Appendix | E

Some Linear Algebra

This appendix derives an auxiliary result used in the proof of Proposition 9.2 in Chapter 9.

MATRICES WITH DOMINANT AVERAGES

An $n \times n$ matrix $\mathbf{A}$ satisfies the *dominant average* condition if in every column the off-diagonal terms are less than the average of the column,

$$\forall i \neq j,\ a_{ij} < \frac{1}{n}\sum_{k=1}^{n} a_{kj} \tag{E.1}$$

and the average of each column is positive,

$$\forall j,\ 0 < \frac{1}{n}\sum_{k=1}^{n} a_{kj} \tag{E.2}$$

Observe that if $\mathbf{A}$ satisfies the dominant average condition and $\mathbf{A}^i$ is obtained by deleting the ith row and ith column of $\mathbf{A}$, then $\mathbf{A}^i$ also satisfies the condition. This is because if from any column an entry that is less than the average is deleted, then the average of the remaining entries increases.

Let $\mathbf{e}^i \in \mathbb{R}^n$ denote the ith unit vector and let $\mathbf{e} = \sum_{i=1}^{n} \mathbf{e}^i$ denote the vector of 1's. Although the same symbols will be used for different n, the sizes of these vectors will be apparent from the context.

Lemma E.1. *Suppose* $\mathbf{A}$ *is an* $n \times n$ *matrix that satisfies the dominant average condition. Then there exists a unique* $\mathbf{x} \gg \mathbf{0}$ *such that*

$$\mathbf{A}\mathbf{x} = \mathbf{e} \tag{E.3}$$

We first show that there is a strictly positive solution to (E.3). The proof is by induction on n.

Step 1: For $n=1$, the fact that there is a strictly positive solution is immediate. Now suppose that the result holds for all matrices of size $n-1$.

Let $\mathbf{A}$ be an $n\times n$ matrix. Define $\mathbf{A}^i$ to be the $(n-1)\times(n-1)$ matrix obtained from deleting the ith row and the ith column of $\mathbf{A}$. From the induction hypothesis, for each $i=1,2,\ldots,n$, there exists an $\mathbf{x}^i \gg \mathbf{0}$ such that

$$\mathbf{A}^i\mathbf{x}^i = \mathbf{e}$$

which is the same as: for all $k\neq i$,

$$\sum_{j\neq i} a_{kj}x_j^i = 1 \tag{E.4}$$

Let

$$\sum_{j\neq i} a_{ij}x_j^i = c_i \tag{E.5}$$

Step 2: Adding the $n-1$ equations (E.4) with (E.5) results in

$$\sum_{j\neq i}\left(\sum_{k=1}^{n} a_{kj}\right)x_j^i = (n-1)+c_i > 0$$

which is positive because of (E.2) and the fact that $\mathbf{x}^i \gg \mathbf{0}$. But now (E.1) implies that

$$\begin{aligned}
c_i &\equiv \sum_{j\neq i} a_{ij}x_j^i \\
&< \sum_{j\neq i}\left(\frac{1}{n-1}\sum_{k\neq i} a_{kj}\right)x_j^i \\
&= \sum_{k\neq i}\left(\frac{1}{n-1}\right)\left(\sum_{j\neq i} a_{kj}x_j^i\right) \\
&= 1
\end{aligned}$$

using (E.4). Thus, $(n-1)+c_i>0$ and $c_i<1$.

Step 3: Since $(n-1)+c_i>0$ and $c_i<1$, for all i, $\frac{1}{1-c_i} > \frac{1}{n}$, so

$$\sum_{i=1}^{n}\frac{1}{1-c_i} > 1 \tag{E.6}$$

Now let $\mathbf{y}^i \in \mathbb{R}^n_+$ be the vector obtained by appending 0 in the ith coordinate to $\mathbf{x}^i \in \mathbb{R}^{n-1}_{++}$. Then (E.4) and (E.5) can be compactly rewritten as follows:

for all i,

$$\mathbf{A}\mathbf{y}^i = \mathbf{e} - (1 - c_i)\,\mathbf{e}^i$$

Dividing through by the positive quantity $(1 - c_i)$ results in

$$\mathbf{A}\left(\frac{1}{1-c_i}\mathbf{y}^i\right) = \frac{1}{1-c_i}\mathbf{e} - \mathbf{e}^i$$

