CS 131 - Spring 2020, Assignment 6 Answers

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March 21, 2020

Tools: $\emptyset, \in, \notin, \subseteq, \subset, \cup, \cap, \exists, \forall, \neg, \vee, \wedge, \iff, \rightarrow, \leftarrow, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \neq, \therefore$

Problem 1.

1.

f)

$$\forall x(\neg W(x) \rightarrow (S(x) \vee V(x)))$$

g)

$$\exists x(\neg W(x) \wedge \neg(S(x) \vee V(x)))$$

h)

$$\forall x(\neg W(x) \rightarrow (S(x) \vee V(x)))$$

i)

$$S(Ingrid) \wedge W(Ingrid)$$

j)

$$\exists x(x \neq Ingrid \wedge S(x))$$

k)

$$\forall x(x \neq Ingrid \rightarrow S(x))$$

2.

e) Proposition. False. Some patients had migraines if and only if they had fainting spells, but not all. Only three of the patients did.

f) Proposition. True. For all the patients, if the patient had migraines and had fainting spells, then they were not given the medication. Of course, if the hypothesis is false, the proposition is true.

g) Proposition. True. There exists a patient (actually two: Gandalf and Bilbo) who took the medication and does not experience fainting spells and does not have migraines.

h) Proposition. False. The statement is translated as: for all patients, if the patient took the medication, then the patient has fainting spells or migraines. However, both Gandalf and Bilbo have taken the medication and do not experience fainting spells or migraines as a result. H is the negation of G.

3.

d)

$$\forall x(P(x) \to M(x))$$

Negation: $\neg\forall x(P(x) \to M(x))$

Applying De Morgan's law: $\exists x(P(x) \land \neg M(x)$

English: There exists a patient who was given the placebo and did not have migraines.

e)

$$\exists x(M(x) \land P(x))$$

Negation: $\neg\exists x(M(x) \land P(x))$

Applying De Morgan's law: $\forall x(\neg M(x) \lor \neg P(x))$

English: Every patient did not experience migraines or did not receive the placebo or both.

4.

b)

$$\exists x\forall y(\neg P(x, y) \lor \neg Q(x, y))$$

c)

$$\forall x\exists y(P(x, y) \land \neg Q(x, y))$$

d)

$$\forall x\exists y((P(x, y) \land \neg P(y, x)) \lor (P(y, x) \land \neg P(x, y)))$$

5.

f)

$$\forall x\exists y(y < x)$$

g)

$$\forall x\exists y((x \neq 0) \to (xy = 1))$$

h)

$$\forall x\exists y\forall z(((x \neq 0) \to (xy = 1)) \land ((z \neq y) \to (xz \neq 1)))$$

6.

f)

$$\forall y \text{ B(Josephine, y)}$$

g)

$$\exists x\forall y(\text{B(Nancy, x)} \land \neg\text{B(Nancy, y)} \land x \neq y)$$

h)

$$\forall x\exists y\exists z((x \neq y) \land (x \neq z) \land (y \neq z) \land \neg B(x, y) \land \neg B(x, z))$$

Problem 2.

1. $B \subseteq A \wedge C \subseteq B$ — Premise
2. $\forall a \in A\ Q(a)$ — Premise
    3. $x \in A \to Q(x)$ — Universal Instantiation, 2
    4. $\forall b \in B\ b \in A$ — Definition of Subset, 1
    5. $x \in B \to x \in A$ — Universal Instantiation, 4
    6. $x \in B \to Q(x)$ — Hypothetical Syllogism, 3,5
    7. $\forall c \in C\ c \in B$ — Definition of Subset, 1
    8. $x \in C \to x \in B$ — Universal Instantiation, 7
    9. $x \in C \to Q(x)$ — Hypothetical Syllogism, 6,8
10. $\forall c \in C\ Q(c)$ — Universal Generalization

    *Similar to Discussion Question

Problem 3.

