MATH 425b ASSIGNMENT 2
SPRING 2016
Prof. Alexander
Due Wednesday February 3.

Chapter 7 #18 and:

(I) Suppose $\mathcal{F}$ is an equicontinuous subset of $C[a,b]$. Show that $\overline{\mathcal{F}}$ (the closure in the uniform metric) is equicontinuous.

(II) Let $\alpha > 0$ and define

$$E = \{f \in C[0,1] : f(0) = 0,\ |f(y) - f(x)| \leq |y - x|^\alpha \text{ for all } x, y\}.$$

(a) Show that E is equicontinuous.
(b) Show that E is a closed subset of $C[0,1]$ (with the uniform metric, as always.)
(c) Show that E is compact (as a subset of $C[0,1]$.)

(III)(a) Let $[0,A]$ be an interval and P a polynomial. Show that for every $\epsilon > 0$ there is a polynomial Q with rational coefficients such that $||P - Q||_\infty < \epsilon$, on $[0,A]$.
(b) Show that $C[0,A]$ has a countable subset which is dense.

(IV) Suppose $\mathcal{A}$ is an algebra of continuous complex-valued functions on a set E, and $\mathcal{A}$ separates points. Show that $\mathcal{A}$ is not equicontinuous.

(V) For this problem, $||f||_\infty$ means $\sup_{x\in\mathbb{R}} |f(x)|$, i.e. the sup is over all of $\mathbb{R}$. Note that the problem would be false if we only looked at the functions on a bounded interval.
(a) Show that if P is a nonconstant polynomial then $||P||_\infty = \infty$.
(b) Show that if a function $f : \mathbb{R} \to \mathbb{R}$ is a uniform limit of polynomials (that is, $||P_n - f||_\infty \to 0$), then f itself is a polynomial.

(VI) Show that a complex-valued function on $[1,1]$ is continuous if and only if it can be written as the uniform limit of a sequence of polynomial functions.

HINTS: (Note (I), (II), (VI) should be easier than (III), (IV), (V).)

(16) Mimic parts of the proof of Theorem 7.25.

(I) Show that if some δ works for all $f \in \mathcal{F}$, then this same δ works for limit points (in the uniform metric) of $\mathcal{F}$.

(II)(a) Use the definition directly.

(III) For (b), use (a).

(IV) If $f \in \mathcal{A}$ satisfies $f(x_1) \neq f(x_2)$ for some x_1, x_2, what happens when you multiply f by a large constant? By taking x_1, x_2 close together (quantify what this means!) you can show equicontinuity fails.

(V)(a) If P has degree k, consider $P(x)/x^k$ as $x \to \infty$.

(b) Consider $P_n - P_m$. Note if f is a polynomial then taking $P_n = f$ for all n is allowed.