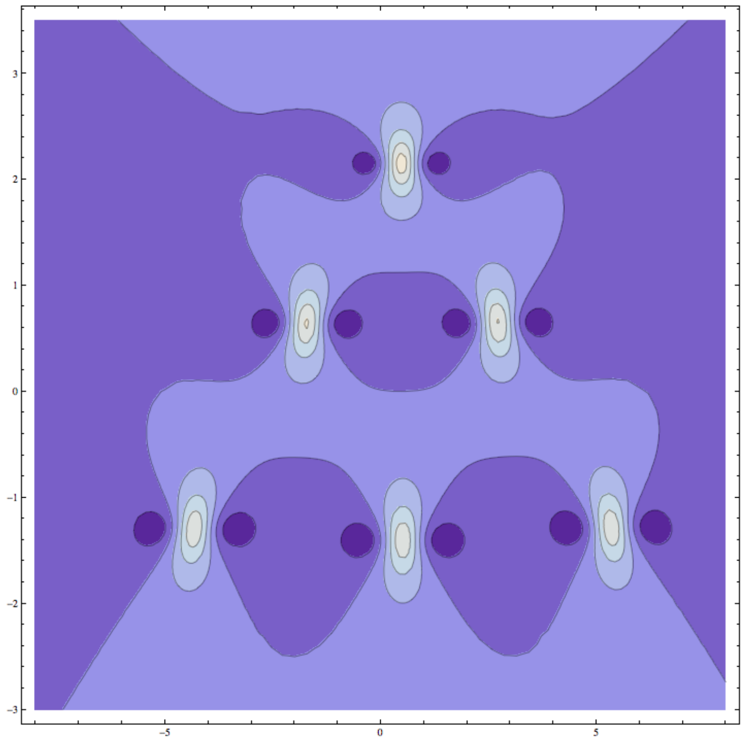

Figure 3: Mathematica contour plot of 3rd order solution for $(a_3, a_5) = (25i/3, 0)$ showing 6 intensity humps.

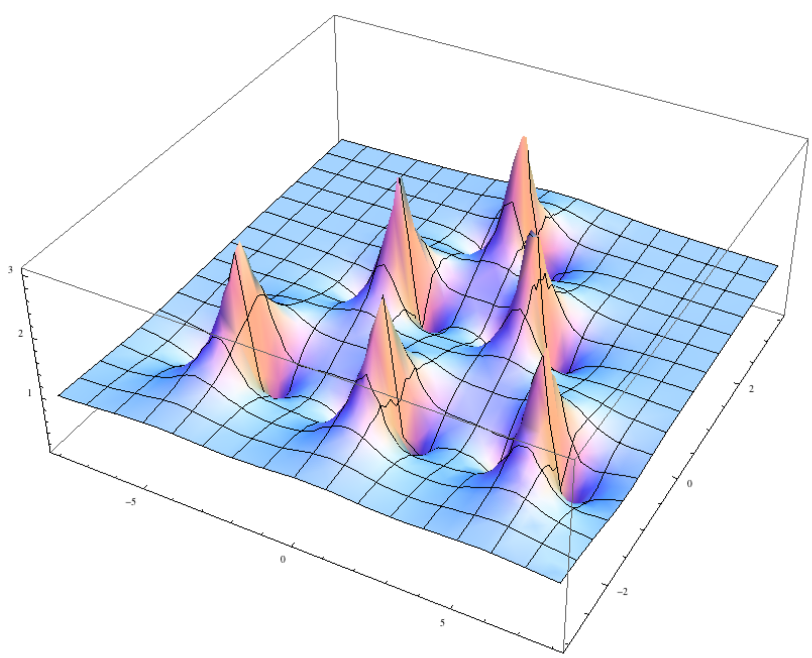

Figure 4: Mathematica plot of 3rd order solution for $(a_3, a_5) = (25i/3, 0)$.