1. $\forall c \in C\ \exists b \in B(g(b) = c)$ — Premise (Definition of Surjective)
2. $\forall b \in B\ \exists a \in A(f(a) = b)$ — Premise (Definition of Surjective)
3. $\exists b(x \in C \to b \in B(g(b) = x))$ — Universal Instantiation, 1
    4. $x \in C$ — Hypothesis
    5. $\exists b(b \in B(g(b) = x))$ — Modus Ponens, 3,4
        6. $y \in B \wedge g(y) = x$ — Existential Instantiation, 5
        7. $y \in B$ — Simplification, 6
        8. $g(y) = x$ — Simplification, 6
        9. $\exists a(y \in B \to a \in A(f(a) = y)$ — Universal Instantiation, 2
            10. $y \in B$ — Hypothesis
            11. $\exists a(a \in A(f(a) = y))$ — Modus Ponens, 9,10
                12. $z \in A \wedge f(z) = y$ — Existential Instantiation, 11
                13. $z \in A$ — Simplification, 12
                14. $f(z) = y$ — Simplification, 12
                15. $g(f(z)) = x$ — Definition of $=$, 8,14
                16. $z \in A \wedge g(f(z)) = x$ — Conjunction, 13,15
            17. $\exists a(a \in A(g(f(a)) = x))$ — Existential Generalization, 12,16
        18. $\exists a(y \in B \to a \in A(g(f(a)) = x)$ — Hypothesis Elimination, 10,17
        19. $\exists a(a \in A(g(f(a)) = x))$ — Modus Ponens, 7,18
    20. $\exists a(a \in A(g(f(a)) = x))$ — Existential Generalization, 6,19
21. $\exists a(x \in C \to a \in A(g(f(a)) = x)$ — Hypothesis Elimination, 4,20
22. $\forall c \in C \exists a \in A(g(f(a)) = c)$ — Universal Generalization, 21
23. $A \to C$ — Definition of Surjective, 22

Problem 4.

1. $A \cap B = \emptyset$	Premise
2. $D = C \cap A$	Premise
3. $E = C \cap B$	Premise
4. $\forall x(x \notin (A \cap B))$	Definition of $\emptyset$, 1
5. $\forall x(x \notin A \vee x \notin B)$	Definition of $\cap$, De Morgan's Law, 4
6. $\exists x(x \in D \cap E)$	Hypothesis (proving by contradiction)
7. $\exists x(x \in D \wedge x \in E)$	Definition of $\cap$, 6
8. $y \in D \wedge y \in E$	Existential Instantiation, 7
9. $y \in D$	Simplification, 8
10. $\forall x(x \in D \rightarrow (C \cap A))$	Definition of Premise, 2
11. $\forall x(x \in D \rightarrow (x \in C \wedge x \in A))$	Definition of $\cap$, 10
12. $y \in D \rightarrow (y \in C \wedge y \in A)$	Universal Instantiation, 11
13. $y \in C \wedge y \in A$	Modus Ponens, 9,12
14. $y \in A$	Simplification, 13
15. $y \in E$	Simplification, 8
16. $\forall x(x \in E \rightarrow (C \cap B))$	Definition of Premise, 3
17. $\forall x(x \in E \rightarrow (x \in C \wedge x \in B))$	Definition of $\cap$, 16
18. $y \in E \rightarrow (y \in C \wedge y \in B)$	Universal Instantiation, 17
19. $y \in C \wedge y \in B$	Modus Ponens, 15,18
20. $y \in B$	Simplification, 19
21. $y \in A \wedge y \in B$	Conjunction, 14,20
22. $\exists x(x \in A \wedge x \in B)$	Existential Generalization, 8,21
23. $\exists x((x \in D \cap E) \rightarrow (x \in A \wedge x \in B))$	Hypothesis Elimination, 6,22
24. $\neg \exists x(x \in D \cap E)$	Modus Tollens, 5,23
25. $\forall x(x \notin D \cap E)$	Definition of Negation, 24
26. $D \cap E = \emptyset$	Definition of Empty Set $\emptyset$, 25

Problem 5.

Following the Zybooks, there doesn't seem to be any sort of indentation scheme so I'm just going to follow how they format it there. I understand that introducing hypotheses mean an indentation in the proof.

1.

d) Predicates:
P(x)=Has a permission slip.
G(x)=Can go on the trip.

Form:
$\forall x(P(x) \rightarrow G(x))$
$\forall x(P(x))$

$\therefore \forall x(G(x))$

1. $\forall x(P(x))$	Hypothesis
2. a is an arbitrary student	Element Definition of Domain
3. $P(a)$	Universal Instantiation, 1,2
4. $\forall x(P(x) \rightarrow G(x))$	Hypothesis
5. $P(a) \rightarrow G(a)$	Universal Instantiation, 2,4
6. $G(a)$	Modus Ponens, 3,5
7. $\forall x(G(x))$	Universal Generalization, 2,6

e) Predicates:
B(x)=Taking Boolean Logic
A(x)=Can take Algorithms

Form:
Larry is a student at the university.
Hubert is a student at the university.
$B(Larry) \wedge B(Hubert)$
$\forall x(B(x) \to A(x))$

