MATH 425b ASSIGNMENT 9 SOLUTIONS
SPRING 2016
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Chapter 10:

(2) For $y \in (2^{-i}, 2^{1-i})$ the only function $\varphi_j(y)$ which may be nonzero is $\varphi_i(y)$. Hence for such y we have $f(x,y) = [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)$ for all x, and then

$$\int f(x,y)\ dx = \varphi_i(y) \int [\varphi_i(x) - \varphi_{i+1}(x)]\ dx = \varphi_i(y)(1-1) = 0.$$

Therefore $\int dy \int f(x,y)\ dx = 0$.

In the other direction, for fixed $i \geq 2$ and $x \in (2^{-i}, 2^{1-i})$ the only functions $[\varphi_j(x) - \varphi_{j+1}(x)]$ which may be nonzero are $[\varphi_i(x) - \varphi_{i+1}(x)]$ and $[\varphi_{i-1}(x) - \varphi_i(x)]$. Hence for such x we have $f(x,y) = [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y) + [\varphi_{i-1}(x) - \varphi_i(x)]\varphi_{i-1}(y)$ for all y, and then

$$\int f(x,y)\, dy = [\varphi_i(x) - \varphi_{i+1}(x)] \int \varphi_i(y)\, dy + [\varphi_{i-1}(x) - \varphi_i(x)] \int \varphi_{i-1}(y)\, dy = \varphi_{i-1}(x) - \varphi_{i+1}(x).$$

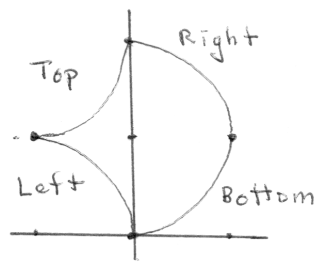
But since $x \in (2^{-i}, 2^{1-i})$ we have $\varphi_{i-1}(x) 0 \varphi_{i+1}(x) = 0$ so $\int f(x,y)\ dy = 0$. Note we have omitted $i = 1$ which corresponds to $x \in (1/2, 1)$. For the case of $x \in (1/2, 1)$, the only function $[\varphi_j(x) - \varphi_{j+1}(x)]$ which may be nonzero is $[\varphi_1(x) - \varphi_2(x)]$. Hence for such x we have $f(x,y) = [\varphi_1(x) - \varphi_2(x)]\varphi_1(y)$ for all y, and then

$$\int f(x,y)\ dy = [\varphi_1(x) - \varphi_2(x)] \int \varphi_1(y)\ dy = [\varphi_1(x) - 0] \cdot 1 = \varphi_1(x).$$

Therefore

$$\int\ dx \int f(x,y)\ dy = \int_{1/2}^1 \varphi_1(x)\ dx = 1.$$

(3)(a) The images of the 4 sides are as shown:



(b) Let $f(x,y)=(y,x)$. Suppose $G_1(x,y)=(g_1(x,y),y)$ and $G_2(x,y)=(x,g_2(x,y))$ are primitive and $G_2\circ G_1=f$. Then $G_1(x,0)=(g_1(x,0),0)$ so

$$(0,x)=(G_2\circ G_1)(x,0)=G_2(g_1(x,0),0)=(g_1(x,0),g_2(g_1(x,0),0))$$

for all x. Thus $g_1(x,0)=0$ for all x, so, by matching the second coordinates, we have $g_2(0,0)=x$ for all x, a contradiction. Similarly we can't have $G_1\circ G_2=f$.

(8) For some $\mathbf{b}$ and some matrix A we have $T(\mathbf{x})=\mathbf{b}+a\mathbf{x}$. We have $\mathbf{b}=T((0,0))=(1,1)$ while the columns of A are $A((1,0))=T((1,0))-\mathbf{b}=(3,2)-(1,1)=(2,1)$ and $A((0,1))=T((0,1))-\mathbf{b}=(2,4)-(1,1)=(1,3)$. Therefore $A=\begin{bmatrix}2&1\\1&3\end{bmatrix}$. Since $T'(\mathbf{x})=A$ for all $\mathbf{x}$, we have Jacobian $J_T=\det A=5$. We have $H=T([0,1]^2)$, and for $\mathbf{u}\in[0,1]^2$, $T_1(\mathbf{u})=1+2u_1+u_2,\ T_2(\mathbf{u})=1+u_1+3u_2$. Hence

$$\begin{aligned}\int_H e^{x-y}\ dx\ dy &= \int_0^1\int_0^1 5e^{u_1-2u_2}\ du_1\ du_2\\ &= \int_0^1 5(e-1)e^{-2u_2}\ du_2\\ &= \frac{5}{2}(e-1)(1-e^{-2}).\end{aligned}$$

(I) We have

$$\begin{aligned}J_1(p)=&\ \text{Jacobian of }(\Phi_2(p),\Phi_3(p))=N_1(p),\\ J_2(p)=&\ \text{Jacobian of }(\Phi_3(p),\Phi_1(p))=N_2(p),\\ J_3(p)=&\ \text{Jacobian of }(\Phi_1(p),\Phi_2(p))=N_3(p),\end{aligned}$$

and so

$$\int_\Phi \omega_f=\int_D [f_1(\Phi(p))J_1(p)+f_2(\Phi(p))J_2(p)+f_3(\Phi(p))J_3(p)]\ dp=\int_D f(\Phi(p))\cdot N(p)\ dp,$$

with the first equality being just the definition of evaluating a differential form.

