# Revision exam 1st year

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**Answer the questions on the back of the page.**
**You are not permitted access to any calculator for this paper.**

Name and surname: ______________________________

Candidate session number: ______________________________

## Section A

1. (maximum mark: 5)

   Find the value of $a$ given that the coefficient of $x^{11}$ in the expansion of $(x^2 + \frac{1}{ax})^{10}$ is 15.

2. (maximum mark: 5)

   It is given that the polynomial $P(x) = x^4 + px^3 - 2x^2 + qx - 3$ touches the $x$-axis at $x = -1$. Find $p$ and $q$ where $p,\ q \in \mathbb{R}$.

3. (maximum mark: 5)

   Consider the complex number $z = \pi^{i-1}$.

   (a) Write the number $\pi$ in the form of $e^a$ where $a \in \mathbb{R}$. (1 mark)

   (b) Hence, giving your answer in the form $p\cos(\ln q)$ where $p,\ q \in \mathbb{R}^+$, find (4 marks)

      1. $\Re(z)$;
      2. $\Re(z + \bar{z})$.

4. (maximum mark: 5)

   Solve for $x$:

$$x^{\ln x} = x \ln x$$

5. (maximum mark: 5)

   Find:

$$\int \sin(\ln x)dx$$

6. (maximum mark: 7)

   Consider the function $f(x) = \dfrac{9}{x^2+x-2}$.

   (a) Determine the domain, asymptotes, stationary points, and intervals where the function is increasing or decreasing. (4 marks)

   (b) Find the equation of the tangent to $y = f(x)$ at the point where $x = 4$. Represent the function and the tangent line on the same graph. (3 marks)

7. (maximum mark: 9)

   Using l'Hôpital's rule, determine the value of

   1. $\lim_{x\to 0} \dfrac{x \sin x}{1-\cos x}$;
   2. $\lim_{x\to\infty} \dfrac{x^n}{e^x}$, $n \in \mathbb{Z}^+$.

8. (maximum mark: 6)

   Find the total area of the regions enclosed by $y = x^3 - 6x + 3$ and $y = x^2 + 3$.

9. (maximum mark: 8)

   Consider the function $f(x) = \cos^{-1}(x)$ for $|x| \leq 1$.

   (a) Sketch the graph of $y = f(x)$ clearly indicating the $y$-intercept and coordinates of the endpoints. (2 marks)

   (b) Show that $\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$. (2 marks)

   (c) Hence or otherwise, solve $\cos^{-1}(x) + \cos^{-1}(x\sqrt{3}) = \dfrac{3\pi}{2}$ for $x \in [-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$. (4 marks)

## Section B

10. (maximum mark: 14)

    (a) Write down an expression for $\cos(\alpha+\beta)$ in terms of $\cos\alpha$, $\cos\beta$, $\sin\alpha$ and $\sin\beta$. (1 mark)

    (b) Hence, show that $\sin\alpha\sin\beta = \dfrac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$. (3 marks)

    (c) Hence, show that $\sin[(k+1)\theta]\sin\frac{\theta}{2} + \sin\frac{k\theta}{2}\sin\frac{(k+1)\theta}{2} = \sin\frac{(k+1)\theta}{2}\sin\frac{(k+2)\theta}{2}$. (4 marks)

(d) Use the principle of mathematical induction to prove that: (6 marks)

$$\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta = \frac{\sin[\frac{1}{2}(n+1)\theta]\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}$$

11. (maximum mark: 13)

(a) Find the derivative of $x^x$ and $x^{x^x}$. (6 marks)

(b) Consider $y = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$. Show that $\ln y = y\ln x$. (2 marks)

(c) Hence find $\frac{d}{dx}(x^{x^{x^{\cdot^{\cdot^{\cdot}}}}})$. (5 marks)

12. (maximum mark: 28)

Let $\mathcal{I} = \int_0^{\frac{\pi}{2}} \frac{\sin x\, dx}{(\sin x + \cos x)^2}$

(a) Show that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ by using an appropriate substitution. (4 marks)

(b) Hence, show that $\mathcal{I} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + \cos x}$. (3 marks)

(c) Let $z = \tan\frac{x}{2}$. Show that $\sin\frac{x}{2} = \frac{z}{\sqrt{1+z^2}}$ and $\cos\frac{x}{2} = \frac{1}{\sqrt{1+z^2}}$. (5 marks)

Hence, show that $\sin x = \frac{2z}{1+z^2}$ and $\cos x = \frac{1-z^2}{1+z^2}$.

(d) Show that $\frac{1}{u^2-a^2} = \frac{1}{2a}\left(\frac{-1}{u+a} + \frac{1}{u-a}\right)$. Hence, show that (4 marks)

$\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$ where $a \in \mathbb{R}$ and $C$ is the constant of integration.

(e) Using the results in (c) and (d) and $z$ as the substitution, show that $\mathcal{I}$ has the form $\frac{1}{p\sqrt{p}}\ln(p^{\frac{q}{p}} + q)$. (6 marks)

(f) Let $\mathcal{J} = \int_0^{\frac{\pi}{2}} \frac{\sin x\, dx}{(\sin x + \cos x)^3}$ and $\mathcal{K} = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sqrt{\tan x}}$. Evaluate $\mathcal{J}$ and $\mathcal{K}$ using previous parts. Hint: A useful substitution for $\mathcal{J}$ is $u = \tan x$ and write $\sin x$ and $\cos x$ in terms of $\sec x$. For $\mathcal{K}$ first show that it is equivalent to $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}\, dx}{\sqrt{\sin x} + \sqrt{\cos x}}$. (6 marks)

**END OF EXAMINATION PAPER**