MORITA EQUIVALENCE AND DIFFERENTIABLE GROUPOIDS

QINGYUN ZENG

Contents

1. Biaction groupoid

Let $\mathcal{G}_1 : X_1 \rightrightarrows X_0$ and $\mathcal{G}_2 : Y_1 \rightrightarrows Y_0$ be differentiable groupoids. Suppose P is a principal $(\mathcal{G}_1, \mathcal{G}_2)$ bibundle (bitosor) such that there exists homeomorphisms $\rho_2 : P/X_1 \to Y_0$ and $\rho_1 : P/Y_1 \to X_0$, i.e. we have a principal $(\mathcal{G}_1, \mathcal{G}_2)$ bibundle,

$$\begin{tikzcd}
X_1 \arrow[d, shift right] \arrow[d, shift left] & P \arrow[dl, "\rho_1"] \arrow[dr, "\rho_2"] & Y_1 \arrow[d, shift right] \arrow[d, shift left] \\
X_0 & & Y_0
\end{tikzcd}$$

then we say P is a $(\mathcal{G}_1, \mathcal{G}_2)$ equivalence, or $\mathcal{G}_1$ and $\mathcal{G}_2$ are *Morita equivalent*. We want to prove the following result.

Proposition 1.1. *If $\mathcal{G}_1$ and $\mathcal{G}_2$ are Morita equivalent, then $B\mathcal{G}_1$ and $B\mathcal{G}_2$ are homotopy equivalent.*

First, form the bimodule structure of P, we can construct a $(\mathcal{G}_1, \mathcal{G}_2)$-*biaction groupoid* $X_1 \times_{t, X_0, \rho_1} P \times_{s, Y_0, \rho_2} Y_1 \rightrightarrows P$ with structure maps

$$\begin{aligned}
&s\big((x,p,y)\big) = x & t\big((x,p,y)\big) &= x \cdot p \cdot y \\
&\rho_1(x \cdot p) = s(x) & \rho_1(p) &= t(x) \\
&\rho_2(p \cdot y) = t(y) & \rho_2(p) &= s(y)
\end{aligned}$$

The $\mathcal{G}_1$ and $\mathcal{G}_2$ commutes under this construction, i.e. $(x \cdot p) \cdot y = x \cdot (p \cdot y)$. Let $(x_1, p_1, y_1), (x_2, p_2, y_2)$ be two composable morphisms, then

$$t\big((x_1, p_1, y_1)\big) = x_1 \cdot p_1 \cdot y_1 = s\big((x_2, p_2, y_2)\big) = p_2$$

then

$$t\big((x_2, p_2, y_2)\big) = x_2 \cdot p_2 \cdot y_2 = x_2 x_1 \cdot p_1 \cdot y_1 y_2$$

Note that $s(x_1) = \rho_1(x_1 \cdot p) = \rho_1(x_1 \cdot p \cdot y_1) = \rho_1(p_2) = t(x_2)$. Similarly $t(y_1) = s(y_2)$. Hence the composition is well defined.

Next, let's consider the map ρ_2. Consider the *pull back groupoid* $\rho_2^* \mathcal{G}_2$ along $\rho_2 : P \to Y_0$.

$$\begin{tikzcd}
P \times_{s,Y_0} Y_1 \times_{t,Y_0} P \arrow[r, "\tilde{\rho_2}"] \arrow[d, shift right] \arrow[d, shift left] & Y_1 \arrow[d, shift right] \arrow[d, shift left] \\
P \arrow[r, "\rho_2"] & Y_0
\end{tikzcd}$$

where

$$P \times_{s,Y_0} Y_1 \times_{t,Y_0} P = \{(p, y, p')|p, p' \in P, x \in X_1, \rho_2(p) = s(y), \rho_2(p') = t(y)\} \tag{1.1}$$

with structure maps

$$s\big((p, y, p')\big) = p \quad t\big((p, y, p')\big) = p'$$
$$(p, y, p')^{-1} = (p', y, p)$$

Note that in fact $t\big((p, y, p')\big) = p' = p \cdot y$. Let (p_1, y_1, p_1') and (p_2, y_2, p_2') be composable morphisms, i.e. $p_1' = p_2$ then

$$(p_1, y_1, p_1') \cdot (p_2, y_2, p_2') = (p_1, y_1 y_2, p_2')$$

Since $t(y_1) = \rho_2(p \cdot y) = \rho_2(p_2) = s(y_2)$, the composition is well defined.

