ASSIGNMENT # 6

MATH 660 , DIFFERENTIAL GEOMETRY

These excercises explore the geometry of $\mathbb{CP}^n$. We use the following notation: S^1 acts via the Hopf action on $\mathbb{S}^{2n+1}(1) \subset \mathbb{C}^{n+1}$ as $x \to e^{i\theta}x$ for $x \in \mathbb{C}^{n+1}$. Then $\mathbb{CP}^n = \mathbb{S}^{2n+1}/\,\mathrm{S}^1$ with its induced Fubini Study metric. $[x] \in \mathbb{CP}^n$ denotes the equivalence class $x \simeq e^{i\theta}x$ and the tangent space $T_{[x]}\mathbb{CP}^n$ is represented by $[(x,v)]$ with $v \in H_x$, i.e. $\langle v, x\rangle = \langle v, ix\rangle = 0$, and equivalence relationship $(x,v) \simeq (e^{i\theta}x, e^{i\theta}v)$.

(1) Show that $\mathbb{CP}^1$ with its Fubini-Study metric is isometric to $\mathbb{S}^2(\frac{1}{2})$.

(2) Compute the full isometry group of the Fubini Study metric on $\mathbb{CP}^n$.
Hint: Use the fact that isometries preserve sectional curvature.

(3) Show that the only isometric quotient of $\mathbb{CP}^n$ is, up to isometry, obtained when $n = 2k-1$ and by dividing by the involution

$$[z_0, z_1, z_2, z_3, \dots, z_{2k-1}, z_{2k}] \to [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \dots, \bar{z}_{2k}, -\bar{z}_{2k}].$$

(4) Show that $J([x,v]) = [x, iv])$ for $[x,v] \in T_{[x]}\mathbb{CP}^n$ is a well defined complex structure and that $\nabla J = 0$, i.e. the Fubini Study metric is Káhler.

(5) Let M be $\mathbb{CP}^n$ with its Fubini Study metric.
  (a) A linear subspace $V \subset T_{[x]}\mathbb{CP}^n$ is called complex if $JV = V$ and totally real if $JV \perp V$. Show that if V is complex, $\exp_{[x]}(V)$ is totally geodesic and isometric to $\mathbb{CP}^k \subset \mathbb{CP}^n$ with its Fubini Study metric. Show that if V is totally real, $\exp_{[x]}(V)$ is totally geodesic and isometric $\mathbb{RP}^k(1) \subset \mathbb{RP}^n(1) \subset \mathbb{CP}^n$.
  (b) Show that these two types of submanifolds are the only ones which are totally geodesic.

Hint: Use (and prove) the fact that for a totally geodesic submanifold $N \subset M$ we have $R(u,v)w \subset T_pN$ if $u,v,w \in T_pN$. You need to derive a formula for $R(u,v)v$ for the Fubini Study metric.