$\therefore A(Larry) \wedge A(Hubert)$

1. Larry is a student at the university.	Hypothesis
2. Hubert is a student at the university.	Hypothesis
3. $B(Larry) \wedge B(Hubert)$	Hypothesis
4. $\forall x(B(x) \to A(x))$	Hypothesis
5. $B(Larry)$	Simplification, 3
6. $B(Larry) \to A(Larry)$	Universal Instantiation, 1,4
7. $A(Larry)$	Modus Ponens, 5,6
8. $B(Hubert)$	Simplification, 3
9. $B(Hubert) \to A(Hubert)$	Universal Instantiation, 2,4
10. $A(Hubert)$	Modus Ponens, 8,9
11. $A(Larry) \wedge A(Hubert)$	Conjunction, 7,10

2.

b) The argument is not valid. To prove an argument is invalid, we have to prove the hypothesis to be true but the conclusion false. Consider a scenario where Suzy did not sell at least 50 boxes of cookies, but she still won a prize. The claim that "every girl scout who sells at least 50 boxes of cookies will get a prize" still holds true. The hypothesis is true and the conclusion is false. Therefore, the argument is not valid.

3.

b) Given:
$\exists x(P(x) \vee Q(x))$
$\exists x(\neg Q(x))$

$\therefore \exists x(P(x))$

	P	Q
a	F	F
b	F	T

$\exists x(P(x) \vee Q(x))$ is true because there exists an x where P(x) OR Q(x) is true, which is Q(b). For the second hypothesis to be true, there must exist a case where x in the domain $\{a, b\}$ is false for $\exists x(\neg Q(x))$ to be true. In a, $\exists x(\neg Q(x))$ is true. Lastly, to prove the invalidity of an argument, $\exists x(P(x))$ must be false. Both a and b are examples since $P(a) = P(b) = F$, $\exists x(P(x))$ is false. Therefore, the argument is invalid.

4.

b) Given:
$\exists x\ Q(x) \land \exists x\ P(x)$

$\therefore \exists x(P(x) \land Q(x))$
This argument is not valid.

	P	Q
a	F	T
b	T	T

The hypothesis, $\exists x\ Q(x) \land \exists x\ P(x)$, is true because there exists in the domain $\{a, b\}$ a case where $Q(a) = Q(b) = T$ and a case where $P(b) = T$. However, the conclusion is false because there does not exist an instance where $P(a) \land Q(a)$ or $P(b) \land Q(b)$ are true. Thus, the argument is not valid.

c) Given:
$\forall x(P(x) \land Q(x))$

$\therefore \forall x\ Q(x) \land \forall x\ P(x)$
This argument is valid.

1. $\forall x(P(x) \land Q(x))$	Hypothesis
2. 'a' is an arbitrary element	Definition of an element
3. $P(a) \land Q(a)$	Universal Instantiation, 1
4. $Q(a) \land P(a)$	Commutative Law, 3
5. $Q(a)$	Simplification, 4
6. $\forall x(Q(x))$	Universal Generalization, 2,5
7. $P(a)$	Simplification, 3
8. $\forall x(P(x))$	Universal Generalization, 2,7
9. $\forall x\ Q(x) \land \forall x\ P(x)$	Conjunction, 6,8

5.

d) Zybooks appears to not make "student in the class" as a function, but rather a part of the domain. For example, "Penelope is a student in the class" is its own hypothesis, which can simply be written as S(Penelope), where S(x) is x is a student in the class.

Predicates:
M(x): x missed class.
D(x): x got a detention.

Form:
$\forall x(M(x) \to D(x))$
Penelope is a student in the class.
$\neg M(Penelope)$

$\therefore \neg D(Penelope)$

This argument is valid. Consider a scenario where Penelope is the only student in the class (2nd hypothesis holds). Suppose Penelope did get a detention, which means that D(Penelope) is True, and therefore the conclusion is false. Now what if she didn't miss class, such that M(Penelope) is false and therefore, the third hypothesis stands. In the first hypothesis, because M(Penelope) is false, where Penelope is the only student in the class, then the first hypothesis must always be true. Therefore, all hypotheses are true and the conclusion is false. As a result, the argument is not valid.

e) Predicates:
M(x): x missed class.
D(x): x got a detention.
A(x): x got an A.

Form:
$\forall x((M(x) \vee D(x)) \rightarrow \neg A(x))$
Penelope is a student in the class.
$A(Penelope)$

$\therefore \neg D(Penelope)$

This argument is valid.

1. $\forall x((M(x) \vee D(x)) \rightarrow \neg A(x))$	Hypothesis
2. Penelope is a student in the class.	Hypothesis
3. $M(Penelope) \vee D(Penelope) \rightarrow \neg A(Penelope)$	Universal Instantiation, 1,2
4. $A(Penelope)$	Hypothesis
5. $\neg(M(Penelope) \vee D(Penelope))$	Modus Tollens, 3,4
6. $\neg M(Penelope) \wedge \neg D(Penelope)$	De Morgan's Law, 5
7. $\neg D(Penelope)$	Simplification, 6