1 Neoclassical Growth

1.1 Model Setup

1. **Technology**: The final output good is produced using labor and capital services as inputs.

$$y_t = F(k_t, n_t)$$

Output can be consumed or invested.

$$y_t = i_t + c_t$$

Investment augments the capital stock which depreciates at a constant rate δ.

$$k_{t+1} = (1-\delta)k_t + i_t$$

$$i_t = k_{t+1} - k_t + \delta k_t$$

2. **Preferences**: There is a large number of identical, infinitely lived households. Preferences of each household are assumed to be representable by a time-separable utility function:

$$u(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t U(c_t)$$

3. **Endowments**: At period 0, each household is born with endowment $\bar{k}_0$. In each subsequent period, each household is endowed with one unit of productive time.

1.2 Social Planner's Problem

1.2.1 Sequential Formulation

$$w(\bar{k}_0) = \max_{\{c_t, k_t, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

s.t.

$$F(k_t, n_t) = c_t + k_{t+1} - (1-\delta)k_t$$

$$c_t \geq 0, k_t \geq 0, 0 \leq n_t \leq 1$$

$$k_0 \leq \bar{k}_0$$

1.2.2 Recursive Formulation

$$v(k) = \max_{0 \le k' \le f(k)} \{U(f(k) - k') + \beta v(k')\}$$

where we have assumed F is homogenous of degree 1. $f(k) = F(k,1) + (1-\delta)k$

1.2.3 Characterizing the Optimal Solution

Euler Equations:

$$\beta U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) = U'(f(k_t) - k_{t+1}), \forall t$$

Transversality Condition:

$$\lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1}) f'(k_t) k_t = 0$$

The transversality condition substitutes for the missing terminal condition. It states that the value of the capital stock k_t, when measured in terms of discounted utility, goes to zero as time goes to infinity. This condition does not require that the capital stock itself converges to zero in the limit, only that the shadow value of the capital stock converges to zero.

1.2.4 Steady State and Modified Golden Rule

In a steady state, $c_t = c_{t+1} = c^*$, from Euler equation we have

$$\beta f'(k^*) = 1.$$

If we deine time discount rate $\rho = \frac{1-\beta}{\beta}$

$$\beta f'(k^*) = 1 + \rho$$

Again,

$$f'(k) = F_k(k,1) + 1 - \delta.$$

The Modified Golden Rule

$$F_k(k^*,1) - \delta = \rho$$

The social planner sets the marginal product of capital, net of depreciation, which is also the real interest rate, equal to the time discount rate.

Golden Rule In steady state, $c = f(k) - k$. The captial stock that maximizes consumption per capita is the Golden Rule k^g, satisfying

$$f'(k^g) = 1$$

$$F_k(k^g, 1) - \delta = 0$$

The social planner find it optimal to set capital $k^* < k^g$ in the long run because he repects the impatience of the representative household.

1.3 Competitive Equilibrium

We assume consumers own all factors of production and rent it out to the firms and they own firms too. We also assume perfect competitive markets where individuals and firms take prices as given.

1.3.1 Arrow-Debreu Competitive Equilibrium

Definition 1.1. *A Competitive Equilibrium (Arrow-Debreu Equilibrium) consists of prices* $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ *and allocations for the firm* $\{k_t^d, n_t^d, y_t\}_{t=0}^{\infty}$ *and the household* $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}$ *such that*

1. *Given* $\{p_t, w_t, r_t\}_{t=0}^{\infty}$*, the allocations of the representative firm* $\{k_t^d, n_t^d, y_t\}_{t=0}^{\infty}$ *solves*

$$\pi = \max_{\{y_t, k_t, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t(y_t - r_t k_t - w_t n_t)$$

$$s.t. \quad y_t = F(k_t, n_t), \forall t \geq 0$$

$$y_t, k_t, n_t \geq 0$$

2. *Given* $\{p_t, w_t, r_t\}_{t=0}^{\infty}$*, the allocations of the representative household* $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}$ *solves*

