Rogue Waves in the Nonlinear Schrödinger Equation

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Figure 0: Rogue wave?

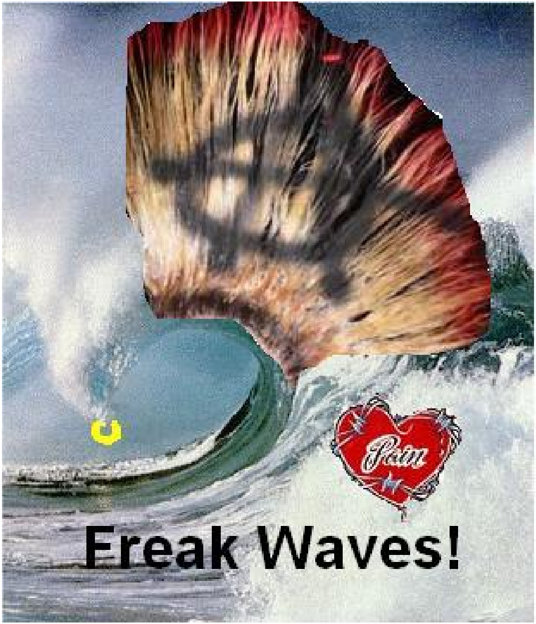

Figure 1: Definitely rogue

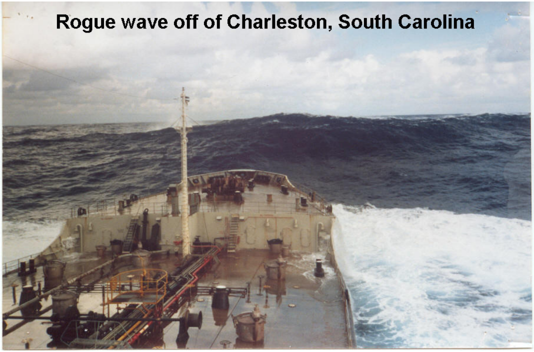

Figure 2: Observed rogue wave

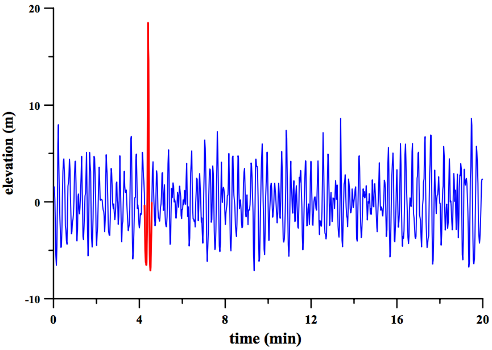

Figure 3: The "Draupner Wave"

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- The structure and nature of these waves is imperative in our understanding of how and why they occur.
- Occur in both water and optical waves.

- In oceanography, the pragmatic approach is to define a rogue wave whenever

$$H/H_s > 2 \quad \text{or} \quad \omega_c/H_s > 1.25 \tag{1.1}$$

where H is the wave height (distance from trough to crest), ω_c is the crest height (distance from mean sea level to crest), and H_s is the significant wave height, here defined as four times the standard deviation of the surface elevation.

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- Known to cause extensive damage, and are even life-threatening, when they come into contact with ocean liners and passenger ships in the open waters.
- Between 1964 and 1994, it is estimated that more than 22 super-carriers have been lost at sea as a direct result of rogue waves (Kharif & Pelinovsky 2003).

Physical mechanisms of rogue wave formation

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- Ohta and Yang (2012) derived general high-order waves in the NLS equation using the bilinear method in the soliton theory, and then further simplified there results to algebraic expressions using Gram determinants and elementary Schur polynomials.

Definition

Let $S_n(\mathbf{x})$ be the elementary Schur polynomial defined by the generating function

$$\sum_{n=0}^{\infty} S_n(\mathbf{x})\lambda^n = \exp\left(\sum_{n=0}^{\infty} x_k\lambda^k\right), \tag{1.3}$$

where $\mathbf{x} = (x_1, x_2, \ldots)$. Such that the first few terms are

$$S_0(\mathbf{x}) = 1, \quad S_1(\mathbf{x}) = 1, \quad S_2(\mathbf{x}) = \frac{1}{2}x_1^2+x_2, \quad S_3(\mathbf{x}) = \frac{1}{6}x_1^3+x_1x_2+x_3, \ldots$$

