Answers Assignment 3

Problem 1

(i) We have

$$Var(\hat{\beta}_m - \hat{\beta}_f) = Var(\hat{\beta}_m) + Var(\hat{\beta}_f) = \sigma_m^2 (X_m' X_m)^{-1} + \sigma_f^2 (X_f' X_f)^{-1}$$

Because the regression coefficients are estimated on independent random samples they are uncorrelated.

(ii) If the CLR assumptions hold then the vector $\hat{\beta}_m - \hat{\beta}_f$ has a multivariate normal distribution with under the null of equal coefficients mean 0 and variance given above. Therefore

$$T = (\hat{\beta}_m - \hat{\beta}_f)' \left(\sigma_m^2 (X_m' X_m)^{-1} + \sigma_f^2 (X_f' X_f)^{-1}\right)^{-1} (\hat{\beta}_m - \hat{\beta}_f)$$

has under the null a chi-square distribution with df equal to the number of regression coefficients in the male or female regression.

(iii) The test statistic is now

$$T = (\hat{\beta}_m - \hat{\beta}_f)' \left(\hat{\sigma}_m^2 (X_m' X_m)^{-1} + \hat{\sigma}_f^2 (X_f' X_f)^{-1}\right)^{-1} (\hat{\beta}_m - \hat{\beta}_f)$$

Rewrite as (to simplify assume that the sample sizes of the male and female samples are equal and equal to n)

$$T = \sqrt{n}(\hat{\beta}_m - \hat{\beta}_f)' \left(\hat{\sigma}_m^2 (X_m' X_m/n)^{-1} + \hat{\sigma}_f^2 (X_f' X_f/n)^{-1}\right)^{-1} \sqrt{n}(\hat{\beta}_m - \hat{\beta}_f)$$

Now

$$\sqrt{n}(\hat{\beta}_m - \hat{\beta}_f) \xrightarrow{d} N(0, \sigma_m^2 (\Sigma_{Xm})^{-1} + \sigma_f^2 (\Sigma_{Xf})^{-1})$$

and by continuous mapping

$$\hat{\sigma}_m^2 (X_m' X_m/n)^{-1} + \hat{\sigma}_f^2 (X_f' X_f/n)^{-1} \xrightarrow{p} \sigma_m^2 (\Sigma_{Xm})^{-1} + \sigma_f^2 (\Sigma_{Xf})^{-1}$$

so that by the Slutsky theorem

$$T \xrightarrow{d} \chi^2(K)$$