Economics 202 Jonathan Levin
Winter 2002 Stanford University

Core Microeconomics III

Overview. This part of the core microeconomics sequence covers non-cooperative game theory, and some of its application to the study of imperfect competition, strategic behavior by firms, the provision of public goods, and market signalling.

Logistics. The class meets Tuesday and Thursday, 9:00–10:50. You can download lecture notes and assignments from the class web page, which is http://www.stanford.edu/class/econ203. I plan to have office hours Tuesdays from 3–5. My office is in the economics building, room 240. You can also make an appointment to meet by emailing me at jdlevin@stanford.edu. The teaching assistant for the class is Peter Coles. His email is pacoles@stanford.edu.

Reading. The main textbook for the class is:

> Mas-Colell, A., M. Whinston and J. Green, *Microeconomic Theory*, OUP.

A few other good references, in no particular order:

> Kreps, D. *A Course in Microeconomic Theory*, Harvester Wheatsheaf.
> Gibbons, R. *Game Theory for Applied Economists*, Princeton.
> Fudenberg, D. and J. Tirole, *Game Theory*, MIT Press
> Tirole, J. *The Theory of Industrial Organization*, MIT Press.
> Schelling, T. *The Strategy of Conflict*, (several editions)
> Dixit, A. and B. Nalebuff, *Thinking Strategically*, Norton.

Kreps' book has a lot of great intuition, and a less formal treatment of game theory than MWG. Gibbon's book is also an excellent, accessible introduction to game theory, and perhaps closest in organization to the lectures. The Fudenberg-Tirole and Tirole books are for those who want to push further in game theory or in applications to industrial organization. Finally, Schelling and Dixit-Nalebuff make terrific bed-time reading for any game theory student.

Assignments. There will roughly five problem sets, and a final exam. Grading will be based 30% on the problem sets, and 70% on the exam. You should feel free (in fact, encouraged) to work together on the assignments, but please write them up individually.

Outline of Topics

1. Normal Form Games (4 lectures)

   Reading Lecture Notes, MWG Ch. 8A–D, 12C.

   Topics: Modeling Strategic Environments, Dominant Strategies, Iterated Dominance and Rationalizability, Nash Equilibrium, Application: Imperfect Competition, Mixed Strategies, Evolutionary Game Theory.

2. Extensive Form Games (3–4 lectures)

   Reading Lecture Notes, MWG Ch. 7A–E, 9A–B, 12E.

   Topics: The Extensive Form, Nash Equilibrium and the Problem of Credibility, Subgame Perfection, Application: Models of Commitment.

3. Repeated Games (2–3 lectures)

   Reading Lecture Notes, MWG Ch. 9 Appendix, 12E, 12 Appendix.

   Topics: Infinite-Horizon Games, One-Step Deviation Principle, Bargaining, Repeated Games, Folk Theorem.

4. Static Games of Incomplete Information (2–3 lectures)

   Reading Lecture Notes, MWG Ch. 8E.

   Topics: Description and Motivation, Bayesian Nash Equilibrium, Examples: Public Goods and Auctions, Purification.

5. Dynamic Games of Incomplete Information (5–6 lectures)

   Reading Lecture Notes, MWG Ch. 9C–D.

   Topics: Perfect Bayesian Equilibrium, Sequential Equilibrium, Signalling Models, Cheap Talk, Reputation.

Useful Math for Microeconomics*

Jonathan Levin Antonio Rangel

September 2001

1 Introduction

Most economic models are based on the solution of optimization problems. These notes outline some of the basic tools needed to solve these problems. It is wort spending some time becoming comfortable with them — you will use them a lot!

We will consider *parametric constrained optimization problems* (PCOP) of the form

$$\max_{x \in D(\theta)} f(x, \theta).$$

Here f is the objective function (e.g. profits, utility), x is a choice variable (e.g. how many widgets to produce, how much beer to buy), $D(\theta)$ is the set of available choices, and θ is an exogeneous parameter that may affect both the objective function and the choice set (the price of widgets or beer, or the number of dollars in one's wallet). Each parameter θ defines a specific problem (e.g. how much beer to buy given that I have \$20 and beer costs \$4 a bottle). If we let Θ denote the set of all possible parameter values, then Θ is associated with a whole class of optimization problems.

In studying optimization problems, we typically care about two objects:

1. The *solution set*

$$x^*(\theta) \equiv \arg\max_{x \in D(\theta)} f(x, \theta),$$

*These notes are intended for students in Economics 202, Stanford University. They were originally written by Antonio in Fall 2000, and revised by Jon in Fall 2001. Leo Rezende provided tremendous help on the original notes. Section 5 draws on an excellent comparative statics handout prepared by Ilya Segal.

that gives the solution(s) for any parameter $\theta \in \Theta$. (If the problem has multiple solutions, then $x^*(\theta)$ is a set with multiple elements).

2. The *value function*

$$V(\theta) \equiv \max_{x \in D(\theta)} f(x, \theta)$$

that gives the value of the function at the solution for any parameter $\theta \in \Theta$ ($V(\theta) = f(y, \theta)$ for any $y \in x^*(\theta)$.)

In economic models, several questions typically are of interest:

1. Does a solution to the maximization problem exist for each θ?

2. Do the solution set and the value function change continuously with the parameters? In other words, is it the case that a small change in the parameters of the problem produces only a small change in the solution?

3. How can we compute the solution to the problem?

4. How do the solution set and the value function change with the parameters?

You should keep in mind that any result we derive for a maximization problem also can be used in a minimization problem. This follows from the simple fact that

$$x^*(\theta) = \arg\min_{x \in D(\theta)} f(x, \theta) \Longleftrightarrow x^*(\theta) = \arg\max_{x \in D(\theta)} -f(x, \theta)$$

and

$$V(\theta) = \min_{x \in D(\theta)} f(x, \theta) \Longleftrightarrow V(\theta) = -\max_{x \in D(\theta)} -f(x, \theta).$$

2 Notions of Continuity

Before starting on optimization, we first take a small detour to talk about continuity. The idea of continuity is pretty straightforward: a function h is continuous if "small" changes in x produce "small" changes in $h(x)$. We just need to be careful about (a) what exactly we mean by "small," and (b) what happens if h is not a function, but a correspondence.

2.1 Continuity for functions

Consider a function h that maps every element in X to an element in Y, where X is the domain of the function and Y is the range. This is denoted by $h : X \to Y$. We will limit ourselves to functions that map $\mathbb{R}^n$ into $\mathbb{R}^m$, so $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$.

Recall that for any $x, y \in \mathbb{R}^k$,

$$\|x - y\| = \sqrt{\sum_{i=1,\ldots,k} (x_i - y_i)^2}$$

denotes the Euclidean distance between x and y. Using this notion of distance we can formally define continuity, using either of following two equivalent definitions:

Definition 1 *A function $h : X \to Y$ is **continuous at** x if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - y\| < \delta$ and $y \in X \Rightarrow \|h(x) - h(y)\| < \varepsilon$.*

Definition 2 *A function $h : X \to Y$ is **continuous at** x if for every sequence x_n in X converging to x, the sequence $h(x_n)$ converges to $f(x)$.*

You can think about these two definitions as tests that one applies to a function to see if it is continuous. A function is continuous if it passes the continuity test at each point in its domain.

Definition 3 *A function $h : X \to Y$ is **continuous** if it is continuous at every $x \in X$.*

Figure 1 shows a function that is not continuous. Consider the top picture, and the point x. Take an interval centered around $h(x)$ that has a "radius" ε. If ε is small, each point in the interval will be less than A. To satisfy continuity, we must find a distance δ such that, as long as we stay within a distance δ of x, the function stays within ε of $h(x)$. But we cannot do this. A small movement to the right of x, regardless of how small, takes the function above the point A. Thus, the function fails the continuity test at x and is not continuous.