(II) The wedge product is

$$\omega\wedge\omega'=(x_1x_2+x_3^2)x_4^2\ dx_4\wedge dx_1\wedge dx_2\wedge dx_3\wedge dx_5-3x_4^3\ dx_4\wedge dx_2\wedge dx_1\wedge dx_3\wedge dx_5.$$

The permutation 41235 requires 3 interchanges, and the permutation 42135 requires 4, so

$$\omega\wedge\omega'=-(x_1x_2x_4^2+x_3^2x_4^2-3x_4^3)\ dx_1\wedge dx_2\wedge dx_3\wedge dx_4\wedge dx_5.$$

Next, $d\omega$ has only one nonzero term:

$$d\omega=\frac{\partial}{\partial x_3}(x_1x_2+x_3^2)\ dx_3\wedge dx_4\wedge dx_1\wedge dx_2=2x_3\ dx_1\wedge dx_2\wedge dx_3\wedge dx_4.$$

(III) Suppose $\omega = \sum_{i=1}^n f_i(x)\ dx_i$. Using the change of variable $t = \varphi(s), dt = \varphi'(s)\ ds$ and the chain rule, we get

$$\begin{aligned}\int_\gamma \omega &= \int_c^d \sum_{i=1}^n f_i(\gamma(t))\gamma_i'(t)\ dt \\ &= \int_a^b \sum_{i=1}^n f_i(\gamma(\varphi(s)))\gamma_i'(\varphi(s))\varphi'(s)\ ds \\ &= \int_a^b \sum_{i=1}^n f_i(\alpha(s))\alpha_i'(s)\ ds \\ &= \int_\alpha \omega. \end{aligned} \tag{1}$$

(IV) One choice is $\gamma(t) = (\cos t, \sin t), t \in [0, \pi]$. This gives

$$\int_\gamma y\ dx = \int_0^\pi \sin t(-\sin t)\ dt = -\frac{\pi}{2}.$$

(V)(b) Suppose $(s,t), (u,v) \in Q$ and $T(s,t) = T(u,v)$, that is, $s - t^2 = u - v^2$ and $s^2 + t = u^2 + v$. Then

$$(*) \qquad (t+v)(t-v) = t^2 - v^2 = s - u, \quad (u+s)(u-s) = u^2 - s^2 = t - v.$$

Substituting the second of these into the first shows that $(t+v)(u+s)(u-s) = s - u$. If $s \neq u$ then this says $(t+v)(u+s) = -1$, but this is impossible since t, v, u, s are nonnegative, so we must have $s = u$. But then the right side of $(*)$ says that $t = v$.

(c) We have $T'(u,v) = \begin{bmatrix} 1 & -2v \\ 2u & 1 \end{bmatrix}$ so $J_T(u,v) = 1 + 4uv$. Let $f(x,y) = x$. Then

$$\int_A x\ dx\ dy = \int_{T(Q)} f(x,y)\ dx\ dy = \int_Q f(\Phi(u,v))|J_T(u,v)|\ du\ dv$$

$$= \int_0^1 \int_0^1 (u - v^2)(1 + 4uv)\ du\ dv = \int_0^1 \int_0^1 (u - v^2 + 4u^2v - 4uv^3)\ du\ dv$$

$$= \int_0^1 (\frac{1}{2} - v^2 + \frac{4}{3}v - 2v^3)\ dv = \frac{1}{3}.$$

(VI) The rotated body is

$$A = \{\mathbf{x} : a \le x \le b, 0 \le R(\mathbf{x}) \le f(x)\} = \{\mathbf{x} : a \le x \le b, y^2 + z^2 \le f(\mathbf{x})\}.$$

We change coordinates from (x,y,z) to $x, r, \theta)$ by $T(x,r,\theta) = (x, r\cos\theta, r\sin\theta)$ so that $R(T(x,r,\theta)) = r$. If we set $Q = \{(x,r,\theta) : a \le x \le b, 0 \le r \le f(x), 0 \le \theta \le 2\pi\}$ then $T(Q) = A$ as a 1-1 $\mathcal{C}'$ map, with

$$T'(x,r,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -r\sin\theta \\ 0 & \sin\theta & r\cos\theta \end{bmatrix} \quad \text{so} \quad J_T(x,r,\theta) = r\cos^2\theta + r\sin^2\theta = r.$$

so the moment is

$$
\begin{aligned}
\int_A R(\mathbf{x})^2 \ d\mathbf{x} &= \int_Q R(T(x,r,\theta))^2 |J_T(x,r,\theta)| \ dr \ d\theta \ dx \\
&= \int_a^b \int_0^{2\pi} \int_0^{f(x)} r^3 \ dr \ d\theta \ dx \\
&= \int_a^b \frac{\pi}{2} f(x)^4 \ dx.
\end{aligned}
$$