Chapter | Four

Qualifications and Extensions

The revenue equivalence principle derived in the previous chapter, Proposition 3.1, is a simple yet powerful result. It constitutes a benchmark of the theory of private value auctions, whereas all other results in the area constitute a departure from the revenue equivalence principle and can be measured against it. Because of its central nature, it is worthwhile to recount the key assumptions underlying the principle:

1. *Independence*—the values of different bidders are independently distributed.
2. *Risk neutrality*—all bidders seek to maximize their expected profits.
3. *No budget constraints*—all bidders have the ability to pay up to their respective values.
4. *Symmetry*—the values of all bidders are distributed according to same distribution function F.

In this chapter we investigate how the revenue equivalence principle is affected when some of these assumptions are relaxed. We first explore the consequences of risk aversion on the part of bidders. We then study the effects of the assumption that bidders have sufficient financial resources to pay any price up to their values. We ask how the revenue equivalence principle holds up in an augmented model in which bidders are subject to budget constraints. Finally, we delve into the important issue of *ex ante* heterogeneity among the bidders. In each case, to isolate the effects of each assumption, we retain the others. For instance, we examine the consequences of risk aversion, retaining the assumptions regarding the independence of values, the lack of budget constraints, and symmetry among the bidders. In the same vein, we examine the consequences of budget constraints in a model with risk-neutral, symmetric bidders with independently distributed values, and we explore the consequences of bidder asymmetries, retaining the independence of the values, and the risk neutrality of the bidders.

An exploration of the consequences of relaxing the first assumption—the independence of the values—is postponed for the moment. It is the focus of Chapter 6, where we consider a more general model that simultaneously relaxes both this assumption and the assumption of private values.

4.1 RISK-AVERSE BIDDERS

We now argue that if bidders are risk-averse, but all other assumptions are retained, the revenue equivalence principle no longer holds. In particular, we retain all our other assumptions: independence of values, symmetry among bidders, and the absence of budget constraints.

Risk neutrality implies that the expected payoff of a bidder is additively separable, it is just the difference between the bidder's expected gain and his expected payment, so the payoff is linear in the payments. This quasi-linearity of a bidder's payoff is crucial in the derivation of the revenue equivalence result and is lost when bidders are not risk neutral.

To examine the consequences of risk aversion, suppose that each bidder has a von-Neumann-Morgenstern utility function $u:\mathbb{R}_+ \to \mathbb{R}$ that satisfies $u(0)=0$, $u'>0$ and $u''<0$. Each bidder now seeks to maximize his or her expected utility rather than expected profits. The main finding is as follows:

Proposition 4.1. *Suppose that bidders are risk-averse with the same utility function. With symmetric, independent private values, the expected revenue in a first-price auction is greater than that in a second-price auction.*

Proof. First, notice that risk aversion makes no difference in a second-price auction: it is still a dominant strategy for each bidder to bid his or her value. Thus, in a second-price auction, the expected price is the same as it would be if bidders were risk neutral.

Let us now examine a first-price auction. Suppose that when bidders are risk averse and have the utility function u, the equilibrium strategies are given by an increasing and differentiable function $\gamma:[0,\omega]\to\mathbb{R}_+$ satisfying $\gamma(0)=0$. If all but bidder 1, say, follow this strategy, then bidder 1 will never bid more than $\gamma(\omega)$. Given a value x, each bidder's problem is to choose $z\in[0,\omega]$ and bid an amount $\gamma(z)$ to maximize his or her expected utility—that is,

$$\max_z G(z)u(x-\gamma(z)) \tag{4.1}$$

where, as before, $G\equiv F^{N-1}$ is the distribution of the highest of $N-1$ values. The first-order condition for this problem is

$$g(z)\times u(x-\gamma(z))-G(z)\times\gamma'(z)\times u'(x-\gamma(z))=0$$

In a symmetric equilibrium, it must be optimal to choose $z=x$. Hence, we get

$$\frac{g(x)u(x-\gamma(x))}{\gamma'(x)}=G(x)u'(x-\gamma(x))$$

which is the same as

$$\gamma'(x) = \frac{u(x-\gamma(x))}{u'(x-\gamma(x))} \times \frac{g(x)}{G(x)} \tag{4.2}$$

With risk neutrality, $u(x) = x$, and (4.2) then yields

$$\beta'(x) = (x - \beta(x)) \times \frac{g(x)}{G(x)}, \tag{4.3}$$

where $\beta(\cdot)$ denotes the equilibrium strategy with risk-neutral bidders.

Next notice that if u is strictly concave and $u(0) = 0$, for all $y > 0$, $[u(y)/u'(y)] > y$. Using this fact, from (4.2) we can derive that

$$\gamma'(x) = \frac{u(x-\gamma(x))}{u'(x-\gamma(x))} \times \frac{g(x)}{G(x)} > (x-\gamma(x)) \times \frac{g(x)}{G(x)} \tag{4.4}$$

Now if $\beta(x) > \gamma(x)$, we have $(x-\gamma(x)) \times g(x)/G(x) > (x-\beta(x)) \times g(x)/G(x)$, and because of (4.4) this implies that $\gamma'(x) > \beta'(x)$.

To summarize, if $\beta(\cdot)$ and $\gamma(\cdot)$ are the equilibrium strategies with risk-neutral and risk-averse bidders, respectively,

$$\beta(x) > \gamma(x) \text{ implies that } \beta'(x) < \gamma'(x) \tag{4.5}$$

It is also easy to check that

$$\beta(0) = \gamma(0) = 0 \tag{4.6}$$

(4.5) and (4.6) imply that for all $x > 0$,

$$\gamma(x) > \beta(x)$$

Thus, in a first-price auction, risk aversion causes an increase in equilibrium bids. Since bids have increased, the expected revenue has also increased. Using Proposition 3.1 and the fact that the expected revenue in a second-price auction is unaffected by risk aversion, we deduce that the expected revenue in a first-price auction is higher than that in a second-price auction. ■

Why does risk aversion lead to higher bids in a first-price auction? Consider a particular bidder—say, 1—with value x. Fix the strategies of all other bidders, and then suppose bidder 1 bids the amount b. Now suppose that this bidder considers decreasing his bid slightly from b to $b - \Delta$. If he wins the auction with this lower bid, this leads to a gain of Δ. A lowering of his bid could, however, cause him to lose the auction. For a risk-averse bidder, the effect of a slightly lower winning bid on his wealth level has a smaller utility consequence than does the possible loss if this lower bid were, in fact, to result in his losing the auction. Compared to a risk-neutral bidder, a risk-averse bidder will thus bid higher. Put another way, by bidding higher, a risk-averse bidder will, as it were, "buy" insurance against the possibility of losing.

Example 4.1. *Constant relative risk aversion (CRRA) utility functions.*

Consider a situation with two bidders who display constant relative risk aversion: Their utility functions are of the form $u(z) = z^{\alpha}$, where α satisfies $0 < \alpha < 1$, so the coefficient of relative risk aversion, $-zu''(z)/u'(z)$, is $1 - \alpha$. Suppose that both values are drawn from the distribution F. It is convenient to define $F_{\alpha} \equiv F^{1/\alpha}$ and notice that F_{α} is also a distribution function with the same support as F. The symmetric equilibrium bidding strategy in a first-price auction is the solution to the differential equation in (4.2). With the specified utility function, (4.2) becomes

$$\gamma'(x)F(x) + \frac{1}{\alpha}\gamma(x)f(x) = \frac{1}{\alpha}xf(x)$$

together with the boundary condition $\gamma(0) = 0$. Using $F(x)^{(1/\alpha)-1}$ as the integrating factor, the solution to this is easily seen to be

$$\gamma(x) = \frac{1}{F_{\alpha}(x)} \int_0^x y f_{\alpha}(y)dy,$$

where $f_{\alpha} = F'_{\alpha}$. This, of course, is of the same form as derived in Proposition 2.2 on page 15.

