Scaling Parameters in Rarefied Flow: Breakdown of the Navier-Stokes equations

Michael Macrossan, Centre for Hypersonics, University of Queensland
Mechanical Engineering, Departmental Report No. 2006/03

March 7, 2006

Abstract

This is an updated version of notes from a seminar delivered in 2004 [7]. Although the text has been revised this update is still in note form. References missing from the original have been added, as well as appendices giving the data taken from two undergraduate theses undertaken at the University of Queensland [6, 8].

In high altitude flight the average spacing between the molecules of the flow gas is not negligible compared to a typical dimension of the flow field. In this case, the gas does not behave like a continuum and its discrete particle nature must be considered. The continuum assumption 'breaks down' and the Navier-Stokes equations can, in theory, no longer be shown to be valid, particularly for high speed flight.

The fundamental equation describing the flow at the particle level is the Boltzmann equation, from which the Euler equations, the Navier-Stokes equations and more accurate Burnett equations may be derived under various assumptions. Various scaling parameters have been suggested to identify the regimes in which these different equations are valid, ranging from the Knudsen number, Tsein's (1946) parameter [9], Cheng's (1961) rarefaction parameter [4], Bird's (1970) breakdown parameter [1], and a form of the viscous interaction parameter derived from shock-boundary layer theory. All these depend primarily on the non-dimensional group $\zeta \equiv U_r \tau_r / L_r$, which must be small for the Navier-Stokes equations to remain valid. U is flow speed, L is a flow length, τ is a collision time, the subscript r denotes reference conditions.

We show how all these parameters may be derived from the Boltzmann equation and interpreted in terms of the ratio of mean time between molecular collision and a characteristic flow time, or as the ratio of typical shear stress to pressure in the flow.

Comparison of Navier-Stokes calculations and direct simulation solutions of the Boltzmann equation show that the Navier-Stokes equations may be adequate for almost rarefied flow.

Cheng's parameter appears to be the best correlation parameter and the best indicator of the validity of the Navier-Stokes equations for high-speed blunt body flow. For slender body flow the viscous interaction parameter based on a reference boundary layer temperature, a 'modified Tsein parameter' similar to Cheng's parameter, appears to be best.

1 Overview

- Importance of mean free path λ, the average distance a typical molecule travelled between collisions. When **Knudsen number** λ/D is large or when the **flow speed is high**, gas is not in 'kinetic equilibrium'.
- No single temperature characterizes the flow ($T_x \neq T_y \neq T_z \neq T_{rot} \neq T_{vib}$); the Navier-Stokes equations are not valid (in theory).
- Many different parameters have been used to characterize high speed and rarefied flow. Consider all the different correlation, or 'continuum breakdown' parameters to show their near equivalence.
- Show how all can be derived from the Boltzmann equation - the fundamental equation describing gas flow at the molecular level
- Quickly show how macroscopic fluid equations, Euler and Navier-Stokes are derived from the Boltzmann equation
- Comparing results of Navier-Stokes calculations with Direct Simulation Monte Carlo (DSMC) 'statistical solutions' of the Boltzmann equation
- Best is a 'viscous interaction parameter' $= C^* \ M_\infty^2/\mathrm{Re}_\infty$ or Cheng's (1961) parameter or modified Tsein (1946) parameter.
- Modification factor $C^* = \mu^* T_\infty/\mu_\infty T^*$ accounts for different $\sim$ collision rates $\sim$ molecular cross-sections $\sim$ viscosity laws for **high energy collisions**

Note that none of the data presented in this report is experimental data. It is all derived from computational fluid dynamics methods, principally the Direct Simulation Monte-Carlo (DSMC) method which is appropriate for rarefied flow.

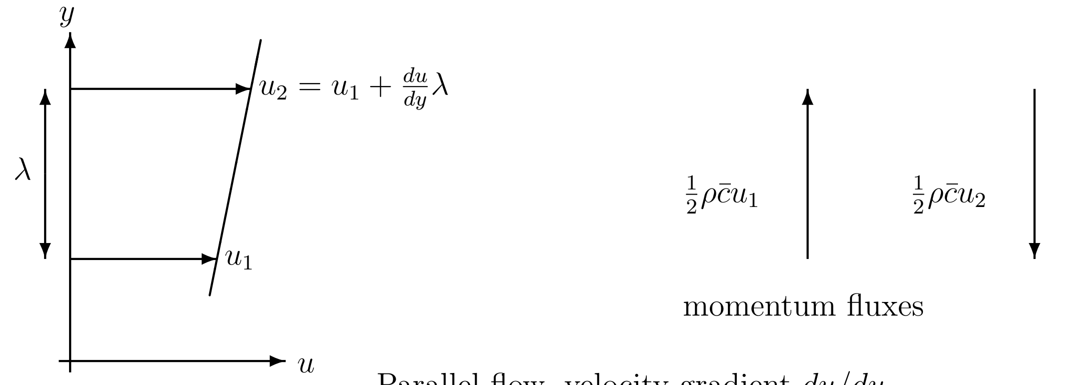
2 Molecular collision rate - viscosity

The usual parameters of hypersonic rarefied flow are the 'Speed ratio', $S = U/sqrt2RT$ and the 'Knudsen number' $Kn = \lambda/L$, where λ is the 'mean free path' (a measure of gas density, temperature and molecular size). These parameters can be related to the Mach number and Reynolds number because of the relationship between the mean free path and the gas viscosity. Fig. 1 shows a gradient of mean velocity in the y-direction in a laminar flow.

