Appendix | D

Affiliated Random Variables

Suppose that the random variables $X_1, X_2, \ldots, X_n$ are distributed on some product of intervals $\mathcal{X} \subset \mathbb{R}^n$ according to the joint density function f. The variables $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ are said to be *affiliated* if for all $\mathbf{x}', \mathbf{x}'' \in \mathcal{X}$,

$$f\left(\mathbf{x}' \vee \mathbf{x}''\right) f\left(\mathbf{x}' \wedge \mathbf{x}''\right) \geq f\left(\mathbf{x}'\right) f\left(\mathbf{x}''\right), \tag{D.1}$$

where

$$\mathbf{x}' \vee \mathbf{x}'' = \left(\max\left(x_1', x_1''\right), \max\left(x_2', x_2''\right), \ldots, \max\left(x_n', x_n''\right)\right)$$

denotes the component-wise maximum of $\mathbf{x}$ and $\mathbf{x}'$, and

$$\mathbf{x}' \wedge \mathbf{x}'' = \left(\min\left(x_1', x_1''\right), \min\left(x_2', x_2''\right), \ldots, \min\left(x_n', x_n''\right)\right)$$

denotes the component-wise minimum of $\mathbf{x}'$ and $\mathbf{x}''$. (See Figure D.1.) If (D.1) is satisfied, then we also say that f is affiliated.[1]

Suppose that the density function $f : \mathcal{X} \to \mathbb{R}_+$ is strictly positive in the interior of $\mathcal{X}$ and twice continuously differentiable. Using (D.1), it is easy to verify that f is affiliated if and only if, for all $i \neq j$,

$$\frac{\partial^2}{\partial x_i \partial x_j} \ln f \geq 0 \tag{D.2}$$

In other words, the off-diagonal elements of the Hessian of $\ln f$ are nonnegative.

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[1]A function g is said to be *supermodular* if $g(\mathbf{x}' \vee \mathbf{x}'') + g(\mathbf{x}' \wedge \mathbf{x}'') \geq g(\mathbf{x}') + g(\mathbf{x}'')$. Thus, f is affiliated if and only if $\ln f$ is supermodular; in other words, f is *log-supermodular*.

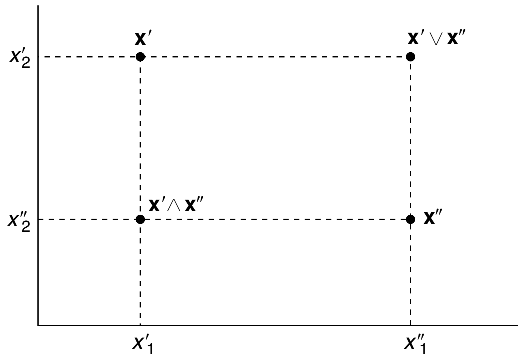

FIGURE D.1 Component-wise maxima and minima.

Suppose that the random variables $X_1, X_2, \ldots, X_n$ are symmetrically distributed and, as in Appendix C, define $Y_1, Y_2, \ldots, Y_{n-1}$ to be the largest, second largest, ..., smallest from among $X_2, X_3, \ldots, X_n$. From (D.1) it follows that if g denotes the joint density of $X_1, Y_1, Y_2, \ldots, Y_{n-1}$, then

$$g\left(x_1, y_1, y_2, \ldots, y_{n-1}\right) = (n-1)!f\left(x_1, y_1, \ldots, y_{n-1}\right)$$

if $y_1 \geq y_2 \geq \cdots \geq y_{n-1}$ and 0 otherwise. Now it immediately follows that

- If $X_1, X_2, \ldots, X_n$ are symmetrically distributed and affiliated, then $X_1, Y_1, Y_2, \ldots, Y_{n-1}$ are also affiliated.

TWO VARIABLES

We now present some special, but important, results concerning affiliation between two variables.

