CS 131 - Spring 2020, Assignment 7 Answers

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Tools: $\emptyset, \in, \notin, \subseteq, \subset, \cup, \cap, \exists, \forall, \neg, \vee, \wedge, \iff, \rightarrow, \leftarrow, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \neq, \therefore, \leq, \geq, \{\}$

Problem 1.

a)

Proof. Let's prove that for all integers n, where $n \geq 0$, $f_0 + f_1 + ... f_n = f_{n+2} - 1$.

Base Case
$n = 0$:

$$f_0 = 0$$
$$f_{n+2} - 1 = 1 - 1 = 0$$
$$f_0 = 0 = 0 = f_{n+2} - 1$$

The statement is true for $n = 0$.
$n = 1$:

$$f_1 = 1$$
$$f_{n+2} - 1 = 2 - 1 = 1$$
$$f_1 = 1 = 1 = f_{n+2} - 1$$

The statement is true for $n = 0$.

Inductive Step
Suppose the statement is true for $n = k$ such that $f_0 + f_1 + ... f_n = k = f_{k+2} - 1$. We will prove the statement is true for $n = k + 1$.

$$f_0 + f_1 + ... + f_k + f_{k+1} = f_{(k+1)+2} - 1$$
$$(f_{k+2} - 1) + f_{k+1} = f_{(k+1)+2} - 1$$
$$f_{(k+1)+2} - 1 = f_{(k+1)+2} - 1$$

This proves that the statement holds true for $n = k + 1$. Proving the base case and the inductive step, by mathematical induction, $f_0 + f_1 + ... f_n = k = f_{k+2} - 1$ is true for all n where $n \geq 0$. ∎

b)

Proof. Let's prove that for every $n \geq 2$, $f_n \geq (1.5)^{n-2}$.

Base Case
$n = 2$:

$$f_2 = 1$$
$$1.5^{n-2} = 1.5^{2-2} = 1.5^0 = 1$$
$$f_2 = 1 \geq 1 = 1.5^{n-2}$$

The statement holds true for $n = 2$.
$n = 3$:

$$f_3 = 2$$
$$1.5^{n-2} = 1.5^{3-2} = 1.5^1 = 1.5$$
$$f_3 = 2 \geq 1.5 = 1.5^{n-2}$$

The statement holds true for $n = 3$.

Inductive Step
Suppose we have that for all $n \geq 2$, there exists k from range 0 through n such that the statement holds true for $n = k$. Therefore, we assume that $f_k \geq (1.5)^{k-2}$. We will prove that $f_{k+1} \geq (1.5)^{(k+1)-2} = (1.5)^{k-1}$. Since $n \geq 2$, then $n - 1 \geq 1$. Therefore, both n and $n - 1$ fall in the range of 0 through n, and by the inductive hypothesis, $f_{k-1} \geq (1.5)^{(k-1)-2}$ and $f_k \geq (1.5)^{k-2}$.

By definition

$$f_{k+1} = f_k + f_{k-1}$$

By the inductive hypothesis

$$f_{k+1} \geq (1.5)^{k-2} + (1.5)^{(k-1)-2} = (1.5)^{k-2} + (1.5)^{k-3}$$
$$f_{k+1} \geq (1.5)^{k-1} \cdot (1.5^{-2} + 1.5^{-1}$$
$$f_{k+1} \geq (1.5)^k \cdot (1.5)^{-2} + (1.5)^k \cdot (1.5)^{-3}$$
$$f_{k+1} \geq (1.5)^k \cdot (1.5)^{-3} \cdot (1.5) + (1.5)^k \cdot (1.5)^{-3}$$
$$f_{k+1} \geq (1.5)^{k-3} \cdot 2.5 \geq (1.5)^{k-3} \cdot (1.5)^2$$
$$f_{k+1} \geq (1.5)^{k-3} \cdot 2.5 \geq (1.5)^{k-1}$$

This concludes the inductive step. By mathematical induction, we have proven the basis step and inductive step. We have proven that $P(n)$ is true for $n = k$ and $n = k + 1$ for all $n \geq 2$. ■

c)

Proof. We will prove that for every $n \geq 0$, $f_n \leq 2^{n-1}$.

