Chapter | Eleven

Bidding Rings

We have looked at an assortment of models with varying features: the auction format, the valuation structure, the informational structure, and so on. The one common feature across all these models has been the assumption that bidders make their decisions independently; that they do not act in a concerted way. In other words, the bidders were assumed to be engaged in a noncooperative game. This chapter explores some issues that arise when a subset, or possibly all, of the bidders act collusively and engage in bid rigging with a view to obtaining lower prices. The resulting arrangement—a *bidding ring*—resembles an industrial cartel and many of the issues surrounding cartels resurface in this context. How can the cartel enforce the agreed upon mode of behavior? How are the gains from collusion to be shared? How should economic agents on the other side of the market—in this case, the seller—respond to the operation of the cartel?

While bidding rings are illegal, they appear to be widely prevalent. Investigations of collusion in auctions constitute a significant component of antitrust activity: over three-quarters of the criminal cases filed in the 1980s under Section 1 of the Sherman Act—the section pertaining to trusts and the illegal restraint of trade—concerned auction markets.

Theoretical models of collusion among bidders involve a mix of cooperative and noncooperative game theory. Difficulties of both a conceptual and technical nature arise, and in order to illustrate the underlying issues in the simplest context, we return to the independent private values model.

Our setup is the same as in Chapter 5—specifically, we assume that bidders' values are private and independently distributed but allow bidders to be asymmetric. Specifically, suppose that each bidder's value is a random variable X_i which is distributed according to the cumulative distribution function F_i over some common interval $[0, \omega]$. The assumption of a common interval is made only for notational convenience and is easily relaxed.

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Even if bidders were *ex ante* symmetric, the presence of a ring would naturally introduce asymmetries between bidders who are members of the ring and those who are not. As we have seen, equilibrium behavior in asymmetric first-price auctions does not lend itself to ready analysis. The equilibrium behavior of bidders in asymmetric second-price auctions, with or without a ring, is, however, quite transparent and allows us to focus our attention on a new set of issues surrounding collusion among bidders.

11.1 COLLUSION IN SECOND-PRICE AUCTIONS

Let $\mathcal{I} \subseteq \mathcal{N}$ be the set of bidders in the *bidding ring* or *cartel*. Without loss of generality, we will suppose that $\mathcal{I} = \{1, 2, \ldots, I\}$ and denote by $\mathcal{N} \setminus \mathcal{I} = \{I+1, I+2, \ldots, N\}$ the set of bidders outside the ring.

For any set of bidders $\mathcal{S} \subseteq \mathcal{N}$, let the random variable $Y_1^{\mathcal{S}}$ denote the highest of the values of the bidders in $\mathcal{S}$. It is in the interests of the bidding ring to make sure that the object goes in the hands of the ring member who values it the most, provided, of course, that it manages to win the object. In other words, in order to maximize profits the bidding ring must allocate the object efficiently among its members. But information regarding their values is privately held by the members and in order to function effectively, a bidding ring needs to gather this information and then to divide the gains from collusion among its members. How, and whether, both tasks can be accomplished is a key question. In what follows, however, we temporarily put aside the question of the internal functioning of a ring—returning to it later—and, assuming that it functions effectively, seek to identify the resulting gains and losses to the various parties.

THE GAINS AND LOSSES FROM COLLUSION

The presence of a ring in a second-price auction does not affect the behavior of bidders who are not members of the ring. It is still a weakly dominant strategy for a bidder $j \notin \mathcal{I}$ to bid his or her value X_j. It is also weakly dominant for the ring to submit a bid equal to the highest value among its members—that is, $Y_1^{\mathcal{I}}$. Equivalently, we may think of the ring as being represented at the auction by the member with the highest value in the ring. The other members submit bids of 0, or if there is a reserve price, they bid at or below this price.

