Auctions with Interdependent Values

In this chapter we simultaneously relax two major assumptions regarding the nature of the information available to the bidders.

Interdependent Values

First, we relax the assumption of private values—that each bidder knows the value of the object to himself—by allowing for the possibility that bidders have only partial information regarding the value, say in the form of a noisy signal. Indeed, other bidders may possess information that would, if known to a particular bidder, affect the value he assigns to the object. The resulting information structure is called one of *interdependent values*. We assume that each bidder has some private information concerning the value of the object. Bidder i's private information is summarized as the realization of the random variable $X_i \in [0, \omega_i]$, called i's *signal*. It is assumed that the value of the object to bidder i, V_i, can be expressed as a function of all bidders' signals and we will write

$$V_i = v_i(X_1, X_2, \ldots, X_N),$$

where the function v_i is bidder i's *valuation* and is assumed to be nondecreasing in all its variables and twice continuously differentiable. In addition, it is assumed that v_i is strictly increasing in X_i.

This specification supposes that the value is completely determined by the signals—that is, there is no remaining uncertainty. This need not be the case, however, and more general formulations can also be accommodated. In a more general setting, suppose that $V_1, V_2, \ldots, V_N$ denote the N (unknown) values to the bidders; $X_1, X_2, \ldots, X_N$ denote the N signals available to the bidders; and S denotes a signal available only to the seller. In that case, we can define

$$v_i(x_1, x_2, \ldots, x_N) \equiv E[V_i \mid X_1 = x_1, X_2 = x_2, \ldots, X_N = x_N]$$

as the expected value to i conditional on all the information available to bidders. For operational purposes, this is the effective value that bidders can use in their calculations.

With either specification, we suppose that $v_i(0,0,\ldots,0)=0$ and that $E[V_i]<\infty$. We continue to assume that bidders are risk neutral: Each bidder maximizes the expectation of $V_i - p_i$, where p_i is the price paid.

Notice that this specification of the values includes, as an extreme case, the private values model of earlier chapters in which $v_i(\mathbf{X}) = X_i$. At the other end of the spectrum is the case of a pure *common value* in which all bidders assign the same value

$$V = v(X_1, X_2, \ldots, X_N)$$

to the object; the valuations of the bidders are identical. Bidders' information consists only of their own signals of course, so while the *ex post* value is common to all, it is unknown to any particular bidder. A special case that is of both analytic and practical interest entails first specifying a distribution for the common value V and then assuming that conditional on the event $V = v$, bidders' signals X_i are independently distributed. Typically, it is also assumed that each X_i is an unbiased estimator of V, so $E[X_i \mid V = v] = v$. This particular specification has been used to model the information structure associated with auctions of oil-drilling leases and is sometimes called the "mineral rights" model.

The interdependence of values complicates the decision problem facing a bidder. In particular, since the exact value of the object is unknown and depends also on other bidders' signals, an *a priori* estimate of this value may need to be revised as a result of events that take place during, and even after, the auction. The reason is that these events may convey valuable information about the signals of other bidders. One such event is the announcement that the bidder has won the auction.

THE WINNER'S CURSE

Prior to the auction the only information available to a bidder—say, 1—is that his own signal $X_1 = x$. Based on this information alone, his estimate of the value is $E[V \mid X_1 = x]$. Now suppose that the object is sold using a sealed-bid first-price auction and consider what happens when and if it is announced that bidder 1 is, in fact, the winner. If all bidders are symmetric and follow the same strategy β, then this fact reveals to bidder 1 that the highest of the other $N-1$ signals is less than x. As a result, his estimate of the value upon learning that he is the winner is $E[V \mid X_1 = x, Y_1 < x]$, which is *less* than $E[V \mid X_1 = x]$. The announcement that he has won leads to a decrease in the estimated value; in this sense, winning brings "bad news." A failure to foresee this effect and take it fully into account when formulating bidding strategies will result in what has been called the *winner's curse*—the possibility that the winner pays more than the value.

The phenomenon is most apparent in a pure common value model in which each bidder's signal $X_i = V + \varepsilon_i$. Suppose that the different ε_i's are independently and identically distributed and satisfy $E[\varepsilon_i] = 0$. Then each bidder's signal is an unbiased estimator of the common value—that is, for all i, $E[X_i \mid V = v] = v$. But now notice that even though each individual signal is an unbiased estimator of the value, the largest of N such signals is not. In fact, since "max" is a convex function, $E[\max X_i \mid V = v] > \max E[X_i \mid V = v] = v$, showing that the expectation of the highest signal, in fact, overestimates the value. A bidder who does not take this fully into account and bids an amount $\beta(X_i)$ which is close to X_i would, upon winning, pay more than the estimated worth of the object. Put another way, bidders may need to shade their bids well below their initial estimates in order to avoid the winner's curse. Note also that the magnitude of the winner's curse increases with the number of bidders in the auction. The news that a signal is the highest of, say, 20 bidders is worse than the news that it is the highest of 10 bidders.

We emphasize that the winner's curse arises only if bidders do not calculate the value of winning correctly and overbid as a result; it does not arise in equilibrium.

NONEQUIVALENCE OF ENGLISH AND SECOND-PRICE AUCTIONS

A second consequence of interdependent values is that it is no longer the case that the English (or open ascending) auction is strategically equivalent to the sealed-bid second-price auction. The difference in the two auction formats is that in an English auction active bidders get to know the prices at which the bidders who have dropped out have done so. This allows the active bidders to make inferences about the information that the inactive bidders had and in this way to update their estimates of the true value. A sealed-bid second-price auction, by its very nature, makes no such information available.

There are two cases in which this information is irrelevant. First, if there are only two bidders, then the English auction is always equivalent to a second-price auction; in this case, when one of the bidders drops out in an English auction, the auction is over. The second case arises if the bidders have private values; in this case, the information gleaned from others is irrelevant.

