ASSIGNMENT # 3

MATH 661 , DIFFERENTIAL GEOMETRY

(1) Let M be a symmetric space with G its isometry group. Then for each point $p \in M$ we get an involutive automorphism of $A_p \colon G \to G \colon g \to \sigma_p \, g\sigma_p$. Show that within the group of automorphisms of G, any two such are inner conjugate. Conversely, any two involutive automorphisms which are inner conjugate give rise to isometric symmetric spaces. What if they are outer conjugate?

(2) Let M be a symmetric space with G the id component of the full isometry group. M is called inner if A_p for some, and hence all, p is inner, i.e. $A = Ad(g)$ for some $g \in G$. What can one say about these elements G?

(3) If $G = \mathrm{SU}(n)$, describe all symmetric space G/H which are inner up to isometry. Show that $\mathrm{SU}(n), n > 2$ has a unique outer automorphism and describe the corresponding symmetric space and that it is not isometric to an inner one. Show that in the case of SU(2) the corresponding "outer" automorphism is actually inner.

(4) Show that $\mathrm{SO}(2n)$ has a unique outer automorphism, but that every automorphism for $\mathrm{SO}(2n{+}1)$ and $\mathrm{Sp}(n)$ is inner, and classify the corresponding symmetric spaces.

(5) Which spaces of constant curvature (1,0, or -1) are symmetric spaces?

(6) to be added...