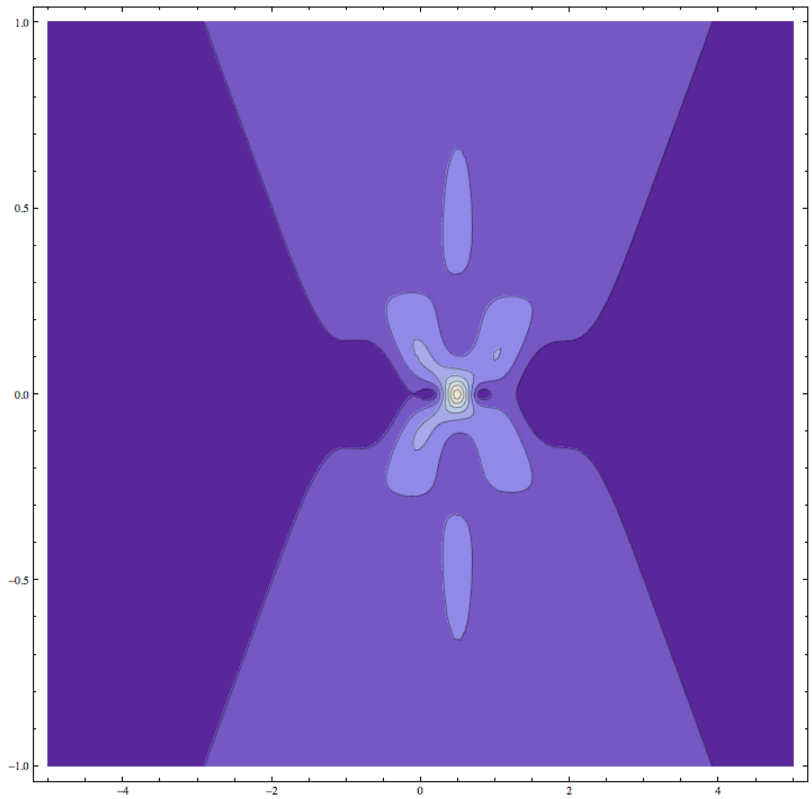

Figure 5: Mathematica contour plot of 3rd order solution for $(a_3, a_5) = (-1/12, -1/240)$ which achieves a maximum amplitude of 9 at $(x, t) = (1/2, 0)$.

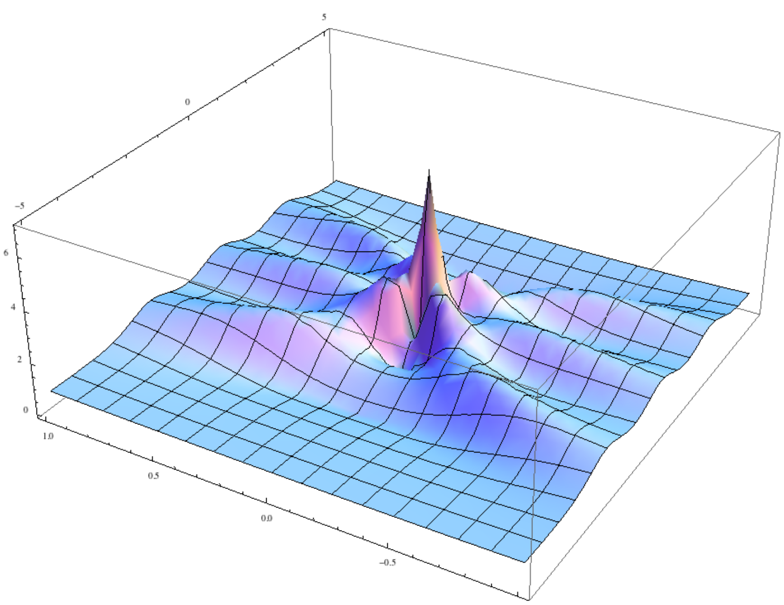

Figure 6: Mathematica plot of 3rd order solution for $(a_3, a_5) = (-1/12, -1/240)$.

