Homotopy theory of derived Lie ∞-groupoids and singular foliations

Qingyun Zeng

University of Pennsylvania

qze@math.upenn.com

June 12, 2019

Overview

Lie groupoids in differential geometry

Lie groupoids are groupoid objects in the category of smooth manifolds Mfd. Let $\mathcal{G} \in \text{LieGrpd}$, then $\mathcal{G} : \mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ where $\mathcal{G}_0, \mathcal{G}_1 \in \text{Mfd}$ and both structure maps are submersions.
Lie goupoids are useful since it is an important tool in studying various areas in geometry and topology, for example, index theory, K-theory, foliations, poisson geometry, gauge theory etc.
Note that Morita equivalences of Lie groupoids corresponds to differentiable stacks.

Foliations

Recall a (regular) **foliation** $\mathcal{F}$ of a manifold M^n is a partition of M into disjoint union of dimension p immersed submanifolds.

Definition (Stefan)

A **singular foliation** $\mathcal{F}$ is a subsheaf of the sheaf of vector fields on M $\mathfrak{X}$ locally finitely generated as an $\mathcal{O}_M$ module and is closed under Lie brackets.

A singular foliation induces a partition of M into (possibly non-equal dimensional) leaves. Note that unlike regular foliations, singular dimensions are not uniquely characterized by leaves.

Lie groupoids and algebroids in foliations

For any foliated manifolds $(M, \mathcal{F})$, we can construct **holonomy groupoids** $\mathcal{G} \rightrightarrows M$ where $\mathcal{G}$ consists of holonomy classes of paths in each leaf.
Holonomy groupoids are Lie groupoids.
Debord(2001) constructed the holonomy groupoids for *almost regular* foliations for which the union of leaves of maximal dimension is a dense open subset of the underlying manifold.
Androulidakis-Skandalis (2006) constructed holonomy groupoids for arbitrary singular foliations, but their generalizations could be highly singular, in particular not Lie groupoids.
On the other hand, every **Lie algebroid** $A \xrightarrow{\rho} TM$ induces a singular foliation $\mathcal{F} = \rho\big(\Gamma(A)\big)$. However, not every foliation is induced from a Lie algebroid.

L_∞-algebroids associated to foliations

Let M be a smooth manifold and $E = (E_{-i})_{0\leq i\leq\infty}$ be a graded vector bundle over M. Let $\mathcal{O}_M$ be the sheaf of C^∞ functions on M. An L_∞-**algebroid** structure on E is a sheaf of L_∞-algebra structures on the sheaf of sections of E with an anchor map $\rho : E_0 \to TM$ such that

1. For $n = 2$ and one of the entry having order 1, we have the Leibniz rule
$$\{x, fy\}_2 = f\{x, y\}_2 + \rho(x)[f]y$$
where $x \in \Gamma(E_0)$, $y \in \Gamma(E)$, $f \in \mathcal{O}_M$. For $n \geq 3$, all brackets $\{\cdots\}_n$ is $\mathcal{O}_M$-linear.
2. E is a dg $\mathcal{O}_M$ module. In addition, $\rho \circ d^{(1)} = 0$.

Remark

L_∞-algebroids are equivalented to non-negatively graded **differential graded manifolds**(NQ-manifolds).

L_∞-algebroids associated to foliations

Let's consider singular foliations $\mathcal{F}$ which admit resolutions by vector bundles $\cdots E_{-2} \to E_{-1} \to \mathcal{F} \to 0$.

Theorem (Laurent-Gengoux, Lavau, Strobl (2018))

For such foliations, there exists a universal L_∞-algebroid whose linear part is the given resolution.

Derived differential geometry

The theory of derived manifolds are developed by Spivak(2008), Borisov-Noel(2011), Joyce(2012), Nuiten(2018) etc.

Definition (Nuiten 2018)

A **derived manifold** is locally modeled on dg $\mathcal{C}^\infty$ rings.