Thus, we conclude that the equilibrium bidding strategy with two bidders with CRRA utility functions $u(z) = z^{\alpha}$ whose values are drawn from the distribution F is the *same* as the equilibrium bidding strategy with two risk-neutral bidders whose values are drawn from the distribution F_{α}. Since $F_{\alpha} \le F$, the expected revenue in a first-price auction with risk-averse bidders is greater than with risk-neutral bidders. The expected revenue in a second-price auction is, of course, unchanged. ▲

Example 4.2. *Constant absolute risk aversion (CARA) utility functions.*

Consider a situation with bidders who exhibit constant absolute risk aversion: Their utility functions are of the form $u(z) = 1 - \exp(-\alpha z)$, where $\alpha > 0$ is the coefficient of absolute risk aversion, $-u''(z)/u'(z)$. Suppose that values are independently distributed according to the function F and let G denote, as usual, the distribution of the highest of $N - 1$ values. First, consider a second-price auction. Consider a bidder with value x who bids z and wins the auction. In a second-price auction, such a bidder faces some uncertainty about the price he or she will pay, since that is determined by the second-highest bid. Suppose that the other bidders are following their equilibrium (and weakly dominant) strategy of bidding their values so that the second-highest bid is Y_1. Let $\rho(x, z)$

be the *risk premium* associated with the "price gamble"; it is the certain amount the bidder would forgo in order to remove the associated uncertainty. Formally,

$$u(x - \rho(x,z)) = E\,[u(x - Y_1) \mid Y_1 < z] \tag{4.7}$$

and CARA implies that we can write $\rho(z) \equiv \rho(x,z)$, since the risk premium depends only on the gamble being faced—which is entirely determined by z—and not on the "wealth level" x. It is optimal for bidder 1 to bid his or her true value and thus

$$x \in \arg\max_z G(z) E\,[u(x - Y_1) \mid Y_1 < z]$$

Using (4.7) this can be rewritten as

$$x \in \arg\max_z G(z) u(x - \rho(z))$$

But this is the same as bidder 1's maximization problem in a *first*-price auction if all other bidders follow the bidding strategy $\gamma = \rho$ (see (4.1)). This implies that for CARA bidders, the equilibrium bidding strategy in a first-price auction is to bid the risk premium associated with "price gamble" in a second-price auction. Finally, since

$$G(x) u(x - \gamma(x)) = G(x) E\,[u(x - Y_1) \mid Y_1 < x]$$

the equilibrium expected utility of a CARA bidder in a first-price auction is the same as his expected utility in a second-price auction. ▲

A key feature of the standard auction model with risk-neutral bidders is that the payoff functions are separable in money. In particular, they are *quasi-linear*—linear in the payments that bidders make—and bidders maximize their expected profits, which are just

$$\text{Expected Value} - \text{Expected Payment}$$

This separation between the expected value and the expected payment is crucial for revenue equivalence principle. Specifically, in the proof of Proposition 3.1 on page 28, this separation leads to equation (3.1) and hence to the conclusion that the expected payments are the same in any standard auction. Risk-averse bidders, on the other hand, maximize

$$\text{Expected Utility of (Value} - \text{Payment)}$$

and since utility is concave, the maximand is no longer linear in the payments that bidders make. The fact that bidders' objective functions are no longer linear in their payments is the reason for the failure of the revenue equivalence principle.

4.2 BUDGET CONSTRAINTS

Until now we have implicitly assumed that bidders face no cash or credit constraints—that is, bidders are able to pay the seller up to amounts equal to their values. In many situations, however, bidders may face financial constraints of one sort or another. In this section we ask how the presence of financial constraints affects equilibrium behavior in first- and second-price auctions and what effect they have on the revenue from these auctions.

We continue with the basic symmetric independent private value setting introduced in the previous chapter. There is a single object for sale and N potential buyers are bidding for the object. As before, bidder i assigns a value of X_i to the object. But now, in addition, each bidder is subject to an absolute *budget* of W_i. In no circumstances can a bidder with value-budget pair (x_i, w_i) pay more than w_i. We also suppose that if bidder i were to bid more than w_i and *default*, then a (small) penalty would be imposed.

Each bidder's value-budget pair (X_i, W_i) is independently and identically distributed on $[0,1] \times [0,1]$ according to the density function f.[1] Bidder i knows the realized value-budget pair (x_i, w_i), and only that other bidders' budget-value pairs are independently distributed according to f. As before, bidders are assumed to be risk neutral, and again we compare first- and second-price auctions. In a substantive departure from the models studied so far, the private information of the bidders is two-dimensional. We will refer to the pair (x_i, w_i) as the *type* of bidder i.

In any auction A (say, a first- or second-price auction), a bidder's strategy is a function of the form $B^A : [0,1] \times [0,1] \to \mathbb{R}$ that determines the amount bid depending on both his value and his budget.

4.2.1 Second-Price Auctions

We begin our analysis by considering second-price auctions. In this case, bidders' equilibrium strategies are straightforward.

Proposition 4.2. *In a second-price auction, it is a dominant strategy to bid according to* $B^{\mathrm{II}}(x, w) = \min\{x, w\}$.

Proof. First, notice that it is dominated to bid above one's budget. Suppose bidder i wins by bidding above his budget. If the second-highest bid is below his budget, then he would have also won by bidding w_i. If the second-highest bid is above his budget, he has to renege, does not get the object, and pays the fine, resulting in a negative surplus.

Second, if $x_i \leq w_i$, then the budget constraint does not bind and the same argument as in the unconstrained situation implies that it is a weakly dominant

[1] The independence holds only across bidders. The possibility that for each bidder the values and budgets are correlated is admitted.

strategy to bid x_i. If $x_i > w_i$, then the same argument shows that bidding w_i dominates bidding less. ■

For every type (x,w), define $x'' = \min\{x,w\}$ and consider the type $\left(x'',1\right)$. Notice that since values never exceed 1, a bidder of type $\left(x'',1\right)$ effectively never faces a financial constraint. But since $\min\{x'',1\} = x'' = \min\{x,w\}$, we have that $B^{\mathrm{II}}(x,w) = B^{\mathrm{II}}(x'',1)$. Thus, in a second-price auction the type $\left(x'',1\right)$ would submit a bid identical to that submitted by type (x,w). The type $\left(x'',1\right)$ is, as it were, the richest member of the family with types (x,w) such that $\min\{x,w\} = x''$. Figure 4.1 depicts the set of types who bid the same in a second-price auction as does type (x,w). This consists of all types on the thin-lined right angle "Leontief iso-bid" curve whose corner lies on the diagonal.

As before, let $m^{\mathrm{II}}(x,w)$ denote the expected payment of a bidder of type (x,w) in a second-price auction. Since $B^{\mathrm{II}}(x,w) = B^{\mathrm{II}}(x'',1)$, we have that

$$m^{\mathrm{II}}(x,w) = m^{\mathrm{II}}(x'',1) \tag{4.8}$$

Now define

$$\mathcal{L}^{\mathrm{II}}(x'') = \left\{(X,W) : B^{\mathrm{II}}(X,W) < B^{\mathrm{II}}(x'',1)\right\} \tag{4.9}$$

to be the set of types who bid less than type $\left(x'',1\right)$ in a second-price auction.

