CS 131 - Spring 2020, Assignment 7 Answers

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Tools: $\emptyset, \in, \notin, \subseteq, \subset, \cup, \cap, \exists, \forall, \neg, \vee, \wedge, \iff, \rightarrow, \leftarrow, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \neq, \therefore, \leq, \geq, \{\}$

Problem 1.

Proof. Suppose there is a circular road that n toll booths and n reward booths. At each toll booth, you have to pay \$1. If you have no money, you are stuck and cannot proceed. However, at each reward booth, you collect \$1. You start with no money, but you can decide where to begin. You must move clockwise. We will prove that for every $n \geq 0$, no matter how the booths are arranged, there is always a starting point that will allow you to go all the way around.

Base Case
Suppose $n = 0$. Let's call the region where there is a possible start region g. If $n = 0$, there are no toll booths or reward booths in this circular road. We are free to proceed with no money for every point on this circular road is a viable starting point, or a region g. Thus, the statement holds true for $n = 0$.
Suppose $n = 1$. If $n = 1$, then we will have one toll booth and one reward booth on the circular road. Then the circle will be divided into two regions. Clockwise-speaking, the region beyond the toll booth to right before the reward booth will be region g and thus, a viable starting point because our first booth will be the reward booth, providing \$1 for the toll booth afterward. Thus, the statement holds true for $n = 1$.

Inductive Step
We know that we begin with \$0. Logically, we then know that in order to pass through any toll booth, we must have at least \$1 or essentially have passed through at least one reward booth. Since reward booths are $+\$1$ and toll booths are $-\$1$, we know that the total number of reward booths passed through at any current position must be greater than the number of toll booths passed.
Let there exist a k such that for all $n = k$ such that $k \geq 0$, there always exists a starting point that will allow you to go all the way around with k toll booths and k reward booths for all booth arrangements so that $P(n) = P(k)$.
We want to prove that for $P(k + 1)$ where $n = k + 1$, there always exists a starting point that allows us to go all the way around with $k + 1$ toll booths and $k + 1$ reward booths.
To prove our inductive case, we can carefully remove two chosen booths. We want to remove a pair of booths consisting of one reward booth and one toll booth such that they are sequential in order (one right after another or next to each other) and that in clockwise rotation, the reward booth comes first before the toll booth. This pair always exists. By contradiction, let there exist no such pair such that there is no region of the circle where the toll booth comes right after the reward booth. Thus, only the reward booth comes after another reward booth. This is a contradiction of the fact for the every n reward booths on the circular road, there must also exist n toll booths. Therefore, there exists a pair of reward-toll booths.
By our assumption, after we remove the reward-toll booth pair, our reach the assumption of k reward booths and k toll booths, which means that there exists a starting point where we can go around. By removing the pair, we are left with k booths of each type, and we assumed this to have a starting point. Now by re-adding that reward-toll booth pairing in that g region, then it guaranteed that there will always be starting point in $k + 1$ booths. Essentially, by locating the valid starting position g in k booths, we can add $+1$ booths of each type to k booths so long as they are a reward booth followed by a toll booth in the g valid region. Therefore, the new valid starting point will simply be the region $g - prime$, which is the g region before the new reward booth added.
This concludes our inductive step. By mathematical induction, we have proven both the base case and

inductive step, proving that there exists a starting point where $n = k$ implies $n = k+1$ for n toll and reward booths in a circular road. ■

Problem 2.

Proof. We will prove by strong induction that any amount of postage worth 8 cents or more can be made from 3-cent or 5-cent stamps. Let $P(n)$ state that a postage of n cents can be made of some combination using 3-cent and 5-cent stamps where $n \geq 8$. A better expression of this is that for any $n \geq 8$, $P(n) = 3a+5b$ where a and b are non-negative integers.

Base Case
Let $n = 8$. $8 = 3(1) + 5(1)$.
Let $n = 9$. $9 = 3(3) + 5(0)$.
Let $n = 10$. $10 = 3(0) + 5(2)$.
The basis step is complete.