If we consider the general rogue wave solutions of the focusing NLS equation

$$iu_t = u_{xx} + 2|u|^2 u \tag{2.1}$$

we find it is invariant under scalings $x \to \alpha x$, $t \to \alpha^2 t$, $u \to u/\alpha$ for any constant $\alpha \in \mathbb{R}$, as well as the Galilean transformation $u(x,t) \to u(x - vt, t)\exp(-ivx/2 + iv^2 t/4)$ (Ohta & Yang 2012). Therefore, we will consider the rogue waves that approach a nonzero constant background at large x and t,

$$u(x,t) \to e^{-2it} \quad \text{as} \quad x,\ t \to \pm\infty.$$

If we apply the variable transformation $u \to ue^{-2it}$, then (2.1) becomes

$$\begin{aligned} i(ue^{-2it})_t &= (ue^{-2it})_{xx} + 2\left|ue^{-2it}\right|^2 \cdot ue^{-2it} \\ i\left(u_t e^{-2it} - 2iue^{-2it}\right) &= e^{-2it}u_{xx} + 2|u|^2 \cdot ue^{-2it} \\ iu_t e^{-2it} &= e^{-2it}u_{xx} + 2|u|^2 \cdot ue^{-2it} - 2ue^{-2it} \\ iu_t &= u_{xx} + 2u\left(|u|^2 - 1\right) \end{aligned} \tag{2.2}$$

where the boundary conditions become

$$u(x,t) \to 1 \quad \text{as} \quad x,\ t \to \pm\infty. \tag{2.3}$$

Main result

The NLS equation (2.1) under the boundary conditions (2.3) has non-singular rational solutions

$$u = \frac{\sigma_1}{\sigma_0}$$

where

$$\sigma_n = \det_{1 \le i,j \le N} \left(m^{(n)}_{2i-1,j-1} \right).$$

The matrix elements in σ_m are defined by

$$m^{(n)}_{ij} = \sum_{v=0}^{\min(i,j)} \Phi^{(n)}_{iv} \Psi^{(n)}_{jv}.$$

Main result, contd

Where

$$\Phi_{iv}^{(n)} = \frac{1}{2^v} \sum_{k=0}^{i-v} a_k S_{i-v-k} \left(\mathbf{x}^+(n) + v\mathbf{s} \right)$$

$$\Psi_{jv}^{(n)} = \frac{1}{2^v} \sum_{l=0}^{j-v} \bar{a}_l S_{j-v-l} \left(\mathbf{x}^-(n) + v\mathbf{s} \right)$$

for a_k for $k = 0, 1, \ldots$ are complex constants, and $\mathbf{x}^{\pm}(n) = (x_1^{\pm}(n), \ldots)$, $\mathbf{s} = (s_1, lots)$ are defined by

$$x_1^{\pm}(n) = x \mp 2it \pm n - \frac{1}{2}, \quad x_k^{\pm} = \frac{x \mp 2^k it}{k!} - r_k, \quad (k \geq 2), \ (2.4)$$

and

$$\sum_{k=1}^{\infty} r_k \lambda^k = \ln\left(\cosh\frac{\lambda}{2}\right) \quad \text{and} \quad \sum_{k=1}^{\infty} s_k \lambda^k = \ln\left(\frac{2}{\lambda}\tanh\frac{\lambda}{2}\right).$$

Deriving Bilinear Form

First we derive the bilinear form of the transformed NLS from (2.1). Let $u = g/f$ for $g \in \mathbb{C}, f \in \mathbb{R}$. Then we have

$$\begin{aligned}
0 &= \left(\frac{g}{f}\right)_{xx} + 2\left(\left|\frac{g}{f}\right|^2 - 1\right)\frac{g}{f} - i\left(\frac{g}{f}\right)_t \\
&= \left(\frac{fg_x - gf_x}{f^2}\right)_x + 2\left(\frac{|g|^2 g}{f^3} - \frac{g}{f}\right) - i\left(\frac{fg_t - gf_t}{f^2}\right) \\
&= \frac{f^2(fg_{xx} - gf_{xx}) - (fg_x - gf_x)2f}{f^4} + 2\left(\frac{|g|^2 g}{f^3} - \frac{g}{f}\right) - i\left(\frac{fg_t - gf_t}{f^2}\right).
\end{aligned}$$