Define

$$F^{\mathrm{II}}(x'') = \int_{\mathcal{L}^{\mathrm{II}}(x'')} f(X,W)\, dXdW \tag{4.10}$$

to be the probability that a type $\left(x'',1\right)$ will outbid *one* other bidder. Notice that this is indeed the distribution function of the random variable $X'' = \min\{X,W\}$. The probability that a type $\left(x'',1\right)$ will actually win the auction is just $(F^{\mathrm{II}}(x''))^{N-1} \equiv G^{\mathrm{II}}(x'')$. In Figure 4.1, $F^{\mathrm{II}}(x'')$ is the probability mass attached to the set of types lying below the lighter of the two right angles.

Now notice that we can write the expected utility of a type $\left(x'',1\right)$ when bidding $B^{\mathrm{II}}(z,1)$ as

$$G^{\mathrm{II}}(z)x'' - m^{\mathrm{II}}(z,1)$$

In equilibrium, it is optimal to bid $B^{\mathrm{II}}(x'',1)$ when the true type is $\left(x'',1\right)$, so in a manner completely analogous to the arguments in Chapter 3 (specifically, Proposition 3.1), we have that

$$m^{\mathrm{II}}(x'',1) = \int_0^{x''} y g^{\mathrm{II}}(y) dy, \tag{4.11}$$

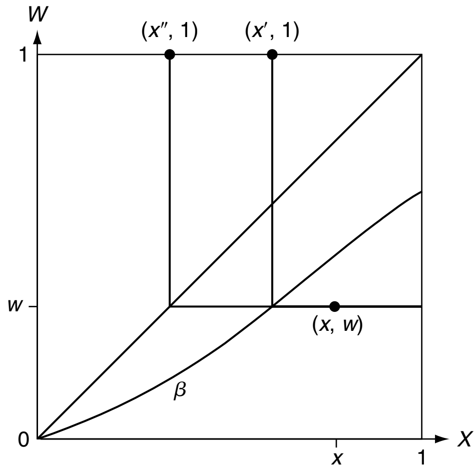

FIGURE 4.1 First- and second-price auctions with budget constraints.

where g^{II} is the density function associated with G^{II}. The *ex ante* expected payment of a bidder in a second-price auction with financial constraints can then be written, in manner completely analogous to (2.7), as

$$\begin{aligned} R^{\mathrm{II}} &= \int_0^1 m^{\mathrm{II}}(x'', 1) f^{\mathrm{II}}(x'')\, dx'' \\ &= E\left[Y_2^{\mathrm{II}(N)}\right], \end{aligned} \tag{4.12}$$

where $Y_2^{\mathrm{II}(N)}$ is the second-highest of N draws from the distribution F^{II}.

4.2.2 First-Price Auctions

First, suppose that in a first-price auction, the equilibrium strategy is of the form

$$B^{\mathrm{I}}(x, w) = \min\{\beta(x), w\} \tag{4.13}$$

for some increasing function $\beta(x)$. In this case, it must be that $\beta(x) < x$, otherwise a bidder of type $x < w$ would deviate by bidding slightly less. Although sufficient conditions in terms of the primitives of the model that guarantee the existence of such an equilibrium can be provided, here we content ourselves with directly assuming that such an equilibrium exists.

As in the case of a second-price auction, for every type (x, w) define x' to be a value such that $\beta(x') = \min\{\beta(x), w\}$ and consider the type $(x', 1)$. As before, a bidder of type $(x', 1)$ effectively never faces a financial constraint. But since $\min\{\beta(x'), 1\} = \beta(x') = \min\{\beta(x), w\}$, we have that $B^{\mathrm{I}}(x, w) = B^{\mathrm{I}}(x', 1)$. Thus, in a first-price auction the type $(x', 1)$ would submit a bid identical to that submitted by type (x, w). Figure 4.1 also depicts the set of types who bid the same in a first-price auction as does type (x, w). This consists of all types on the thick line right-angle "Leontief iso-bid" curve whose corner lies on the curve β.

Now define

$$\mathcal{L}^{\mathrm{I}}(x') = \left\{(X, W) : B^{\mathrm{I}}(X, W) < B^{\mathrm{I}}(x', 1)\right\} \tag{4.14}$$

and define m^{I}, F^{I}, and G^{I} in a fashion completely analogous to the corresponding objects for a second-price auction. Exactly the same reasoning shows that

$$E\left[R^{\mathrm{I}}\right] = E\left[Y_2^{\mathrm{I}(N)}\right] \tag{4.15}$$

where $Y_2^{\mathrm{I}(N)}$ is the second-highest of N draws from the distribution F^{I}.

4.2.3 Revenue Comparison

To compare the expected payments in the two auctions, notice that since for all x, $\beta(x) < x$, the definitions of $\mathcal{L}^{\mathrm{II}}(x)$ and $\mathcal{L}^{\mathrm{I}}(x)$ in (4.9) and (4.14), respectively, imply that $\mathcal{L}^{\mathrm{I}}(x) \subset \mathcal{L}^{\mathrm{II}}(x)$. See Figure 4.1. Now (4.10) implies that for all x, $F^{\mathrm{I}}(x) \leq F^{\mathrm{II}}(x)$ and a strict inequality holds for all $x \in (0, 1)$. We have thus argued that F^{I} stochastically dominates F^{II}. This implies that

$$E\left[Y_2^{\mathrm{I}(N)}\right] > E\left[Y_2^{\mathrm{II}(N)}\right]$$

Thus, we have shown the following:

Proposition 4.3. *Suppose that bidders are subject to financial constraints. If the first-price auction has a symmetric equilibrium of the form* $B^{\mathrm{I}}(x, w) = \min\{\beta(x), w\}$, *then the expected revenue in a first-price auction is greater than the expected revenue in a second-price auction.*

At an intuitive level, Proposition 4.3 results from the fact that budget constraints are "softer" in first-price auctions than in second-price auctions. Given a situation with budget constraints, consider a hypothetical situation in which each bidder's value is $Z_i = \min\{X_i, W_i\}$ and there are no budget constraints. Since value-budget pairs are independently and identically distributed across bidders, the revenue equivalence principle applies to the hypothetical situation, so the second-price and first-price auctions yield the same expected revenue—say, R. Now returning to the original situation with budget constraints, Proposition 4.2 implies that the revenue from the second-price auction in the original situation is also R; bids in the two are identical for every realization. The revenue in a first-price auction is greater than R because the comparison is between a situation in which bidders have values $X_i \geq Z_i$ and budgets $W_i \geq Z_i$ and a hypothetical situation in which they have values Z_i but no budget constraints.

4.3 ASYMMETRIES AMONG BIDDERS

In this section, we consider situations in which bidders are *ex ante* asymmetric: different bidders' values are drawn from different distributions. Asymmetries among bidders do not affect bidding behavior in second-price auctions; it is still a weakly dominant strategy for each bidder to bid his or her value. In a first-price auction, however, asymmetries lead to numerous complications. First, although an equilibrium exists (see Appendix G), unlike in the case of symmetric bidders, a closed form expression for the bidding strategies is not available, making a comparison with the second-price auction rather difficult. Second, the allocations under the two auctions are quite different: The second-price auction is efficient, whereas the first-price auction is not—and as a result, the two are no longer revenue equivalent. Indeed, as we will see, no general ranking of the revenues can be obtained.

