Chapter | Seventeen

Packages and Positions

This chapter continues the study of auctions of multiple nonidentical objects in a private value setting. Of course, with private values, the VCG mechanism results in efficient allocations. This chapter explores whether, and under what circumstances, VCG outcomes can be achieved via open ascending auctions.

We study two specific settings for which positive results are available. In the first, there are multiple objects but these are considered to be gross substitutes[1] by all bidders. As a benchmark, recall that with mutiple identical objects with diminishing marginal values there exists an open ascending auction—the Ausubel auction—that replicates VCG outcomes. It will be shown that a generalization of this result holds for nonidentical objects that are substitutes. The generalization requires that bidders be allowed to place bids not only on individual objects but also on combinations of objects or *packages*.

In the second setting, there are also multiple nonidentical objects to be sold but each buyer can use at most one. Moreover, all bidders rank these objects in the same manner. It is then natural to think of the objects as *positions* in a queue or a list.

In each case, there is a "natural" ascending auction that replicates VCG outcomes. The two main propositions of this chapter can be viewed as "restricted domain" results. If we think of $\mathcal{X}$ as the set of all possible value vectors for the objects on sale, then each result concerns a particular restriction of $\mathcal{X}$—in one case, to those values satisfying the gross substitutes condition; and in the other case, to those values conforming to the interpretation of the objects as positions.

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[1]A precise definition is provided below.

17.1 PACKAGE AUCTIONS

As in the last chapter, there is a set of K distinct objects for sale and N buyers. Each buyer i has a private value $x^i(S)$ for each package $S \subseteq K$. We suppose that for all i, $x^i(\emptyset) = 0$ and denote by $\mathbf{x}^i$ the vector $\left(x^i(S)\right)_{S\subseteq K}$ of package values of buyer i.

An *allocation* $\mathbf{S} = \left\langle S^1, S^2, \ldots, S^N \right\rangle$ is an ordered collection of N packages, which forms a partition—that is, $\cup_{i\in\mathcal{N}} S^i = K$ and for all $i \neq j$, $S^i \cap S^j = \emptyset$. Buyer i is allocated package S^i, so in an allocation (i) every object is allocated to some buyer, and (ii) no object is allocated to more than one buyer.

17.1.1 The Ascending Auction

The rules of the Ausubel-Milgrom ascending auction are as follows. Bidding proceeds in discrete rounds and bids are denominated in terms of a smallest monetary unit, say ε.[2]

In any round, say t, each bidder can place a bid $b_t^i(S)$ on any package $S \subseteq K$. Within a round, all bids are placed simultaneously. After each round, the auctioneer chooses a set of *provisionally winning bids*. These are the bids that maximize the total revenue subject to constraints that (1) each bidder is associated with at most one provisionally winning bid; and (2) the same object cannot be allocated to two bidders. Specifically, at the end of round t the seller chooses an allocation $\mathbf{S}_t = \left\langle S_t^1, S_t^2, \ldots, S_t^N \right\rangle$ that maximizes the provisional revenue, that is,

$$\mathbf{S}_t = \arg\max_{\mathbf{S}} \sum_{i\in\mathcal{N}} b_t^i\left(S^i\right)$$

(A fixed tie-breaking rule is invoked in case there are ties.) Bidders who are provisionally awarded some objects—that is, if $S_t^i \neq \emptyset$—are called *provisional winners*.

At the end of round t, the provisional *payoff* of bidder i is

$$\pi_t^i = x^i\left(S_t^i\right) - b_t^i\left(S_t^i\right)$$

if he is among the provisional winners ($S_t^i \neq \emptyset$) and zero otherwise.

As long as the highest bid on at least one package was raised in round t, round $t+1$ proceeds in the same manner. If the highest bid on no package was raised in round t, the auction comes to an end. In that case, the provisional winning bids become final.

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[2]The discreteness of the rounds and the bids distinguishes this auction from the ascending auctions considered in earlier chapters.