By multiplying through by f^3, and then grouping by g, f we have

$$
\begin{aligned}
0 &= f(f_{xx} - gf_{xx}) - 2(fg_x - gf_x) + 2|g|^2 g - gf^2 - if(fg_t - gf_t) \\
&= f((fg_{xx} - gf_{xx}) - i(fg_t - gf_t)) + g\left(2|g|^2 - f^2 - 2f\frac{g_x}{g} - 2f_x\right) \\
&= f(D_x^2 - iD_t)(gf) + g((D_x^2 + 2)(f^2)) - 2|g|^2)
\end{aligned}
$$

so we have the desired bilinear form

$$\left.\begin{array}{rl} (D_x^2+2)f\cdot f & = 2|g|^2 \\ \text{and} \;\; (D_x^2-iD_t)g\cdot f & = 0, \end{array}\right\} \tag{3.1}$$

where D is Hirota's bilinear differential operator such that $D_x(fg) = f_x g - g_x f$. Then for h another complex variable, we consider the $2+1$-dimensional generalization of the above bilinear form

$$\left.\begin{array}{rl} (D_x D_y+2)f\cdot f & = 2gh \\ \text{and} \;\; (D_x^2-iD_t)g\cdot f & = 0. \end{array}\right\} \tag{3.2}$$

Solutions to equations (3.2) under the conditions

$$(\partial_x + \partial_y)f = Cf, \qquad \text{where} \quad f \in \mathbb{R} \quad \text{and} \quad h = \overline{g} \tag{3.3}$$

for C some constant then also satisfy the bilinear NLS equations, since then $gh = |g|^2$ and $D_x D_y(f) = D_x^2(f)$.
Next, we verify Lemma 3.1 of Ohta and Yang (2012) to begin constructing Gram determinant solutions to the $2 + 1$-dimensional bilinear system.

Lemma

Let $m_{ij}^{(n)}, \phi_i^{(n)}, \psi_j^{(n)}$ be functions of x_1, x_2, x_{-1} satisfying the following differential and difference relations,

$$
\left.
\begin{array}{rrl}
 & \partial_{x_1} m_{ij}^{(n)} & = \phi_i^{(n)} \psi_j^{(n)}, \\
 & \partial_{x_2} m_{ij}^{(n)} & = \phi_i^{(n+1)} \psi_j^{(n)} + \phi_i^{(n)} \psi^{(n-1)}, \\
 & \partial_{x_{-1}} m_{ij}^{(n)} & = -\phi_i^{(n-1)} \psi_j^{(n+1)}, \\
 & m_{ij}^{(n+1)} & = m_{ij}^{(n)} + \phi_i^{(n)} \psi_j^{(n+1)}, \\
 \text{and} & \partial_{x_k} \phi_i^{(n)} & = \phi_i^{(n+k)}, \quad \partial_{x_k} \psi_j^{(n)} = -\psi_j^{(n-k)}, \ (k = 1, 2, -1).
\end{array}
\right\}
$$

Lemma, contd

Then the determinant,

$$\tau_n = \det_{1 \le i,j \le N} \left(m_{ij}^{(n)} \right),$$

satisfies the bilinear equations,

$$\left.\begin{aligned} (D_{x_1} D_{x_{-1}} - 2)\tau_n \cdot \tau_n &= -2\tau_{n+1}\tau_{n-1} \\ \text{and } (D_{x_1}^2 - D_{x_2})\tau_{n+1} \cdot \tau_n &= 0. \end{aligned}\right\}$$

Proof. We then verify

$$\partial_{x_1}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n)} \\ -\psi_j^{(n)} & 0 \end{vmatrix},$$

$$\partial_{x_1}^2\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n+1)} \\ -\phi_j^{(n)} & 0 \end{vmatrix} + \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n)} \\ \phi_j^{(n-1)} & 0 \end{vmatrix},$$

$$\partial_{x_2}\tau_n = \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n+1)} \\ -\phi_j^{(n)} & 0 \end{vmatrix} - \begin{vmatrix} m_{ij}^{(n)} & \phi_i^{(n)} \\ \phi_j^{(n-1)} & 0 \end{vmatrix},$$

$$\vdots$$

So we have

$$
\begin{aligned}
(\partial_{x_1}\partial_{x_{-1}} - 1)\tau_n \times \tau_n &= \partial_{x_1}\tau_n \times \partial_{x_{-1}}\tau_n - (-\tau_{n-1})(-\tau_{n+1}), \\
\tfrac{1}{2}(\partial_{x_1}^2 - \partial_{x_2})\tau_{n+1} \times \tau_n &= \partial_{x_1}\tau_{n+1} \times \partial_{x_1}\tau_n - \tau_{n+1}\tfrac{1}{2}(\partial_{x_1}^2 + \partial_{x_2})\tau_n,
\end{aligned}
$$