Note that the category of dg $\mathcal{C}^\infty$ rings forms a tractable model category, which presents an ∞-category $\mathcal{C}^\infty\mathrm{Alg}$.
Nuiten(2018) showed that derived L_∞-algebroids forms a *semi-model category*, where weak equivalences are L_∞ quasi-isomorphisms and fibrations are degreewise surjections. Denote the associated ∞-category by $\mathrm{dL}_\infty\mathrm{Algd}$.

Derived Lie ∞-groupoids

Let's consider simplicial objects in derived manifolds $X_\bullet : \Delta^{op} \to \mathrm{dMfd}$. Let $X_\bullet$, $Y_\bullet$ be two simplicial derived manifolds, a map $f : X_\bullet \to Y_\bullet$ is a *Kan fibration* if

$$X_k \to Y_k \times_{Y(\Lambda^i[k])} X(\Lambda^i[k])$$

is a surjective subersion for all $0 \le i \le k$ and $k \ge 1$. We call $X_\bullet$ a **derived Lie ∞-groupoid** if the canonical map $X_\bullet \to *$ is a Kan fibration. Denote the category of derived Lie ∞-groupoid by $\mathrm{dLie}_\infty\mathrm{Grpd}$.

Derived Lie ∞-groupoids

Recall that a **category of fibrant objects**(CFO) consists of two extraordinary classes of morphisms: fibrations and weak equivalences, which gives a weaker version of a model category but still permits many operations of homotopy theory.

Rogers-Zhu(2018) showed that Lie ∞-groupoids in Banach manifolds forms an *incomplete category of fibrant objects*, where the 'incomplete' mainly comes from the fact that the category of Banach manifolds lacks of many limits.

Let's consider dMfd equipped with a Grothendieck topology generated by surjective submesions. We have an embedding of $\mathrm{y} : \mathrm{dLie}_\infty\mathrm{Grpd} \to s\mathrm{PSh}(\mathrm{dMfd})$ induced from the Yonena embedding.

Proposition (Z.)

$\mathrm{dLie}_\infty\mathrm{Grpd}$ *forms a category of fibrant objects, where fibrations are Kan fibrations and weak equivalences are given by stalkwise weak equivalences (in* $s\mathrm{Set}$*).*

Back to singular foliations

We already know how to present a singular foliation by Lie algebroids and L_∞ algebroids. Debord(2008) showed that *almost injective* Lie algebroids are integrable, and in fact they integrates to the holonomy groupoids of the foliations induced by the original algebroids.

Question

What is the corresponding picture for (derived) L_∞-algebroids and Lie ∞-groupoids?

Laurent-Gengoux, Lavau, Strobl (2018) showed that if $(E_\bullet, Q)$ be a universal Lie ∞-algebroid of a singular foliation $\mathcal{F}$, the 1-truncated groupoid of $(E_\bullet, Q)$ is a universal cover of the connected component of the manifold of units of the holonomy groupoid in the sense of Androulidakis-Skandalis.

Integrating L_∞-algebroids

Severa-Siran(2019) used Sullivan's integrating functor

$$\mathrm{Hom}_{cdga}\left(CE(E_\bullet, Q), \Omega(\Delta^n, d)\right)$$

which integrates L_∞-algebroids to Lie ∞-groupoids, which generalizes Henriques(2008)'s results in integrating L_∞-algebras.
This suggests us to apply the integrating functor to derived L_∞-algebroids.
Rogers and Zhu(2018) looked at the integrating function from Lie n-algebras to Lie n-groups and relates the homotopy theory of Lie n-algebras (CFO) to the homotopy theory of Lie n-groups (iCFO). This inspires us to consider how integration functor relates the semi-model structure on $\mathrm{dL}_\infty\mathrm{Algd}$ and CFO on $\mathrm{dLie}_\infty\mathrm{Grpd}$.
Goal: *canonically construct derived Lie ∞-groupoids for 'nice' singular foliations (e.g. admits resolution) which lifts the structure of holonomy groupoids.*

The End