Inductive Step
Suppose there is some k that exists such that $8 \leq n \leq k$ for $k \geq 10$, $P(k) = 3a + 5b$. We must prove that $P(k + 1)$ is true. By the inductive hypothesis, we know that $n = k - 2 \geq 8$ to be true so we can apply $P(k - 2)$ as it is true.

$$k - 2 = 3a + 5b$$

To get to the $k + 1$ term, we can add one 3-cent stamp such that

$$k - 2 + 3 = 3a + 5b + 3$$

$$k + 1 = 3a + 5b + 3$$

In fact, we can know use this equation to prove that the base cases can make any $n \geq 8$-cent stamp.

$$k + 1 = 3(a + 1) + 5b$$

Essentially, from $P(8)$, $P(9)$, and $P(10)$, or $k - 2$, $k - 1$, and k respectively, we can make any $n \geq 8$ stamp with a multiple of 3. $17 = 8+9$, $39 = 9+30$, $100 = 10+90$ – all of which are base cases added by a multiple of 3. This concludes the inductive step. We have proved both the basis step and the inductive step, displaying that $P(k)$ and $P(k - 1)$ and $P(k - 2)$ implies $P(k + 1)$, and thus, by mathematical strong induction, the statement $P(n)$ holds true for all $n \geq 8$. ■

Problem 3.

Proof. Suppose there is a two-player game as follows: there are two piles of matches. Initially, both piles contain n matches. Players will alternate moves where a player removes some positive number of matches from one of the two piles. The player who removes the last match wins (this game sounds like Nim). We will demonstrate that the second player to go always has a winning strategy by proof of strong induction. Let $P(n)$ where $n \geq 1$ represent that if there are n matches in each pile and player 1 moves first, player 2 (the second player) always has a winning strategy.

Base Case
Let $n = 1$. We have 1 match in pile A and 1 match in pile B. The first player must take the one match from either pile, leaving the second player to take the one match from the other pile not selected by the first player. Thus, player 2 always wins.
Let $n = 2$. There are a few different ways the game can be played out, but always leaving second player with a guaranteed win. Suppose player 1 picks 1 match from pile A (or B, it doesn't matter). Then player 2 can pick 1 match from pile B (or A, player 2 picks 1 from the opposite pile of player 1). This results in the same situation as $n = 1$, thus, player 2 wins. If player 1 begins by taking 2 matches from pile A or B, then player 2 wins by taking 2 matches from the opposite pile, and thus, always winning.

Inductive Step

Our inductive hypothesis is that $P(x)$ is true for $1 \leq x \leq k$ where k is some number of matches in each pile A and B and that x is any integer number of matches taken from the pile. We assume to be true that for any value of x and k where $1 \leq x \leq k$, player 2 will always have a winning strategy. For strong induction, we must assume that all scenarios before $k+1$ are true and that we prove $P(k+1)$ to be true.

To prove that player 2 will win for $x = k+1$, we can apply the hypothesis inequality to make a general statement.

$$1 \leq k - y + 1 < k + 1$$

where y is the number of removed matches from range $1 \leq y \leq k$. In general, player 1 can remove y number of matches from pile A (without loss of generality since it could be either pile) that has $k+1$ matches. This means that there are $k - y + 1$ matches remaining after player 1's turn. For player 2 (the second player) to come out with a winning strategy, player 2 will have to remove the same number of matches y from pile B so that there are the same number of matches in both piles $k - y + 1$. We know that $1 \leq k - y + 1 < k + 1$. By our inductive hypothesis, we assume that for any value of matches remaining in piles A and B such that it is between 1 and k inclusively, the statement holds true. Therefore, by removing any value y matches from $k+1$, we are left with a number of matches between 1 and k inclusively which we assume to be true.

The other case to consider is what if player 1 takes all $k+1$ matches from pile A? Then it is simple, player 2 must take all $k+1$ matches from pile B to win.

This concludes our inductive step. By the principles of strong mathematical induction, we have proven the basis steps and also the inductive steps, which utilized the assumed hypothesis of $P(k)$ to imply that $P(k+1)$ to be true for any and all value of n matches where $n \geq 1$. Thus, player 2 always has a winning strategy. ■

Problem 4.

1. Number of 5-subset in which 3 computers with the file is selecting the 2 of 37 without the file. 37 choose 2.

2. There are $2^8 = 256$ unique 8-bit binary strings. There are only two options that do not have at least two consecutive 0's or two consecutive 1's: 10101010 and 01010101. Thus, there are 254 8-bit strings which have at least two consecutive 0's or two consecutive 1's.