The bottom figure illustrates the second definition of continuity. To meet this requirement at the point x, it must be the case that for *every* sequence x_n converging to x, the sequence $h(x_n)$ converges to $h(x)$. But consider

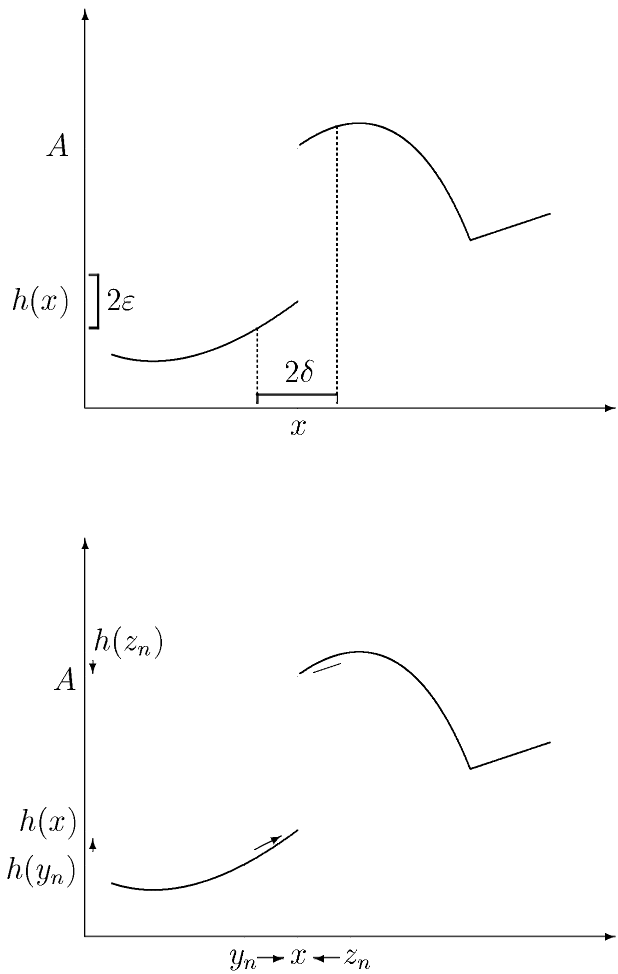

Figure 1: Testing for Continuity.

the sequence z_n that converges to x from the right. The sequence $h(z_n)$ converges to the point A from above. Since $A > h(x)$, the test fails and h is not continuous. We should emphasize that the test must be satisfied *for every sequence*. In this example, the test is satisfied for the sequence y_n that converges to x from the right.

In general, to show that a function is continuous, you need to argue that one of the two continuity tests is satisfied at every point in the domain. If you use the first definition, the typical proof has two steps:

- Step 1: Pick any x in the domain and any $\varepsilon > 0$.
- Step 2: Show that there is a $\delta_x(\varepsilon) > 0$ such that $\|h(x) - h(y)\| < \varepsilon$ whenever $\|x - y\| < \delta_x(\varepsilon)$. To show this you have to give a formula for $\delta_x(\cdot)$ that guarantees this.

The problems at the end should give you some practice at this.

2.2 Continuity for correspondences

A correspondence ϕ maps points x in the domain $X \subseteq \mathbb{R}^n$ into sets in the range $Y \subseteq \mathbb{R}^m$. That is, $\phi(x) \subseteq Y$ for every x. This is denoted by $\phi : X \rightrightarrows Y$. Figure 2 provides a couple of examples. We say that a correspondence is:

- non-empty-valued if $\phi(x)$ is non-empty for all x in the domain.
- convex if $\phi(x)$ is a convex set for all x in the domain.
- compact if $\phi(x)$ is a compact set for all x in the domain.

For the rest of these notes we assume, unless otherwise noted, that correspondences are non-empty-valued.

Intuitively, a correspondence is continuous if small changes in x produce small changes in the set $\phi(x)$. Figure 3 shows a continuous correspondence. A small move from x to x' has a small effect since $\phi(x)$ and $\phi(x')$ are approximately equal. Not only that, the smaller the change in x, the more similar are $\phi(x)$ and $\phi(x')$.

Unfortunately, giving a formal definition of continuity for correspondences is not so simple. With functions, it's pretty clear that to evaluate the effect of moving from x to a nearby x' we simply need to check the distance between the point $h(x)$ and $h(x')$. With correspondences, we need to make a

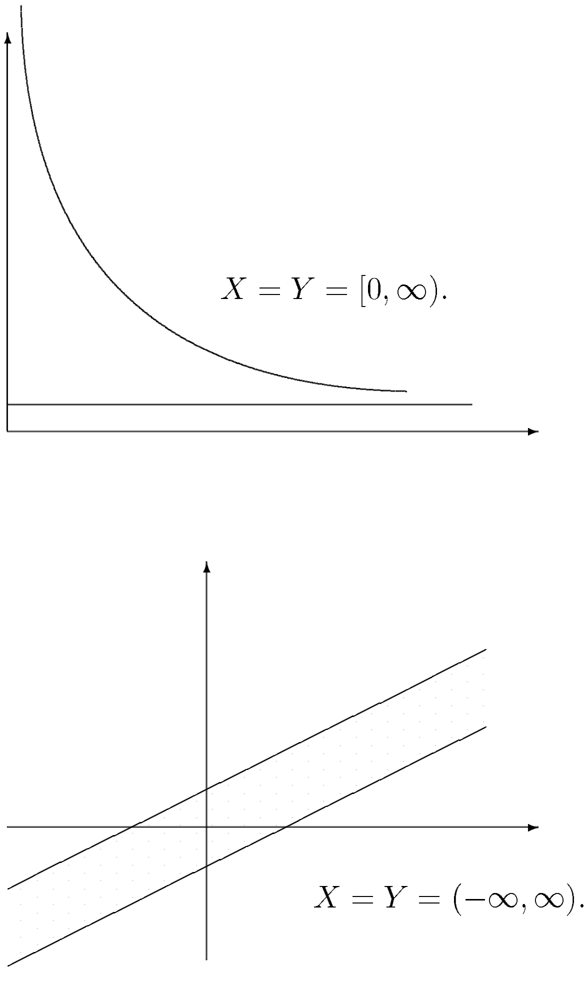

Figure 2: Examples of Correspondences.

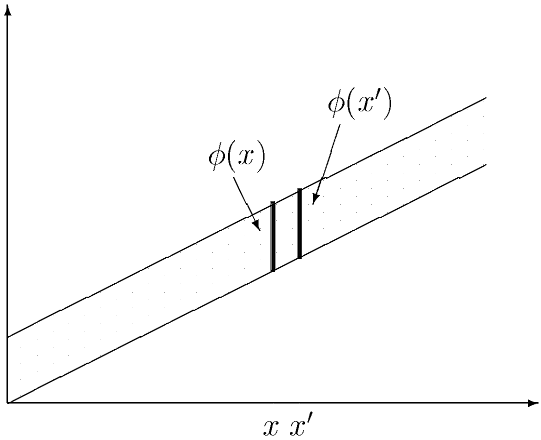

Figure 3: A Continuous Correspondence.

comparison between the sets $\phi(x)$ and $\phi(x')$. To do this, we will need two distinct concepts: upper and lower semi-continuity.

Definition 4 *A correspondence $\phi : X \rightrightarrows Y$ is **lower semi-continuous (lsc) at x** if for each open set G meeting $\phi(x)$, there is an open set $U(x)$ containing x such that if $x' \in U(x)$, then $\phi(x') \cap G \neq \emptyset$.*[1] *A correspondence is **lower semi-continuous** if it is lsc at every $x \in X$.*

Lower semi-continuity captures the idea that any element in $\phi(x)$ can be "approached" from all directions. That is, if we consider some x and some $y \in \phi(x)$, lower semi-continuity at x implies that if one moves a little way from x to x', there will be some $y' \in \phi(x')$ that is close to y.

As an example, consider the correspondence in Figure 4. It is not lsc at x. To see why, consider the point $y \in \phi(x)$, and let G be a very small interval around y that does not include $\hat{y}$. If we take any open set $U(x)$ containing x, then it will contain some point x' to the left of x. But then $\phi(x') = \{\hat{y}\}$ will contain no points near y (i.e. will not intersect G).

On the other hand, the correspondence in Figure 5 *is* lsc. (One of the exercises at the end is to verify this.) But it still doesn't seem to reflect an intuitive notion of continuity. Our next definition formalizes what is wrong.

[1] Recall that a set $S \subset \mathbb{R}^n$ is open if for every point $s \in S$, there is some ε such that every point s' within ε of s is also in S.

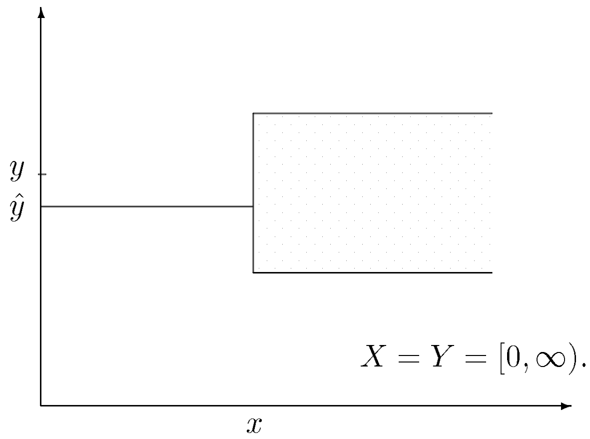

Figure 4: A Correspondence that is usc, but not lsc.

Definition 5 *A correspondence* $\phi : X \rightrightarrows Y$ *is* ***upper semi-continuous (usc) at*** x *if for each open set* G *containing* $\phi(x)$*, there is an open set* $U(x)$ *containing* x *such that if* $x' \in U(x)$*, then* $\phi(x') \subset G$*. A correspondence is* ***upper semi-continuous*** *if it is usc at every* $x \in X$*, and also compact-valued.*

Upper semi-continuity captures the idea that $\phi(x)$ will not "suddenly contain new points" just as we move past some point x. That is, if one starts at a point x and moves a little way to x', upper semi-continuity at x implies that there will be no point in $\phi(x')$ that is not close to some point in $\phi(x)$.

