MATH 425b ASSIGNMENT 3
SPRING 2016
Prof. Alexander
Due Wednesday February 17.

Rudin Chapter 7 #20, 21; Chapter 8 #4, 6 and:

(A) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series. An example was given in lecture to show that it's possible for such a series to have radius of convergence 1, with $f(x)$ staying bounded as $x \to 1$, yet the series not converging for $x = 1$. This problem shows that is not possible if the coefficients a_n are nonnegative. Suppose all $a_n \geq 0$, and show that the following are equivalent:

(a) $f(1) = \sum_{n=1}^{\infty} a_n$ converges;
(b) The power series converges uniformly on $[0, 1]$;
(c) f is bounded on $[0, 1)$.

(B) Show that

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for all } x \in (-1, 1).$$

(C)(a) Let $\mathcal{A}_1$ be an algebra of real-valued functions on $[0, 1]^2$ which contains the functions $f(x, y) = e^y$ and $g(x, y) = 1/(x + 2)$. Does the uniform closure of $\mathcal{A}_2$ necessarily include the function $h(x, y) = \sin xy$?

(b) Let $\mathcal{A}_2$ be the algebra consisting of all polynomials f on $[0, 1]$ satisfying $f'(\frac{1}{2}) = 0$. Is $\mathcal{A}_2$ dense in $C[0, 1]$?

(D) For $\alpha \in \mathbb{R}$ and $n \geq 0$ and integer, define

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \text{ for } n \geq 1; \quad \binom{\alpha}{0} = 1; \quad S(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

Note if α is positive integer $\geq n$ then this is the same as the usual definition, so this extends the definition to non-integer and negative α.

(a) Show that the radius of convergence of the series $S(x)$ is 1.
(b) Show that $S'(x) = \alpha S(x) - x S'(x)$ for all $x \in (-1, 1)$.
(c) Show that $\frac{d}{dx} \log S(x) = \alpha/(1 + x)$, and then that $S(x) = (1 + x)^\alpha$. This is an extension of the binomial expansion to non-integer α.

HINTS:

(20) You may find Exercise 2 in Chapter 6 useful–you may take it as given.

(21) Notice that in the Stone-Weierstrass Theorem (7.32), the functions are real-valued, so that theorem is not violated by this problem!

You need to find a continuous function f on K with $\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\ d\theta \neq 0$. What's a simple sort of function that has a nonzero integral?

(4) Try to do (a), (c), (d) without l'Hopital's Rule:
(a) Find a way to use Theorem 8.6b.
(c) Use (b).
(d) Use (c).

(6)(a) Plugging in the right x, y gives you $f(0)$. Then show $\log f(x)$ has a constant derivative.
(b) Four steps: (i) Show $g(mx) = mg(x)$ for positive integers m. (ii) Prove an analog of (a) for $g(x/n)$ for positive integers n. (iii) Use (i), (ii) to relate $g(m/n)$ to $g(1)$. (iv) Use (iii) and continuity of g to complete the proof.

(A) Show (a) $\implies$ (b) $\implies$ (c), and "not (a)" $\implies$ "not (c)."

(B) Take the derivative of both sides. You may assume the calculus fact that the derivative of $\log x$ is $1/x$ for $x > 0$.

(C)(b) $\mathcal{A}_2$ includes constants, and functions of form $(x - \frac{1}{2})^m Q(x)$, where $Q(x)$ is any polynomial; for what powers m is this true?

(D)(a) The Root or Ratio Test may work better than the usual formula.
(b) You'll have to manipulate things a bit. How is $\binom{\alpha}{n}$ related to $\binom{\alpha}{n-1}$? Try changing the index using $m = n - 1$.
(c) This should be relatively straightforward from (b)—recall the formula for derivative of a log.