- Molecules move ($\uparrow$ or $\downarrow$) with **thermal** speed $\bar{c} \approx \sqrt{2RT}$
- Molecules transport their momentum over a distance $\lambda =$ **mean free path**
- Collision time $t_c = \lambda/\bar{c}$, collision rate $\nu = \bar{c}/\lambda$, $\bar{c}$ is the mean thermal (random) speed of molecules.
- Shear stress $\tau_{xy} =$ net x-momentum flux
- $\tau_{xy} \approx \frac{1}{2}\rho\bar{c}\,(u_2 - u_1) = \left[\frac{1}{2}\rho\bar{c}\lambda\right] \times \frac{du}{dy}$

Viscosity is related to the mean free path, the mean free time (both of which depend on the molecular size) and can be expressed in a number of ways:

$$\mu \approx \frac{1}{2}\rho\bar{c}\lambda = \frac{1}{2}\rho\bar{c}^2 t_c \approx \rho R T t_c = p t_c$$



Parallel flow, velocity gradient du/dy

Figure 1: The net transport of parallel momentum down a gradient of mean flow velocity in a laminar is related to the mean distance travelled by molecules between collisions.

It is important to note that the mean free path λ is defined in rest frame of gas; it may be expressed as $\lambda = \frac{2\mu}{\rho\bar{c}} \sim \frac{\mu}{\rho\sqrt{T}}$; it is a 'state property'. The Knudsen number $\mathrm{Kn} = \lambda/L$, where $L =$ body dimension is a measure of rarefaction; when the Knudsen number is large the gas cannot be expected to behave as a continuum. Although Kn may be expressed in terms of the Mach number and Reynolds number

$$\mathrm{Kn} = \frac{\lambda}{L} \sim \frac{\mu}{\rho\bar{c}L} \sim \frac{U/\bar{c}}{\rho UL/\mu} \sim \frac{\mathrm{M}}{\mathrm{Re}}$$

it is not a flow property (the flow speed in the Mach number and Reynolds number cancel in the expression for Knudsen number).

3 Sphere Drag - DSMC calculations

Fig. 2 shows DSMC data for the drag coefficient on spheres (as calculated by DSMC by various authors [5, 6, 8]) for a range of Knudsen numbers. The 'cold-wall' condition ($T_w \approx T_\infty$) was applied in all calculations. The 'free-molecular' drag coefficient (i.e. for $\mathrm{Kn} \to \infty$ all freestream molecules hit the surface, with their velocity unchanged by collisions with other molecules) is a strong function of Mach number. The data can be 'collapsed' (or correlated) by expressing the drag coefficient as a fraction of the free-molecular value C_{Df} which varies from 2.2 - 2.5 for this data. The second part of Fig. 2 shows the data expressed as C_D/C_{Df} as a function of Knudsen number. Although the scatter is less reduced it is clear that the freestream Knudsen number is not a good correlation parameter for this data; in other words the data does not 'collapse' into a single-valued (or close to single-valued) empirical function $C_d/C_{Df} = f\,(Kn_\infty)$.

Various parameters have been suggested as measures of when rarefaction effects make the Navier-Stokes equations are invalid. We now review these parameters and investigate whether these parameters can provide a better correlation parameter than the Knudsen number.

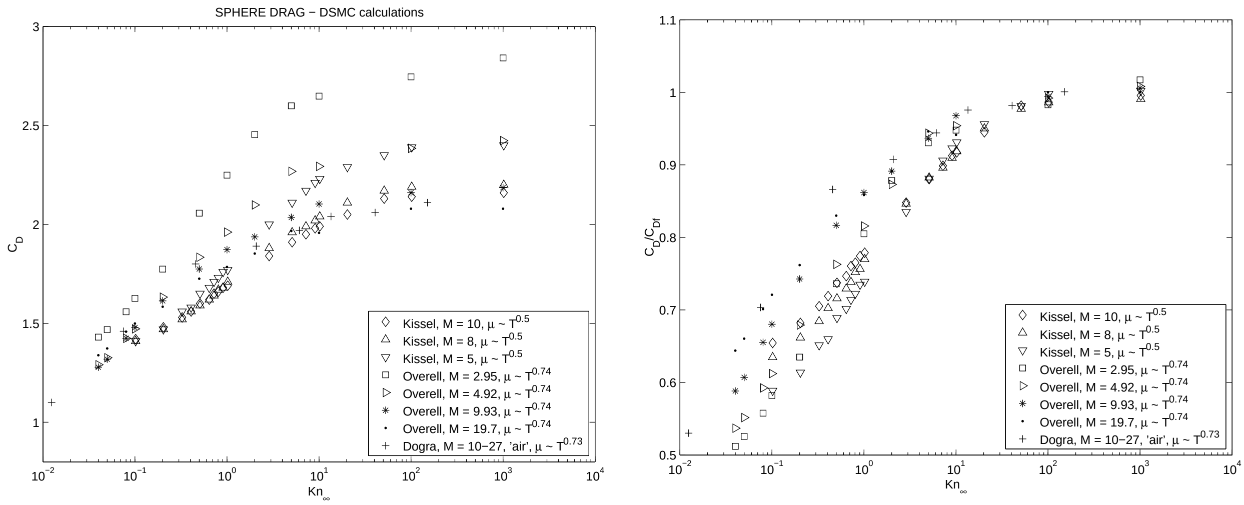
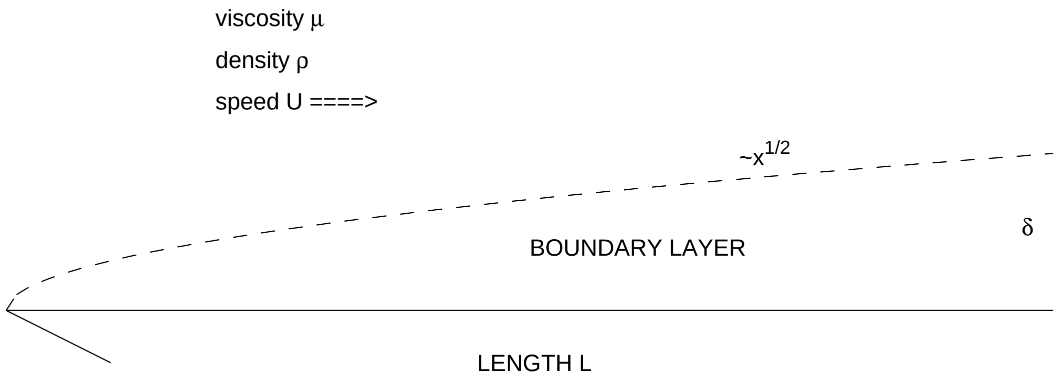


Figure 2: Drag coefficient for spheres in hypersonic flow as calculated by DSMC, for different gas models. In all cases the sphere wall temperature is equal to the free-stream static temperature ($T_w \approx T_\infty$). The data from Dogra *et al.* [5] is for a multi-species air model. The data of Kissel [6] and Overell [8] is for a pure gas with ratio of specific heats $\gamma = 5/3$. Overell's calculations of drag on a sphere took account of windward pressures and stresses only.