We begin by exploring the nature of equilibrium bidding behavior in first-price auctions. To keep the analysis at a relatively simple level, we concentrate on the case of two bidders.

4.3.1 Asymmetric First-Price Auctions with Two Bidders

Suppose there are two bidders with values X_1 and X_2, which are independently distributed according to the functions F_1 on $[0, \omega_1]$ and F_2 on $[0, \omega_2]$, respectively. Suppose for the moment that there is an equilibrium of the first-price auction in which the two bidders follow the strategies β_1 and β_2, respectively. Further suppose that these are increasing and differentiable and have inverses $\phi_1 \equiv \beta_1^{-1}$ and $\phi_2 \equiv \beta_2^{-1}$, respectively.

It is clear that $\beta_1(0) = 0 = \beta_2(0)$, since it would be dominated for a bidder to bid more than the value. Moreover, $\beta_1(\omega_1) = \beta_2(\omega_2)$ since otherwise, if say, $\beta_1(\omega_1) > \beta_2(\omega_2)$, then bidder 1 would win with probability 1 when his value is ω_1 and would pay more than he needs to—he could increase his payoff by bidding slightly less than $\beta_1(\omega_1)$. Let

$$\bar{b} \equiv \beta_1(\omega_1) = \beta_2(\omega_2) \tag{4.16}$$

be the common highest bid submitted by either bidder.

Given that bidder $j = 1, 2$ is following the strategy β_j, the expected payoff of bidder $i \neq j$ when his value is x_i and he bids an amount $b < \bar{b}$ is

$$\begin{aligned} \Pi_i(b, x_i) &= F_j\big(\phi_j(b)\big)\,(x_i - b) \\ &= H_j(b)\,(x_i - b) \end{aligned}$$

where $H_j(\cdot) \equiv F_j\big(\phi_j(\cdot)\big)$ denotes the distribution of bidder j's bids.

The first-order condition for bidder i requires that for all $b < \bar{b}$,

$$h_j(b)\,(\phi_i(b) - b) = H_j(b) \tag{4.17}$$

where $j \neq i$ and, as usual, $h_j(b) \equiv H_j'(b) = f_j(\phi_j(b))\,\phi_j'(b)$ is the density of j's bids. This can be rearranged so that

$$\phi_j'(b) = \frac{F_j(\phi_j(b))}{f_j(\phi_j(b))} \frac{1}{(\phi_i(b) - b)} \tag{4.18}$$

A solution to the system of differential equations in (4.18)—one for each bidder—together with the relevant boundary conditions constitutes an equilibrium of the first-price auction. Unfortunately, an explicit solution can be obtained only in some special cases—an example is given later—and so instead, we deduce some properties of the equilibrium strategies indirectly. To do this, we make some assumptions regarding the specific nature of the asymmetries.

WEAKNESS LEADS TO AGGRESSION

Suppose that bidder 1's values are "stochastically higher" than those of bidder 2. In particular, we will make the stronger assumption that the distribution F_1 *dominates* F_2 *in terms of the reverse hazard rate*—that is, $\omega_1 \geq \omega_2$ and for all $x \in (0, \omega_2)$,

$$\frac{f_1(x)}{F_1(x)} > \frac{f_2(x)}{F_2(x)} \tag{4.19}$$

Reverse hazard rate dominance is further discussed in Appendix B where it is also shown that it implies that F_1 stochastically dominates F_2—that is, $F_1(x) \leq F_2(x)$. If (4.19) holds, we will call bidder 1 the "strong" bidder and bidder 2 the "weak" bidder. (A simple class of examples in which the two distributions can be ordered according to the reverse hazard rate consists of distributions satisfying $F_1(x) = (F_2(x))^\theta$ for some $\theta > 1$.)

We now show that the weak bidder will bid more aggressively than the strong bidder in the sense that for any fixed value, the bid of the weak bidder will be higher than the bid of the strong bidder.

Proposition 4.4. *Suppose that the value distribution of bidder* 1 *dominates that of bidder* 2 *in terms of the reverse hazard rate. Then in a first-price auction, the "weak" bidder* 2 *bids more aggressively than the "strong" bidder* 1*—that is, for any* $x \in (0, \omega_2)$,

$$\beta_1(x) < \beta_2(x)$$

Proof. First, notice that if there exists a c such that $0 < c < \overline{b}$ and $\phi_1(c) = \phi_2(c) \equiv z$, then (4.18) and (4.19) imply that

$$\phi_2'(c) = \frac{F_2(z)}{f_2(z)} \frac{1}{(z-c)} > \frac{F_1(z)}{f_1(z)} \frac{1}{(z-c)} = \phi_1'(c)$$

Since $\phi_i'(c) = 1/\beta_i'(z)$, this is equivalent to saying that if there exists a z such that $\beta_1(z) = \beta_2(z)$, then $\beta_1'(z) > \beta_2'(z)$. In other words, if the curves β_1 and β_2 ever intersect, the former is steeper than the latter and this implies that they intersect at most once.

We will argue by contradiction. So suppose that there exists an $x \in (0, \omega_2)$ such that $\beta_1(x) \geq \beta_2(x)$. Then, either β_1 and β_2 do not intersect at all so $\beta_1 > \beta_2$ everywhere, or they intersect only once at some value $z \in (0, \omega_2)$ and for all x such that $z < x < \omega_2$, $\beta_1(x) > \beta_2(x)$. In either case, for all x close to ω_2, $\beta_1(x) > \beta_2(x)$.

Now notice that if $\omega_1 > \omega_2$, then from (4.16) $\beta_1(\omega_1) = \beta_2(\omega_2)$, so $\beta_1(\omega_2) < \beta_2(\omega_2)$. This contradicts the fact that for all x close to ω_2, $\beta_1(x) > \beta_2(x)$.

Next suppose $\omega_1 = \omega_2 \equiv \omega$. If we write $\beta_1(\omega) = \beta_2(\omega) = \overline{b}$, then in terms of the inverse bidding strategies we have that for all b close to $\overline{b}$, $\phi_1(b) < \phi_2(b)$. This implies that for all b close to $\overline{b}$,

$$H_1(b) = F_1(\phi_1(b)) \leq F_2(\phi_2(b)) = H_2(b)$$

and since $H_1(\overline{b}) = 1 = H_2(\overline{b})$, it must be that $h_1(b) > h_2(b)$. Now using (4.17) we obtain that for all b close enough to $\overline{b}$,

$$\phi_1(b) = \frac{H_2(b)}{h_2(b)} + b > \frac{H_1(b)}{h_1(b)} + b = \phi_2(b),$$

which is a contradiction. ■

We know that F_1 stochastically dominates F_2 so that bidder 1's values are stochastically higher. At the same time, Proposition 4.4 shows that for any given value, bidder 2 bids higher than does bidder 1. What can be said about the distributions of bids, H_1 and H_2? Notice that since for all $b \in (0, \overline{b})$, $\phi_1(b) > \phi_2(b)$, it now follows from (4.17) and (4.18) that

$$\frac{H_2(b)}{h_2(b)} = \phi_1(b) - b > \phi_2(b) - b = \frac{H_1(b)}{h_1(b)}$$

so the distribution of bids of the strong bidder H_1 dominates the distribution of bids of the weak bidder H_2 in terms of the reverse hazard rate. Thus, under the hypotheses of Proposition 4.4, H_1 also stochastically dominates H_2.