$$\max_{\{c_t, i_t, x_{t+1}, k_t, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$s.t. \quad \sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t n_t) + \pi$$

$$x_{t+1} = (1-\delta)x_t + i_t, \forall t \geq 0$$

$$0 \leq n_t \leq 1, 0 \leq k_t \leq x_t, \forall t \geq 0$$

$$c_t, x_{t+1} \geq 0, \forall t \geq 0$$

$$x_0 given$$

3. Markets clear.

$$y_t = c_t + i_t \quad (Goods\,Market)$$

$$n_t^d = n_t^s \quad (Labor\,Market)$$

$$k_t^d = k_t^s \quad (Capital\,Services\,Market)$$

1.3.2 Characterizing the Competitive Equilibrium

Firm:

$$\pi = \max_{\{y_t,k_t,n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t(y_t - r_t k_t - w_t n_t)$$

$$s.t. \quad y_t = F(k_t, n_t), \forall t \geq 0$$

$$y_t, k_t, n_t \geq 0$$

F.O.Cs:

$$r_t = F_k(k_t, n_t)$$

$$w_t = F_n(k_t, n_t)$$

In equilibrium, $\pi_t = 0$. It is the consequence of perfect competition and the assumption that the production function F exhibits constant returns to scale.

$$r_t = F_k(k_t, 1) = f'(k_t) - (1 - \delta)$$

Households:

$$\max_{\{c_t,i_t,x_{t+1},k_t,n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$s.t. \quad \sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t n_t) + \pi$$

$$x_{t+1} = (1 - \delta)x_t + i_t$$

Obviously, $n_t = 1$, $k_t = x_t$. F.O.Cs:

$$\beta^t U'(c_t) = \mu p_t$$

$$\beta^{t+1} U'(c_{t+1}) = \mu p_{t+1}$$

$$\mu p_t = \mu(1 - \delta + r_{t+1}) p_{t+1}$$

Euler Equation:

$$\frac{\beta U'(c_{t+1})}{U'(c_t)} = \frac{p_{t+1}}{p_t} = \frac{1}{1 + r_{t+1} - \delta}$$

or

$$\frac{(1-\delta+r_{t+1})\beta U'(c_{t+1})}{U'(c_t)} = 1$$

or

$$\frac{f'(k_{t+1})\beta U'(f(k_{t+1})-k_{t+2})}{U'(f(k_t)-k_{t+1})} = 1$$

The same as the social planners' problem.

Transversality Condition:

$$\lim_{t\to\infty} p_t k_{t+1} = 0$$

and

$$\lim_{t\to\infty} p_t k_{t+1} = \frac{1}{\mu}\lim_{t\to\infty} \beta^t U'(f(k_t)-k_{t+1})f'(k_t)k_t$$

This is the same TVC as the social planners' problem.

1.3.3 Sequencial Competitive Equilibrium

Definition 1.2. *A sequential market equilibrium is a sequence of prices $\{w_t, r_t\}_{t=0}^{\infty}$, allocations for therepresentative household $\{c_t, k_{t+1}^s\}_{t=0}^{\infty}$ and allocations for the representative firm $\{n_t^d, k_{t+1}^d\}_{t=0}^{\infty}$ such that*

1. *Given k_0 and $\{w_t, r_t\}_{t=0}^{\infty}$, allocations for the representative household $\{c_t, k_{t+1}^s\}_{t=0}^{\infty}$ solve the household maximization problem*

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$s.t. \quad c_t + k_{t+1} - (1-\delta)k_t = w_t + r_t k_t$$

$$c_t, k_{t+1} \geq 0$$

$$k_0 \, given$$

2. *For each $t \geq 0$, given (w_t, r_t) the firm allocation (n_t^d, k_t^d) solves the firms' maximization problem.*

$$\max_{k_t, n_t \geq 0} F(k_t, n_t) - w_t n_t - r_t k_t$$

3. *Markets clear: for all $t \geq 0$*

$$n_t^d = 1$$

$$k_t^d = k_t^s$$

$$F(k_t^d, n_t^d) = c_t + k_{t+1}^s - (1-\delta)k_t^s$$

1.3.4 Recursive Competitive Equilibrium

Definition 1.3. *A recursive competitive equilibrium is a value function* $v : R_+^2 \to R$ *and policy function* $C, G : R_+^2 \to R_+$ *for the representative household, pricing functions* $w, r : R_+ \to R_+$ *and an aggregate law of motion* $H : R_+ \to R_+$ *such that*

