Appendix | G

Existence of Equilibrium in First-Price Auctions

In this appendix we explore the issue of the existence of a pure strategy Bayesian-Nash equilibrium in first-price auctions. Our setting is one of independent private values. Of course, when bidders are symmetric, an equilibrium in closed form can be derived as in Proposition 2.2 on page 15. But when bidders are asymmetric, such a closed form solution is not readily available and we are thus interested in determining whether an equilibrium exists at all. For example, in Chapter 4, we studied some properties of equilibria in this setting (see Proposition 4.4 on page 47, for instance) but assumed that there was an equilibrium in which bidders' strategies were increasing functions of their values. We also computed equilibrium strategies explicitly for the case when there were two bidders with uniformly distributed values over differing supports.

The question of whether an equilibrium exists in general—regardless of the distributions or the number of bidders—is somewhat involved. Most of the difficulties stem from the fact that bidders' payoffs in an auction are discontinuous in the amounts bid. For instance, in a two-bidder first-price auction, if bidder 1 bids b_1, then his payoff is zero as long as $b_2 > b_1$, but is typically positive if $b_2 = b_1$. Standard results on the existence of equilibrium in games assume that players' payoff functions are continuous, so these cannot be directly applied. Moreover, even under the assumption of continuity, these results usually conclude only that a *mixed* strategy equilibrium—in which players randomize—exists.

In this appendix we outline a method of proof that separates the problem into two components. In the first, and key, step it is assumed that bidders may only bid in discrete amounts—there is some minimum bid increment, say, one cent. Somewhat remarkably with this restriction, it is possible to formulate the question of existence of equilibrium in a way that standard tools can be brought to bear on the problem. Thus, it is possible to show that with discrete bids, there exists a *pure* strategy equilibrium in which each bidder's strategy is a nondecreasing function of his or her value (see Proposition G.1). The second step shows that a limit of these equilibria, as the minimum bid increment shrinks, is a pure strategy equilibrium of the auction in which bids are not restricted. In the interests of space, here we do not prove the second step; rather we only indicate some of the attendant difficulties.

EQUILIBRIUM WITH DISCRETE BIDS

Suppose that there are N bidders with independently distributed values. Bidder i's value X_i is distributed over the interval $\mathcal{X}_i = [0, \omega_i]$ according to the distribution function F_i with associated density f_i.

Let $\omega = \max_i \omega_i$. Fix an integer T and define

$$\mathcal{B}^T = \left\{ \frac{t}{T}\omega : t = 0, 1, \ldots, T \right\}$$

to be the finite set of allowable bids. Thus, there is a minimum bid increment of ω/T. In what follows, we will use the notation

$$b^t \equiv \frac{t}{T}\omega$$

A strategy for bidder i in an auction with discrete bids is a function $\beta_i : \mathcal{X}_i \to \mathcal{B}^T$. Fix the strategies β_j of bidders $j \neq i$ and let $H_i(b^t)$ denote the probability that i will win with a bid of b^t. Formally, for $t = 0, 1, \ldots, T$,

$$\begin{aligned} H_i(b^t) = {} & \text{Prob}\left[\max_{j \neq i} \beta_j(X_j) \leq b^{t-1}\right] \\ & + \frac{1}{k+1}\text{Prob}\left[\max_{j \neq i} \beta_j(X_j) = b^t\right], \end{aligned} \tag{G.1}$$

where k is the number of *other* bidders who bid exactly b^t. The first term comes from events in which i is the outright winner. The second term comes from events in which there is more than one bid at b^t and the winner is determined at random from among those with the highest bid. Because bids are discrete, ties occur with positive probability. Notice that $H_i(\cdot)$ is a nondecreasing function.

A bid $b_i \in \mathcal{B}^T$ is a *best response at* x_i by bidder i if it maximizes his expected payoff against $\boldsymbol{\beta}_{-i}$, that is, if for all $b \in \mathcal{B}^T$,

$$H_i(b_i)(x_i - b_i) \geq H_i(b)(x_i - b) \tag{G.2}$$

Denote by $BR_i(x_i)$ the set of best responses at x_i.