Affiliation

We also relax the assumption that bidders' information is independently distributed by allowing for the possibility that bidders' signals are correlated. Thus, the joint density of the bidders' signals, $f(\mathbf{X})$, need not be a product of densities of individual signals, $f_i(X_i)$. In fact, we will assume that the signals $X_1, X_2, \ldots, X_N$ are positively *affiliated*. Affiliation is a strong form of positive correlation and roughly means that if a subset of the X_i's are all large, then this makes it more likely that the remaining X_j's are also large. While a formal definition and a more detailed discussion may be found in Appendix D, for the purposes of this chapter, the following three implications of affiliation are sufficient.

First, define, as usual, the random variables $Y_1, Y_2, \ldots, Y_{N-1}$ to be the largest, second-largest, ... , smallest from among $X_2, X_3, \ldots, X_N$. If the variables $X_1, X_2, \ldots, X_N$ are affiliated, then the variables $X_1, Y_1, \ldots, Y_{N-1}$ are also affiliated.

Second, let $G(\cdot \mid x)$ denote the distribution of Y_1 conditional on $X_1 = x$. Then the fact that X_1 and Y_1 are affiliated implies that if $x' > x$, then $G(\cdot \mid x')$ dominates $G(\cdot \mid x)$ in terms of the reverse hazard rate—that is, for all y,

$$\frac{g(y \mid x')}{G(y \mid x')} \geq \frac{g(y \mid x)}{G(y \mid x)} \tag{6.1}$$

Third, if γ is any increasing function, then $x' > x$ implies that

$$E\left[\gamma(Y_1) \mid X_1 = x'\right] \geq E\left[\gamma(Y_1) \mid X_1 = x\right] \tag{6.2}$$

6.1 THE SYMMETRIC MODEL

As in the case of independent private values, it is instructive to begin by considering symmetric situations. As before, we will first derive symmetric equilibrium strategies in the three auction formats: second-price, English, and first-price. We will then compare the expected revenues from these auctions.

In a model with independent private values, bidders are symmetric if their values are drawn from the same distribution. With interdependent values and affiliated signals, however, there are two aspects to symmetry. The first concerns the symmetry of the valuations v_i and the second concerns the symmetry of the distribution of signals.

It is assumed that all signals X_i are drawn from the same interval $[0, \omega]$ and that the valuations of the bidders are symmetric in the following sense. For all i, we can write these in the form

$$v_i(\mathbf{X}) = u\left(X_i, \mathbf{X}_{-i}\right)$$

and the function u, which is the same for all bidders, is symmetric in the last $N-1$ components. This means that from the perspective of a particular bidder, the signals of other bidders can be interchanged without affecting the value. For instance, when $N = 3$, the value to bidder 1 depends on his or her own signal and the signals of bidders 2 and 3, but if the signals of the other bidders were interchanged, then the value would not be affected. Thus, for all x, y, and z it is the case that $u(x, y, z) = u(x, z, y)$.

It is also assumed that the joint density function of the signals f, defined on $[0, \omega]^N$, is a symmetric function of its arguments and the signals are affiliated.

Define the function

$$v(x, y) = E\left[V_1 \mid X_1 = x, Y_1 = y\right] \tag{6.3}$$

to be the expectation of the value to bidder 1 when the signal he or she receives is x and the highest signal among the other bidders, Y_1, is y. Because of symmetry this function is the same for all bidders and from (6.2), v is a nondecreasing function of x and y. We will, in fact, assume that v is strictly increasing in x. Moreover, since $u(\mathbf{0}) = 0$, $v(0,0) = 0$. The function v plays an important role in what follows.

For the symmetric model with interdependent values, we proceed in a manner parallel to the development of Chapter 2 concerning the symmetric independent private values model. As in Chapter 2, we first derive symmetric equilibrium strategies in the three common formats: the second-price, English, and first-price auctions. We then compare the three formats by directly computing the expected revenues in each.

6.2 SECOND-PRICE AUCTIONS

We begin by deriving a symmetric equilibrium in a second-price sealed-bid auction.

Proposition 6.1. *Symmetric equilibrium strategies in a second-price auction are given by:*

$$\beta^{\mathrm{II}}(x) = v(x,x)$$

Proof. Suppose all other bidders $j \neq 1$ follow the strategy $\beta \equiv \beta^{\mathrm{II}}$. Bidder 1's expected payoff when his signal is x and he bids an amount b is

$$\begin{aligned}\Pi(b,x) &= \int_0^{\beta^{-1}(b)} (v(x,y) - \beta(y))\, g(y \mid x)\, dy \\ &= \int_0^{\beta^{-1}(b)} (v(x,y) - v(y,y))\, g(y \mid x)\, dy\end{aligned}$$

where $g(\cdot \mid x)$ is the density of $Y_1 \equiv \max_{i \neq 1} X_i$ conditional on $X_1 = x$.

Since v is increasing in the first argument, for all $y < x$, $v(x,y) - v(y,y) > 0$ and for all $y > x$, $v(x,y) - v(y,y) < 0$. Thus, Π is maximized by choosing b so that $\beta^{-1}(b) = x$ or equivalently, by choosing $b = \beta(x)$. ■

The nature of the bidding strategies in Proposition 6.1 can be understood as follows. A bidder, say 1, with signal x is asked to bid an amount $\beta^{\mathrm{II}}(x)$ such that if he were to just win the auction with that bid—if the highest competing bid, and hence the price, were also $\beta^{\mathrm{II}}(x)$—he would just "break even." This is because if the highest competing bid were $\beta^{\mathrm{II}}(x)$, then bidder 1 would infer that $Y_1 = x$ and the expected value of the object conditional on this new piece of information would be $E[V_1 \mid X_1 = x, Y_1 = x] = v(x,x) = \beta^{\mathrm{II}}(x)$.

Proposition 6.1 applies, of course, to the special case of private values (where $v(x,x) = x$) and in those circumstances the equilibrium strategy is weakly dominant. With general interdependent values, however, the strategy β^{II} identified above is not a dominant strategy.

The equilibrium identified above is unique in the class of symmetric equilibria with an increasing strategy. To see this, suppose that β is an increasing symmetric equilibrium strategy. Writing $\Pi(b,x)$ as in the proof of Proposition 6.1 and optimizing over b, the first-order condition immediately implies that $\beta(x)$ must equal $v(x,x)$. We will see in Chapter 8, however, that even symmetric second-price auctions may have other, asymmetric equilibria.

