ASSIGNMENT # 8

MATH 660 , DIFFERENTIAL GEOMETRY

(1) Let $N_i \subset M^n$ be two submanifolds of dimension n_i.
  (a) If $N_1 \cap N_2 = \emptyset$, and $c\colon [0,a] \to M$ is a shortest connection from N_1 to N_2, show that c is a geodesic that meets the submanifolds N_i orthogonally at the endpoints.
  (b) Assume that γ is a geodesic from N_1 to N_2 that meets the submanifolds orthogonally. Develop a second variation formula for variational vector fields V along c which are tangent to N_i at the endpoints.
  (c) Show that every such vectorfield V comes from a variation of curves c_s which all start on N_1 and end on N_2.

(2) Let M^n be a complete Riemannian manifold.
  (a) If M has positive sectional curvature, and N_i are 2 compact totally geodesic submanifolds with $n_1 + n_2 \geq n$, show that N_1 and N_2 must intersect.
  (b) Give an example that the dimension assumption in (a) is necessary.
  (c) Show that 2 compact minimal hypersurfaces in a manifold with positive Ricci curvature must intersect.

(3) Fix $p \in M$ and for each unit vector $v \in T_pM$ let $t(v)$ be the first conjugate point along the geodesic $\exp(tv)$. Show that t is continuous.
  Hint: Show it is upper and lower continuous, and for one case use the index form.

(4) Let M be a complete simply connected manifold with non-positive sectional curvature.
  (a) In class we showed that $|d(\exp)_p)_v(w)| \geq |w|$ for all $v, w \in T_pM$. If we have equality, show that $\sec(\gamma'(t), E_w) = 0$ for all $t \leq 1$, where $\gamma(t) = \exp(tv)$ and E_w is the parallel vector field along γ with $E_w(0) = w$. Furthermore, E_w is also a Jacobi field.
  (b) Assume you have a geodesic triangle defined by 3 geodesics $\gamma_i,\ i = 1, 2, 3$ with angles at the vertices given by $\alpha_i,\ i = 1, 2, 3$. In class we showed that $\sum \alpha_i \leq \pi$. If you have equality, show that the triangle is the boundary of a flat totally geodesic surface.
  Hint for (a): If A is symmetric endomorphism with $\langle Av, v\rangle \geq 0$ for all v, and v_0 a vector with $\langle Av_0, v_0\rangle = 0$, show that $Av_0 = 0$.
  Hint for (b): The proof in class was one vertex and angle at a time. Discuss equality at one vertex first, and then use a second vertex as well.

(5) Let M be a simply connected complete Riemannian manifold with non-positive sectional curvature
  (a) If γ is a geodesic and p a point not on γ, show that $f(t) = d^2(p, \gamma(t))$ is a strictly convex function, and that there is a unique point $\gamma(t_0)$ closest to p..
  (b) Show that for all $p \in M$ and $r > 0$, the ball $B_r(p)$ is strictly convex, i.e. any geodesic connecting 2 points in B, completely lies in B.