MATH 425b SAMPLE MIDTERM EXAM 2
Spring 2016
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The midterm will again be open book. You can use Rudin, your lecture notes, your homework and solutions, but no other books or published materials.

(1) Let E be a *connected* open set in $\mathbb{R}^n$, and suppose $f : E \to \mathbb{R}^m$ is differentiable in E, and suppose $f'(\mathbf{x}) = 0$ (meaning it's the 0 transformation) for all $\mathbf{x} \in E$. According to the Corollary to 9.19, if B is a ball in E, then f is constant on B. Show that in fact f is constant on all of E. HINT: Connected means that any nonempty set G which is both open and closed in E is all of E. Find such a G described in terms of f.

(2) Let (X, d) be a metric space and suppose X is compact. Let $\varphi : X \to X$ satisfy $d(\varphi(x), \varphi(y)) < d(x, y)$ for all $x \neq y$ in X. Show that φ has a unique fixed point in X.

HINT: φ need not be a contraction map—for that you need $d(\varphi(x), \varphi(y)) \leq cd(x, y)$, with $c < 1$. So instead of applying Theorem 9.23, use the hypothesis directly. Consider properties of the function $g(x) = d(x, \varphi(x))$. What property would guarantee existence of a fixed point? Consider the infumum of g.

(3) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ has the form $f(\mathbf{x}) = T\mathbf{x} + g(\mathbf{x})$ where T is a linear transformation and $|g(\mathbf{x})| \leq c|\mathbf{x}|^\alpha$ for some $c > 0$ and $\alpha > 1$.

(a) Show that f is differentiable at $\mathbf{0}$ and find $f'(\mathbf{0})$.

(b) Show that (a) may be false for any n if we allow $\alpha = 1$, by considering $f(\mathbf{x}) = |\mathbf{x}|$. (Here $T = 0$ and the dimension $m = 1$.)

(4) The two equations

$$\begin{aligned} -4x^2 + y^2 + z^2 &= 4 \\ yz - 2xy &= 0 \end{aligned}$$

define a curve in $\mathbb{R}^3$ which passes through $(1, 2, 2)$.

(a) Show that this curve can be parametrized using z as the parameter, at least in a neighborhood of $(1, 2, 2)$. This means writing the curve as $(h_1(z), h_2(z), z)$ for some h_1, h_2 that are $\mathcal{C}'$ functions on an interval containing $z = 2$.

(b) For which of the other two variables (x or y) can we be sure the parametrization could also be done using that variable instead of z? (Prove, if we know the parametrization can be done. If it can't necessarily be done, you don't have to prove one way or the other, just say how you know.)