MATH 425a ASSIGNMENT 7
FALL 2015 Prof. Alexander
Due Friday October 30.

Rudin Chapter 3 #11c, Chapter 4 #2, 4, plus the problems (I)–(VII) below:

(I) If $\sum a_n$ converges, show that $\sum \frac{a_n}{n}$ converges. (Do not assume $a_n \geq 0$. The Comparison Test won't work.)

(II) Prove *directly from the definition of limit* that $\lim_{x\to 0} \frac{x}{1-x} = 0$.

(III) Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is not continuous at any point $p \in \mathbb{R}$.

(IV) Determine (with proof, or disproof by example) whether each statement is true or false.
(a) There is a rearrangement of the series $\sum_{n=1}^{\infty}(-1)^n n^{-1}$ which converges to 3.14159.
(b) If a series $\sum_n a_n$ has the property that all of its rearrangements converge, it is absolutely convergent.
(c) If a series $\sum_n a_n$ has the property that some rearrangement converges, then it itself converges.

(V)(a) Let $f : \mathbb{R} \to \mathbb{R}$. Show that if f is continuous, then $|f|$ is continuous.
(b) If $|f|$ is continuous, must f be continuous? Prove or disprove.

(VI) Determine whether or not the following limits exist.
(a) $\lim_{x\to 0} \cos(1/x)$
(b) $\lim_{x\to 0} x \sin(1/x)$.

(VII) Let X, Y be metric spaces, $E \subset X$, $f : E \to Y$ a function, p a limit point of E, and $q \in Y$.
(a) Suppose $d(f(x), q) \leq 5d(x, p)$ for all $x \in E$. Show that $\lim_{x\to p} f(x) = q$.
(b) Suppose f is real-valued (that is, $Y = \mathbb{R}$), $f(x) \leq c$ for all $x \in E$, and $\lim_{x\to p} f(x)$ exists. Show that $\lim_{x\to p} f(x) \leq c$.

HINTS:

For all problems, do not use calculus tools like l'Hospital's Rule which we haven't discussed yet.

(I) This is very short if you cite the right theorem.

(II) Use a lower bound for the denominator $1 - x$, valid when x is near 0.

(III) Find an ϵ for which no δ "works."

(VII)(b) Proof by contradiction works well here.