It is instructive to find the equilibrium bidding strategies explicitly in an example. In the example that follows, there is a common value and conditional on that value, bidders' signals are independently distributed. In other words, it is an instance of the "mineral rights" model.

Example 6.1. *Suppose that there are three bidders with a common value V for the object that is uniformly distributed on* $[0,1]$. *Given $V=v$, bidders' signals X_i are uniformly and independently distributed on* $[0,2v]$.

As usual, $\mathbf{X}=(X_1,X_2,X_3)$ and it is convenient to define the random variable $Z\equiv\max\{X_1,X_2,X_3\}$.

The density of X_i conditional on $V=v$ is $1/2v$ on the interval $[0,2v]$, so the joint density of $(V,\mathbf{X})$ is $1/8v^3$ on the set $\{(V,\mathbf{X})\mid\forall i, X_i\leq 2V\}$. Now notice that the only information about V that knowledge of X_1,X_2,X_3 provides is that $V\geq\frac{1}{2}Z$. (See Figure 6.1.) Thus, the joint density of $\mathbf{X}$ is

$$\begin{aligned} f(x_1,x_2,x_3) &= \int_{\frac{1}{2}z}^{1}\frac{1}{8v^3}\,dv \\ &= \frac{4-z^2}{16z^2} \end{aligned}$$

where $z=\max\{x_1,x_2,x_3\}$. Thus, the density of V conditional on $\mathbf{X}=\mathbf{x}$ is the same as the density of V conditional on $Z=z$, so

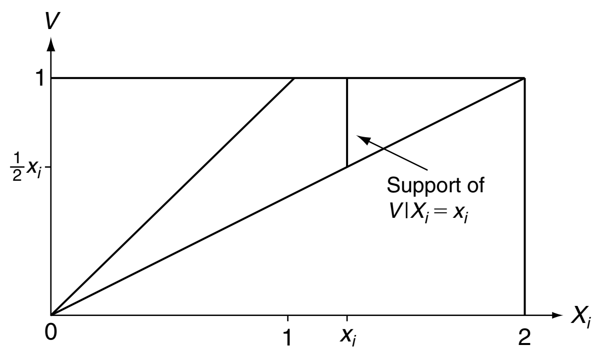

FIGURE 6.1 Support of $V\mid X_i=x_i$ in Example 6.1.

$$\begin{aligned} f(v \mid \mathbf{X}=\mathbf{x}) &= f(v \mid Z=z) \\ &= \frac{1}{8v^3} \times \frac{16z^2}{4-z^2} \end{aligned}$$

on the interval $\left[\frac{1}{2}z, 1\right]$. Thus,

$$\begin{aligned} E[V \mid \mathbf{X}=\mathbf{x}] &= E[V \mid Z=z] \\ &= \int_{\frac{1}{2}z}^{1} vf(v \mid \mathbf{X}=\mathbf{x})\, dv \\ &= \frac{2z}{2+z} \end{aligned}$$

Notice that since $Y_1 = \max\{X_2, X_3\}$ and $Z = \max\{X_1, X_2, X_3\}$, $Z = \max\{X_1, Y_1\}$

$$\begin{aligned} v(x,y) &= E[V \mid X_1 = x, Y_1 = y] \\ &= E[V \mid Z = \max\{x,y\}] \\ &= \frac{2\max\{x,y\}}{2+\max\{x,y\}} \end{aligned}$$

Thus, from Proposition 6.1 we obtain

$$\begin{aligned} \beta^{\mathrm{II}}(x) &= v(x,x) \\ &= \frac{2x}{2+x} \end{aligned}$$

▲

6.3 ENGLISH AUCTIONS

In sealed-bid auctions, a bidder's strategy determines the amount of his bid as a function of his private information. In an English, or open ascending, auction, additional information—the identities of the bidders who drop out and the prices at which they do so—becomes available. In the symmetric model considered here, the exact identities of the bidders who drop out are not relevant, but the prices at which bidders dropped out are. Similarly, the identities of the bidders who are active are not relevant, but the number of active bidders is.

The term *English auction* encompasses a variety of open ascending price formats, which differ in their precise rules. In one format the price is raised by the bidders themselves as they outbid each other. We adopt a formulation of the rules that is particularly convenient for analytic purposes. An auctioneer sets the price at zero and gradually raises it. The current price is observed by all and bidders signal their willingness to buy by raising a hand, holding up a sign, or pushing a button that controls a light. The important aspect is that this action is witnessed by all, so at any time the set of active bidders—those who signal their willingness to buy at the current price—is commonly known. Bidders may drop out at any time, but once they do so cannot reenter the auction at a higher price. The auction ends when there is only one active bidder.

A symmetric equilibrium strategy in an English auction is thus a collection $\boldsymbol{\beta} = (\beta^N, \beta^{N-1}, \ldots, \beta^2)$ of $N-1$ functions $\beta^k : [0,1] \times \mathbb{R}_+^{N-k} \to \mathbb{R}_+$, for $1 < k \leq N$, where $\beta^k(x, p_{k+1}, \ldots, p_N)$ is the price at which bidder 1 will drop out if the number of bidders who are still active is k, his own signal is x, and the prices at which the other $N-k$ bidders dropped out were $p_{k+1} \geq p_{k+2} \geq \cdots \geq p_N$.

Now consider the following strategies for the bidders. When all bidders are active, let

$$\beta^N(x) = u(x, x, \ldots, x) \tag{6.4}$$

and notice that $\beta^N(\cdot)$ is a continuous and increasing function.

Suppose that bidder N, say, is the first to drop out at some price p_N and let x_N be the unique signal such that $\beta^N(x_N) = p_N$ (since $\beta^N(\cdot)$ is continuous and increasing there exists a unique such x_N). When some bidder drops out at a price p_N, let the remaining $N-1$ bidders who are still active follow the strategy

$$\beta^{N-1}(x, p_N) = u(x, \ldots, x, x_N),$$

where $\beta^N(x_N) = p_N$. The function $\beta^{N-1}(\cdot, p_N)$ is also continuous and increasing.

