# CS 131 - Spring 2020, Assignment 4 Answers

Andy Vo

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Tools: $\emptyset, \in, \notin, \subseteq, \subset, \cup, \cap, \exists, \forall, \neg, \vee, \wedge, \iff, \rightarrow, \leftarrow$

## Problem 1

**a)**

$$\underline{P \subseteq Q \iff \overline{Q} \subseteq \overline{P}}$$

*Firstly*, we prove

$$\underline{P \subseteq Q \rightarrow \overline{Q} \subseteq \overline{P}}$$

1. Definition of Subset ($\subseteq$), premise

$$(x \in P) \rightarrow (x \in Q)$$

2. Contrapositive, 1

$$\neg(x \in Q) \rightarrow \neg(x \in P)$$

3. Definition of Complement, 2

$$(x \in \overline{Q}) \rightarrow (x \in \overline{P})$$

4. Definition of Subset, 3

$$\overline{Q} \subseteq \overline{P}$$

*Secondly*, we prove

$$\underline{\overline{Q} \subseteq \overline{P} \rightarrow P \subseteq Q}$$

1. Definition of Subset ($\subseteq$), premise

$$(x \in \overline{Q}) \rightarrow (x \in \overline{P})$$

2. Definition of Complement, 1

$$\neg(x \in Q) \rightarrow \neg(x \in P)$$

3. Definition of Contrapositive, 2

$$(x \in P) \rightarrow (x \in Q)$$

4. Definition of Subset, 3

$$P \subseteq Q$$

**b)** Proving "If $P \subseteq R$ and $Q \subseteq \overline{R}$, then $Q \subseteq \overline{P}$."

1. Premise

$$P \subseteq R$$

2. Premise

$$Q \subseteq \overline{R}$$

3. Definition of $\subseteq$, 1

$$(x \in P) \rightarrow (x \in R)$$

4. Contrapositive, 3

$$\neg(x \in R) \rightarrow \neg(x \in P)$$

5. Definition of $\subseteq$, 2

$$(x \in Q) \to (x \in \overline{R})$$

6. Conditional Identity Law, 5

$$\neg(x \in Q) \vee (x \in \overline{R})$$

7. Conditional Identity Law, 4

$$\neg\neg(x \in R) \vee \neg(x \in P)$$

8. Double Negation Law, 7

$$(x \in R) \vee \neg(x \in P)$$

9. Resolution, 6 8

$$\neg(x \in Q) \vee \neg(x \in P)$$

10. Conditional Identity Law, 9

$$(x \in Q) \to \neg(x \in P)$$

11. Definition of Complement, 10

$$(x \in Q) \to (x \in \overline{P})$$

12. Definition of $\subseteq$, 11

$$Q \subseteq \overline{P}$$

**c)** Proving "If $c$ is such that $c \in P \cup Q$ and $c \in \overline{Q} \cap R$, then $c \in P \cap R$."

1. Premise

$$c \in P \cup Q$$

2. Definition of $\cup$, 1

$$(c \in P) \vee (c \in Q)$$

3. Premise

$$c \in \overline{Q} \cap R$$

4. Definition of $\cap$, 3

$$(c \in \overline{Q}) \wedge (c \in R)$$

5. Simplification, 4

$$c \in \overline{Q}$$

6. Disjunctive Syllogism, 2 5

$$c \in P$$

7. Simplification, 4

$$c \in R$$

8. Conjunction, 6 7

$$(c \in P) \wedge (c \in R)$$

9. Definition of $\cap$

$$c \in P \cap R$$

# Problem 2

The former set equation (Equation 1) is not always true. The latter set equation (Equation 2) is always true. An example to demonstrate this is as follows.

$$A = 1, 2$$

$$B = 2, 3$$

From Equation 1, $P(A \cup B) = 1, 2, 1, 2, 2, 3, 1, 3, 1, 2, 3$. However, $P(A) \cup P(B) = 1, 2, 3, 1, 2, 2, 3$. The reasoning for this is because of the difference between a power set of $A \cup B$ to a power set of $A$ and a power set of $B$. The union ($\cup$) operator combines all of the elements on the left hand side of Equation 1 and then creates the power set. However, on the right hand side, the power sets are created prior to the union. As a result, elements of the power set $1, 3$ and $1, 2, 3$ are not formed.

