Note 4 with HW4

Bayesain Equilibrium with uncertainty about the types of players

In Bayesian equilibrium, each player has a payoff matrix to consider.
The strategy of the other player looks like a mixed strategy. Computing the payoff is the same as computing the payoff when the other player adopts a mixed strategy. The optimal action is an optimal response to the mixed strategy. The distribution of the mixed strategy is determined by the prior distribution of the other player types, and the strategy chosen by each type.

An example: A simultaneous game. GF with the low quality dairy firm, the payoff matrix is

$$\begin{array}{ccc} & & h & l \\ diary & h & (1,1) & (1,3) \\ & l & (5,1) & (0,0) \end{array} = (u_1(a_{1h}, a_2), u_2(a_{1h}, a_2)),$$

with high quality dairy firm, the payoff matrix is

$$\begin{array}{ccc} & & h & l \\ diary & h & (1,1) & (1,3) \\ & l & (5,1) & (5,3) \end{array} = (u_1(a_{1l}, a_2), u_2(a_{1l}, a_2)).$$

The belief is that both types are equally likely.

Equilibrium with complete information: with high quaility dairy firm, the only equilibrium is (l,l). With low quality firm, there are two pure strategy equilibrium, one mixed strategy equilibrium $(\frac{1}{3}, \frac{1}{5})$, the probability is that of using the high test.
Checking whether there is Bayesian equilibrium with GF using the pure strategy h: each type of diary firm chooses l, as an optimal response. The payoff of GF choosing h is 1, the payoff of GF deviating to the strategy l is 1.5, which is higher than choosing h. Hence there is no Bayesian equilibrium in which GF chooses the pure strategy h.
Checking whether there is Bayesian equilibrium with GF using the pure strategy l: low quality dairy firm chooses h, while high quality firm chooses l. The payoff of GF choosing l is 3. The payoff of GF deviating to the strategy h (but assuming no change in the dairy firm strategy) is 1. Hence GF choosing l, while dl choosing h, dh choosing l is a Bayesian Nash equilibrium.
Here the idea is that the payoff is computed from the belief about the types of players you are dealing with. You also need to take into account that different firm may have different optimal response to your strategy. For any strategy a_2 GF chooses, each dairy firm responds optimally to a_2, and may choose the same or different optimal response. The strategy a_2 is optimal if any other response a_2' (but no change in dairy firm strategy) yields no better payoff.

Formally, a pure strategy Bayesian equilibrium is one in which there is a strategy a_2^* for the GF, and one strategy a_{1h}^* for high quality firm, one a_{1l}^* for low quality firm. The choice of a_{1h}^* should be optimal for the high quality firm

$$u_1(a_{1h}^*, a_2^*) \geq u_1(a_{1h}, a_2^*) \text{ for all } a_{1h}.$$

Similarly, we must have a_{1l}^* optimal for low quality firm:

$$u_1(a_{1l}^*, a_2^*) \geq u_1(a_{1l}, a_2^*) \text{ for all } a_{1l}.$$

With the initial belief about the likelihood of the dairy firms $(\pi_h, 1-\pi_h)$, GF choice of a_2^* should also be optimal under the belief and given the choice of a_{1h}^*, a_{1l}^* :

$$\pi_h u_2(a_{1h}^*, a_2^*) + (1-\pi_h) u_2(a_{1l}^*, a_2^*) \geq \pi_h u_2(a_{1h}^*, a_2) + (1-\pi_h) u_2(a_{1l}^*, a_2) \text{ for all } a_2.$$

The idea applies to mixed strategies as well. Let p_h be the probability of high test for low quality dairy firm. Note that the high quality firm will always use the pure strategy l because that is the dominant strategy. Let q be the probability of high test for GF. A mixed strategy Bayesian Nash equilibrium of the above example, $(l, p_l^*), q^*, 0 < p_l^* < 1, 0 < q^* < 1,$ should satisfy the following properties

$$q^* u_1(h,h) + (1-q^*) u_1(h,l) = q^* u_1(l,h) + (1-q^*) u_1(l,l)$$

for the low quality dairy firm, and

$$\pi_h u_2(l,h) + (1-\pi_h)[p_h^* u_2(h,h) + (1-p_h^*) u_2(l,h)] = \pi_h u_2(l,l) + (1-\pi_h)[p_h^* u_2(h,l) + (1-p_h^*) u_2(l,l)]$$

It can be checked that there is no Nash equilibrium of the example. The reason is because the payoff from dealing with the high quality firm will dominate the payoff so that GF will use only pure strategy in equilibrium.

