Practice paper HL

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June 2025

Paper 1

Section A

1. [Maximum mark: 5]

Consider the probability density function $f(x) = \dfrac{x^2}{9}$ where $x \in [0, 3]$.

a) Check that $f(x)$ is a valid probability density function. [2]

b) Find $P(X \in [0, 1])$. [3]

2. [Maximum mark: 5]

Use the principle of mathematical induction to prove that:

$$\prod_{i=1}^{n}(1 - \frac{1}{i+1}) = \frac{1}{n+1}$$

3. [Maximum mark: 5]

Let $f'(x) = \dfrac{1-x}{1+x}$. Given that $f(0) = 1$, find $f(x)$

4. [Maximum mark: 5]

Find the shortest distance between the skew lines $x = t$, $y = 1 - t$, $z = 2 + t$ and $3 - s$, $-1 + 2s$, $z = 4 - s$ where $s, t \in \mathbb{R}$.

5. [Maximum mark: 5]

Find $|z|$ given that $3|z + 1| = |z + 9|$, where $z \in \mathbb{C}$.

6. [Maximum mark: 8]

Given the function $f(x) = \dfrac{3x^2 + x - 2}{x - 1}$. Find the axes intercepts and the asymptotes.

7. [Maximum mark: 7]

Solve for x:

$$\log_{27}(\frac{1}{x}) + \log_3(x^4) = \log_3 10$$

8. [Maximum mark: 7]

Find the derivative of $f(x) = \sin x$ from first principles.

9. [Maximum mark: 8]

Find:

$$\int \frac{dx}{1 + \cos^2 x}$$

Section B

10. [Maximum mark: 16]

In an RL-circuit, the current I amps changes according to the differential equation

$$L\frac{dI}{dt} + RI = E$$

L is the induction in henrys, R is the resistance in ohms, E is the voltage drop in volts, t is the time in seconds.

a) Find a particular solution for $I(t)$ if $I(0) = 0$ amps. [8]

b) By considering I as $t \to \infty$, find the limiting current. [4]

c) Find the time required for the current to reach 99 percent of its limiting value. [4]

11. [Maximum mark: 22]

Laplace transforms provide a useful link between improper integrals and differential equations. The Laplace transform of a function $f(x)$ is defined as $F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty (e^{-sx} f(x))dx$.

a) Show that: [6]

$$\mathcal{L}\{e^{ax}\} = \frac{1}{s-a}, s > a$$
$$\mathcal{L}\{x\} = \frac{1}{s^2}, s > 0$$
$$\mathcal{L}\{\sin ax\} = \frac{a}{s^2+a^2}, s > 0$$

b) Show that: [7]

$$\mathcal{L}\{f'(x)\} = s\mathcal{L}\{f(x)\} - f(0)$$
$$\mathcal{L}\{f''(x)\} = s^2\mathcal{L}\{f(x)\} - sf(0) - f'(0)$$

c) Consider the differential equation $f''(x) + f(x) = x, f(0) = 0, f'(0) = 2$
Show that:

$$\mathcal{L}\{g(x) + h(x)\} = \mathcal{L}\{g(x)\} + \mathcal{L}\{h(x)\}$$

Hence, show that:

$$\mathcal{L}\{f(x)\} = \frac{1}{s^2} + \frac{1}{s^2+1}$$

and find a possible solution function $f(x)$, and check your answer. [9]

12. [Maximum mark: 17]

a) Write out the power series of $\frac{1}{1+x}$ and of $\frac{1}{1+x^2}$. [4]

b) Hence, find the Maclaurin series for $\tan^{-1} x$. [4]

c) Write the first three non-zero terms in the Maclaurin series for $\tan^{-1} 2x$ and $e^{\tan x}$, and find: [9]

$$\lim_{x \to 0} \frac{\tan^{-1}(2x)}{e^{\tan x} - 1}$$