ASSIGNMENT # 9

MATH 660 , DIFFERENTIAL GEOMETRY

(1) Let $\gamma\colon [0,a] \to M$ be a geodesic and $L(c) = \int_0^a |c'(t)|dt$ be the length of a curve. Let c_s be a proper variation (i.e. end points fixed) with $c_0 = \gamma$, $V = \frac{d(c_s)}{ds}_{|s=0}$ its variational vector field, and $L(s) = L(c_s)$. Show that $L'(0) = 0$ and

$$L''(0) = \int_0^a \langle V_\perp', V_\perp' \rangle - \langle R(V_\perp, \gamma')\gamma', V_\perp \rangle dt$$

where $V_\perp = V - \langle V, \gamma' \rangle \gamma' / |\gamma'|^2$ is the component of V orthogonal to γ'.

(2) Prove the Morse Schönberg comparison theorem:
Let $\gamma\colon [0,a] \to M^n$ be a normal geodesic and $\sec \geq \delta$. Show that if $L(\gamma) = a > k\frac{\pi}{\sqrt{\delta}}$, for some positive integer k, then the index of γ satisfies $\operatorname{ind}(\gamma) > k(n-1)$. Thus, by the index theorem, γ has at least $k(n-1)$ conjugate points (counted with multiplicity). What is the corresponding statement for $\sec \leq \delta$?
Hint: Compare the index form of γ with the index form of a geodesic $\bar{\gamma}$ in a space of constant curvature δ.

(3) Let $N^k \subset M^n$ be a submanifold and $\gamma\colon [0,a] \to M$ a geodesic with $\gamma(0) \in N$, $\gamma'(0)$ orthogonal to N, and $\gamma(a) = p$.
(a) If γ has no focal points for $t \leq a$, show that there exists a neighborhood U of γ such that any curve from N to p, which is completely contained in U, is longer than γ unless it is a reparametrization of γ.
(b) If there exists a focal point $\gamma(t_0)$ with $t_0 < a$, show that there exist nearby curves from N to p which are shorter than γ. (You need to use the formula for the index form I in problem (3) below).
Hint: For part (a) you need to use the Gauss Lemma from problem (1) in the next Assignment # 10.

(4) Recall the following: Let γ be a geodesic as in problem (3) above and define the index form as

$$I(V,W) = \int_0^a \langle V', W' \rangle - \langle R(V, \gamma')\gamma', W \rangle dt + \langle S_{\gamma'}(V), W \rangle_{|t=0}.$$

Then for any variation c_s from N to p with variational vector field V, $E''(0) = I(V,V)$. Show that I is positive definite iff γ has no focal points. (You need to first determine the null space of I.)

(5) Let $N \subset \mathbb{R}^3$ be a paraboloid $z = y^2$. Determine the focal points along any geodesic starting normal to N. Up to what value of t are the parallel hypersurfaces $N_t = \{\exp(tn_p) \mid p \in N\}$ smooth (where n is the inward pointing normal).

(6) Show that if M has sectional curvature $\sec \leq 0$, then a geodesic in M has no focal points.