which is the bilinear form of the $2+1$-dimensional NLS, as desired. □

Since we write

$$m_{ij}^{(n)} = \int^{x_1} \phi_i^{(n)} \psi_j^{(n)} dx_1$$

the determinant τ_n is the Gram determinant solution. By defining

$$f = \tau_0, \quad g = \tau_1, \quad h = \tau_{-1}, \tag{3.11}$$

these are the Gram determinant solutions for the $2+1$-dimensional bilinear system where

$$x_1 = x,\ x_2 = -it, \text{ and } x_{-1} = -y. \tag{3.12}$$

The following Lemma shows that by choosing the matrix elements appropriately in τ_n, we have solutions that also satisfy the reduction condition.

Lemma

Define matrix elements

$$m_{ij}^{(n)} = A_i B_j m^{(n)}\Big|_{p=1,q=1}$$

and

$$m^{(n)} = \frac{1}{p+q}\left(-\frac{p}{q}\right)^n e^{\xi+\eta}, \quad \xi = px_1 + p^2 x_2, \quad \eta = qx_1 - q^2 x_2,$$

where A_i, B_j are differential operators with respect to p, q defined as

$$\begin{aligned}
A_0 &= a_0, \\
A_1 &= a_0 p\partial_p + a_1, \\
A_2 &= \frac{a_0}{2}(p\partial_p)^2 + a_1 p\partial_p + a_2, \\
&\vdots
\end{aligned}$$

Lemma, contd

$$
\begin{aligned}
B_0 &= b_0, \\
B_1 &= b_0 q\partial_q + b_1, \\
B_2 &= \frac{b_0}{2}(q\partial_q)^2 + b_1 q\partial_q + b_2, \\
&\vdots
\end{aligned}
$$

where a_k, b_l are constants. Then the determinant

$$
\tau_n = \det_{1\le i,j\le N}\left(m^{(n)}_{2i-1,2j-1}\right) = \begin{vmatrix} m^{(n)}_{1,1} & \cdots & m^{(n)}_{1,2N-1} \\ \vdots & \ddots & \vdots \\ m^{(n)}_{2N-1,1} & \cdots & m^{(n)}_{2N-1,2N-1} \end{vmatrix}
$$

satisfies the NLS bilinear equations.

If we combine the complex conjugate conditions of (3.3) with (3.11) we have that

$$\tau_0 \in \mathbb{R} \quad \text{and} \quad \tau_{-1} = \overline{\tau}_1 \in \mathbb{C}. \tag{3.31}$$

Thus combining (3.31) with (3.12), we can satisfy the above condition by taking the parameters a_k and b_k of Lemma 3.2 to be complex conjugates of one another such that $b_k = \overline{a}_k$. With this condition in mind we know that the rational solution from the main result is non-singular since $u = g/f = \tau_1/\tau_0$ where $f = \tau_0$.

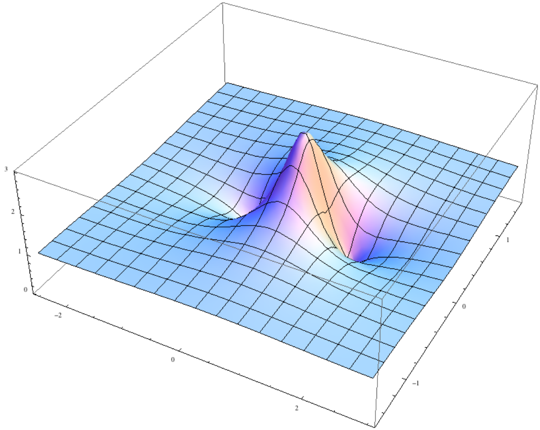

Figure 4: Mathematica plot of 1st order rogue wave solution.