PROXY BIDDING

Instead of analyzing the ascending auction directly, it is more convenient, and substantially easier, to study a "direct mechanism" version of the auction in which each bidder submits a value vector $\mathbf{z}^i$ to a proxy who then bids on his behalf in a prespecified manner. Of course, bidders are free to misreport—that is, to submit a $\mathbf{z}^i \neq \mathbf{x}^i$, which differs from his true value. The seller's choices of provisionally winning bids are made in the same manner as just specified.

Let $b^i_{t-1}(S)$ be bidder i's bid on the package S at the end of round $t-1$ and, as previously, let $\mathbf{S}_{t-1} = \langle S^1_{t-1}, S^2_{t-1}, \ldots, S^N_{t-1} \rangle$ be the provisional allocation chosen by the seller at the end of round $t-1$. The *minimum bid* that i can place on package S in round t is $\underline{b}^i_t(S) = b^i_{t-1}(S) + \varepsilon$, if $S \neq S^i_{t-1}$; and $\underline{b}^i_t(S) = b^i_{t-1}(S)$, otherwise.

Suppose that bidder i reported a value vector $\mathbf{z}^i$ to the proxy. Then in round t, the proxy bids on behalf of i as follows:

- The proxy first determines packages $\hat{S}^i_t$ that would be most profitable for i at the minimum bids according to the reported values (breaking ties arbitrarily), that is,

$$\hat{S}^i_t \in \arg\max_S \left(z^i(S) - \underline{b}^i_t(S)\right)$$

- The proxy then bids the minimum bid on the profit maximizing packages $\hat{S}^i_t$ (there may be more than one) and leaves all other bids unchanged; that is

$$\beta^i_t(S \mid \mathbf{z}^i) = \begin{cases} \underline{b}^i_t(S) & \text{if } S = \hat{S}^i_t \text{ and } z^i(S) \geq \underline{b}^i_t(S) \\ b^i_{t-1}(S) & \text{otherwise} \end{cases}$$

The *ascending proxy auction* results when proxy bidding takes place as described above and the only choice an actual bidder faces is what value vector $\mathbf{z}^i$ to report to his proxy. Our goal is to identify circumstances under which reporting truthfully—that is, reporting $\mathbf{z}^i = \mathbf{x}^i$—constitutes an equilibrium.

17.1.2 Gross Substitutes

We now introduce a new condition on buyers' values. This condition will be sufficient to guarantee that the ascending proxy auction has an equilibrium that replicates VCG outcomes.

Suppose that each good $a \in K$ has a price p_a. The *demand* of buyer i at prices $\mathbf{p} = (p_a)_{a \in K}$ is

$$D^i(\mathbf{p}) = \arg\max_{S \subseteq K} \left(x^i(S) - \sum_{a \in S} p_a \right)$$

which consists of those packages that maximize the buyer's net profit when the packages are purchased at the item-by-item prices given by $\mathbf{p}$. Notice that there may be more than one package with this property.

Bidder i satisfies the *gross substitutes* condition if for any two price vectors $\mathbf{p}' \geq \mathbf{p}$, for every $S \in D^i(\mathbf{p})$ there exists an $S' \in D^i(\mathbf{p}')$ such that the set $\{a : a \in S,\ p_a = p'_a\}$ is a subset of S'. The condition requires that a rise in the price of an object (or a set of objects) does not decrease the demand for other objects. This condition is clearly satisfied if all objects are identical and bidders have declining marginal values.

The gross substitutes condition is stronger than the substitutes condition (16.6) introduced in the previous chapter (a problem at the end of this chapter asks the reader to verify this). It has some nontrivial implications regarding how the objects in K should be "optimally" allocated across potential buyers.

Given a value vector $\mathbf{x}$ and a subset of bidders $\mathcal{I} \subseteq \mathcal{N}$, define

$$w(\mathcal{I}) = \max_{\mathbf{S}} \sum_{i \in \mathcal{I}} x^i(S)$$

to be the maximum *surplus* attainable by the bidders in $\mathcal{I}$ from an optimal allocation of all the objects.

