SE212

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Contents

Chapter 1

Module 2 - Propositional Logic

1.1 Definitions

Proof Theory are methods that manipulate strings of symbols base on pattern matching. There may be multiple ways to prove a formula.

Sound If $p_1, p_2, \ldots, p_n \models q$ (proof), then $p_1, p_2, \ldots, p_n \vdash q$ (valid)

Complete If $p_1, p_2, \ldots, p_n \vdash q$ (valid) then $p_1, p_2, \ldots, p_n \models q$ (proof)

Boolean Valuation is a function v from the set of formulas in propositional logic to the set T_r. Boolean valuation is also called a model or an interpretation.

- $v(false) = F, v(true) = T$
- $v(\neg p) = NOT(v(p))$
- For the connectives:
  - $v(p \wedge q) = v(p) AND v(q)$
  - $v(p \vee q) = v(p) OR v(q)$
  - $v(p \Rightarrow q) = v(p) IMP v(q)$
  - $v(p \Leftrightarrow q) = v(p) IFF v(q)$

Satisfiability If a formula is **satisfiable**, then there exists a Boolean valuation v such that $v(p) = T$

Tautologies A propositional formula p is a **tautology** or **valid** if $v(p) = T$ for all Boolean valuations v. When a formula q is a tautology, we write

$$\vdash q$$

Logical Implication A formula p **logically implies** a formula q iff for all Boolean valuations v, if for all premises $v(p_i) = T$, then $v(q) = T$, meaning

$$p \vdash q \text{ which is equivalent to } \vdash p \Rightarrow q$$

Contradiction A propositional formula a is a **contradiction** if $v(a) = F$ for all Boolean valuations v

Contingent A **contingent** is one that is neither a **tautology** nor a **contradiction**

Logical Equivalence Two formulas are **logically equivalent** iff their equivalence is a tautology i.e. $v(p) = v(q)$ for all v.

$$p \leftrightarrow q \text{ which also means } \vdash p \Leftrightarrow q$$

$\leftrightarrow$ Logical equivalence
$\Leftrightarrow$ Material equivalence

Consistency A collection of formulas is **consistent** if there exists a boolean valuation where all the formulas can be true simultaneously. If a set of formulas in the antecedent of an implication, they can be used to prove a contradiction.

Transformational Proof Rules

Comm	$p \wedge q \Longleftrightarrow q \wedge p$
Lem	$p \vee \neg \Longleftrightarrow true$
Contr	$p \wedge \neg p \Longleftrightarrow false$
Impl	$p \Rightarrow q \Longleftrightarrow \neg p \vee q$
Idemp	$p \wedge p \Longleftrightarrow p$
Neg	$\neg(\neg p) \Longleftrightarrow p$
Simp1	$p \wedge true \Longleftrightarrow p$
Assoc	$p \wedge (q \wedge r) \Longleftrightarrow (p \wedge q) \wedge r$
Dm	$\neg(p \wedge q) \Longleftrightarrow \neg \vee \neg q$
Distr	$p \vee (q \wedge r) \Longleftrightarrow (p \vee q) \wedge (p \vee r)$
Contrapos	$p \Rightarrow q \Longleftrightarrow \neg q \Rightarrow \neg p$
Equiv	$p \Leftrightarrow q \Longleftrightarrow (p \Rightarrow q) \wedge (q \Rightarrow p)$
Simp2	$p \vee (p \wedge q) \Longleftrightarrow p$

Rules

1. **Rule of substitution** substituting an equivalent for a subformula
2. **Rule of transitivity** If $p \Longleftrightarrow q$ and $q \Longleftrightarrow r$, then $p \Longleftrightarrow r$

Literal is a propositional letter or the negation of a proposition letter

Conjunctive Normal Form is a conjunction (AND) of clauses, where a clause is a dusjunction (OR) of literal or a single literal

Disjunctive Normal Form is a disjunction (OR) of clauses, where a clause is a conjunction (AND) of literals or a single literal

Argument is a collection of formulas, one of which reffered to as the conclusion, is justified by the others, referred to as the premises.

