MATH 425b ASSIGNMENT 1
SPRING 2016
Prof. Alexander
Due Monday January 25.

Rudin Chapter 7 #3, 6, 8 and:

(I) Let $g_n(x) = nxe^{-nx^2}$, $0 \le x \le 1$.
(a) Show that g_n converges pointwise on $[0,1]$ to some g, and find g.
(b) Show that $\int_0^1 g_n(x)\,dx \not\to \int_0^1 g(x)\,dx$.

(II) Define the functions $f_n : (-\pi, 3\pi) \to \mathbb{R}$ by

$$f_n(x) = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots \pm \frac{\sin nx}{n}.$$

It can be shown (you don't need to prove it!) that this sequence of functions converges pointwise to

$$f(x) = \begin{cases} \frac{x}{2}, & x \in (-\pi, \pi), \\ 0, & x = \pi \\ \frac{x}{2} - \pi, & x \in (\pi, 3\pi). \end{cases}$$

Show that this convergence cannot possibly be uniform.

(III)(a) Suppose X is a set, (Y, d) is a metric space, and $\psi : X \to Y$ is a bijection. Show that $\rho(x, y) = d(\psi(x), \psi(y))$ defines a metric on X.
(b) Show that

$$\nu(x, y) = \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right|$$

defines a metric on $\mathbb{R}$, and $\nu(x_n, x) \to 0 \iff |x_n - x| \to 0$.
(c) Let $A = \{a_1, a_2, \ldots\}$ be a countable set and let $\mathcal{F}$ be the set of all real-valued functions on A. For $f, g \in \mathcal{F}$ define

$$\rho(f, g) = \sum_{k=1}^{\infty} 2^{-k} \left| \frac{f(a_k)}{1 + |f(a_k)|} - \frac{g(a_k)}{1 + |g(a_k)|} \right|.$$

Show that ρ is a metric on $\mathcal{F}$, and $f_n \to f$ pointwise on $A \iff \rho(f_n, f) \to 0$. (As a side note, no such metric for pointwise convergence exists when A is uncountable.)

(IV) The Corollary on p. 152 concerns integrating uniformly converging series of functions. Show that if the covergence is only pointwise, the Corollary fails. In other words, find a series $f(x) = \sum_{n=1}^{\infty} f_n(x)$ which converges pointwise on some $[a, b]$, but

$$\int_a^b f\,dx \ne \sum_{n=1}^{\infty} \int_a^b f_n\,dx.$$

(V) Let $\alpha > 0$ and define

$$E = \{f \in C[0,1] : f(0) = 0,\ |f(y) - f(x)| \le |y - x|^\alpha \text{ for all } x, y\}.$$

(a) Show that E is equicontinuous.
(b) Show that E is a closed subset of $C[0,1]$ (with the uniform metric, as always.)

(VI) Let $\{f_1, .., f_n\}$ be a fixed set of continuous functions on $[a, b]$. Let $\mathcal{F}$ be the class of all functions of the form $\sum_{i=1}^n c_i f_i$ with $|c_i| \le 1$ for all i. Show that $\mathcal{F}$ is equicontinuous.

HINTS:

(3) Because of Problem 2, the functions you choose must be unbounded. You can find examples with $f_n = g_n$.

(6) Split the series into two separate ones with numerators x^2 and n.

(8) This is a fairly quick one, if you apply the right theorem from Ch. 7.

(II) Don't try to do it using epsilons and deltas! Find a theorem you can cite. Sketch the graph of f.

(III)(a) What you're really doing here is transferring the metric from Y to X using the bijection.
(b) Is part (a) relevant?

(IV) For g_N, g from exercise (I), make a series $\sum_{n=1}^\infty f_n$ for which the Nth partial sum is $\sum_{n=1}^N f_n = g_N$. Consider $g_N - g_{N-1}$.

(V)(b) Use Theorem 3.2d.