ASSIGNMENT # 7

MATH 660 , DIFFERENTIAL GEOMETRY

(1) Let $u, v, z \in T_pM$ and $f(t,s)$ a 2-parameter family with $f(0,0) = p,\ f_t(0,0) = u,\ f_s(0,0) = v$. Define $z(s_0) \in T_pM$ as the parallel translate of z around the closed curve which is the image under f of the boundary of the square $0 \le t \le s_0,\ 0 \le s \le s_e$. Show that
$$R(u,v)z = \lim_{s \to 0} \frac{z(s) - z}{s^2}$$
Hint: Define a vector field $Z(t,s)$ along f by parallel translating z first along the t parameter curve $s = 0$ and then along the s parameter curve to $f(t,s)$ and then consider $R(f_t, f_s)Z$.

(2) Let M^2 be a 2-dimensional Riemannian manifold.
  (a) Show that the conjugate points along a geodesic are isolated and have multiplicity one.
  (b) Let γ be a geodesic and assume that $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ. Show that nearby geodesics intersect γ near $\gamma(t_0)$ but after t_0.
  (c) Let $C_k = \{v \in T_pM \mid \exp(v)$ is the k-th conjugate point to p along the geodesic $t \to \exp(tv)\}$. Show that $C_k \subset T_pM$ is a smooth curve. Is $\exp(C_k)$ smooth?

(3) Let $\gamma(t),\ 0 \le t \le t_0$ be a geodesic without self intersections such that $\gamma(t)$ is not conjugate to $\gamma(0)$ for all $t \le t_0$. Show that γ is locally minimizing, i.e. there exists a neighborhood U of $\mathrm{Im}(\gamma)$ such that any curve in U from $\gamma(0)$ to $\gamma(t_0)$ has length $\ge L(\gamma)$ and equal length iff it agrees with γ up to parametrization.

(4) Consider a product metric on $M \times N$.
  (a) Show that the sectional curvature of a 2-plane spanned by (X_1, X_2) and (Y_1, Y_2) is given by the mean value
$$\frac{|X_1 \wedge Y_1|^2 \sec(X_1, Y_1) + |X_2 \wedge Y_2|^2 \sec(X_2, Y_2)}{|(X_1, Y_1) \wedge (X_2, Y_2)|^2}$$
where $\sec(X_1, Y_1) = 0$ if X_1 and Y_1 are linearly dependent.
  (b) Show that $M \times N$ contains a flat totally geodesic surface.
  (c) Show that the product metric on $\mathbb{S}^n(r) \times \mathbb{S}^m(s)$ has $\sec \ge 0$ and $\mathrm{Ric} > 0$ and becomes Einstein for appropriate r and s.

(5) Describe the first conjugate locus to $p \in M$, both as a subset of T_pM and of M, for the product metric on $\mathbb{S}^n(1) \times \mathbb{S}^m(1)$ for all $n, m \ge 1$.

(6) Use Jacobi fields to give a simple proof of the following facts:
  (a) All distance spheres $\partial B_r(p)$ are orthogonal to the geodesics starting at p.
  (b) The Gauss curvature of a metric in polar coordinates $ds^2 = dt^2 + f(t,\theta)^2 d\theta^2$ is given by $-f_{tt}/f$.