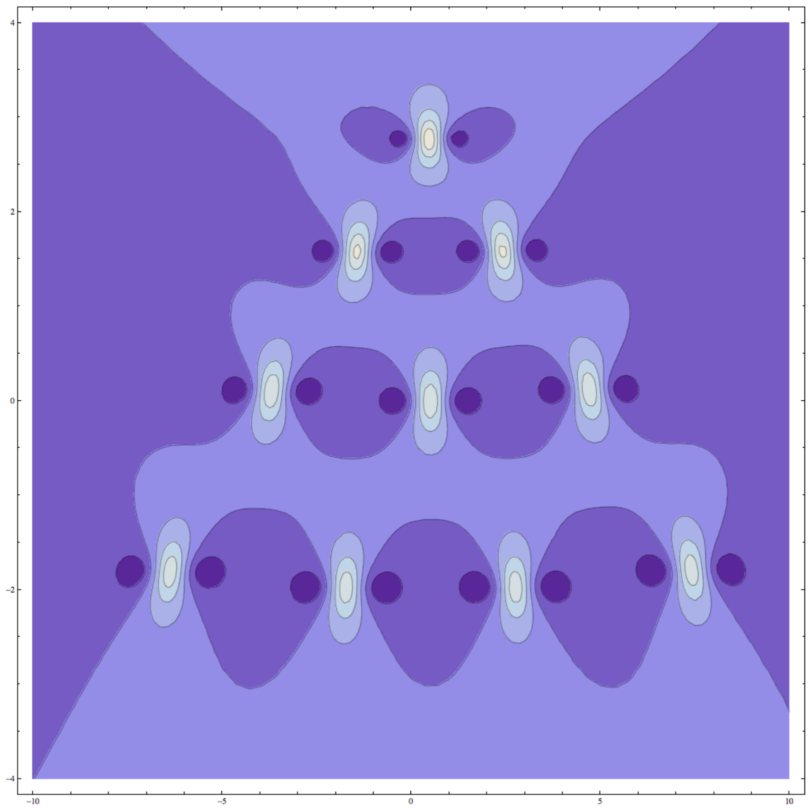

Figure 7: Mathematica contour plot of 4rd order solution for $(a_3, a_5, a_7) = (25i/3, 0, 0)$ showing 10 intensity humps.

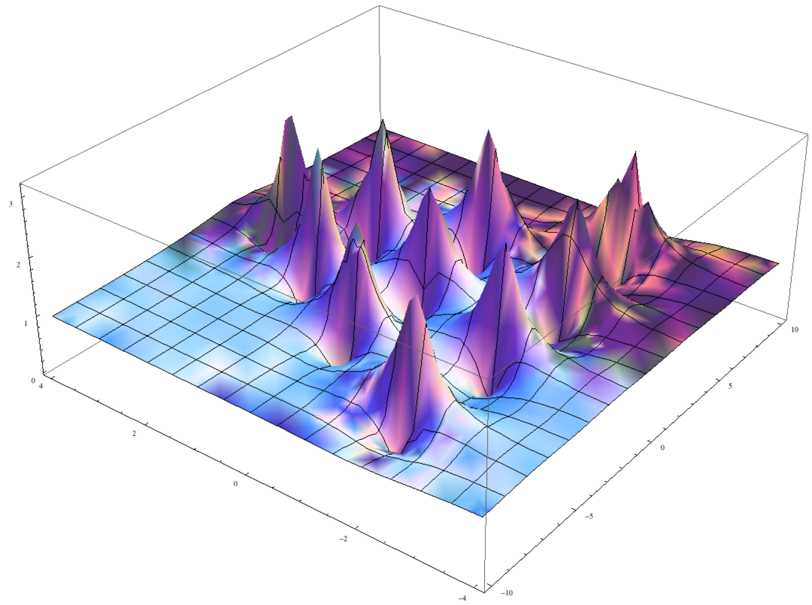

Figure 8: Mathematica plot of 4rd order solution for $(a_3, a_5, a_7) = (25i/3, 0, 0)$.

