Econ 501, SPRING 2016

RATIKA NARAG, PhD

Introduction

- Problem with the Solow model: ad-hoc assumption of constant saving rate
- Ramsey or Cass-Koopmans model: differs from the Solow model only because it explicitly models the consumer side and endogenizes savings
- Determining the behavior of saving/consumption as the result of optimal intertemporal choices by individual households

Introduction

- The Ramsey-Cass-Koopmans model (Neoclassical growth model) is a benchmark "growth model" with endogenous saving rate. It is widely used in growth, business cycles and asset pricing theory
- Objectives
  - To find the optimal level of output
  - To determine how the output should be allocated between consumption and investment

- Use the RCK model to make 3 points:
  - assumption of exogenous savings rate not a central determinant of the major implications of the Solow model. The RCK model also exhibits an optimal balanced-growth path
  - The optimal savings rate depends on preference and technology parameters in particular ways
  - The balanced growth path is “dynamically efficient”
- You should focus on <u>understanding</u> these points rather than following the mathematical computations

Optimal Growth Model: The Ramsey-Cass-Koopmans Model

- Basic setup of Ramsey model was described by Ramsey in 1928
- Dynamics were developed by Cass and Koopmans in a growth context in 1965
- Beyond its use as a basic growth model, also a workhorse for many areas of macroeconomics (growth, business cycles, asset pricing theory)

Model Set Up

- Time is discrete and index by $t = 0, 1, 2, \ldots, \infty$
- Firms produce homogeneous output and rent productive factors labor and capital
- Households hold labor and assets and receive rental payments. They consume and save out of incomes obtained.

Structure of the Markets

- There exist rental markets for labor, L, and capital services, K. Prices are w and R respectively.
- Assets (debt) market where households can borrow and lend at the interest rate r
- The market for final output is competitive. Final output, Y , can be consumed, C, or invested, I
- The price in this market is normalized to unity

Supply of Inputs

- The supply of capital, K_t, evolves over time as the result of investment decisions, $K_{t+1} = (1-\delta)K_t+I_t$, where δ is the depreciation rate
- Regarding labor supply, we assume that at time t there is a measure of identical households or individuals N_t. We assume inelastic individual supply of labor (as in the Solow model), $L_t = N_t$
- Assume no population growth for simplicity

Firms

- There is a large number of identical firms that have access to a neoclassical production function
- Assumption of constant returns, the scale of firms is of no consequence
- To simplify we consider that there is one representative firm
- The firm's flow of profit at any point in time is,

  Profit = F(K,L) – WL – RK

Firms

- The objective of the firm is to max the PV of profits
- But there are no intertemporal elements in the firms max problem. Thus, the firm maximizes current profits at every point in time, taking as given the price of output and rental rates.
- FOC: $$R_t = F_K(K_t, L_t)$$ $$w_t = F_L(K_t, L_t)$$

Firms

- In per capita terms: $y_t \equiv f(k_t)$
- Euler Theorem: $$R_t = f'(k_t) > 0$$ $$w_t = f(k_t) - k_t f'(k_t) > 0$$
- Note that factor payments exhaust output. Pure profits are zero and there is no incentive for entry or exit of competitive firms

Households

- Single, representative consumer with preferences over an infinite stream of consumption $c = \{c_t\}_{t=0}^{\infty}$ given by a time separable utility function: $$U_0 = u(c) = \sum_{t=0}^{\infty} \beta^t U(c_t)$$
- β is the time discount factor, $0 < \beta < 1$
- The discount factor measures household's impatience: the more impatient, the less weight to future utilities relative to current one (smaller β).

Preferences

- $u(c)$ is strictly increasing, concave, twice continuously differentiable with derivatives u' and u'', and satisfies the following *Inada* type assumptions:

  $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$

- An important implication of time separability is that the MU of consumption at date t $\frac{\partial U(c)}{\partial c_t} = \beta^t u'(c_t)$ is independent of the level of consumption at any other date

---

- Households hold assets a_t (including, e.g.,one-period bonds) which give a one-period return r_t, and are endowed with one unit of labor.
- Since there is no disutility from work, individual labor supply is inelastic and unity. So labor income is given by the wage rate, w_t.
- Thus the budget constraint is,

$$c_t + a_{t+1} = w_t + (1+r_t)a_t, a_0 \, given$$

all t = 0, 1, 2, ...