3.1 Tsein's parameter

Tsein (1946) [9] proposed a Knudsen number based not on a typical body dimension L but the thickness δ of the boundary layer on the body. Thus, $\mathrm{Kn}_\delta = \lambda/\delta$. Flat plate length L. Boundary layer thickness δ.

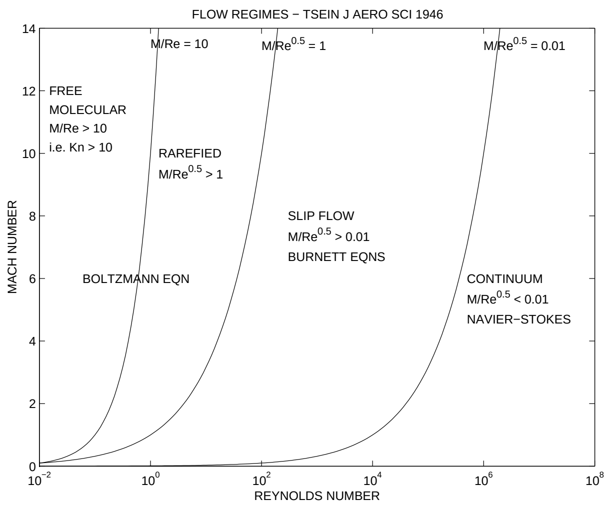


$$\delta \propto \frac{L}{\sqrt{\mathrm{Re}_L}}, \quad \mathrm{Re}_L = \frac{\rho U L}{\mu}, \quad Re_\delta = \frac{\rho U \delta}{\mu} \propto \sqrt{\mathrm{Re}_L}$$

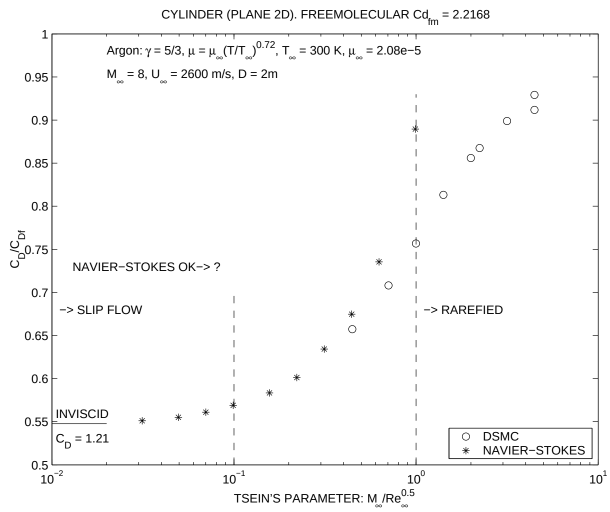
For flat plate flow, the Knudsen number based on δ can be expressed in terms of the freestream Mach and Reynolds numbers: $\mathrm{Kn}_\delta = \lambda/\delta = \dfrac{\mathrm{M}}{\mathrm{Re}_\delta} \sim \dfrac{\mathrm{M}}{\sqrt{\mathrm{Re}_L}}$

We may refer to the non-dimensional group $\mathrm{M}_\infty/\mathrm{Re}_\infty^{1/2}$ as Tsein's parameter. Tsein classified flows according tot he following scheme

- Continuum (Navier-Stokes) $\mathrm{M}_\infty/\mathrm{Re}_\infty^{1/2} < 0.01$



(a) Flow regimes



(b) Drag on a cylinder in cross-flow

Figure 3: (a) Tsein's parameter (1946) used to define flow regimes. The region labelled slip flow is now usually tackled by DSMC, rather than the Burnett equations. (b) Plane 2D flow round a circular cylinder as calculated by DSMC and the Navier-Stokes equations. Rarefaction increases as Tsien's parameter $M_\infty/\sqrt{\mathrm{Re}_\infty}$ increases. The Navier-Stokes calculations appear to be valid for $\mathrm{M}_\infty/\mathrm{Re}_\infty^{1/2} < 0.1$.

- Slip flow (Burnett equations) $0.01 < \mathrm{M}_\infty/\mathrm{Re}_\infty^{1/2} < 1$
- Rarefied flow $\mathrm{M}_\infty/\mathrm{Re}_\infty^{1/2} > 1$ and $M_\infty/Re_\infty < 10$
- Free-molecular flow $\mathrm{M}_\infty/\mathrm{Re}_\infty > 10$.

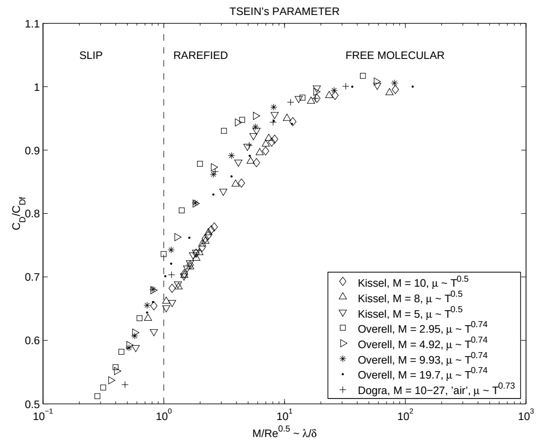
Note that the condition for free-molecular flow comes from Tsein's parameter for Re << 1, for which $\delta \approx L$, $\mathrm{Kn}_\delta \approx \mathrm{Kn}_L = \mathrm{M}/\mathrm{Re}_L \equiv Kn_\infty$. In other words, Tsein's parameter reduces to the ordinary Knudsen number for free-molecular (large Knudsen number) flow.