As an example, the correspondence in Figure 4 *is* usc at x. To see why, imagine an open interval $\phi(x)$ that encompasses $\phi(x)$. Now consider moving a little to the left of x to a point x'. Clearly $\phi(x') = \{\hat{y}\}$ is in the interval. Similarly, if we move to a point x' a little to the right of x, then $\phi(x')$ will be inside the interval so long as x' is sufficiently close to x.

On the other hand, the correspondence in Figure 5 is not usc at x. If we start at x (noting that $\phi(x) = \{\hat{y}\}$), and make a small move to the right to a point x', then $\phi(x')$ suddenly contains many points that are not close to $\hat{y}$. So this correspondence fails to be upper semi-continuous.

We now combine upper and lower semi-continuity to give a definition of continuity for correspondences.

Definition 6 *A correspondence* $\phi : X \rightrightarrows Y$ *is* ***continuous at x*** *if it is usc and lsc at* x*. A correspondence is* ***continuous*** *if it is both upper and lower*

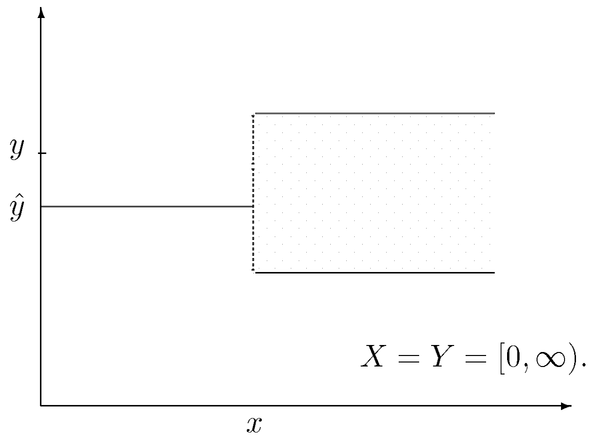

Figure 5: A Correspondence that is lsc, but not usc.

semi-continuous.

As it turns out, our notions of upper and lower semi-continuity both reduce to the standard notion of continuity if ϕ is a single-valued correspondence, i.e. a function.

Proposition 1 *Let $\phi : X \rightrightarrows Y$ be a single-valued correspondence, and $h : X \to Y$ the function given by $\phi(x) = \{h(x)\}$. Then,*

$$\phi \textit{ is continuous} \Leftrightarrow \phi \textit{ is usc} \Leftrightarrow \phi \textit{ is lsc} \Leftrightarrow h \textit{ is continuous.}$$

You can get some intuition for this result by drawing a few pictures of functions and checking the definitions of usc and lsc.

3 Properties of Solutions: Existence, Continuity, Uniqueness and Convexity

Let's go back to our class of parametric constrained optimization problems (PCOP). We're ready to tackle the first two questions we posed: (1) under what conditions does the maximization problem has a solution for every parameter θ?, and (2) what are the continuity properties of the value function $V(\theta)$ and the solution set $x^*(\theta)$?

The following Theorem, called the *Theorem of the Maximum*, provides an answer to both questions. We will use it time and time again.

Theorem 2 (Theorem of the Maximum) *Consider the class of parametric constrained optimization problems*

$$\max_{x \in D(\theta)} f(x, \theta)$$

defined over the set of parameters Θ. *Suppose that (i)* $D : \Theta \rightrightarrows X$ *is continuous (i.e. lsc and usc) and compact-valued, and (ii)* $f : X \times \Theta \to \mathbb{R}$ *is a continuous function. Then*

1. $x^*(\theta)$ *is non-empty for every* θ*;*
2. x^* *is upper semi-continuous (and thus continuous if* x^* *is single-valued);*
3. V *is continuous.*

We will not give a formal proof, but rather some examples to illustrate the role of the assumptions.

Example What can happen if D is not compact? In this case a solution might not exist for some parameters. Consider the example $\Theta = [0, 10]$, $D(\theta) = (0, 1)$, and $f(x, \theta) = x$. Then $x^*(\theta) = \emptyset$ for all θ.

Example What can happen if D is lsc, but not usc? Suppose that $\Theta = [0, 10]$, $f(x, \theta) = x$, and

$$D(\theta) = \begin{cases} \{0\} & \text{if } \theta \leq 5, \\ [-1, 1] & \text{otherwise} \end{cases}.$$

The solution set is given by

$$x^*(\theta) = \begin{cases} \{0\} & \text{if } \theta \leq 5, \\ \{1\} & \text{otherwise} \end{cases},$$

which is a function, but not continuous. The value function is also discontinuous.

Example What can happen if D is usc, but not lsc? Suppose that $\Theta = [0, 10]$, $f(x, \theta) = x$, and

$$D(\theta) = \begin{cases} \{0\} & \text{if } \theta < 5, \\ [-1, 1] & \text{otherwise} \end{cases}$$

The solution set is given by

$$x^*(\theta) = \begin{cases} \{0\} & \text{if } \theta < 5, \\ \{1\} & \text{otherwise} \end{cases}$$

which once more is a discontinuous function.

In the last two examples, the goal of the maximization problem is to pick the largest possible element in the constraint set. So the solution set potentially can be discontinuous if (and only if) the constraint set changes abruptly.

Example Finally, what can happen if f is not continuous? Suppose that $\Theta = [0, 10]$, $D(\theta) = [\theta, \theta + 1]$, and

$$f(x, \theta) = \begin{cases} 0 & \text{if } x < 5 \\ 1 & \text{otherwise} \end{cases}.$$

The solution set is given by

$$x^*(\theta) = \begin{cases} [\theta, \theta + 1] & \text{if } \theta < 4 \\ [5, \theta + 1] & \text{if } 4 \leq \theta < 5 \\ [\theta, \theta + 1] & \text{otherwise} \end{cases}$$

and the value function is given by

$$V(\theta) = \begin{cases} 0 & \text{if } \theta < 4 \\ 1 & \text{if } 4 \leq \theta < 5 \\ 1 & \text{otherwise} \end{cases}$$

As it can be easily seen from Figure 3, x^* is not usc and V is not continuous at $\theta = 4$.

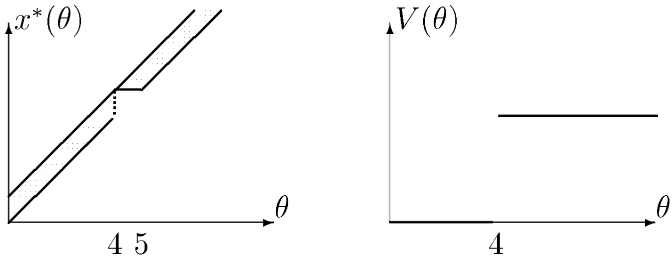

Figure 6: Failure of the Maximum Theorem when f is discontinuous.

You might wonder if the second result in the theorem can be strengthened to guarantee that x^* is continuous, and not just usc. In general, the answer is no, as the following example illustrates.

Example Consider the problem of a consumer with linear utility choosing between two goods. Her objective function is $U(x) = x_1 + x_2$, and her optimization problem is:

$$\begin{array}{c} \max_x U(x) \\ \text{s.t. } p \cdot x \leq 10. \end{array}$$

Let $P = \{ p \in \mathbb{R}^2_{++} \mid p_1 = 1 \text{ and } p_2 \in (0, \infty) \}$ denote the set of possible prices (the parameter set).

The solution (the consumer's demand as a function of price), is given by

$$x^*(p) = (x_1^*(p), x_2^*(p)) = \left\{ \begin{array}{cl} \{(0, 10/p_2)\} & \text{if } p_2 < 1, \\ \{ x \mid p \cdot x = 10 \} & \text{if } p_2 = 1, \\ \{(10, 0)\} & \text{otherwise} \end{array} \right. .$$

Figure 7 graphs x_1^* as a function of p_2. Demand is not continuous, since it explodes at $p_2 = 1$. However, as the theorem states, it is usc.

The Theorem of the Maximum identifies conditions under which optimization problems have a solution, and says something about the continuity of the solution. We can say more if we know something about the curvature of the objective function. This next theorem is the basis for many results in price theory.