3. The length of "SUBSETS" is 7. There are 3 S's, 1 U, 1 B, 1 E, and 1 T. This gives us a total of $\frac{7!}{3! \cdot 1! \cdot 1! \cdot 1! \cdot 1!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$ different ways to permute the letters in "SUBSETS".

4. The school cook has to cook each of the ten meals twice so that she can cook each meal the same number of times in the 20 days. For each of the ten different meals, the two meal 1's have twenty different days to be chosen from, or 20 choose 2. For the two meal 2's, there are 18 days to choose from so 18 choose 2, and it goes on and on such that there are $\frac{20 \cdot 19}{2} \cdot \frac{18 \cdot 17}{2} \cdot \ldots \cdot \frac{2 \cdot 1}{2} = \frac{20!}{2^{10}}$ different possibilities.

5. In the race, suppose there are 7 runners whose times are between 6 and 7 minutes, which allots for 60 seconds. Can we be certain that there were a pair of racers who came in with less than nine seconds apart? We will do a small "proof by contradiction." Let's say runner 1 comes in at 6 minutes flat. Runner 2 comes in 9 seconds later at 6:09, and the trend keeps going for the 7 runner.
   Runner 1: 6:00
   Runner 2: 6:09
   Runner 3: 6:18
   Runner 4: 6:27
   Runner 5: 6:36
   Runner 6: 6:45
   Runner 7: 6:54
   There is certainty that there exists two runners whose times are less than nine seconds apart. However, if there are eight runners we require $(8-1) \cdot 9 = 63$ seconds to ensure that all eight runners are nine seconds apart. If all eight racers are to be between 6 and 7 minutes, then it is certain that at least 2 runners will come in with times that are less than nine seconds apart.

6. number of people selected to make sure that there are at least 20 born in the same month k(b-1)+1 To determine the number that must be selected to make sure that there are at least 20 who are born in the same month, we apply the contrapositive formula $k(b-1)+1$ and set k = target set size = 12 months, b = desired size = 20 -¿ 20-1 = 19, which means we need at least 229 people.

7. The pigeonhole principle states that if m items are put into n containers and that $m > n$ and there must be at least one of m items in every n container, then there exists at least one container with more than one item.
   From the set $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14$, we can partition it into 7 sets, each with two numbers: $1, 14, 2, 13, 3, 12, 4, 11, 5, 10, 6, 9, 7, 8$. Selecting 8 numbers and 7 partitions. Regardless of which number will be the eighth number, that eighth number can be summed with one of the seven numbers chosen before to sum up to 15. This is because there are only seven partitions (each that sum up to 15), and we must choose 8 numbers ($8 > 7$).

8. a) With 5 books and 20 students in the class and given the restriction that there can at most one child per book and the books are the same, we have $\binom{20}{5}$ ways of distributing books.
   b) With different books and no child gets more than one book, there are $P(20, 5)$ different ways of distributing the books.
   c) If the books are all the same and there are no restrictions on the number of books that can be given to any child, there $\binom{n+m-1}{m-1}$, where $m = 20$ and $n = 5$, so $\binom{24}{19}$ different ways to distribute the books.

9. a) The coefficient is $\frac{25!}{9!2!5!7!2!}$.
   b) There are $\binom{29}{25}$ different terms.

10. For the number of permutations of S where $S = 1, 2, ..., 100$ in which the number 1 is next to at least one even number, there are 50 odd and 50 even numbers is S. Say 1 is in the front, then there are $50 \cdot 98!$ different permutations. If 1 is in the back, then there are $50 \cdot 98!$ different permutations. If 1 is placed at neither the front or end of S, then it is the subtraction of all numbers by permuting 1 surrounded by two odd numbers: $99! - P(49, 2) \cdot 97!$. Total is $7448 \cdot 97! + 99!$.

11. *Proof.* Suppose $\binom{n}{k}$ is the number of k-subsets of some set with n elements where $n, k \in \mathbb{Z}$. To show that for any positive integer n, $\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n+1} + ... + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}$, we can rearrange the left-hand side as $\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$ based on manipulation of a combinatoric identity. Then, we know that the LHS and RHS are equal because of the Binomial Theorem. ■