Why is it that the weak bidder bids more aggressively than does the strong bidder? To gain some intuition, it is useful to see why the opposite is impossible—that is, it cannot be that the strong bidder bids more aggressively than does the weak bidder. If for all x, $\beta_1(x) > \beta_2(x)$, then certainly the distribution H_1 of competing bids facing the weak bidder is stochastically higher than the distribution H_2 of competing bids facing the strong bidder. It is easy to see that all else being equal, a bidder who faces a stochastically higher distribution of bids—in the sense of reverse hazard rate dominance—will bid higher. It is also true that for a particular bidder, all else being equal, a higher realized value will lead to a higher bid. Now consider a particular bid b and suppose that $\beta_1(x_1) = \beta_2(x_2) = b$.

Since by assumption, the strong bidder bids more aggressively, it must be that the value at which the strong bidder bids b is lower than the value at which the weak bidder bids b—that is, $x_1 < x_2$. This means that, relative to the strong bidder, the weak bidder faces *both* a stochastically higher distribution of competing bids—H_1 versus H_2—and has a higher value—x_2 versus x_1. Since both forces cause bids to be higher, if it were optimal for the strong bidder to bid b when his value is x_1, it cannot be optimal for the weak bidder to bid b when his value is x_2. Thus, we have a contradiction.

Put another way, in equilibrium the two forces must balance each other. The weak bidder faces a stochastically higher distribution of competing bids than does the strong bidder, but the value at which any particular bid b is optimal for the weak bidder is lower than it is for the strong bidder.

ASYMMETRIC UNIFORM DISTRIBUTIONS

Equilibrium bidding strategies in asymmetric first-price auctions can be explicitly derived if the two value distributions are uniform but with differing supports. Specifically, suppose bidder 1's value X_1 is uniformly distributed on $[0, \omega_1]$ and bidder 2's value X_2 is uniformly distributed on $[0, \omega_2]$ and that $\omega_1 \geq \omega_2$. Then $F_i(x) = x/\omega_i$ and $f_i(x) = 1/\omega_i$, so the first-order condition (4.17) can be simplified as follows: for $i = 1, 2$ and $j \neq i$, for all $b \in (0, \overline{b})$,

$$\phi_i'(b) = \frac{\phi_i(b)}{\phi_j(b) - b} \tag{4.20}$$

which is equivalent to

$$\left(\phi_i'(b) - 1\right)\left(\phi_j(b) - b\right) = \phi_i(b) - \phi_j(b) + b$$

Adding the two equations for $i = 1, 2$ results in

$$\frac{d}{db}\left((\phi_1(b) - b)(\phi_2(b) - b)\right) = 2b$$

and integrating this, we obtain

$$(\phi_1(b) - b)(\phi_2(b) - b) = b^2 \tag{4.21}$$

(The constant of integration is zero since $\phi_i(0) = 0$.) Since $\phi_i\left(\overline{b}\right) = \omega_i$,

$$\left(\omega_1 - \overline{b}\right)\left(\omega_2 - \overline{b}\right) = \overline{b}^2$$

so that

$$\overline{b} = \frac{\omega_1 \omega_2}{\omega_1 + \omega_2} \tag{4.22}$$

Now, using (4.21), the equations in (4.20) can be rewritten as follows: for $i=1,2$,

$$\phi_i'(b) = \frac{\phi_i(b)\,(\phi_i(b)-b)}{b^2} \tag{4.23}$$

with the advantage that in this form they are separable in the variables.

We now undertake a change of variables by defining $\xi_i(b)$ implicitly by

$$\phi_i(b) - b = \xi_i(b)b \tag{4.24}$$

so

$$\phi_i'(b) - 1 = \xi_i'(b)b + \xi_i(b)$$

With this substitution, the differential equation (4.23) becomes

$$\xi_i'(b)b + \xi_i(b) + 1 = \xi_i(b)\,(\xi_i(b)+1)$$

or

$$\frac{\xi_i'(b)}{\xi_i(b)^2 - 1} = \frac{1}{b}$$

the solution to which is easily verified to be

$$\xi_i(b) = \frac{1-k_i b^2}{1+k_i b^2},$$

where k_i is a constant of integration. Using (4.24) this becomes

$$\phi_i(b) = \frac{2b}{1+k_i b^2} \tag{4.25}$$

and since $\phi_i\left(\overline{b}\right) = \omega_i$, where $\overline{b}$ is defined in (4.22), we obtain that the constant of integration

$$k_i = \frac{1}{\omega_i^2} - \frac{1}{\omega_j^2} \tag{4.26}$$

The bidding strategies, obtained by inverting (4.25), are

$$\beta_i(x) = \frac{1}{k_i x}\left(1 - \sqrt{1 - k_i x^2}\right) \tag{4.27}$$

It is routine to verify that these form an equilibrium.[2] Figure 4.2 depicts the equilibrium bidding strategies when $\omega_1 = \frac{4}{3}$ and $\omega_2 = \frac{4}{5}$.

[2]Although we do not verify this here, it can be shown that this is the only equilibrium.

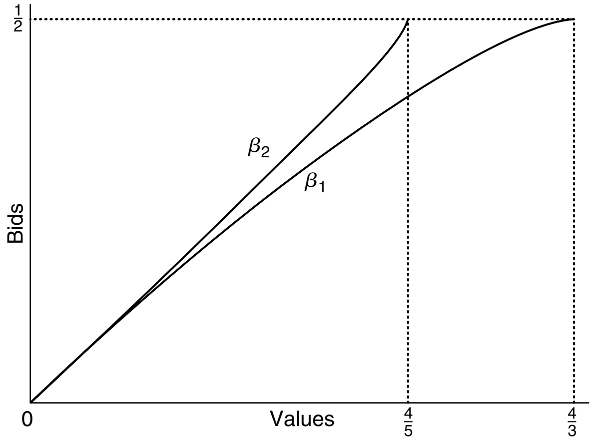

FIGURE 4.2 Equilibrium of an asymmetric first-price auction.

4.3.2 Revenue Comparison

We first use the equilibrium strategies derived previously to show that for some distributions, the revenue from a first-price auction may exceed that from a second-price auction.

Example 4.3. *With asymmetric bidders, the expected revenue in a first-price auction may exceed that in a second-price auction.*

As a special case of the example with values drawn from different uniform distributions, let $\alpha \in [0,1)$ and suppose that bidder 1's value X_1 is uniformly distributed on the interval $[0, 1/(1-\alpha)]$, whereas bidder 2's value X_2 is uniformly distributed on the interval $[0, 1/(1+\alpha)]$. (In the example depicted in Figure 4.2, $\alpha = 1/4$.)

We will compare the expected revenues accruing from a second-price auction to those from a first-price auction when $\alpha > 0$. Notice that when $\alpha = 0$, the situation is symmetric and the two auctions yield the same expected revenue.

REVENUE IN THE SECOND-PRICE AUCTION

It is a dominant strategy to bid one's value in a second-price auction and thus the distribution of the selling price in a second-price auction is

$$L_{\alpha}^{\mathrm{II}}(p) = \mathrm{Prob}\left[\min\{X_1, X_2\} \leq p\right],$$

where $p \in \left[0, \frac{1}{1+\alpha}\right]$. We have

$$\begin{aligned} L_{\alpha}^{\mathrm{II}}(p) &= F_1(p) + F_2(p) - F_1(p)F_2(p) \\ &= (1-\alpha)p + (1+\alpha)p - (1-\alpha)(1+\alpha)p^2 \\ &= 2p - \left(1-\alpha^2\right)p^2, \end{aligned}$$

which is *increasing* in α. Thus, in a second-price auction the expected selling price when $\alpha > 0$ is *lower* than the expected selling price when $\alpha = 0$.

