1 CRRA Utility

Constant Relative Risk Aversion (or Power Utility or Isoelastic Utility)

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

where $\gamma > 0$ and $\gamma \neq 1$.

$$u'(c) = c^{-\gamma}$$

$$u''(c) = -\gamma c^{-\gamma-1}$$

Special cases:

$$\underset{\gamma\to 1}{Lim} \frac{c^{1-\gamma}-1}{1-\gamma} = \underset{\gamma\to 1}{Lim} \frac{-c^{1-\gamma}\ln c}{-1} = \ln c$$

2 Discrete-Time Dynamic Optimization

2.1 Equivalence Results

$$V^*(x(0)) = \underset{\{x(t)\}_{t=0}^{\infty}}{sup} \sum_{t=0}^{\infty} \beta^t U(x(t), x(t+1))$$

subject to

$$x(t+1) \in G(x(t)) \quad \forall \quad t \geq 0,$$

$$x(0) \quad given.$$

Under some assumptions, solving this is equivalent to solving the following problem.

$$V(x) = \underset{y\in G(x)}{sup} \{U(x,y) + \beta V(y)\}, \quad \forall x \in X.$$

The basic idea of dynamic programming is to turn the sequence problem into a functional equation; that is, to transform the problem into one of finding a function rather than a sequence.

2.2 Solving the Problem

$$V(x) = \sup_{y \in G(x)} \{U(x,y) + \beta V(y)\}, \quad \forall x \in X.$$

F.O.C Equation: (Control Variables)

$$\frac{\partial U(x,y^*)}{\partial y} + \beta V'(y^*) = 0$$

Envelope Equation: (State Variables)

$$V'(x) = \frac{\partial U(x,y^*)}{\partial x}$$

Here, we use F.O.C and E.E. to derive the Euler Equation. This approach is different from the direct Lagrangian method. However, it will give us the same Euler equation.

Euler Equation:

$$\frac{\partial U(x(t),x^*(t+1))}{\partial y} + \beta \frac{\partial U(x^*(t+1),x^*(t+2))}{\partial x} = 0$$

Transversality Condition:

$$\lim_{t \to \infty} \quad \beta^t \frac{\partial U(x^*(t),x^*(t+1))}{\partial x} \cdot x^*(t) = 0$$