The following result is key.

Lemma 17.1. *Suppose all bidders satisfy the gross substitutes condition. Then the corresponding surplus function w is submodular—that is, if $\mathcal{I} \subset \mathcal{J} \subset \mathcal{N}$ and $i \notin \mathcal{J}$, then*

$$w(\mathcal{I} \cup i) - w(\mathcal{I}) \geq w(\mathcal{J} \cup i) - w(\mathcal{J})$$

The lemma says that the inclusion of a particular buyer i in a set of buyers is subject to diminishing returns—the additional surplus accruing from including i decreases as the set gets larger.

We omit a formal proof. There are two reasons. First, a complete proof requires some additional machinery which would lead us too far afield. Second, proofs are easily available in readings indicated in the chapter notes.

VCG PAYOFFS

Our goal is to compare equilibrium outcomes of the proxy auction with VCG outcomes. To this end, it will be convenient to write the payoffs in the VCG mechanism in terms of the surplus function w. Notice that $w(\mathcal{N}) = W(\mathbf{x})$, as defined in (16.2) and that $w(\mathcal{N} \setminus i) = W(\mathbf{0}, \mathbf{x}^{-i})$, as defined in (16.4). From these identities it follows that the VCG *payoff* (not the payment) of buyer i can be written as

$$\overline{\pi}^i = w(\mathcal{N}) - w(\mathcal{N} \setminus i) \tag{17.1}$$

and represents i's marginal contribution to the total surplus.

17.1.3 Equilibrium of the Proxy Auction

We are now ready to prove the main result of this section. We suppose that the minimum bid increment, ε, is small and can be neglected.

Proposition 17.1. *Suppose that all bidders satisfy the gross substitutes condition. Then truthful reporting is an* ex post *equilibrium of the ascending proxy auction in which the allocations and payments are the same as in the VCG mechanism.*

Proof. We first establish that truthful reporting in the ascending proxy auction leads to VCG outcomes. We then show that truthful reporting is indeed an *ex post* equilibrium.

Suppose all bidders report their values truthfully. We claim that for all i, the final payoff $\pi_T^i \geq \overline{\pi}^i$ (up to ε), his VCG payoff. Suppose to the contrary that in some round t, $\pi_t^i < \overline{\pi}^i$. Then bidder i must be a provisional winner. To see this, suppose that $\mathcal{I}$ is the set of provisional winners in round t and $i \notin \mathcal{I}$. Then the seller's revenue in round t is

$$\begin{aligned} w(\mathcal{I}) - \sum\nolimits_{j\in\mathcal{I}} \pi_t^j &< w(\mathcal{I}) - \sum\nolimits_{j\in\mathcal{I}} \pi_t^j + \overline{\pi}^i - \pi_t^i \\ &= w(\mathcal{I}) - \sum\nolimits_{j\in\mathcal{I}\cup i} \pi_t^j + w(\mathcal{N}) - w(\mathcal{N}\setminus i) \\ &\leq w(\mathcal{I}) - \sum\nolimits_{j\in\mathcal{I}\cup i} \pi_t^j + w(\mathcal{I}\cup i) - w(\mathcal{I}) \\ &= w(\mathcal{I}\cup i) - \sum\nolimits_{j\in\mathcal{I}\cup i} \pi_t^j, \end{aligned}$$

where the inequality in the third line follows from the submodularity of w. Thus, the seller would be better off by including i in the set of provisional winners. But this contradicts the fact that in every round, the seller chooses an allocation that maximizes his provisional revenue. Thus, there is no incentive for player i to bid in a way that results in a provisional payoff π_t^i that is less than $\overline{\pi}^i$ (more accurately, $\overline{\pi}^i - \varepsilon$ because bids are discrete). Hence, we have that for all i, $\pi_T^i \geq \overline{\pi}^i$ (up to ε).