Fig. 3 (a) is an adaptation of a figure from Tsien's paper showing these various regimes. Of particular interest is the limit $M/\sqrt{\mathrm{Re}} < 0.01$ for the validity of the Navier-Stokes equations which was set by Tsien by considering when the Burnett equations became significantly different from the Navier-Stokes equations.

Fig. 3 (b) shows some Navier-Stokes and DSMC results for drag coefficient on a cylinder in cross-flow (the plane 2D equivalent of sphere drag). The Navier-Stokes results approach the inviscid limit for low values of Tsein's parameter. The results suggest that for this flow the Navier-Stokes equations are valid for $\mathrm{M}_\infty/\mathrm{Re}_\infty^{1/2} < 0.1$ which is well into the the regimes which Tsein characterized as 'slip flow'. Figure 4 shows Tsein's parameter is a better correlation parameter than the Knudsen number for the hypersonic sphere drag data.

3.2 'Breakdown Parameter'

Bird's 'breakdown parameter' has been used to identify flow regions where the Navier-Stokes equations are invalid ('break down'). If is a form of Knudsen



Tsein's parameter used as a correlation parameter for the sphere drag data shown in Fig. 2. This parameter correlates the data in the in Tsein's 'slip regime' $M_\infty/\sqrt{Re_\infty} < 1$ better than the Knudsen. However, Tsein's parameter is not a noticeably better correlation parameter than the Knudsen number in the rarefied regime ($M_\infty/Re_\infty^{1/2} > 1$)

Figure 4: Tsein's rarefaction parameter $M_\infty/\sqrt{Re_\infty}$ used as a correlation parameter.

number where the characteristic length is derived from the gradients of the flow structures. Local non-equilibrium (i.e. multi-temperatures) when breakdown parameter

$$B \sim M\frac{\lambda}{\rho/|\nabla\rho|} > 0.02. \quad \text{For } \rho/|\nabla\rho| \sim D, \quad B \sim \mathrm{MKn} \sim \frac{\mathrm{M}^2}{\mathrm{Re}} \equiv \mathrm{Kn}_\delta^2$$

Take conservative estimate $B = 0.01$ = 'Breakdown of equilibrium' when Tsein's parameter $\mathrm{Kn}_\delta^2 > 0.01$, $\mathrm{Kn}_\delta > 0.10$, i.e. **Navier-Stokes may be valid well into slip regime**

3.3 Viscous Interaction Parameter

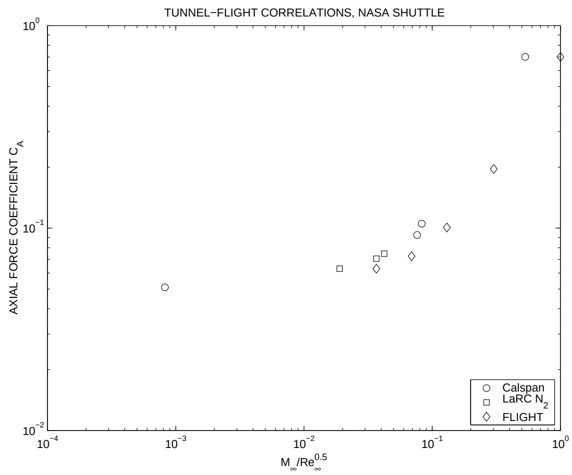


Figure 5: Data from Wilhite *et al.* AIAA Paper 84-0406

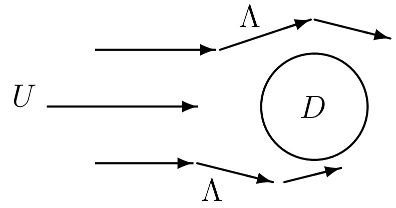
$\bar{V}^2 = C^*\mathrm{M}_\infty^2/\mathrm{Re}$. $C^* = \mu^* T_\infty/\mu_\infty T^*$. T^* = characteristic T of boundary layer. For $C^* \approx 1$, $\bar{V}^2 = \mathrm{Kn}_\delta^2$
i.e. Tsein's parameter. Wind tunnel/Flight Data. Consider C^* later.

3.4 Collision Number

By analogy with the Damköhler number of chemically reaction flow (the ratio of flow time to chemical reaction time), and by analogy with Bird's breakdown parameter, we can expect the flow to depend on the average number of collisions per molecule as the flow passes the body.

Rarefaction parameter
(inverse 'collision number')

$$\begin{aligned} P &= \frac{\text{coll. time}}{\text{flow time}} = \frac{\lambda/\bar{c}}{D/U} = \frac{\lambda}{D}\frac{U}{\bar{c}} \approx \mathrm{Kn} \times \mathrm{M} \\ &= \frac{\mathrm{M}^2}{\mathrm{Re}} = \mathrm{Kn}_\delta^2 \end{aligned}$$



Typical path of molecules
$\Lambda \approx$ mean free path
(in body fame of reference)
$P = t_c/(D/U) = Ut_c/D = \Lambda/D$

4 Boltzmann Equation

Distribution function f: $fd\mathbf{v}$ = fraction of molecule velocities $\mathbf{v} \rightarrow \mathbf{v} + d\mathbf{v}$

$$\text{Evoultion of } f(\mathbf{v}, \mathbf{r}, t): \quad \tfrac{\partial nf}{\partial t} + \mathbf{v} \cdot \tfrac{\partial nf}{\partial \mathbf{r}} = \Delta\left[nf\right]_{coll}$$

$\mathbf{r}$ = position, t = time, $n(\mathbf{r}, t)$ = number density (molecules/m^3)
Bhatnagar-Gross-Krook (BGK) approximation to collision term

$$\Delta\left[nf\right]_{coll} = \frac{n}{\theta_c}\left(f_e - f\right)$$

f_e = Maxwell distribution
θ_c = local characteristic time = p/μ ($\sim$ collision time t_c).