Proceeding recursively in this way, for all k such that $2 \leq k < N$ suppose that bidders $N, N-1, \ldots, k+1$ have dropped out of the auction at prices p_N, $p_{N-1}, \ldots, p_{k+1}$, respectively. Let the remaining k bidders who are still active follow the strategy

$$\beta^k(x, p_{k+1}, \ldots, p_N) = u(x, \ldots, x, x_{k+1}, \ldots, x_N) \tag{6.5}$$

where $\beta^{k+1}(x_{k+1}, p_{k+2}, \ldots, p_N) = p_{k+1}$.

We will argue that these strategies constitute an equilibrium of the English auction, but before doing so formally it is worthwhile to understand the nature of the bidding strategies. Suppose that bidders $k+1, k+2, \ldots, N$ have dropped out, so only k bidders are still active. Because the strategies are revealing, the signals $x_{k+1}, x_{k+2}, \ldots, x_N$ of the bidders who have dropped out become known to the other bidders. Consider a particular bidder—say, 1—with signal x, and suppose the other bidders are following β^k. Bidder 1 evaluates whether or not he should drop out at the current price p and does the following "mental calculation." He asks what would happen if he were to win the good at the current price p. Now the only way this can happen is if *all* the other $k-1$ bidders drop out at p. In that case, bidder 1 would infer that each of their signals were equal to a y such that $\beta^k(y, p_{k+1}, \ldots, p_N) = p$. The value of the object would then be inferred to be $u(x, y, \ldots, y, x_{k+1}, x_{k+2}, \ldots, x_N)$. It is worth continuing in the auction if and only if the inferred value of the object exceeds the current price p. Thus, the strategy calls for bidder 1 to continue until the price is such that if he were to win the object at that price he would just break even.

Proposition 6.2. *Symmetric equilibrium strategies in an English auction are given by $\boldsymbol{\beta}$ defined in (6.4) and (6.5).*

Proof. Consider bidder 1 with signal $X_1 = x$ and suppose that all other bidders follow the strategy $\boldsymbol{\beta}$. As usual, define $Y_1, Y_2, \ldots, Y_{N-1}$ to be the largest, second-largest,..., smallest of $X_2, X_3, \ldots, X_N$, respectively.

Suppose that the realizations of $Y_1, \ldots, Y_N$, denoted by $y_1, \ldots, y_N$, respectively, are such that bidder 1 wins the object if he also follows the strategy $\boldsymbol{\beta}$. Then it must be that $x > y_1$. The price that bidder 1 pays is the price at which the bidder with the second-highest signal, y_1, drops out, and from (6.5) this is just $u(y_1, y_1, y_2, \ldots, y_N)$. Since $x > y_1$, the payoff to bidder 1 upon winning is

$$u(x, y_1, y_2, \ldots, y_N) - u(y_1, y_1, y_2, \ldots, y_N) > 0$$

Bidder 1 cannot affect the price he pays and winning yields a positive payoff. Thus, he cannot do better than to follow $\boldsymbol{\beta}$ also.

Next, suppose that the realizations of $Y_1, \ldots, Y_N$ are such that bidder 1 does not win the object by also following $\boldsymbol{\beta}$. Then it must be that $x < y_1$. If bidder 1 does not drop out and wins the auction, then again it must be at a price of $u(y_1, y_1, y_2, \ldots, y_N)$. But since $x < y_1$, the price now exceeds the *ex post* value of the object to 1. So bidder 1 cannot do better than to drop out as specified by $\boldsymbol{\beta}$. ■

The equilibrium strategy $\boldsymbol{\beta}$ defined in (6.4) and (6.5) is quite remarkable in that it depends only on the valuation functions u and not on the underlying distribution of signals f: For any given u satisfying the assumptions made here, if $\boldsymbol{\beta}$ is an equilibrium for some joint density function f, then the same $\boldsymbol{\beta}$ would remain an equilibrium if the signals were distributed according to some other density function, say $g \neq f$. In other words, the strategies form an *ex post* equilibrium: For any realization of signals the play prescribed by $\boldsymbol{\beta}$ forms a Nash equilibrium of the complete information game that results if the signals were commonly known. (See Appendix F.) This also means that the equilibrium strategy $\boldsymbol{\beta}$ has an important "no regret" feature—that is, for any realization of the signals the bidders have no cause to regret the outcome even if, after the fact, all signals were to become publicly known. In sharp contrast, once there are three or more bidders, bidders playing the symmetric equilibrium of the second-price auction, identified in the previous sections, may suffer from regret after the fact. The equilibrium of the second-price auction is not an *ex post* equilibrium. Nor is the symmetric equilibrium of the first-price auction an *ex post* equilibrium.

The symmetric equilibrium of the English auction has the strong no regret property because, in fact, in the course of the auction the signals of all other bidders are revealed to the winner, so he does not regret winning. On the other hand, bidders who drop out do not regret losing because if they were to win, it would be at a price that is too high. In Chapter 9, we undertake a further exploration of the English auction in cases where the valuation functions need not be symmetric.

6.4 FIRST-PRICE AUCTIONS

We now derive the equilibrium bidding strategies in a first-price auction, beginning, as usual, with a heuristic derivation.

Suppose all other bidders $j \neq 1$ follow the increasing and differentiable strategy β. Clearly, it does not pay for bidder 1 to bid less than $\beta(0)$ or more than $\beta(\omega)$.

As defined earlier, let $G(\cdot \mid x)$ denote the distribution of $Y_1 \equiv \max_{i \neq 1} X_i$ conditional on $X_1 = x$ and let $g(\cdot \mid x)$ be the associated conditional density function. The expected payoff to bidder 1 when his signal is x and he bids an amount $\beta(z)$ is

$$\begin{aligned}\Pi(z,x) &= \int_0^z (v(x,y) - \beta(z))\, g(y \mid x)\, dy \\ &= \int_0^z v(x,y)\, g(y \mid x)\, dy - \beta(z) G(z \mid x)\end{aligned}$$

The first-order condition is

$$(v(x,z) - \beta(z))\, g(z \mid x) - \beta'(z)\, G(z \mid x) = 0$$

At a symmetric equilibrium, the optimal $z = x$, so setting $z = x$ in the first-order condition, we obtain the differential equation:

$$\beta'(x) = (v(x,x) - \beta(x)) \frac{g(x \mid x)}{G(x \mid x)} \tag{6.6}$$

The differential equation (6.6) is only a necessary condition. We must also have that for all x, $v(x,x) - \beta(x) \geq 0$, since otherwise a bid of 0 would be better. Since, by assumption, $v(0,0) = 0$, it is the case that $\beta(0) = 0$. Thus, associated with (6.6) we have the boundary condition $\beta(0) = 0$.