Adding the n such equation systems, one for each i yields

$$\mathbf{A}\left(\sum_{i=1}^{n}\frac{1}{1-c_i}\mathbf{y}^i\right) = \left(\sum_{i=1}^{n}\frac{1}{1-c_i}\right)\mathbf{e} - \mathbf{e}$$

or equivalently,

$$\mathbf{A}\sum_{i=1}^{n}\frac{1}{K(1-c_i)}\mathbf{y}^i = \mathbf{e}$$

where $K = \left[\left(\sum_{i=1}^{n}\frac{1}{1-c_i}\right) - 1\right] > 0$ from (E.6). Since each $\mathbf{y}^i \geq \mathbf{0}$ with only the ith component equal to zero, and $(1 - c_i) > 0$ we determine that

$$\mathbf{x} = \sum_{i=1}^{n}\frac{1}{K(1-c_i)}\mathbf{y}^i \gg \mathbf{0}$$

is a solution to the system (E.3).

Thus, there is a strictly positive solution to (E.3).

Step 4: We now verify that the solution is unique by arguing that $\det \mathbf{A} \neq 0$, and hence $\mathbf{x} = \mathbf{A}^{-1}\mathbf{e}$. Again, the proof is by induction on n.

For $n = 1$ it is immediate that the solution is unique. Now suppose that for all matrices of size $n - 1$, there is a unique solution to the system. Let $\mathbf{A}$ be of size n and let $\mathbf{x} \gg \mathbf{0}$ be such that $\mathbf{A}\mathbf{x} = \mathbf{e}$.

If $\mathbf{A}$ is singular, then there exists a column, say the kth, which is a linear combination of the other $n - 1$ columns—that is, for all $j \neq k$ there exists a z_j such that

$$\forall i,\ a_{ik} = \sum_{j \neq k} a_{ij} z_j \tag{E.7}$$

and since $a_{kk} > 0$, not all the z_j can be zero.

Of course, (E.3) is equivalent to

$$\forall i,\ \sum_{j=1}^{n} a_{ij} x_j = 1$$

and substituting from (E.7) yields that

$$\forall i, \ \sum_{j \neq k} a_{ij} \left(z_j x_k + x_j \right) = 1 \tag{E.8}$$

As before, let $\mathbf{A}^k$ be the $(n-1) \times (n-1)$ matrix obtained from $\mathbf{A}$ by eliminating the kth row and the kth column of $\mathbf{A}$. From the induction hypothesis, there exists a unique $\mathbf{y} \gg \mathbf{0}$ such that $\mathbf{A}^k \mathbf{y} = \mathbf{e}$, which is equivalent to

$$\forall i \neq k, \ \sum_{j \neq k} a_{ij} y_j = 1 \tag{E.9}$$

Since the solution is unique, comparing (E.9) and the equations in (E.8) for $i \neq k$ implies that $\forall j \neq k, z_j x_k + x_j = y_j$, and the kth equation in (E.8) can be rewritten as

$$\sum_{j \neq k} a_{kj} y_j = 1 \tag{E.10}$$

Step 5: Now adding the $n-1$ equations in (E.9) and dividing by $n-1$ results in

$$\sum_{j \neq k} \left(\frac{1}{n-1} \sum_{i \neq k} a_{ij} \right) y_j = 1 \tag{E.11}$$

But (E.1) implies that

$$\forall j, \ a_{kj} < \frac{1}{n-1} \sum_{i \neq k} a_{ij} \tag{E.12}$$

and since $y_j > 0$, (E.12) implies that

$$\sum_{j \neq k} a_{kj} y_j < 1$$

contradicting (E.10). Thus, $\mathbf{A}$ is not singular and $\mathbf{Ax} = \mathbf{e}$ has a unique solution. ■

The dominant average condition may be weakened as follows. An $n \times n$ matrix $\mathbf{A}$ satisfies the *dominant weighted average* condition if there exist positive weights $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $\sum_i \lambda_i = 1$ such that

$$\forall i \neq j, \ a_{ij} < \sum_{k=1}^{n} \lambda_k a_{kj}$$

and

$$\forall j,\ 0 < \sum_{k=1}^{n} \lambda_k a_{kj}$$

The conclusion of Lemma E.1 follows under this weaker condition.

Suppose **A** is a matrix that satisfies the *dominant diagonal* condition and for all $i \neq j$, $a_{ij} \leq 0$. Then **A** satisfies the dominant weighted average condition.