1. *Given the functions w,r and H, the value function v solves the Bellman equation and C, G are the associated policy functions.*

$$v(k, K) = \max_{c,k' \geq 0}\{U(c) + \beta v(k', K')\}$$

$$s.t \quad c + k' = w(K) + (1 + r(K) - \delta)k$$

$$K' = H(K)$$

2. *The pricing functions satisfy*

$$w(K) = F_l(K, 1)$$

$$r(K) = F_k(K, 1).$$

3. *Consistency*

$$H(K) = G(K, K).$$

4. *For all* $K \in R_+$

$$c(K, K) + G(K, K) = F(K, 1) + (1 - \delta)K.$$

2 Pure Exchange OLG

2.1 Model Setup

- Time is discrete, $t = 1, 2, 3, \ldots$. The economy lives forever.
- We denote generation t's endowment of the consumption good (e_t^t, e_{t+1}^t) and consumption allocation (c_t^t, c_{t+1}^t).
- In period 1 there is an initial old generation 0 that has endowment e_1^0 and consumes c_1^0.
- In some models, the initial old generation 0 has an amount of outside money m.

- Preferences of individuals:
$$u_t(c) = U(c_t^t) + \beta U(c_{t+1}^t),$$
Preferences of initial old generation:
$$u_0(c) = U(c_1^0).$$

2.1.1 Arrow-Debreu Equilibrium

Definition 2.1. *Given m, an Arrow-Debreu Equilibrium is an allocation $\hat{c}_1^0$, $\{(\hat{c}_t^t, \hat{c}_{t+1}^t)\}_{t=1}^{\infty}$ and prices $\{p_t\}_{t=1}^{\infty}$ such that*

1. *Given $\{p_t\}_{t=1}^{\infty}$, for each $t \geq 1$, $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ solves*
$$\max_{(\hat{c}_t^t, \hat{c}_{t+1}^t) \geq 0} \ u_t(c_t^t, c_{t+1}^t)$$
$$s.t. \quad p_t c_t^t + p_{t+1} c_{t+1}^t \leq p_t e_t^t + p_{t+1} e_{t+1}^t$$
2. *Given p_1, $\hat{c}_1^0$ solves*
$$\max_{c_1^0} \ u_0(c_1^0)$$
$$s.t. \quad p_1 c_1^0 \leq p_1 e_1^0 + m$$
3. *For all $t \geq 1$ (Resource Balance or Goods Market Clearing)*
$$\hat{c}_t^{t-1} + \hat{c}_t^t = e_t^{t-1} + e_t^t \quad \forall t \geq 1.$$

2.1.2 Sequential Equilibrium

Definition 2.2. *Given m, a Sequential Equilibrium is an allocation $\hat{c}_1^0$, $\{(\hat{c}_t^t, \hat{c}_{t+1}^t), \hat{s}_t^t\}_{t=1}^{\infty}$ and interest rates $\{r_t\}_{t=1}^{\infty}$ such that*

1. *Given $\{r_t\}_{t=1}^{\infty}$, for each $t \geq 1$, $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)$ solves*
$$\max_{(\hat{c}_t^t, \hat{c}_{t+1}^t) \geq 0, s_t^t} \ u_t(c_t^t, c_{t+1}^t)$$
s.t.
$$c_t^t + s_t^t \leq e_t^t$$
$$c_{t+1}^t \leq e_{t+1}^t + (1 + r_{t+1}) s_t^t$$