The solution to the differential equation (6.6) together with the boundary condition $\beta(0) = 0$, as stated in the next proposition, constitutes a symmetric equilibrium.

Proposition 6.3. *Symmetric equilibrium strategies in a sealed-bid first-price auction are given by*

$$\beta^{\mathrm{I}}(x) = \int_0^x v(y,y)\, dL(y \mid x)$$

where

$$L(y\,|\,x)=\exp\left(-\int_y^x \frac{g(t\,|\,t)}{G(t\,|\,t)}\,dt\right) \tag{6.7}$$

Proof. First, note that $L(\cdot\,|\,x)$ can be thought of as a distribution function with support $[0,x]$. To see this recall that, because of affiliation (see Appendix D), for all $t>0$,

$$\frac{g(t\,|\,t)}{G(t\,|\,t)} \geq \frac{g(t\,|\,0)}{G(t\,|\,0)}$$

and so

$$\begin{aligned}
-\int_0^x \frac{g(t\,|\,t)}{G(t\,|\,t)}\,dt &\leq -\int_0^x \frac{g(t\,|\,0)}{G(t\,|\,0)}\,dt \\
&= -\int_0^x \frac{d}{dt}\,(\ln G(t\,|\,0))\,dt \\
&= \ln G(0\,|\,0)-\ln G(x\,|\,0) \\
&= -\infty
\end{aligned}$$

Applying the exponential function to both sides implies that for all x, $L(0\,|\,x)=0$. Moreover, $L(x\,|\,x)=1$ and $L(\cdot\,|\,x)$ is nondecreasing. Thus, $L(\cdot\,|\,x)$ is a distribution function.

Next, notice that if $x'>x$, then for all y, $L\big(y\,|\,x'\big)\leq L(y\,|\,x)$. In other words, the distribution $L\big(\cdot\,|\,x'\big)$ stochastically dominates the distribution $L(\cdot\,|\,x)$. Since $v(y,y)$ is an increasing function of y, this means that $\beta\equiv\beta^{\mathrm{I}}$ is an increasing function of x.

Now consider a bidder who bids $\beta(z)$ when his signal is x. The expected profit from such a bid can be written as

$$\Pi\,(z,x)=\int_0^z (v(x,y)-\beta(z))\,g(\,y\,|\,x)\,dy$$

since β is increasing.

Differentiating with respect to z yields

$$\begin{aligned}
\frac{\partial\Pi}{\partial z} &= (v(x,z)-\beta(z))\,g(z\,|\,x)-\beta'(z)\,G(z\,|\,x) \\
&= G(z\,|\,x)\left[(v(x,z)-\beta(z))\,\frac{g(z\,|\,x)}{G(z\,|\,x)}-\beta'(z)\right]
\end{aligned}$$

If $z<x$, then since $v(x,z)>v(z,z)$ and because of affiliation,

$$\frac{g(z\,|\,x)}{G(z\,|\,x)} > \frac{g(z\,|\,z)}{G(z\,|\,z)}$$

we obtain that

$$\frac{\partial \Pi}{\partial z} > G(z \mid x)\left[(v(z,z) - \beta(z))\frac{g(z \mid z)}{G(z \mid z)} - \beta'(z)\right] = 0$$

using (6.6). Similarly, if $z > x$, then $\frac{\partial \Pi}{\partial z} < 0$. Thus, $\Pi(z,x)$ is maximized by choosing $z = x$. ■

Proposition 6.3 is, of course, a generalization of Proposition 2.2. When values are private, $v(y,y) = y$. Also, when signals are independent, $G(\cdot \mid x)$ does not depend on x, so we can write $G(\cdot \mid x) \equiv G(\cdot)$. Thus,

$$\begin{aligned} L(y \mid x) &= \exp\left(-\int_y^x \frac{g(t)}{G(t)}\,dt\right) \\ &= \frac{1}{G(x)} G(y), \end{aligned}$$

so $\beta^{\mathrm{I}}(x)$ in the previous derivation reduces to $E[Y_1 \mid Y_1 < x]$, the equilibrium bid with private values.

Again, it is instructive to find the equilibrium bidding strategy explicitly in the context of a relatively simple example in which values are interdependent and signals are affiliated.

Example 6.2. *Suppose S_1, S_2, and T are uniformly and independently distributed on* $[0,1]$. *There are two bidders. Bidder* 1 *receives the signal $X_1 = S_1 + T$, and bidder* 2 *receives the signal $X_2 = S_2 + T$. The object has a common value*

$$V = \frac{1}{2}(X_1 + X_2)$$

for both the bidders.

Even though S_1, S_2, and T are independently distributed, the random variables X_1 and X_2 are affiliated. Moreover, since there are only two bidders, $Y_1 = X_2$. The joint density of X_1 and Y_1 is given in Figure 6.2 and from this it may be calculated that for all $x \in [0,2]$,

$$\frac{g(x \mid x)}{G(x \mid x)} = \frac{2}{x}$$

and for all $y \in [0,x]$,

$$L(y \mid x) = \frac{y^2}{x^2}$$

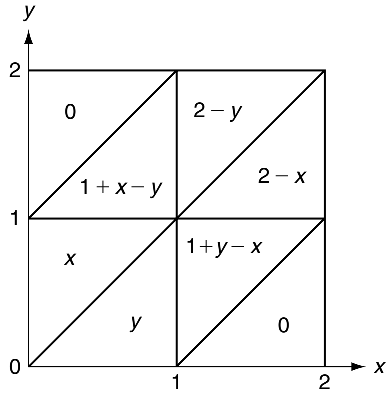

FIGURE 6.2 Joint density of X_1 and Y_1 in Example 6.2.

