MATH 425a SAMPLE FINAL EXAM
Fall 2015 Prof. Alexander

(1)(a) Establish convergence or divergence: $\sum \frac{n+1}{n^3+1}$.

(b) Show that $\frac{3^n}{n!} \to 0$ as $n \to \infty$.

(c) Find a power α for which $\lim_{n\to\infty} n^\alpha(\sqrt{n+1} - \sqrt{n})$ is in $(0, \infty)$, and find the value of this limit. HINT: What is the standard way to simplify an expression involving a difference of two square roots?

(2) Let X and Y be metric spaces and $f : X \to Y$ a function. Let us call a subset U of X *nice* if $f(U) \subset D$ for some closed bounded set D in Y.

You may take as given the following fact: A finite union of bounded sets is bounded.

(a) Show that if U and V are nice, then $U \cup V$ is nice.

(b) Suppose X is compact, and every $x \in X$ has a "nice" neighborhood N_x. Show that the entire image $f(X)$ is contained in a closed bounded set. HINT: Use compactness of X.

(3) Suppose $f : X \to Y$ is a continuous 1-to-1 function, $E \subset X$ and x is a limit point of E. Show that $f(x)$ is a limit point of $f(E)$.

(4) Suppose f is continuous on $[a, b]$ and let $G(x) = \int_x^b f(t)\, dt$.

(a) Show that $G'(x) = -f(x)$ for all x. HINT: Don't redo the proof of 6.20—use 6.20.

(b) Suppose $2\int_a^x f(t)\, dt + 3\int_x^b f(t)\, dt = 0$ for all $x \in [a, b]$. Show that $f(x) = 0$ for all $x \in [a, b]$.

(5) Suppose f is differentiable on $[a, b]$, $0 \in (a, b)$, $f(0) = 0$, and f' is nondecreasing.

(a) Show that f' is continuous. HINT: Think about types of discontinuities.

(b) Show that the tangent line to f at 0 lies on or below the graph of f on $[a, 0]$. In other words, if $y = g(x)$ is the tangent line, then $g(x) \leq f(x)$.

(6) For a partition $P = \{x_0, \ldots, x_n\}$ of an interval $[a, b]$, define $\mathrm{mesh}(P) = \max_{i\leq n} \Delta x_i$. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $\{P_k\}$ is a sequence of partitions of $[a, b]$ with $\mathrm{mesh}(P_k) \to 0$.

(a) Show that $U(P_k, f) - L(P_k, f) \to 0$ as $k \to \infty$. HINT: What property does f have, stronger than just continuity?

(b) Does it follow from (a) that $U(P_k, f) \to \int_a^b f\, dx$? Explain. (The same is true for $L(P_k, f)$ but you need not prove it.)

(7) Suppose f_n and f are functions from E to $\mathbb{R}$, and each of these functions is bounded.

(a) If $f_n \to f$ uniformly, show that

$$\sup_{x\in E} f_n(x) \to \sup_{x\in E} f(x) \quad \text{as } n \to \infty.$$

(b) Give and example to show that the conclusion of (a) can be false if we only assume pointwise converence. HINT: It's simplest to take $f \equiv 0$.