2. Given r_1, $\hat{c}_1^0$ solves

$$\max_{c_1^0} \; u_0(c_1^0)$$
$$s.t. \quad c_1^0 \leq e_1^0 + (1+r_1)m$$

3. For all $t \geq 1$ (Resource Balance or Goods Market Clearing)

$$\hat{c}_t^{t-1} + \hat{c}_t^t = e_t^{t-1} + e_t^t \quad \forall t \geq 1.$$

Remark:

$$s_{t+1}^{t+1} = (1+r_{t+1})s_t^t$$
$$s_1^1 = (1+r_1)m$$
$$s_t^t = \Pi_{\tau=1}^t (1+r_\tau)m$$

Proposition 2.1. *Equivalence between the two equilibriums.*

$$p_1 = \frac{1}{1+r_1}$$
$$p_{t+1} = \frac{p_t}{1+r_{t+1}}$$

2.2 Offer Curve Analysis

2.2.1 Basic Equation

$$y(p_t, p_{t+1}) = c_t^t(p_t, p_{t+1}) - w_1$$
$$z(p_t, p_{t+1}) = c_{t+1}^t(p_t, p_{t+1}) - w_2$$

By Budget Constraint:

$$\frac{z(p_t, p_{t+1})}{y(p_t, p_{t+1})} = -\frac{p_t}{p_{t+1}}$$

By Market Clearing:

$$y(p_t, p_{t+1}) + z(p_{t-1}, p_t) = 0$$

Initial Condition:

$$z_0(p_1, m) = \frac{m}{p_1}$$

By Optimality (MRS):

$$\frac{p_t}{p_{t+1}} = -\frac{U'(w_1)}{\beta U'(w_2)}$$

Autarkic Interest rate:

$$1 + \bar{r} = \frac{U'(w_1)}{\beta U'(w_2)}$$

When $\bar{r} < 0$, it is the Samuelson Case, when $\bar{r} \geq 0$, it is the classical case. (Remember the Graph.)

Proposition 2.2. *Blasko and Shell (1990)*

Autarkic equilibrium is Pareto optimal if and only if

$$\sum_{t=1}^{\infty} \Pi_{\tau=1}^{t}(1 + r_{\tau+1}) = +\infty$$

2.3 Features of OLG

2.3.1 Positive Valuation of Money

Proposition 2.3. *In pure exchange economies with a finite number of infinitely lived agents there cannot be an equilibrium in which outside money is valued.*

But in the OLG model, these pieces of paper can help achieve an intertemporal allocation of consumption goods that dominates the autarkic allocation. Without the outsie asset, this economy can do nothing else to but remain in the possibly dismal state of autarky. This is why the social contrivance of money is so useful in this economy. PAYG can achieve the same as money.

2.3.2 A Continuum of Equilibria

2.4 Social Security: Pay As You Go

Simple OLG model without money and the population is growing at constant rate n. New Resource Constraint:

$$c_t^{t-1} + (1+n)c_t^t = e_t^{t-1} + (1+n)e_t^t$$

The social security system is modeled as follows: the young pay social security taxes of $\tau \in [0, w_1)$ and receive social security benefits b when old. We assume that the social security system balances its budget in each period, so that benefits are given by

$$b = \tau(1+n).$$

Autarkic equilbirum $(w_1 - \tau, w_2 + \tau(1+n))$ with interest rate

$$1 + r = \frac{U'(w_1 - \tau)}{\beta U'(w_2 + \tau(1+n))}.$$

For any $\tau > 0$, the initial old receives a windfall transfer of $\tau(1+n)$ and benefits. For all other generations,

$$V(\tau) = U(w_1 - \tau) + \beta U(w_2 + \tau(1+n))$$
$$V'(\tau) = -U'(w_1 - \tau) + \beta U'(w_2 + \tau(1+n))(1+n)$$
$$V'(0) = -U'(w_1) + \beta U'(w_2)(1+n)$$

$V'(0) > 0$ if and only if

$$n > \frac{U'(w_1)}{\beta U'(w_2)} - 1 = \bar{r}$$

$\bar{r}$ is the autarkic interest rate before. Hence the introduction of a pay-as-you-go social security system is welfare improving if and only if the population growth rate exceeds the equilibrium autarkic interest rate , i.e., in the Samuelson Case where autarky is not a Pareto Optimal allocation. The social security has the same function as money in our economy: it is a social institution that transfers resources between generations that do not trade among each other in equilibrium. It may generate allocations that are Pareto superior in the case where the individual private marginal rate of substitution $1 + \bar{r}$ falls short of the social intertemporal rate of transformation 1+n.