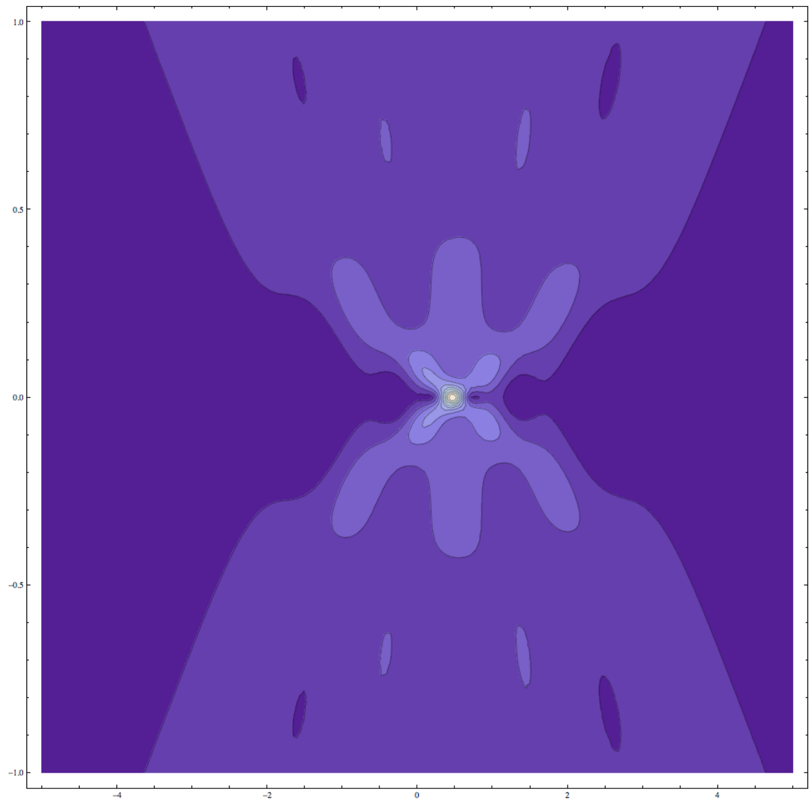

Figure 9: Mathematica contour plot of 4rd order solution for
$(a_3, a_5, a_7) = (-1/12, -1/240, 0)$, which achieves a maximum amplitudeof 9 also at $(1/2, 0)$.

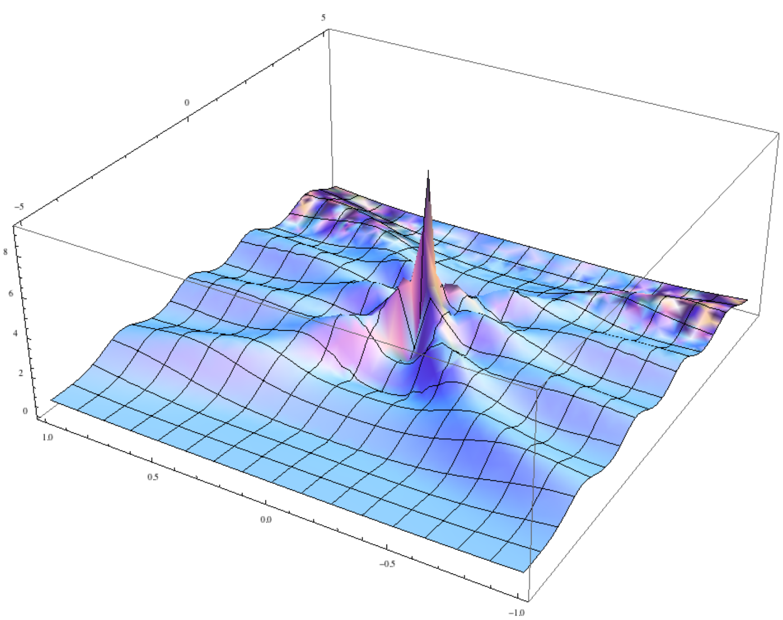

Figure 10: Mathematica plot of 4rd order solution for $(a_3, a_5, a_7) = (-1/12, -1/240, 0)$.

