ASSIGNMENT # 3

MATH 660 , DIFFERENTIAL GEOMETRY

(1) (a) Show that the curvature of the metric $dr^2 + f^2(r,\theta)d\theta^2$ is given by
$$-\frac{1}{f}\frac{\partial^2 f}{\partial r^2}$$
.

(b) On a surface we can introduce polar coordinates (r,θ) at p by
$$(r,\theta) \to \exp_p(r\cos\theta\, e_1 + r\sin\theta\, e_2)$$
where e_1, e_2 is an orthonormal basis of T_pM. Show that the metric in polar coordinates takes on the form $dr^2 + f^2(r,\theta)d\theta^2$, and is smooth for $r > 0$.

(c) Show that locally every function on a surface is the curvature of some metric.

(2) (a) Show that the curvature of the metric $e^\phi(dx^2 + dy^2)$ is given by $\dfrac{-\triangle\phi}{2e^\phi}$ where $\triangle\phi = \frac{\partial\phi}{\partial x^2} + \frac{\partial\phi}{\partial y^2}$. These coordinates are called isothermal coordinates and one can show that they always exist locally.

(b) Show that the curvature of hyperbolic space $\frac{dx^2+dy^2}{y^2}$, defined on the upper half space $y > 0$ is -1.

(c) What is the curvature of $\frac{dx^2+dy^2}{y}$, defined on the upper half space $y > 0$? Is this metric complete?

(3) (a) Do DoCarmo page 77 No 1 a)–d) (geodesics on a surface of revolution).

(b) Describe the behaviour of all geodesics on the following surfaces of revolution, where $f(s)$ is the distance to the axis of rotation:
   (i) $f(s) = e^s$
   (ii) $f(s) = \cosh s$
   (iii) $f(s) = \frac{1}{1+s^2}$

(c) In each of the above examples describe all closed geodesics.

(4) (a) Show that in a surface of revolution, if the curve that is rotated is parametrised by arclength and $f(s)$ is the distance to the axis of rotation, then the Gauss curvature is $-f''/f$.

(b) Show that the Gauss curvature of $ds^2 + e^{2s}d\theta^2$ is -1. Can this be realized as the metric on a surface of revolution?

(c) Classify *all* surfaces of revolution (up to congruence) with constant Gauss curvature $+1, 0, -1$ and draw their picture in $\mathbb{R}^3$.

(5) A surface of revolution is obtained by rotating $z = x^n$ around the $z-$axis. For what values of n do the geodesics hit every meridian infinitely many times.

(6) Let A, B be two isometries of M and $p \in M$ a point such that $Ap = Bp$ and $dA_p = dB_p$. Show that $A = B$.