The sizes of the social security system for which the resulting equilibrium allocation is Pareto optimal is such that the resulting autarkic equilibrium interest rate is at least equal to the population growth rate. However, for any $\tau > \tau^*$ satisfying $\tau \leq w_1$ generates a Pareto optimal allocation, too. (A little less from that benefit the representative generation but hurt the initial old.)

2.5 Ricardian Equivalence

The Ricardian Equivalence hypothesis is a theorem that holds in a fairly wide class of models. Consider the simple infinite horizon pure exchange model and introduce a government that has to finance a given exogenous stream of government expenditures (in real terms) denoted by $\{G_t\}_{t=1}^{\infty}$. These government expenditures do not yield any utility to the agents. The government has initial outstanding real debt of B_1.

$$\sum_{i \in I} b_1^i = B_1$$

2.5.1 Arrow-Debreu Equilibrium

Given a sequence of government spending $\{G_t\}_{t=1}^{\infty}$ and initial government debt B_1 and $(b_1^i)_{i\in I}$ and Arrow-Debreu equilibrium are allocations $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}$, prices $\{\hat{p}_t\}_{t=1}^{\infty}$ and taxes $\{(\tau_t^i)_{i\in I}\}_{t=1}^{\infty}$ such that

1. Given prices $\{\hat{p}_t\}_{t=1}^{\infty}$ and taxes $\{(\tau_t^i)_{i\in I}\}_{t=1}^{\infty}$ for all $i \in I$, $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}$ solves

$$\max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(c_t^i)$$

$$s.t. \quad \sum_{t=1}^{\infty} \hat{p}_t (c_t + \tau_t^i) \leq \sum_{t=1}^{\infty} \hat{p}_t e_t^i + \hat{p}_1 b_1^i$$

2. Given prices $\{\hat{p}_t\}_{t=1}^{\infty}$ the tax policy satisfies

$$\sum_{t=1}^{\infty} \hat{p}_t G_t + \hat{p}_1 B_1 = \sum_{t=1}^{\infty} \sum_{i\in I} \hat{p}_t \tau_t^i$$

3. For all $t \geq 1$

$$\sum_{i\in I} \hat{c}_t^i + G_t = \sum_{i\in I} e_t^i$$

Theorem 2.1. *Take as given a sequence of government spending $\{G_t\}_{t=1}^{\infty}$ and initial government debt B_1, $(b_1^i)_{i\in I}$. Suppose that allocations $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}$, prices $\{\hat{p}_t\}_{t=1}^{\infty}$ from an Arrow-Debreu equilibrium. Let $\{(\hat{\tau}_t^i)_{i\in I}\}_{t=1}^{\infty}$ be an arbitrary alternative tax system satisfying*

$$\sum_{t=1}^{\infty} \hat{p}_t \tau_t^i = \sum_{t=1}^{\infty} \hat{p}_t \hat{\tau}_t^i \; \forall i \in I$$

Then $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}$, $\{\hat{p}_t\}_{t=1}^{\infty}$ and $\{(\hat{\tau}_t^i)_{i\in I}\}_{t=1}^{\infty}$ form an Arrow-Debreu equilibrium.

2.5.2 Sequential Equilibrium

Borrowing Constrait is cruical for the validity of Ricardian Equivalence.