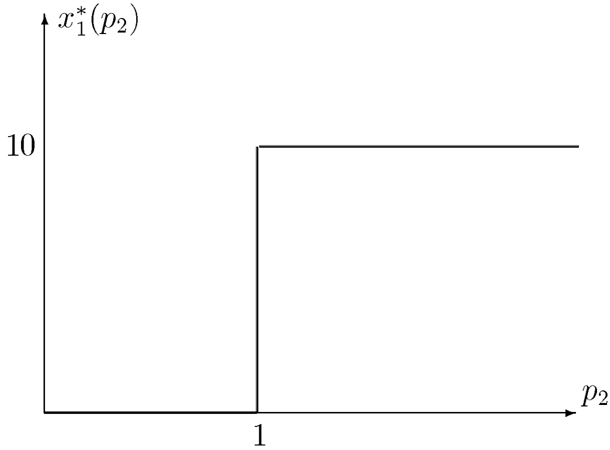

Figure 7: Demand is not Continuous.

Theorem 3 *Consider the class of parametric constrained optimization problems*

$$\max_{x \in D(\theta)} f(x, \theta)$$

defined over the convex set of parameters Θ. *Suppose that (i)* $D : \Theta \rightrightarrows X$ *is continuous and compact-valued, and (ii)* $f : X \times \Theta \to \mathbb{R}$ *is a continuous function.*

1. *If* $f(\cdot, \theta)$ *is a quasi-concave function in* x *for each* θ*, and* D *is convex-valued, then* x^* *is convex-valued.*[2]

2. *If* $f(\cdot, \theta)$ *is a strictly quasi-concave function in* x *for each* θ*, and* D *is convex-valued, then* $x^*(\theta)$ *is single-valued.*

3. *If* f *is a concave function in* (x, θ) *and* D *is convex-valued, then* V *is a concave function and* x^* *is convex-valued.*

4. *If* f *is a strictly concave function in* (x, θ) *and* D *is convex-valued, then* V *is strictly concave and* x^* *is a function.*

[2] Recall that a function $h : X \to \mathbb{R}$ is quasi-concave if its upper contour sets $\{x \in X : h(x) \geq k\}$ are convex sets. That is, if $h(x) \geq k$ and $h(x') \geq k$ implies that $h(tx+(1-t)x') \geq k$ for all $x \neq x'$ and $t \in (0, 1)$. We say that h is strictly quasi-concave if the last inequality is strict.

Proof. (1) Suppose that $f(\cdot, \theta)$ is a quasi-concave function in x for each θ, and D is convex-valued. Pick any $x, x' \in x^*(\theta)$. Since D is convex-valued, $x^t = tx + (1-t)x' \in D(\theta)$ for all $t \in [0,1]$. Also, by the quasi-concavity of f we have that

$$f(x^t, \theta) \geq f(x, \theta) = f(x', \theta).$$

But since $f(x, \theta) = f(x', \theta) \geq f(y, \theta)$ for all $y \in D(\theta)$, we get that $f(x^t, \theta) \geq f(y, \theta)$ for all $y \in D(\theta)$. We conclude that $x^t \in x^*(\theta)$, which establishes the convexity of x^*.

(2) Suppose that $f(\cdot, \theta)$ is a strictly quasi-concave function in x for each θ, and D is convex-valued. Suppose towards, a contradiction, that $x^*(\theta)$ contains two distinct points x and x' ; i.e., it is not single-valued at θ. As before, D is convex-valued implies that $x^t = tx + (1-t)x' \in D(\theta)$ for all $t \in (0,1)$. But then strict quasi-concavity of $f(\cdot, \theta)$ in x implies that

$$f(x^t, \theta) > f(x, \theta) = f(x', \theta),$$

which contradicts the fact that x and x' are maximizers in $D(\theta)$.

(3) Suppose that f is a concave function in (x, θ) and that D has a convex graph. Pick any θ, θ' in Θ and let $\theta^t = t\theta + (1-t)\theta'$ for some $t \in [0,1]$. We need to show that $V(\theta^t) \geq tV(\theta) + (1-t)V(\theta')$. Let x and x' be solutions to the problems θ and θ' and define $x^t = tx + (1-t)x'$. By the definition of the value function and the concavity of f we get that

$$\begin{aligned}
V(\theta^t) &\geq f(x^t, \theta^t) \qquad \text{(by definition of } V(\theta^t)) \\
&= f(tx + (1-t)x', t\theta + (1-t)\theta') \qquad \text{(by definition of } x^t, \theta^t) \\
&\geq tf(x, \theta) + (1-t)f(x', \theta') \qquad \text{(by concavity of } f) \\
&= tV(\theta) + (1-t)V(\theta'). \qquad \text{(by definition of } V(\theta) \text{ and } V(\theta')).
\end{aligned}$$

Also, since $f(\cdot, \cdot)$ concave in $(x, \theta) \Rightarrow f(\cdot, \theta)$ quasi-concave function in x for each θ, the proof that x^* is convex-valued follows from (1).

(4) The proof is nearly identical to (3). *Q.E.D.*

4 Characterization of Solutions

Our next goal is to learn how to actually solve optimization problems. We will focus on a more restricted class of problems given by:

$$\max_{x \in \mathbb{R}^n} f(x, \theta)$$

subject to

$$g_k(x, \theta) \le b_k \qquad \text{for } k = 1, \ldots, K,$$

where $f(\cdot, \theta)$, $g_1(\cdot, \theta)$, $\ldots$, $g_K(\cdot, \theta)$ are functions defined on $\mathbb{R}^n$ or on an open subset of $\mathbb{R}^n$.

In this class of problems, the constraint set $D(\theta)$ is given by the intersection of K inequality constraints. There may be any number of constraints (for instance, there could be more than n or more constraints than choice variable), but it is important that at least some values of x satisfy the constraints (so the choice set is non-empty). Figure 8 shows an example with 4 constraints, and $n = 2$ (i.e. $x = (x_1, x_2)$ and the axes are x_1 and x_2).

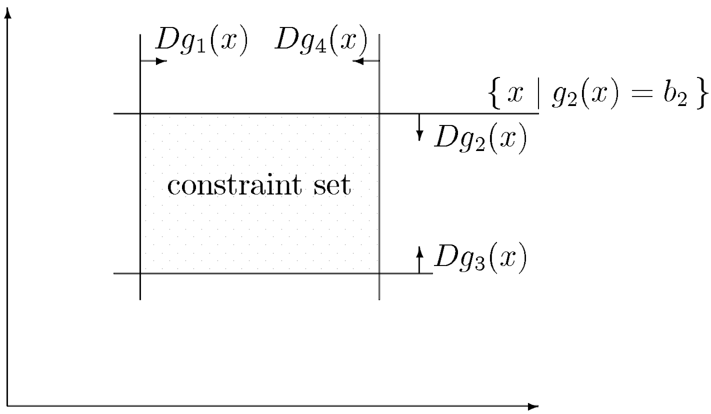

Figure 8: A Constraint Set Given by 4 Inequalities.

Note that this class of problems includes the cases of equality constraints ($g_k(x, \theta) = b_k$ is equivalent to $g_k(x, \theta) \le b_k$ and $g_k(x, \theta) \ge b_k$) and non-negativity constraints ($x \ge 0$ is equivalent to $-x \le 0$). Both of these sorts of constraints arise frequently in economics.

Each parameter θ defines a separate constrained optimization problem. In what follows, we focus on how to solve one of these problems (i.e. how to solve for $x^*(\theta)$ for a given θ).

We first need a small amount of notation. Let $h : \mathbb{R}^n \to \mathbb{R}$ be any differentiable real valued function. The derivative, or gradient, of h, $Dh : \mathbb{R}^n \to \mathbb{R}^n$, is a vector-valued function

$$Dh(x) = \left(\frac{\partial h(x)}{\partial x_1}, \ldots, \frac{\partial h(x)}{\partial x_n} \right).$$

As illustrated in Figure 9, the gradient has a nice graphical interpretation: it is a vector that is orthogonal to the level set of the function, and thus points in the direction of maximum increase. (In the figure, the domain of h is the plane (x_1, x_2); the curve is a level curve of h. Movements in the direction L leave the value of the function unchanged, while movements in the direction of $Dh(x)$ increase the value of the function at the fastest possible rate).

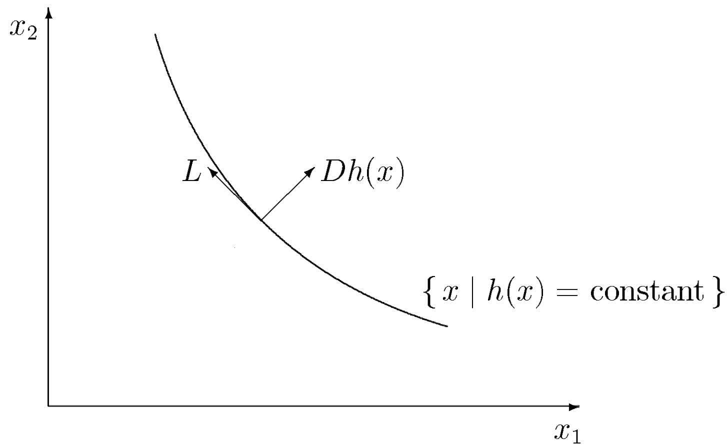

Figure 9: Illustration of Gradient.

