ASSIGNMENT # 2

MATH 661 , DIFFERENTIAL GEOMETRY

(1) Define the curvature operator $\hat{R}\colon \Lambda^2 T \to \Lambda^2 T$ where T is the tangent space at a point by the formula $\langle \hat{R}(x \wedge y), z \wedge w\rangle = -\langle R(x,y)z, w\rangle$ and show it is a symmetric endomorphism and that $\hat{R} > 0$ (or $\hat{R} \geq 0$), implies that sec > 0 (resp. sec ≥ 0).

Show that the converse is not true by verifying that for the Fubini Study metric on $\mathbb{CP}^n$ with complex structure J and Kaehler form ω one has:

$$\hat{R}(\eta) = \eta + J^*(\eta) - 2\langle \eta, \omega\rangle \omega$$

(2) Show that a homogeneous metric on a homogeneous space is complete.

(3) Define a metric g_t on $\mathbb{S}^{2n+1}$ such that $\mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is a Riemannian submersion with totally geodesic fibers of length $2\pi t$ and metric on the base the Fubini Study metric. Thus g_1 is the round sphere metric. Show that this Riemannian submersion has totally geodesic fibers and that for $0 < t \leq 1$, we have $0 < t \leq \sec \leq 4 - 3t$ (No matter how much we shrink the fiber, the curvature stays positive !).

(4) Show that the Fubini Study metric on $\mathbb{CP}^n$ in its usual (complex) inhomogeneous coordinates has the form

$$ds^2 = \frac{dz d\bar{z}}{1 + |z|^2} - \sum_{\alpha,\beta} \frac{\bar{z}_\alpha z_\beta dz_\alpha d\bar{z}_\beta}{(1 + |z|^2)^2}$$

You need to first make sense out of this. What if $n = 1$?

(5) Show that the Fubini Study metric on $\mathbb{CP}^n$ in normal coordinates (t, θ), t positive real and $\theta \in \mathbb{S}^{2n-1}(1) \subset T_p\mathbb{CP}^n$ is given by

$$ds^2 = dt^2 + \frac{1}{4}\sin^2(2t) d\theta^2_{|F} + \sin^2(t) d\theta^2_{|F^\perp}$$

where F is tangent to the Hopf fibration $\mathbb{S}^{2n-1}(1) \to \mathbb{CP}^{n-1}$ in the tangent space and $F^\perp$ orthogonal to it. You again need to first make sense out of this. Interpret this metric on a ball of radius $\pi/2$.

(6) Show that a Kähler metric with positive holomorphic sectional curvature is simply connected.

(7) Show that a Kähler metric with constant holomorphic sectional curvature 1 is isometric to $\mathbb{CP}^n$ with its Fubini Study metric.