Suppose the random variables X and Y have a joint density $f : [0, \omega]^2 \to \mathbb{R}$. If X and Y are affiliated, then for all $x' \geq x$ and $y' \geq y$,

$$f(x', y)f(x, y') \leq f(x, y)f(x', y')$$

or equivalently that

$$\frac{f(x, y')}{f(x, y)} \leq \frac{f(x', y')}{f(x', y)} \tag{D.3}$$

Let $F(\cdot \mid x) \equiv F_Y(\cdot \mid X = x)$ denote the conditional distribution of Y given $X = x$ and, as usual, let $f(\cdot \mid x) \equiv f_Y(\cdot \mid X = x)$ denote the corresponding density function. Then (D.3) is equivalent to

$$\frac{f\left(y' \mid x\right)f(x)}{f(y \mid x)f(x)} \leq \frac{f\left(y' \mid x'\right)f(x')}{f(y \mid x')f(x')}$$

and so

$$\frac{f(y \mid x')}{f(y \mid x)} \leq \frac{f(y' \mid x')}{f(y' \mid x)} \tag{D.4}$$

Thus, we determine that if X and Y are affiliated, then for all $x' \geq x$, the *likelihood ratio*

$$\frac{f(\cdot \mid x')}{f(\cdot \mid x)}$$

is increasing and this is referred to as the *monotone likelihood ratio property*.

In the language of Appendix B, (D.4) implies that for all $x' \geq x$, $F(\cdot \mid x')$ *dominates* $F(\cdot \mid x)$ *in terms of the likelihood ratio*. Likelihood ratio dominance was the strongest stochastic order considered in Appendix B and the results derived there immediately imply that if X and Y are affiliated, then the following properties hold:

- For all $x' \geq x$, $F(\cdot \mid x')$ dominates $F(\cdot \mid x)$ in terms of the *hazard rate*; that is,

$$\lambda(y \mid x') \equiv \frac{f(y \mid x')}{1 - F(y \mid x')} \leq \frac{f(y \mid x)}{1 - F(y \mid x)} \equiv \lambda(y \mid x)$$

  or equivalently, for all y, $\lambda(y \mid \cdot)$ is nonincreasing.
- For all $x' \geq x$, $F(\cdot \mid x')$ dominates $F(\cdot \mid x)$ in terms of the *reverse hazard rate*; that is,

$$\sigma(y \mid x') \equiv \frac{f(y \mid x')}{F(y \mid x')} \geq \frac{f(y \mid x)}{F(y \mid x)} \equiv \sigma(y \mid x)$$

  or equivalently, for all y, $\sigma(y \mid \cdot)$ is nondecreasing.
- For all $x' \geq x$, $F(\cdot \mid x')$ (first-order) *stochastically dominates* $F(\cdot \mid x)$; that is,

$$F(y \mid x') \leq F(y \mid x)$$

  or equivalently, for all y, $F(y \mid \cdot)$ is nonincreasing.

All of these results extend in a straightforward manner to the case where the number of conditioning variables is more than one. Suppose $Y, X_1, X_2, \ldots, X_n$ are affiliated and let $F_Y(\cdot \mid \mathbf{x})$ denote the distribution of Y conditional on $\mathbf{X} = \mathbf{x}$. Then, using the same arguments as above, it can be deduced that for all $\mathbf{x}' \geq \mathbf{x}$, $F_Y(\cdot \mid \mathbf{x}')$ dominates $F_Y(\cdot \mid \mathbf{x})$ in terms of the likelihood ratio. The other dominance relationships then follow as usual.

CONDITIONAL EXPECTATIONS OF AFFILIATED VARIABLES

Suppose X and Y are affiliated. The fact that $F(y \mid \cdot)$ is nonincreasing implies in turn that the expectation of Y conditional on $X = x$, $E[Y \mid X = x]$, is a nondecreasing function of x. In other words, the "regression line" of Y against X has a nonnegative slope. Thus, X and Y are nonnegatively correlated.

Also, the same fact implies that if γ is a nondecreasing function, then $E[\gamma(Y) \mid X = x]$ is a nondecreasing function of x. More generally,

- If $X_1, X_2, \ldots, X_n$ are affiliated and γ is a nondecreasing function, then for all i,

$$E\left[\gamma(\mathbf{X}) \mid x_1' \leq X_1 \leq x_1'', x_2' \leq X_2 \leq x_2'', \ldots, x_n' \leq X_n \leq x_n''\right]$$

  is a nondecreasing function of x_i' and x_i''.

NOTES ON APPENDIX D

The affiliation inequality in (D.1) is known as a version of *total positivity* in the statistics literature (Karlin and Rinott, 1980). More specifically, a vector random variable $\mathbf{X}$ that satisfies (D.1) is said to be "MTP_2" (*multivariate total positivity*) and the implications of this have been extensively studied. Closely related is the notion of *association* and the "FKG inequality" (see Shaked and Shanthikumar, 1994). The term affiliation appears to have been coined by Milgrom and Weber (1982) and the appendix to their paper is a convenient reference for results useful in auction theory.