MATH 425b ASSIGNMENT 5 SOLUTIONS
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Prof. Alexander

Chapter 9:

(I)(a) If $y \in T(U), y \neq 0$, then $y = Tx$ for some $x \in U$ with $x \neq 0$, so $|y| = |Tx| \leq |Tx|/|x| \leq ||T||$, meaning $y \in B(0, ||T||)$. Thus $T(U) \subset \overline{B}(0, ||T||)$. On the other hand, if $r < ||T||$, then there exists $x \neq 0$ with $|Tx|/|x| > r$, so for $z = x/|x|$ we have $z \in U$ but $Tz \notin \overline{B}(0, r)$. This shows $T(U) \not\subset \overline{B}(0, r)$, meaning $\overline{B}(0, ||T||)$ is the smallest ball containing $T(U)$.

For the conorm, since $\mathfrak{m}(T) > 0$ we cannot have $x \neq 0, Tx = 0$. Therefore T is one-to-one, and thus onto (since $m = n$.) Suppose $y \in \overline{B}(0, \mathfrak{m}(T)), y \neq 0$. Since T is onto, we have $y = Tz$ for some $z \in \mathbb{R}^n, z \neq 0$. Then

$$\mathfrak{m}(T) \geq |y| = |Tz| \geq \mathfrak{m}(T)|z|,$$

where the last inequality is from the definition of $\mathfrak{m}(T)$, so $|z| \leq 1$, that is, $z \in U$, meaning $y \in T(U)$. This shows $\overline{B}(0, \mathfrak{m}(T)) \subset T(U)$. On the other hand, if $r > \mathfrak{m}(T)$, then by definition of infimum there exists $x \neq 0$ with $|Tx|/|x| < r$. This means we can find $c > 1$ with $c|Tx|/|x| < r$. Let $w = cx/|x|$, so $|w| = c > 1$, that is, $w \notin U$. Since T is one-to-one, this means $Tw \notin T(U)$. But we have $|Tw| = c|Tx|/|x| < r$, so $Tw \in \overline{B}(0, r)$. This shows $\overline{B}(0, r) \not\subset T(U)$. It follows that $\mathfrak{m}(T)$ is the radius of the largest ball contained in $T(U)$.

(b) We prove the contrapositive: if T is not invertible then $Tx = 0$ for some $x \neq 0$, so $|Tx|/|x| = 0$, which shows that $\mathfrak{m}(T) = 0$.

(II)(a) Given $\epsilon > 0$ we have

$$|y - x| < \frac{\epsilon}{||T||} \quad \text{implies} \quad |Ty - Tx| = |T(y - x)| \leq ||T|| \; |y - x| < \epsilon.$$

Thus T is uniformly continuous.

(b) Let $f_n(x) = \frac{1}{n} \sin nx, \ x \in (a, b)$. Then $f_n \to 0$ uniformly and $T(0) = 0$, but we claim that $T(f_n) \not\to T(0) = 0$, which means T is not continuous. In fact $(Tf_n)(x) = f_n'(x) = \cos nx$, and for sufficiently large n there exist points of form $2\pi k/n$ in (a, b) with k an integer (such points exists provided $2\pi/n < b - a$.) At these points, $f_n'(2\pi k/n) = \cos 2\pi k = 1$. This shows $T(f_n)$ does not converge to 0 in $C(a, b)$, i.e. it does not converge uniformly to 0. Thus as claimed, T is not continuous.

But T is linear since $(cf + g)' = cf' + g'$.

(III)(a) We have $|\mathbf{u}|^2 = x^2 + y^2 = 1$, so $y^2 = 1 - x^2$, and $|A\mathbf{u}|^2 = (2x + 3y)^2 + (x - 6y)^2 = 5x^2 + 45y^2 = 5x^2 + 45(1 - x^2) = 45 - 40x^2$. This is maximized when x^2 is as small as possible, that is, $x = 0$, and $y = \pm 1$, giving $|A\mathbf{u}|^2 = 45$. Thus $\|A\| = \sup_{|\mathbf{u}|=1} |A\mathbf{u}| = \sqrt{45}$, and this is achieved by $\mathbf{u} = (0, \pm 1)$.

(b) $|A\mathbf{u}|/|\mathbf{u}|$ is not changed if $\mathbf{u}$ is multiplied by a nonzero constant, so the maximizing vectors are all nonzero multiples of the vectors $\mathbf{u} = (0, \pm 1)$ found in part (a), that is, all vectors $(0, y)$ with $y \neq 0$.

(c) $1/\|A\| = 1/\sqrt{45}$ and $\|A^{-1}\| = 1/\sqrt{5}$, so they are not equal.

(IV)(a) By the Cauchy-Schwarz inequality, $|Ax \cdot y| \leq |Ax|\ |y|$, so

$$\sup_{x,y\neq 0} \frac{|Ax \cdot y|}{|x|\ |y|} \leq \sup_{x,y\neq 0} \frac{|Ax|\ |y|}{|x|\ |y|} = \sup_{x\neq 0} \frac{|Ax|}{|x|} = ||A||.$$

(b) Considering only x with $Ax \neq 0$ and only $y = Ax$ we get

$$\sup_{x,y\neq 0} \frac{|Ax \cdot y|}{|x|\ |y|} \geq \sup_{x:x\neq 0 \text{ and } Ax\neq 0} \frac{|Ax \cdot Ax|}{|x|\ |Ax|} = \sup_{x:x\neq 0 \text{ and } Ax\neq 0} \frac{|Ax|}{|x|}.$$

But including also vectors $x \neq 0$ with $Ax = 0$ does not change the last sup, since the value of $|Ax|/|x|$ is 0 for these x anyway, and thus the last sup is the same as

$$\sup_{x\neq 0} \frac{|Ax|}{|x|} = ||A||.$$

(V) If $T\mathbf{x}_1 \neq T\mathbf{x}_2$ then $T(\mathbf{x}_1 - \mathbf{x}_2) \neq 0$ so by assumption $\mathbf{x}_1 - \mathbf{x}_2 \neq 0$, that is, $\mathbf{x}_1 \neq \mathbf{x}_2$. This shows T is one-to-one.