4.1 Boltzmann Eq $\rightarrow$ Macroscopic fluid equations

Multiply B.E. by $Q = [m, m\mathbf{v}, m\mathbf{v}^2/2]$, integrate over all velocities, integrate over finite volume V, with surface S

$$\begin{aligned} \int\!\!\int Q\frac{\partial nf}{\partial t}d\mathbf{v}dV + \int\!\!\int Q\mathbf{v}\cdot\frac{\partial nf}{\partial \mathbf{r}}d\mathbf{v}dV &= \int\!\!\int Q\Delta\left[nf\right]_{coll}d\mathbf{v}dV \\ \frac{\partial}{\partial t}\int\left[\int nfQd\mathbf{v}\right]dV + \int\left[\int nfQc_n d\mathbf{v}\right]dS &= 0 \text{ (for } \mathbf{any} f) \end{aligned}$$

Conserved quantities: $[\int nfQd\mathbf{v}] = [\rho, \rho\mathbf{u}, \rho\left(\mathbf{u^2/2} + \mathbf{C_v T}\right)]$ for **any** f
Fluxes: $[\int nfQc_n d\mathbf{v}]$, depends on form of f (on surface S)

1. $f = f_e$ **Euler** Fluxes
2. $f = f_e\left(1 + \Phi_{CE}\right)$ **Navier-Stokes** fluxes
3. $f = f_e\left(1 + \Phi_{CE} + \Phi_B\right)$ Burnett Fluxes

5 Non-dimensional Boltzmann Equation

Select L_{r}, U_{r} = characteristic length, speed, (n_{r}, T_r) = reference state
$\tau_{\mathrm{r}} = \mu_{\mathrm{r}} / (n_{\mathrm{r}} k T_{\mathrm{r}})$, where $\mu_{\mathrm{r}} = \mu(T_{\mathrm{r}})$ = reference viscosity
Non-dimensional values: $\hat{\mathbf{r}} = \mathbf{r}/L_{\mathrm{r}}$, $\hat{\mathbf{v}} = \mathbf{v}/U_{\mathrm{r}}$, $\hat{n} = n/n_{\mathrm{r}}$, $\hat{t} = tU_{\mathrm{r}}/L_{\mathrm{r}}$, $\hat{f} = fU_{\mathrm{r}}^3$

$$\begin{aligned}
\frac{\partial \hat{n}\hat{f}}{\partial \hat{t}} + \hat{\mathbf{v}} \cdot \frac{\partial \hat{n}\hat{f}}{\partial \hat{\mathbf{r}}} &= \left(\frac{L_{\mathrm{r}}}{U_{\mathrm{r}}\theta_{\mathrm{r}}}\right) \left(\frac{\theta_{\mathrm{r}}}{\theta_c} \frac{n}{n_{\mathrm{r}}}\right) \left(\hat{f}_e - \hat{f}\right) \\
&= \left(\frac{L_{\mathrm{r}}}{U_{\mathrm{r}}\theta_{\mathrm{r}}}\right) \left(\frac{\mu_{\mathrm{r}}}{\mu} \frac{T}{T_{\mathrm{r}}}\right) \hat{n}^2 \left(\hat{f}_e - \hat{f}\right) \\
&= (\zeta\, C)^{-1} \hat{n}^2 \left(\hat{f}_e - \hat{f}\right) \\
\text{where } \zeta &= \frac{U_{\mathrm{r}}\theta_{\mathrm{r}}}{L_{\mathrm{r}}} \text{ and } C = \frac{\mu_{\mathrm{r}} T}{\mu T_{\mathrm{r}}}
\end{aligned}$$

For the following two conditions (which will be examined in more detail later):

1. $\mu/\mu_{\mathrm{r}} \sim T/T_{\mathrm{r}}$
2. non-dimensional boundary conditions are the same between two flows

Then both flows are governed by the same non-dimensional Boltzmann equation

$$\frac{\partial \hat{n}\hat{f}}{\partial \hat{t}} + \hat{\mathbf{v}} \cdot \frac{\partial \hat{n}\hat{f}}{\partial \hat{\mathbf{r}}} = \zeta^{-1} \hat{n}^2 \left(\hat{f}_e - \hat{f}\right)$$

The parameter $\zeta = \theta_{\mathrm{r}} / (L_{\mathrm{r}}/U_{\mathrm{r}})$ is a non-dimensional collision time (the same as the parameter P, the inverse 'collision number'). The limiting cases of collision rate are

- $\zeta \to \infty$: Collision rate zero $\to$ maximum non-equilibrium
- $\zeta \to 0$: Infinite collision rate $\to$ equilibrium

Thus

- ζ is a rarefaction parameter
- ζ could be a **correlation** parameter

5.1 Collisionless flow

Non-dimensional collision time $\zeta \to \infty$

$$\frac{\partial \hat{n}\hat{f}}{\partial \hat{t}} + \hat{\mathbf{v}} \cdot \frac{\partial \hat{n}\hat{f}}{\partial \hat{\mathbf{r}}} = 0$$

COLLISIONLESS BOLTZMANN EQUATION

Typical solution is highly non-equilibrium distribution function.

5.2 Limiting case $\zeta \to 0$

Non-dimensional collision time $\zeta \to 0$. Infinite collision rate

$$\hat{f}_e - \hat{f} \quad = \quad \frac{\zeta}{\hat{n}^2}\left[\frac{\partial \hat{f}}{\partial \hat{t}} + \hat{\mathbf{v}} \cdot \frac{\partial \hat{f}}{\partial \hat{\mathbf{r}}}\right] \to 0$$

Result[1] $f \to f_e$ was 'built-in' to BGK approximation, but
same result follows from full collision term of Boltzmann equation.