Now suppose that there is a bidder i such that $\pi_T^i > \overline{\pi}^i$. Then

$$\begin{aligned} w(\mathcal{N}) &= \pi_T^0 + \pi_T^i + \sum\nolimits_{j\neq i} \pi_T^j \\ &> \pi_T^0 + w(\mathcal{N}) - w(\mathcal{N}\setminus i) + \sum\nolimits_{j\neq i} \pi_T^j \end{aligned}$$

which implies that the sum of the payoffs of the seller and all bidders other than i is less than $w(\mathcal{N}\setminus i)$; that is,

$$\pi_T^0 + \sum\nolimits_{j\neq i} \pi_T^j < w(\mathcal{N}\setminus i) \tag{17.2}$$

On the other hand, the seller's revenue

$$\begin{aligned}
\pi_T^0 &= \max_{\mathbf{S}} \sum\nolimits_{j\in\mathcal{N}} \beta_T^j(S^j \mid \mathbf{x}^j) \\
&= \max_{\mathbf{S}} \sum\nolimits_{j\in\mathcal{N}} \max\left(x^j(S^j) - \pi_T^j, 0\right) \\
&= \max_{\mathbf{S}} \max_{\mathcal{I}\subseteq\mathcal{N}} \sum\nolimits_{j\in\mathcal{I}} \left(x^j(S^j) - \pi_T^j\right) \\
&= \max_{\mathcal{I}\subseteq\mathcal{N}} \max_{\mathbf{S}} \sum\nolimits_{j\in\mathcal{I}} \left(x^j(S^j) - \pi_T^j\right) \\
&= \max_{\mathcal{I}\subseteq\mathcal{N}} \left(w(\mathcal{I}) - \sum\nolimits_{j\in\mathcal{I}} \pi_T^j\right)
\end{aligned}$$

and so, in particular, for $\mathcal{I} = \mathcal{N}\setminus i$, this implies that

$$\pi_T^0 + \sum\nolimits_{j\neq i} \pi_T^j \geq w(\mathcal{N}\setminus i) \tag{17.3}$$

contradicting (17.2). We have thus argued that truthful reporting leads to the VCG outcome.

In order to show that truthful reporting constitutes an *ex post* equilibrium, consider a bidder i and suppose that all bidders $j \neq i$ report truthfully. We know from (17.3) that independent of what bidder i submits as his value vector, the seller's revenue $\pi_T^0 \geq w(\mathcal{N}\setminus i) - \sum_{j\neq i} \pi_T^j$. The right-hand side is a lower bound on the seller's revenue because it can be obtained by ignoring bidder i and including all other bidders in the set of winners. On the other hand, since the total payoff of all the participants—the bidders and the seller—can never exceed $w(\mathcal{N})$, the seller's revenue $\pi_T^0 \leq w(\mathcal{N}) - \sum_{j\in\mathcal{N}} \pi_T^j$. This means that i's payoff from any misreporting of values cannot exceed $w(\mathcal{N}) - w(\mathcal{N}\setminus i) = \overline{\pi}^i$. But this is exactly bidder i's payoff from truthful bidding and so reporting a value vector $\mathbf{z}^i \neq \mathbf{x}^i$ cannot be profitable. ■

17.2 POSITION AUCTIONS

Consider a situation in which K positions have be allocated to $N > K$ bidders. Bidders' preferences over positions are identical—each prefers position 1 to position 2 to position 3 and so on. Specifically, suppose that each position k has an inherent value α_k and $\alpha_1 > \alpha_2 > \cdots > \alpha_K > 0$. The α_k's are fixed and commonly known. The value to bidder i of being in position k is $\alpha_k x^i$ where x^i is the realization of i's private value X^i. For simplicity, we assume that each X^i is independently distributed on $[0,1]$ according to the distribution function F^i.

Consider the following auction format. Each of the bidders submits a bid b^i and suppose we rename the bidders so that $b^1 > b^2 > \cdots > b^N$. Bidder 1 is then allocated position 1, bidder 2 is allocated position 2, and so on. Bidders $K+1$ to N are not allocated any positions. The payment rule is as follows. Bidder 1 pays b^2, bidder 2 pays b^3 ... and bidder K pays b^{K+1}. For obvious reasons, this auction is called a (sealed-bid) *generalized second-price* (or GSP) auction.