A bidding ring generates profits for its members, of course, by suppressing competition. Specifically, instead of N effective bids, only $N - I + 1$ effective bids are submitted since only one member of the cartel—the one with the highest value in the ring—submits a serious bid by bidding according to his or her value. The rest submit nonserious bids by bidding at or below the reserve price. The ring's profits come from the fact that, in certain circumstances, the price paid by a winning bidder from the ring is lower than it would be if there were no ring. Specifically, suppose that one of the ring members $i \in \mathcal{I}$ has a value X_i that is the highest of all bidders in the ring or otherwise—that is, $X_i = Y_1^{\mathcal{N}}$. Assuming that $X_i > r$, in the absence of a ring, this bidder would pay an amount equal to

$P_i = \max\left\{Y_1^{\mathcal{N}\setminus i}, r\right\}$ for the object. But if he were part of a functioning ring, then his fellow members in $\mathcal{I}$ would bid at most r, so he would pay only

$$\widehat{P}_{\mathcal{I}} \equiv \max\left\{Y_1^{\mathcal{N}\setminus\mathcal{I}}, r\right\}$$

Thus, the expected payments of ring members are lower than they would be if the ring did not exist.

For a fixed reserve price r, let $m_i(x_i)$ denote the expected payment of bidder i with value x_i when there is no ring operating and all bidders behave noncooperatively. Likewise, let $\widehat{m}_i(x_i)$ denote i's expected payment when there is a ring. As we have argued, for all ring members $i \in \mathcal{I}$, and for all $x_i > r$, $\widehat{m}_i(x_i) < m_i(x_i)$. Then

$$t_i(x_i) \equiv m_i(x_i) - \widehat{m}_i(x_i) \tag{11.1}$$

represents the contribution of bidder i to the ring's expected profits when his value is x_i. The total *ex ante* expected profits of the ring—the spoils of collusion, as it were—amount to

$$t_{\mathcal{I}} \equiv \sum_{i\in\mathcal{I}} E\left[t_i(X_i)\right]$$

What about bidders who are not members of the ring? In market contexts, the presence of a cartel usually exerts a positive externality on firms who are not part of the cartel. Cartels result in higher prices and these benefit all sellers. In the present context, however, the bidding ring exerts no externality whatsoever on bidders who are not part of the ring. First, the probability that a bidder who is not a member of the ring will win the object is the same whether or not the ring is functioning; in both cases it is just the probability that she has the highest value among all bidders. Furthermore, the price that a bidder $j \notin \mathcal{I}$ would pay in the event that she wins is

$$\max\left\{Y_1^{\mathcal{I}}, Y_1^{\mathcal{N}\setminus\mathcal{I}\setminus j}, r\right\} = \max\left\{Y_1^{\mathcal{N}\setminus j}, r\right\}$$

the same as the price she would pay if there were no ring. Since for all bidders who are not part of the ring, neither the probability of winning nor the price upon winning is affected, the expected payments in the two situations are the *same*—that is, for all $j \notin \mathcal{I}$ and x_j,

$$\widehat{m}_j(x_j) = m_j(x_j)$$

For the same reasons, the profits of these bidders are also unaffected. Since the profits of bidders outside the cartel are unaffected by its presence, the gains accruing to the cartel as a whole are equal to the loss suffered by the seller.

This reasoning also leads to the conclusion that the gains from collusion increase as the size of the ring increases. Specifically, consider the addition of another member to the ring $\mathcal{I}$ and, without loss of generality, suppose that the new ring consists of bidders in $\mathcal{J} = \{1, 2, \ldots, I, I+1\}$. To see that this increases ring profits, first consider a bidder $i \in \mathcal{I}$ who is a member of the "original" ring and has a value $X_i > r$ that is the highest of all bidders. With the larger ring, his fellow members in $\mathcal{J}$ would bid at most r and since

$$\widehat{P}_{\mathcal{J}} = \max\left\{Y_1^{\mathcal{N}\backslash\mathcal{J}}, r\right\} \leq \max\left\{Y_1^{\mathcal{N}\backslash\mathcal{I}}, r\right\} = \widehat{P}_{\mathcal{I}}$$

the price he would pay would be lower than the price paid when the ring consisted only of bidders in $\mathcal{I}$. Thus, the price paid by any member of the original ring $i \in \mathcal{I}$ is lower when a bidder is added to the ring (with positive probability, the price is strictly lower). For bidder $I+1$, certainly, joining the ring also means that the price she would pay upon winning is lower than otherwise. Thus, the profits of the ring increase as the membership increases. Once again, this increase comes solely at the expense of the seller.