REVENUE IN THE FIRST-PRICE AUCTION

Since $\omega_1 = 1/(1-\alpha)$ and $\omega_2 = 1/(1+\alpha)$, from (4.22) it follows that the highest amount that either bidder bids is $\overline{b} = \frac{1}{2}$. Moreover, the constants of integration in (4.26) are $k_1 = -4\alpha$ and $k_2 = 4\alpha$. Using (4.25) the inverse bidding strategies in equilibrium are: for all $b \in \left[0, \frac{1}{2}\right]$,

$$\phi_1(b) = \frac{2b}{1-4\alpha b^2}$$
$$\phi_2(b) = \frac{2b}{1+4\alpha b^2}$$

The distribution of the equilibrium prices in a first-price auction is thus

$$L_\alpha^{\mathrm{I}}(p) = \mathrm{Prob}\left[\max\left\{\beta_1(X_1), \beta_2(X_2)\right\} \le p\right],$$

where $p \in \left[0, \frac{1}{2}\right]$. We have

$$\begin{aligned} L_\alpha^{\mathrm{I}}(p) &= F_1(\phi_1(p)) \times F_2(\phi_2(p)) \\ &= (1-\alpha)\,\frac{2p}{1-4\alpha p^2} \times (1+\alpha)\,\frac{2p}{1+4\alpha p^2} \\ &= \frac{\left(1-\alpha^2\right)(2p)^2}{1-\alpha^2(2p)^4}, \end{aligned}$$

which is *decreasing* in α. Thus, in a first-price auction the expected selling price when $\alpha > 0$ is *higher* than the expected selling price when $\alpha = 0$.

For $\alpha = 0$, the expected selling price in the two auctions is the same. An increase in α leads to a decrease in the expected price in the second-price auction and an increase in the expected price in a first-price auction.

We have thus shown that in this example, for all $\alpha > 0$, the expected selling price in a first-price auction is *greater* than that in a second-price auction. (More generally, it can be shown that with asymmetric uniformly distributed values, the first-price auction is revenue superior for all ω_1 and ω_2.) ▲

A second example shows that the opposite ranking is also possible.

Example 4.4. *With asymmetric bidders, the expected revenue in a second-price auction may exceed that in a first-price auction.*

Suppose that bidder 1's value X_1 is distributed according to $F_1(x) = x - 1$ over $[1,2]$ and bidder 2's value is distributed according to $F_2(x) = \exp\left(\frac{1}{2}x - 1\right)$

over $[0,2]$. (Note that we are departing from our assumption that lowest possible value for each bidder is 0. Also, F_2 has a mass point at 0.)

It may be verified that the bidding strategies $\beta_1(x) = x - 1$ and $\beta_2(x) = \frac{1}{2}x$ for the two bidders, respectively, constitute an equilibrium of the first-price auction. The bids range between 0 and 1 and it is routine to see that the distribution of prices in a first-price auction is

$$L^{\mathrm{I}}(p) = p\exp(p-1)$$

The expected revenue in a first-price auction $E\left[R^{\mathrm{I}}\right] \simeq 0.632$.

On the other hand, in a second-price auction the selling prices are distributed according to

$$L^{\mathrm{II}}(p) = \begin{cases} \exp\left(\frac{1}{2}p - 1\right) & p \leq 1 \\ (p-1) + (2-p)\exp\left(\frac{1}{2}p - 1\right) & p > 1 \end{cases}$$

over $[0,2]$. The expected revenue in a second-price auction $E\left[R^{\mathrm{II}}\right] \simeq 0.662$.

Thus, in this example, the expected selling price in a first-price auction is *less* than that in a second-price auction. ▲

4.3.3 Efficiency Comparison

As noted earlier, it is a weakly dominant strategy for a bidder to bid his or her value in a second-price auction—recall that this is true even when bidders are asymmetric—so the winning bidder is also the one with the highest value. Thus, the second-price auction is always *ex post* efficient under the assumption of private values.

In contrast, asymmetries inevitably lead to inefficient allocations in a first-price auction. Suppose that there are two bidders and (β_1, β_2) is an equilibrium of the first-price auction such that both strategies are continuous and increasing. Because the bidders are asymmetric—their values are drawn from different distributions—it will be the case that $\beta_1 \neq \beta_2$. Without loss of generality, suppose that $\beta_1(x) < \beta_2(x)$, and since both the strategies are continuous, for small enough $\varepsilon > 0$, it is also the case that $\beta_1(x+\varepsilon) < \beta_2(x-\varepsilon)$. This, of course, means that with positive probability the allocation is inefficient, since bidder 2 would win the auction even though he has a lower value than does bidder 1.

For future reference we record these observations as follows:

Proposition 4.5. *With asymmetrically distributed private values, a second-price auction always allocates the object efficiently, whereas with positive probability, a first-price auction does not.*

4.4 RESALE AND EFFICIENCY

In the previous section we saw that asymmetries among bidders lead to inefficient allocations in first-price auctions—with positive probability the winner of the auction is not the person who values the object the most. Achieving an efficient allocation may well be an important, or even primary, policy goal of the seller, especially if the seller is a government undertaking the privatization of some public asset. This seems to imply that such a seller should use an efficient auction—with private values, say a second-price auction—even if, as we have seen, it may bring lower revenues than an inefficient one, say a first-price auction. An argument against this point of view, in the Chicago school vein, is that even if the outcome of the auction is inefficient, postauction transactions among buyers—resale—will result in an efficient final allocation. Absent any transaction costs, the asset will be transferred into the hands of the person who values it the most. The conclusion is that the choice of the auction form is irrelevant to the efficiency question and one may as well select the auction format on other grounds—say, revenue. In this section we ask whether this is indeed the case. Does resale automatically lead to efficiency?

To examine the resale question in the simplest possible setting, consider the basic setup of the previous section. There are two bidders with values X_1 and X_2, which are independently distributed according to the functions F_1 and F_2, respectively, and for notational ease suppose that these have a common support $[0, \omega]$. The bidders are asymmetric, so $F_1 \neq F_2$. Suppose in addition that $E[X_1] \neq E[X_2]$, a condition that will hold generically.

In this context, let us first put forward the argument that a first-price auction followed by resale will lead to efficiency. Suppose β_1 and β_2 are equilibrium bidding strategies in the first-price auction and we know that these are increasing. Further, suppose that at the conclusion of the auction, both the bids—winning and losing—are publicly announced. This means that if the bids were b_1 and b_2, then at the conclusion of the auction, it would be commonly known that the buyers have values $x_1 = \beta_1^{-1}(b_1)$ and $x_2 = \beta_2^{-1}(b_2)$, respectively. If $b_1 > b_2$ but $x_1 < x_2$, the outcome of the auction would be inefficient, but since the values would be commonly known, so would the fact that there are some unrealized gains from trade. In particular, knowing that $x_1 < x_2$, bidder 1 could offer to resell the object to bidder 2 at some price between the two values. This would mean that ultimately the object ends up in the right hands.

This line of reasoning seems so simple as to be beyond question. It fails, however, to take into account that rational buyers will behave differently during the auction once they correctly foresee future resale possibilities. Let us see why.

