MATH 425b MIDTERM EXAM 2
April 1, 2016
Prof. Alexander

Problem	Points	Score
1	16	
2	27	
3	29	
4	28	
Total	100	

Last Name: ______________________________

First Name: ______________________________

USC ID: ______________________________

Signature: ______________________________

Notes:

(1) Use the backs of the sheets if you need more room.

(2) If you can't do part (a) of a problem, you can assume it and do part (b).

(3) Problems (3b) and (4b) marked with $*$ are probably a bit harder than the others.

(4) Recall that the inverse of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(1)(16 points) Suppose X is a complete metric space and $\varphi : X \to X$ is "almost a contraction", satisfying $d(\varphi(y), \varphi(x)) \leq d(y, x)$ for all x, y. Let $x_0 \in X$ and define inductively $x_{n+1} = \varphi(x_n), n \geq 1$. Suppose some subsequence $\{x_{n_k}\}$ converges to a fixed point x^* of φ. Show that the full sequence $\{x_n\}$ converges to x^*.

HINT: What properties does the sequence $\{d(x_n, x^*)\}$ have, thanks to the "almost contraction" property and the inductive definition?

(2)(27 points) Consider the system of equations

$$f_1(x, y, z) = x^2 + y^2 + z^2 - 4 = 0$$

$$f_2(x, y, z) = (x - 2)^2 + y^2 + z^2 - 4 = 0,$$

which are satisfied at $(x_0, y_0, z_0) = (1, \sqrt{2}, 1)$.

(a) Show that there is a neighborhood of (x_0, y_0, z_0) in which we can solve these equations for (x, y), that is, we can express $(x, y) = u(z)$. Also, find $u'(z_0)$.

(b) Show that there is no neighborhood of (x_0, y_0, z_0) in which we can solve these equations for (y, z), that is, no way to express $(y, z) = v(x)$. Explain why this doesn't contradict the Implicit Function Theorem. HINT: You can't get this from the theorem used for (a). Instead, if (x, y, z) satisfies the two equations, what can you conclude about x? It may help to think geometrically about the two equations.

(3)(29 points) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are each differentiable at $\mathbf{x}$, with $f'(\mathbf{x}) = A_f, g'(\mathbf{x}) = A_g$. Prove the following directly from the definition 9.11 of derivative (that is, not using the description in terms of partial derivatives as matrix entries):

(a) $(f + g)'(\mathbf{x}) = A_f + A_g$

(b*) For the dot product, $(f \cdot g)'(\mathbf{x}) = f(\mathbf{x})A_g + g(\mathbf{x})A_f$. Note the products on the right are matrix products: $f(\mathbf{x}), g(\mathbf{x})$ are $1 \times m$ and A_f, A_g are $m \times n$.

HINT: Note that for a column vector $\mathbf{h}$, the matrix product $f(\mathbf{x})A_g\mathbf{h}$ is the same as the dot product $f(\mathbf{x}) \cdot A_g\mathbf{h}$ or $A_g\mathbf{h} \cdot f(\mathbf{x})$, since $f(\mathbf{x})$ is a row matrix and $A_g\mathbf{h}$ is a column.

(4)(28 points)(a) Let A be an $n \times n$ matrix, and recall that $\|A\| = \sup\{|Ax| : |x| = 1\}$. Show that there is an x where this sup is achieved.

(b*) Let D be an $n \times n$ diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Express $\|D\|$ in terms of $\lambda_1, \ldots, \lambda_n$. (With proof!)

HINT: You don't need calculus techniques for maximimizing here. Just think about how you would choose x to maximize $|Dx|^2$ given that $|x|^2 = 1$, and determine what the resulting maximum is.