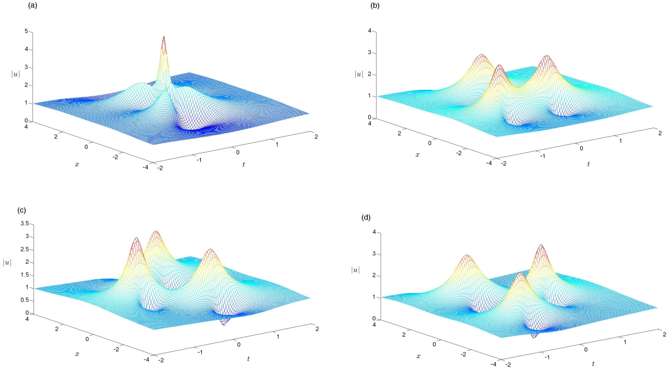

Figure 5: 2nd order wave solutions plotted in MATLAB for the following values of a_3: (a) -1/12, (b) 5/3, (c) $-5i/2$, and (d) $5i/2$.

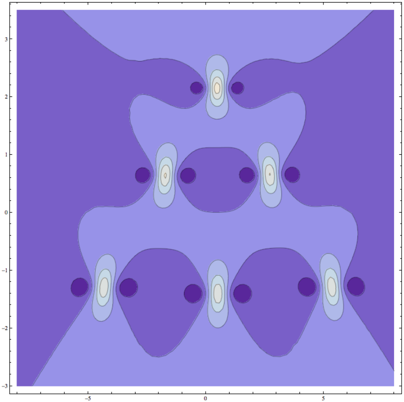

Figure 6: Mathematica contour plot of 3rd order solution for $(a_3, a_5) = (25i/3, 0)$ showing 6 intensity humps.

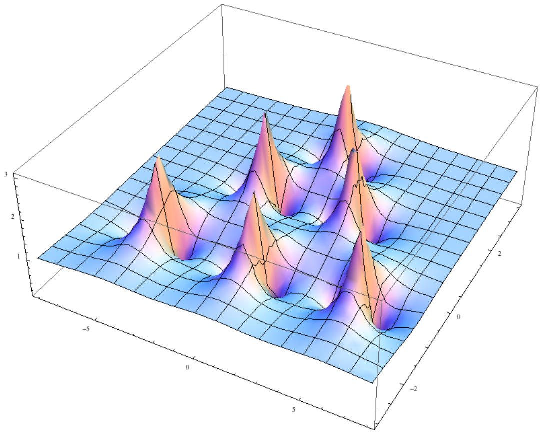

Figure 7: Mathematica plot of 3rd order solution for $(a_3, a_5) = (25i/3, 0)$.

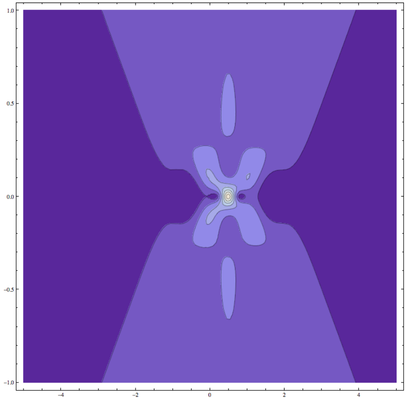

Figure 8: Mathematica contour plot of 3rd order solution for $(a_3, a_5) = (-1/12, -1/240)$ which achieves a maximum amplitude of 9 at $(x, t) = (1/2, 0)$.

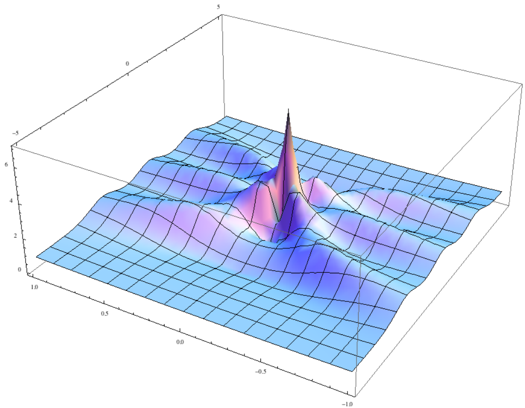

Figure 9: Mathematica plot of 3rd order solution for $(a_3, a_5) = (-1/12, -1/240)$.

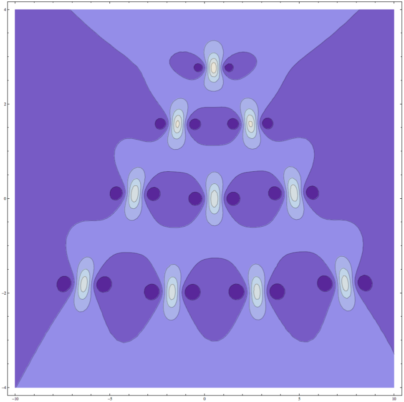

Figure 10: Mathematica contour plot of 4rd order solution for $(a_3, a_5, a_7) = (25i/3, 0, 0)$ showing 10 intensity humps.