Deductive means the conclusion of an argument is wholly justified by the premises

Inductive arguments conclude more general new knowledge from a small number of particular facts or observations.

Valid means all Boolean valuations where the premises have the value T, the conclusion has the truth value T

Invalid means there is one Boolean valuation where premises is true but the conclusion is false

Natural Deducation is a collection of rules called inference rules which allows us to infer new formulas from given formulas.

Inference rule is a primitive valid argument form. Each inference rule enables the elimination or the introduction of a logical connective

Syllogism is a kind of logical argument that applies deductive reasoning to arrive at a conclusion based on two or more propositions that are asserted or assumed to be true.

Summary of Natural Deduction Rules

- p, q (and i) $p \wedge q$
- pq (and e) p
- p (or i) $p \vee q$
- $p \vee r$ (cases) $casep\{...q\}caser\{...q\}q$
- assume r { ... q } (imp i) $r \Rightarrow q$
- $p \Rightarrow q, p$ (imp e) q
- disprove r { ... false } (raa) $\neg r$
- $p, \neg p$ (not e) q
- p (not not i) $\neg\neg p$
- $\neg\neg p$ (not not e) p
- $p \Rightarrow q, q \Rightarrow p$ (iff i) $p \Leftrightarrow q$
- $p \Leftrightarrow q$ (iff e) $p \Rightarrow q$
- $p \vee q$, $\neg p$ (or e) q
- (lem) $p \vee \neg p$

Summary of Semantic Tableaux Rules

- $p \wedge q$ (and nb) p, q
- $\neg(p \wedge q)$ (not and br) 1. $\neg p$ 2. $\neg q$
- $p \vee q$ (or br) 1. p 2. q
- $\neg(p \vee q)$ (not or nb) $\neg p$, $\neg q$
- $p \Rightarrow$ (imp br) 1. $\neg p$ 2. q
- $\neg(p \Rightarrow q)$ (not imp nb) p, $\neg q$
- $\neg\neg p$ (not not nb) p
- $p \Leftrightarrow q$ (iff br) 1. $p \wedge q$ 2. $\neg p \wedge \neg q$
- $\neg(p \Leftrightarrow q)$ (not iff br) 1. $p \wedge \neg q$ 2. $\neg p \wedge q$

Chapter 2

Module 3 - Predicate Logic

2.1 Definitions

Predicate is a symbol denoting the meaning of an attribute of an object or the meaning of a relationship between two or more objects

Unary, binary, N ary predicate takes 1, 2 or n number of objects as arguments

Constant is a symbol denoting a particular object

Quantifiers

forall $\forall$, predicate holds for all elements (every, all, for all). The formula must be true for all substitutions of an object in the domain for the quantified variable

exists $\exists$, predicate holds for some element (at least one, some, there exists). For an existentially quantified variable, the formula must be true for some substitution of an object in the domain for the quantified variable

Terms

- Every constant is a term
- Every variable is a term
- If $t_1, t_2, t_3...t_n$ are terms and f is an n ary function symbol, then $f(t_1, t_2, ...t_n)$
- Nothing is a term
- Terms describe objects

Scope of a quantifier is a subformula over which the quantifier applies in the given formula. Without brackets, we assume the scope of the quantifier extends to the right end of the formula

Bound Variable is an occurence of a variable that is within the scope of a quantifier over that variable. A variable is bound to the closest quantifier to the left of its name whose scope it is within. All variable occurrences that are bound to the same quantifier represent the same object

Free , a variable is free if it does not fall within the scope of a quantifier for that variable

Closed A wff is closed if it contains no free variables (we work with closed wff)

Type is a set of objects describing the possible values of a variable. These values are constant of that type. We assume types are non empty, i.e. at least one element of any type exists. We won't need types unless explicitly instructed.

