ASSIGNMENT # 5

MATH 660 , DIFFERENTIAL GEOMETRY

(1) Let M be a Riemannian manifold
   (a) Show that the limit of minimizing geodesics is minimizing in the following sense: If γ_i and γ are geodesics with $\gamma_i(0) \to \gamma(0)$ and $\gamma_i'(0) \to \gamma'(0)$ show that $\gamma_i(t) \to \gamma(t)$ for all t. If γ_i is minimizing, so is γ.
   (b) Show that if M is non-compact and complete, then there exists at least one ray, i.e., a normal geodesic $\gamma\colon [0,\infty) \to M$ with $d(\gamma(0), \gamma(t)) = t$ for all $t \geq 0$.
   (c) A line is a geodesic $\gamma\colon (-\infty, \infty) \to M$ that is minimizing on any interval. Give an example of a non-compact complete manifold that contains no line.

(2) Show that for a compact hypersurface in $M^n \subset \mathbb{R}^{n+1}$, there exists a point p where $\sec(\sigma) > 0$ for all 2-planes $\sigma \subset T_pM$. What can you say if the codimension is bigger than one?

(3) Show that if $M^n \subset \mathbb{R}^{n+1}$ is a compact submanifold with positive sectional curvature, then M is diffeomorphic to $\mathbb{S}^n$.

(4) Show that if M^2 is a surface in $\mathbb{R}^3$ and $p \in M$ with Gauss curvature $G(p) > 0$, then near p the surface M lies one one side of the tangent plane (you also need to define what that means). What if $G(p) \geq 0$. What can you say if $G(p) < 0$ or $G(p) \leq 0$? Same question for a compact hyper surface in $\mathbb{R}^{n+1}$.

(5) Show that on a 3-dimensional manifold the Ricci curvature determines the sectional curvature

(6) Show that for a hypersurface in $\mathbb{R}^n$, positive sectional curvature implies that the curvature operator $\hat{R}\colon \Lambda^2 T_pM \to \Lambda^2 T_pM$ is positive definite.
   Hint: Show that $\hat{R}$ is diagonal in a basis of principal curvatures.