If you have taken economics before, you may have learned to solve constrained optimization problems by forming a Lagrangian:

$$\mathcal{L}(x, \lambda, \theta) = f(x, \theta) + \sum_{k=1}^{K} \lambda_k \left(b_k - g_k(x, \theta) \right),$$

and maximizing the Lagrangian with respect to (x, λ). Here, we go a bit deeper to show exactly when and why the Lagrangian approach works.

To do this, we take two steps. First, we identify *necessary conditions* for a solution, or conditions that must be satisfied by any solution to a constrained optimization problem. We then go on to identify *sufficient conditions* for a solution, or conditions that if satisfied by some x guarantee that x is indeed a solution.

Our first result, the famed *Kuhn-Tucker Theorem*, identifies necessary conditions that must be satisfied by any solution to a constrained optimization problem. To state it, we need the following definition.

Definition 7 *Consider a point x that satisfies all of the constraints, i.e. $g_k(x,\theta) \leq b_k$ for all k. We say* ***constraint*** *k* ***binds at*** *x if $g_k(x,\theta) = b_k$, and is* ***slack*** *if $g_k(x,\theta) < b_k$. If $B(x)$ denotes the set of binding constraints at point x, then* ***constraint qualification holds at*** *x if the vectors in the set $\{Dg_k(x,\theta) | k \in B(x)\}$ are linearly independent.*

Theorem 4 (Kuhn-Tucker) *Suppose that for the parameter θ, the following conditions hold: (i) $f(\cdot,\theta)$, $g_1(\cdot,\theta)$, ..., $g_K(\cdot,\theta)$ are continuously differentiable in x; (ii) $D(\theta)$ is non-empty; (iii) x^* is a solution to the optimization problem; and (iv) constraint qualification holds at x^*. Then*

1. *There exist non-negative numbers λ_1, ..., λ_K such that*

$$Df(x^*,\theta) = \sum_{k=1}^{K} \lambda_k Dg_k(x^*,\theta),$$

2. *For $k = 1,\ldots,K$,*

$$\lambda_k(b_k - g_k(x^*,\theta)) = 0.$$

In practice, we often refer to the expression in (1) as the first-order condition, and (2) as the complementary slackness conditions. Conditions (1) and (2) together are referred to as the *Kuhn-Tucker conditions.* The numbers λ_k are *Lagrange multipliers.* The theorem tells us these multipliers must be non-negative, and equal to zero for any constraint that is not binding (binding constraints typically have positive multipliers, but not always).

The Kuhn-Tucker theorem provides a partial recipe for solving optimization problems: simply compute all the pairs (x,λ) such that *(i)* x satisfies all of the constraints, and *(ii)* (x,λ) satisfies the Kuhn-Tucker conditions. Or in other words, one identifies all the solutions (x,λ) for the system:

$$g_1(x,\theta) \leq b_1,$$
$$\vdots$$
$$g_K(x,\theta) \leq b_K,$$

$$Df(x,\theta) = \sum_{k=1}^{K} \lambda_k Dg_k(x,\theta),$$

and

$$\lambda_1(b_1 - Dg_1(x,\theta)) = 0,$$
$$\vdots$$
$$\lambda_K(b_K - Dg_K(x,\theta)) = 0.$$

Since there are $n + 2K$ equations and only $n + K$ unknowns, you might be concerned that this system has no solution. Fortunately the Maximum Theorem comes to the rescue. If its conditions are satisfied, we know a maximizer exists and so must solve our system of equations.[3]

What is the status of the Lagrangian approach you might have learned previously? If you applied it correctly (remembering that the λ_k's must be non-negative), you would have come up with exactly the system of equations above as your first-order conditions. So the Kuhn-Tucker gives us a short-cut: we can apply it directly without even writing down the Lagrangian!

Because the Kuhn-Tucker Theorem is widely used in economic problems, we include a proof (intended only for the more ambitious).

Proof (of the Kuhn-Tucker Theorem).[4] The proof will use the following basic result from linear algebra: If $\alpha_1, \ldots, \alpha_j$ are linearly independent vectors in $\mathbb{R}^n$, with $j \leq n$, then for any $b = (b_1, \ldots, b_j) \in \mathbb{R}^j$ there exists $x \in \mathbb{R}^n$ such that

$$Ax = b,$$

where A is the $j \times n$ matrix that given by

$$\begin{pmatrix} - & \alpha_1 & - \\ & \cdots & \\ - & \alpha_j & - \end{pmatrix}.$$

To prove the KT Theorem we need to show that the conditions of the theorem imply that if x^* solves the optimization problem, then: (1) there are numbers $\lambda_1, \ldots, \lambda_K$ such that

$$Df(x^*,\theta) = \sum_{k=1}^{K} \lambda_k Dg_k(x^*,\theta),$$

[3]Also note that if the constraint qualification holds at x^*, then there are at most n binding constraints. It is easy to see why. The vectors $Dg_k(x^*,\theta)$ are vectors in $\mathbb{R}^n$ and we know from basic linear algebra that at most n vectors in $\mathbb{R}^n$ can be linearly independent.

[4]This proof is taken from a manuscript by Kreps, who attributes it to Elchanan Ben-Porath.

(2) the numbers λ_1, ..., λ_K are non-negative, and (3) the complementary slackness conditions are satisfied at x^*.

In fact, we can prove (1) and (3) simultaneously by establishing that there are numbers λ_k for each $k \in B(x^*)$ such that

$$Df(x^*, \theta) = \sum_{k \in B(x^*)} \lambda_k Dg_k(x^*, \theta).$$

Then (3) follows because the slackness conditions are automatically satisfied for the binding constraints, and (1) follows because we can set $\lambda_k = 0$ for the non-binding constraints.

So let's prove this first. We know that the vectors in the set $\{Dg_k(x^*, \theta) | k \in B(x^*)\}$ are linearly independent since the constraint qualification is satisfied at x^*. Now suppose, towards a contradiction, that there are no numbers λ_1, ..., λ_K for which the expression holds. Then we know that the set of vectors $\{Df(x^*, \theta)\} \cup \{Dg_k(x^*, \theta) | k \in B(x^*)\}$ is linearly independent. In turn, this implies that there are at most $n-1$ binding constraints. (Recall that there can be at most n linearly independent vectors in $\mathbb{R}^n$.) But then, by our linear algebra result, we get that there exists $z \in \mathbb{R}^n$ such that

$$Df(x^*, \theta) \cdot z = 1 \qquad \text{and} \qquad Dg_k(x^*, \theta) \cdot z = -1 \quad \text{for all } k \in B(x^*).$$

Now consider the effect of moving from x^* to $x^* + \epsilon z$, for $\epsilon > 0$ but small enough. By Taylor's theorem for ϵ small enough all of the constraints are satisfied at $x^* + \epsilon z$. The constraints that are slack are no problem, and for the ones that are binding the change makes them slack. Also, by Taylor's theorem, $f(x^* + \epsilon z, \theta) > f(x^*, \theta)$, a contradiction to the optimality of x^*.

Now look at (2). Suppose that the first order condition holds, but that one of the multipliers is negative, say λ_j. Pick $M > 0$ such that $-M\lambda_j > \sum_{k \in B(x^*), k \neq j} \lambda_k$. Again, since the set of vectors $\{Dg_k(x^*, \theta) | k \in B(x^*)\}$ is linearly independent, we know that there exists $z \in \mathbb{R}^n$ such that

$$Dg_j(x^*, \theta) \cdot z = -M \qquad \text{and} \qquad Dg_k(x^*, \theta) \cdot z = -1 \quad \text{for all } k \in B(x^*), k \neq j.$$

By the same argument than before, for ϵ small enough all of the constraints are satisfied at $x^* + \epsilon z$. Furthermore, by Taylor's Theorem

$$\begin{aligned} f(x^* + \epsilon z, \theta) &= f(x^*, \theta) + \epsilon Df(x^*, \theta) \cdot z + o(\epsilon) \\ &= f(x^*, \theta) + \epsilon \sum_{k \in B(x^*)} \lambda_k Dg_k(x^*, \theta) \cdot z + o(\epsilon) \end{aligned}$$

$$
\begin{aligned}
&= f(x^*, \theta) + \epsilon \left[-M\lambda_j - \sum_{k \in B(x^*), k \neq j} \lambda_k \right] + o(\epsilon) \\
&> f(x^*, \theta);
\end{aligned}
$$

where the last inequality holds as long as ϵ is small enough. Again, this contradicts the optimality of x^*. *Q.E.D.*

Figure 10 provides a graphical illustration of the Kuhn-Tucker Theorem for the case of one constraint. In the picture, the two outward curves are level curves of $f(\cdot, \theta)$, while the inward curve is a constraint (any eligible x must be inside it). Consider the point $\hat{x}$ that is not optimal, but at which the constraint is binding. Here the gradient of the objective function cannot be expressed as a linear combination (i.e., a multiple given that there is only one constraint) of the gradient of the constraint. Now consider a small movement towards point A in a direction perpendicular to $Dg_1(\widehat{x})$ which is feasible since it leaves the value of the constraint unchanged. Since this movement is not perpendicular to $Df(\widehat{x})$ it will increase the value of the function. By contrast, consider a similar movement at x^*, where $Dg_1(x^*)$ and $Df(x^*)$ lie in the same direction. Here any feasible movement along the constraint also leaves the objective function unchanged.