For future reference it is useful to summarize the main conclusions reached so far. These are

- The operation of a bidding ring does not affect the payments or profits of bidders outside the ring.
- An increase in the membership of the ring increases ring profits, so the most effective ring is one that includes all bidders.

11.1.1 Efficient Collusion

The arguments made thus far have all assumed that the bidding ring submitted only one serious bid, and this was the highest of the values of its members. To do this, however, the ring has to induce its members to truthfully reveal private information regarding their values. In addition, cartel profits have to be shared among the members—the spoils have to be divided—in a way that the members would wish to participate in its workings. In other words, the bidding ring faces a mechanism design problem akin to those considered in Chapter 5. Indeed, the mechanism design perspective offers many insights into the workings of bidding rings.

Consider a bidding ring $\mathcal{I}$ and a *ring center*, which coordinates the activities of the ring. The mechanism designer—in this case, the center—must choose an internal mechanism that allocates the right to represent the cartel in the auction to the member with the highest value and determines the payment that each $i \in \mathcal{I}$ is asked to make to the center. These payments may be negative, since it may be necessary for the center to make transfers to members of the ring other than the winner. It is in the interests of the center, acting on behalf of the ring, to allocate the representation right efficiently. Temporarily, suppose that the internal mechanism is incentive compatible so that the ring member with the highest value wins the right to represent the cartel in the auction. It is natural to call this *efficient collusion*.

For any bidder $i \in \mathcal{N}$, let $Q_i^r(\mathbf{x})$ denote the probability that bidder $i \in \mathcal{N}$ will get the object in a second-price auction with a reserve price of r if the vector of values of all the bidders is $\mathbf{x}$. Since the cartel is represented by the ring member with the highest value, this is just the probability that $X_i > \max\left\{Y_1^{\mathcal{N}\setminus i}, r\right\}$, neglecting any ties. But this probability is the same as it would be if there were *no* bidding ring and all bidders behaved noncooperatively. Put another way, if we think of the auction in which a bidding ring operates efficiently as just another mechanism, say $(\mathbf{Q}^r, \widehat{\mathbf{M}})$, then the resulting allocation rule is the *same* as in an auction without any collusion—that is, $(\mathbf{Q}^r, \mathbf{M})$. But since the two mechanisms have the same allocation rule, the revenue equivalence principle (Proposition 5.2 on page 66) applies. This means that the expected payments of any bidder in the two mechanisms—with the operation of the ring and without—differ by at most a constant. Formally, the revenue equivalence principle implies that for all $i \in \mathcal{N}$ there exist constants t_i such that for all x_i,

$$m_i(x_i) - \widehat{m}_i(x_i) = t_i \tag{11.2}$$

The difference in the two expected payments t_i then represents the *gains* from collusion accruing to members of the ring.

We have already argued that bidders who are not members of the ring are unaffected by its operation, so for all $j \notin \mathcal{I}$ and x_j,

$$m_j\left(x_j\right) - \widehat{m}_j\left(x_j\right) = 0$$

and that for all ring members $i \in \mathcal{I}$ and x_i,

$$m_i(x_i) - \widehat{m}_i(x_i) = t_i \geq 0$$

This leads to the following:

Proposition 11.1. *In a second-price auction with a reserve price, the expected gains from efficient collusion obtained by a ring member depend on the ring member's identity but not on his or her value. Furthermore, the operation of the ring does not affect any bidder outside the ring.*

In a second-price auction the cartel agreement—only the member with the highest value submits a serious bid; all others bid at or below the reserve price—is self-enforcing in the following sense. A member of the cartel who does not have the highest value has no incentive to cheat on the agreement and bid higher since doing this cannot possibly be profitable. If such a member were to win the object, it would be at a price that exceeds his own value. This is because there is going to be at least one bid in the auction that exceeds his value: that of the cartel representative.