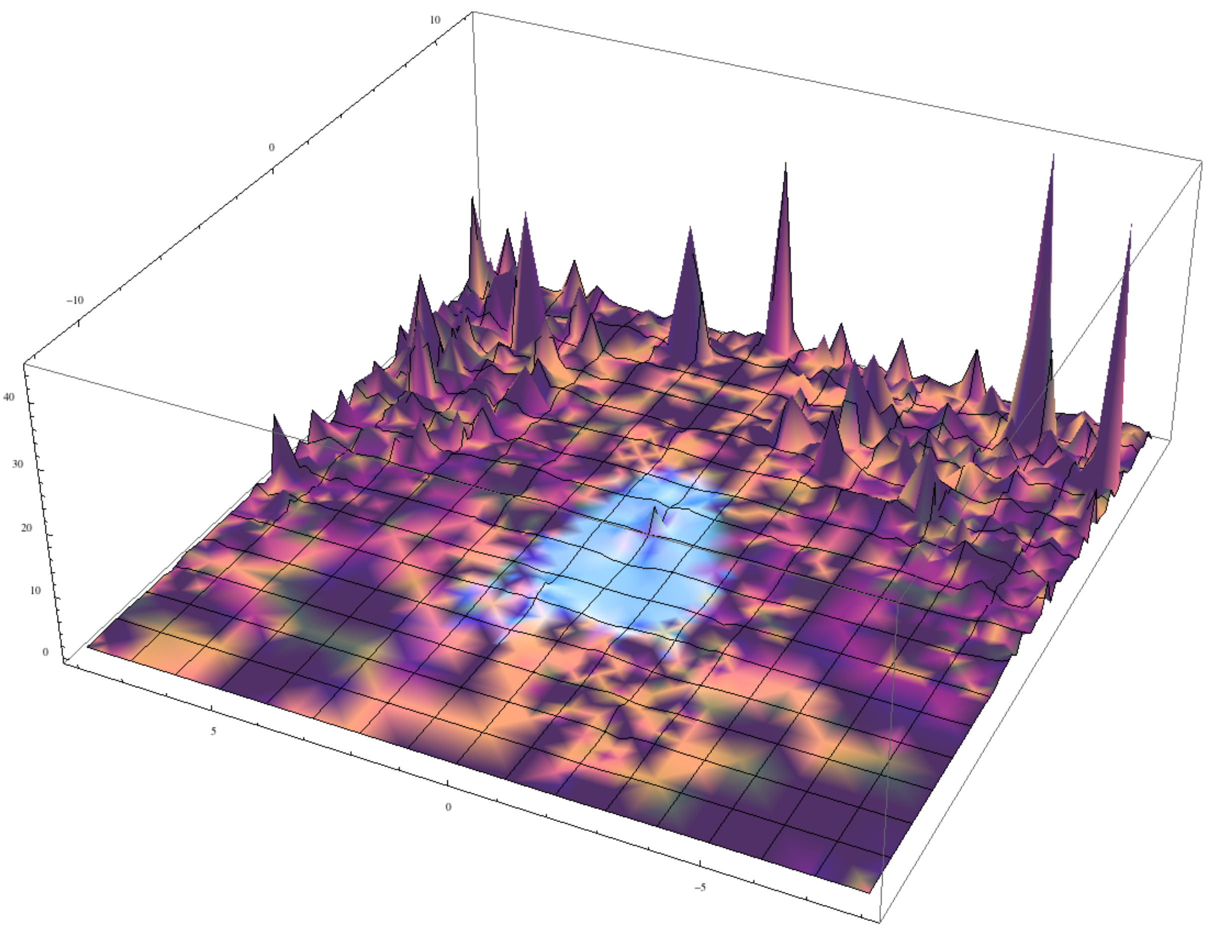

Figure 11: Mathematica plot of 5th order solution for $(a_3, a_5, a_7, a_9) = (-1/12, -1/240, 0, 0)$ showing the extreme instability of such high order solutions. At $(x, t) = (1/2, 0)$ this achieves a maximum amplitude of 11.

Appendix A: Mathematica Code

First Order Code

Remove[“Global`*”]

$n = 1;$

$m = 2 * n;$

$\text{bound} = m - 1;$

$\text{Scoeff} = \text{Series}\left[\text{Log}\left[\frac{2}{\lambda}\text{Tanh}\left[\frac{\lambda}{2}\right]\right], \{\lambda, 0, m\}\right];$

$s[1] = 1; \text{Do}[s[j] = \text{Coefficient}[\text{Scoeff}, \lambda, j], \{j, m\}];$

(*$s[1]$

$s[2]$

$s[3]$

$s[4]$*)

$\text{Rcoeff} = \text{Series}\left[\text{Log}\left[\text{Cosh}\left[\frac{\lambda}{2}\right]\right], \{\lambda, 0, m\}\right];$

$r[1] = 1; \text{Do}[r[i] = \text{Coefficient}[\text{Rcoeff}, \lambda, i], \{i, m\}];$

$x^{+}[1] = x - 2 * i * t + \frac{1}{2};$

$x^{-}[1] = x + 2 * i * t - \frac{3}{2};$

$\text{Do}\left[x^{+}[y] = \frac{x - 2^{y} * i * t}{y!} - r[y] + b * s[y], \{y, 2, m\}\right];$