Similarly, we define the pull back groupoid $\rho_1^* \mathcal{G}_1 = P \times_{t,X_0} X_1 \times_{s,X_0} P \rightrightarrows P$ and we have the similar diagram

$$\begin{tikzcd}
X_1 \arrow[d, shift right] \arrow[d, shift left] & P \times_{t,X_0} X_1 \times_{s,X_0} P \arrow[l, "\tilde{\rho_1}"'] \arrow[d, shift right] \arrow[d, shift left] \\
P & X_0 \arrow[l, "\rho_1"]
\end{tikzcd}$$

Lemma 1.2. *We claim that*

$$\rho_1^* \mathcal{G}_1 \quad \simeq \quad X_1 \times_{X_0} P \times_{Y_0} Y_1 \rightrightarrows P \quad \simeq \quad \rho_1^* \mathcal{G}_1 \tag{1.2}$$

Note that $P \times_{t,X_0} X_1 \times_{s,X_0} P = \rho_1^* \mathcal{G}_1$, $P \times_{s,Y_0} Y_1 \times_{t,Y_0} P = \rho_2^* \mathcal{G}_2$, *and* $X_1 \times_{X_0} P \times_{Y_0} Y_1 \rightrightarrows P$ *is the bi-action groupoid.*

Proof. Let's consider $P \times_{s,Y_0} Y_1 \times_{t,Y_0} P \rightrightarrows P$ first. By definition,

$$\rho_2^* \mathcal{G}_2 = \{(p, y, p')|p, p' \in P \text{ such that } \rho_2(p) = s(y), \rho_2(p') = t(y)\} \tag{1.3}$$

Now define a map $\phi : X_1 \times_{X_0} P \times_{Y_0} Y_1 \to P \times_{Y_0} Y_1 \times_{Y_0} P$ by

$$\phi : (x, p, y) \mapsto (p, y, x \cdot p \cdot y). \tag{1.4}$$

Let (x_1, p_1, y_1) and (x_2, p_2, y_2) be composable, so $t\big((x_1, p_1, y_1)\big) = x_1 \cdot p_1 \cdot y_1 = s\big((x_2, p_2, y_2)\big) = p_2$. Then

$$\begin{aligned}
\phi\big((x_1, p_1, y_1)\big) \cdot \phi\big((x_2, p_2, y_2)\big) &= (p_1, y_1, x_1 \cdot p_1 \cdot y_1) \cdot (p_2, y_2, x_2 \cdot p_2 \cdot y_2) \\
&= (p_1, y_1, x_1 \cdot p_1 \cdot y_1) \cdot (x_1 \cdot p_1 \cdot y_1, y_2, x_2 x_1 \cdot p_1 \cdot y_1 y_2) \\
&= (p_1, y_1 y_2, x_2 x_1 \cdot p_2 \cdot y_1 y_2) \\
&= \phi\big((x_1, p_1, y_1) \cdot (x_2, p_2, y_2)\big)
\end{aligned}$$

since $t(y_1) = \rho_2(x_1 \cdot p_1 \cdot y_1) = \rho_2(p_2) = s(y_2)$.

Next, define $\psi : P \times_{Y_0} Y_1 \times_{Y_0} P \to X_1 \times_{X_0} P \times_{Y_0} Y_1$ by

$$\psi : (p, y, p') = \big(x(p, y, p'), p, y\big) \tag{1.5}$$

where $x(p, y, p') \in X_1$ is given by

$$t(x) = \rho_1(p) \quad s(x) = \rho_1(p')$$

Note that $x(p, p')$ is uniquely determined by p and p' since the $\mathcal{G}_1$ action on P is free. Let (p_1, y_1, p_1') and (p_2, y_2, p_2') be composable, i.e. $p_1' = p_1 \cdot y_1 = p_2$ and $t(y_1) = \rho_2(p_1 \cdot y_1) = s(y_2)$. Then

$$\psi\big((p_1, y_1, p_1')\big) \cdot \psi\big((p_2, y_2, p_2')\big) =(x_1(p_1, p_1'), p_1, y_1) \cdot (x_2(p_2, p_2'), p_2, y_2)$$