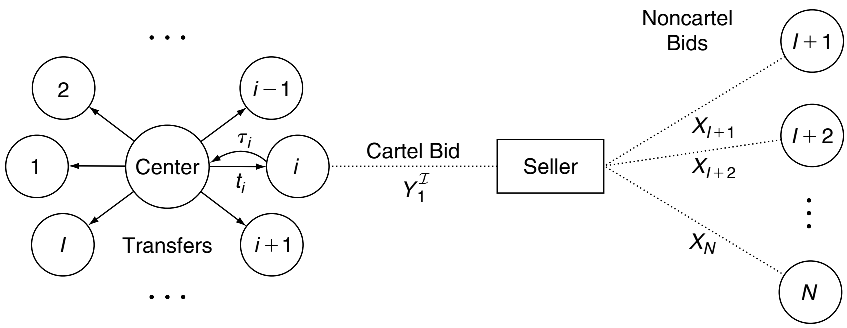

FIGURE 11.1 A bidding ring.

PREAUCTION KNOCKOUTS

One method of ensuring efficient collusion is for the ring center itself to conduct, prior to the actual sale, an auction among members of the ring. The winner of this auction, called a preauction knockout (PAKT), wins the right to represent the bidding ring at the main auction. A preauction knockout is akin to a party primary in the context of a political election; it selects the ring's "nominee" for the main auction.

The PAKT is also conducted under second-price rules, and its workings are as follows. First, each member of the ring is asked to reveal his or her private value to the center. The member reporting the highest value then represents the ring at the main auction and, if he wins the auction, pays the center an amount

$$\tau_i = P_i - \widehat{P}_{\mathcal{I}},$$

which is the difference in the price he would have paid if there were no ring and the price actually paid. This amount is easily determined if all ring members report their values truthfully. Figure 11.1 is a schematic representation of a bidding ring using a PAKT.

The center makes a lump-sum transfer to each ring member i of t_i (as defined in (11.2)). By definition, the center's budget is balanced in expected terms. In particular realizations, however, the sum of the transfers made by the center may exceed or fall short of what it receives. Certainly, there are circumstances in which the ring is unable to obtain the object, so there are no receipts while the lump-sum payments to the ring members have still to be made. Thus, the workings of a PAKT require that the ring center act as a banker who can finance the deficits and claim the surpluses.

The PAKT is incentive compatible—no ring member can do better than to report his true value to the center. Indeed, it is a weakly dominant strategy for every ring member to report truthfully. This is because the second-price PAKT requires the winning member, if and when he obtains the object, to pay the second-highest of *all* N values. So from each ring member's perspective, the situation is the same as in an ordinary second-price auction.

It is also individually rational—no member of the ring can do better by not participating since, no matter what the bidder's value, the gains from participating are positive. Its major weakness is that it needs an outside agency to finance its workings. It is not an *ex post* balanced budget mechanism.

A BALANCED BUDGET MECHANISM

Is efficient collusion feasible without the need for outside financing? To address this question it is useful to think of the bidding ring as facing an allocation problem: Who should be awarded the "right to represent" the cartel at the main auction? We will refer to this right as the *ticket*, thereby avoiding confusion with the object being offered for sale at the main auction. The ticket has a positive imputed value for each ring member—the expected gain from participating in the main auction. As a starting point, suppose that the center sells the ticket by means of a genuine second-price auction in which the winning member actually pays the center the second-highest imputed value for the ticket. In this case, the ring center would end up with a surplus for sure. But the second-price auction is the same as a Vickrey-Clarke-Groves (VCG) mechanism, introduced in Chapter 5, specialized to an auction context. Thus, if the ticket is allocated using the VCG mechanism, the center would run a surplus. Now Proposition 5.6 (see page 77) can be directly applied—if the VCG mechanism runs a surplus, there exists an efficient, incentive compatible, individually rational mechanism that also balances the budget in an *ex post* sense. The proof of Proposition 5.6, in fact, provides an explicit construction of such a mechanism.