From Proposition 6.3 the equilibrium bidding strategy in a first-price auction is

$$\begin{aligned}\beta^{\mathrm{I}}(x) &= \int_0^x v(y,y)\,dL(y\,|\,x)\\ &= \frac{2}{3}x\end{aligned}$$

using the fact that $v(x,y) = \frac{1}{2}(x+y)$. ▲

6.5 REVENUE COMPARISONS

We now examine the performance of the three auctions studied here by comparing the expected revenues resulting in symmetric equilibria of each. Our main finding will be that under the assumption that signals are affiliated, the English auction outperforms the second-price auction, which in turn, outperforms the first-price auction. The reasons underlying these results are explored in the next chapter.

6.5.1 English versus Second-Price Auctions

Proposition 6.4. *The expected revenue from an English auction is at least as great as the expected revenue from a second-price auction.*

Proof. Recall from Proposition 6.1 that symmetric equilibrium strategies in a second-price auction are given by $\beta^{\mathrm{II}}(x) = v(x,x)$, where $v(x,y)$ is defined in (6.3). Thus, we have that, if $x > y$,

$$\begin{aligned}v(y,y) &= E\left[u\left(X_1,Y_1,Y_2,\ldots,Y_{N-1}\right)\,|\,X_1 = y, Y_1 = y\right]\\ &= E\left[u\left(Y_1,Y_1,Y_2,\ldots,Y_{N-1}\right)\,|\,X_1 = y, Y_1 = y\right]\\ &\leq E\left[u\left(Y_1,Y_1,Y_2,\ldots,Y_{N-1}\right)\,|\,X_1 = x, Y_1 = y\right],\end{aligned} \tag{6.8}$$

where the last inequality follows from the fact that u is increasing in all its arguments and all signals are affiliated. The expected revenue in a second-price auction can be written as

$$\begin{aligned} E\left[R^{\mathrm{II}}\right] &= E\left[\beta^{\mathrm{II}}(Y_1) \,|\, X_1 > Y_1\right] \\ &= E\left[v(Y_1, Y_1) \,|\, X_1 > Y_1\right] \\ &\leq E\left[E\left[u\left(Y_1, Y_1, Y_2, \ldots, Y_{N-1}\right) \,|\, X_1 = x, Y_1 = y\right] \,|\, X_1 > Y_1\right] \\ &= E\left[u\left(Y_1, Y_1, Y_2, \ldots, Y_{N-1}\right) \,|\, X_1 > Y_1\right] \\ &= E\left[\beta^2\left(Y_1, Y_2, \ldots, Y_{N-1}\right)\right] \\ &= E\left[R^{\mathrm{Eng}}\right] \end{aligned}$$

where we have used (6.8) and, as defined in Proposition 6.2, β^2 is the strategy used in an English auction when only two bidders remain. The price at which the second-to-last bidder drops out, $\beta^2\left(Y_1, Y_2, \ldots, Y_{N-1}\right)$, is, of course, the price paid by the winning bidder. ■

We postpone discussion of this result, and the reasons underlying it, until the next chapter. For the moment, notice that the English auction yields a strictly higher revenue than a second-price auction only if values are interdependent *and* signals are affiliated. With private values, the two are equivalent. The same is true if signals are independent.

6.5.2 Second-Price versus First-Price Auctions

Proposition 6.5. *The expected revenue from a second-price auction is at least as great as the expected revenue from a first-price auction.*

Proof. The payment of a bidder with signal x upon winning the object in a first-price auction is just his bid $\beta^{\mathrm{I}}(x)$, where β^{I} is defined in Proposition 6.3. The expected payment of a bidder with signal x upon winning the object in a second-price auction is $E[\beta^{\mathrm{II}}(Y_1) \,|\, X_1 = x, Y_1 < x]$, where β^{II} is defined in Proposition 6.1. We will show that the former is no greater than the latter. Since in both auctions the probability that a bidder with signal x will win the auction is the same—it is just the probability that x is the highest signal—this will establish the proposition.

From Proposition 6.1,

$$\begin{aligned} E\left[\beta^{\mathrm{II}}(Y_1) \,|\, X_1 = x, Y_1 < x\right] &= E\left[v(Y_1, Y_1) \,|\, X_1 = x, Y_1 < x\right] \\ &= \int_0^x v(y, y)\, dK(y \,|\, x) \end{aligned}$$

and where for all $y < x$,

$$K(y \mid x) \equiv \frac{1}{G(x \mid x)} G(y \mid x) \tag{6.9}$$

and notice that $K(\cdot \mid x)$ is a distribution function with support $[0,x]$.

Now recall from Proposition 6.3 that

$$\beta^{\mathrm{I}}(x) = \int_0^x v(y,y)\, dL(y \mid x),$$

where $L(\cdot \mid x)$, defined in (6.7), is also a distribution function with support $[0,x]$.

We will argue that for all $y < x$, $K(y \mid x) \leq L(y \mid x)$ or, in other words, that $K(\cdot \mid x)$ stochastically dominates $L(\cdot \mid x)$. Since v is an increasing function of its arguments, this will complete the proof of the proposition.

To verify the stochastic dominance, note that because of affiliation, for all $t < x$, $G(\cdot \mid x)$ dominates $G(\cdot \mid t)$ in terms of the reverse hazard rate (see (6.1)), so for all $t < x$,

$$\frac{g(t \mid t)}{G(t \mid t)} \leq \frac{g(t \mid x)}{G(t \mid x)}$$

and thus for all $y < x$,

$$\begin{aligned}
-\int_y^x \frac{g(t \mid t)}{G(t \mid t)}\, dt &\geq -\int_y^x \frac{g(t \mid x)}{G(t \mid x)}\, dt \\
&= -\int_y^x \frac{d}{dt} (\ln G(t \mid x))\, dt \\
&= \ln G(y \mid x) - \ln G(x \mid x) \\
&= \ln \left(\frac{G(y \mid x)}{G(x \mid x)} \right)
\end{aligned}$$

Applying the exponential function to both sides, we obtain that for all $y < x$,

$$L(y \mid x) \geq K(y \mid x)$$

and this completes the proof. ■

The next example illustrates the workings of Proposition 6.5 by explicitly calculating the expected revenues in the second- and first-price auctions in the setting of Example 6.2.

