ASSIGNMENT # 10

MATH 660 , DIFFERENTIAL GEOMETRY

(1) Let $N^k \subset M^n$ be a compact submanifold and

$$\nu_\epsilon = \{v \in T_pM \mid p \in N, \text{ and } v \perp T_pN \text{ with } |v| \le \epsilon\}$$

   (a) Show that there exists an $\epsilon > 0$ such that $\exp^\perp \colon \nu_\epsilon \to M$ (where $\exp^\perp(v) = \exp_p(v)$ for $v \in T_pM$) is a diffeomorphism onto its image. In this case $T_\epsilon(N) = \exp^\perp(\nu_\epsilon)$ is called an ϵ tubular neighborhood of N.
   (b) Prove an analogue of the Gauss Lemma: The geodesic $\exp^\perp(tv)$ meets $\partial T_\delta(N)$ orthogonally for all $\delta \le \epsilon$.
   (c) If $\gamma(t) = \exp^\perp(tv),\ t \le a$ is a geodesic in a tubular neighborhood, show that γ is the shortest connection from N to $\gamma(a)$.

(2) Let $\pi \colon M \to B$ be a submersion with M compact such that the fibers are constant distance apart: If $F_p = \pi^{-1}(p)$ are the fibers of π, then $d(x, F_q) = d(y, F_q)$ for all $x, y \in F_p$ for all $q \ne p$. Show that π is a Riemannian submersion.

   Hint: Choose p, q near each other such that F_q lies in an ϵ tubular neighborhood of F_p. Using problem (1) show that there is a unique minimal geodesic γ_x from x to F_q, which meets F_p as well as F_q orthogonally. Next show that $\gamma_x(t)$ and $\gamma_y(t)$ are contained in the same fiber for all $x, y \in F_p$ and all t.

(3) Show that for every metric on $\mathbb{S}^2$ the cut locus and the first conjugate locus must intersect.

(4) Compute the cut locus of a point, and its homeomorphism type, on a square and on a non square flat torus, and for a flat metric on the Klein bottle.

(5) On what compact surface can there be a metric where the cut locus of a point is a circle. For each surface exhibit a metric where you can compute the cut locus of a point.

(6) Let M be a torus of revolution (with specified radii) and p a point on the outer geodesic. Find out as much as you can say about the cut locus of p. How does the cut point of the outer geodesic relate to its first conjugate point?