MATH 425a SAMPLE MIDTERM EXAM 1
Fall 2015
Prof. Alexander

You will be asked to prove one or more of the following theorems on the exam:

1.33cde, 2.12, 2.19, 2.24, 2.27a, 2.34, 2.35.

Your proof does not have to be the same as what's in the text, just a correct proof (but don't cite a theorem in your proof that appears in the text AFTER the theorem you're proving.)

(1)(a) Prove Theorem 2.20: If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

(b) State the converse of this theorem. Is it true? (No formal proof needed for this one, just a comment as to why it is true, or why false.)

(2)(a) State the contrapositive of the statement, "G open implies $x \notin G$."

(b) Suppose G is a bounded open set in $\mathbb{R}$, and let $x = \sup G$. Show that $x \notin G$. HINT: Assume G is bounded, and prove either the statement in (a) or its contrapositive.

(3)(a) For $\epsilon > 0$, a subset E of a metric space K is called ϵ-*approximating* if for every $x \in K$ there is a $y \in E$ with $d(x, y) < \epsilon$. Show that if K is compact, then for every $\epsilon > 0$ there exists a *finite* $E \subset K$ which is ϵ-approximating. HINT: If E is ϵ-approximating, what do you know about the collection of neighborhoods $N_\epsilon(y), y \in E$?

(b) Show that K has a dense subset which is at most countable. HINT: Use (a), and consider multiple different values of ϵ. Recall that a set E is *dense* in K if every point of K is a point, or a limit point, of E, or equivalently, if every neighborhood of every point of K contains a point of E.

Do EITHER PROBLEM 4 OR PROBLEM 5.

(4)(a)(4 points) State what it means for a point p to be an *interior point* of a set E.

(b)(14 points) Show that for sets A, B in a metric space, $A^\circ \cap B^\circ \subset (A \cap B)^\circ$. (Here A° denotes the interior of A.)

(c)(9 points) Consider the corresponding statement for an infinite sequence of sets: $\cap_{i \geq 1} A_i^\circ \subset (\cap_{i \geq 1} A_i)^\circ$. Give an example to show this can be false. HINT: Intervals in $\mathbb{R}$ can work.

(5) Let $\mathcal{A} = \{G \subset \mathbb{R} : G \text{ is open}\}$. Define $I : \mathbb{R} \to \mathcal{A}$ by letting $I(x)$ be the interval $(x, x+1)$.

(a)(7 points) Is I 1-to 1? Onto? Explain.

(b)(10 points) Show that $\mathcal{A}$ is uncountable.

(c)(10 points) Let $\mathcal{B} = \{(a, b) \subset \mathbb{R} : a, b \text{ rational}\}$ be the set of all open intervals with rational endpoints. Show that $\mathcal{B}$ is countable.