The balanced budget mechanism from Proposition 5.6 is incentive compatible, but truth-telling is not a dominant strategy. In contrast, a PAKT balances the budget only in expected terms but has the property that truth-telling is a dominant strategy. In designing a mechanism to facilitate efficient collusion, the ring center thus faces a trade-off.

11.1.2 Reserve Prices in the Face of Collusion

Collusion among a subset of bidders reduces the profits of the seller, so it is natural to ask how she might respond to the presence of a bidding ring. But what recourse does the seller have? Assuming that the collusion cannot be detected by the antitrust authorities and that the seller cannot make a credible case for the presence of a bidding ring, the only instrument left in her hands with which to counter the actions of the bidding ring is to set a reserve price. Here we explore the issue of the optimal reserve price when the seller is aware that a bidding ring consisting of members in the set $\mathcal{I}$ is in operation and likely to act in concert in an upcoming auction.

From the seller's perspective, the presence of a ring that operates efficiently is equivalent to a situation with $N-I+1$ bidders with values $Y_1^{\mathcal{I}}, X_{I+1}, X_{I+2}, \ldots, X_N$. This is because with efficient collusion, the members of the bidding ring submit only one serious bid of $Y_1^{\mathcal{I}}$, the highest value among its members; the rest of the $N-I$ bidders bid their values. The object is sold if and only if the highest of these, which is the same as $Y_1^{\mathcal{N}}$, is greater than the reserve price r. If, in addition,

$$Z^{\mathcal{I}} \equiv \text{2nd highest of } \left\{Y_1^{\mathcal{I}}, X_{I+1}, X_{I+2}, \ldots, X_N\right\} \tag{11.3}$$

is also greater than r, then the object is sold for $Z^{\mathcal{I}}$; otherwise, it is sold at the reserve price. Thus, if $Y_1^{\mathcal{N}} \geq r$, and the object is sold, the price obtained by the seller is

$$\widehat{P} = \max\left\{Z^{\mathcal{I}}, r\right\}$$

Let $H^{\mathcal{I}}$ denote the distribution function of $Z^{\mathcal{I}}$ with the density $h^{\mathcal{I}}$. It is also convenient to define G to be the distribution of $Y_1^{\mathcal{N}}$, so

$$G(y) = \prod_{j \in \mathcal{N}} F_j(y)$$

and g to be the corresponding density.

Now the expected selling price that the seller receives can be written as

$$r\left(H^{\mathcal{I}}(r) - G(r)\right) + \int_r^{\omega} z h^{\mathcal{I}}(z)\, dz$$

The first term in the expression above comes from the event that the object is sold at the reserve price. This happens when the highest value from $\{Y_1^{\mathcal{I}}, X_{I+1}, X_{I+2}, \ldots, X_N\}$, which is just $Y_1^{\mathcal{N}}$, exceeds r but the second-highest value, $Z^{\mathcal{I}}$, does not. The second term comes from the event that the second-highest value, $Z^{\mathcal{I}}$, itself exceeds r, so the object is sold at a price equal to $Z^{\mathcal{I}}$.

Assuming that there is an interior maximum, the optimal reserve price $r^* > 0$ must satisfy the following first-order condition:

$$H^{\mathcal{I}}\left(r^*\right) - G(r^*) - r^* g\left(r^*\right) = 0 \tag{11.4}$$

The optimal reserve price is thus determined implicitly by (11.4). It may be verified that when there is no cartel and bidders are symmetric, the optimal reserve price is the same as that determined in Chapter 2.

