MATH 425b ASSIGNMENT 5
SPRING 2016
Prof. Alexander
Due Friday March 4.

(I) The *conorm* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$\mathfrak{m}(T) = \inf \left\{ \frac{|T\mathbf{x}|}{|\mathbf{x}|} : x \neq 0 \right\}.$$

Let U be the closed unit ball in $\mathbb{R}^n$.

(a) Suppose $n = m$. Show that the norm of T is the radius of the smallest ball that contains $T(U)$.

(b) Suppose $n = m$. Show that the conorm of T is the radii of the largest ball contained in $T(U)$.

(c) Show that if $T : \mathbb{R}^n \to \mathbb{R}^n$ has conorm $\mathfrak{m}(T) > 0$, then T is invertible.

(II)(a) Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear. Show that T is uniformly continuous.

(b) In contrast, let (a, b) be an interval and define $T : \mathcal{C}'(a, b) \to C(a, b)$ by $T(f) = f'$. (Here $\mathcal{C}'(a, b)$ is the space of differentiable functions on (a, b).) Show that T is linear but not continuous. Here the distance in both spaces is given by the sup norm.

(III) Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -6 \end{bmatrix}$.

(a) Find $\|A\|$, and give the vector $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ which maximizes $|A\mathbf{u}|$ subject to $|\mathbf{u}| = 1$.

(b) What vector(s) $\mathbf{u}$ maximize $|A\mathbf{u}|/|\mathbf{u}|$ over all $\mathbf{u} \neq 0$?

(c) It can be shown that $A^{-1} = \frac{1}{15}\begin{bmatrix} 6 & 3 \\ 1 & -2 \end{bmatrix}$, and $\|A^{-1}\| = 1/\sqrt{5}$ (you need not do this!) Does $\|A^{-1}\| = 1/\|A\|$?

(IV) Let A be an $n \times n$ matrix. (a) Show that

$$\sup_{x,y \neq 0} \frac{|Ax \cdot y|}{|x|\ |y|} \leq ||A||.$$

Here $Ax \cdot y$ is the dot product.

(b) Prove the reverse inequality:

$$\sup_{x,y \neq 0} \frac{|Ax \cdot y|}{|x|\ |y|} \geq ||A||.$$

(V) Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and the only solution of $T\mathbf{x} = 0$ is $\mathbf{x} = 0$. Show that T is one-to-one.

HINTS:

(I)(a) Show that $T(U) \subset \overline{B}(0, ||T||)$. Then show that if $r < \|T\|$, there exists $z \in U$ with $Tz \notin \overline{B}(0, r)$. Mostly this is an exercise in using the definition of $\|T\|$.

(b) First, why is T a bijection when $\mathfrak{m}(T) > 0$? The rest of the proof is rather tricky, but give it a try.

(c) If T is not invertible, what do you know about solutions of $Tx = 0$?

(II)(b) Continuity would mean $f_n \to f$ implies $T(f_n) \to T(f)$.

(III)(a) You want to maximize $|A\mathbf{u}|^2$ subject to $|\mathbf{u}|^2 = 1$. Write this in terms of x, y and substitute for one of the variables, so you're just maximizing over x or over y. The calculations should not be very messy.

(IV)(b) Compare the sup to a particular choice of y.

(V) This should be very short.