However, if we change the belief to $\pi_h = 0.1,$ the presence of high quality firm is unlikely, and we can have a mixed strategy Bayesian Nash equilibrium (in homework).

Perfect Bayesian Equilibrium:

Bayesain Equilibrium with updating of beliefs

An Example with Uncertainty

We will continue with the diary firm vs. GE example above. We still maintain the assumption that General Foods knows which test has been used by the diary firm. However, there is uncertainty about what kind of diary firm GE is facing.

Suppose that there is a diary firm dh whose quality is so good that there is no need to conduct the high test, and low test will be used by both all the time. This is reflected by the payoff vectors (1,1),(5,1),(1,3),(5,3) corresponding to (h,h),(l,h),h,l),(l,l) test strategies (by convention, we always list diary first then General Food in the vectors). We assume that the diary firm dh has no other market opportunities, and will always sell to General Foods. When it does, both will choose the low test.

Suppose that there is another diary firm dl which has the same strategic interaction with General Foods as the certainty example earlier with the payoffs (1,1),(5,1),(1,3),(0,0) for (h,h),(l,h),(h,l),(l,l) respectively. However the diary firm dl can make profit $5 processing its own milk.

If General Foods knows which firm it is dealing with, the response is simple. When dh sells its product, General Foods and dh will choose low test and earns profit $3, $5 respectively. but if it is dl, then when it sells to GF, it chooses low test and earns profit $5, which is the same as making its own milk product without selling to GF. Hence the equilibrium is for the dairy firm to sell to FG with some probability, and GF chooses high test in equilibrium.

However, GF does not know which type of diary firm it faces. It has a belief. We assume that the initial belief is equally likely for dh, dl. In other words, each occurs with probability 0.5.

We claim that it is a perfect Bayesian equilibrium for General Foods to choose low test always, and only dh firm sells to General Foods and chooses low test as well. The diary firm dl processes the milk itself. When dl sells to GF (out-of-equilibrium strategy), it will choose high test (because of low test choice of GF). GF out–of-equilibrium belief of the diary firm making high test choice is that it is dealing with dl, and low test is an optimal response in this case.

Checking the equilibrium conditions: 1. Verify that GE strategy of ll (low test always) is optimal, subject to updated beliefs; 2. verify that dh strategy is optimal; 3. verify that dl strategy is optimal

In equilibrium when the milk is sold to General Foods, it must be the dh firm. The two diary firms are distinguished by their different strategies, and reveal themselves in the market place by its actions. By choosing to sell, it reveals that it is a high quality firm.

The updated information is complete information in equilibrium. The equilibrium expected profit for General Foods is 0.5*3=1.5.

We call this a separating equilibrium.

Suppose that we change the initial belief, and make the low quality firm very likely. For instance, let the probability of dl be 0.9, and the probability of dh is 0.1. The diary firm dl will always choose low test if it anticipates high test by General Foods, but it is indifferent between selling or processing it on its own in this case because the profit is $5 either way. We assume that when there is indifference, the diary firm chooses randomly between selling or processing on its own, thus each with probability 0.5.

We now show that it is a perfect Bayesian equilibrium for General Foods to choose high test. The dh firm always sells and chooses low test, and the dl firm sells with probability 0.5 and chooses low test.

To see why this is a perfect Bayesian equilibrium, note that when milk is sold to General Foods, it could be from dh or dl. Both diary firms choose low test.

The updated belief of General Foods is that with probability 0.1 it is from dh, and with probability 0.5*0.9 it is from dl. The total probability is 0.1+0.45=0.55.

The conditional probability that it is from dh is 0.1/0.55=2/11. The conditional probability that it is from dl is 0.45/0.55=9/11.

Based on this updated belief, conditional on buying the milk from either of the firm, if General Foods chooses low test the expected profit is 3*2/11=6/11. The expected profit from choosing the high test is 1. Therefore high test is the optimal strategy based on the updated belief.

The equilibrium profit for General Foods is 0.55. This is also the probability of buying milk from either firm multiplied by the profit $1. This is a pooling equilibrium in which GE cannot distinguish whether it is dealing with dh or dl diary firm.

Unobserved Actions

Now back to the game we discussed first in which: The diary can sell the dry milk to General Foods, but it chooses whether to test high or low. If it tests high, and General Foods tests high, the payoff is (1,1). If it tests high, but General Foods tests low, the payoff is (1,3), reflecting the cost saving of the low test. If it tests low, and General Foods tests high, the payoff is (5,1), reflecting cost saving and higher revenue potential of selling it to General Foods. It it tests low, and General Foods also tests low, the ensuing tort liability reduces their profits to (0,0). The payoffs are (3,0) when the diary firm uses its own milk for production.

We now assume that the strategy of the diary firm is not observable by General Foods. (fig 3.2 in the text book). Since the choice of strategy is part of the equilibrium concept, the belief regarding the choice of actions is part of the equilibrium, and has to be specified in a perfect Bayesian equilibrium. The belief should also be consistent with the optimal strategies (mixed strategies in this case) of the players (sequential rationality).

We now specify an equilibrium: The diary firm processes the milk itself. GF believes that with probability q=0.4 the diary firm tests high when it sells milk to GF (off-equilibrium belief). With this belief, GF tests low.

	h	l
h	(1,1)	(1,3)
l	(5,1)	(0,0)

The belief regarding the strategy choice of the diary firm can be any belief that is consistent with sequential rationality. For example, if we change the belief to q=0.45 or 0.35 or any q higher than 1/3, it is still a perfect Bayesian equilibrium. In equilibrium, the milk is not sold to GF, and any belief is not contradicted. If we change the belief to q=0.3, then it is no longer an equilibrium because GF finds it better to test high instead.

There is another perfect Bayesian equilibrium of the game: The diary firm sells the milk to GF. GF believes that the diary firm tests low when it sells milk to GF. With this belief, GF tests high. This is the only efficient equilibrium.

Equilibrium refinement: adding restrictions to out-equilibrium beliefs to eliminate certain perfect Bayesian equilibrium. For the above example, in equilibrium the diary firm does not sell to GE. The out-of-equilibrium belief (probability 0.4 that the diary firm has high test) is unjustified because high test can only lead to profit $1 which is dominated by $3. Therefore, GE should not have the belief that when the diary firm does sell to it, high test will not be used with positive probability.

If we require that no dominated strategy should be used. Then the only perfect Bayesian equilibrium is: The diary firm sells the milk to GF. GF believes that the diary firm tests low when it sells milk to GF. GF tests high. This is called a refinement of equilibrium. With this refinement the equilibrium is unique.

If the own processing option for the diary firm is $0.5, while the other data and assumptions remain the same, then the refinement no longer works to narrow the equilibria to the efficient one.

In fact, the following is a perfect Bayesian Nash equilibrium: the diary firm sells to GF, and chooses high test with probability 1/3. GF buys milk from the diary firm, and chooses high test with probability 1/5. In equilibrium, both firms make expected profit $1.

This is not an efficient equilibrium as the expected profit only adds up to $2, while the efficient one has total profit $6.

To eliminate the inefficient equilibria, the two firms can contract with each other, exchange information, and require the diary firm to test low, while GF tests high. To compensate GF for its lower profit, the contract term can specify a payment of $2 to GF. In this equilibrium with contract, the diary firm makes $3, while GF makes $3. Both firms will be happy with the arrangement.

A model of verifiable information with no social benefit

First we look at an example in which the information has no social benefit. However, players may have incentive to incur costs to acquire the information.

The seller wants to sell a house. The value of the house is affected by the condition of the furnace which may need to be replaced. A house that needs a new furnace is worth $180. A house that does not is worth $200. There is no social benefit from knowing the information.