To model the situation outlined above carefully, we need to be more specific about how resale actually takes place—that is, how the buyer and the seller settle on a price. Suppose that after learning what the losing bid was, the winner of the auction—and the new owner of the object—may, if he so wishes, resell the object to the other bidder by making a one-time take-it-or-leave-it offer. Notice that all bargaining power in this transaction resides with the (new) seller—that is, the winner of the auction. In particular, if bidder 1 wins the auction and subsequently learns that $x_1 < x_2$, then he can offer to sell the object to bidder 2 at a price $p = x_2$ (or just below), and this offer will be accepted. Bidder 1 will then make a profit of $x_2 - b_1$, whereas bidder 2's profit will be 0 (or just above). Of course, this is one of many possible ways in which the price may be determined. It is particularly simple and has the virtue that it ensures efficiency if the values are commonly known, so in some sense, it makes the best case for resale since there are no underlying transaction costs or delays. As an alternative, one could consider a situation in which the buyer makes a take-it-or-leave-it offer to the seller without affecting any of what follows.

Our main finding is that there cannot be an equilibrium of a first-price auction with resale in which the outcome of the auction completely reveals the values. Thus, the prospects of postauction resale make an efficient allocation impossible.

Suppose to the contrary that the first-price auction with resale has an efficient equilibrium. The equilibrium must specify both how bidders bid in the auction and what they do in the post-auction resale stage. Let β_1 and β_2 denote the bidding strategies in the first-price auction, and suppose that these are *increasing* with inverses $\phi_1 \equiv \beta_1^{-1}$ and $\phi_2 \equiv \beta_2^{-1}$. In the resale stage, if the announced bids b_1 and b_2 are such that $b_i > b_j$ but $x_i < \phi_j(b_j)$, then i makes a take-it-or-leave-it offer to sell the object to j at a price of $\phi_j(b_j)$. This is accepted if and only if $x_j \geq \phi_j(b_j)$. Otherwise, no offer is made.

The assumption that β_1 and β_2 are invertible means that after the winning and losing bids are announced, the values will become commonly known. Thus, if there are any unrealized gains from trade, resale will take place and the object will be allocated efficiently.

As a first step, notice that, as in the previous section, the bidding strategies must agree both at the lower and the upper end of the support of values—that is, $\beta_1(0) = 0 = \beta_2(0)$ and for some $\overline{b}$,

$$\beta_1(\omega) = \overline{b} = \beta_2(\omega) \tag{4.28}$$

Now suppose bidder 2 behaves according to the prescribed equilibrium strategy. Suppose bidder 1 has value x_1 but behaves as if his value were z_1—that is, he bids an amount $\beta_1(z_1)$ in equilibrium and in the resale stage also follows the given equilibrium strategy as if his value were z_1. Bidder 1's overall expected payment in equilibrium when he behaves as if his value is z_1 is[3]

$$\begin{aligned} m_1(z_1) &= F_2(\phi_2\beta_1(z_1))\,\beta_1(z_1) \\ &\quad - \int_{z_1}^{\phi_2\beta_1(z_1)} \max\{z_1, x_2\} f_2(x_2)\, dx_2 \end{aligned} \tag{4.29}$$

[3]We write $\phi_2\beta_1(z_1)$ to denote $\phi_2(\beta_1(z_1))$.

The first term is bidder 1's expected payment to the seller. The second term is the result of monetary transfers between the bidders in the event that resale takes place. If $z_1 < X_2 < \phi_2\beta_1(z_1)$, then bidder 1 wins the auction and resells the object to bidder 2 at a price of $X_2 = \max\{z_1, X_2\}$. On the other hand, if $\phi_2\beta_1(z_1) < X_2 < z_1$, then bidder 1 loses the auction but purchases the object from bidder 2 at a price of $z_1 = \max\{z_1, X_2\}$.

The final allocation is efficient, so if bidder 1 behaves as if his value is z_1, the probability that he will get the object is just $F_2(z_1)$. Following the reasoning in the proof of the revenue equivalence principle (Proposition 3.1 on page 28), the expected payoff to bidder 1 from behaving as if his value is z_1 when it is actually x_1 is

$$F_2(z_1)\,x_1 - m_1(z_1)$$

In equilibrium, it is not optimal for bidder 1 to deviate, so we must have

$$F_2(x_1)\,x_1 - m_1(x_1) \geq F_2(z_1)\,x_1 - m_1(z_1)$$

The first-order condition for this optimization results in the differential equation

$$m_1'(x_1) = x_1 f_2(x_1)$$

and since $m_1(0) = 0$, it follows that

$$m_1(x_1) = \int_0^{x_1} x_2 f_2(x_2)\,dx_2 \tag{4.30}$$

Setting $z_1 = x_1$ in (4.29) and equating this with (4.30), we obtain that a necessary condition for a first-price auction followed by resale to be efficient is that for all x_1,

$$\begin{aligned} F_2(\phi_2\beta_1(x_1))\beta_1(x_1) - \int_{x_1}^{\phi_2\beta_1(x_1)} \max\{x_1, x_2\} f_2(x_2)\,dx_2 \\ = \int_0^{x_1} x_2 f_2(x_2)\,dx_2 \end{aligned} \tag{4.31}$$

Equation (4.31) says that the expected payment of bidder 1 in an equilibrium of the first-price auction with resale that is efficient—the expression on the left-hand side—is the same as that in an efficient *second*-price auction—the expression on the right-hand side. Indeed, it is just a version of the revenue equivalence principle extended to the asymmetric case: since the equilibrium outcomes of a first-price auction with resale are the same as those of a second-price auction, the expected payments in the two must be the same.[4]

[4]A general revenue equivalence principle for asymmetric situations is derived in Chapter 5 and can also be used to deduce equation (4.31).

Now $\beta_1(\omega)=\overline{b}$, so $\phi_2\beta_1(\omega)=\omega$. Setting $x_1=\omega$ in (4.31), we obtain

$$\overline{b}=E[X_2]$$

But interchanging the roles of bidder 1 and 2, we similarly obtain that

$$\overline{b}=E[X_1]$$

and since $E[X_1]\neq E[X_2]$, this is a contradiction. We have thus argued that in an asymmetric first-price auction followed by resale, the bidding strategies β_1 and β_2 *cannot* be increasing everywhere. Bidders' equilibrium behavior in the auction cannot reveal their values completely, so resale transactions must take place under incomplete information.

From here it is a short step to see that because of this incomplete information, reaching an efficient allocation in all circumstances is impossible. In the interests of space, we only sketch the argument. Suppose the equilibrium strategies β_1 and β_2 are continuous but only nondecreasing (we have already ruled out the possibility that they are increasing). In other words, the strategies involve some "pooling"—that is, for at least one of the bidders—say, bidder 2—there is an interval of values $[x_2',x_2'']$ such that for all $x_2\in[x_2',x_2'']$, the equilibrium bid $\beta_2(x_2)$ is the same, say b_2. We will now argue that pooling is also incompatible with efficient resale.