Satisfiability A predicate logic is satisfiable iff there exists an interpretation I that satisfies the formula. An interpretation I satisfies a predicate logic formula A iff $M[[A]] = T$ when evaluted I

Tautology A predicate logic formula is a tautology iff every interpretation satisfies the formula, $\vdash$ P

Variable capture means a free variable becomes bound after the substitution

Genuine varaible is a free variable such that the universal quantification of it yields a formula that is true. It represents any value

Unknown varaible is a free variable such that the existential quantification of it yields a formula that is true. It represents a specific (but unknown) value

Chapter 3

Module 4 - Theories

3.1 Definitions

Enthymeme is an argument that contains a hidden premise, that is, an argument that contains unstated premises that are obviously true

Theory

- A set of statements or principles devised to explain a group of facts or phenomena, especially one that has been repeatedly tested or is widely accepted and can be used to make predictions about natural phenomena
- A set of theorems that constitute a systematic view of a branch of mathematics

Leibnizs Law if $t_1 == t_2$ is a theorem, then so is $P[t_1/x] \Leftrightarrow P[t_2/x]$
Lebinizs Law is generally referred to as the ability to substitute equals for equals

Normal interpretations are interpretations in which the symbol == is interpreted as equality on the objects of the domain.

Induction

- $P(0)$ is called the base case
- $P(k)$ is called the induction hypothesis
- $\forall k \wedge P(k) \Rightarrow P(suc(k))$ is called the induction proof or inductive step

Deduction is showing a conclusion follows frmo the stated premises using rules of inference

Philosophical induction is the process of deriving general principles from particular observations

Recursive function is one that is defined in terms of itself and certain terminating clauses

Chapter 4

Module 5 - Sets

4.1 Definitions

Set is a collection of element or members

Set enumeration List the item in the set

Set comprehension Define a set by using a predicate

Z notation $\{\langle term \rangle \bullet \langle signature \rangle | \langle predicate \rangle\}$

Term is a term in predicate logic i.e an expression using variables and functions that returns an object. Term can be omitted if it is just a variable and we have a signature

Signature lists the variables used in term and their types. If we are not using types, the signature can be omitted, but we need one of the term or the signature

Predicate is any wff formula in predicate logic with the variables used in term as free variables in the formula. If no predicate is needed, i.e the formula true would be used, then it can be omitted

Empty set is written $\emptyset$

Universal Set consists of all the objects of concern in any discussion. Often, the universal set is an appropriate choice for a type. It is denoted as U

Single set is a set consisting of only one element

Power set of a set is the set of all of its subsets. P is the function that returns the power set of a set. Usually we leave off the brackets around the argument of this function.

Set Theory Summary

- Types as sets: $forall x : B \bullet P(x) \Leftrightarrow forall x \bullet x \in B \Rightarrow P(x)$
- Set comprehension: $x \in \{y \bullet y : S | P(y)\} \Leftrightarrow x \in S \wedge P(x)$
- Empty set: $\forall x \bullet \neg(x \in \emptyset)$
- Set equality: $D == B \Leftrightarrow (\forall x \bullet x \in D \Leftrightarrow x \in B)$
- Subset: $D \subseteq B \Leftrightarrow (\forall x \bullet x \in D \Rightarrow x \in B)$
- Proper Subset: $D \subset B \Leftrightarrow D \subseteq B \wedge \neg(D == B)$
- Power Set: $PD == \{B | B \subseteq D\}$
- Set Union: $D \cup B == \{x | x \in D \vee x \in B\}$
- Set intersection: $D \cap B == \{x | x \in D \wedge x \in B\}$
- Absolute Complement: $D' == \{x | x \in U \wedge \neg(x \in D)\}$
- Set difference: $D - B == \{x | x \in D \wedge \neg(x \in B)\}$

Disjoint Two sets D and B are disjoint if their intersection is empty