Example 6.3. *As in Example 6.2, suppose that S_1, S_2, and T are uniformly and independently distributed on $[0,1]$. Bidder 1 receives the signal $X_1 = S_1 + T$, and bidder 2 receives the signal $X_2 = S_2 + T$. The object has a common value*

$$V = \frac{1}{2}(X_1 + X_2)$$

to both the bidders.

In this example, $v(x,y) = \frac{1}{2}(x+y)$, so in a second-price auction the equilibrium bidding strategy is

$$\beta^{\mathrm{II}}(x) = x$$

The expected revenue in a second-price auction is thus

$$\begin{aligned} E\left[R^{\mathrm{II}}\right] &= E\left[\min\{X_1, X_2\}\right] \\ &= E\left[\min\{S_1, S_2\}\right] + E[T] \\ &= \frac{5}{6} \end{aligned}$$

In a first-price auction, we know from Example 6.2 that the equilibrium bidding strategy is

$$\beta^{\mathrm{I}}(x) = \frac{2}{3}x$$

The expected revenue in a first-price auction is thus

$$\begin{aligned} E\left[R^{\mathrm{I}}\right] &= E\left[\max\left\{\frac{2}{3}X_1, \frac{2}{3}X_2\right\}\right] \\ &= \frac{2}{3}E\left[\max\{S_1, S_2\}\right] + \frac{2}{3}E[T] \\ &= \frac{7}{9} \end{aligned}$$

and so $E\left[R^{\mathrm{II}}\right] > E\left[R^{\mathrm{I}}\right]$. ▲

The conclusions of Propositions 6.4 and 6.5 are summarized as follows:

Proposition 6.6. *In the symmetric model with interdependent values and affiliated signals, the English, second-price, and first-price auctions can be ranked in terms of expected revenue as follows:*

$$E\left[R^{\mathrm{Eng}}\right] \geq E\left[R^{\mathrm{II}}\right] \geq E\left[R^{\mathrm{I}}\right]$$

EQUILIBRIUM BIDDING AND THE WINNER'S CURSE

At the beginning of the chapter we pointed out that a bidder in a first-price auction needs to shade his bid, relative to the estimated value based on his own signal alone, by a factor large enough to avoid the winner's curse. We now verify that the equilibrium bidding strategies in a first-price auction indeed have this feature. Observe that

$$\begin{aligned} \beta^{\mathrm{I}}(x) &= \int_0^x v(y,y)\, dL(y \mid x) \\ &\leq \int_0^x v(y,y)\, dK(y \mid x) \end{aligned}$$

$$\begin{aligned} &< \int_0^x v(x,y)\, dK(y\,|\,x) \\ &= E[V_1\,|\,X_1 = x, Y_1 < x], \end{aligned}$$

where $L(\cdot\,|\,x)$ is defined in (6.7), $K(\cdot\,|\,x)$ is defined in (6.9). The proof of Proposition 6.5 shows that $K(\cdot\,|\,x)$ stochastically dominates $L(\cdot\,|\,x)$, and this implies the first inequality. The second inequality follows from the fact that $v(\cdot,y)$ is increasing. Thus, we have shown that the equilibrium bid in a first-price auction is less than the expected value conditional on winning; such behavior shields bidders from the winner's curse.

The same is true in a sealed-bid second-price auction. The expected selling price upon winning is

$$\begin{aligned} E\left[\beta^{\mathrm{II}}(Y_1)\,|\,X_1 = x, Y_1 < x\right] &= \int_0^x v(y,y)\, dK(y\,|\,x) \\ &< E[V_1\,|\,X_1 = x, Y_1 < x] \end{aligned}$$

as noted earlier. Thus, equilibrium bidding strategies in a second-price auction are also immune to the winner's curse.

6.6 EFFICIENCY

In the context of the symmetric model used in this chapter, all three of the auction forms considered here—second-price, English, and first-price—have symmetric and increasing equilibria. This means that the winning bidder is the one with the highest *signal*. An auction allocates efficiently if the bidder with the highest *value* is awarded the object and the bidder with the highest signal need not be the one with the highest value.

Example 6.4. *Symmetric equilibria may be inefficient.*

For instance, if the valuations in a two-bidder symmetric situation are

$$\begin{aligned} v_1(x_1,x_2) &= \tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 \\ v_2(x_1,x_2) &= \tfrac{2}{3}x_1 + \tfrac{1}{3}x_2 \end{aligned}$$

then $v_1 > v_2$ if and only if $x_2 > x_1$. Thus, in this example, the bidder with the higher signal is the one with the *lower* value, so all three auction forms, almost always, allocate the object inefficiently. The reason is that each bidder's signal has a greater influence on the other bidder's valuation than it does on his own valuation. ▲

We will say that the valuations satisfy the *single crossing condition* if for all i and $j \neq i$ and for all $\mathbf{x}$,

$$\frac{\partial v_i}{\partial x_i}(\mathbf{x}) \geq \frac{\partial v_j}{\partial x_i}(\mathbf{x}) \tag{6.10}$$

The single crossing condition—so named because it implies that, keeping all other signals fixed, i's valuation as a function of i's signal x_i is steeper than j's valuation, so the two cross at most once—will play a significant role in later chapters.

Recall that in the symmetric model with interdependent values, the value to i is written as

$$v_i(\mathbf{x}) = u\left(x_i, \mathbf{x}_{-i}\right)$$

and it is assumed that the function u is symmetric in its last $N-1$ arguments. Let u_1' denote the partial derivative of u with respect to its first argument and, in general, let u_j' denote the partial derivative of u with respect to its jth argument. In the symmetric case, the single crossing condition reduces to requiring that for all $j \neq 1$, $u_1' \geq u_j'$, and since u is symmetric in the last $N-1$ arguments, it is enough to suppose that $u_1' > u_2'$.