$\text{Do}\left[x^{-}[z] = \frac{x + 2^{z} * i * t}{z!} - r[z] + b * s[z], \{z, 2, m\}\right];$

$x^{+}[4];$

$\text{Schur}^{+} = \left(\sum_{j=1}^{m}\left(\frac{1}{j!}\right)\left(\sum_{k=1}^{m} x^{+}[k] * \lambda \hat{\ } k\right)^{j}\right);$

$\text{Collect}\left[\text{Schur}^{+}, \lambda\right];$

$S^{+}[0] = 1;$

$\text{Do}\left[S^{+}[i] = \text{Coefficient}\left[\text{Schur}^{+}, \lambda, i\right], \{i, m\}\right];$

$\text{Schur}^{-} = \left(\sum_{j=1}^{m}\left(\frac{1}{j!}\right)\left(\sum_{k=1}^{m} x^{-}[k] * \lambda \hat{\ } k\right)^{j}\right);$

$\text{Collect}\left[\text{Schur}^{-},\lambda\right];$

$S^{-}[0]=1;$

$\text{mat1}=\text{Table}\left[\sum_{b=0}^{\text{Min}[i,j]}\left(\frac{1}{2^b}\left(\sum_{k=0}^{i-b}a[k]*S^{+}[i-b-k]\right)*\frac{1}{2^b}\left(\sum_{l=0}^{j-b}\bar{a}[l]*S^{-}[j-b-l]\right)\right),\right.$
$\{i,1,\text{bound},2\},\{j,1,\text{bound},2\}];$

$x^{+}[1]=x^{+}[1]-1;$

$x^{-}[1]=x^{-}[1]+1;$

$\text{Schur}^{+}=\left(\sum_{j=1}^{m}\left(\frac{1}{j!}\right)\left(\sum_{k=1}^{m}x^{+}[k]*\lambda^\wedge k\right)^{j}\right);$

$\text{Collect}\left[\text{Schur}^{+},\lambda\right];$

$S^{+}[0]=1;$

$\text{Do}\left[S^{+}[i]=\text{Coefficient}\left[\text{Schur}^{+},\lambda,i\right],\{i,m\}\right];$

$\text{Schur}^{-}=\left(\sum_{j=1}^{m}\left(\frac{1}{j!}\right)\left(\sum_{k=1}^{m}x^{-}[k]*\lambda^\wedge k\right)^{j}\right);$

$\text{Collect}\left[\text{Schur}^{-},\lambda\right];$

$S^{-}[0]=1;$

$\text{Do}\left[S^{-}[i]=\text{Coefficient}\left[\text{Schur}^{-},\lambda,i\right],\{i,m\}\right];$

$S^{+}[2];$

$\text{mat0}=\text{Table}\left[\sum_{b=0}^{\text{Min}[i,j]}\left(\frac{1}{2^b}\left(\sum_{k=0}^{i-b}a[k]*S^{+}[i-b-k]\right)*\frac{1}{2^b}\left(\sum_{l=0}^{j-b}\bar{a}[l]*S^{-}[j-b-l]\right)\right),\right.$
$\{i,1,\text{bound},2\},\{j,1,\text{bound},2\}];$

$a[0]=1;a[1]=0;\text{Do}[a[2*i]=0,\{i,1,m\}];$

$\bar{a}[0]=1;\bar{a}[1]=0;\text{Do}\left[\bar{a}[2*i]=0,\{i,1,m\}\right];$

$u=\text{Det}[\text{mat1}]/\text{Det}[\text{mat0}];$

$(*t=0;x=1/2;a[3]=-25i/3;\bar{a}[3]=25i/3;a[5]=0;\bar{a}[5]=0;\text{Simplify}[u]$
$*)$

```
Simplify[Apart[u]]
(*a[3] = −25 ∗ i/3; ā[3] = 25 ∗ i/3; a[5] = 0; ā[5] = 0; *)
(*Plot3D[Abs[u], {x, −8, 8}, {t, −3, 3}, PlotRange → Full]*)
NMaximize[Abs[u], {{x, 0.45, 0.55}, t}]
```

