MATH 425a ASSIGNMENT 4 SOLUTIONS
FALL 2015 Prof. Alexander

These solutions are for the individual use of Math 425a students and are not to be distributed outside that group.

Rudin Chapter 2:

(12) Let $\{G_\alpha, \alpha \in A\}$ be an open cover of K. Since $0 \in K$, we have $0 \in G_{\alpha_0}$ for some α_0. Since G_{α_0} is open, there is a neighborhood $N_\epsilon(0) \subset G_{\alpha_0}$. Since $1/n \to 0$, there exists N such that $n \geq N \implies 1/n \in N_\epsilon(0)$. For each $n = 1, \dots, N-1$, since $1/n \in K$, there exists G_{α_n} such that $1/n \in G_{\alpha_n}$. Thus $G_{\alpha_0} \cup G_{\alpha_1} \cup \cdots \cup G_{\alpha_{N-1}}$ contains 0 and all points $1/n$, that is, $\{G_{\alpha_0}, \dots, G_{\alpha_{N-1}}\}$ is a finite subcover of K. This shows K is compact.

(14) $\{(\frac{1}{n}, 1) : n \geq 1\}$ is one example. For any finite subcollection $\{(\frac{1}{n}, 1) : n \in B\}$, if m is the largest index in B, then $\cup_{n \in B}(\frac{1}{n}, 1) = (\frac{1}{m}, 1) \neq (0, 1)$, so there is no finite subcover.

(16) Since $\sqrt{2}, \sqrt{3}$ are irrational, the complement of E in Q is

$$\big((-\infty, -\sqrt{3}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{3}, \infty)\big) \cap \mathbb{Q}.$$

This set is the intersection with $\mathbb{Q}$ of an open set in $\mathbb{R}$, so it is open in $\mathbb{Q}$. This shows that E is closed in $\mathbb{Q}$. E is also bounded since $|x| \leq \sqrt{3}$ for all $x \in E$. But the set E is not closed in $\mathbb{R}$ (for example, $\sqrt{3}$ is a limit point of E not contained in E) so E is not compact in $\mathbb{R}$. This is equivalent to E being non-compact in $\mathbb{Q}$.

(22) Let $\mathbb{Q}^k = \mathbb{Q} \times \cdots \times \mathbb{Q}$ be the set of all points of $\mathbb{R}^k$ with rational coordinates. Let $\epsilon > 0$ and $x \in \mathbb{R}^k$. For each coordinate i, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q_i \in \mathbb{Q}$ with $|x_i - q_i| < \epsilon/\sqrt{k}$. Letting $q = (q_1, \dots, q_k)$ we then have

$$|x - q| = \left(\sum_{i=1}^{k} (x_i - q_i)^2 \right)^{1/2} \leq \left(k \frac{\epsilon^2}{k} \right)^{1/2} = \epsilon.$$

This shows $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$.

Handout:

(A)(i) Let p be a limit point of $N_r(x)$ and let $\epsilon > 0$. Then by definition of limit point, there is a point y of $N_r(x)$ in $N_\epsilon(p)$. Therefore $d(p, x) \leq d(p, y) + d(y, x) < \epsilon + r$. Since ϵ is arbitrary, this shows $d(p, x) \leq r$. Thus both $N_r(x)$ and its limit points are contained in $\{y : d(x, y) \leq r\}$.

(ii) In the metric space $\mathbb{Z}$, we have $N_1(0) = \{0\}$ which is a closed set, so $\overline{N_1(0)} = \{0\}$. But $\{x \in \mathbb{Z} : d(x, 0) \leq 1\} = \{-1, 0, 1\}$ so they are not the same.

(B) Since each $x \in E$ is isolated, there exists a radius $r(x)$ such that $E \cap N_{r(x)}(x) = \{x\}$. Since each $x \in N_{r(x)}(x)$, the collection $\{N_{r(x)}(x) : x \in E\}$ forms an open cover of E. Let $\{N_{r(x_1)}(x_1), \ldots, N_{r(x_m)}(x_m)\}$ be any finite subcollection. Then $E \cap \big(\cup_{i=1}^m N_{r(x_i)}(x_i) \big) = \{x_1, \ldots, x_m\}$, which is finite, so it isn't all of E. This means no finite subcollection can cover E, that is, the original collection has no finite subcover. This shows E is not compact.

(C)(i) Let $\{G_\alpha, \alpha \in A\}$ be an open cover of $L \cup M$. Since this is also an open cover of each of the compact sets L and M individually, there is a finite subcover of L, say $\{G_\alpha : \alpha \in B\}$, and a finite subcover of M, say $\{G_\alpha : \alpha \in C\}$. Then $\{G_\alpha : \alpha \in B \cup C\}$ is a finite subcover of $L \cup M$. Thus $L \cup M$ is compact.

(ii) Since K is closed, so is $D \cap K$, so $D \cap K$ is a closed subset of a compact set, so $D \cap K$ is compact by 2.35. By assumption $D \cap K^c$ is compact, so by (i), $D = (D \cap K) \cup (D \cap K^c)$ is compact.

(D) Since G_j is open, G_j^c is closed and bounded in $\mathbb{R}$, hence it is compact. Since $G_1^c \supset G_2^c \supset \ldots$, it follows from the Corollary after 2.36 that $\cap_{j \geq 1} G_j^c \neq \emptyset$. Therefore $\cup_{j \geq 1} G_j = \big(\cap_{j \geq 1} G_j^c \big)^c \neq \mathbb{R}$.

(E)(i) Compact because it is closed (the only limit point is 0 which is in the set) and bounded (all points are in $[0, 1]$.)

(ii) Not compact because it isn't bounded—it contains points $(x, 1/x)$ for arbitrarily large x.

(iii) Compact because it is closed and bounded.