Now suppose that the cartel is enlarged with the addition of another bidder so that it grows in size from $\mathcal{I}$ to $\mathcal{J} = \mathcal{I} \cup \{I+1\}$. For the same reasons as noted earlier, the operation of this larger ring is equivalent in the eyes of the seller, to a second-price auction with $N - I$ bidders whose values are $Y_1^{\mathcal{J}}, X_{I+2}, X_{I+3}, \ldots, X_N$, respectively. Analogous to (11.3), let

$$Z^{\mathcal{J}} \equiv \text{2nd highest of } \left\{Y_1^{\mathcal{J}}, X_{I+2}, X_{I+3}, \ldots, X_N\right\} \tag{11.5}$$

and let $H^{\mathcal{J}}$ be the distribution of $Z^{\mathcal{J}}$ with density $h^{\mathcal{J}}$. The expected selling price when the reserve price is set at r is now

$$r\left(H^{\mathcal{J}}(r)-G(r)\right)+\int_r^{\omega} zh^{\mathcal{J}}(z)\,dz$$

Comparing $Z^{\mathcal{J}}$ to $Z^{\mathcal{I}}$, we see that the only circumstances in which the two are different are when either

$$(i)\ Z^{\mathcal{I}}=X_{I+1}>Y_1^{\mathcal{N}\backslash\mathcal{J}};\text{ or}$$
$$(ii)\ Z^{\mathcal{I}}=Y_1^{\mathcal{I}}<X_{I+1}$$

In the first case, bidder $I+1$ had the second-highest value, but once he joins the cartel the second-highest value becomes

$$Z^{\mathcal{J}}=Y_1^{\mathcal{N}\backslash\mathcal{J}}<X_{I+1}=Z^{\mathcal{I}}$$

In the second case, the "value" of the cartel $Y_1^{\mathcal{I}}$ was the second-highest value, while bidder $I+1$ won; once bidder $I+1$ joins it, the cartel wins and the new price, the second-highest value, becomes

$$Z^{\mathcal{J}}=Y_1^{\mathcal{N}\backslash\mathcal{J}}<Y_1^{\mathcal{I}}=Z^{\mathcal{I}}$$

In all other circumstances, $Z^{\mathcal{J}}=Z^{\mathcal{I}}$.

The fact that $Z^{\mathcal{J}}\leq Z^{\mathcal{I}}$ implies, of course, that the distribution $H^{\mathcal{I}}$ stochastically dominates $H^{\mathcal{J}}$. The derivative of the expected selling price with the larger cartel $\mathcal{J}$, evaluated at the reserve price that is optimal for the smaller cartel $\mathcal{I}$, is

$$H^{\mathcal{J}}\left(r^*\right)-G(r^*)-r^*g\left(r^*\right)\geq 0$$

using (11.4) and the fact that $H^{\mathcal{I}}\left(r^*\right)\leq H^{\mathcal{J}}\left(r^*\right)$. Thus, the optimal reserve price r^{**} when the seller faces a larger cartel $\mathcal{J}$ must be at least as large as r^*. We state this finding formally as follows:

Proposition 11.2. *The addition of a bidder to a bidding ring causes the optimal reserve price for the seller to increase. In particular, the optimal reserve price with a bidding ring is always greater than the optimal reserve price with no ring.*

The increase in the optimal reserve price resulting from a cartel is illustrated in the following simple example.

Example 11.1. *Suppose that there are two bidders with values that are uniformly and independently distributed on* $[0,1]$.

We saw in Chapter 2 that without a cartel the optimal reserve price was $r^* = \frac{1}{2}$.

Now suppose that both bidders are members of a cartel and suppose that the seller sets a reserve price of r. The cartel would bid $\max\{X_1, X_2\}$ and would purchase the good for r if this bid exceeds r. The seller's expected profit is

$$r \times \operatorname{Prob}[\max\{X_1, X_2\} > r] = r\left(1 - r^2\right)$$

which is maximized by setting $r^{**} = \frac{1}{\sqrt{3}} > \frac{1}{2} = r^*$. ▲