- We can add up all the budget constraints into a single restriction

$$\sum_{t=0}^{\infty}\prod_{s=0}^{t}\left(\frac{1}{1+r_s}\right)c_t = a_0 + \sum_{t=0}^{\infty}\prod_{s=0}^{t}\left(\frac{1}{1+r_s}\right)w_t$$

- Need to impose a no Ponzi game condition (the agent cannot roll over debt indefinitely or that debt does not explode)

$$\lim_{t\to\infty}\prod_{s=0}^{t}(1+r_s)^{-1}a_{t+1} = 0$$

- This is the interteomporal budget constraint
- The present value of consumption is equal to total wealth which is the sum of non-human wealth and human wealth (the present value of labor income)

$$PDV\,(c) = a_0 + PDV\,(w)$$

- intertemporal constraint represents the feasible choices because of the possibility to borrow and lend. Thus the consumer can spend more or less than current income provided that the intertemporal constraint is satisfied.

- Assume perfect foresight by households so that they know both current and future values of w_t and r_t and take them as given
- Households maximize the value of utility by choice of sequences c_t and at that satisfy the budget constraint
- Solution is characterized by the budget constraint and the Euler equation:

$$u'(c_t) = \beta(1 + r_{t+1})u'(c_{t+1})$$

- Assume CRRA utility: $u(c) = \frac{c^{1-\theta}}{1-\theta}$
- Euler Equation: $\left(\frac{c_{t+1}}{c_t}\right)^{\theta} = \beta(1 + r_{t+1})$
- We add another condition that rules out the optimality of excessive accumulation of assets: transversality condition

$$\lim_{t\to\infty} \beta^t u'(c_t)a_t = 0$$

Equilibrium

- The equilibrium for the economy is characterized as time-paths of quantities (c_t, k_t, y_t) and prices (r_t, w_t, R_t) such that agents behave optimally and all markets clear (demand=supply)
- Optimization by firms:

$$R_t = f'(k_t)$$
$$w_t = f(k_t) - k_t f'(k_t)$$

- Optimization by households:

$$c_t + a_{t+1} = w_t + (1 + r_t)a_t$$
$$\left(\frac{c_{t+1}}{c_t}\right)^{\theta} = \beta(1 + r_{t+1})$$
$$\lim_{t\to\infty} \beta^t u'(c_t)a_t = 0, a_0 \, given$$

- No-arbitrage in assets markets: the returns from capital must equalize the return from debt instrument $R - \delta = r$

- Market clearing:
  - Asset Market: Since agents are assumed to be identical, in equilibrium borrowing and lending through bonds must be zero. In Equilibrium there is neither lending nor borrowing. $k_t = a_t$
  - Goods Markets: $I_t + C_t = F(K_t, L_t)$

$$K_{t+1} - (1-\delta)k_t + C_t = F(K_t, L_t)$$
$$k_{t+1} - (1-\delta)k_t + c_t = f(k_t)$$

- Note that, by Walras' Law, the last equation is redundant with the household's budget constraint

- After some rearranging, the equilibrium is completely described by the following set of conditions:

$$k_{t+1} + c_t = f(k_t) + (1-\delta)k_t$$
$$\frac{c_{t+1}}{c_t} = \left[\beta\left(1 + f'(k_{t+1}) - \delta\right)\right]^{1/\theta}$$
$$\lim_{t\to\infty} \beta^t u'(c_t)k_t = 0$$
$$k_0 > 0$$

The Planner's Problem and Welfare

- Want to study the optimal choices that a central planner that cares about the utility of households would impose if he had the chance to do so (Ramsey (1928))
- The problem can be formulated as:

$$Max \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$s.t. k_{t+1} + c_t = f(k_t) + (1-\delta)k_t$$
$$k_0 > 0$$
$$k_t, c_t \geq 0$$

Write the Langrangian

$$L = \sum_{t=0}^{\infty} \beta^t U(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \right]$$

where for each date t, $\lambda_t \geq 0$ denotes the multiplier on the resource constraint

