ASSIGNMENT # 12

MATH 660 , DIFFERENTIAL GEOMETRY

(1) Show that a Riemannian manifold on which the isometry group acts transitively is complete. Furthermore, show that every geodesic loop is a closed geodesic.

Hint for part 2: Prove and use the fact that the vector field associated to a one parameter group of isometries is a Jacobi field when restricted to any geodesic.

(2) Let M be a compact even dimensional manifold with $0 < \sec \leq 1$. Show that the injectivity radius satisfies $i(M) \geq \pi/2$.

(3) Compute the cut locus of a lens space $\mathbb{S}^3/\mathbb{Z}_k$ where $\mathbb{Z}_k \subset S^1 \subset \mathbb{C}$ acts via multiplication on each complex coordinate on $\mathbb{S}^3 \subset \mathbb{C}^2$. Show in particular that it is independent of the point.

(4) Let $N \subset M^n$ be a hypersurface, γ a normal geodesic starting orthogonal to N, and t_0 its first focal point. Furthermore, assume that $\mathrm{Ric}_M \geq (n-1)\delta > 0$. If N is minimal, show that $t_0 \leq \frac{\pi}{2\sqrt{\delta}}$. If it is not minimal, but has mean curvature H, show that $t_0 \leq \frac{1}{\sqrt{\delta}} \arctan(\frac{\sqrt{\delta}}{H})$.

(5) We showed in class that $\mathrm{Ric} \geq (n-1)\delta$ implies that $vol(B_r(p)) \leq vol(B_r^\delta(p_o))$ where $B_r^\delta(p_o)$ is a ball in constant curvature δ space. Show that the analogous statement for $\mathrm{Ric} \leq (n-1)\delta$ does not hold.

Hint: Fubini Study metric on $\mathbb{CP}^n$ and use the computation of the Jacobi fields from Problem (2) in Assignment # 11.

(6) Let γ be a geodesic in M. Show that along the geodesic γ, focal points come before conjugate points.

Left over from last 2 weeks

(7) Compute the cut locus of any(!) flat metric on the 2-torus and the Klein bottle.

(8) Show that the n-dimensional paraboloid $\{x_1, \ldots, x_n, \sum_i x_i^2\} \subset \mathbb{R}^{n+1}$ has positive sectional curvature.

(9) (generalized Morse Schoenberg) Let M^n and $\tilde{M}^n$ be two manifolds with $\sec_{\tilde{M}} \geq \sec_M$ and γ and $\tilde{\gamma}$ be two normal geodesics in M resp. $\tilde{M}$ with $L(\gamma) = L(\tilde{\gamma})$. Show that $ind(\tilde{\gamma}) \geq ind(\gamma)$.

Use this result to prove both directions of Problem (6) in Assignment # 11.

Left over from last week is also Problem (2) and (3) from Assignment # 11.