$1 + \frac{-2+8it}{1+8t^2-2x+2x^2}$

$\{3., \{x \to 0.5, t \to -7.083385313162492$*^-9$\}\}$

Fourth Order

```
Remove["Global`*"]

n = 4;
m = 2 ∗ n;
bound = m − 1;

Scoeff = Series[Log[(2/λ) Tanh[λ/2]], {λ, 0, m}];
s[1] = 1; Do[s[j] = Coefficient[Scoeff, λ, j], {j, m}];
(*s[1]
s[2]
s[3]
s[4]*)
Rcoeff = Series[Log[Cosh[λ/2]], {λ, 0, m}];
r[1] = 1; Do[r[i] = Coefficient[Rcoeff, λ, i], {i, m}];

x+[1] = x − 2 ∗ i ∗ t + 1/2;
x−[1] = x + 2 ∗ i ∗ t − 3/2;
Do[x+[y] = (x − 2^y ∗ i ∗ t)/y! − r[y] + b ∗ s[y], {y, 2, m}];
Do[x−[z] = (x + 2^z ∗ i ∗ t)/z! − r[z] + b ∗ s[z], {z, 2, m}];
```

$x^+[4];$

$\text{Schur}^+ = \left(\sum_{j=1}^{m} \left(\frac{1}{j!}\right) \left(\sum_{k=1}^{m} x^+[k] * \lambda\wedge k\right)^j\right);$

$\text{Collect}\left[\text{Schur}^+, \lambda\right];$

$S^+[0] = 1;$

$\text{Do}\left[S^+[i] = \text{Coefficient}\left[\text{Schur}^+, \lambda, i\right], \{i, m\}\right];$

$\text{Schur}^- = \left(\sum_{j=1}^{m} \left(\frac{1}{j!}\right) \left(\sum_{k=1}^{m} x^-[k] * \lambda\wedge k\right)^j\right);$

$\text{Collect}\left[\text{Schur}^-, \lambda\right];$

$S^-[0] = 1;$

$\text{mat1} = \text{Table}\left[\sum_{b=0}^{\text{Min}[i,j]} \left(\frac{1}{2^b} \left(\sum_{k=0}^{i-b} a[k] * S^+[i-b-k]\right) * \frac{1}{2^b} \left(\sum_{l=0}^{j-b} \bar{a}[l] * S^-[j-b-l]\right)\right),\right.$
$\left.\{i, 1, \text{bound}, 2\}, \{j, 1, \text{bound}, 2\}\right];$

$x^+[1] = x^+[1] - 1;$

$x^-[1] = x^-[1] + 1;$

$\text{Schur}^+ = \left(\sum_{j=1}^{m} \left(\frac{1}{j!}\right) \left(\sum_{k=1}^{m} x^+[k] * \lambda\wedge k\right)^j\right);$

$\text{Collect}\left[\text{Schur}^+, \lambda\right];$

$S^+[0] = 1;$

$\text{Do}\left[S^+[i] = \text{Coefficient}\left[\text{Schur}^+, \lambda, i\right], \{i, m\}\right];$

$\text{Schur}^- = \left(\sum_{j=1}^{m} \left(\frac{1}{j!}\right) \left(\sum_{k=1}^{m} x^-[k] * \lambda\wedge k\right)^j\right);$

$\text{Collect}\left[\text{Schur}^-, \lambda\right];$

$S^-[0] = 1;$

$\text{Do}\left[S^-[i] = \text{Coefficient}\left[\text{Schur}^-, \lambda, i\right], \{i, m\}\right];$

$S^+[2];$

$\mathbf{mat0} = \mathbf{Table}\left[\sum_{b=0}^{\mathrm{Min}[i,j]}\left(\frac{1}{2^b}\left(\sum_{k=0}^{i-b} a[k] * S^{+}[i-b-k]\right) * \frac{1}{2^b}\left(\sum_{l=0}^{j-b} \bar{a}[l] * S^{-}[j-b-l]\right)\right),\right.$
$\{i, 1, \mathrm{bound}, 2\}, \{j, 1, \mathrm{bound}, 2\}];$

$a[0] = 1; a[1] = 0; \mathrm{Do}[a[2 * i] = 0, \{i, 1, m\}];$

$\bar{a}[0] = 1; \bar{a}[1] = 0; \mathrm{Do}\left[\bar{a}[2 * i] = 0, \{i, 1, m\}\right];$