Budget Constraints for individual:

$$c_t^i + \frac{b_{t+1}^i}{1 + r_{t+1}} \leq e_t^i - \tau_t^i + b_t^i$$

$$b_t^i \geq -a_t^i(r, e^i, \tau)$$

Budget Constraints for Government:

$$G_t + B_t = \sum_{i\in I} \tau_t^i + \frac{B_{t+1}}{1+r_{t+1}}$$
$$B_t \geq -A_t(r, G, \tau)$$

Given a sequence of government spending $\{G_t\}_{t=1}^{\infty}$ and initial government debt B_1, $(b_1^i)_{i\in I}$. A sequential Markets equilibrium is allocations $\{(\hat{c}_t^i, \hat{b}_{t+1}^i)_{i\in I}\}_{t=1}^{\infty}$, interest rates $\{\hat{r}_{t+1}\}_{t=1}^{\infty}$ and government policies $\{\tau_t^i, B_{t+1}\}_{t=1}^{\infty}$ such that

1. Given interest rates $\{\hat{r}_{t+1}\}_{t=1}^{\infty}$ and taxes $\{(\tau_t^i)_{i\in I}\}_{t=1}^{\infty}$ for all $i \in I$, $\{(\hat{c}_t^i, \hat{b}_{t+1}^i)_{i\in I}\}_{t=1}^{\infty}$ solves
$$\max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(c_t^i)$$
$$s.t. \quad c_t^i + \frac{b_{t+1}^i}{1+r_{t+1}} \leq e_t^i - \tau_t^i + b_t^i$$
$$b_t^i \geq -a_t^i(r, e^i, \tau)$$
2. Given interest rates $\{\hat{r}_{t+1}\}_{t=1}^{\infty}$, the tax policy satisfies
$$G_t + B_t = \sum_{i\in I} \tau_t^i + \frac{B_{t+1}}{1+r_{t+1}}$$
$$B_t \geq -A_t(r, G, \tau)$$
3. For all $t \geq 1$
$$\sum_{i\in I} \hat{c}_t^i + G_t = \sum_{i\in I} e_t^i$$
$$\sum_{i\in I} \hat{b}_{t+1}^i = B_{t+1}$$

Natural Borrowing Limit:

$$an_t^i(\hat{r}, e, \tau) = \sum_{\tau=1}^{\infty} \frac{e_{t+\tau}^i - \tau_{t+\tau}^i}{\Pi_{j=t+1}^{t+\tau-1}(1+\hat{r}_{j+1})}$$

It is the amount that at given sequence of interest rates, the consumer can maximally repay, by setting consumption to zero in each period.

For Government:

$$An_t^i(\hat{r}, e, \tau) = \sum_{\tau=1}^{\infty} \frac{\tau_{t+\tau}^i}{\Pi_{j=t+1}^{t+\tau-1}(1+\hat{r}_{j+1})}$$

Theorem 2.2. *For an Arrow-Debreu equilibrium, there exists a corresponding sequential markets equilibrium with natural borrowing limits. Reversely, for a sequencetial market equilibrium with natural borrowing limits, there exists a corresponding Arrow-Debreu equilibrium.*

A corollary of the result is the Ricardian Equivalence theorem for sequential markets with natural debt limits.

The Ricardian Equivalence theorem rests on several important assumptions.

- There are perfect capital markets. If consumers face biding borrowing constraints or incomplete markets, the Ricardian Equivalence fails.
- All taxes are lump-sum. Non-lump sum taxes may distort relative prices and hence a change in the timing of taxes may have real effects.
- A change from one ot another tax system is assumed to not redistribute wealth among agents.

2.6 Barro's Operative Bequest OLG Model

Barro argues that under certain conditions finitely lived agents will behave as if the had infinite lifetime. An OLG economy with two period-lived agents and operative bequest motives is formally equivalent to an infinitely lived agent model.

- Endowment $e_t^t = w$ when young and 0 when old.
- Government: no expenditures and maintain an intitial debt B.

  $$\frac{B}{1+r} + \tau = B$$

  Thus,

  $$\tau = \frac{rB}{1+r} (< w)$$

- Budget constraint of a representative generation:

  $$c_t^t + \frac{a_t^t}{1+r} = w - r$$

  $$c_{t+1}^t + \frac{a_{t+1}^t}{1+r} = a_t^t + a_t^{t-1}$$

  $$a_{t+1}^t \geq 0$$

- Budget constraint of initial old:
$$c_1^0 + \frac{a_1^0}{1+r} = B$$
- Life time endowment:
$$e_t = w + \frac{a_t^{t-1}}{1+r} - \tau$$
- Preferences of generation t
$$u_t(c_t^t, c_{t+1}^t, a_{t+1}^t) = U(c_t^t) + \beta U(c_{t+1}^t + \alpha V_{t+1}(e_{t+1}))$$
$V_{t+1}(e_{t+1})$ is the maximal utility generation t+1 can attain with lifetime resources e_{t+1}.