- <u>Necessary conditions for an optimum (FOC):</u>

$$\frac{\partial L}{\partial c_t} = 0 \Leftrightarrow \beta^t U'(c_t) = \lambda_t \qquad \text{....(1)}$$

$$\frac{\partial L}{\partial k_{t+1}} = 0 \Rightarrow \lambda_t = \lambda_{t+1}[f'(k_{t+1}) + 1 - \delta] \qquad \text{....(2)}$$

as well as the resource constraints

- Transversality condition:

$$\lim_{t\to\infty} \lambda_t k_{t+1} = \lim_{t\to\infty} \beta^t U'(c_t) k_{t+1} = 0$$

This requires that asymptotically the shadow value of more capital is zero. This is the natural infinite-horizon equivalent of the requirement that $k_{T+1} = 0$ in a model with a finite horizon T

Euler Equation

- Combining (2) with (1): Get <u>Euler equation</u>

$$\frac{\beta U'(c_{t+1})}{U'(c_t)} = \frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{1 + f'(k_{t+1}) - \delta} \qquad \text{....(3)}$$

- Requires the equality of the marginal rate of substitution (MRS) between consumption today and tomorrow with the physical marginal rate of transformation (MRT)

- An equilibrium is consumption c_t and capital k_t that solve the coupled system of non-linear difference equations

$$U'(c_t) = \beta U'(c_{t+1})[1 + f'(k_{t+1}) - \delta] \quad \text{....(4)}$$

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \quad \text{....(5)}$$

with 2 boundary conditions, the given initial condition k_0 and the transversality condition

- Note that these conditions are just the equilibrium conditions derived earlier
- Therefore, the competitive equilibrium leads to the social optimum (decentralized and social planner's optimization)
- Not surprising since in a competitive economy without distortions or externalities the Welfare Theorems must hold

Steady State of the Ramsey-Cass-Koopmans Model

- Steady-state is given by numbers $\bar{c} = c_t = c_{t+1}$ and $\bar{k} = k_t = k_{t+1}$
- These solve

$$1 = \beta\left[1 + f'(\bar{k}) - \delta\right] \quad \text{....(6)}$$

$$\bar{c} + \bar{k} = f(\bar{k}) + [1-\delta]\bar{k} \quad \text{....(7)}$$

- Equation (6) implies we can solve for steady-state capital stock independent of consumption

$$f'(\bar{k}) = \frac{1}{\beta} - 1 + \delta = \rho + \delta \qquad \text{….(8)}$$

where the parameter ρ such that $\beta \equiv \frac{1}{1+\rho}$ is the time discount rate

- Hence at steady-state, the capital stock is such that the net marginal product of capital is equal to the discount rate

An Example

- Cobb Douglas production function

$$f(k) = Ak^{\alpha}$$

$$\bar{k} = \left(\frac{\alpha A}{\rho + \delta}\right)^{\frac{1}{1-\alpha}}$$

- Clearly, more patience (lower ρ) tends to increase capital accumulation and so increase $\bar{k}$ Similarly, lower depreciation δ or more capital intensity in production (higher α) raise $\bar{k}$

Steady State Consumption and Investment

- Once the steady state capital stock is computed, the associated consumption level can be backed out from the resource constraint

$$\bar{c} = f(\bar{k}) - \delta \bar{k} \qquad \text{….(9)}$$

- $\delta\bar{k}$ stands for steady state investment

Qualitative Dynamics

- Euler equation: consumption will be growing along an optimal path whenever

$$c_{t+1} > c_t \Leftrightarrow \frac{U'(c_t)}{U'(c_{t+1})} > 1$$

$$\Leftrightarrow \; \beta\left[1 + f'(k_{t+1}) - \delta\right] > 1$$

$$\Leftrightarrow f'(k_{t+1}) > \frac{1}{\beta} - 1 + \delta = \rho + \delta = f'(\bar{k})$$

$$\Leftrightarrow k_{t+1} < \bar{k}$$

- Whenever the capital stock will be less than its steady state value, the real interest rate will be high relative to the time discount rate so the representative consumer will find it optimal to defer consumption so as to invest in capital accumulation thereby enjoying higher consumption tomorrow relative to today

