MATH 425b NOTES ON PARSEVAL'S THEOREM 8.16
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These notes are on the meaning of Parseval's Theorem.

In k-dimensional (complex) Euclidean space $\mathbb{C}^k$, the unit coordinate vectors $e_1, .., e_k$ form an orthonormal system. If we express vectors in terms of their coordinates in this system:

$$\mathbf{u} = (a_1, \ldots, a_k) = \sum_{i=1}^{k} a_i \mathbf{e}_i, \quad \mathbf{v} = (b_1, \ldots, b_k) = \sum_{i=1}^{k} b_i \mathbf{e}_i,$$

then we can express lengths and dot products in terms of the coordinates:

$$|\mathbf{u}|^2 = \sum_{i=1}^{k} |a_i|^2, \quad \mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{k} a_i \overline{b_i}.$$

One question is, does the analogous thing work in L^2? In other words, if on $[-\pi, \pi]$ we have Fourier series

$$f(x) \sim \sum_{n \in \mathbb{Z}} c_n e^{inx}, \quad g(x) \sim \sum_{n \in \mathbb{Z}} \gamma_n e^{inx},$$

can we express lengths and inner products in L^2 as if the c_n's and γ_n's were coordinates? In other words, is it true that

$$\|f\|_{L^2}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{n \in \mathbb{Z}} |c_n|^2$$

and

$$\langle f, g \rangle_{L^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx = \sum_{n \in \mathbb{Z}} c_n \overline{\gamma_n}?$$

(In each of these, the first equality is the definition, and we want to know if the second one holds.) Parseval's Theorem says that yes, this is true, provided $f, g \in \mathcal{R}$. It's actually true more generally than that, but this is the version Rudin gives.

Parseval's Theorem also says that as long as $f \in \mathcal{R}$, the Fourier series really does "represent" the function f in the sense that

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx},$$

provided we interpret this in the sense of convergence of partial sums in L^2 distance: this means that for the Nth partial sum $s_N(f, x) = \sum_{n=-N}^{N} c_n e^{inx}$ we have

$$d_2(s_N(f, \cdot), f) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |s_N(f, x) - f(x)|^2 \, dx \right)^{1/2} \to 0 \quad \text{as } N \to \infty.$$

The proof of this last part is based on these steps:

(1) Since $f \in \mathcal{R}$, there is a continuous function h close to f, that is, $d_2(f, h)$ is small. We get this h by choosing a "good" partition of $[-\pi, \pi]$ and making h a piecewise linear function that matches f at the partition points x_i.

(2) The continous h can be approximated uniformly by a trigonometric polynomial P, of some degree N, by 7.33. The approximation is then also good in L^2 distance. Therefore P is close to f, in L^2 distance.

(3) Among all trigonometric polynomials having the same degree N that P has, the closest trigonometric polynomial to f is $s_N(f, \cdot)$, by 8.11, so it's even closer to f than P is. Since this can be arbitrarily close, it follows that $d_2(s_N(f, \cdot), f) \to 0$.