MATH 425b ASSIGNMENT 2 SOLUTIONS
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Chapter 7

(18) We claim that $\{F_n\}$ is equicontinuous. Proof: By assumption there exists $M < \infty$ such that $|f_n(x)| \leq M$ for all x and n. Then for $x < y$ in $[a, b]$,

$$|F_n(y) - F_n(x)| = \left| \int_x^y f_n(t)\ dt \right| \leq \int_x^y |f_n(t)|\ dt \leq \int_x^y M\ dt = M|y - x|.$$

Thus given $\epsilon > 0$, we have $|y - x| < \epsilon/M$ implies $|F_n(y) - F_n(x)| < \epsilon$, which proves the claim. Then by Theorem 7.25, $\{F_n\}$ has a uniformly converging subsequence.

(I) Suppose $\mathcal{F}$ is equicontinuous and $f \in \overline{\mathcal{F}}$. Then there exists a sequence $\{f_n\} \subset \mathcal{F}$ with $f_n \to f$ uniformly. Let $\epsilon > 0$. By equicontinuity, there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies |f_n(y) - f_n(x)| < \epsilon \text{ for all } n \geq 1.$$

Then

$$d(x, y) < \delta \implies |f(y) - f(x)| = \lim_n |f_n(y) - f_n(x)| \leq \epsilon,$$

so the same δ "works" for f. Thus this δ "works" for all $f \in \overline{\mathcal{F}}$, so $\overline{\mathcal{F}}$ is equicontinuous.

(II)(a) Given $\epsilon > 0$ let $\delta = \epsilon^{1/\alpha}$. Then

$$f \in E, |y - x| < \delta \implies |f(y) - f(x)| \leq |y - x|^\alpha < \delta^\alpha = \epsilon,$$

i.e. this δ "works" uniformly over E. This shows E is equicontinuous.

(b) Suppose $f_n \in E$ for all n and $f_n \to f$ uniformly. Then $f(0) = \lim_n f_n(0) = 0$ and for all $x, y \in [0, 1]$,

$$|f(x) - f(y)| = \lim_n |f_n(x) - f_n(y)| \leq |y - x|^\alpha,$$

so $f \in E$.

(III)(a) Suppose $P(x) = \sum_{n=0}^N a_n x^n$, and let $\epsilon > 0$. For each $n \leq N$ there exists a rational q_n with $|a_n - q_n| < \epsilon/NA^n$. Let $Q(x) = \sum_{n=0}^N q_n x^n$. Then

$$|P(x) - Q(x)| \leq \sum_{n=0}^N |a_n - q_n||x|^n \leq \sum_{n=0}^N \frac{\epsilon}{NA^n} A^n = \epsilon$$

for all $x \in [0, A]$, so $||P - Q||_\infty < \epsilon$.

(b) Let D be the set of all polynomial functions on $[0, A]$ with rational coefficients. Let $\epsilon > 0$. By the Weierstrass theorem, given $f \in C[0, A]$ there is a polynomial P with $||f - P||_\infty < \epsilon/2$. By (a) there is a polynomial $Q \in E$ with $||P - Q||_\infty < \epsilon/2$. Then $||f - Q||_\infty < \epsilon$, which shows D is dense in $C[0, A]$. There is a one-to-one correspondence between D and the countable set $\mathbb{Q}^{N+1}$ (where $\mathbb{Q}$ = rationals), by pairing $\sum_{n=0}^{N} q_n x^n$ with $(q_0, .., q_N)$, so D is countable.

(IV) Let $\epsilon > 0$. We want to show that no δ "works" for this ϵ, so let $\delta > 0$. Since E has a limit point p, there exist infinitely points of E within distance $\delta/2$ of p; taking any 2 of these points we get $x, y \in E$ with $d(x, y) < \delta$. Since $\mathcal{A}$ separates points, there exists $f \in \mathcal{A}$ with $f(x) - f(y) = c \neq 0$. Let $b > \epsilon/|c|$. Then $bf \in \mathcal{A}$, and we have $d(x, y) < \delta$ but $|bf(x) - bf(y)| = b|f(x) - f(y)| = b|c| > \epsilon$. Thus δ does not "work" for all functions in $\mathcal{A}$. Since δ is arbitrary, $\mathcal{A}$ is not equicontinuous.

(V)(a) Let $k \geq 1$ be the degree of P: $P(x) = a_k x^k + a_{k-1} x^{k-1} + \dots$ with $a_k \neq 0$. Then

$$\frac{P(x)}{x^k} = a_k + a_{k-1} x^{-1} + \cdots \to a_k \quad \text{as } x \to \infty,$$

so for large x, $\left|\frac{P(x)}{x^k}\right| \geq \frac{1}{2}|a_k|$ and therefore $|P(x)| \geq \frac{1}{2}|a_k| x^k \to \infty$ as $x \to \infty$. Thus $||P||_\infty = \infty$.

(b) For all $n \neq m$ we have

$$||P_n - P_m||_\infty \leq ||P_n - f||_\infty + ||f - P_m||_\infty,$$

which is finite if n, m are large (since $||P_n - f||_\infty \to 0$), so by (a), $P_n - P_m$ must be a constant, call it c_{mn}. But then for fixed m and x we get

$$\lim_n c_{mn} = \lim_n (P_n(x) - P_m(x)) = f(x) - P_m(x),$$

so $\lim_n c_{mn}$ is some finite c and $f(x) = P_m(x) + c$, meaning f is a poynomial.

(VI) Suppose f is continuous, that is, $f \in C[-1, 1]$. By the Weierstrass Theorem 7.26, f is a uniform limit of a sequence of polynomials. Conversely, suppose f is a uniform limit of polynomials. Since polynomials are continuous, by 7.12 f is continuous.