Base Case
$n = 0$:

$$f_0 = 0$$
$$2^{n-1} = 2^{0-1} = 2^{-1} = (1/2)$$
$$f_0 = 0 \leq (1/2) = 2^{n-1}$$

The statement holds true for $n = 0$.
$n = 1$:

$$f_1 = 1$$
$$2^{n-1} = 2^{1-1} = 2^0 = 1$$
$$f_1 = 1 \leq 1 = 2^{n-1}$$

The statement holds true for $n = 1$.

Inductive Step
Assume f_k is true for $n = k$ such that $f_k \leq 2^{k-1}$. We also know that $n = k - 1$ is true such that $f_{k-1} \leq 2^{(k-1)-1}$. We will prove $n = k + 1$ for $f_{k+1} \leq 2^{(k+1)-1} = 2^k$.

By definition

$$f_{k+1} \leq f_k + f_{k-1}$$

By the inductive hypothesis

$$f_{k+1} \leq 2^{k-1} + 2^{k-2}$$

$$f_{k+1} \leq 2^{k-2} \cdot 3$$

$$f_{k+1} \leq 2^{k-2} \cdot 3 \leq 2^{k-2} \cdot 4$$

$$f_{k+1} \leq 2^{k-2} \cdot 3 \leq 2^{k-2} \cdot 2^2$$

$$f_{k+1} \leq 2^{k-2} \cdot 3 \leq 2^k$$

This concludes the inductive step. By mathematical induction, since both the base case is true and the inductive step is true such that $n = k$ and $n = k - 1$ implies $n = k + 1$ is true, then n is true for all $n \geq 0$ for $f_n \leq 2^{n-1}$. ∎

d)

Proof. We will prove that if $f_n = X$ for some $X > 0$, then

$$1 + \log_2 X \leq n \leq 2(1 + \log_2 X)$$

Let's consider each inequality separately. Firstly, let's consider

$$1 + \log_2 X \leq n$$

First, let's subtract both sides by 1.

$$\log_2 X \leq n - 1$$

If we raise both sides by the power of 2, we get

$$2^{\log_2 X} \leq 2^{n-1}$$

$$2^{\log_2 X} \leq 2^{n-1}$$

$$X \leq 2^{n-1}$$

Since $X = f_n$, and we have proven that for every $n \geq 0$ that $f_n \leq 2^{n-1}$ in part C, the same can be said for X by substitution. Therefore, given that we have already proven $f_n \leq 2^{n-1}$ for $n \geq 0$, we know that it is true for $X > 0$ for $X \leq 2^{n-1}$ and that $1 + \log_2 X \leq n$ is true for $X > 0$.

Let's consider $n \leq 2(1 + \log_2 X)$. By logarithmic rules, we can simplify this to be

$$n \leq 2 + 2\log_2 X$$

$$n - 2 \leq 2\log_2 X$$

$$(n-2)/2 \leq \log_2 X$$

Raising both sides by the power of 2, we get

$$2^{(n-2)/2} \leq X$$

Referring to part b, we proved that $f_n \geq (1.5)^{n-2}$. By logarithmic rules, we can rewrite this as

$$f_n \geq (1.5)^{2^{(n-2)/2}}$$

$$f_n \geq (2.25)^{(n-2)/2}$$

Since $f_n \geq (2.25)^{(n-2)/2}$ and $(2.25)^{(n-2)/2} \geq 2^{(n-2)/2}$, the transitive property holds that $f_n \geq 2^{(n-2)/2}$. X is a substitute variable for f_n. Therefore, by proving both sides of the inequality, the statement holds true that $1 + \log_2 X \leq n \leq 2(1 + \log_2 X)$. ∎

Problem 2.

Proof. We will prove that for any integer $n \geq 1$, if $a > b \geq 1$ and $Euclid(a, b)$ takes n iterations, then $a \geq f_{n+2}$ and $b \geq f_{n+1}$ where f_n is the n^{th} term of the Fibonacci sequence.