The study of such situations is motivated by the sale of advertising slots by Internet search engines like Google and Yahoo!. Typing a keyword, say "flowers," in a search engine produces K links to providers of flower delivery services. The link positioned at the top of the page receives more clicks from those searching for flowers than the link below it in the second position, which receives more clicks than the link in the third position, and so on. The parameter α_k can be thought of as the number of clicks associated with having a link in position k and x^i can be thought of as the value to provider i of a single click.

Before embarking on a study of the GSP and its variants, it is instructive to examine the workings of the VCG mechanism when positions are to be allocated. The VCG mechanism asks each bidder to report a value z^i and then allocates the positions to maximize the total surplus. It is a weakly dominant strategy to report truthfully, that is, $z^i = x^i$. If we rename the bidders so that $x^1 > x^2 > \cdots > x^N$, then social surplus is obviously maximized by allocating position 1 to bidder 1, allocating position 2 to bidder 2 and so on. For any bidder i, such that $i \leq K$, the VCG payment is

$$\overline{M}^i(\mathbf{x}) = \sum_{j=i+1}^{K+1} \left(\alpha_{j-1} - \alpha_j\right) x^j, \tag{17.4}$$

where $\alpha_{K+1} = 0$. This is because by reporting x^i instead of 0, bidder i obtains position i. This pushes all those in positions $k \geq i$ one position lower. The preceding sum is then simply the externality exerted by bidder i on the other bidders. A bidder $i > K$, of course, is not allocated a position and pays nothing.

Instead of studying the sealed-bid GSP, we will study a closely related open ascending auction. The *generalized English* auction bears the same relationship to the GSP as the English auction bears to the second-price auction. Specifically, an auctioneer starts at zero and gradually raises the price. As in the standard English auction, at a price of zero, all bidders are active and indicate this by pressing buttons or raising hands. Subsequently, a particular bidder i may decide to drop out of the auction at a price b^i, which we will call i's "bid." Once a bidder drops out he cannot become active again. The auction ends when there is only one bidder remaining, say bidder 1. By renaming the bidders and neglecting ties, suppose again that $b^2 > b^3 > \cdots > b^N$. Then bidder 1 is allocated position 1 and pays b^2 for this, bidder 2 is allocated position 2 and pays b^3 and so on. All bids are in "per unit" terms. Thus the total amount paid by bidder $i \leq K$ for obtaining position i is $\alpha_i b^{i+1}$.

Some equilibrium properties of the generalized English auction are given in the following result.

Proposition 17.2. *The generalized English auction has a symmetric* ex post *equilibrium in which the allocations and payments are the same as in the VCG mechanism.*

Proof. As in Chapter 6, a bidding strategy in the generalized English auction is defined by a collection $\boldsymbol{\beta} = (\beta_N, \beta_{N-1}, \ldots, \beta_2)$ of $N-1$ functions $\beta_k : [0,1] \times \mathbb{R}_+^{N-k} \to \mathbb{R}_+$, for $1 < k \leq N$, where $\beta_k\left(x, p^{k+1}, \ldots, p^N\right)$ is the price at which a bidder will drop out if the number of bidders who are still active is k, his own signal is x, and the prices at which the other $N-k$ bidders dropped out were $p^{k+1} \geq p^{k+2} \geq \cdots \geq p^N$. Note that since values are private, the identities of the bidders who dropped out are irrelevant.