2.6.1 Ricardian Experiment

Government increase initial government debt by ΔB and repay this additional debt by levying higher taxes on the first young generation. In Barro economy, in order to repay ΔB, the taxes for the young have to increase by
$$\Delta\tau = \Delta B.$$
The optimal choices fro c_1^0 and e_1 do not change since
$$c_1^0 + e_1 = w + B - \tau.$$
The initial old receives additional transfer of bonds of ΔB and increase its bequests a_1^0 by $(1+r)\Delta B$ so that the lifetime resources to their descendants unchanged. Ricardian Equivalence is restored.

3 OLG with Production

3.0.1 Model Setup

Diamond 1965

- Population:

  N_t^t is the population of young at date t; N_t^{t-1} is the population of old at date t.

  Normalize $N_0^0 = N_1^0 = 1$, $N_t^t = (1+n)^t$, n is the rate of population growth.

Thus, total population at date t is:

$$N_t^t + N_t^{t-1} = (1+n)^t(1+\frac{1}{1+n})$$

- Preferences:

$$u(c_t^t, c_{t+1}^t) = u(c_t^t) + \beta u(c_{t+1}^t).$$

For initial old:

$$u(c_1^0) = \beta u(c_1^0).$$

- Endowment:

Generation t ($t \geq 1$) has one unit of production time when young, zero when old.

Each member of initial old owns k.

$$(1+n)k_1 = K_1$$

- Technology:

$$y_t = \frac{Y_t}{L_t} = \frac{F(K_t, L_t)}{L_t} = F(\frac{K_t}{L_t}, 1) = f(k_t)$$

f is twice continuously differentiable strictly concave in k and satisfied Inada condition.

3.1 Sequential Market Equilibrium

Definition 3.1. *A sequential market equilibrium is an allocation for households $\hat{c}_1^0$, $\{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^{\infty}$; an allocation for firms $\{\hat{K}_t, \hat{L}_t\}_{t=1}^{\infty}$ and prices $\{\hat{r}_t, \hat{w}_t\}_{t=1}^{\infty}$ such that*

1. *For all $t \geq 1$, given $\hat{r}_t, \hat{w}_t$ $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)$ solves*

$$\max_{c_t^t, c_{t+1}^t \geq 0} u(c_t^t) + \beta u(c_{t+1}^t)$$

$$s.t.\ c_t^t + s_t^t \leq \hat{w}_t$$

$$c_{t+1}^t \leq (1 + \hat{r}_{t+1} - \delta)s_t^t$$

2. *Given $\bar{K}_1$ and $\hat{r}_1$, initial old $\hat{c}_1^0$ solves*

$$\max_{c_1^0 \geq 0} \beta u(c_1^0)$$

$$s.t.\ c_1^0 \leq (1 + \hat{r}_1 - \delta)\bar{K}_1$$

3. Given $(\hat{r}_t, \hat{w}_t)$, Firms choose $(\hat{K}_t, \hat{L}^t)$

$$\max_{K_t, L_t \geq 0} F(K_t, L_t) - \hat{r}_t K_t - \hat{w}_t L_t$$

4. Markets clear.

$$L_t = 1 \; \forall t$$

$$N_t^t \hat{c}_t^t + N_t^{t-1} \hat{c}_t^{t-1} + \hat{K}_{t+1} - (1-\delta)\hat{K}_t = F(\hat{K}_t, \hat{L}_t)$$

$$N_t^t \hat{s}_t^t = \hat{K}_{t+1}$$

Remarks:

$$\hat{K}_{t+1} - (1-\delta)\hat{K}_t = N_t^t \hat{s}_t^t + N_t^{t-1} \hat{s}_t^{t-1}$$

$$(1+n)^t \hat{s}_t^t = \hat{K}_{t+1} = (1+n)^{t+1} k_{t+1}$$

$$s_t^t = (1+n) k_{t+1}$$

3.2 Steady State

Definition 3.2. *A steady state equilibrium is $(\bar{k}, \bar{s}, \bar{c}_1, \bar{c}_2, \bar{r}, \bar{w})$ such that the sequence $\hat{c}_1^0$, $\{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^{\infty}$, $\{(\hat{K}_t, \hat{L}_t)\}_{t=1}^{\infty}$, $\{(\hat{r}_t, \hat{w}_t)\}_{t=1}^{\infty}$ defined by for all t,*

$$\hat{c}_t^t = \bar{c}_1$$

$$\hat{c}_{t+1}^t = \bar{c}_2$$

$$\hat{s}_t^t = \bar{s}$$

$$\hat{r}_t = \bar{r}$$

$$\hat{w}_t = \bar{w}$$

$$\hat{K}_t = \bar{k} N_t^t$$

$$\hat{L}_t = N_t^t$$

is an equilibrium for an initial condition $k_1 = \bar{k}$.

3.3 Characterization of Equilibrium

Firm's maximization:

$$r_t = F_k(K_t, L_t) = F_k(\frac{K_t}{L_t}, 1) = f'(k_t)$$
$$w_t = f(k_t) - f'(k_t)k_t$$

Household Euler equation:

$$u'(w_t - s_t^t) = \beta(1 + r_{t+1} - \delta)u'((1 + r_{t+1} - \delta)s_t^t)$$

Implicit function:

$$s_t^t = s(w_t, r_{t+1}) = s(f(k_t) - f'(k_t)k_t, f'(k_{t+1}))$$
$$k_{t+1} = \frac{1}{1+n}s(f(k_t) - f'(k_t)k_t, f'(k_{t+1}))$$

Assume there is a unique steady state (positive), then a necessary and sufficient condition for locally stability at unique k^* is

$$|\frac{-s_w(w(k^*), r(k^*))f''(k^*)k^*}{1 + n - s_r(w(k^*), r(k^*))f''(k^*)}| < 1$$

For monotonic adjustment:

$$0 < \frac{-s_w(w(k^*), r(k^*))f''(k^*)k^*}{1 + n - s_r(w(k^*), r(k^*))f''(k^*)} < 1.$$

3.4 Dynamic Inefficiency

Assume the last equation holds, we have a unique positive steady state with monotone dynamics (convergence for any initial k_1). In steady state,

$$\bar{c}_1 + \frac{\bar{c}_2}{1+n} = f(\bar{k}) - (n + \delta)\bar{k}$$
$$\frac{d\bar{c}}{d\bar{k}} < 0 \Longleftrightarrow f'(\bar{k}) < n + \delta$$

Resource constraint:

$$\bar{c}_1 + \frac{\bar{c}_2}{1+n} = f(\bar{k}) - (n + \delta)\bar{k} = w(\bar{k}) + f'(\bar{k})\bar{k} - (n + \delta)\bar{k}$$

Intertemperal constraint:

$$c_t^t + \frac{c_{t+1}^t}{1 + r_{t+1} - \delta} = w_t$$

Dynamic Inefficiency:

- Capital stock is too high.
- Marginal product if outweighted by the cost of replacing depreciation capital and equipping new workers.
- $\frac{1}{1+n}$ is social rate of transmation of future in terms of current consumption.
- $\frac{1}{1+\bar{r}-\delta}$ is private market price of future in terms of current consumption.
- Oversaving:
$$r < n + \delta \Longleftrightarrow f'(k) < n + \delta \Longleftrightarrow \frac{1}{1+\bar{r}-\delta} > \frac{1}{1+n}.$$
- Golden rule capital stock that maximizes SS per capita consumption:
$$f'(\bar{k}^*) = n + \delta$$