We first claim that there exists an $x_1\in(x_2',x_2'')$ such that $\beta_1(x_1)\geq b_2$. Otherwise, for all x_1 such that $\beta_1(x_1)\geq b_2$, $x_1\geq x_2''$. Let x_1'' be the smallest value for bidder 1 such that $\beta_1(x_1)=b_2$, and we know that $x_1''\geq x_2''$. Now we must have $b_2<x_1''$, since otherwise bidder 2 with value x_2'' would never bid b_2. The most that she can gain from winning the auction is $\max\{x_1'',x_2''\}=x_1''$, so she would never bid more than this amount. Since $b_2<x_1''$, for ε small enough, $b_2<x_1''-\varepsilon$. But now notice that by bidding $b_2+\varepsilon$ instead of below b_2, bidder 1 with value $X_1=x_1''-\varepsilon$ would gain a discrete amount—winning against all values X_2 in $[x_2',x_2'']$ to which he was losing previously—at an infinitesimal cost. This is a profitable deviation, so we have reached a contradiction.

Thus, there exists an $x_1\in(x_2',x_2'')$ such that $\beta_1(x_1)\geq b_2$. If the realized values are x_1 and some $x_2\in(x_1,x_2'']$, then bidder 1 wins the auction, but $x_1<x_2$. The announcement of b_2 reveals to bidder 1 only that $X_2\in[x_2',x_2'']$. He would then make a take-it-or-leave-it offer p that maximizes his expected profit. But such an offer would of necessity satisfy $p>x_1$ and so would be rejected with positive probability even though efficiency dictates that the object be transferred from bidder 1 to bidder 2. Thus, any equilibrium with pooling is inefficient.

If the resale price is determined instead by the buyer making a take-it-or-leave-it offer, the argument is virtually the same as the preceding one, except that the resale price is $\min\{X_1,X_2\}$ instead of $\max\{X_1,X_2\}$. All the other steps are identical.

We have thus argued the following:

Proposition 4.6. *With asymmetric bidders, a first-price auction followed by resale (at a take-it-or-leave-it price offered by one of the parties) does not result in efficiency.*

Proposition 4.6 casts doubt on the argument that resale will inevitably lead to efficiency. A seller whose goal is to ensure that the object ends up in the hands of the person who values it the most cannot rely on the "market" to do the job. The appropriate choice of an auction format remains very relevant—in order to assure efficiency, it is best to use an efficient auction.

PROBLEMS

4.1. (Risk-averse bidders) There are two bidders with private values which are distributed independently according to the uniform distribution $F(x)=x$ over $[0,1]$. Both bidders are risk-averse and have utility functions $u(z)=\sqrt{z}$. Find symmetric equilibrium bidding strategies in a first-price auction.

4.2. (Increase in risk aversion) Consider an N-bidder first-price auction where each bidder's value is distributed according to F. All bidders are risk averse with a utility function u that satisfies $u(0)=0,\ u'>0,\ u''<0$. Show that if one changed the utility function of all bidders from $u(z)$ to $\phi(u(z))$, where ϕ is an increasing and concave function satisfying $\phi(0)=0$, then this would lead to a higher symmetric equilibrium bidding strategy.

4.3. (Asymmetric first-price auction) Suppose that bidder 1's value X_1 is distributed according to $F_1(x)=\frac{1}{4}(x-1)^2$ over $[1,3]$ and bidder 2's value is distributed according to $F_2(x)=\exp\left(\frac{2}{3}x-2\right)$ over $[0,3]$.

a. Show that $\beta_1(x)=x-1$ and $\beta_2(x)=\frac{2}{3}x$ constitute equilibrium bidding strategies in a first-price auction.

b. Compare the expected revenues in a first- and second-price auction.

4.4. (Equilibrium with reserve price) Suppose that bidder 1's value X_1 is distributed uniformly on $[0,2]$ while bidder 2's value X_2 is distributed uniformly on $\left[\frac{3}{2},\frac{5}{2}\right]$. The object is sold via a first-price auction with a reserve price $r=1$. Verify that $\beta_1(x)=\frac{x}{2}+\frac{1}{2}$ and $\beta_2(x)=\frac{x}{2}+\frac{1}{4}$ constitute equilibrium strategies.

4.5. (Discrete values) Suppose that there is no uncertainty about bidder 1's value and $X_1=2$ always. Bidder 2's value, X_2, is equally likely to be 0 or 2.

a. Find equilibrium bidding strategies in a first-price auction. (Note that since values are discrete, the equilibrium will be in mixed strategies.)

b. Compare the revenues in a first- and second-price auction.

CHAPTER NOTES

The result that with symmetric risk-averse bidders the revenue from the first-price auction is greater than that in a second-price auction is due to Holt (1980), who studied the analogous problem in the context of bidding for procurement contracts. The key to Proposition 4.1, of course, is the fact that in a first-price auction, risk aversion causes bidders to increase their bids. The result that CARA bidders are indifferent between the first- and second-price auctions is due to Matthews (1987).

The material on auctions with budget constraints is based on Che and Gale (1998). This paper provides a sufficient condition on the primitives that guarantees that the first-price auction has an equilibrium of the sort hypothesized in Proposition 4.3. It also considers situations where the financial constraints need not be in the form of an absolute budget but may be somewhat more flexible. For instance, it may be that bidders face borrowing constraints so that they can borrow larger amounts only at higher marginal costs.

Vickrey (1961) himself pointed out that asymmetries among bidders may lead to inefficient allocations in a first-price auction. He studied a two-bidder asymmetric example in which bidder 1's value, a, was commonly known and bidder 2's value was uniformly distributed on $[0, 1]$. In this case, an equilibrium of the first-price auction involves randomization on the part of bidder 1, and Vickrey (1961) showed that depending on the value of a, the revenue from a first-price auction could be better or worse than that from a second-price auction. The derivation of equilibrium bidding strategies in first-price auctions with asymmetric uniformly distributed values is due to Griesmer, Levitan, and Shubik (1967). Plum (1992) has extended this to the more general class of asymmetric "power" distributions. Cheng (2006) identifies another family of distributions for which equilibrium strategies in an asymmetric first-price auction can be explicitly derived. It can be shown that an equilibrium exists in general under weak conditions on the distributions of values. Appendix G outlines some results in this direction. With a view to applications, Marshall, Meurer, Richard, and Stromquist (1994) have developed numerical techniques to compute bidding strategies in asymmetric auctions.

Maskin and Riley (2000a) have studied the equilibrium properties of asymmetric first-price auctions in more detail and have derived some sufficient conditions for one or the other auction to be revenue superior. Example 4.3 is taken from their paper, while Example 4.4 is due to Cheng (private communication, 2007).

Gupta and Lebrun (1999) have also studied a model of a first-price auction with resale but reach very different conclusions from those reached here. In particular, they do not find that resale will lead to inefficiency. This discrepancy is easily accounted for. Gupta and Lebrun (1999) assume that regardless of the outcome, at the conclusion of the auction the *values* of the bidders are publicly announced. Since after the auction is over, bidders are completely informed of each other's values, resale is inevitably efficient. The auctioneer, however, has no direct knowledge of bidders' values, so it is not clear how this is to be implemented. In contrast, in our model, we assumed that only the *bids* were announced and showed that if bidders take this into account, the bids would not reveal the values completely.

Garratt and Tröger (2006) study the possibility that a bidder with a known value of 0 may speculatively bid in a first-price auction when resale is present. Hafalir and Krishna (2008) construct equilibria of first-price auctions with resale and show that revenue from a first-price auction with resale exceeds that from a second-price auction. This result holds for all asymmetric distributions which satisfy a regularity condition (see Chapter 5 for a definition).

Haile (2003) studies a model of resale in which, at the time of bidding, buyers have only partial information regarding the true value. After the auction is over they receive a further signal that determines their actual values. The discrepancy between the estimated values at the time of the auction and the true values realized after the auction creates a motive for resale.