- Similarly,

$$k_{t+1} > k_t \Leftrightarrow f(k_t) + (1-\delta)k_t - c_t > k_t$$

$$\Leftrightarrow f(k_t) - \delta k_t > c_t$$

- The capital stock grows whenever there is any output left over once consumption and depreciation have been taken out

Modified Golden Rule Point

- Recall in steady state

$$1 = \beta\left[1 + f'(\bar{k}) - \delta\right]$$

Or,
$$f'(\bar{k}) = \frac{1}{\beta} - 1 + \delta = \rho + \delta$$

Thus, the modified golden rule is given by:

$$f'(k_{gold}) - \delta = \rho \quad \text{….(10)}$$

i.e. the MP of capital, net of depreciation equals the time discount rate

- Because the model assumed

$$\beta < 1$$

Comparing to the golden rule: $f'(k_{gold})=\delta$, Ramsey capital is below the golden rule

$$\bar{k} < k_{gold}$$

- There is no possibility of a dynamically inefficient steady state with capital over-accumulation (unlike Solow)
- If agents discount, they do not want to maximize S-S consumption and would rather consume a little more at the expense of the long run

- The capital stock in the Ramsey model is below the golden rule by an amount that depends on the rate of time preference
- The impatience reflected in ρ means that it is not optimal to reduce current consumption in order to attain the higher golden-rule consumption level

Govt. in the Ramsey Model

- Explore how fiscal policy (public spending and taxes) affects the behavior of the equilibrium variables
- Main implications:
  - Govt. demand for goods and services does not contribute to the economy's level of output per capita
  - Ricardian Equivalence
  - Distortionary taxes have a negative impact on aggregate economic activity

Public Spending and Crowding out

- Assume government spending is financed through lump sum (non distortionary) taxes, τ_t, borne by HH
- Government's budget constraint: $\tau_t = g_t$
- HH budget constraint:

$$c_t + a_{t+1} = w_t + (1 + r_t)a_t - \tau_t$$

- Firm's behavior is not affected by the presence of the public sector
- So,

$$r_t = f'(k_t) - \delta$$
$$w_t = f(k_t) - k_t f'(k_t)$$

Public Spending and Crowding out

- Verify that the HH's optimal choice is characterizes by the Euler equation (not affected by the government)
- Thus, equilibrium is described by:

$$\frac{U'(c_t)}{U'(c_{t+1})} = \beta[1 + f'(k_{t+1}) - \delta]$$

$$k_{t+1} + c_t + g_t = f(k_t) + (1-\delta)k_t$$

Plus the initial condition for k_0 and the TVC.

Example

- Preferences: $u(c)=\frac{c^{1-\theta}}{1-\theta}, \theta>0$
- In equilibrium:

$$\frac{c_{t+1}}{c_t}=\left[\beta\left(1+f'\left(k_{t+1}\right)-\delta\right)\right]^{\frac{1}{\theta}}$$

$$k_{t+1}+c_t+g_t=f\left(k_t\right)+\left(1-\delta\right)k_t$$

- At the steady state:

$$f'(\bar{k})=\delta+\rho$$

$$\bar{c}=f(\bar{k})-\delta\bar{k}-g$$

- Steady state capital stock and output are independent of government spending
- However, there is full crowding out: government spending reduces private consumption on a one-to-one basis

---

A permanent increase in g

- Graphically, a permanent increase in g shifts out the $\Delta k=0$ curve downwards by the magnitude of the change in g
- The economy adjusts instantaneously through a downward jump of c to its new steady state value.
- Since the HH perceive that will be permanently "poorer" by the magnitude of τ=g, they decide to adjust immediately its consumption pattern by this amount
- No dynamic effect on capital accumulation

A temporary increase in g

- When g increases for a short while (and HH perceive it to be so), then consumption falls but by less that the increase in g
- Thus, capital falls initially until it returns to the original saddle path towards steady state
- Intuition: a temporary increase in g and thus in τ, does not affect substantially the life-time wealth of the HH. Since HH prefer smooth consumption patterns, they decide to spread the fall in consumption over several periods by also reducing savings and capital accumulation until the economy converges back to the stead state.