ζ is a **rarefaction** parameter: continuum: $\zeta = 0$, free molecular $\zeta = \infty$

5.3 Rarefaction parameter $\zeta \sim P \sim$ Tsein's parameter

Reference quantities from the freestream

- $L_{\mathrm{r}} = D$ (body dimension), $U_{\mathrm{r}} = U_\infty$, $\tau_{\mathrm{r}} = \mu_\infty / p_\infty \approx \lambda_\infty / \bar{c}_\infty$

$$\zeta_\infty = \frac{U_{\mathrm{r}} \tau_{\mathrm{r}}}{L_{\mathrm{r}}} = \frac{U_\infty}{\bar{c}_\infty}\frac{\lambda_\infty}{D} \sim \mathrm{M}_\infty \mathrm{Kn}_\infty \sim \frac{\mathrm{M}_\infty^2}{\mathrm{Re}_\infty} \sim P_\infty \sim \mathrm{Kn}_\delta^2$$

$$\textbf{Note: } \zeta_\infty = \frac{\mu_\infty}{p_\infty}\frac{U}{D} = \frac{\mu_\infty U/D}{p_\infty} \sim \frac{\text{shear stress}}{\text{pressure}}$$

5.4 Summary: Rarefaction Parameter ζ_∞

- ζ ~collision time/flow time – inverse collision number (# collisions past body)
- $\sqrt{\zeta} \sim$ mean free path/boundary layer thickness $\mathrm{Kn}_\delta = \lambda/\delta$ **Tsein**'s parameter
- $\zeta \sim$ shear stress/pressure
- Regimes (for hypersonic flow)

Continuum	0	$<$	ζ	$<$	0.0001	Navier-Stokes
Slip	0.0	$<$	ζ	$<$	0.02	Navier-Stokes?
Slip	0.02	$<$	ζ	$<$	1	Burnett Equations, DSMC
Rarefied	1	$<$	ζ	$<$	5-20	Boltzmann Eq., DSMC
Free molecular	20	$<$	ζ	$<$	∞	Collisionless Boltzmann

6 Non-dimensional boundary condition

Assume diffuse reflection at surface

- Reflected with thermal speed at wall temperature $c_{\mathrm{W}} \sim (2RT_{\mathrm{W}})^{\frac{1}{2}}$
  $\hat{c}_{\mathrm{W}} = c_{\mathrm{W}}/U_{\mathrm{ref}} = S_{\mathrm{W}}^{-1}$, where $S_{\mathrm{W}} = S_\infty \left(\frac{T_\infty}{T_{\mathrm{W}}}\right)^{\frac{1}{2}}$, $S_\infty = U_\infty / \sqrt{2RT_\infty}$

[1] If $\hat{n} \to 0$ there may be local non-equilibrium regions embedded in continuum solution

- Reflected particle travels (average) distance λ_{W} before collision [10, 11, 3] $\hat{\lambda}_{\mathrm{W}} = \lambda_{\mathrm{W}}/D == f\left(Kn_\infty, S_\infty, T_w/T_\infty\right)$ and the viscosity law $\mu = \mu\left(T\right)$.

Similarity depends on ζ (S_∞ or M_∞, and Re_∞) and T_{W}/T_∞, and viscosity law.

6.1 $C \equiv \left(\mu/\mu_{\mathbf{r}}\right)\left(T_{\mathbf{r}}/T\right)$ term

With reference state = freestream. Non-dimensional Boltzmann equation

$$\frac{\partial \hat{n}\hat{f}}{\partial \hat{t}} + \hat{\mathbf{v}} \cdot \frac{\partial \hat{n}\hat{f}}{\partial \hat{\mathbf{r}}} = \left(\zeta_\infty C\right)^{-1}\left(\hat{n}\hat{f}_e - \hat{n}\hat{f}\right)$$

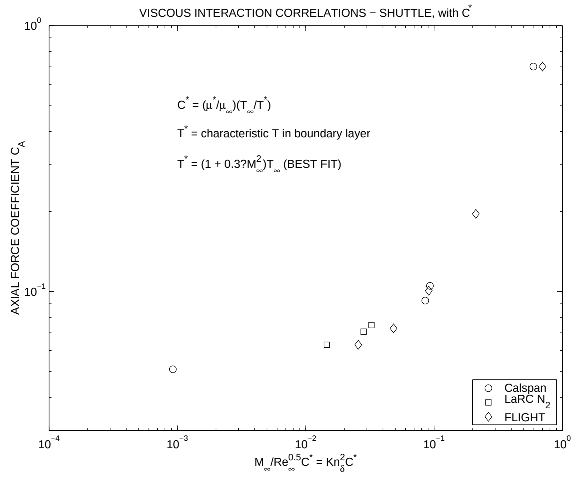
$C = \left(\mu/\mu_\infty\right)\left(T_\infty/T\right)$ varies throughout flow

if C is same for two flows at some characteristic region of flow, **then**
C (flow 1) $\approx C$ (flow 2) for all regions of flow (we hope!)

e.g. Viscous interaction parameter $\bar{V}^2 = C^* \mathrm{M}_\infty^2/\mathrm{Re}_\infty \sim C^*\zeta \sim C^*\mathrm{Kn}_\delta^2$
$C^* = \left(\mu^*/\mu_\infty\right)\left(T_\infty/T^*\right)$, $T^*/T_\infty = 1 + 0.3?M_\infty^2$
T^* = characteristic temperature in boundary layer

If $\bar{V}^2 \sim C^*\mathrm{Kn}_\delta^2$ same for two flows, non-dimensional flows $\approx$ similar[2].

