MATH 425a ASSIGNMENT 10
FALL 2011 Prof. Alexander
Due Thursday December 10, at noon.

Rudin Chapter 7 #1, 2, plus the problems (A)–(D) below, are to be turned in.

Chapter 7 #3, 9 are also suggested, as good practice for the exam.

You can turn this in by sliding it under my office door, or taking it to the Math Department office, KAP 104. No late homework accepted for this one. Problem 2 and maybe 1, (A), (B), (D) should be relatively "quick" ones, the type that most often appears on exams.

(A) Suppose $h : [0, 1] \to \mathbb{R}$ is continuous with $h(0) = 0$. Let $g_n(x) = h(x)e^{-nx}$. Show that $g_n \to 0$ uniformly.

(B) Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is uniformly continuous for all n, and $f_n \to f$ uniformly. Show that f is uniformly continuous.

(C)(a) Suppose $f_n : [a, b] \to \mathbb{R}$ with each f_n having at most 2 discontinuities, and suppose $f_n \to f$ uniformly. Show that f has at most 2 discontinuities.

(b) Give an example to show (a) is false if we replace "at most 2" with "at least 2". (This means some of the discontinuities disappear in the limit f.)

(D) Suppose f_n, f are functions from a finite set E to $\mathbb{R}$. If $f_n \to f$ pointwise, show that $f_n \to f$ uniformly.

HINTS:

(1) First show that there is an N for which $\{f_n, n \geq N\}$ is uniformly bounded. Then show that you can actually include $f_1, \ldots, f_{N-1}$ as well. Also, note that the assumption of "bounded functions" means that for each n there is an M_n such that $|f_n(x)| \leq M_n$ for all x; in other words, the bound M_n may depend on n. You want to show there is a bound M for all the functions f_n, which does not depend on n.

(2) For the second statement, use Exercise 1.

(3) Problem 2 shows that the sequences $\{f_n\}$ and $\{g_n\}$ cannot both have what property? Also, try $f_n(x) = f(x) + a_n$ for some f and some sequence $a_n \to 0$.

(9) Just do the first part, not the converse. You can bound $|f_n(x_n) - f(x)|$ by comparing both $f_n(x_n)$ and $f(x)$ to some third quantity. (In other words, bound the difference between that third quantity and each of those two function values.)

(A) Loosely speaking, $|g_n(x)|$ will be small if either (i) $|h(x)|$ is small and e^{-nx} is bounded, or (ii) $|h(x)|$ is bounded and e^{-nx} is small. Given $\epsilon > 0$, to show $|g_n(x)| < \epsilon$ for all $x \in [0, 1]$, you can divide $[0, 1]$ into two intervals, one where (i) is true and the other where (ii) is true.

(B) Given $\epsilon > 0$, suppose that you have a δ that "works" for $\epsilon/3$ for the function f_n, for some n. Show that if n is large enough, then δ also "works" for ϵ, for the function f.

(C) For a given x, if there are infintely many f_n's (and therefore a subsequence f_{n_k}) which are all continuous at x, what does that tell you about continuity of f at x? So if f is discontinuous at x, what does this say about how many of the f_n's can be discontinuous at x?