ASSIGNMENT # 4

MATH 660 , DIFFERENTIAL GEOMETRY

(1) (a) Show that every connected totally geodesic submanifold of $\mathbb{R}^n$ (flat metric) is contained in some linear (affine) subspace.
   (b) Show that every connected totally geodesic submanifold of $S^n(1)$ is contained in some great k-sphere $S^k(1) \subset S^n(1)$.
   (c) If $A : M \to M$ is an isometry, show that $\mathrm{Fix}(A) = \{p \in M | A(p) = p\}$ is a union of totally geodesic submanifolds.
   Hint: Prove and use the fact that $A(\exp_p v) = \exp_{A(p)}(dA_p(v))$.
   (d) Give an example where $\mathrm{Fix}(A)$ has components of different dimensions.

(2) Let M be a surface in $\mathbb{R}^3$.
   (a) Assume that M is ruled, i.e. for every point $p \in M$, there exists a line of $\mathbb{R}^3$ through p which is contained in M. What can you say about the second fundamental form and the Gauss curvature of M ?
   (b) If a surface in $\mathbb{R}^3$ is given by a parametrization $(x_1, x_2) \to f(x_1, x_2) \in \mathbb{R}^3$, show that the Gauss curvature is equal to $\det(h_{ij})/\det(g_{ij})$, where $g_{ij} = \langle f_{x_i}, f_{x_j}\rangle$ and $h_{ij} = \langle f_{x_i,x_j}, N\rangle$ with $N = (f_{x_1} \times f_{x_2})/||f_{x_1} \times f_{x_2}||^2$ a normal to the surface.
   (c) Compute the Gauss curvature of the following examples:
      (i) the tangent surface of a curve $\gamma(t)$ in $\mathbb{R}^3$: $f(t,s) = \gamma(t) + s\gamma'(t)$.
      (ii) the hyperboloid $f(t,s) = (\cos s \mp t \sin s, \sin s \pm t \cos s, t)$.

(3) Consider the hyperbolic metric $ds^2 = \frac{dx^2+dy^2}{y^2} = \frac{dzd\bar{z}}{\mathrm{Im}(z)^2}$ on the upper half space $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \subset \mathbb{C}$ with Gauss curvature -1.
   (a) Show that the fractional linear transformations $z \to \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$, are isometries of ds^2.
   (b) Compute the full isometry group of ds^2. (Hint:What are the isometries fixing i.)
   (c) Show that the geodesics are precisely the circles perpendicular to the x-axis.
   (d) Show that ds^2 is complete.
   (e) What are the distance circles in this metric with center $i \in \mathbb{C}$.

(4) Show that the metric $ds^2 = \frac{dx_1^2+\ldots+dx_n^2}{(1+\frac{c}{4}(x_1^2+\ldots+x_n^2))^2}$ has constant sectional curvature equal to c, and is complete on the domain where it is defined.

(5) Show that if $M^n \subset \mathbb{R}^{n+1}, n \geq 3$, has constant positive sectional curvature, then it is a portion of a round sphere. What if $n = 2$?
   Hint: First show that all points are umbilic, i.e. all principal curvatures are the same.

(6) (a) Show that the catenoid $x(u, v) = (\cosh v \cos u, \cosh v \sin u, v)$ is a surface of revolution and a minimal surface (in fact the only one).
   (b) Show that the helicoid $y(u, v) = (\sinh v \cos u, \sinh v \sin u, u)$ is a ruled surface and a minimal surface (in fact the only one).
   (c) Show that $(\cos t)x + (\sin t)y$ is a one parameter family of minimal surfaces and that they are all isometric to each other.