The single crossing condition ensures that the *ex post* values of different bidders will be ordered in the same way as their signals. To see this, suppose that $x_i > x_j$, and define $\boldsymbol{\alpha}(t) = (1-t)\left(x_j, x_i, \mathbf{x}_{-ij}\right) + t\left(x_i, x_j, \mathbf{x}_{-ij}\right)$ to be the line joining the points $\left(x_j, x_i, \mathbf{x}_{-ij}\right)$ and $\left(x_i, x_j, \mathbf{x}_{-ij}\right)$. Using the fundamental theorem of calculus for line integrals, we can write

$$u\left(x_i, x_j, \mathbf{x}_{-ij}\right) = u\left(x_j, x_i, \mathbf{x}_{-ij}\right) + \int_0^1 \nabla u(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)\, dt$$

where

$$\begin{aligned} \nabla u(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) &= u_1'(\boldsymbol{\alpha}(t))\left(x_i - x_j\right) + u_2'(\boldsymbol{\alpha}(t))\left(x_j - x_i\right) \\ &\geq 0 \end{aligned}$$

since $x_i > x_j$ and $u_1' \geq u_2'$. So, $x_i > x_j$ implies that bidder i's value, $u\left(x_i, x_j, \mathbf{x}_{-ij}\right)$, is greater than or equal to bidder j's value, $u\left(x_j, x_i, \mathbf{x}_{-ij}\right)$.

Thus, we obtain the following:

Proposition 6.7. *With symmetric, interdependent values and affiliated signals, suppose the single crossing condition is satisfied. Then the second-price, English, and first-price auctions all have symmetric equilibria that are efficient.*

PROBLEMS

6.1. (Affiliation) Suppose there are two bidders who receive private signals X_1 and X_2 which are jointly distributed over the set

$$S = \left\{(x_1, x_2) \in [0,1]^2 : \sqrt{x_1} \leq x_2 \leq (x_1)^2\right\}$$

with a uniform density. The bidders attach a common value $V = \frac{1}{2}(X_1 + X_2)$ to the object.

a. Find symmetric equilibrium bidding strategies in both a first-price and a second-price auction.

b. Calculate the expected revenues from both auctions and show that the revenue in a second-price auction is greater than that in a first-price auction.

6.2. (Lack of affiliation) Suppose there are two bidders whose private values X_1 and X_2 are jointly distributed over the set

$$S = \left\{ (x_1, x_2) \in [0,1]^2 : x_1 + x_2 \leq 1 \right\}$$

with a uniform density. Show that a first-price auction with this information structure does *not* have a monotonic pure strategy equilibrium.

6.3. (Bidding gap) Consider a common value second-price auction with two bidders. The bidders receive signals X_1 and X_2, respectively, and these are independently and uniformly distributed on $[0,1]$. Each bidder's value for the object is $V = \frac{1}{2}(X_1 + X_2)$.

a. If the seller sets a reserve price $r > 0$, show that there is no bid. Explain why there is no bid in the neighborhood of the reserve price.

b. Find the optimal reserve price.

6.4. (Common value auction with Pareto distribution) Suppose that two bidders are competing in a first-price auction for an object with common value V, which is distributed according to a Pareto distribution $F_V(v) = 1 - v^{-2}$ over $[1, \infty)$. Prior to bidding, each bidder receives a signal X_i whose distribution, conditional on the realized common value $V = v$, is uniform over $[0, v]$; that is, $F_{X_i|V}(x_i \mid v) = (x_i / v)$. Conditional on $V = v$, the signals X_1 and X_2 are independently distributed. Verify that the following strategy constitutes a symmetric equilibrium of the first-price auction

$$\beta(x) = \tfrac{2}{3} \max\{x, 1\} \left(1 + \max\{x, 1\}^{-2}\right)$$

6.5. (Discrete values and signals) Consider a common value first-price auction with two bidders. The common value V can take on only two values, 0 and 1. Prior to the auction, each bidder i receives a signal X_i which can also take on only two values, 0 and 1. The joint distribution of the three random variables V, X_1 and X_2 is:

$$\Pr[V = v, X_1 = x_1, X_2 = x_2] = \begin{cases} 2/9 & \text{if } x_1 = x_2 = v \\ 1/18 & \text{if } x_1 = x_2 \neq v \\ 1/9 & \text{if } x_1 \neq x_2 \end{cases}$$

Show that the following strategy constitutes a symmetric equilibrium.

a. A bidder with signal $X_i = 0$ bids $E[V \mid X_1 = 0, X_2 = 0] = \frac{1}{5}$.

b. A bidder with signal $X_i = 1$ bids randomly over the interval $\left[\frac{1}{5}, \frac{8}{15}\right]$ according to the distribution

$$G(b) = \frac{4}{5}\left(\frac{5b-1}{4-5b}\right)$$

(Since the signals and values are discrete, the equilibrium is in mixed strategies.)

CHAPTER NOTES

Auctions for offshore oil drilling leases led to the development of the pure common value model, specifically the so-called "mineral rights" model. Moving beyond Vickrey's private value setting, early work on characterizing an equilibrium of first-price auctions in this alternative informational setting was carried out by Wilson (1969) and extended by Ortega-Reichert (1968) (Wilson's article was available as a working paper in 1966). Wilson (1977) obtained an explicit expression for the equilibrium bidding strategy in a first-price auction in the same model. The presence of the winner's curse as an empirical fact was pointed out by Capen, Clapp, and Campbell (1971) in the context of bidding for offshore oil drilling leases.

The symmetric equilibrium of a second-price auction in a common value setting was derived in a paper by Milgrom (1981).

The symmetric model with interdependent values and affiliated signals—encompassing both the common value and private values models—was introduced by Milgrom and Weber (1982). This paper, now a classic, generalized all known equilibrium characterizations to date for the first-price, second-price, and English auctions and derived the general revenue rankings among the three formats. The main results of this chapter are almost entirely based on this paper.

Avery (1998) has studied an alternative model of the English auction in which bidders can call out prices. The equilibrium described in Proposition 6.2 survives in such a model, but there are other equilibria that involve "jump bidding," in which bidders sometime raise the price by a discrete amount. Avery (1998) shows, however, that the revenue to the seller in such equilibria cannot exceed the revenue from the equilibrium derived in Proposition 6.2.

Example 6.1 belongs to a special class of situations, studied by Harstad and Levin (1985), in which the equilibrium strategy in a second-price auction can be obtained by means of iterative elimination of weakly dominated strategies.