1 Endogenous Growth

1.1 AK Model

$$F(K,L) = AK, \quad A > 1$$

$$f(K/L) = Ak, \quad A > 1$$

Remarks:

- Constant marginal product of capital violates Inada condition. The return to accumulation never goes to zero.
- There is no labor in the production function.
- The factor payments to capital per worker exhaust output. $rk = Ak = y$
- It is the reduced form of a class of models.

An AK Model with CES preferences:

$$max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

s.t.

$$c_t + k_{t+1} - (1-\delta)k_t = Ak_t$$

k_0 given.
Euler Equation:

$$c_t^{-\sigma} = \beta(A + 1 - \delta)c_{t+1}^{-\sigma}$$

Hence,

$$\frac{c_{t+1}}{c_t} = [\beta(A+1-\delta)]^{\frac{1}{\sigma}} := g_c$$

$$g_c > 1$$

$$\frac{c_t}{k_t} = \{(A+1-\delta) - [\beta(A+1-\delta)]^{\frac{1}{\sigma}}\} \quad \in \quad (0,1)$$

$$[\beta(A+1-\delta)]^{\frac{1}{\sigma}} > 1 > [\beta(A+1-\delta)]^{\frac{1-\sigma}{\sigma}}$$

$$c_0 = (A+1-\delta-g_c)k_0 \quad > \quad 0$$

- There is no steady state and optimal consumption growth is constant.
- Balanced growth is consistent with optimality.
- From feasibility condition, g_i, g_k, g_y are equal to g_c.

$$\frac{c_t}{k_t} + \frac{k_{t+1}}{k_t} = A + 1 - \delta$$

- If g_k is increasing, then $g_c < g_k$, TVC violates.
- If g_k is decreasing, then $g_c > g_k$, nonegativity of c violates.
- There is no transitional dynamics. A Balanced Growth Path is immediately attained.
- The lifetime utility on BGP is also bounded. A sufficient condition for this is

$$\beta[\beta(A + 1 - \delta)]^{\frac{1-\sigma}{\sigma}} < 1$$

Optimal sustained growth is possible on AK model without resorting to exogenous technological change. The main feature is that the marginal productivity of capital does not diminish. Growth rate in this model is a function of undelying parameters of preferences and technology.

1.2 Lucas Human Capital Model

1.2.1 One Sector

This model attempts to think about how decisions about knowledge accumulation lead to endogenous growth. Labor is replaced in the production function by a reproducible asset, human capital.

Social Planner's problem

$$max \quad \sum_{t=0}^{\infty} \beta^t \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

s.t.

$$c_t + K_{t+1} + H_{t+1} - (1 - \delta_K)K_t - (1 - \delta_H)H_t = K_t^{\alpha} H_t^{1-\alpha}$$

1.2.2 Two Sector

Social Planner's Problem

$$max \quad \sum_{t=0}^{\infty} \beta^t \frac{c^{1-\sigma}-1}{1-\sigma}$$

s.t.

$$c_t + K_{t+1} - (1-\delta_K)K_t = K_t^{\alpha}(\phi_t H_t)^{1-\alpha}$$

$$H_{t+1} - H_t = A(1-\phi_t)H_t$$

This problem is very similar to the social planners's problem of the Romer model in the next subsection. We will show how to solve this kind of model in detail there.

1.3 The Romer Model: Expanding Product Variety

Notes from Angeletos, MIT Econ 451 and Professor Betts

1.3.1 Model Setup

- There are three sectors: one for the final good sector, one for intermediate goods, and one for R&D.
- The final good sector is perfectly competitive and thus makes zero profits. Its output is used either for consumption or as input in each of the other two sectors.
- The intermediate good sector is monopolistic. There is product differentiation. Each intermediate producer is a quasi-monopolist with repect to his own product and thus **enjoys positive profits**. To become an intermediate producer, however, you must first acquire a blueprint from the R&D sector. A blueprint is the technology or know-how for transforming final goods to differentiated intermediate inputs.
- The R&D sector is competitive. Researchers produce blueprints. Blueprints are protected by perpetual patents. Innovators auction their blueprints to a large number of potential buyers, thus absorbing all the profits of the intermediate good sector. But there is free entry in the R&D sector, which drive net profits in that sector to zero as well.

1.3.2 Final Good Sector

The final good sector is perfectly competitive. Firms are price takers. The technology for final goods is given by a neoclassical production function of labor L and a composite factor χ:

$$Y_t = F(\chi_t, L_{1t}) = A(L_{1t})^{1-\alpha}(\chi_t)^{\alpha}.$$

The composite factor is given by a CES aggregator of intermediate inputs:

$$\chi_t = [\int_0^{N_t} (X_{t,j})^{\epsilon} dj]^{1/\epsilon},$$

where N_t denotes the number of different intermediate goods available in period t and $X_{t,j}$ denotes the quantity of intermediate input j employed in period t.

When we assume $\epsilon = \alpha$, which implies

$$Y_t = A(L_{1t})^{1-\alpha} \int_0^{N_t} (X_{t,j})^{\alpha} dj. \tag{1.1}$$

With this assumption, the marginal product of each intermediate input is independent of the quantity of other intermediate inputs:

$$\frac{\partial Y_t}{\partial (X_{t,j})} = \alpha A (\frac{L_{1t}}{X_{t,j}})^{1-\alpha}.$$

More generally, intermediate inputs could be either complements or substitutes.
We will interpret intermediate inputs as capital goods and therefore let aggregate capital be given by the aggregate quantity of intermediate inputs:

$$K_t = \int_0^{N_t} X_{t,j} dj.$$

If $X_{t,j} = X$ for all j and t, then

$$Y_t = A{L_{1t}}^{1-\alpha} N_t X^{\alpha}$$

$$K_t = N_t X_t,$$

implying

$$Y_t = A(N_t L_{1t})^{1-\alpha}(K_t)^{\alpha}. \tag{1.2}$$

or in intesive form, $y_t = A(N_t)^{1-\alpha}(k_t)^{\alpha}$. Therefore, to the extent that all intermediate inputs are used in the same quantity, the technology is linear in knowledge N and capital K. Therefore, if both N and K grow at a constant rate, as we will show to be the case in equilibrium, the economy will exhibit long run growth, as **in an AK model**.

Final good firms solve

$$max \quad Y_t - w_t L_{1t} - \int_0^{N_t} (p_{t,j} X_{t,j}) dj$$

where w_t is the wage rate and $p_{t,j}$ is the price of interediate good j.

FOCs:

$$w_t = \frac{\partial Y_t}{\partial L_{1t}} = (1-\alpha)\frac{Y_t}{L_{1t}} = (1-\alpha)A\int_0^{N_t} (\frac{X_{t,j}}{L_{1t}})^\alpha dj$$

if we assume symmetry,

$$w_t = (1-\alpha)AN_t(\frac{X_{t,j}}{L_{1t}})^\alpha$$

and

$$p_{t,j} = \frac{\partial Y_t}{\partial X_{t,j}} = \alpha A(\frac{L_{1t}}{X_{t,j}})^{1-\alpha}$$

for all j.

1.3.3 Intermediate Good Sector

The intermediate good sector is monopolistic. Firm understand that they face a downward sloping demand for their output. The producer of intermediate good j solves

$$max \quad \Pi_{t,j} \quad = \quad p_{t,j} X_{t,j} - \theta(X_{t,j})$$

subject to the demand curve

$$X_{t,j} = L_{1t}(\frac{\alpha A}{p_{t,j}})^{\frac{1}{1-\alpha}}$$

where $\theta(X)$ represents the cost of producing X in terms of final-good units. The price numeraire is the final good. We will let the cost funciton be linear:

$$\theta(X) = X.$$

The implicit assumption behind this linear specification is that technology of producing intermediate goods is identical to the technology of producing final goods. Equivalently, you can think of intermediate good producers buying final goods and transforming them to intermediate inputs. What gives them the know-how for this transformation is precisely the blueprint they hold.