Base Case
$n = 1$:
If $n = 1$, then b is at least 1 by definition, and a is at least 2 since a and b are integers and $a > b$.

$$f_{n+2} = 2$$

$$a = 2 \geq 2 = f_{n+2}$$

$$f_{n+1} = 1$$

$$b = 1 \geq 1 = f_{n+1}$$

Thus, the statement holds true for $n = 1$.

Inductive Step
Assume that for some k such that $n = k$ such that it holds true for $k \geq 1$, if $a > b \geq 1$ and $Euclid(a, b)$ takes k iterations, then $a \geq f_{k+2}$ and $b \geq f_{k+1}$. We will prove that $P(k+1)$ holds true for all $n = k+1$ such that if c and d are integers such that $c > d \geq 1$ and $Euclid(c, d)$ takes $k+1$ iterations, then $c \geq f_{k+3}$ and $d \geq f_{k+2}$.

The Euclidean algorithm for GCD gives us $a = bq_0 + r_0$ where r is a non-negative remainder and q is a quotient integer such that $q \geq 1$. Given the recursive nature of GCD, $b = r_0q_0 + r_1$, $r_0 = r_1q_2 + r_2$, etc. for k iterations. For $Euclid(c, d)$, the first step of the Euclidean algorithm is $c = aq_0 + r_0$ and the second step is $d = r_0q_0 + r_1$. Since it takes $k+1$ steps to compute $Euclid(c, d)$ and k steps to compute $Euclid(d, r_0)$. Another way to explain it is that it takes steps 2 through $k+1$ to compute $Euclid(d, r_0)$. By the induction hypothesis, the smallest integers for k steps are f_{k+2} and f_{k+1} such that $b \geq f_{k+2}$ and $r_0 \geq f_{k+1}$. To prove that $c \geq f_{k+3}$, we know that $c = aq_0 + r_0$ and $q \geq 1$. Thus, $c \geq d + r_0 \geq f_{k+2} + f_{k+1} = f_{k+3}$.

This concludes the inductive step. We have proved that for any integer $n \geq 1$, if $a > b \geq 1$ and $Euclid(a, b)$ takes n iterations, then $a \geq f_{n+2}$ and $b \geq f_{n+1}$ where f_n is the n^{th} term of the Fibonacci sequence through mathematical induction, proving the basis step and inductive step. ∎

Problem 3.

a)

Proof. Let's prove the corollary that for all $n \geq 1$, if $a > b \geq 1$ and $b < f_{n+1}$, then $Euclid(a, b)$ takes fewer than n iterations using contradiction. By contradiction, let's assume $Euclid(a, b)$ takes n or more iterations. From the theorem in problem 2, if $Euclid(a, b)$ takes n or more iterations, then $a \geq f_{n+2}$ and $b \geq f_{n+1}$ where f_n is the n^{th} term of the Fibonacci sequence. This directly contradicts our assumption in the corollary that $b < f_{n+1}$. By contradiction then, $Euclid(a, b)$ must take fewer than n iterations. ∎

b)

Proof. We will prove the corollary that for all $n \geq 2$, if $a < f_n$ or $b < f_n$, then $Euclid(a, b)$ takes strictly fewer than n iterations. There are some cases to consider here given that we know $a, b \geq 0$. We will not need to consider the case where $a > b \geq 1$ because it takes fewer than n iterations as proven in the previous corollary. However, let's examine other cases:

Case 1: $b = 0$. ∎

Problem 4.

Proof. Suppose you have $n \in \mathbb{N}$ posters, and that for each poster, it can go on either the wall or in the closet. To figure out how many possible sets of posters we can have on the wall, let's begin to break down the logic of poster sets. When we have 0 posters, the following sets can be made: 0 on the wall and 0 in the closet; a total of 1 set. When there is one poster, it can be on the wall and not in the closet, or it can be in the closet and not on the wall, which totals 2 sets. With two posters, there are four possible sets: 0(wall) and 2(closet), poster A on the wall and poster B in the closet, vice versa, or 2(wall) and 0(closet). Following this pattern, we can make the claim that $P(n) = 2^n$ where n is the number of posters and P is a function of n that tells us how many different possible sets of posters we can have on the wall.

