Answers to HW5:

1. (a) Given the bidding behavior of the other bidder, a buyer with value v bidding x has the probability of winning 2x, and profit after winning v-x, hence the expected profit is 2x(v-x).

The optimal x is determined by maximizing the expected profit and we have the first order condition 2v-4x=0, yielding the optimal bid amount x=0.5v.

This means that if the other bidder uses the bidding strategy b(v)=0.5v in the first-price auction, then the buyer will also use the same bidding strategy. Thus the symmetric bidding strategy b(v)=0.5v is a Bayesian Nash equilibrium strategy in the sealed bid first-price auctions.

(b) Compute the revenue by the formmula

$$2\int_0^4 b(v)F(v)dF(v) = 2\int_0^4 0.5v * 0.25vd(0.25v) = \frac{1}{16}\int_0^4 v^2dv = \frac{4^3}{3 * 16} = \frac{4}{3}.$$

2. (a) The bidding strategy $b(v) = 0.5v$ has the inverse bidding function $\phi(b) = 2b$. The winning probability of a buyer with value v bidding b is b. The profit after winning is $v - b$. Hence the expected profit of bidding b is

$$(v - b)b.$$

Taking the partial derivative with respect to b, we get the optimal bid $b = 0.5v$.

(b) The seller's expected total revenue is given by

$$2\int_0^2 b(v)F(x)dF(v) = 2\int_0^2 0.5v * 0.5vd(0.5v) = 0.25\int_0^2 v^2dv = \frac{1}{4}\frac{1}{3}2^3 = \frac{2}{3}.$$

3. (a) For second-price auctions, when a buyer with value v bids b, the winning probability is $2b$. When he wins, he pays a price uniformly distributed between $[0,b]$ hence the expected price to pay is $0.5b$. Hence the expected profit bidding b is

$$(v - 0.5b)2b.$$

Taking the derivative with respect to b, and set it to 0, we get

$$-0.5 * 2b + 2v - b = 0,$$

and we get the optimal bid $b = v$. Hence the bidding strategy $b(v) = v$ is an equilibrium.

(b) A buyer with value v wins with probability $0.5v$, and pays the expected price (expected second highest bid) $0.5v$ after winning. Hence the revenue from one buyer is

$$\int_0^2 0.5v * 0.5v dF(v) = 0.25 \int_0^2 v^2 0.5 dv = \frac{1}{3}.$$

hence the total revenue is also $\frac{2}{3} = 2 * \frac{1}{3}$, the same as that of the first-price auction.