Since $t\big((x_1(p_1, p_1'), p_1, y_1)\big) = x_1 \cdot p_1 \cdot y_1$, $\rho_2(x_1 \cdot p_1 \cdot y_1) = \rho_2(p_1 \cdot y_1) = \rho_2(p_2) = s(y_2)$. Hence they are composable and we get

$$\begin{aligned}(x_1(p_1, p_1'), p_1, y_1) \cdot (x_2(p_2, p_2'), p_2, y_2) =&(x_2x_1, p_1, y_1y_2)\\ =&\psi\big(p_1, y_1y_2, x_2x_1 \cdot p_1 \cdot y_1y_2\big)\end{aligned}$$

By our construction, $t(x_1) = \rho_1(p_1)$ and $s(x_1) = \rho_1(p_1') = \rho_1(x_1 \cdot p_1) = \rho_1(x_1 \cdot p_1 \cdot y_1)$. Similarly $s(x_2) = \rho_1(p_2') = \rho_1(x_2 \cdot p_2 \cdot y_2)$. Since both $\mathcal{G}_1, \mathcal{G}_2$ actions are free, we get $p_1' = x_1 \cdot p_1 \cdot y_1$ and $p_2' = x_2 \cdot p_2 \cdot y_2$. Hence

$$\psi\big((p_1, y_1y_2, x_2x_1 \cdot p_1 \cdot y_1y_2)\big) = \psi\big((p_1, y_1y_2, p_2'\big).$$

Finally, let's show both $\psi \circ \phi$ and $\phi \circ \psi$ are homotopic to identities. First

$$\psi \circ \phi\big((x, p, y)\big) = \psi\big((p, y, x \cdot p \cdot y)\big) = (x, p, y) \tag{1.6}$$

since $t(x) = \rho_1(p)$ and $s(x) = \rho_1(x \cdot p \cdot y) = \rho_1(x \cdot p)$.

Conversely,

$$\phi \circ \psi\big((p, y, p')\big) = (x(p, p'), p, y) = (p, y, p') \tag{1.7}$$

since $(x \cdot p \cdot y = p'$ as verified before.

Therefore, the groupoid $\rho_2^*\mathcal{G}_2$ is equivalent to $X_1 \times_{X_0} P \times_{Y_0} Y_1 \rightrightarrows P$. The proof of $\rho_1^*\mathcal{G}_1$ is similar. □

Denote the groupoid $X_1 \times_{X_0} P \times_{Y_0} Y_1 \rightrightarrows P$ by $\mathcal{P}$. Now it suffices to show $B\mathcal{G}_1 \simeq B\mathcal{P} \simeq B\mathcal{G}_2$. More generally, we have

Lemma 1.3. *Let $\mathcal{G} : X_1 \rightrightarrows X_0$ be a differentiable groupoid. Suppose $p : M \to X_0$ be a surjective submersion, then the classifying space of the pullback groupoid $p^*\mathcal{G}$ is homotopic to $B\mathcal{G}$.*

Proof. First, let's recall a basic result by Quillen:

Theorem 1.4 (Quillen Theorem A)**.** *If C, C' are topological categories and $f : \mathsf{C} \to \mathsf{C}'$ is a continuous functor, if $B(d \downarrow f)$ is contractible for any object d in C', then the induced maps $Bf : B\mathsf{C} \to B\mathsf{C}'$ is a homotopic equivalence.*

We have the following diagram

$$\begin{tikzcd} P \times_{s,Y_0} Y_1 \times_{t,Y_0} P \arrow[r, "\tilde{\rho_2}"] \arrow[d, shift right] \arrow[d, shift left] & Y_1 \arrow[d, shift right] \arrow[d, shift left] \\ P \arrow[r, "\rho_2"] & Y_0 \end{tikzcd}$$

