1 Lucas Tree Model

Notes from Kaiji Chen

- Previously, our focus is optimal resource allocation across time given resources constraint(in a social planner's problem) or individual agents' budget constraint(in a competitive equilibrium).
- Alternatively, given equilibrium quantities and demand function, we can back out equilibrium prices.
- In particular, take consumption process as given, solve for the equilibrium prices of given financial assets that transfer resource across time and different states.

1.1 Model Setup

- The economy is inhabited by a large number of identical agents.
- Endowment: The only durable good in the economy is a set of trees, which are equal in the number to the number of people in the economy. Each agent starts life at time zero with one tree. Each period, each tree yields fruit or dividends in the amount d_t to its owner at the beginning of period t.
- Technology: Fruits cannot be stored. Dividends are exogenous and follow a stochastic process.
- Preferences: agents in this economy consume a single good, which is fruit.

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

  where $u(\cdot)$ is concave, strictly increasing and twice continuous differentiable.
- Market structure: A competitive market in trees. Ownership of a tree at the beginning of period t entitles the owner to receive the dividend in period t and to have the right to sell the tree at price p_t in terms of consumption good.

Since all agents are identical in terms of preferences and endowment, we can assume there is a representative agent in this economy. In equilibrium, there is only 1 tree (supply). Our purpose is to find the price of the tree that will make the aggregate demand of the tree equal to 1.

Representative agent's problem in a competitive market:

$$\max_{\{c_t,s_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + p_t s_{t+1} = (p_t + d_t)s_t$$

$$s_0 = 1, \quad given$$

Lagrangian:

$$L = E_0[\sum_{t=0}^{\infty} \beta^t u(c_t) - \sum_{t=0}^{\infty} \beta^t \lambda_t (c_t + p_t s_{t+1} - (p_t + d_t)s_t)]$$

F.O.C:

$$\frac{\partial (L|I_t)}{\partial c_t} = E_t[\beta^t u'(c_t) - \beta^t \lambda_t] = 0$$

i.e.

$$u'(c_t) = \lambda_t$$

$$\frac{\partial (L|I_t)}{\partial s_{t+1}} = -E_t[\beta^t \lambda_t p_t - \beta^{t+1} \lambda_{t+1}(p_{t+1} + d_{t+1})] = 0$$

i.e.

$$E_t[\beta \lambda_{t+1}(p_{t+1} + d_{t+1})] = \lambda_t p_t$$

Euler:

$$\beta E_t[u'(c_{t+1})(p_{t+1} + d_{t+1})] = p_t u'(c_t)$$

Remarks:

- The utility cost of buying one tree today is $p_t u'(c_t)$.
- Tomorrow, each tree delivers payoff $p_{t+1} + d_{t+1}$, which is a random variable.
- The shadow value of the tree, discounted to today, is $\beta E_t[u'(c_{t+1})(p_{t+1} + d_{t+1})]$.
- At the margin, an investor is indifferent between buying an additional unit of tree.

T.V.C: (?)

$$\lim_{k \to \infty} E_t \beta^k u'(c_{t+k}) p_{t+k} s_{t+k} = 0$$

Equilibrium Price:

$$p_t = E_t \beta \frac{u'(d_{t+1})}{u'(d_t)}(p_{t+1} + d_{t+1})$$

$$p_t u'(d_t) = E_t \sum_{j=1}^{\infty} \beta^j u'(d_{t+j}) d_{t+j} + \lim_{k\to\infty} E_t \beta^k u'(c_{t+k}) p_{t+k}$$

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(d_{t+j})}{u'(d_t)} d_{t+j}$$

Special Case 1: log uitlity ($u(c) = ln(c)$)

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{d_t}{d_{t+j}} d_{t+j} = \frac{\beta}{1-\beta} d_t$$

Intuition: Higher d_t, higher p_t. When d_t is high, consumer tends to transfer resources from today to tomorrow by purchasing the tree. The demand is high, price is high. When $d_t + j$ becomes larger, income effect pushes up c_t, thus push demand for asset down; substitution effect implies a higher return for investment and thus push demand for asset up. When the utility is log, the two effects cancel out.