Consider the following strategy (which depends only on the last drop-out price p^{k+1}, and not on any of the earlier drop-out prices). If $k > K$, then

$$\beta_k\left(x, p^{k+1}\right) = x \tag{17.5}$$

and if $k \leq K$, then

$$\beta_k\left(x, p^{k+1}\right) = x - \frac{\alpha_k}{\alpha_{k-1}}\left(x - p^{k+1}\right) \tag{17.6}$$

For $k \leq K$, the prescribed drop-out price $\beta_k\left(x, p^{k+1}\right)$ is such that a bidder with value x is indifferent between winning position k at the price p^{k+1} and winning position $k-1$ at the price $\beta_k\left(x, p^{k+1}\right)$. Also, note that β_k is an increasing function of x.

To see why this constitutes an equilibrium, consider a bidder with value x and suppose all other bidders $j \neq k$ follow the strategy just specified. Rename the bidders according to their realized values so that $x^1 > x^2 > \cdots > x^N$ and let $x^i = x$. (Once again, we are neglecting ties.)

There are two cases to consider. First, suppose $i > K$. By following the prescribed strategy, bidder i will not win any position. If he deviates and wins position K, then it will be at a price of $x^K > x^i$ and so such a deviation is not profitable. The same is true if he deviates and wins any other position.

Second, suppose $i \leq K$. By following the prescribed strategy, bidder i will win position i. If he deviates and wins position $i-1$, then this will be at a price p^i such that the bidder with value x^{i-1} is just indifferent between winning position i at price p^{i+1} and winning position $i-1$ at price p_i. But since $x^i < x^{i-1}$, this means that bidder strictly prefers to win i at price p^{i+1}. The same is true if he deviates and wins any other position.

Since none of the reasoning above relied on the distribution of values and thus was of an *ex post* nature, we have established that the prescribed strategies constitute an *ex post* equilibrium.

The allocation is efficient and it only remains to verify that the payments in the generalized English auction, say M^i, are the same as in the VCG mechanism. Again, suppose $x^1 > x^2 > \cdots > x^N$. Then for all $i > K$, $M^i = 0$, which is the same as in the VCG mechanism. Next, note that $M^K = \alpha_K x^{K+1}$, which is also the same as in the VCG mechanism. For $i < K$, the equilibrium strategy (17.6) implies that bidder $i+1$ drops out at a price $\beta_{i+1}\left(x^{i+1}, p^{i+2}\right) \equiv p^{i+1}$ such that

$$p^{i+1} = x^{i+1} - \frac{\alpha_{i+1}}{\alpha_i}\left(x^{i+1} - p^{i+2}\right)$$

which can be rearranged so that

$$\alpha_i p^{i+1} = \left(\alpha_i - \alpha_{i+1}\right) x^{i+1} + \alpha_{i+1} p^{i+2}$$

or equivalently,

$$M^i = \left(\alpha_i - \alpha_{i+1}\right) x^{i+1} + M^{i+1}$$

Thus M^i satisfies the formula in (17.4) and so is identical to $\overline{M}^i$, the VCG payment. ■

PROBLEMS

17.1. (Inefficiency without package bidding) Suppose that there are two objects, a and b, for sale and two bidders with the following values

	a	b	ab
$\mathbf{x}^1$	y	z	2
$\mathbf{x}^2$	2	2	2

where y and z are parameters that lie between 0 and 1. Argue that an ascending auction format in which bidders can only bid on a and b individually, and not on the package ab, cannot allocate efficiently. (Note: Without package bidding, the price of the package ab is necessarily the sum of the prices of the individual objects a and b.)

17.2. (Gross substitutes) Show that if bidder i with value vector $\mathbf{x}^i$ satisfies the gross substitutes condition (defined on page 241), then $\mathbf{x}^i$ satisfies the substitutes condition (defined in (16.6)). Equivalently, show that the gross substitutes condition implies that $x^i(S)$ is submodular.