With a little abuse of notations, we denote the function $\rho_2^*\mathcal{G}_2 \to \mathcal{G}_2$ also by ρ_2. Now let's look at the comma category $y \downarrow \rho_2$. The object of $y \downarrow \rho_2$ consists of pairs (x, v) such that $v : y \to \rho_2(x)$ in Y_1. Morphisms between (x, v) and $x', v')$ are maps $w : x \to x'$ such that $\rho_2(w)v = v'$, i.e. the commutative digram

$$\begin{tikzcd} & y \arrow[dl, "v"'] \arrow[dr, "v'"] & \\ \rho_2(x) \arrow[rr, "\rho_2(w)"] & & \rho_2(x') \end{tikzcd}$$

First we will show that $y \downarrow \rho_2$ has an initial object. Let $p \in P$ be any element such that $\rho_2(p) = y$, then we claim $(p, \mathbb{1}_y) \in \mathsf{ob}(y \downarrow \rho_2)$ is initial. Let (x, v) be arbitrary. Since $v : y \to \rho_2(x)$, we have the following diagram

$$\begin{tikzcd} & y \arrow[dl, "\mathbb{1}_y"'] \arrow[dr, "v"] & \\ y = \rho_2(p) \arrow[rr, "v"] & & \rho_2(x) \end{tikzcd}$$

Since ρ_2 is a submersion, we can lift $v : \rho_2(p) \to \rho_2(x)$ to $\tilde{v} : p \to x$ in $\rho_2^*\mathcal{G}_2$. Hence our claim is verified. Next, we have

Theorem 1.5 (Segal 1967)**.** *If C, C' are topological categories and $F_0, F_1 : \mathsf{C} \to \mathsf{C}'$ are continuous functors, and $F : F_0 \to F_1$ is a morphism of functors, then the induced maps $BF_0, BF_1 : B\mathsf{C} \to B\mathsf{C}'$ are homotopic.*

As a consequence, any functor which is a left or right adjoint induced a homotopy on the classifying spaces. Since there is an adjunction $[0] \rightleftarrows \mathsf{C}$, C has an initial object implies $B\mathsf{C}$ is contractible. Therefore, $B(y \downarrow \rho_2)$ is contractible for any $y \in Y_0$, which implies that $B\rho_2^*\mathcal{G}_2$ is homotopic to $B\mathcal{G}_2$. □

Now we see that $B\mathcal{G}_1 \simeq B\mathcal{P} \simeq B\mathcal{G}_2$.

2. APPLICATIONS

Let $(V_1, F_1), (V_2, F_2)$ be two foliations with groupoid holonomy $\mathcal{G}_1$ and $\mathcal{G}_2$. An application of f of C^∞ class is given of V_1/F_1 in V_2/F_2 is given by its graph G_f which is a variety (not necessarily separated) of class C^∞ equipped with a C^∞ application $r : G_f \to V_1$ and $s : G_f \to V_2$, and actions of $\mathcal{G}_1$ and $\mathcal{G}_2$ commutes so that $r : G_f \to V_1$ is a principal groupoid fibration $\mathcal{G}_2$, that is, r is surjective for all $x, y \in G_f$ such that if $r(x) = r(y)$, then there exists a unique $\gamma \in CG_2$ with $x\gamma = y$, and the action of $\mathcal{G}_2$ is proper.

Recall that if we have $x \in G_f, \gamma_2 \in \mathcal{G}_2$ such that $s(x) = r(\gamma_2)$ and $x\gamma_2 \in G_f$ with $r(x\gamma_2) = r(x)$, $s(x\gamma_2) = s(\gamma_2)$. If $\gamma_1 \in \mathcal{G}_1$ with $s(\gamma_1) = r(x)$, then $\gamma_1 x \in G_f$, $r(\gamma_1 x) = r(x)$, $s(\gamma_1 x) = s(x)$, and $\gamma_1 x \gamma_2 = \gamma_1(x\gamma_2)$.

The differentiability of the actions by $\mathcal{G}_1$ and $\mathcal{G}_2$ is defined as follows:

Let $\mathcal{G}_1 \times_{V_1} G_f = \{(\gamma_1, x) | r(x) = s(\gamma_1)\}$ and $G_f \times_{V_2} \mathcal{G}_2 = \{(x, \gamma_2) | r(\gamma_2) = s(x)\}$. These are varieties as $s : \mathcal{G}_1 \to V_1$, $r : \mathcal{G}_2 \to V_2$ are submersions. The actions $(\gamma_1, x) \mapsto \gamma_1 x$ and $(x, \gamma_2) \mapsto x\gamma_2$ are C^∞.