Special Case 2: Risk Neutrality ($u(c) = c$) Efficient Market Hypothesis:

$$E_t \frac{p_{t+1} + d_{t+1}}{p_t} = \frac{1}{\beta} - 1$$

2 Equity Premium Puzzle

2.1 Model Setup

$$E_0[\sum_{t=0}^{\infty} \beta^t u(c_t)], \quad 0 < \beta < 1,$$

$$u(c, \alpha) = \frac{c^{1-\alpha}}{1-\alpha}, 0 < \alpha < \infty.$$

Euler:

$$p_t u'(c_t) = \beta E_t[(p_{t+1} + y_{t+1}) u'(c_{t+1})]$$

For Stocks:

$$1 = \beta E_t[\frac{u'(c_{t+1})}{u'(c_t)} R_{e,t+1}]$$

For Bonds:

$$1 = \beta E_t[\frac{u'(c_{t+1})}{u'(c_t)} R_{f,t+1}]$$

$$E_t(R_{e,t+1}) = R_{f,t+1} + cov_t\{\frac{-u'(c_{t+1}), R_{e,t+1}}{E_t[u'(c_{t+1})]}\}.$$

The equity premium $E_t(R_{e,t+1}) - R_{f,t+1}$, can be easily computed. Expected asset returns equal the risk-free rate plus a premium for bearing risk, which depends on the covariance of the asset returns with the marginal utility of consumption. Assets that covary positively with consumption, that is, assets that pay off in states when consumption is high and marginal utility is low, command a high premium because these assets destabilize consumption.

The question now is: **Is the magnitude of the covariance between the assets and the marginal utility of consumption large enough to justify the observed 6pp equity premium in U.S. equtiy markets?**

Additional Assumption:

- The growth rate of consumption, $x_{t+1} \equiv c_{t+1}/c_t$, is i.i.d. .
- The growth rate of dividends, $z_{t+1} \equiv y_{t+1}/y_t$, is i.i.d. .
- (x_t, z_t) are jointly lognormally distributed.

$$E_t(R_{e,t+1}) == \frac{E_t(z_{t+1})}{\beta E_t(z_{t+1} x_{t+1}^{-\alpha})} = \frac{e^{\mu_z + 1/2\sigma_z^2}}{\beta e^{\mu_z - \alpha\mu_x + 1/2(\sigma_z^2 + \alpha^2\sigma_x^2 - 2\alpha\sigma_{x,z})}}$$

$$R_{f,t+1} = \frac{1}{\beta E_t(x_{t+1}^{-\alpha})} = \frac{1}{\beta e^{-\alpha\mu_x + 1/2\alpha^2\sigma_x^2}}$$

$$lnE_t(R_{e,t+1}) = -ln\beta + \alpha\mu_x - \frac{1}{2}\alpha^2\sigma_x^2 + \alpha\sigma_{x,z}$$

$$lnR_f = -ln\beta + \alpha\mu_x - \frac{1}{2}\alpha^2\sigma_x^2$$

$$lnE(R\_e) - lnR_f = \alpha\sigma_{x,z} = \alpha\sigma_x^2$$

The last equality holds if we impose $x = z$, a restriction that the return on equity be perfectly correlated with the growth rate of consumption. σ_x^2 is 0.00125

2.2 Efforts to Solve the Puzzle

2.2.1 Modifying the Time-and-State-Separable Utility Function

A restriction imposed by CRRA preferences is that the coefficient of risk aversion is rigidly linked to the elasticity of intertemporal substitution; one is the reciprocal of the other. The implication is that if an individual is averse to variation of consumption in different states at a particular point in time, then she or he will be averse to consumption variation over time. There is no a priori reason that this must be so.

Epstein and Zin, Generalized Expected Utility

$$U_t = [c_t^{1-\rho} + \beta(E_t U_{t+1}^{1-\alpha})^{(1-\rho)/(1-\alpha)}]^{1/1-\rho}.$$

2.2.2 Habit Formation

Constantinides 1990.

This formuation assumes that utility is affected not only by current consumption but also by past consumption. It captures the notion that utility is a decreasing function of past consumption and marginal utility is an increasing function of past consumption.

Utility:

$$U(c) = E_t \sum_{s=0}^{\infty} \beta^s \frac{(c_{t+s} - \lambda c_{t+s-1})^{1-\alpha}}{1-\alpha}, \quad \lambda > 0.$$

This preference ordering makes the agent extremely averse to consumption risk even when the risk aversion is small. For small changes in consumption, changes in marginal utility can be large. It can solve address the risk-free rate puzzle because the induced aversion to consumption risk increases the demand for bonds, thereby reducing the risk-free rate.

I omit other explanations such as Idiosyncratic and Uninsurable Income Risk, Disaster States and Survivorship Bias, Borrowing Constraints, Liquidity Premium and Taxes.

A nice survey would be

Mehra, R. (2003). The equity premium: why is it a puzzle?(corrected). Financial Analysts Journal, 59(1), 5469.