11.2 COLLUSION IN FIRST-PRICE AUCTIONS

The operation of bidding rings in first-price auctions introduces some new elements. First, unlike in a second-price auction, the cartel agreement in a first-price auction is *not* self-enforcing and, hence, is somewhat fragile. To see this simply, consider an all-inclusive cartel. Assuming that the highest value exceeds the reserve price set by the seller, such a cartel will try to obtain the object at the reserve price by submitting only one bid at this level and ensuring that no other bid exceeds this amount. But now consider a bidder whose value is greater than the reserve price but is not the highest. Such a bidder has the incentive to cheat on the cartel agreement and, by submitting a bid that just exceeds the reserve price, win the object. This suggests that second-price auctions are more susceptible to collusive practices—they have a built-in enforcement mechanism—than are first-price auctions. The analysis of cartels in the context of first-price auctions must postulate some enforcement mechanism that operates outside the model. Apart from physical coercion—always a possibility, especially given the criminal nature of the activity—the agreement may be enforced by repeated play. The same cartel may be involved in many auctions and cheating in one may be deterred by the threat of expulsion and the consequent loss of a share of the cartel's future profits.

Second, even if the bidders are *ex ante* symmetric, the operation of a cartel naturally introduces asymmetries among bidders. While this did not affect bidding behavior in second-price auctions—it was still a dominant strategy to bid one's value—it does affect behavior in first-price auctions. In particular, bidders not in the cartel face a different decision problem if there is a cartel in operation than if there is not. We have already seen that the analysis of bidding behavior in asymmetric first-price auctions is more problematic; as a result, our understanding of bidding rings in this context is more limited.

In what follows, we suppose that bidders are *ex ante* symmetric—that is, their values are drawn independently from the same distribution F. We also suppose that the cartel is all-inclusive. These two assumptions together ensure symmetry among bidders.

The behavior of an all-inclusive cartel is clear. It should submit only one serious bid—at the reserve price. But in order to operate efficiently, it still needs to determine what the highest valuation is. One option is to employ a first-price preauction knockout.

A FIRST-PRICE PAKT

In a first-price PAKT, a bid is an offer to pay *all* other members of the bidding ring that amount. The winner of the PAKT then represents the cartel at the main auction, obtaining the good at the reserve price. What does a symmetric equilibrium of the first-price PAKT look like? The derivation is somewhat simpler if we suppose that the reserve price $r = 0$ and can be easily extended to the general case.

Proposition 11.3. *Symmetric equilibrium strategies in a first-price sealed-bid PAKT among all the bidders are given by*

$$\beta(x) = \frac{1}{N} E\left[Y_1^{(N)} \mid Y_1^{(N)} < x\right]$$

Proof. Suppose all other bidders follow the strategy β and suppose that bidder 1 with value x reports z. The expected profits from doing this are

$$\Pi(z,x) = \underbrace{G(z)\left[x - (N-1)\,\beta(z)\right]}_{\text{Gain from winning PAKT}} + \underbrace{\int_z^1 \beta(y) g(y)\, dy}_{\text{Gain from losing PAKT}}$$

Differentiating with respect to z

$$\begin{aligned}\frac{\partial \Pi}{\partial z} &= g(z)\left[x - N\beta(z)\right] - (N-1)\, G(z)\beta'(z) \\ &= (N-1)\, F(z)^{N-2}\left[f(z)x - Nf(z)\beta(z) - F(z)\beta'(z)\right]\end{aligned}$$

But

$$\beta(z) = \frac{1}{F(z)^N}\int_0^z yF(y)^{N-1} f(y)\, dy$$

and differentiating this results in

$$\beta'(z) = \frac{f(z)}{F(z)} z - N\frac{f(z)}{F(z)}\beta(z)$$

which can be rearranged as

$$\beta'(z)\, F(z) + N\beta(z) f(z) = f(z) z$$

This implies that

$$\begin{aligned}\frac{\partial \Pi}{\partial z} &= (N-1)\,F(z)^{N-2}\left[f(z)x - Nf(z)\beta(z) - F(z)\beta'(z)\right]\\ &= (N-1)\,F(z)^{N-2}f(z)\,(x-z)\end{aligned}$$

and thus it is optimal to choose $z=x$. ■

In the special case of an all-inclusive cartel, the first-price PAKT is an effective mechanism for eliciting private information while ensuring that the budget is always balanced.

