Notatki do ćwiczeń

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1 Brakety

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \end{pmatrix} = \sum_i \psi_i |i\rangle, \quad \text{baza } \{|i\rangle\}\text{: } \textbf{ortonormalna: } \langle i|j\rangle = \delta_{ij}, \textbf{zupełna: } \sum_i |i\rangle\langle i| = \mathbb{1}; \tag{1}$$

$$\langle\psi| = \begin{pmatrix} \psi_1^* & \psi_2^* & \psi_3^* & \dots \end{pmatrix} = \sum_i \psi_i^* \langle i|; \tag{2}$$

$$\langle\psi|\phi\rangle = \begin{pmatrix} \psi_1^* & \psi_2^* & \psi_3^* & \dots \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \end{pmatrix} = \left(\sum_i \psi_i^* \langle i|\right) \left(\sum_j \phi_j |j\rangle\right) = \sum_{ij} \psi_i^* \phi_i \delta_{ij} = \sum_i \psi_i^* \phi_i \tag{3}$$

$$|\psi\rangle\langle\phi| = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \end{pmatrix} \begin{pmatrix} \phi_1^* & \phi_2^* & \phi_3^* & \dots \end{pmatrix} = \begin{pmatrix} \psi_1\phi_1^* & \psi_1\phi_2^* & \psi_1\phi_3^* & \dots \\ \psi_2\phi_1^* & \psi_2\phi_2^* & \psi_2\phi_3^* & \dots \\ \psi_3\phi_1^* & \psi_3\phi_2^* & \psi_3\phi_3^* & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4}$$

2 Operatory

Element macierzowy operatora:

$$A_{nm} = \langle n|\hat{A}|m\rangle \tag{5}$$

Operator w bazie $\{|n\rangle\}$:

$$\hat{A} = \sum_{nm} \langle n|A|m\rangle |n\rangle\langle m| = \sum_{nm} A_{nm} |n\rangle\langle m| = \begin{pmatrix} \langle 1|A|1\rangle & \langle 1|A|2\rangle & \langle 1|A|3\rangle & \dots \\ \langle 2|A|1\rangle & \langle 2|A|2\rangle & \langle 2|A|3\rangle & \dots \\ \langle 3|A|1\rangle & \langle 3|A|2\rangle & \langle 3|A|3\rangle & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{6}$$

Działanie operatora na ket:

$$\hat{A}|\psi\rangle = \left(\sum_{nm} A_{nm} |n\rangle\langle m|\right) \left(\sum_k \psi_k |k\rangle\right) = \sum_n \underbrace{\sum_m A_{nm}\psi_m}_{(\hat{A}|\psi\rangle)_n} |m\rangle = |\psi'\rangle \tag{7}$$

Wartość oczekiwana w stanie $|\psi\rangle$:

$$\langle A\rangle_\psi = \langle A\rangle = \langle\psi|A|\psi\rangle = \sum_{nm} A_{nm}\langle\psi|n\rangle\langle m|\psi\rangle = \sum_{nm}\sum_{ij} A_{nm}\psi_i^*\delta_{in}\psi_j\delta_{mj} = \sum_{nm}\psi_n^* A_{nm}\psi_m \tag{8}$$

Zagadnienie własne:

$$\hat{A}|a\rangle = a|a\rangle \tag{9}$$

Operator w bazie stanów własnych:

$$\hat{A} = \sum_{aa'}\langle a|\hat{A}|a'\rangle|a\rangle\langle a'| = \sum_{aa'} a\delta_{aa'}|a\rangle\langle a'| = \sum_a a|a\rangle\langle a| \tag{10}$$

3 (Anty-)commutator

Własności komutatora $[A,B] = AB - BA$ and anty-komutatora $\{A,B\} = AB + BA$:

1. $[\alpha A + \beta B, C] = \alpha[A,C] + \beta[B,C]$, gdzie α, β to stałe;
2. $[A,B] = -[B,A]$, $\{A,B\} = \{B,A\}$;
3. $[AB,C] = A[B,C] + [A,C]B$;
4. $[AB,C] = A\{B,C\} - \{A,C\}B$.

4 Kwanotwanie kanoniczne

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad [\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0; \tag{11}$$

$$\text{operator położenia: } \hat{x}_i = x_i; \tag{12}$$

$$\text{operator pędu: } \hat{p}_i = -i\hbar\frac{\partial}{\partial x_i}; \tag{13}$$

5 Zasada nieoznaczoności

Dla hermitowskich operatorów: $A = A^\dagger, B = B^\dagger$ ($\Delta A = \sqrt{(A - \langle A\rangle)^2}$):

$$\Delta A\,\Delta B \geq |\langle \tfrac{1}{2i}[A,B]\rangle| \tag{14}$$

6 Zmiana bazy—transformacje

Dane są dwie zupełne, ortonormalne bazy $\{|i\rangle\}|_{i=1,2,3,\ldots}$ oraz $\{|\widetilde{i}\rangle\}$ oraz transformacja unitarna U:

$$U|i\rangle = |\widetilde{i}\rangle, \quad U = \sum_j |\widetilde{j}\rangle\langle j| \tag{15}$$

Elementy macierzowe U:

$$U_{ij} = \langle i|\left(\sum_{j'}|\widetilde{j'}\rangle\langle j'|\right)|j\rangle = \langle i|\widetilde{j}\rangle \tag{16}$$

Postać U w bazie $|i\rangle$

$$U = \sum_{ij} U_{ij}|i\rangle\langle j| = \sum_{ij}\langle i|\widetilde{j}\rangle|i\rangle\langle j| = \begin{pmatrix} \langle 1|\widetilde{1}\rangle & \langle 1|\widetilde{2}\rangle & \langle 1|\widetilde{3}\rangle & \ldots \\ \langle 2|\widetilde{1}\rangle & \langle 2|\widetilde{2}\rangle & \langle 2|\widetilde{3}\rangle & \ldots \\ \langle 3|\widetilde{1}\rangle & \langle 3|\widetilde{2}\rangle & \langle 3|\widetilde{3}\rangle & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{17}$$

Transformacja stanu $|\psi\rangle = \sum_i \psi_i |i\rangle$:

$$|\psi\rangle = \sum_i \underbrace{\left(\sum_j |\widetilde{j}\rangle\langle\widetilde{j}|\right)}_{\mathbb{1}} \psi_i |i\rangle = \sum_j \sum_i \underbrace{\langle\widetilde{j}|i\rangle}_{U^\dagger_{ji}} \psi_i |\widetilde{j}\rangle = \sum_j \left(U^\dagger|\psi\rangle\right)_j |\widetilde{j}\rangle \tag{18}$$

Transformacja U jest unitarna i zachowuję normę $|\psi\rangle$:

$$UU^\dagger = \left(\sum_i |\widetilde{i}\rangle\langle i|\right)\left(\sum_j |j\rangle\langle\widetilde{j}|\right) = \sum_{ij} |\widetilde{i}\rangle \delta_{ij} \langle\widetilde{j}| = \mathbb{1}; \quad \langle\widetilde{\psi}|\widetilde{\psi}\rangle = \langle\psi|U^\dagger U|\psi\rangle = \langle\psi|\psi\rangle \tag{19}$$

Transformacja operatora: