CS 131 - Spring 2020, Assignment 5 Answers

Andy Vo

March 5, 2020

Tools: $\emptyset, \in, \notin, \subseteq, \subset, \cup, \cap, \exists, \forall, \neg, \vee, \wedge, \iff, \rightarrow, \leftarrow, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

Problem 1

1. The relation is reflexive, symmetric, and transitive. Therefore it is also an equivalence relation. To prove reflexive properties, $\forall x \in A$, we know that fRf because $f(x) = f(x)$, $(f, f) \in R$. The same can be applied to function g. $\forall x \in A$, we know that gRg because $g(x) = g(x)$, and therefore $(f, f) \in R$. The relation is symmetric. Because $(f, g) \in R$, we have to prove that $(g, f) \in R$. Essentially, we have $f(1) = g(1)$ and we have to prove $g(1) = f(1)$, and by the property of "=", it is true. Lastly, the relation is transitive because let $((f, g), (g, h)) \in R$. We want to prove that $(f, h) \in R$. Since $f(1) = g(1)$, and $g(1) = h(1)$, we have $f(1) = h(1)$.

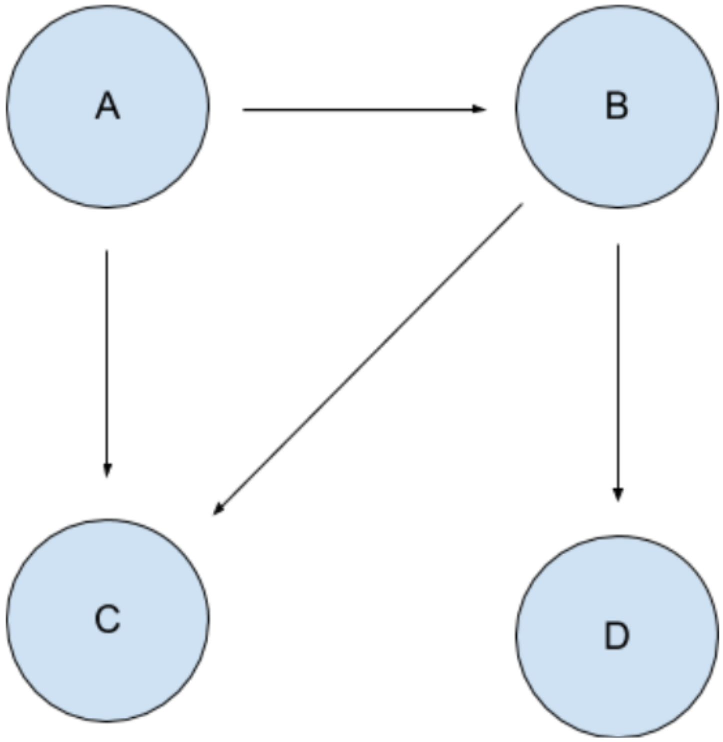
2. The relation is reflexive and symmetric, but not transitive. To prove reflexive properties, we prove that $(f, f) \in R$ and $(g, g) \in R$ is true since $f(0) = f(0)$ or $f(1) = f(1)$, and $g(0) = g(0)$ or $g(1) = g(1)$. By letting $(f, g) \in R$, we want to show that $(g, f) \in R$, which we know to be true because $f(0) = g(0)$ and $g(0) = f(0)$. Also, $f(1) = g(1)$ and $g(1) = f(1)$. However, this relation is not transitive because suppose $((f, g), (g, h)) \in R$.

3. The relation is symmetric.

4. The relation is reflexive, symmetric, and transitive.

Problem 2

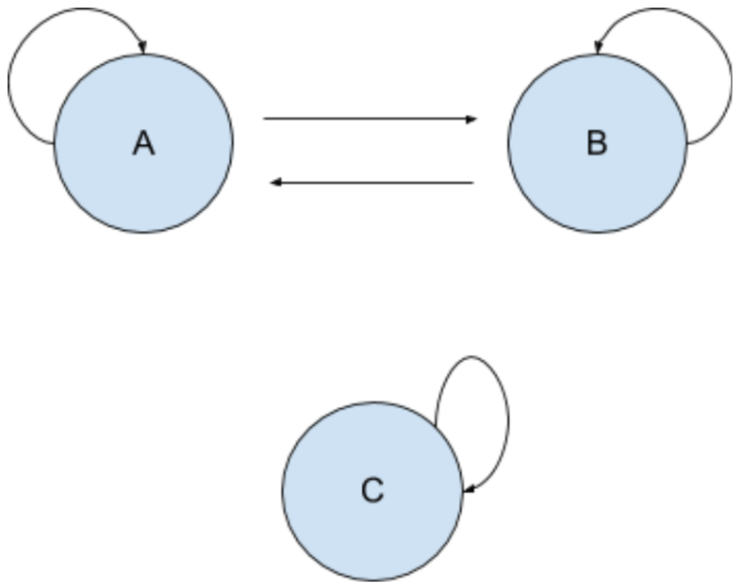
a) The difference between Transitivity and Vahideanness is the directionality of the relations. For Transitivity, one element points to another aRb, and the "another" element points a third element bRc, following a linear direction aRc. Vahideanness, on the other hand, has an element pointing at two other elements. It begins with an element pointing to another element aRb. That same first element a, however, also points to a third element aRc. Vahideanness is not following the typical logic of a linear relation. Furthermore, it only creates a one-way relation from the second element to the third bRc and not cRb. Of course, by the Commutative Property, $\forall a, b, c \in A aRb \wedge aRc \rightarrow bRc$ can be written as $\forall a, b, c \in A$, $aRc \wedge aRb \rightarrow cRb$. When viewing a digraph representation, one can confuse the two without context. Below, however, is an digraph example of a Vahidean relation on a, b, c, d which is not transitive. In this relation, we have aRb, aRc, and bRd. Therefore, by Vahidean logic, cRd. If this was transitive, we would also have to see aRd because, depending on how you look at it, we have aRb and bRd, or aRc and cRd.



b) No. But by the Commutative Law, $aRb \wedge aRc$ can also be written as $aRc \wedge aRb$. Therefore, yes in a sense, technically $\forall a, b, c \in A$ $aRb \wedge aRc \rightarrow cRb$. However, I think it is not an implication of Vahideanness. Applying the Commutative Law changes the order. For the transitive property, if $aRb \wedge bRc$ is true, the Transitive Property does not say cRa. Instead, it says aRc. There is a significant difference between aRc and cRa. The direction of the relation is what matters here. As a result, when we ask "is $\forall a, b, c \in A$ $aRb \wedge aRc \rightarrow cRb$?" The answer is no. It should be $\forall a, b, c \in A$ $aRb \wedge aRc \rightarrow bRc$ and $\forall a, b, c \in A$ $aRc \wedge aRb \rightarrow cRb$.

c) To prove that a relation R is an equivalence relation if it is Reflexive and Vahidean, then we must use the Reflexive and Vahidean properties to also prove Symmetrical and Transitive properties. Let's say we have $\forall a, b, c \in A$ on $A = \mathbb{R}$. By the Reflexive property, $(a, a) \in R$, $(b, b) \in R$, and $(c, c) \in R$ because $a = a$, $b = b$, and $c = c$. Essentially, aRa, bRb, and cRc. To prove Symmetry, we can apply the Vahidean property with the Reflexive property. Essentially, by Vahideanness, $\forall a, b \in A$ $aRb \wedge aRa \rightarrow bRa$ and $bRa \wedge bRb \rightarrow aRb$ which proves Symmetry between for $(a, b) \in R$. By $\forall a, c \in A$ $aRc \wedge aRa \rightarrow cRa$ and $cRa \wedge cRc \rightarrow aRc$ proving Symmetry for $(a, c) \in R$. We can also apply the same idea on b and c to prove $(b, c) \in R$. For the Transitive property, Reflexive and Symmetric properties imply transitivity. $\forall a, b \in A$ $aRa \wedge aRb \wedge bRbbRa$. By this logic, a and b are transitive with one another, and this logic can be used to prove transitivity between a and c, and b and c. Since we have proved that relation R is reflexive, symmetric, and transitive, it is an equivalence relation.

Problem 3



a) The image above depicts Alan's friendship relation. This relation was reached by the following logic. Given that there are three people, someone must not be friends with another person. Since there are only three people and we must prove friendship, the person who is not the friend (or also the enemy) is not friends with the second person either. In this case, A is friends with B and B is friends with A. However, since "Not everyone is friends with everyone", A is not friends with C and neither is B friends with C.

"The enemy of my enemy is my friend." This is proven by the fact that A is not friends with C and B is not friends with C. As a result, A is friends with B and B is friends with A.

"The enemy of my friend is my enemy." If A is friends with B, and B is enemies with C, then A is enemies with C. Same logic applies with B to A.

Furthermore, we can prove that friendship is reflexive. We can apply logic onto C as well. If enemy of C is A and the enemy of A is C, then C is friends with C. The same applies with C and B. As a result, all 3 are friends with themselves.

b) Alan will think that "friendship" is an equivalence relation because having "friendship" implies being friends with yourself (reflexive) and sharing a reciprocal relation (symmetric), which implies a transitive relation ($aRa \wedge aRb \wedge bRb \wedge bRa$). On a set of 100 people, we can have a maximum of 50 friendships given that a friendship requires two people and if we assume the following: friendship is only between two people so no third or "other" person can be friends with another group of two. If everybody is friends with one another in this group of 100, then we have one giant equivalence class.

Problem 4

Prove $(X \to Y \wedge Z \to T) \to ((X \wedge Z) \wedge (Y \wedge T))$

1. premise

$$X \to Y$$

2. premise

$$Z \to T$$

3. Hypothesis

$$X$$

4. Hypothesis

$$Z$$

5. Modus Ponens 1,3

$$Y$$

6. Modus Ponens 2,4

$$T$$

7. Conjunction 3,4

$$X \wedge Z$$

8. Conjunction 5,6

$$Y \wedge T$$

9. Hypothesis Elimination 3,8

$$(X \wedge Z) \wedge (Y \wedge T)$$