PROBLEMS

11.1. (Maximal loss from collusion) Consider a second-price auction with $N \geq 2$ bidders. Each bidder's private value X_i is independently and uniformly distributed accordingly on $[0,1]$.

- **a.** First, suppose bidders bid individually—that is, there is no bidding ring. As a benchmark, find the expected revenue of the seller Π^* if he sets an optimal reserve price $r^* > 0$ (as in Chapter 2).
- **b.** Now suppose that the N bidders form a perfectly functioning bidding ring. Find the expected revenue of the seller Π^{**} if he sets an optimal reserve price $r^{**} > 0$ in the face of such collusion.
- **c.** Show that for all n, the optimal revenue with collusion is at least one-half of the optimal revenue without collusion; that is, $\Pi^{**} > \frac{1}{2}\Pi^*$.

11.2. (Collusion in first-price auctions) Consider a first-price auction with three bidders. Bidder 1's value $X_1 = \frac{3}{4}$ with probability $\frac{3}{4}$ and $X_1 = \frac{1}{2}$ with probability $\frac{1}{4}$. Bidders 2 and 3 have fixed and commonly known values. Specifically, $x_2 = 1$ and $x_3 = \frac{1}{4}$.

- **a.** Find an equilibrium of the first-price auction when the three bidders act independently. (Note: Since values are discrete, this will be in mixed strategies.)
- **b.** Now suppose that bidders 1 and 2 form a cartel. While the cartel cannot control the bids submitted by its members, it can arrange transfers and recommend bids. Further, suppose that the values of its members become commonly known among the cartel once it is formed.
  - **i.** Find an equilibrium with the cartel, assuming that bidder 3 acts independently.
  - **ii.** Is it possible for the cartel to ensure that only one member submits a bid?

11.3. (PAKT) A single object is to be sold via a second-price auction to two bidders whose private values X_i are drawn independently from the uniform distribution on $[0,1]$. Suppose that the bidders form a cartel. Find the equilibrium bidding strategies in the preauction knockout (PAKT).

11.4. (Collusion-proof mechanism) A single object is to be sold to two bidders with private values drawn independently from the uniform distribution on $[0,1]$. The following mechanism is used to sell the object. Each bidder i submits a bid b_i. Suppose that $b_i > b_j$. Then the loser, bidder j, is asked to pay a fixed amount $\frac{1}{3}$ to the seller. The winner, bidder i, is awarded the object and asked to pay b_i to the losing bidder j. (If there is a tie, either bidder is assigned the role of a winner.)

a. Find a symmetric equilibrium of this mechanism assuming that the bidders act noncooperatively.

b. Can the two bidders gain by forming a cartel and colluding against the seller?

CHAPTER NOTES

The analysis of collusion in second-price auctions was initiated by Graham and Marshall (1987) in a symmetric independent private values model. Their model and results were later extended to accommodate asymmetric bidders by Mailath and Zemsky (1991). Most of the material in this chapter is based on these two papers.

Collusion in first-price auctions and the first-price PAKT were analyzed by McAfee and McMillan (1992). Robinson (1985) discusses the relative susceptibility of the different auction formats—second-price, first-price, and English—to collusion. In more recent work, Marshall and Marx (2007) have studied collusion in both first- and second-price auctions when the cartel is not all-inclusive. If the cartel is unable to control the bids submitted by its members the first-price auction is less vulnerable to collusion than is the second-price auction. Problem 11.2 is based on their paper.

Che and Kim (2006) have shown, quite remarkably, that there exist mechanisms which are immune to collusion by the bidders. The idea is to use mechanisms that in effect sell the object to the cartel for a fixed price equal to the expected revenue in, say a second-price auction without collusion. Problem 11.4 is based on their paper.

A general survey of the area, emphasizing many open questions, has been written by Hendricks and Porter (1989). The statistics on the level of antitrust activity that relates to bid rigging reported on page 157 are taken from a U.S. Department of Justice study that is quoted by Hendricks and Porter (1989).