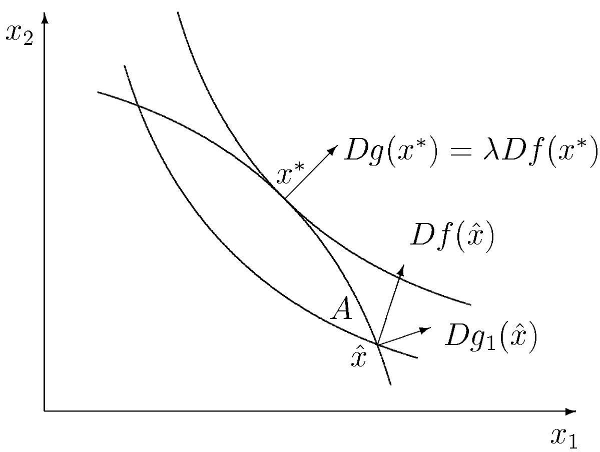

Figure 10: Illustration of Kuhn-Tucker Theorem.

It is easy to see why the multiplier of a binding constraint has to be non-negative. Consider the case of one constraint illustrated in Figure 11.

We cannot have a negative multiplier because then a movement towards the interior of the constraint set that makes the constraint slack would increases the value of the function — a violation of the optimality of x^*.

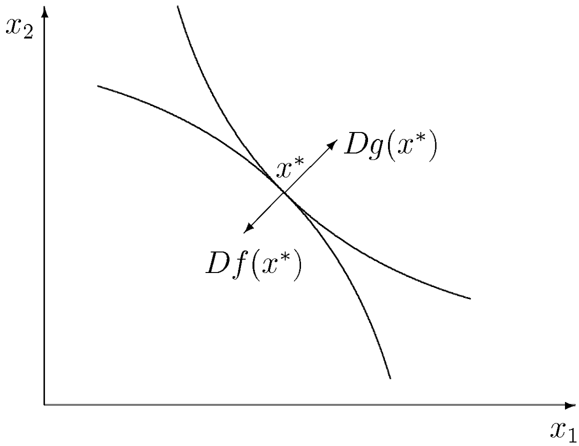

Figure 11: Why λ is Non-Negative.

It is important to emphasize that if the constraint qualification is not satisfied, the Kuhn-Tucker recipe might fail. Consider the example illustrated in Figure 12. Here x^* is clearly a solution to the problem since it is the only point in the constraint set. And yet, we cannot write $Df(x^*)$ as a linear combination of $Dg_1(x^*)$ and $Dg_2(x^*)$.

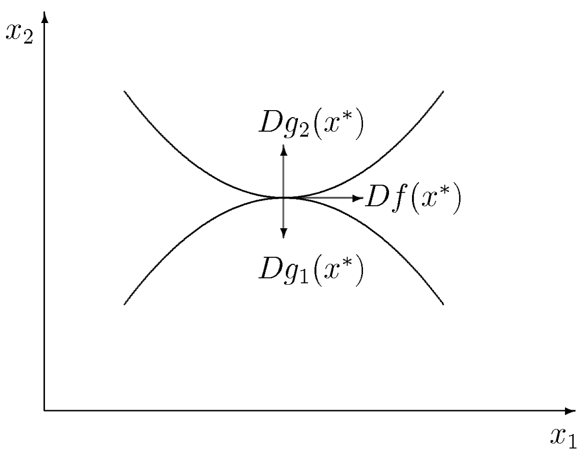

Figure 12: Why Constraint Qualification is Required.

Fortunately, in many cases it will be easy to verify that constraint qualification is satisfied. For example, we may know that at the solution only one constraint is binding. Or we may have only linear constraints with linearly independent gradients. It is probable that in most problems you encounter, establishing constraint qualification will not be an issue.

Why does the Kuhn-Tucker Theorem provide only a partial recipe for solving constrained optimization problems? The reason is that there may be solutions (x, λ) to the Kuhn-Tucker conditions that are not solutions to the optimization problem (i.e. the Kuhn-Tucker conditions are necessary but not sufficient for x to be a solution). Figure 13 provides an example in which x satisfies the Kuhn-Tucker conditions but is not a solution to the problem, the points x^* and x^{**} are the solution.

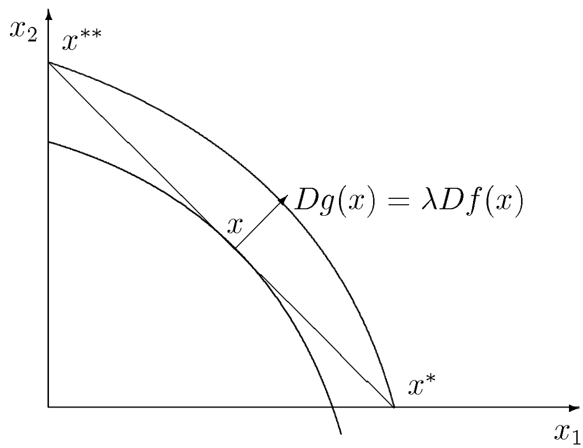

Figure 13: KT conditions are not sufficient.

To rule out this sort of situation, one needs to check second-order, or sufficient conditions. In general, checking second-order conditions is a pain. One needs to calculate the Hessian matrix of second derivatives and test for negative semi-definiteness, a rather involved procedure. (The grim details are in Mas-Colell, Whinston and Green (1995).) Fortunately, for many economic problems the following result comes to the rescue.

Theorem 5 *Suppose that the conditions of the Kuhn-Tucker Theorem are satisfied and that (i) $f(\cdot, \theta)$ is quasi-concave, and (ii) $g_1(\cdot, \theta), \ldots, g_K(\cdot, \theta)$ are quasi-convex.*[5] *Then any point x^* that satisfies the Kuhn-Tucker conditions*

[5]Recall that quasi-concavity of $f(\cdot, \theta)$ means that the upper countour sets of f (i.e. the

is a solution to the constraint optimization problem.

Proof. We will use the following fact about quasi-convex functions. A continuously differentiable function $g : \mathbb{R}^n \to \mathbb{R}$ is quasi-convex if and only if

$$Dg(x) \cdot (x - x') \leq 0 \quad \text{whenever} \quad g(x\prime) \leq g(x).$$

Now to the proof. Suppose that x^* satisfies the Kuhn-Tucker conditions. Then there are multipliers λ_1 to λ_K such that

$$Df(x^*, \theta) = \sum_{k=1}^{K} \lambda_k Dg_k(x^*, \theta).$$

But then, for any x with that satisfies the constraints we get that

$$Df(x^*, \theta)(x - x^*) = \sum_{k=1}^{K} \lambda_k Dg_k(x^*, \theta)(x - x^*) \leq 0.$$

The last inequality follows because $\lambda_k = 0$ if the constraint is slack at at x^*, and $g_k(x, \theta) \leq g_k(x^*, \theta)$ if it binds.

To conclude the proof note that for concave f we know that

$$f(x, \theta) \leq f(x^*, \theta) + Df(x^*, \theta)(x - x^*).$$

Since the second term on the right is non-positive we can conclude that $f(x, \theta) \leq f(x^*, \theta)$, and thus x^* is a solution to the problem. *Q.E.D.*

One reason this theorem is valuable is that its conditions are satisfied by many economic models. When the conditions are met, we can solve the optimization problem in a single step by solving the Kuhn-Tucker conditions. If the conditions fails, you will need to be much more careful in solving the optimization problem.

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set of points x such that $f(x, \theta) \geq c$) are convex. (See the footnote above). Note also that quasi-convexity of $g_1(\cdot, \theta)$, $\ldots$, $g_K(\cdot, \theta)$ implies that the constraint set $D(\theta)$ is convex — this is useful for applying the characterization theorem from the previous section.

4.1 Non-negativity Constraints

A common variation of the optimization problem above arises when x is required to be non-negative (i.e. $x \geq 0$ or $x \in \mathbb{R}^n_+$). This problem can be written as

$$\max_{x \in \mathbb{R}^n_+} f(x, \theta)$$

subject to

$$g_k(x, \theta) \leq b_k \quad \text{for all} \quad k = 1, \ldots, K.$$

Given that this variant comes up often in economic models, it is useful to explicitly write down the Kuhn-Tucker conditions.