Note that then G_f is foliated by $F = (dr)^{-1}(F_1)$, and the quotient space V_1/F_1 is isomorphic to G_f/F by r and the action s defines a homomorphism of the holonomy groupoid of the laminated graph $G_f, F)$ with values in $\mathcal{G}_2$. Therefore, we can regard f as a homomorphism of the holonomy groupoid of a desingularization of V_1/F_1 in $\mathcal{G}_2$ (or in groupoid of any desingularization).

An equivalent way of defining an application f of V_1/F_1 in V_2/F_2 is given by the notion of a cocycle of $\mathcal{G}_1$ with values in $\mathcal{G}_2$. Let $\{\Omega_i\}$ be an open covering of V_1 with C^∞ functions $g_{ij} : \mathcal{G}_{1,i}^j \to \mathcal{G}_2$, where

$$\mathcal{G}_{1,i}^j = \{\gamma \in \mathcal{G}_1, r(\gamma) \in \Omega_i \text{ and } s(\gamma) \in \Omega_j\} \tag{2.1}$$

such that $g_{ji}(\gamma^{-1}) = g_{ij}(\gamma)^{-1}$ for $\gamma \in G_{1,i}^j$. If $\gamma' \in G_{1,k}^j$ and $s(\gamma) = r(\gamma')$, $s(g_{i,j}(\gamma)) = r(g_{j,k}(\gamma'))$, then

$$g_{i,k}(\gamma\gamma') = g_{i,j}(\gamma) g_{j,k}(\gamma'). \tag{2.2}$$

Hence $\mathcal{G}_1' = \coprod_{i,j} \Omega_{1,i}^j$ is a Groupoid, which is equivalent to $\mathcal{G}_1$, and g is a homomorphism of $\mathcal{G}_1$ to $\mathcal{G}_2$. Note that if $g_{i,j}$ is a cocycle, then $r(g_{i,j}(\gamma))$ only depends on i and $r(\gamma)$. Define $f_i : \Omega_i \to V_2$ by $f_i(r(\gamma)) = r(g_{ij}(\gamma))$. By replacing the covering by refinement, we can assume Ω_i's are open trivialization of transversals $T_{1,i}$ and $f(\Omega_i)$ are included in the transversals $T_{2,i}$. Let $T_1 = \coprod_i T_{1,i}$ and $T_2 = \coprod T_{2,i}$. Let

$$g_{i,j}' : \mathcal{G}_{1,T_{1,j}}^{T_{1,i}} \to \mathcal{G}_{2,T_{2,j}}^{T_{2,i}} \tag{2.3}$$

be the projected transversal of the restriction of g_{ij} to $\mathcal{G}_{1,T_{1,j}}^{T_{1,i}}$. Note that T_j is a transverse of (V_i, F_j), and T_1 is faithful (i.e. meets all leaves of (V_1, F_1), and $g_{i,j}'$ defines a C^∞ homomorphism $g' : \mathcal{G}_{1,T_1}^{T_1} \to \mathcal{G}_{1,T_2}^{T_2}$.)

Proposition 2.1. *An C^∞ application $f : V_1/F_1 \to V_2/F_2$ is defined by the following equivalent data*

1. *A $\mathcal{G}_1$ principal G_f over V_1 with structure groupoid $\mathcal{G}_2$.*
2. *A cocycle of $\mathcal{G}_1$ in $\mathcal{G}_2$.*
3. *A homomorphism $\mathcal{G}_1' \to \mathcal{G}_2'$ where $\mathcal{G}_j'$ is equivalent to $\mathcal{G}_j$.*
4. *A homomorphism $\phi : \mathcal{G}_{1,T_1}^{T_1} \to \mathcal{G}_{1,T_2}^{T_2}$ where T_j is a faithful transverse of (V_j, F_j).*

References

1. Segal, Graeme. Classifying spaces and spectral sequences, *Publications Mathmatiques de l'IHS*. 34 (1968): 105-112
2. Moerdijk, I., Mrcun, J . *Introduction to Foliations and Lie Groupoids*, Cambridge Studies in Advanced Mathematics, (2003)

Current address: Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104

E-mail address: `qze@math.upenn.edu`