

# GROUP THEORY PRACTICE PROBLEMS 1

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## 1. BASIC DEFINITION

**Problem 1.1.** *Prove that if  $G$  is an abelian group, then for all  $a, b \in G$  and all integers  $n$ ,  $(a \cdot b)^n = a^n \cdot b^n$ .*

**Problem 1.2.** *If  $G$  is a group such that  $(a \cdot b)^2 = a^2 \cdot b^2$  for all  $a, b \in G$ , show that  $G$  must be abelian.*

**Problem 1.3.** *If  $G$  is a finite group, show that there exists a positive integer  $N$  such that  $a^N = e$  for all  $a \in G$ .*

**Problem 1.4.** (1) *If the group  $G$  has three elements, show it must be abelian.*  
(2) *Do part (1) if  $G$  has four elements.*  
(3) *Do part (2) if  $G$  has four elements*

**Problem 1.5.** *Show that if every element of the group  $G$  is its own inverse, then  $G$  is abelian.*

**Problem 1.6.** *If  $G$  is a group of even order, prove it has an element  $a \neq e$  satisfying  $a^2 = e$ .*

**Problem 1.7.** *For any  $n > 2$  construct a non-abelian group of order  $2n$ . (Hint: imitate the relations in  $S_3$ .)*

## 2. SUBGROUPS

**Problem 2.1.** *If  $G$  has no nontrivial subgroups, show that  $G$  must be finite of prime order.*

**Problem 2.2.** (1) *If  $H$  is a subgroup of  $G$ , and  $a \in G$ . Let  $aHa^{-1} = \{aha^{-1} | h \in H\}$ . Show that  $aHa^{-1}$  is a subgroup of  $G$ .*

(2) *If  $H$  is finite, what is the order of  $aHa^{-1}$ ?*

**Problem 2.3.** *Write out all the right cosets of  $H$  in  $G$  where*

(1)  *$G = \langle a \rangle$  is a cyclic group of order 10 and  $H = \langle a^2 \rangle$  is the subgroup of  $G$  generated by  $a^2$ .*

(2)  $G$  as in part (1),  $H = \langle a^5 \rangle$  is the subgroup of  $G$  generated by  $a^5$ .

**Problem 2.4.** If  $a \in G$ , define  $N(a) = \{x \in G | xa = ax\}$ . Show that  $N(a)$  is a subgroup of  $G$ .  $N(a)$  is usually called the **normalizer** or **centralizer** of  $a$  in  $G$ .

**Problem 2.5.** If  $H$  is a subgroup of  $G$ , then by the **centralizer**  $C(H)$  of  $H$  we mean the set  $\{x \in G | xh = hx \ \forall h \in H\}$ . Prove that  $C(H)$  is a subgroup of  $G$ .

**Problem 2.6.** The **center**  $Z(G)$  of a group  $G$  is defined by  $Z(G) = \{z \in G | zx = xz \ \forall x \in G\}$ . Prove that  $Z(G)$  is a subgroup of  $G$ . Can you recognize  $Z$  as  $C(T)$  for some subgroup  $T$  of  $G$ ?

**Problem 2.7.** If  $H$  is a subgroup of  $G$ , let  $N(H) = \{a \in G | aHa^{-1} = H\}$ . Prove that

- (1)  $N(H)$  is a subgroup of  $G$ .
- (2)  $H \subset N(H)$ .

We call  $N(H)$  the **normalizer** of  $H$  in  $G$ .

**Problem 2.8.** If  $a \in G$  and  $a^m = e_G$ , prove that the order of  $a$  divides  $m$ .

### 3. HOMOMORPHISMS

**Problem 3.1.** Let  $G$  be a finite abelian group of order  $\text{ord}(G)$  and suppose the integer  $n$  is relatively prime to  $\text{ord}(G)$ . Prove that every  $g \in G$  can be written as  $g = x^n$  with  $x \in G$ . (HINT: Consider the mapping  $\phi : G \rightarrow G$  defined by  $\phi(y) = y^n$ , and prove this mapping is an isomorphism of  $G$  onto  $G$ .)

**Problem 3.2.** Let  $G$  be the dihedral group defined as  $\{x, y | x^2 = e, y^n = e, xy = y^{-1}x\}$ . Prove

- (1) The subgroup  $N = \{e, y, y^2, \dots, y^{n-1}\}$  is normal in  $G$ .
- (2) That  $G/N \simeq W$ , where  $W = \{1, -1\}$  is the group under the multiplication of the real numbers.

**Problem 3.3.** Prove that a group of order 9 is abelian.

**Problem 3.4.** If  $G$  is a non-abelian group of order 6, prove that  $G \simeq S_3$ .

**Problem 3.5.** If  $G$  is abelian and if  $N$  is any subgroup of  $G$ , prove that  $G/N$  is abelian.

**Problem 3.6.** Let  $G$  be the group of all nonzero complex numbers under multiplication and let  $\bar{G}$  be the group of all real  $2 \times 2$  matrices of the form  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ , where not both  $a$  and  $b$  are 0, under matrix multiplication. Show that  $G$  and  $\bar{G}$  are isomorphic by exhibiting an isomorphism of  $G$  onto  $\bar{G}$ .

### REFERENCES

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