To do this, we write the constraint $x \in \mathbb{R}^n_+$ as n constraints:

$$h_1(x) = -x_1 \leq 0, ..., h_n(x) = -x_n \leq 0,$$

and apply the Kuhn-Tucker Theorem. For completeness, we give a formal statement.

Theorem 6 *Consider the constrained optimization problem with non-negativity constraints. Suppose that for the parameter θ, the following conditions hold: (i) $f(\cdot, \theta)$, $g_1(\cdot, \theta)$, $\ldots$, $g_K(\cdot, \theta)$ are continuously differentiable; (ii) $D(\theta)$ is non-empty; (iii) x^* is a solution to the optimization problem; (iv) constraint qualification holds at x^* for all the constraints (including any binding non-negativity constraints). Then*

1. *There are numbers λ_1, $\ldots$, λ_I such that*

$$\frac{\partial f(x^*, \theta)}{\partial x_j} + \mu_j = \sum_{k=1}^{I} \lambda_k \frac{\partial g_k(x^*, \theta)}{\partial x_j} \quad \textit{for all} \quad j = 1, \ldots, n$$

2. *For $k = 1, \ldots, I$,*

$$\lambda_k (b_k - g_k(x^*, \theta)) = 0.$$

3. *For $j = 1, \ldots, n$,*

$$\mu_j x_j^* = 0.$$

The proof is a straightforward application of the Kuhn-Tucker theorem. Note that the conditions in (3) are simply complementary slackness conditions for the non-negativity constraints. The multiplier μ_j of the j-th non-negativity constraint is zero whenever $x_j^* > 0$.

4.2 Equality Constraints

Another important variant on our optimization problem arises when we have equality, rather than inequality, constraints. Consider the problem:

$$\max_{x \in \mathbb{R}^n} f(x, \theta)$$

subject to

$$g_k(x, \theta) = b_k \qquad \text{for all } k = 1, \ldots, K.$$

To make sure that the constraint set is non-empty we assume that $K \leq n$.

As discussed before, this looks like a special case of our previous result since $g_k(x, \theta) = b_k$ can be rewritten as $g_k(x, \theta) \leq b_k$ and $-g_k(x, \theta) \leq b_k$. Unfortunately, we cannot use our previous result to solve the problem in this way. At the solution both sets of inequality constraints must be binding, which implies that the constraint qualification cannot be satisfied. (Why?) Thus, our recipe does not work.

The following result, known as the *Lagrange Theorem*, provides the recipe for this case.

Theorem 7 *Consider a constrained optimization problem with $K \leq n$ equality constraints. Suppose that for the parameter θ, the following conditions hold: (i) $f(\cdot, \theta)$, $g_1(\cdot, \theta)$, ..., $g_K(\cdot, \theta)$ are continuously differentiable; (ii) $D(\theta)$ is non-empty; (iii) x^* is a solution to the optimization problem; (iv) the following constraint qualification holds at x^**

$$\mathit{Rank} \begin{pmatrix} - & Dg_1(x^*, \theta) & - \\ & \ldots & \\ - & Dg_I(x^*, \theta) & - \end{pmatrix} = I.$$

Then there are numbers λ_1, ..., λ_I such that

$$Df(x^*, \theta) = \sum_{k=1}^{I} \lambda_k Dg_k(x^*, \theta)$$

This result is very similar, but not identical. First, there are no complementary slackness conditions since all of the constraints are binding. Second, the multipliers can be positive or negative. The proof in this case is more complicated and is omitted.

4.3 Simplifying Constrained Optimization Problems

In many problems that you will encounter you will know which constraints are binding and which ones are not. For example, in the problem

$$\max_{x \in \mathbb{R}^n} f(x,\theta) \qquad \text{subject to} \quad g_k(x,\theta) \le b_k \quad \text{for all } k = 1, \ldots, K$$

you may know that constraints 1 to $K-1$ are binding, but constraint K is slack. You might then ask if it is possible to solve the original problem by solving the simpler problem

$$\max_{x \in \mathbb{R}^n} f(x,\theta) \qquad \text{subject to} \quad g_k(x,\theta) = b_k \quad \text{for all } k = 1, \ldots, K-1.$$

This transformation is very desirable because the second problem (which has equality constraints, and fewer equations) is typically much easier to solve.

Unfortunately, this simplification is not always valid, as Figure 14 demonstrates. At the solution x^* one of the constraints is not binding. However, if we eliminate that constraint we can do better by choosing $\hat{x}$. So elimination of the constraint changes the value of the problem.

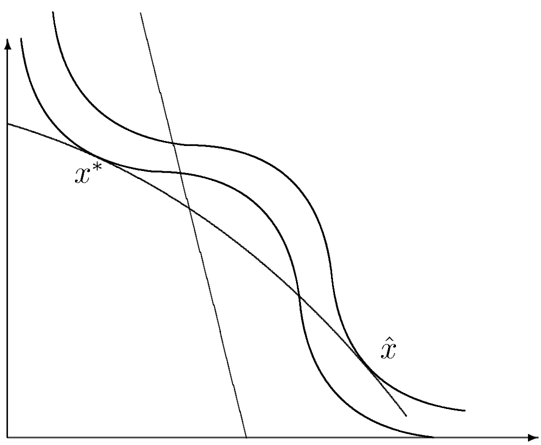

Figure 14: An indispensable non-binding constraint.

Fortunately, however, the simplification is valid in a particular set of cases that includes many economic models.

Theorem 8 *Consider the maximization problem*

$$\max_{x \in \mathbb{R}^n} f(x,\theta) \qquad \text{subject to} \quad g_k(x,\theta) \le b_k \quad \text{for all } k = 1, \ldots, I.$$

Suppose that (i) $f(\cdot,\theta)$ is strictly quasi-concave; (ii) $g_1(\cdot,\theta)$, ..., $g_K(\cdot,\theta)$ are quasi-convex; (iii) $g_k(\cdot,\theta)$ for $k=1,\ldots,B$ are binding constraints at the solution, and (iv) $g_k(\cdot,\theta)$ for $k=B+1,\ldots,K$ are slack constraints at the solution. Then x^* is a solution if and only if is a solution to the modified problem

$$\max_{x\in\mathbb{R}^n} f(x,\theta) \qquad \text{subject to } g_k(x,\theta)=b_k \qquad \text{for all } k=1,\ldots,B.$$

Proof. Conditions *(i)* and *(ii)* imply that the optimization problem has a unique solution. Call it x^*. Suppose, towards a contradiction, that there is a point $\hat{x}$ that satisfies the constraints of the second problem and for which $f(\hat{x},\theta)>f(x^*,\theta)$. Then because $f(\cdot,\theta)$ is strictly quasi-concave, $f(x^t,\theta)>f(x^*,\theta)$ for all $x^t=t\widehat{x}+(1-t)x^*$ and $t\in(0,1)$. Furthermore, by strict quasi-convexity of the constraints, x^t satisfies all of the constraints of hte first problem for t close enough to zero. But then x^* cannot be a solution to the first problem, a contradiction. *Q.E.D.*

This theorem will allow us to transform most of our problems into the simpler case of equality constraints. For this reason, the rest of the notes will focus on this case.

5 Comparative Statics

Many economic questions can be phrased in the following way: how do endogenous variables respond to changes in exogenous variables? These types of questions are called *comparative statics*.

Comparative statics questions can be qualitative or quantitative. We may just want to know if $x^*(\cdot)$ or $V(\cdot)$ increase, decrease, or are unaffected by θ. Alternatively, we may care about exactly how much or how quickly $x^*(\cdot)$ and $V(\cdot)$ will change with θ.

These questions might seem straightforward. After all, if we know the formulas for $x^*(\cdot)$ or $V(\cdot)$, all we have to do is compute the derivative, or check if the function is increasing. However, things often are not so simple. It may be hard to find an explicit formula for $x^*(\cdot)$ or $V(\cdot)$. Or we may want to know if $x^*(\cdot)$ is increasing as long as the objective function $f(\cdot)$ satisfies a particular property.

This section discusses the three important tools that are used for comparative statics analysis: (1) the Implicit Function Theorem, (2) the Envelope Theorem, and (3) Topkis's Theorem on monotone comparative statics.

5.1 The Implicit Function Theorem

The *Implicit Function Theorem* is an invaluable tool. It gives sufficient conditions under which, at least locally, there is a differentiable solution function to our optimization problem $x^*(\theta)$. And not only that, it gives us a formula to compute the derivative without actually solving the optimization problem!

Given that you will use the IFT in other applications, it is useful to present it in a framework that is more general than our class of PCOP. Consider a system of n equations of the form

$$h_k(x,\theta) = 0 \quad \text{for all } k = 1,\ldots,n$$

where $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ and $\theta = (\theta_1,\ldots,\theta_s) \in \mathbb{R}^s$. Suppose that the functions are defined on an open set $X \times T \subset \mathbb{R}^n \times \mathbb{R}^s$. We can think of x as the variables and θ as the parameters. To simplify the notation below define $h(x,\theta) = (h_1(x,\theta),\ldots,h_n(x,\theta))$.