Lemma G.1. *For any* $\boldsymbol{\beta}_{-i}$ *and* $0 < x_i' < x_i''$,

$$\min BR_i\left(x_i''\right) \geq \max BR_i\left(x_i'\right)$$

Proof. Let $b_i' = \max BR_i\left(x_i'\right)$. By definition, for all $b < b_i'$ such that $b \in \mathcal{B}^T$,

$$H_i\left(b_i'\right)\left(x_i' - b_i'\right) \geq H_i(b)\left(x_i' - b\right)$$

which can be rearranged as

$$\left(H_i\left(b_i'\right) - H_i(b)\right) x_i' \geq H_i\left(b_i'\right) b_i' - H_i(b)\, b \tag{G.3}$$

Now notice that for all $b < b_i'$ we must have $H_i\left(b_i'\right) - H_i(b) > 0$. Since H_i is nondecreasing, $b < b_i'$ implies that $H_i\left(b_i'\right) - H_i(b) \geq 0$, but if $H_i\left(b_i'\right) - H_i(b) = 0$, then b_i' cannot be a best response—a bid of $b < b_i'$ has the same chances of winning while it decreases the amount bid if bidder i wins.

Now (G.3) implies that for $x_i'' > x_i'$, for all $b < b_i'$ such that $b \in \mathcal{B}^T$,

$$\left(H_i\left(b_i'\right) - H_i(b)\right) x_i'' > H_i\left(b_i'\right) b_i' - H_i(b)\, b$$

Thus, when the value is x_i'', it is strictly better to bid b_i' than to bid a smaller amount. This implies that any best response when the value is x_i'' is at least as large as b_i', so $\min BR_i\left(x_i''\right) \geq b_i'$. ■

A bidding strategy for bidder i, $\beta_i : \mathcal{X}_i \to \mathcal{B}^T$ is said to be a *best response against* $\boldsymbol{\beta}_{-i}$ if for all x_i, $\beta_i(x_i)$ is a best response when his value is x_i.

The import of Lemma G.1 is that if the strategy $\beta_i : \mathcal{X}_i \to \mathcal{B}^T$ is a best response, then it is a nondecreasing function with a finite number of discontinuities—it is a "step function." Thus, we can find T points in $[0, \omega_i]$, say $\alpha_i^1 \leq \alpha_i^2 \leq \cdots \leq \alpha_i^T$ such that

$$\beta_i(x_i) = b^t \text{ if } \alpha_i^t < x_i < \alpha_i^{t+1}, \tag{G.4}$$

where by convention, we set $\alpha_i^0 \equiv 0$ and $\alpha_i^{T+1} \equiv \omega_i$. Note that we have said nothing about what happens at the points α_i^t themselves—$\beta_i\left(\alpha_i^t\right)$ is either b^{t-1} or b^t—but since there are only a finite number of such points, the bids at these points do not affect a bidder's expected payoff. Thus, except perhaps at a finite number of points, any β_i that is a best response is completely determined by the vector $\boldsymbol{\alpha}_i = \left(\alpha_i^1, \alpha_i^2, \ldots, \alpha_i^T\right)$. In other words, any β_i that is a best-response—and thus a step function—can be represented by a finite dimensional object, the vector $\boldsymbol{\alpha}_i$.

Given a bidding strategy β_i that is nondecreasing, if $\boldsymbol{\alpha}_i$ is such that (G.4) holds, we will write $\boldsymbol{\alpha}_i \leftrightarrow \beta_i$ to denote that β_i can be equivalently represented by

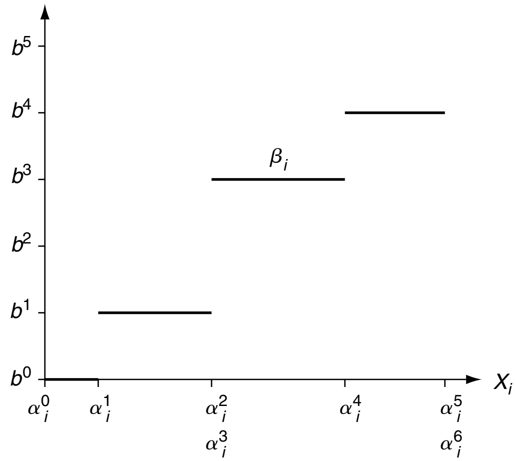

FIGURE G.1 A best response with discrete bids.

$\boldsymbol{\alpha}_i$ and vice versa. In that case, we will refer to $\boldsymbol{\alpha}_i$ itself as the "bidding strategy" of bidder i.

See Figure G.1 for an illustration when $T = 5$. In the figure, there is no x_i such that $\beta_i(x_i) = b^2$, so $\alpha_i^2 = \alpha_i^3$. Similarly, there is no x_i such that $\beta_i(x_i) = b^5$, so $\alpha_i^5 = \alpha_i^6 \equiv \omega_i$.

