# 1 Discrete-Time Dynamic Optimization

Acemoglu, D. (2008). Chapter 6, Introduction to Modern Economic Growth. Princeton University Press.

## 1.1 Equivalence Results

### 1.1.1 Problem 6.2

$$V^*(x(0)) = \underset{\{x(t)\}_{t=0}^{\infty}}{sup} \sum_{t=0}^{\infty} \beta^t U(x(t), x(t+1))$$

subject to

$$x(t+1) \in G(x(t)) \quad \forall \quad t \geq 0,$$

$$x(0) \quad given.$$

### 1.1.2 Problem 6.3

$$V(x) = \underset{y \in G(x)}{sup} \{U(x, y) + \beta V(y)\}, \quad \forall x \in X.$$

The basic idea of dynamic programming is to turn the sequence problem into a functional equation; that is, to transform the problem into one of finding a function rather than a sequence.

**Assumption 1.1.** *$G(x)$ is nonempty for all $x \in X$; and for all $x(0) \in X$ and $x \in \Phi(x(0))$, $Lim_{n\to\infty} \sum_{t=0}^{\infty} \beta^t U(x(t), x(t+1))$ exists and is finite.*

**Theorem 1.1.** *Equivalence of Values*

*Suppose Assumption 1.2.1 holds. Then for any $x \in X$, any solution $V^*(x)$ to Problem 6.2 is also a solution to Problem 6.3. Moreover, any solution $V(x)$ to Problem 6.3 is also a solution to Problem 6.2, so that $V^* = V(x)$ for all $x \in X$.*

**Theorem 1.2.** *Principle of Optimality*

*Suppose Assumption 1.2.1 holds. Let $x^* \in \Phi(x(0))$ be a feasible plan that attains $V^*(x(0))$ in Problem 6.2. Then*

$$V^*(x^*(t)) = U(x^*(t), x^*(t+1)) + \beta V^*(x^*(t+1))$$

*for* $t = 0, 1, \ldots,$ *with* $x^*(0) = x(0)$.

*Moreover, if any* $x^* \in \Phi(x(0))$ *satisfies the equation above, then it attains the optimal value in Problem 6.2.*

**Assumption 1.2.** *$X$ is a compact subset of $R^K$, $G$ is nonempty-valued, compact-valued, and continuous. Moreover, $U : X_G \to R$ is continuous.*

**Theorem 1.3.** *Existence and Uniqueness of Solutions*

*Suppose that Assumption 1.2.1 and 1.2.2 hold. Then there exists a* ***unique*** *continuous and bounded function $V : X \to R$ that satisfies equation in Problem 6.3. Moreover, for any $x(0) \in X$, an optimal plan $x^* \in \Phi(x(0))$ exists.*

# 2 Deriving the Euler Equation with Lagrangian

When it is hard to substitute the constraint into the objecctive function or in complex models with many states and controls, it is more convinient to apply the Lagrangian method in the recursive setting. For example, when $c, x, y$ are jointly determined by a complex function, it would then be hard to write the function form of U explicitly as x and $y$.

$$V(x) = \max_{c,y} [U(c) + \beta V(y)]$$

s.t.

$$g(c, x, y) = 0$$

Thus,

$$L = U(c) + \beta V(y) + \lambda g(c, x, y)$$

F.O.C Equation: (Control Variables)

$$c : \quad \frac{\partial L}{\partial c} = U'(c) + \lambda g_1(c, x, y) = 0$$

$$y : \quad \frac{\partial L}{\partial y} = \beta V'(y) + \lambda g_3(c, x, y) = 0$$

Envelope Equation: (State Variables)

$$x : \quad V'(x) = \frac{\partial L}{\partial x} = \lambda g_2(c, x, y)$$

Thus,

$$V'(y) = \frac{1}{\beta}\frac{U'(c)}{g_1(c,x,y)}g_3(c,x,y)$$

$$V'(x) = -\frac{U'(c)}{g_1(c,x,y)}g_2(c,x,y)$$

Advance it $V'(x)$ by one period, we can have another expression of $V'(y)$.

This approach is equilivalent to using the Implicit Function Theorem.

$$g(c,x,y) = 0$$

Thus,

$$\frac{dc}{dy} = -\frac{g_3(c,x,y)}{g_1(c,x,y)}$$
$$\frac{dc}{dx} = -\frac{g_2(c,x,y)}{g_1(c,x,y)}$$

F.O.C Equation: (Control Variables)

$$\frac{\partial U(x,y)}{\partial y} + \beta V'(y) = U'(c)\frac{dc}{dy} + \beta V'(y) = 0$$

Thus,

$$V'(y) = \frac{1}{\beta}\frac{U'(c)}{g_1(c,x,y)}g_3(c,x,y)$$

Envelope Equation: (State Variables)

$$V'(x) = \frac{\partial U(x,y)}{\partial x} = U'(c)\frac{dc}{dx}$$

Thus,

$$V'(x) = -\frac{U'(c)}{g_1(c,x,y)}g_2(c,x,y)$$