Whether a house needs a new furnace depends on its age and how heavily it was used in the past. The seller can gather this information and convey it to the buyer by tracking down the original sales receipt and heating bills incurred since then.

We assume that the seller does not know whether a new furnace is needed without acquiring the necessary information. Only the seller has the ability to acquire this information at some cost. With no acquisition of information, both know only that it is equally likely whether a new furnace is needed.

Three legal regimes will be studied. Under different legal regimes, we may have different types of (perfect Bayesian) equilibrium, and different incentives to acquire information. (1) voluntary regime (2) mandatory disclosure (2) mandatory acquisition of information and disclosure.

Under regime (1), we have voluntary acquisition of information and voluntary disclosure of information after acquisition.

Assume that the cost of acquiring information is under $10. We claim that it is a perfect Bayesian equilibrium for the seller to acquire information. Unraveling holds, and there is complete revelation of information in this equilibrium.

If a sellers finds out there is no need for a new furnace, this information will be revealed, and he gets $200 for the house. If the seller finds out there is a need for a new furnace, he will remain silent, but will only get $180 for the house because of unraveling.

Before learning the condition of the furnace, it expects to get $190 (because good or bad furnace is equally likely) The unraveling result also helps us to justify some rules that forbid disclosure of information.

The value of information in this situation is $10 which is the difference of $190 (when informed) and $180 (when uninformed, and paid as if there is a bad furnace).

Since the cost of information is less than its value $10, it is optimal for the seller to acquire information.

Note that if the seller knows the condition of the furnace, but need to acquire (collect) information to convince the buyer, then the value of acquisition is $20 instead. This is because only the good furnace seller wants to collect information. Once he collects the information to show it is a good furnace, it is worth 200. If he does not collect information, it will be regarded as bad, and the price is then $180. The difference is $20. If the cost of collecting information is less than $20, then the seller with a good furnace will collect information. The seller with a bad furnace will not collect information.

Another equilibrium to consider is one in which no seller acquires information. When the cost of acquisition exceeds $5, we claim that it is an equilibrium. In this case, no one has information, and the seller expects to receive $190. If a seller deviates by acquiring information, he receives expected price $195. But the cost of acquisition exceeds $5, therefore, it is not optimal to deviate.

With no acquisition he already receives $190. This is better than acquiring information.

A third type of equilibrium exists when there are sellers with different costs of acquiring information. Some acquire information, and others don’t. The buyer does not know what kinds of sellers she is dealing with. Furthermore, the buyer does not observe whether a seller acquires information or not. The buyer only knows the reported outcome (or silence) from the seller.

In this kind of equilibrium, a seller may become silent because he does not acquire information, and hence no information to report. A seller may also becomes silent when he acquires information, and find out it is bad information.

Updating of belief is important in this context. Assume that half of the sellers have low costs ($4) and half of the sellers have high cost ($8) of acquisition.

If only the low cost sellers acquire information, and the high cost sellers don’t, then with probability ¼, the furnace is good, and this information is revealed to the buyer.

The remaining ¾ remain silent. This is because with probability ¼ , the seller acquire information but finds bad furnace, and remain silent. With probability ½, the seller remains silent because there is no information to report.

Conditional on receiving no report from the seller, the buyer’s belief that the furnace is good with probability 1/3, and bad with probability 2/3. (The prior belief before updating is that it equally likely). The price the buyer is willing to pay is 180*2/3 +200*1/3= 186.67.

Before acquiring information, a seller expects to receive $200 or $186.67 with equal probability. This amount is equal to 200*1/2 +186.67*1/2 = 193.33. The difference of $193.33 (with information) and $186.67 (no information) is $6.67 (the value of information).

For the low cost sellers, the value of information ($6.67) is bigger than the cost of information ($4), Hence it is optimal for the low cost sellers to acquire information.

For the high cost sellers, the value of information is smaller than the cost of information ($8). Hence it is optimal for the high cost sellers not to acquire information.

This is also the assumption we take to compute the numbers. The assumption is consistent with equilibrium behavior.

Therefore it is a perfect Bayesian equilibrium for high cost sellers not to acquire information, and low cost sellers to acquire information. When the sellers find good information, he reports it. When the information is bad, he remains silent as does when he does not acquire information.