FOCs:

$$p_{t,j} = \frac{1}{\alpha}$$

for the optimal price, and

$$X_{t,j} = A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L_1$$

for the optimal supply.

The resulting maximal profits are

$$\Pi_{t,j} = (p-1)xL_1 = (\frac{1-\alpha}{\alpha}) A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L_1$$

The price is higher than the marginal cost, the gap representing the mark-up that intermediate-good firms charge to the final good firms.

1.3.4 The Innovation Sector

Romer(1990) model. Building on the Shoulders' of Giants. L is the total labor time and constant and

$$L_{1t} = \phi_t L$$

is devoted to final goods sector at t.

$$L_{2t} = (1 - \phi_t)L$$

is devoted to the innovation sector. The production for the innovation secotor:

$$N_{t+1} - N_t = \kappa L_{2t} N_t$$

Designing a new variety requires a cost of $\frac{1}{\kappa N_t}$ of labor. Thus, the cost of innovation

$$\frac{w_t}{\kappa N_t}.$$

On the other hand, the value of these new blueprints is $V \Delta N$, where $V = \pi L_{1t}/R$. The net profit for a research firm are thus given by

$$profit_{R\&D} = (V - \frac{w_t}{\kappa N_t}) \Delta N$$

Free entry in the sector of producing blueprints imposes $profit_{R\&D} = 0$, or equivalently

$$V = \frac{w_t}{\kappa N_t}$$

1.3.5 Households

Households solve

$$max \quad \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

s.t.

$$c_t + a_{t+1} = w_t L + (1 + R_t) a_t$$

Euler:

$$\frac{c_{t+1}}{c_t} = [\beta(1 + R_{t+1})]^{1/\sigma}$$

1.3.6 Resource Constraints

Final goods are used either for consumption by households C_t, or for production of intermediate goods in the intermediate sector $K_t = \int_0^{N_t} X_{t,j} dj$. It is not used in the production for new blueprints in the innovation sector in this version of model. The resourse constraint of the economy is therefore given by

$$C_t + K_{t+1} - (1 - \delta) K_t = Y_t$$

where $C_t = c_t L$ and $Y_t = A(N_t \phi_t L)^{1-\alpha} (K_t)^{\alpha}$.

Labor are separated between innovation sector and final goods production sector.

$$L_{1t} + L_{2t} = L$$

1.3.7 Characterizing the Competitive Equilibrium

Notes from CREI, Barcelona

Guess and Verify the existence of a Balance Growth Path equilibrium. Assume

$$\frac{c_{t+1}}{c_t} = \frac{N_{t+1}}{N_t} = \frac{K_{t+1}}{K_t} = 1 + \gamma.$$

From Innovation sector,

$$L_{2t} = \frac{N_{t+1} - N_t}{\kappa N_t} = \frac{\gamma}{\kappa}$$

$$L_{1t} = L - L_{2t} = L - \frac{\gamma}{\kappa}$$

Combining the formula for the value of innovation with free entry condition we infer

$$\Pi/R = V = w_t/\kappa N_t.$$

It follows that the equilibrium interest rate is

$$R = \kappa \Pi N_t/w_t = \frac{\alpha\kappa}{A} L_{1t} = \frac{\alpha\kappa L - \alpha\gamma}{A}.$$

From Euler Equation:

$$1 + \gamma = [\beta(1 + R)]^{1/\sigma}$$

γ can be solved.

In continuous case the solution is

$$\gamma = \frac{\kappa\alpha L - \rho}{\alpha + \sigma}.$$

ρ is the continuous discount factor.

1.3.8 Characterizing the Social Planner's Problem

Notes from Professor Betts's Notes

$$max \quad \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1 - \sigma}$$

subject to (for consistency with Professor Betts's class note, A = 1)

$$C_t + K_{t+1} - (1 - \delta)K_t = (N_t \phi_t L)^{1-\alpha} (K_t)^{\alpha}$$

$$N_{t+1} = N_t + \kappa(1 - \phi_t) L N_t$$