Ricardian Equivalence

- Whether public spending is financed though taxes or debt is of no relevance
- Main modification to the model:
  - Govt.'s budget may not be balances. The govt. can borrow or led to make up the difference between g and τ. Govt.'s budget constraint:

$$d_{t+1} - d_t + \tau_t = g_t + r_t d_t$$

  - HH budget constraint:

$$c_t + a_{t+1} + d_{t+1} = w_t + (1+r_t)(a_t + d_t) - \tau_t$$

- Notice the two assets (government and private) are perfect substitutes
- Firm's behavior is again not affected by the presence of the public sector

$$r_t = f'(k_t) - \delta$$
$$w_t = f(k_t) - k_t f'(k_t)$$

- HH's optimal choice is characterized by the Euler equation:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\theta} = [\beta(1+r_{t+1})]$$

- Combining and noting that k=a+d, the equilibrium is described by

$$\frac{c_{t+1}}{c_t} = \left[\beta\left(1 + f'(k_{t+1}) - \delta\right)\right]^{\frac{1}{\theta}}$$

$$k_{t+1} + c_t + g_t = f(k_t) + (1-\delta)k_t$$

Plus the initial condition for k and the TVC.

- Fiscal policy only appears through the path of expenditures g, while the method of financing is irrelevant for the allocation of resources

- Intuition: HH know that they will eventually pay with taxes for current spending. If taxes are not raised today, they will be raised tomorrow. Public debt just changes the distribution of taxes over time but not is total value.

- Notice that the intertemporal budget constraint of the representative household is not affected by the sequence of public debt and taxes.

- With govt., the intertemporal constraint for the HH: PDV(c) =a_0+d_0+PDV(w)-PDV(τ)

- The intertemporal budget constraint for the govt.: d_0+PDV(g)=PDV(τ)

- Combining: PDV(c) =a_0+PDV(w)-PDV(g)

- All that matters for the HH's behavior is the present value of expenditures, irrespective of how the govt. decides to pay for it. Even if consumer observe low taxes today, they rationally expect to have to pay in the future.

Recent Controversy

- *""Ricardian Equivalence," which is the theorem that stimulus does not work in a well-functioning economy…."* John Cochrane (2011), The Grumpy Economist blog
- *"There have been a lot of shockingly bad performances among macroeconomists in this crisis; but if I had to pick the one that is most startling, it is the way freshwater economists have demonstrated that they don't understand one of their own doctrines, that of Ricardian equivalence...How could anyone who thought about this for even a minute-let alone someone with an economics training-get this wrong? And yet as far as I can tell almost everyone on the freshwater side of this divide did get it wrong, and has yet to acknowledge the error.." Paul Krugman (2009).* "A Note On The Ricardian Equivalence Argument Against Stimulus" NYT, December 26, 2011

- Two major positions currently exist in macroeconomics:
  - Keynesians (after John M. Keynes, 1936)
  - Classicals (after Robert Lucas, 1972)

- Keynesian position:
  - Without reservations: countercyclical fiscal policy should be used ... in extreme situations
  - In the current context monetary policy is in a liquidity trap (not efficient) ... .fiscal policy ought to be used
  - "Fiscal Multiplier" larger than 1: 1 dollar in public expenditure, more than 1 dollar in GDP; The multiplier only works in the short term
- For the Classical school in macroeconomics:
  - Fiscal policy does not work and should never be used to manage short term business cycles
  - Not even ... in extreme situations like the one we experienced in 2007-2009

- For Ricardian Equivalence to hold, we need for:
  - people to be able to borrow and lend as much as they want at the market interest rate
  - people to be your standard non-myopic economic agents
  - people to care only about the subjective utility of their descendants
  - everybody to leave a positive bequest to each of their descendants
  - everybody to have some descendants
  - nobody immigrates or emigrates

- In other words, the principle only applies when taxes are lump sum, when all agents are equal, when agents make no mistakes in forecasting the future
- It does not apply whenever:
  - Households are heterogeneous, not all affected in the same way by the tax cut: redistribution exists in reality
  - Taxes are distortionary (VAT, income taxes, etc.)
  - The additional debt raised by the government is not paid back within the lifetime of every household
  - Credit markets are not perfect: government and households face different borrowing constraints and different borrowing costs
  - Agents do not have rational expectations: mistakes about forecasting the future are common