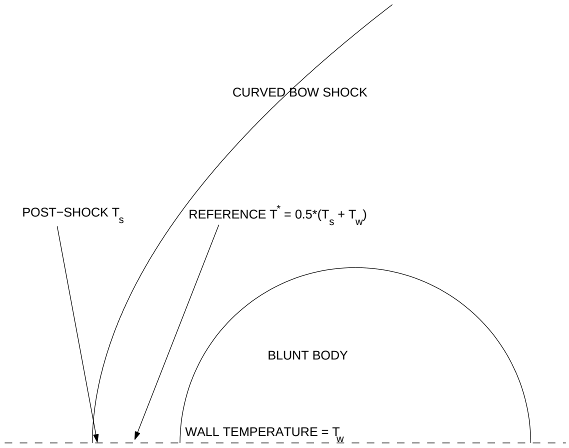
Wilhite *et al.* [12] tried various estimates of the characteristic temperature in the boundary layer on the underside of the NASA shuttle during reentry, and used the viscous interaction parameter to correlate wind twin and flight test data.



6.2 Cheng's (1961) parameter $K_{\mathbf{C}}^2$: Blunt body

Cheng's parameter appears to be a modification of the viscous interaction parameter for blunt body flow. Thick shock - thick boundary layer: T^* in the merged shock/boundary layer, between shock and body.

[2] The interaction between the growth of the boundary layer on a flat plate, and the shock formed as the freestream is turned by this boundary layer



$K_{\mathrm{C}}^2 \equiv$ inverse viscous interaction parameter

$$1/K_{\mathrm{C}}^2 = C^* \frac{U_\infty \mu_\infty}{p_\infty D} \sim C^* \frac{\mathrm{M}_\infty^2}{\mathrm{Re}}$$

Modified Tsein

$$K_{\mathrm{C}}^2 \sim \left(C^* \mathrm{Kn}_\delta^2\right)^{-1} = (C^* \zeta_\infty)^{-1}$$

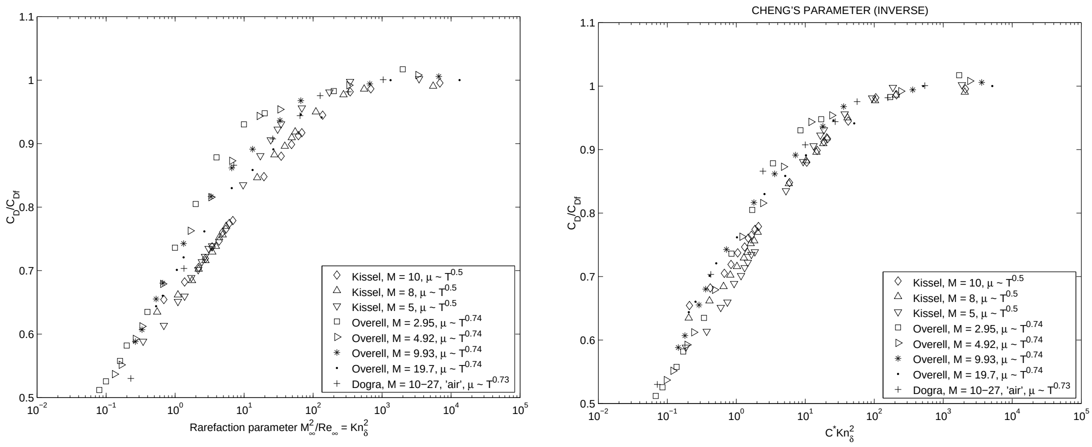


Figure 6: Cheng's parameter account for the different collision cross-section in the characteristic region → a modified (improved) correlation parameter.

7 Conclusions

- Navier-Stokes equations can 'work' in slip regime (more evidence needed)
- Knudsen number is a state parameter, not a flow parameter, not good correlation parameter in high speed flow.
- Many parameters are nearly equivalent

$$\zeta \sim \frac{\text{coll. time}}{\text{flow time}} \sim \frac{\text{Shear}}{\text{Pressure}} \sim \frac{\mathrm{M}^2}{\mathrm{Re}} \sim \mathrm{M\ Kn} \sim \bar{V}^2 \sim \mathrm{Kn}_\delta^{-2}$$

- Best[3] is $C^*\ \mathrm{M}_\infty^2/\mathrm{Re}_\infty$ where $C^* = \mu^* T_\infty / \mu_\infty T^*$:
  1. For high energy collisions C^* accounts for different collision rates $\equiv$ different collision cross-sections $\equiv$ different viscosity laws.

[3] Equivalent to Cheng's parameter, or modified Tsein parameter.

2. For a given configuration, must select (guess) the appropriate characteristic T^*. The boundary layer temperature (flat plate) or post-shock merged layer temperature (blunt bodies).

References

[1] G. A. Bird. Breakdown of translational and rotational equilibrium in gaseous expansions. *A.I.A.A. Journal*, 8(11):1998–2003, 1970.

[2] G. A. Bird. *Molecular Gas Dynamics and the Direct Simulation of Gas Flows*. Clarendon, Oxford, 1994.

[3] G. P. Cathcart and M. N. Macrossan. Aerodynamic drag reduction for satellites in low earth orbits. *A.I.A.A. Journal*, 31:826–831, 1993.

[4] H. K. Cheng. Hypersonic shock-layer theory of the stagnation region at low Reynolds number. Report IRN 13539152, Cornell Aeronautical Laboratory, Proc 1961 heat Transfer and Fluid Mechanics Institute, 1961.

[5] V. K. Dogra, R. G. Wilmoth, and J. N. Moss. Aerodynamics of a 1.6 meter diameter sphere in hypersonic rarefied flow. *A.I.A.A. Journal*, 30(7):1789–1794, July 1992.

[6] H. Kissel. CFD investigation of rarefied flow conditions in the etst section of a low density, hypervelocity expansion-tube dump tank. Technical Report No. 2003/08, Department of Mechanical Engineering, Unversity of Queensland, St. Lucia 4072, Australia, http://mech.uq.edu.au/˜michael/ugtheses/kissel_harald_2003.pdf, 2003.

