ASSIGNMENT # 1

MATH 661 , DIFFERENTIAL GEOMETRY

(1) Let $f\colon M \to B$ be a submersion and g a metric on M such that the fibers are constant distance apart, i.e., $d(x, F_q)$ is independent of $x \in F_p$ for all $p, q \in B$. Show that there exists a metric on B such that f is a Riemannian submersion. Conversely, show that for every Riemannian submersion the fibers are constant distance apart.

(2) Determine the full isometry group of $\mathbb{CP}^n$ with its Fubini Study metric obtained from the Riemannian submersion $\mathbb{S}^{2n+1}(1) \to \mathbb{CP}^n$.

(3) Determine all totally geodesic submanifolds of $\mathbb{CP}^n$ with its Fubini Study metric.

(4) Show that multiplication by i defines a complex structure J on $\mathbb{CP}^n$. Show that $\nabla J = 0$, i.e. if V is parallel along a curve, then JV is parallel as well.

(5) Develop the geometry of $\mathbb{HP}^n$ as the base of the Riemannian submersion $\mathbb{S}^{4n+1}(1) \to \mathbb{HP}^n$. Show that the sectional curvature again lies between 1 and 4 and that $\mathbb{HP}^1$ is isometric to $\mathbb{S}^4(\frac{1}{2})$.

(6) Show that $\mathbb{CP}^n$ and $\mathbb{HP}^n$ are Einstein metrics.