$u = \mathrm{Det}[\mathrm{mat1}]/\mathrm{Det}[\mathrm{mat0}];$

$(*t = 0; x = 1/2; a[3] = -25i/3; \bar{a}[3] = 25i/3; a[5] = 0; \bar{a}[5] = 0; \mathrm{Simplify}[u]$

$*)$

$u;$

$a[3] = -1/12; \bar{a}[3] = -1/12; a[5] = -1/240; \bar{a}[5] = -1/240;\ a[7] = -1/4480;\ \bar{a}[7] = -1/4480;$

$(*\mathrm{ContourPlot}[\mathrm{Abs}[u], \{x, -8, 8\}, \{t, -3, 3\}, \mathrm{PlotRange} \to \mathrm{Full}]*)$

$\mathrm{NMaximize}[\mathrm{Abs}[u], \{\{x, 0.5, 0.55\}, t\}]$

$\{8.99953, \{x \to 0.501058, t \to -1.0670060986818555\text{*\^{}-10}\}\}$

Appendix B: MATLAB Code

Second Order

```
%M395Project.m
%Reagan & Roma
%Spring 2012

clear all
close all
clc

a3 = -1/12; % this parameter generates the max peak of order 1
Z = zeros(100);
for w = 1:100
    for j = 1:100
        %X = 0; T = 0; z = 0;
        X = 8*w/100-4;
        T = 4*j/100 - 2;
        phi=(24*((3*X - 6*X^2 + 4*X^3 - 2*X^4 - 48*T^2 + 48*X*T^2 -
```

```
         48*X^2*T^2 - 160*T^4)+ i*T*(-12 + 12*X - 16*X^3 + 8*X^4 +
          32*T^2 - 64*X*T^2 + 64*X^2*T^2 + 128*T^4)+ 6*a3*(1 - 2*X +
           X^2 - 4*i*T + 4*i*X*T - 4*T^2)+ 6*conj(a3)*(-X^2 + 4*i*X*T + 4*T^2 )));

        psi=(9 - 36*X + 72*X^2 - 72*X^3 + 72*X^4 - 48*X^5 +
         16*X^6)+96*T^2*(3 + 3*X - 4*X^3 +2*X^4) +
          384*T^4*(5 - 2*X+2*X^2)+1024*T^6 +
           24*(a3+conj(a3))*(3*X^2 - 2*X^3 - 12*T^2
            +24*X*T^2)+48*i*(a3 - conj(a3))*(3*T +6*X*T
             - 6*X^2*T + 8*T^3)+144*a3*conj(a3);
        z = 1+(phi/psi);
        Z(w,j) = z;
    end
end
mesh(-2:4/100:2-4/100,-4:8/100:4-8/100,abs(Z))

max(max(abs(Z)))
```

References

[] Dysthe, K., Krogstad, H.E., & Müller, P. 2008 Oceanic rogue waves. *Annual Review of Fluid Mechanics* **40**: 287-310. (doi:10.1146/annurev.fluid.40.111406.102203)

[] Hirota, R. 2004 *The direct method in soliton theory.* Cambridge, UK: Cambridge University Press. (doi:10.1017/CBO9780511543043)

[] Jimbo, M. & Miwa, T. 1983 Solitons and infinite dimensional Lie algrebras. *Publ. RIMS Kyoto Univ.* **19**, 943-1001. (doi:10.2977/prims/1195182017)

[] Kharif, C. & Pelinovsky, E. 2003 Physical mechanisms of the rogue wave phenomenon. *Eur. F Mech. B* **22**: 603-634.

[] Ohta, Y. & Yang, J. 2012 General rogue waves in the NLS equation. *The Royal Society.* (doi: 10.1098/rspa.20110640)

[] Peregrine, D.H. 1983 Water waves, nonlinear Schrödinger equations and their solutions. *J. Aust. Math. Soc. B* **25**, 16-43. (doi:10.1017/S0334270000003891)

[] Sato, M. 1981 Soliton equations as dynamical systems on an infinite dimensional Grassman manifolds. *RIMS Kokyuroku* **439**, 30-46.

[] Solli, D. R., Ropers, C., Koonath, P. & Jalali, B. 2007 Optical rogue waves. *Nature* **450**, 1054-1057. (doi: 10.1038/nature06402)

[] Zakharov, V. E. & Shabat, A. B. 1972 Exact theory of two-dimensional self focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys. JETP* **34**, 62-69.