MATH 425b TAKE-HOME FINAL EXAM
Spring 2016
Prof. Alexander

The exam is due Monday May 9 at 8:00 am at the in-class final. I'll be available during the week of May 2–6 during normal office hours or by appointment. One exception: My usual Monday 1-3 pm office hour will be moved to 2:30–4:30 pm.

What is allowed: Use of Rudin, your lecture notes, homework and midterms, and homework and midterm solutions.

What is not allowed: Use of other books or other published or internet materials; consulting with each other or with anyone except me.

(1)(33 points) Let K be a compact metric space and for $c > 0$ define the sets of Lipschitz functions

$$\mathrm{Lip}_c(K) = \{f : K \to \mathbb{R} : |f(x) - f(y)| \le cd(x,y) \text{ for all } x, y \in K\}, \quad \mathrm{Lip}(K) = \cup_{c=1}^{\infty} \mathrm{Lip}_c(K).$$

As usual, $C(K)$ denotes the set of all continuous functions on K, endowed with the uniform (sup) metric.

(a) Show that $\mathrm{Lip}(K)$ is an algebra, that is, if $f, g \in \mathrm{Lip}(K)$ and $a \in \mathbb{R}$ then $af, f+g$ and fg are in $\mathrm{Lip}(K)$.

(b) Show that $\mathrm{Lip}(K)$ is dense in $C(K)$, that is, the closure of $\mathrm{Lip}(K)$ is $C(K)$.

(c) Show that $F_{c,M} = \{f \in \mathrm{Lip}_c(K) : ||f||_\infty \le M\}$ is a compact subset of $C(K)$,for each $c, M > 0$. HINT: We proved a criterion for a subset of $C(K)$ to be compact.

(d) Show that $C(K)$ is *separable*, meaning it has a countable dense subset. HINT: Use (b), (c). What is the union of all the $F_{c,M}$'s? Also recall the theorem we proved after Theorem 7.23: every compact set has a countable dense subset.

(2)(22 points) For vectors $\mathbf{v}^{1)}, \dots, \mathbf{v}^{(n-1)} \in \mathbb{R}^n$, we define $\mathcal{H}(\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n-1)}) \in \mathbb{R}^n$ to be the "formal determinant" of the $n \times n$ matrix

$$\begin{bmatrix} e_1 & v_1^{(1)} & \cdot & \cdot & v_1^{(n-1)} \\ \cdot & & & & \\ \cdot & & & & \\ e_n & v_n^{(1)} & \cdot & \cdot & v_n^{(n-1)} \end{bmatrix},$$

that is, the columns after the first one are $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n-1)}$. This means the coefficient of each e_i is $(-1)^{i-1}$ times the determinant of the $(n-1) \times (n-1)$ submatrix obtained by deleting the row and column containing e_i.

Then in $\mathbb{R}^n$, the normal vector N to an $(n-1)$-surface $\Phi : D \to \mathbb{R}^n$ at $\Phi(\mathbf{u})$ is obtained by applying this operation to the tangent vectors: the normal is defined to be $\mathcal{H}(\Phi'(\mathbf{u})\hat{e}_1, \dots, \Phi'(\mathbf{u})\hat{e}_{n-1})$, where $\hat{e}_1, \dots, \hat{e}_{n-1}$ are the unit coordinate vectors in $\mathbb{R}^{n-1}$. The determinant of the submatrices are then Jacobians, so $N = (N_1, \dots, N_n)$ with

$$N_i = (-1)^{i-1} \frac{\partial(\Phi_1, \dots, \Phi_{i-1}, \Phi_{i+1}, \dots, \Phi_n)}{\partial(u_1, \dots, u_{n-1})}.$$

Given a $\mathcal{C}'$ vector field $F = (F_1, \dots, F_n)$ in $\mathbb{R}^n$, we can associate to it an $(n-1)$-form $\omega^{(F)}$ by

$$\omega^{(F)} = \sum_{k=1}^{n} \epsilon_k F_k(\mathbf{x})\ dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \dots dx_n, \tag{1}$$

where each ϵ_k is 1 or -1. Also, in $\mathbb{R}^n$ the divergence of F is defined to be the function

$$(\operatorname{div} F)(\mathbf{x}) = \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_k}(\mathbf{x}).$$

(a) Show that the vector $\mathcal{H}(\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n-1)})$ is orthogonal to each of $\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n-1)}$. HINT: The dot product $\mathcal{H}(\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n-1)}) \cdot \mathbf{v}^{(i)}$ is the same as the determinant of what matrix?

(b) What choice of the ϵ_k's do we need to make in (1) so that

$$d\omega^{(F)} = (\operatorname{div} F)\, dx_1 \wedge \cdots \wedge dx_n\,?$$

Your answer should be something like, all $\epsilon_k = 1$, or $\epsilon_k = (-1)^k$, or similar.

(c) Let D be (the trace of) an affine n-simplex in $\mathbb{R}^n$, with positively oriented boundary ∂D. Show that

$$\int_{\partial D} F \cdot N = \int_D (\operatorname{div} F).$$

(For $n = 3$ this is the usual Divergence Theorem.) We can view D as parametrized by the identity map σ; then the left side is formally a shorthand for the integral of the $(n-1)$-chain $\partial\sigma = \tau_0 + \cdots + \tau_k$, with each $\tau_i : Q^{n-1} \to \mathbb{R}^n$ a parametrization of one face of ∂D, and the integral is

$$\sum_{i=0}^{n} \int_{Q^{n-1}} F(\tau_i(\mathbf{u})) \cdot N(\tau_i(\mathbf{u}))\ du_1 \cdots du_{n-1}.$$

The right side is an ordinary integral over a subset of $\mathbb{R}^n$, since we are parametrizing D by the identity map.