Complex numbers exercises

June 2025

Answer the questions on the back of the page.

Name and section: ______________________________

Instructor's name: ______________________________

Section A

1. (maximum mark: 9)
   (a) Write i in polar and Euler's form. (1 mark)
   (b) Find the value of i^i. (3 marks)
   (c) Hence, write i^{i^i} in polar form. (5 marks)

2. (maximum mark: 6)
   The complex number $z \in \mathbb{C}$ satisfies the equation $i(z+2) = 1 - 2z$, where $i = \sqrt{-1}$. Solve for z.

3. (maximum mark: 4)
   Let $z_1 = a(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ and $z_2 = b(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$.
   Express $(\dfrac{z_1}{z_2})^3$ in cartesian form.

4. (maximum mark: 5)
   $(z - 2i)$ is a factor of $2z^3 - 3z^2 + 8z - 12$. Find the other two factors.

5. (maximum mark: 5)
   Let $z \in \mathbb{C}$ and $|z + 16| = 4|z + 1|$, find $|z|$.

6. (maximum mark: 6)

   The numbers $z,\ w \in \mathbb{C}$ satisfy the equations:

$$\frac{w}{z} = 2i$$

$$z^* - 3w = 5 + 5i$$

   Find z and w in cartesian form.

7. (maximum mark: 8)

   Consider the complex geometric series $e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + ...$

   (a) Find an expression for z, the common ratio of this series. Show that $|z| = \dfrac{1}{2}$. (2 marks)

   (b) Write down an expression for the sum to infinity of this series. (1 mark)

   (c) Express your answer to part (b) in terms of $\sin\theta$ and $\cos\theta$, and show that: (5 marks)

$$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + ... = \frac{4\cos\theta - 2}{5 - 4\cos\theta}$$

8. (maximum mark: 7)

   Consider $w = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.

   (a) Show that w is a solution of the equation $z^5 - 1 = 0$. (1 mark)

   (b) Simplify $(w-1)(w^4 + w^3 + w^2 + w + 1)$, (2 marks)

   and hence show that $w^4 + w^3 + w^2 + w + 1 = 0$.

   (c) Hence or otherwise, show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. (4 marks)

9. (maximum mark: 5)

   Consider the complex numbers $z = 1 + 2i$ and $w = 2 + ai$ where $a \in \mathbb{R}$. Find a when

   (a) $|w| = 2|z|$;

   (b) $\Re(zw) = 2\Im(zw)$.

Section B

10. (maximum mark: 14)

   If z is a non-zero complex number, define $\mathcal{L}(z)$ by the equation

$$\mathcal{L}(z) = \ln|z| + i\arg z,\ \arg z \in [0, 2\pi]$$

   (a) Show that when z is a positive real number, $\mathcal{L}(z) = \ln z$. (2 marks)

(b) Use the equation to calculate: (12 marks)

1. $\mathcal{L}(-1)$
2. $\mathcal{L}(1-i)$
3. $\mathcal{L}(-1+i)$

Hence show that the property $\mathcal{L}(z_1 z_2) = \mathcal{L}(z_1) + \mathcal{L}(z_2)$ does not hold for all values of z_1 and z_2.

11. (maximum mark: 17)

It is given that $u = 1 + \sqrt{3}i$ and $v = 1 - i$.

(a) Express u and v in polar form, and find $u^3 v^4$. (5 marks)

The complex numbers u and v can be represented in the Argand diagram via the point A and the point B respectively.

(b) Draw the Argand diagram and place the points A and B. (3 marks)

The point A is rotated $\frac{\pi}{2}$ anticlockwise with respect to the origin O and the point B is rotated $\frac{\pi}{2}$ clockwise, converting to the points A' and B' respectively.

(c) Calculate the area of the triangle $OA'B'$. (5 marks)

Given that u and v are the roots of the equation $z^4 + bz^3 + c^2 + dz + e = 0$, where $b, c, d, e \in \mathbb{R}$,

(d) Find b, c, d and e. (4 marks)

12. (maximum mark: 24)

(a) Use the principle of mathematical induction to prove that: (7 marks)

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx, \ n \in \mathbb{Z}^+$$

Let $z = \cos\theta + i\sin\theta$, where $\theta \in (-\frac{1}{4}, \frac{1}{4})$.

(b) Find z^3 using the binomial theorem and show that: (6 marks)

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

and

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

(c) Hence, show that: (6 marks)

$$\tan\theta = \frac{\sin 3\theta - \sin\theta}{\cos 3\theta + \cos\theta}$$

(d) Given that $\sin\theta = \frac{1}{3}$, find the exact value of $\tan 3\theta$. (5 marks)