HOMEWORK 4

Econ 501: Macroeconomic Analysis and Policy

Spring 2016

1. First order conditions for C_t, L_t and K_{t+1}.
   a) Write the Lagrangian and get the FOC wrt the relevant variables:

$$c_t : \frac{1}{c_t} - \lambda_t = 0$$

$$l_t : -\frac{\theta}{1-l_t} + (1-\alpha)\lambda_t e^{z_t} k_t^{\alpha} l_t^{-\alpha} = 0$$

$$k_{t+1} : -\lambda_t + \beta E_t \lambda_{t+1}\left(\alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + (1-\delta)\right) = 0$$

   b) Equilibrium conditions:

$$w_t = \frac{\theta c_t}{1-l_t}$$

$$1 = \beta E_t \frac{c_t}{c_{t+1}}\left(r_{t+1} + (1-\delta)\right)$$

$$y_t = e^{z_t} k_t^{\alpha} l_t^{1-\alpha}$$

$$i_t = k_{t+1} - (1-\delta)k_t$$

$$r_t = \alpha \frac{y_t}{k_t}$$

$$w_t = (1-\alpha)\frac{y_t}{l_t}$$

$$y_t = c_t + i_t$$

   c) From above, we can get the non-stochastic steady state:

$$1 = \beta\left(r + (1-\delta)\right) \Rightarrow r = \frac{1}{\beta} + \delta - 1$$

$$\frac{y}{k} = \frac{\frac{1}{\beta} + \delta - 1}{\alpha}$$

$$\frac{y}{l} = \left(\frac{y}{k}\right)^{\frac{\alpha}{\alpha-1}} = \left(\frac{\frac{1}{\beta} + \delta - 1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$$

$$w = (1-\alpha)\frac{y}{l} = (1-\alpha)\left(\frac{\frac{1}{\beta} + \delta - 1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$$

$$\frac{i}{y} = \frac{\delta k}{y} = \frac{\delta\alpha}{\frac{1}{\beta} + \delta - 1}$$

$$\frac{c}{y} = 1 - \frac{i}{y} = 1 - \frac{\delta\alpha}{\frac{1}{\beta} + \delta - 1}$$

$$(1-\alpha)\frac{y}{l} = \frac{\theta c}{1-l} \Rightarrow \frac{(1-\alpha)}{\theta}\frac{1-l}{l} = \frac{c}{y} = 1 - \frac{\delta\alpha}{\frac{1}{\beta} + \delta - 1}$$

$$l = \frac{1}{1 + \frac{\theta}{1-\alpha}\frac{c}{y}}$$

2.

(a) First we solve a one period problem. Use budget constraint to substitute for c_t in the utility function. Since there is only one period, we can drop the time subscript.

$$Maxu = \ln(wl) + \frac{b(1-l)}{1-\gamma}$$

$$F.O.C.: \frac{1}{l} = b(1-l)$$

which implicitly determines labor and we see that it does not depend on real wage. We know that c = wl; if we divide both sides of the equation for l by w, we will get

$$\frac{1}{c} = \frac{b}{w^{1-\gamma}(w-c)^{\gamma}}$$

which implicitly determines consumption as a function of real wage.

(b) Now we solve the two-period case. From the budget constraint

$$C_{2,t+1} = (w_t l_{1,t} - c_t)(1 + r_{t+1}) + w_{t+1} l_{2,t+1}$$

Substitute it into the utility function:

$$Maxu = \ln(c_{1,t}) + \frac{b(1-l_{1,t})}{1-\gamma} + e^{-\rho} \ln(w_t l_{1,t} - c_{1,t})(1 + r_{t+1}) + w_{t+1} l_{2,t+1} + e^{-\rho} \frac{b^{(1-l_{2,t+1})^{1-\gamma}}}{1-\gamma}$$

F.O.C.:

$$\frac{c_{2,t+1}}{c_{1,t}} = e^{-\rho}(1 + r_{t+1})$$

$$b(1-l_{1,t})^{-\gamma} = e^{-\rho} \frac{w_t(1 + r_{t+1})}{c_{2,t+1}}$$

$$b(1-l_{2,t+1})^{-\gamma} = \frac{w_{t+1}}{c_{2,t+1}}$$

Consumption path can be found from the first of the FOC's and the budget constraint. To find the path of labor we can divide the third condition by the second one to find

$$\frac{(1-l_{1,t})}{(1-l_{2,t+1})} = \left[\frac{1}{e^{-\rho}(1 + r_{t+1})} \frac{w_{t+1}}{w_t}\right]^{\frac{1}{\gamma}}$$

If relative wage in period t+1 rises ($w_{t+1}=w_t$ rises), then relative demand for leisure in period t rises. If the interest rate r rises then the relative demand for leisure in period t falls. It is also straightforward from the last equation that the responsiveness of the relative demand for leisure to the changes of relative wage and interest rate is higher the higher is $1/\gamma$. Remember from the beginning of the semester that $1/\gamma$ is the intertemporal elasticity of substitution. The higher is the elasticity of substitution, the larger is the response of the relative demand for leisure to the change in relative wage or interest rate.

3.

(a) Interest rate is equal to the marginal product of capital minus the rate of depreciation. MPK in this case is equal to A and the equation describing the dynamics of capital stock implies that there is no depreciation. Therefore the interest rate is equal to A.

(b) Assume $\rho = A$.

Euler Equation: $MU(c_t) = E_t\{MU(c_{t+1})\}$

$1-2\theta c_t = 1-2\theta E_t\{c_{t+1}\}$

$c_t = E_t\{c_{t+1}\}$

(c) We are now trying to find a value of consumption at each period of time that would satisfy Euler equation and all other equations of our model. we should first try the linear functional form

$$C_t=\alpha+\beta K_t+\gamma e_t$$

Euler equation must hold. So we will find α, β and γ using method of undetermined coefficients equating LHS and RHS of Euler equation.

$$\text{LHS: } \alpha+\beta K_t+\gamma e_t$$

$$\text{RHS: } E_t\{\alpha+\beta K_{t+1}+\gamma e_{t+1}\}$$

From the equation of motion of capital

$$K_{t+1}=K_t+AK_t+e_t-\alpha-\beta K_t-\gamma e_t$$

We also know that E t{et+1}=ϕ et because e follows random walk. Note also that nothing else is random on the RHS after we substitute for Kt+1. We can substitute these results to get

$$\text{RHS}= \alpha(1-\beta)+ \beta\,(1+A- \beta)K_t+ \beta\,(1-\gamma)\,e_t +\gamma\phi\,e_t$$
$$= \alpha(1-\beta)+ \beta\,(1+A- \beta)K_t+ [\beta\,(1-\gamma) +\gamma\phi]\,e_t$$

The only way RHS and LHS can be equal for all values of K and e is when the coefficients in front of them and the constants are equal on both sides. Thus

$$\alpha=(1- \beta)$$
$$\beta= \beta\,(1+A- \beta)$$
$$\gamma= \beta(1-\gamma)+ \gamma\phi$$

Two sets of solutions are possible.

$$\alpha=0$$
$$\beta= A$$
$$\gamma= A\,/1+A-\phi$$

and

$$\alpha=\text{anything}$$
$$\beta= 0$$
$$\gamma= 0$$

The second one can be ignored since it does not make economic sense. Therefore our solutions for consumption and capital are

$$C_t=AK_t + (A/1+A-\phi)e_t$$
$$K_{t+1} =K_t+ (1/1+A-\phi)\,e_t$$