We prove $P(A \cap B) = P(A) \cap P(B)$ with two formal logical proofs.

$$S \in P(A) \iff S \subseteq A$$

In the first, we prove the following:

$$\underline{P(A \cap B) \subseteq P(A) \cap P(B)}$$

1. Definition of a Power Set, premise

$$S \in P(A \cap B) \iff S \subseteq A \cap B$$

2. Definition of a Subset ($\subseteq$), 1

$$(x \in S) \to (x \in A \cap B)$$

3. Definition of Intersection ($\cap$), 2

$$(x \in S) \to ((x \in A) \land (x \in B))$$

4. Conditional Identity Law, 3

$$\neg(x \in S) \lor ((x \in A) \land (x \in B))$$

5. Distributive Law, 4

$$(\neg(x \in S) \lor (x \in A)) \land (\neg(x \in S) \lor (x \in B))$$

6. Conditional Identity Law, 5

$$((x \in S) \to (x \in A)) \land ((x \in S) \to (x \in B))$$

7. Definition of a Subset ($\subseteq$), 6

$$(S \subseteq A) \land (S \subseteq B)$$

8. Definition of a Power Set, 6

$$P(A) \land P(B)$$

9. Definition of an Intersection ($\cap$), 7

$$P(A) \cap P(B)$$

In the second, we prove the following:

$$\underline{P(A) \cap P(B) \subseteq P(A \cap B)}$$

1. Definition of a Power Set, premise

$$(S \in P(A)) \cap (S \in P(B)) \iff (S \subseteq A) \cap (S \subseteq B)$$

2. Definition of a Subset, 1

$$((x \in S) \to (x \in A)) \cap ((x \in S) \to (x \in B))$$

3. Definition of an Intersection ($\cap$), 2

$$((x \in S) \to (x \in A)) \land ((x \in S) \to (x \in B))$$

4. Conditional Identity Law, 3

$$(\neg(x \in S) \lor (x \in A)) \land (\neg(x \in S) \lor (x \in B))$$

5. Distributive Law, 4

$$\neg(x \in S) \lor ((x \in A) \land (x \in B))$$

6. Conditional Identity Law, 5

$$(x \in S) \to ((x \in A) \land (x \in B))$$

7. Definition of Intersection, 6

$$(x \in S) \to (x \in A \cap B)$$

8. Definition of Subset, 7

$$S \subseteq A \cap B$$

9. Definition of Power Set, 8

$$P(A \cap B)$$

## Problem 3

"$A$ is an empty set" ($A = \emptyset$) can be written with quantifiers as the following:

$$\forall x(x \notin A)$$

and

$$\neg\exists x(x \in A)$$

## Problem 4

**a)**

1. $J(A, B) = 0/6 = 0$; $d_J(A, B) = 1 - 0 = 1$
2. $J(A, B) = 5/7$; $d_J(A, B) = 1 - 5/7 = 2/7$

**b)**

1. The Jaccard similarity of two sets is the number of elements in intersection divided by the number of elements in union. If we take the Jaccard similarity of the same set, the number of elements in intersection is the same number of elements in union, and as a result, $J(A, B)$ is 1. The Jaccard distance is 1 - $J(A, B)$ and as a result, $d_J(A, B)$ will always be 0 of the same set.
2. $J(A, B) = J(B, A)$ because regardless of the order, by the Commutative Law, $A \cap B = B \cap A$ and $A \cup B = B \cup A$. Since their Jaccard similarities will always be the same, their Jaccard distances will also be the same as well since you subtract from the same value.
3. If $A = B$, meaning set A contains the same values as set B, then it is just like 1. By taking the Jaccard similarity of the "same" set, the number of elements in intersection will always equal the number of elements in union. Thus, the Jaccard similarity will always be 1.