[7] M. N. Macrossan. Scaling parameters in rarefied flow and the breakdown of the Navier-Stokes equations. Technical Report No: 2004/09, Department of Mechanical Engineering, University of Queensland, St Lucia 4072, Australia, 2004.

[8] P. Overell. Numerical simulation of rarefied flow over a hemispherical body using direct simulation monte carlo method and investigation of correlation parameters to define the flow. Bachelor of Engineering thesis, Scool of Engineering, Univesity of Queensland, http://mech.uq.edu.au/˜michael/ugtheses/overell_peter_2003.pdf, 2003.

[9] H. S. Tsein. Superaerodynamics, mechanics of rarefied gases. *J. Aero. Sci.*, 13:342, 1946.

[10] D. L. Whitfield. *Drag on bodies in rarefied high-speed flow*. PhD thesis, The University of Tennessee, U.S.A., December 1971.

[11] D. L. Whitfield. Mean free path of emitted molecules and correlation of sphere drag data. *A.I.A.A. Journal*, 11(12):1666–1670, 1973.

[12] A. W. Wilhite, J. P. Arrington, and R. S. McCandless. Performance aerodynamics of aero-assisted orbital vehicles. A.I.A.A. Paper 84-0406, 1984.

A Limiting drag coefficients

The drag coefficient for a sphere with a diffusely reflection surface at temperature T_w is given by Bird [2] (Eq. 7.71). It is

$$C_D = \frac{2S_\infty^2 + 1}{\pi^{1/2} S_\infty^3} \exp\left(-S_\infty^2\right) + \frac{4S_\infty^4 + 4S_\infty^2 - 1}{2S_\infty^4} \mathrm{erf}\left(\right) + \frac{2\pi^{1/2}}{3S_\infty}\left(\frac{T_w}{T_\infty}\right)^{1/2}$$

The drag coefficient for a sphere in the continuum limit, according to the modified Newtonian approach[4] is

$$C_d = \left[\frac{1}{2}\left(\gamma+1\right) M_\infty^2\right]^{\frac{\gamma}{\gamma-1}} \left[\frac{2\gamma M_\infty^2 - \left(\gamma-1\right)}{\gamma+1}\right]^{\frac{-1}{\gamma-1}} \left[\gamma M_\infty^2\right]^{-1}$$

- Experimental (Air) $C_{Dc} \approx 0.92 + 0.02 \exp\left[-\left(M-2\right)/1.75\right]$

B Kissel's data

Drag coefficient. DSMC. Kissel [6]

$T_w = T_\infty$, Diffuse surface reflection. H. Kissel [6]

$\gamma = 7/5$				$\gamma = 5/3$			
Kn	M 5	M 8	M 10	Kn	M 5	M 8	M 10
0.01	1.24	1.27	1.25	0.01	1.2	1.22	1.24
0.03	1.35	1.37	1.40	0.03	1.32	1.33	1.35
0.05	1.37	1.39	1.41	0.05	1.34	1.36	1.35
0.10	1.41	1.41	1.42	0.10	1.39	1.38	1.4
0.20	1.47	1.47	1.48	0.20	1.47	1.46	1.47
0.32	1.56	1.52	1.53	0.32	1.55	1.50	1.53
0.40	1.58	1.56	1.56	0.40	1.51	1.55	1.55
0.50	1.65	1.59	1.6	0.50	1.63	1.59	1.59
0.63	1.68	1.62	1.62	0.63	1.67	1.64	1.63
0.71	1.71	1.64	1.65	0.71	1.69	1.65	1.65
0.79	1.73	1.67	1.66	0.79	1.73	1.67	1.66
0.89	1.76	1.68	1.68	0.89	1.75	1.69	1.67
1.00	1.77	1.71	1.69	1.00	1.78	1.71	1.69
2.82	2.00	1.88	1.84	2.82	1.98	1.86	1.84
5.01	2.11	1.96	1.91	5.01	2.07	1.93	1.91
7.08	2.17	1.99	1.95	7.08	2.14	1.99	1.94
8.91	2.21	2.02	1.98	8.91	2.16	2.01	1.96
10.0	2.23	2.04	1.99	10.0	2.18	2.02	1.97
50.1	2.35	2.17	2.13	20.0	2.25	2.10	2.03
100	2.39	2.19	2.14	50.1	2.32	2.15	2.12
1000	2.4	2.2	2.16	100	2.32	2.15	2.14
1000	2.35	2.2	2.15	1000	2.35	2.20	2.15

[4] That is the Newtonian pressure coefficient $2\sin^2\theta$ is reduced by a constant ratio for all surface angles θ, so that the 'modified Newtonian' stagnation point pressure matches the Rayleigh-Pitot stagnation pressure.

C Overell's data

Drag coefficient. DSMC. $\gamma = 7/5$

$T_w = T_\infty$, Diffuse surface reflection. P. Overell [8]

Kn	M 2.95	M 4.92	M 9.83	M 19.66
0.0010	1.365	1.229	1.171	1.163
0.0019	1.311	1.260	1.190	1.165
0.0048	1.338	1.257	1.198	1.176
0.0097	1.398	1.292	1.223	1.215
0.0145	1.401	1.308	1.252	1.241
0.0193	1.440	1.340	1.283	1.1275
0.0290	1.470	1.380	1.335	1.400
0.0387	1.515	1.411	1.376	1.443
0.0483	1.555	1.449	1.421	1.480
0.0773	1.650	1.557	1.534	1.571
0.0966	1.722	1.609	1.592	1.616
0.1933	1.879	1.785	1.738	1.708
0.4832	2.178	2.004	1.911	1.860
0.9663	2.381	2.143	2.017	1.924
1.9327	2.599	2.294	2.086	1.997
4.8316	2.752	2.479	2.192	2.120
9.6633	2.804	2.507	2.265	2.110
96.633	2.907	2.606	2.327	2.241
966.33	3.009	2.649	2.354	2.242