ASSIGNMENT # 11

MATH 660 , DIFFERENTIAL GEOMETRY

(1) Assume that M is a symmetric space, i.e. for all $p \in M$, there exists an isometry f such that $f(p) = p$ and $d(f)_p = -\operatorname{Id}$. Show that this implies that $\nabla R = 0$, which in turn implies that, along every geodesic γ, $R(X,Y)Z$ is parallel if X, Y, Z are parallel along γ.

(2) Compute the conjugate (with multiplicity) and cut points for the Fubini-Study on $\mathbb{CP}^n$.
Hint: For the conjugate points you may want to use problem (1).

(3) Let G be a compact Lie group.
   (a) Show that it carries a biinvariant metric, i.e. a metric where L_g and R_g are isometries for all $g \in G$.
   (b) Show that its sectional curvature is given by $\sec(u,v) = \frac{\|\,[u,v]\,\|^2}{4\|\,u\wedge v\,\|^2}$ for all $u, v \in \mathfrak{g} \simeq T_eG$.
   (c) Show that it is a symmetric space and relate the conjugate points along $\exp(tu)$ with the eigenvalues of ad_u^2 where $\operatorname{ad}_u(v) = [u,v]$ for $u \in \mathfrak{g}$. Show that all conjugate points have even multiplicity.

(4) In the following Table of 4-manifolds fill in what one knows about each category, i.e. do such (complete) examples exits, if yes give one, and if no explain why.

	$\mathbb{S}^4$	$\mathbb{S}^3 \times \mathbb{S}^1$	$\mathbb{S}^2 \times \mathbb{S}^2$	$\mathbb{RP}^2 \times \mathbb{S}^2$	$\mathbb{RP}^2 \times \mathbb{RP}^2$	T^4	$\mathbb{R}^4$
$\sec > 0$							
$\sec \geq 0$							
$\sec \leq 0$							
$\sec < 0$							
$Ric > 0$							

(5) Let M be a manifold with $\sec \leq \mu$ and γ a geodesic in M. Assume that J is a Jacobi field along γ with $J(0) = J(\frac{\pi}{\sqrt{\mu}}) = 0$. Show that $J(t) = \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu}\, t) E_{J'(0)}(t)$. Does the corresponding statement hold when $\sec \geq \mu > 0$?

(6) Prove the Morse Schönberg comparison theorem:
Let $\gamma \colon [0,a] \to M^n$ be a normal geodesic and $\sec \geq \delta$. Show that if $L(\gamma) = a > k\frac{\pi}{\sqrt{\delta}}$, for some positive integer k, then the index of γ satisfies $\operatorname{ind}(\gamma) \geq k(n-1)$. Thus, by the index theorem, γ has at least $k(n-1)$ conjugate points (counted with multiplicity). What is the corresponding statement for $\sec \leq \delta$?
Hint: Compare the index form of γ with the index form of a geodesic $\bar{\gamma}$ in a space of constant curvature δ.