ASSIGNMENT # 2

MATH 660 , DIFFERENTIAL GEOMETRY

(1) (a) If M is a manifold such that around each point there exists a coordinate neighborhood in which g_{ij} is a constant matrix, show that M is locally isometric to $\mathbb{R}^n$.
(b) If there exists a coordinate neighborhood around each point such that $\Gamma_{ij}^k = 0$, show that M is locally isometric to $\mathbb{R}^n$.

(2) (a) Let ∇ be the connection on $\mathbb{R}$ defined by $\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = \frac{\partial}{\partial t}$ where t is the natural coordinate on $\mathbb{R}$. Compute all geodesics of ∇ and give a formula for the geodesic flow $\gamma(t, p, v)$. Show that the connection is not complete.
(b) Show that this also defines a connection on $S^1 = \mathbb{R}/\mathbb{Z}$ and that it is also not complete.

(3) Give an elementary proof that a Riemannian metric on a compact manifold is geodesically complete by first showing that the integral curves of a vector field on a compact manifold are defined for all time, and then show that the geodesic flow on TM is tangent to the unit tangent bundle $UM = \{v \in TM | \, \|v\| = 1\}$ and that UM is compact.

(4) Let γ be a curve on a surface parametrized by arc length, which is fixed by an isometry A, i.e. $A(\gamma(t)) = \gamma(t)$ for all t and $A \neq id$. Show that γ must be a geodesic by showing that it is locally minimizing. Illustrate with some examples.

(5) Let a metric be given by $dx^2 + f^2(x, y)dy^2$ with $f > 0$. Show that the curves $y =$constant are geodesics, in fact minimizing geodesics (for all values of x).

(6) (a) Let Γ act properly discontinously and isometrically on a simply connected Riemannian manifold M. Describe the isometry group of M/Γ in terms of the isometry group of M and the deck group Γ.
(b) Compute the isometry group of the lens space $\mathbb{S}^{2n-1}/\mathbb{Z}_k$ where $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ and $\mathbb{Z}_k$ acts as $(z_1, \dots, z_n) \to (\xi z_1, \dots, \xi z_n)$ for each k-th root of unity ξ.