Answers for HW 8

1. (a) The optimal quantity choice by the consumer is determined by the first order condition

$$\theta(4-2q)=p,$$

which gives us

$$q=2-\frac{p}{2\theta}.$$

The optimal price is determined by the first order condition

$$(2-\frac{p}{2\theta})-(p-1)\frac{1}{2\theta}$$

giving us the solution for optimal price

$$p=0.5+2\theta.$$

Hence the price for the higher demander is $p_H=20.5$, and for the low demander is $p_L=10.5$. The quantities are $q_H=2-\frac{20.5}{20}, q_L=2-\frac{10.5}{10}$. Profit from high demander is $19.5(2-\frac{20.5}{20})$, and profit from low demander is $9.5(2-\frac{10.5}{10})$.

The maximum expected profit is

$$\beta(p^*(\theta_L)-c)q^*(\theta_L)+(1-\beta)(p^*(\theta_H)-c)q^*(\theta_H)$$

$$=0.5*9.5(2-\frac{10.5}{10})+0.5*19.5(2-\frac{20.5}{20})=14.019.$$

(b) The expected demand is

$$0.5(2-\frac{p}{10})+0.5(2-\frac{p}{20})=2-\frac{3p}{40}$$

Hence the expected profit of the monopolist is

$$(p-1)(2-\frac{3p}{40})$$

Take the derivative of the above with respect to p, we get the first order condition

$$2-\frac{3}{40}p-(p-1)\frac{3}{40}=0,$$

and we have the optimal $p=\frac{83}{6}$. You can also get the same answer using the following formula from the notes.

$$p=c-\frac{D(p)}{D'(p)}=1+\frac{2-\frac{3p}{40}}{\frac{3}{40}}=1+\frac{80-3p}{3},$$

or

$$p=\frac{83}{6}.$$

The expected profit is

$$(\frac{83}{6}-1)(2-\frac{3}{40}\frac{83}{6}) = 12.352$$

(c) The profit in (a) is in fact higher. The reason is that the profit in (b) is constrained by the same price for each consumer. The profit in (a) is not. Clearly constrained optimal profit is smaller.

2. (a) In the optimal two-part pricing, the price per unit is set at $p = 1$. At this price, the quantities demanded have been computed earlier as

$$q_H = 2 - \frac{1}{20}, q_L = 2 - \frac{1}{10}.$$

The demand functions are $q = 2 - \frac{p}{2\theta}$, with the price axis intercept $p = 4\theta$. Hence the fixed fees charged are given by the

$$\begin{aligned} T_H &= \frac{1}{2}(4\theta_H - 1)q_H = \frac{39}{2}(2-\frac{1}{20}), \\ T_L &= \frac{1}{2}(4\theta_L - 1)q_L = \frac{19}{2}(2-\frac{1}{10}). \end{aligned}$$

Hence the expected profit is

$$0.5T_H + 0.5T_L = \frac{39}{4}(2-\frac{1}{20}) + \frac{19}{4}(2-\frac{1}{10}) = 28.038.$$

(b) If the same price and fee for each consumer, then $D_L(p) = 2 - \frac{p}{10}, D(p) = 2 - \frac{3p}{40}$

$$S_L(p) = \theta_L v(D_L(p)) - p(2-\frac{p}{10}) = 5(4(2-\frac{p}{10})-(2-\frac{p}{10})^2) - p(2-\frac{p}{10}) = \frac{1}{20}p^2 - 2p + 20.$$

The optimal price is given by

$$p = c - \frac{D(p)+S_L'(p)}{D'(p)} = 1 + \frac{2-\frac{3p}{40}+\frac{1}{10}p-2}{\frac{3}{40}} = 1 + \frac{1}{3}p,$$

and we have $p^* = 1.5$. The maximum expected profit is given by

$$S_L(p^*) + (p^*-c)D(p^*) = \frac{1}{20}(1.5)^2 - 2*1.5 + 20 + 0.5(2-\frac{3*1.5}{40}) = 18.056.$$

(c) The profit in (a) is higher because the profit in (b) is constrained by a single two-part pricing system, but (a) is not. Note that the profits in (a),(b) here are greater than the profits in problem 1.

