MATH 425b SAMPLE MIDTERM EXAM 1
Spring 2016
Prof. Alexander

(1) Suppose the Fourier series of $f(x)$ is $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, and it converges pointwise as well as in L^2. Show that f is real-valued if and only if $c_{-n} = \overline{c_n}$ for all n.

(2)(a) Let $f(x) = |x+1|^{3/2}$, so $f''(x) = \frac{3}{4}|x+1|^{-1/2}$ for all $x \neq -1$ (you may take this as given.) Suppose f is given by some power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ in a neighborhood of $x = 0$. Show that this power series cannot converge for any $x > 1$. HINT: No computations are required, just reasoning.

(b) Suppose $u_1, .., u_{n+1}$ are orthonormal vectors or functions. Show that u_{n+1} cannot be in the linear span of $u_1, .., u_n$, that is, we cannot have $u_{n+1} = \sum_{i=1}^{n} c_i u_i$ with $c_1, .., c_n$ scalars.

(3) Suppose X is a totally bounded metric space, that is, for every $\delta > 0$ there exists a finite set $D_\delta \subset X$ which is δ-dense. (δ-dense means that for every $x \in X$ there is a $y \in D_\delta$ with $d(x,y) < \delta$.) Suppose $f_n \to f$ pointwise on all the sets D_δ, and $\{f_n, n \geq 1\}$ is equicontinuous. Show that $f_n \to f$ uniformly on all of X.

(4)(a) Let $\mathcal{A}_1$ be the algebra of all polynomial functions on $[0,1]$ with only even powers, that is, having form $P(x) = a_0 + a_2 x^2 + \cdots + a_{2n} x^{2n}$. Show that $\mathcal{A}_1$ is dense in $C[0,1]$. HINT: Each $P(x)$ in $\mathcal{A}_1$ can be written as $Q(x^2)$ for some other polynomial Q. What happens if $Q(x)$ is close to $f(x^\alpha)$ for some appropriately chosen α? Why does such a Q exist?

(b) Let $\mathcal{A}_2$ be the algebra of all even polynomial functions on $[-1,1]$ (that is, all polynomials with $P(x) = P(-x)$ for all x.) Show that the closure of $\mathcal{A}_2$ in $C[-1,1]$ (with the uniform metric) consists of all even continuous functions. HINT: Use (a).