Base Case
$n = 0$: The least number of posters we can have is zero.
There is only one way to arrange zero posters.

$$P(0) = 1$$

This follows the formula:

$$2^0 = 1$$

Therefore, the claim holds for $n = 0$.

$$P(0) = 1 = 1 = 2^0$$

Inductive Step
Suppose there are some k posters for $k \in \mathbb{N}$ such that $n = k$ where $P(k) = 2^k$. We will prove that for every additional poster, $n = k + 1$, there are 2^{k+1} different sets or arrangements of posters on the wall.
If there are k posters, we assumed that there are 2^k arrangements/sets of posters on the wall. If there are $k+1$ posters, then we have $2^k + 2^k$ arrangements of posters. Simplifying this, we get $2^k + 2^k = 2 \cdot 2^k = 2^k + 1$ possible sets of posters on the wall.
This concludes our inductive step. By mathematical induction, since both the basis step is true and inductive step is true such that $n = k$ implies $n = k + 1$, then $P(n) = 2^n$ is true for all $n \in \mathbb{N}$. ■

Problem 5.

Proof. We will prove

$$(\sum_{i=1}^{n} i)^2 = \sum_{i=1}^{n} i^3$$

In English, we will prove that the square of a sum of n equals the sum of cubed n terms.

Base Case
$n = 1$: *not sure if this is a typo because the HW assignment says $n \geq 0$, but the lower bound is 1.

$$(\sum_{i=1}^{n} i)^2 = (1)^2 = 1$$

$$\sum_{i=1}^{n} i^3 = (1)^3 = 1$$

$$(\sum_{i=1}^{n} i)^2 = (1)^2 = 1 = 1 = \sum_{i=1}^{n} i^3 = (1)^3$$

The statement holds true for $n = 1$.

Inductive Step To better manipulate the square of sums, we can rewrite $(\sum_{i=1}^{n} i)^2$ as the arithmetic sum formula.

$$\frac{n(n+1)}{2}$$

$$(\frac{n(n+1)}{2})^2$$
$$\frac{n^2(n+1)^2}{4}$$
Furthermore, the formula for the sum of i^3 for n terms is $\frac{n^2(n+1)^2}{4}$.
Suppose for some integer k where $n = k$, the statement holds true such that the square of sums is $P(k) = \frac{k^2(k+1)^2}{4}$. We will prove $n = k+1$ such that the square of the sum of $k+1$ terms is $P(k+1) = \frac{(k+1)^2(k+2)^2}{4}$.
A way to express the square of the sum of $k+1$ terms is by adding the next cube value to the square of sums of k terms.

$$\begin{aligned}
P(k+1) &= P(k) + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4}
\end{aligned}$$

This concludes the inductive step. By mathematical induction, we have proven the basis step and the inductive step, proving that for all $n \geq 1$ terms, the square of a sums is equal to the sum of cubed terms.
■

Problem 6.

Suppose there is some sequence of positive real numbers a_1,...,a_n with the property that for all j such that $1 \leq j \leq n$, $(\sum_{i=1}^{j} a_i)^2 = \sum_{i=1}^{j} a_i^3$. We will prove that for all i such that $1 \leq i \leq n$, $a_i = i$.

Base Case
$n = 1$: We know that $a_1^2 = a_1^3$. Therefore, a_1 has to be 1 since $a_1 > 0$ (positive real number). Since $n = 1$, we know $1^2 = 1^3$.

Inductive Step Assume that we have a sequence with length k where $n = k$ such that a_1,...,a_k. This satisfies the property that the sequence has to be 1,2,...,k. Now we have to prove that by having some sequence s_1,s_2,...,s_k,s_{k+1} where $n = k+1$ that the statement holds true for $a_i = i$. We are showing that if $a_k = k$, then $a_{k+1} = k+1$.
We know that $a_{k+1} = a_k + (k+1)$.