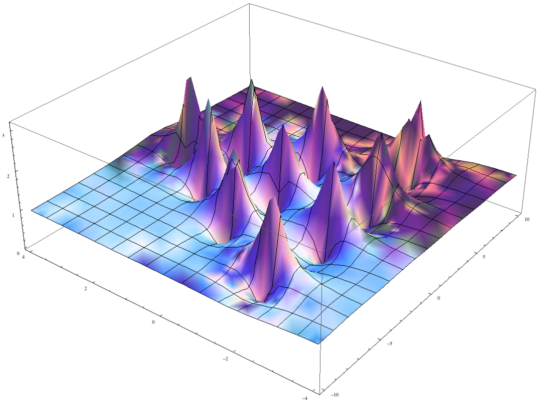

Figure 11: Mathematica plot of 4rd order solution for $(a_3, a_5, a_7) = (25i/3, 0, 0)$.

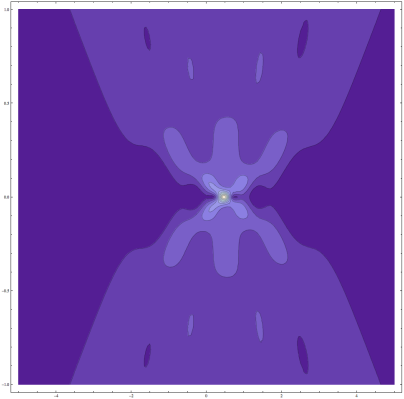

Figure 12: Mathematica contour plot of 4rd order solution for $(a_3, a_5, a_7) = (-1/12, -1/240, 0)$, which achieves a maximum amplitude of 9 also at $(1/2, 0)$.

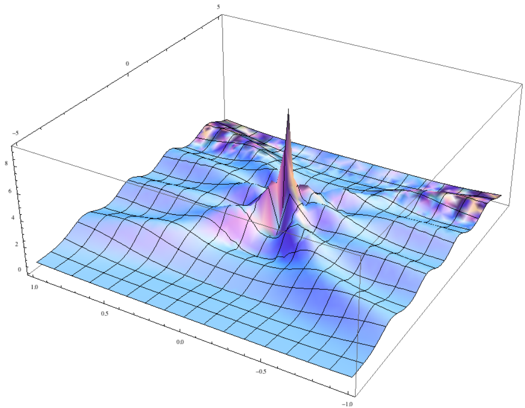

Figure 13: Mathematica plot of 4rd order solution for $(a_3, a_5, a_7) = (-1/12, -1/240, 0)$.

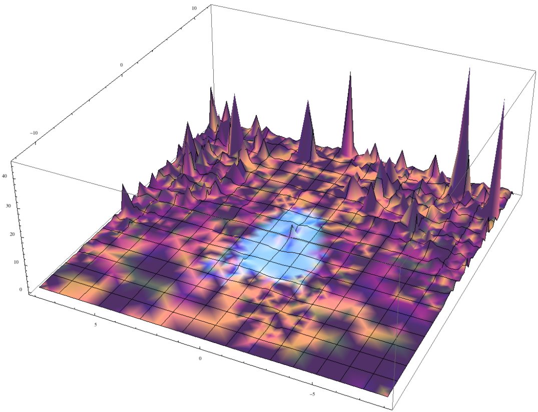

Figure 14: Mathematica plot of 5th order solution for $(a_3, a_5, a_7, a_9) = (-1/12, -1/240, 0, 0)$ showing the extreme instability of such high order solutions. At $(x, t) = (1/2, 0)$ this achieves a maximum amplitude of 11.

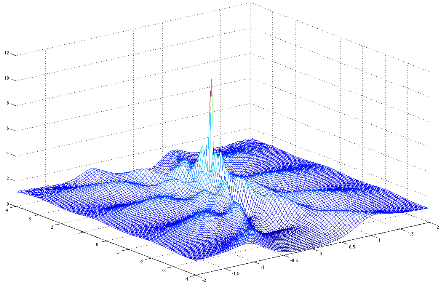

Figure 14: Matlab plot of 5th order solution for $(a_3, a_5, a_7, a_9) = (-1/12, -1/240, 0, 0)$ with higher accuracy, showing that Mathematica had numerical issues.