3. The consumer has the utility function

$$\theta v(q) - T$$

where θ can be high θ_H or low θ_L. Consider the example, $\theta_L = 3, \theta_H = 4, \beta = 0.5$. Assume that $c = 1$. Let $v(q) = q - \frac{1}{2}q^2$. When you exclude the low demander, the optimal policy is to set per unit price at $P = c = 1$. The fixed fee Z is set at the high demander consumer surplus at $P = 1$, we have the

$$Z = S_H(P) = \theta_H v(D_H(P)) - PD_H(P),$$

where $D_H(P) = 1 - \frac{1}{4}P = \frac{3}{4}, \theta_H = 4$. Hence

$$Z = 4v(\frac{3}{4}) - \frac{3}{4} = 4(\frac{3}{4} - \frac{9}{32}) - \frac{3}{4} = \frac{9}{8}.$$

The profit for the monopolist is then the expected sum of revennes the fixed fee only (because there is no profit from per-unit charge).

$$0.5Z = \frac{9}{16} = 0.562\,5.$$

The profit we had in case (d) of the note is 0.670 14. We don't get a higher profit by excluding the low demanders.

4. (a) The maximization problem becomes

$$\max 0.5(T_L - q_L) + 0.5(T_H - q_H) \tag{1}$$

subject to

$$6(4q_H - q_H^2) - T_H = 6(4q_L - q_L^2) - T_L$$
$$5(4q_L - q_L^2) - T_L = 0$$

We get

$$T_L = 5(4q_L - q_L^2),$$
$$T_H = 6(4q_H - q_H^2) - (4q_L - q_L^2)$$

Substitute into the profit function, we have the unconstrained maximization problem: Maximize

$$0.5(19q_L - 5q_L^2) + 0.5(23q_H - 6q_H^2 - 4q_L + q_L^2)$$

Take the derivative with respect to q_H, q_L respectively, we get

$$23 - 12q_H = 0, 15 - 8q_L = 0.$$

Hence we have $q_H = \frac{23}{12}, q_L = \frac{15}{8}$. From these, we can compute

$$T_H = 6(4 * \frac{23}{12} - (\frac{23}{12})^2) - 4 * \frac{15}{8} + (\frac{15}{8})^2 = \frac{3835}{192} = 19.974.$$

$$T_L = 20 * \frac{15}{8} - 5(\frac{15}{8})^2 = \frac{1275}{64} = 19.922$$

Hence the optimal menu of contract is to offer two packages. The one intended for the high demander is $T_H = 19.974, q_H = \frac{23}{12}$. The one intended for the low demander is $T_L = 19.922, q_L = \frac{8}{15}$.

(b) The optimal profit for the monopolist is

$$0.5(T_L - q_L) + 0.5(T_H - q_H) = 0.5(19.922 - \frac{15}{8}) + 0.5(19.974 - \frac{23}{12}) = 18.052.$$

(c) To translate the package into fixed fees and prices charged per unit, we compute prices first. From the demand $q_H = 2 - \frac{p_H}{12}$, we have $p_H = 1$. From the demand $q_L = 2 - \frac{p_L}{10}$, we have $p_L = 1.25$. The fixed fee for the high demander is $T_H - p_H q_H = 19.974 - \frac{23}{12} = 18.057$. The fixed fee for the low demander is $T_L - p_L q_L = 19.922 - 1.25 * \frac{15}{8} = 17.578$. Hence the package for the high demander is a fixed fee 18.057 and a per unit charge $p_H = 1$. The package for the low demander has a lower fixed fee 17.578, but higher per unit charge $p_L = 1.25$.

HW8

Q.8

(a)

$$\mathrm{b(v)} = \mathrm{v} - \frac{1}{F(v)}\int_r^v F(x)^{N-1}\ dx$$

$$= \mathrm{v} - \frac{1}{v^2}\int_0^v x^2\ dx$$

$$= \mathrm{v} - \frac{1}{3v^2}(v^3 - 0)$$

$$= \frac{2}{3}\mathrm{v}$$

(b)

$$\text{revenue} = \mathrm{N}\int_r^\beta b(v)\,F(v)^{N-1}\ dF(v)$$

$$= 2\int_0^1 \frac{2}{3}v(v^2)\ dv^2$$

$$= 2\int_0^1 \frac{2}{3}v(v^2)2v\ dv$$

$$= \ \frac{8}{3}\int_0^1 v^4\,dv$$

$$= \frac{8}{3}\times\frac{1}{5}(1^5 - 0^5)$$

$$= \frac{8}{15}$$

(c)

$$\text{revenue} = \ \ \mathrm{N}\int_r^\beta v\,f(v)[1 - F(v)]dv$$

$$= 2\int_0^1 v(2v)[1 - v^2]\ dv$$

$$= 2\int_0^1 2v^2 - 2v^4\ dv$$

$$= 2\ \ \left[\frac{2}{3} - \frac{2}{5}\right]$$

$$= \frac{8}{15}$$