Given the strategies $\boldsymbol{\alpha}_{-i} = (\boldsymbol{\alpha}_j)_{j \neq i}$ of the other bidders, let $\Gamma_i(\boldsymbol{\alpha}_{-i})$ denote the set of best response strategies for bidder i, consisting of vectors $\boldsymbol{\alpha}_i \in [0, \omega_i]^T$. By this we mean, of course, that there is a β_i that is a best response to $\boldsymbol{\beta}_{-i}$ and $\boldsymbol{\alpha}_i \leftrightarrow \beta_i$ and for all $j \neq i$, $\boldsymbol{\alpha}_j \leftrightarrow \beta_j$. The mapping $\Gamma_i(\cdot)$ assigns to every element in $\times_{j \neq i}[0, \omega_j]^T$ a subset of $[0, \omega_i]^T$, and we will refer to it as bidder i's *best-response correspondence*. Some properties of a bidder's best response correspondence will prove useful.

First, note that for all $\boldsymbol{\alpha}_{-i}$, the set of best responses $\Gamma_i(\boldsymbol{\alpha}_{-i})$ is a convex set. This is equivalent to saying that if some b_i is a best response at x_i' and also at x_i'', then for all $\lambda \in [0, 1]$ it is also a best response at $\lambda x_i' + (1 - \lambda) x_i''$ and this follows trivially from (G.2).

Second, note that if $(\boldsymbol{\alpha}_i^n, \boldsymbol{\alpha}_{-i}^n)$ is a sequence converging to $(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_{-i})$ and for all n, $\boldsymbol{\alpha}_i^n \in \Gamma_i(\boldsymbol{\alpha}_{-i}^n)$, then $\boldsymbol{\alpha}_i \in \Gamma_i(\boldsymbol{\alpha}_{-i})$. For all n and j let β_j^n be such that $\boldsymbol{\alpha}_j^n \leftrightarrow \beta_j^n$ and let β_j be such that $\boldsymbol{\alpha}_j \leftrightarrow \beta_j$. Let $H_i^n(\cdot)$ be defined according to (G.1) when bidders $j \neq i$ use strategies β_j^n and similarly, let $H^i(\cdot)$ be defined according to (G.1) when bidders $j \neq i$ use strategies β_j. Since $\boldsymbol{\alpha}_{-i}^n \to \boldsymbol{\alpha}_{-i}$, we have that for all $j \neq i$, $\beta_j^n \to \beta_j$, so $H_i^n(\cdot) \to H_i(\cdot)$ at every point of $\mathcal{B}^T$. Now it is routine to verify that if for all n, β_i^n is a best response to $\boldsymbol{\beta}_{-i}^n$, and $(\beta_i^n, \boldsymbol{\beta}_{-i}^n) \to (\beta_i, \boldsymbol{\beta}_{-i})$, then β_i is a best response to $\boldsymbol{\beta}_{-i}$. This establishes that $\boldsymbol{\alpha}_i \in \Gamma_i(\boldsymbol{\alpha}_{-i})$.

Kakutani Fixed Point Theorem. Let $\mathcal{Z}$ be a nonempty, compact and convex set and let Γ be a correspondence that maps every element $\mathbf{z} \in \mathcal{Z}$ to a nonempty subset of $\mathcal{Z}$. The Kakutani *fixed point theorem* states that if (1) Γ is *convex valued*—that is, for all $\mathbf{z}$, $\Gamma(\mathbf{z})$ is convex, and (2) Γ has a *closed graph*—that is, $(\mathbf{y}^n, \mathbf{z}^n) \to (\mathbf{y}, \mathbf{z})$ and for all n, $\mathbf{y}^n \in \Gamma(\mathbf{z}^n)$ implies that $\mathbf{y} \in \Gamma(\mathbf{z})$, then there exists a $\mathbf{z}^*$ such that $\mathbf{z}^* \in \Gamma\left(\mathbf{z}^*\right)$. Such a $\mathbf{z}^*$ is called a fixed point of Γ.