The buyer believes that it is good furnace when it is reported (verifiable information), and is willing to pay $200. When the seller remains silent, the buyer is willing to pay $186.67 (with the updated belief that the furnace is bad with probability 2/3).

Homework Assignment 4

Due Feb 8

1. In the General Foods vs Valley Lea, assume that the choice of tests by the diary firm is observable by General Foods, but the identity of the high or low quality firm is not known. There two kinds of diary firm as described in the note above. The initial belief of General Foods is that both high or low quality firms are equally likely. We want to know whether it is a perfect Bayesian equilibrium for General Foods to use the high test when the milk is sold by the diary firm. The assumptions and payoff structures are the same as in the note. The only difference is the initial belief.

(a) What is the optimal strategy of dh or dl when it knows that General Food uses high test?

(b) When the diary firm offers to sell to General Foods, what is the updated belief about the probability of high quality or low quality diary firms?

(c) What is the optimal strategy of General Foods when it updates the information?

(d) Explain intuitively why the high test choice by General Foods is not optimal.

2. In the General Foods vs Valley Lea, assume that the choice of tests by the diary firm is observable by General Foods, but the identity of the high or low quality firm (dh, dl) is not known. The initial belief of General Foods is that the probability of dh is p, and the probability of dl is $1-p$. Assume that $p < 0.2$. Show that it is a perfect Bayesian equilibrium for General Foods to choose high test. The dh firm always sells, and the dl firm sells with probability 0.5 and chooses low test.

3. In the General Foods vs Valley Lea, assume that the choice of tests by the diary firm is unobservable by General Foods. In this model, there is only one kind of diary firm. The own processing option for the diary firm yields profit \$10. Assume that we have instead the following payoff matrix from the combination of test strategies: row is for the diary firm

	low	high
low	(0,0)	(15,2)
high	(2,10)	(2,2)

(a) Determine whether it is a perfect Bayesian equilibrium for the diary firm to process its own milk. Assume that the out-of-equilibrium belief of GF is that the diary firm chooses high test with probability 0.3. What is the corresponding GF choice of strategy under this belief system?

(b) Can the refinement idea be used to determine the unique equilibrium? Explain the idea.

4. Under regime (1), we have voluntary acquisition of information and voluntary disclosure of information after acquisition. The information has no social value, and is verifiable. Assume that the cost of acquiring information is \$8 for high cost sellers, and \$4 for low cost sellers. The probability of a low cost seller is p (and that of a high cost seller is 1-p). The house is worth \$200 when it has a good furnace, and \$180 when it has a bad furnace. A seller does not know whether the furnace is good (neither does the buyer). The probability of finding a good furnace when the seller incurs the cost and acquires information is q. After the acquisition of information, the seller can either report the information or hide it from the buyer pretending that he has no information. This acquisition of information is not observable by the buyer. Assume that the buyer believes that only low cost sellers acquire information.

(a) When the seller acquires information and finds a bad furnace, it is better for him to hide the information and pretends that there is no information available. Explain why.

(b) What is the updated belief regarding the condition of the furnace when there is no information provided by the seller?

(c) What is the value of information?

5. In this problem, we want to illustrate how the value of information may change when the buyer has a different belief. Assume instead that the buyer believes that all sellers acquire information, other things remain the same as in the problem 1 above.

(a) When the seller acquires information and finds a bad furnace, it is not better for him to hide the information and pretends that there is no information available. Explain why.

(b) What is the updated belief regarding the condition of the furnace when there is no information provided by the seller?

(c) What is the value of information?

6. In the simultaneous game in example B above, assume that the probability of the high quality firm is not very likely, $\pi_h = 0.1$. We are interested in the possibility of a mixed strategy Bayesian Nash equilibrium of the model.

(a) Write down the two equations that need to be satisfied for the mixed strategy equilibrium.

(b) Solve the two equations in (a). Is there a solution? Give the answer if there is one.

(c) Compare the equilibrium in (b) to the mixed strategy equilibrium of the complete information case when GF knows that the dairy firm is low quality firm. Give an interpretion of your results.