17.3. (Complements) Suppose that the objects in $K = \{a_1, a_2, b_1, b_2\}$ are sold via the ascending proxy auction. There are five interested bidders Bidder 1 has use only for objects a_1 and b_1; bidders 2 and 3 have use only for objects a_2 and b_2; bidder 4 has use only for b_1 and b_2; and bidder 5 has use only for objects a_1 and a_2. Specifically, the values attached by the bidders to these bundles are

$$x^1(a_1 b_1) = 10$$
$$x^2(a_2 b_2) = 20$$
$$x^3(a_2 b_2) = 25$$
$$x^4(b_1 b_2) = 10$$
$$x^5(a_1 a_2) = 10$$

All other combinations (or packages) are valued at zero. (The specification is the same as in Problem 16.2.)

a. Show that the gross substitutes condition is violated.

b. Show that it is not an equilibrium for each bidder to report truthfully to his proxy.

17.4. (Identical objects) Consider a situation in which K identical units are to be allocated among N identical buyers. Each buyer has the same K-dimensional value vector $\mathbf{x}$, where $x(k)$ denotes the total value from obtaining k units. Suppose that the units have decreasing marginal values—that is, $x(k+2) - x(k+1) \leq x(k+1) - x(k)$. An allocation is then simply a vector of the form $\mathbf{s} = \left(s^1, s^2, \ldots, s^N\right)$, where s^i is the number of units awarded to buyer i. For any $I \leq N$, define the surplus function $w(I) = \max_{\mathbf{s}} \sum_{i=1}^{I} x(s^i)$. Show that w is submodular, that is, for any $I < J < N$

$$w(I+1) - w(I) \geq w(J+1) - w(J)$$

(Note: This is just a specialization of Lemma 17.1 to the case of identical objects and identical buyers.)

17.5. (GSP) Three positions are to be assigned among four bidders. The positions have "click" values of $\alpha_1 > \alpha_2 > \alpha_3$. Each of the four bidders have "per-click" values of $x^1 > x^2 > x^3 > x^4$. These values are commonly known and so we are considering a situation with complete information.

a. Find an equilibrium of the sealed-bid generalized second-price auction (GSP).

b. Is the equilibrium unique? If not, characterize all equilibria of the GSP.

CHAPTER NOTES

The ascending auction with package bidding was introduced and analyzed by Ausubel and Milgrom (2002) in an important paper. The material in the first section is based entirely on their work, as are the problems. In this chapter we concentrated on the relationship between the truthful equilibria of the ascending proxy auction and VCG outcomes. In addition to this, in their paper, Ausubel and Milgrom (2002) also develop some deep connections between VCG outcomes and the core of the cooperative game defined by the surplus function w. For instance, it turns out that the VCG payoff is the largest payoff an agent (either a bidder or the seller) can obtain in any core allocation. With gross substitutes, the vector of VCG payoffs itself lies in the core.

The gross substitutes condition originates in neoclassical general equilibrium theory (with divisible goods). In that context, it is a strong condition—when applied at the aggregate level it guarantees the uniqueness and stability of Walrasian equilibrium. The version of the condition used here—with indivisibilities—was introduced by Kelso and Crawford (1982) in the context of a worker-firm matching problem. Proofs of the result that the gross substitutes condition implies that the surplus function w is submodular may be found in Ausubel and Milgrom (2002) and in the book by Milgrom (2004). The latter contains a very readable account of the theory of package auctions.

While the ascending package auction (and its proxy version) seems like a very natural mechanism, it places substantial computational demands on both the seller and the bidders. For instance, in each round, the seller is asked to compute the set of provisionally winning bids and bidders are asked to compute the package that is most profitable. These computations are no less complex than those required by the VCG mechanism.

The allocation of nonidentical objects when the gross substitutes condition fails is problematic. While the VCG mechanism achieves efficiency, it has other weaknesses. For instance, the resulting revenue may be very low (as indicated in Problem 16.1). Moreover, ascending auctions do not perform well in these circumstances (see Problem 17.3).

The sealed-bid generalized second-price auction for the sale of positions was studied by Edelman, Ostrovsky, and Schwarz (2007) and Varian (2007). Both analyses were in the context of a model with complete information and emphasize the connection to the VCG mechanism. Edelman, Ostrovsky, and Schwarz (2007) also considered the generalized English auction and the material in this chapter is largely based on their analysis.