Existence of Equilibrium. In our context, we can define $\mathcal{Z} = \times_i [0, \omega_i]^T$ and $\Gamma(\boldsymbol{\alpha}) = \times_i \Gamma_i\left(\boldsymbol{\alpha}_{-i}\right)$, where each Γ_i is i's best response correspondence. As just argued, Γ is convex valued and has a closed graph. The Kakutani fixed point theorem then implies that there exists an $\boldsymbol{\alpha}^*$ such that $\boldsymbol{\alpha}^* \in \Gamma\left(\boldsymbol{\alpha}^*\right)$. If we define β_i^* as in (G.4), that is, $\boldsymbol{\alpha}_i^* \leftrightarrow \beta_i^*$, then $\boldsymbol{\beta}^* = \left(\beta_1^*, \beta_2^*, \ldots, \beta_N^*\right)$ constitutes an equilibrium of the first-price auction with discrete bids. We have thus established the following:

Proposition G.1. *Suppose all bids must lie in the set $\mathcal{B}^T$. Then there exists an equilibrium of the first-price auction in which all bidders follow nondecreasing strategies.*

TAKING LIMITS

In the previous section, bidders were restricted to use strategies with a minimum bid increment of ω/T. We argued that with this restriction, for all T, there exists a pure strategy equilibrium, say $\boldsymbol{\beta}^*(T)$, in which each bidder's strategy is a nondecreasing function of his or her value. Here we are interested in examining what happens as the restriction is removed—that is, as T approaches infinity. A detailed treatment of this question is somewhat involved, so we only indicate the path to be followed.

First, it can be shown that there exists a subsequence of pure strategy equilibria $\boldsymbol{\beta}^*(T)$ that converges to a vector of strategies, say $\boldsymbol{\beta}^*$, in the auction in which bids are unrestricted. Moreover, the convergence is uniform almost everywhere. The strategies in $\boldsymbol{\beta}^*$ are, of course, all nondecreasing also.

Second, it can be argued that $\boldsymbol{\beta}^*$ constitutes an equilibrium of the auction with unrestricted bids. If for all i, the limiting strategy β_i^* were strictly increasing, this would follow immediately. This is because, in that case, ties would occur with probability zero. The argument that this is indeed the case uses the fact that the sequence $\boldsymbol{\beta}^*(T)$ converges to $\boldsymbol{\beta}^*$ uniformly almost everywhere and is rather involved. The interested reader may consult the readings mentioned in the notes that follow.

NOTES ON APPENDIX G

This appendix is based on the work of Athey (2001), in which she establishes a general result for the existence of pure strategy equilibria in games with incomplete information. The key insight is that as long as the conclusion of Lemma G.1 holds, the arguments leading up to Proposition G.1 can be applied unchanged and hence an equilibrium exists. A consequence of her result is that a pure strategy equilibrium can be shown to exist under a wide variety of circumstances, for example, with risk aversion or affiliated private values. For asymmetric interdependent values with affiliated signals, however, this technique is applicable to only the case of two bidders. Reny and Zamir (2004) have, however, succeeded in developing an existence result in this environment that applies with an arbitrary number of bidders.

Other papers that establish the existence of an equilibrium in first-price auctions using discrete approximation techniques—either by discretizing the set of values or, as above, the set of possible bids—include Lebrun (1996) and Maskin and Riley (2000b). Reny (1999) establishes a general existence result for a class of discontinuous games. This can also be applied to first-price auctions.

Jackson and Swinkels (2005) exhibit some very general existence results in private value settings by using somewhat different techniques. Their approach is to show that an equilibrium exists in an auxiliary game in which tie-breaking is endogenously chosen and then to show that the tie-breaking rule is, in fact, irrelevant.

An alternative approach is to write down a set of necessary conditions that a pure strategy equilibrium in increasing strategies must satisfy. For instance, if there are only two bidders, the necessary first-order conditions are, as derived in (4.18) on page 47, are that the inverse bidding strategies ϕ_1 and ϕ_2 must satisfy for $i = 1, 2$, and $j \neq i$,

$$\phi_j'(b) = \frac{F_j\left(\phi_j(b)\right)}{f_j\left(\phi_j(b)\right)} \frac{1}{(\phi_i(b) - b)} \tag{G.5}$$

together with the boundary conditions that for $i = 1, 2$, $\phi_i(0) = 0$. The fundamental theorem of differential equations provides sufficient conditions for the existence and uniqueness of the solution to such a system. A difficulty with this approach is that precisely where the boundary condition holds—at $b = 0$—the right-hand side of (G.5) has a $\frac{0}{0}$ form, so it is undefined. This means that the fundamental theorem of differential equations, which requires that the right-hand side satisfy a Lipschitz condition at the boundary point, cannot be directly applied. These difficulties can be overcome, however, as shown by Plum (1992) and Lebrun (1999).