MATH 425b SAMPLE FINAL EXAM
Spring 2016
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50-60% of the exam will be on material after Midterm 2, and 40-50% on topics already covered on the midterms. These percentages apply to the combined in-class and take-home exams, so one exam might have a significantly higher or lower percentage.

In Chapter 10, it is hard to specify exactly the sections covered because some topics I discussed but not in full depth. Here is a good approximation: 10.1–10.37, and 10.42, 10.48, 10.49, 10.51.

(1) These are 3 separate problems about forms; the assumptions for each problem don't apply to the other problems.

(a) Let $\omega = f(x)\ dx_1 \wedge dx_2 + g(x)\ dx_3 \wedge dx_4$ be a 2-form in some $\Omega \subset \mathbb{R}^n$, with f, g continuous. Under what conditions on f, g does $\omega \wedge \omega \neq 0$?

(b) Suppose $\omega = f(x)\ dx_I + g(x)\ dx_J$ is a k-form, and k is odd. Show that $\omega \wedge \omega = 0$. HINT: (53) in Chapter 10. (The result is actually true for general k-forms—I used a form here with two terms just to keep things simpler.)

(c) Suppose ω is an exact k-form in some open $E \subset \mathbb{R}^n$. Show that $\omega \wedge d\beta$ is exact for every form β in E. HINT: This is very short if you apply the right theorem.

(2) Let $f^{(k)}$ and f be functions in L^2 having Fourier coefficients $\{c_n^{(k)}, n \in \mathbb{Z}\}$ and $\{c_n, n \in \mathbb{Z}\}$ respectively.

(a) Show that $f^{(k)} \to f$ in L^2 (that is, $||f^{(k)} - f||_2 \to 0$) if and only if $\sum_{n \in \mathbb{Z}} |c_n^{(k)} - c_n|^2 \to 0$ as $k \to \infty$.

(b*)(Harder problem) Suppose that for each $n \in \mathbb{Z}$, we have $c_n^{(k)} \to c_n$ as $k \to \infty$, and suppose there is a sequence $\{b_n, n \in \mathbb{Z}\}$ such that $|c_n^{(k)}| \leq b_n$ for all k, and $\sum_{n \in \mathbb{Z}} b_n^2 < \infty$. Show that $f^{(k)} \to f$ in L^2. HINT: Use (a). Decompose $\sum_{n \in \mathbb{Z}}$ into $\sum_{n \in [-N,N]} + \sum_{n \notin [-N,N]}$, for some appropriate N. One of these two sums can be made small using $\sum_{n \in \mathbb{Z}} b_n^2 < \infty$.

(3) Let K be a compact metric space and for $c > 0$ define the sets of Lipschitz functions

$$\mathrm{Lip}_c(K) = \{f : K \to \mathbb{R} : |f(x) - f(y)| \leq c d(x, y) \text{ for all } x, y \in K\}, \quad \mathrm{Lip}(K) = \cup_{c=1}^{\infty} \mathrm{Lip}_c(K).$$

An example of a nonconstant Lipschitz function in a general metric space is $f(x) = d(x, x_0)$ for some fixed x_0. As usual, $C(K)$ denotes the set of all continuous functions on K, endowed with the uniform (sup) metric.

(a) Show that $\mathrm{Lip}(K)$ is an algebra, that is, if $f, g \in \mathrm{Lip}(K)$ and $a \in \mathbb{R}$ then $af, f + g$ and fg are in $\mathrm{Lip}(K)$.

(b) Show that $F_{c,M} = \{f \in \mathrm{Lip}_c(K) : ||f||_\infty \leq M\}$ is a compact subset of $C(K)$, for each $c, M > 0$. HINT: We proved a criterion for a subset of $C(K)$ to be compact.

(4)(a) Let $\omega = (x^2y^2 + z^2)\, dx \wedge dy + z^3\, dx \wedge dz$ in $\mathbb{R}^3$. Show that for every cube $A = [-a, a]^3$ centered at the origin, we have

$$\int_{\partial A} \omega = 0. \tag{1}$$

HINT: It is not necessary to parametrize ∂A to do this.

(b) For what other points (x_0, y_0, z_0) does (1) remain true for all cubes A centered at (x_0, y_0, z_0)?