MATH 425b IN-CLASS FINAL EXAM
May 9, 2016
Prof. Alexander

Last Name: ______________________________

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Signature: ______________________________

Problem	Points	Score
1	38	
2	18	
3	16	
4	28	
Total	95	

Notes:

(1) Use the backs of the sheets if you need more room.

(2) If you can't do part (a) of a problem, you can assume it and do part (b).

(3) Points total 100 but 95 is the maximum score. This means the first 5 points you miss don't count against you.

(1)(38 points) In $\mathbb{R}^3$, suppose

$$\omega = f(\mathbf{x})\ dx_1 \wedge dx_2, \qquad \lambda \text{ is a } \mathcal{C}' \text{ 1-form,} \tag{1}$$

with f also $\mathcal{C}'$.

(a) Show that $d\omega = 0$ if and only if f is a function of x_1, x_2 only (it doesn't depend on x_3.)

(b) Suppose $d\lambda = 0$, i.e. λ is closed. Let $\beta : [0,1] \to \mathbb{R}^3$ and $\gamma : [0,1] \to \mathbb{R}^3$ be two $\mathcal{C}''$ curves with the same endpoints: $\beta(0) = \gamma(0) = \mathbf{a}$, $\beta(1) = \gamma(1) = \mathbf{b}$. Show that $\int_\beta \lambda = \int_\gamma \lambda$. This means the integral does not depend on the path, but only on the endpoints $\mathbf{a}$ and $\mathbf{b}$. HINT: Almost no calculation is needed. Instead, what else do you know about λ, from the fact that λ is closed?

(c) Let g be a $\mathcal{C}'$ function on $[0,1]^2$, and parametrize the graph of g by $\Phi(u_1, u_2) = (u_1, u_2, g(u_1, u_2))$ for $(u_1, u_2) \in [0,1]^2$. For ω from (1) above, if $d\omega = 0$, show that $\int_\Phi \omega$ takes the same value for all such g. HINT: Use the parametrization to express $\int_\Phi \omega$ as an ordinary double integral. Also use (a).

(d) Let $g(u_1, u_2) = u_1 + u_2^2$, let Φ be as in part (c), and define the 2-form $\xi = x_2 x_3\ dx_1 \wedge dx_3$. Calculate $\int_\Phi \xi$. HINT: No tricks here, just a basic calculation.

(2)(18 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{x}$ with $f(\mathbf{x}) \neq 0$. Show directly from the definition of derivative that the square f^2 is differentiable at $\mathbf{x}$, and express the derivative of f^2 in terms of $f'(\mathbf{x})$ and $f(\mathbf{x})$.

HINT: "Directly from the definition" means don't use the Chain Rule or other theorems. What do you need to show about the quantity

$$f(\mathbf{x}+\mathbf{h})^2 - f(\mathbf{x})^2 - 2f(\mathbf{x})f'(\mathbf{x})\mathbf{h}\,?$$

The assumption that f is differentiable tells you that

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = f'(\mathbf{x})\mathbf{h} + (\text{what??}).$$

(3)(16 points) We discussed in lecture that the functions $\varphi_n(x) = \frac{1}{\sqrt{\pi}} \sin nx, n \geq 1$ form an orthonormal system on $[-\pi, \pi]$.

(a) Show that if g is an even real-valued function on $[-\pi, \pi]$, then g is orthogonal to φ_n for all $n \geq 1$.

(b) Let f be Riemann integrable and let c_n be the nth Fourier coefficient of f relative to $\{\varphi_n\}$ (see (66), (67) in Chapter 8 for the definition.) Does the analog of Parseval's Theorem necessarily hold for this orthonormal system? In other words, is it always true that, as in (85) in Chapter 8,

$$\sum_{n=1}^{\infty} c_n^2 = \int_{-\pi}^{\pi} f(x)^2 \, dx \, ?$$

Prove or give a counterexample. HINT: Is part (a) relevant?

(4)(28 points) Let $\Psi = \sigma^1 + \sigma^2$ be an affine k-chain in $\mathbb{R}^n$, with $\sigma^1 = [p_0, \ldots, p_k]$ and $\sigma^2 = [q_0, \ldots, q_k]$. Recall that trace($\sigma^1$) is the range of σ^1 as a mapping, which is a "triangular" region with corners $p_0, \ldots, p_k$, and similarly for σ^2. Assume that σ^1 is nondegenerate, in the sense that the Jacobian $\frac{\partial(\sigma^1_{i_1}, \ldots, \sigma^1_{i_k})}{\partial(u_1, \ldots, u_k)}$ (which is constant since σ^1 is affine) is not 0, for some $i_1, \ldots, i_k$, and the same for σ^2.

(a*) Suppose trace(σ^1) $\neq$ trace(σ^2). Show that there exists a k-form ω with $0 \neq \int_{\sigma^1} \omega + \int_{\sigma^2} \omega$.

HINT: Use the fact that for every neighborhood U of any point $\mathbf{x}_0$, there exists a non-negative continuous function f which is positive near $\mathbf{x}_0$, and 0 outside U (we know this from the partitions of unity proof.) You can then make a k-form $\omega = f(\mathbf{x})dx_I$ for some I, for which ω is 0 outside of U.

(b) Suppose that $\int_\Psi \omega = \int_{\sigma^1} \omega + \int_{\sigma^2} \omega = 0$ for every k-form ω. Show that $\sigma^2 = -\sigma^1$.

HINT: σ^1 and σ^2 having the same trace means $q_0, \ldots, q_k$ is just a permutation of the points $p_0, \ldots, p_k$. What you need to show is that the permutation is an odd one, so that $\sigma^2 = -\sigma^1$.