Consider a solution $\hat{x}$ of the system at $\hat{\theta}$. We say that the system can be *locally solved at* $(\hat{x},\hat{\theta})$ if for some open set A of parameters that contains $\hat{\theta}$, and for some open set B of variables that contains $\hat{x}$, there exists a *uniquely determined* function $\eta : A \to B$ such that for all $\theta \in A$

$$h_k(\eta(\theta),\theta) = 0 \quad \text{for all } k = 1,\ldots,n.$$

We call η an "implicit" solution of the system. (Note: if $h(\cdot,\cdot)$ come from the first-order conditions of an optimization problem, then $\eta(\theta) = x^*(\theta)$).

Theorem 9 *Consider the system of equations described above and suppose that (i) $h_k(\cdot)$ is continuously differentiable at (x,θ) for all k; (ii) $\hat{x}$ is a solution at $\hat{\theta}$; and (iii) the matrix $D_x h(\hat{x},\hat{\theta})$ is non-singular:*

$$rank \begin{pmatrix} - & D_x h_1(\hat{x},\hat{\theta}) & - \\ & \ldots & \\ - & D_x h_n(\hat{x},\hat{\theta}) & - \end{pmatrix}_{n\times n} = n$$

Then:

1. *The system can locally be solved at* $(\hat{x},\hat{\theta})$

2. *The implicit function* η *is continuously differentiable and*

$$D_\theta \eta(\hat{\theta}) = -[D_x h(\hat{x},\hat{\theta})]^{-1} D_\theta h(\hat{x},\hat{\theta}).$$

This result is extremely useful! Not only does it give you sufficient conditions under which there is a unique (local) solution, it also gives you sufficient conditions for the differentiability of the solution and a shortcut for computing the derivative. You don't need to know the solution function, only the solution at $\hat{\theta}$.

If there is only 1 variable and 1 parameter, the formula for the derivative takes the simpler form

$$\frac{\partial \eta(\hat{\theta})}{\partial \theta} = -\frac{h_\theta(\hat{x},\hat{\theta})}{h_x(\hat{x},\hat{\theta})}.$$

A full proof of the implicit function theorem is beyond the scope of these notes. However, to see where it comes from, consider the one-dimensional case and suppose that $\eta(\theta)$ is the unique differentiable solution to $h(x,\theta) = 0$. Then $\eta(\theta)$ is implicity given by

$$h(\eta(\theta),\theta) = 0.$$

Totally differentiating with respect to θ (use the chain rule), we get

$$\frac{\partial h(\eta(\theta),\theta)}{\partial x} \cdot \frac{\partial \eta(\theta)}{\partial \theta} + \frac{\partial h(\eta(\theta),\theta)}{\partial \theta} = 0.$$

Evaluating this at $\theta = \hat{\theta}$ and $\eta(\hat{\theta}) = \widehat{x}$, and re-arranging, gives the formula above.

In optimization problems, the implicit function theorem is useful for computing how a solution will change with the parameters of a problem. Consider the class of constrained optimization problems with equality constraints defined by an open parameter set $\Theta \subset \mathbb{R}$ (so there is a single parameter). Lagrange's Theorem provides an *implicit* characterization of the problem's solution.

Recall that any solution (x^*,λ) must solve the following $n+K$ equations:

$$\frac{\partial f(x^*,\theta)}{\partial x_j} - \sum_{k=1}^{K} \lambda_k \frac{\partial g_k(x^*,\theta)}{\partial x_j} = 0 \quad \text{for all } j = 1,\ldots,n$$

$$g_k(x^*, \theta) - b_k = 0 \quad \text{for all } k = 1, \ldots, K.$$

Let $(x^*(\theta), \lambda(\theta))$ be the solution to this system of equations, and suppose (as we will often do) that $x^*(\theta)$ is a function. The first part of the IFT says the solution $(x^*(\theta), \lambda(\theta))$ is differentiable if when we take the derivative of the left hand side of each equation with respect to x and λ (i.e. we find $D_x h$ and $D_\lambda h$), we get something that is not equal to zero when evaluated at $(x^*(\theta), \lambda(\theta))$. The second part of the IFT provides a formula for the derivative of $\partial x^*(\theta)/\partial\theta$ and $\partial\lambda(\theta)/\partial\theta$. Very useful!

In closing this section, it is worth mentioning a few subtleties of the IFT.

1. The IFT guarantees that the system has a unique *local* solution at $(\hat{x}, \hat{\theta})$, but not necessarily that $\widehat{x}$ is a unique *global* solution given $\hat{\theta}$. As shown in figure 15, the second statement is considerably stronger. At $\hat{\theta}$ the system has 3 solutions x', x'', and x'''. A small change in from $\hat{\theta}$ to $\tilde{\theta}$ changes each solution slightly. Since $D_x h(x, \theta)$ is non-zero at each solution — call them $\eta'(\theta)$, $\eta''(\theta)$ and $\eta'''(\theta)$ — the IFT applies to each individually. It guarantees that each is differentiable and gives the formula for the derivative. However, one needs to be a little careful when there are multiple solutions. In this example,

$$D_\theta \eta'(\theta) \neq D_\theta \eta''(\theta) \neq D_\theta \eta'''(\theta),$$

so each solution reacts differently to changes in the parameter θ.

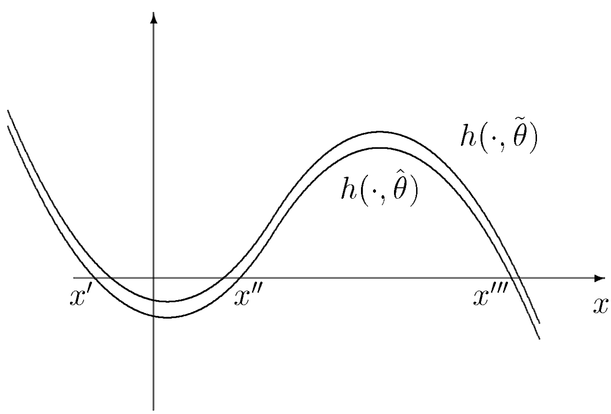

Figure 15: IFT: Solution may not be unique globally.

2. The IFT applies only when $D_x h(x,\theta)$ is non-singular. Figure 16 illustrates why. At $\hat{\theta}$ the system has two solutions: x' and x''. However, if we changes the parameter to any $\theta > \hat{\theta}$, the system suddenly gains an extra solution. By contrast, if we change the parameters to any $\theta < \hat{\theta}$, the system suddenly has only one solution. If we tried to apply the IFT, we would find that $D_x h(x', \hat{\theta}) = 0$, and that the system does not have a unique local solution in a neighborhood around $(x', \hat{\theta})$.

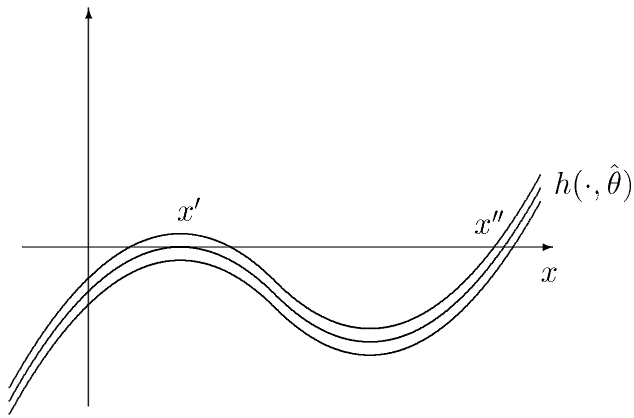

Figure 16: IFT: Solution may not be locally unique where $D_x h$ is singular.

5.2 Envelope Theorems

Whereas the IFT allows us to compute the derivative of $x^*(.)$, the *Envelope Theorem* can be used to compute the derivative of the value function.

Once more, consider the class of constrained optimization problems with equality constraints defined by a parameter set Θ.

Theorem 10 (Envelope Theorem) *Consider the maximization problem*

$$\max_{x \in R^n} f(x, \theta) \text{subject to } g_k(x, \theta) = b_k \text{for all } k = 1, ..., K.^6$$

and suppose that (i) $f(\cdot)$, $g_1(\cdot)$, ..., $g_K(\cdot)$ are continuously differentiable in (x, θ); and (ii) $x^*(.)$ is a differentiable function in an open neighborhood A of $\widehat{\theta}$. Then

1. $V(\cdot)$ is differentiable in A.

2. For $i = 1, ..., s$

$$\frac{\partial V(\widehat{\theta})}{\partial \theta_i} = \frac{\partial f(x^*(\widehat{\theta}), \widehat{\theta})}{\partial \theta_i} - \sum_{k=1}^{K} \lambda_k \frac{\partial g_k(x^*(\widehat{\theta}), \widehat{\theta})}{\partial \theta_i},$$

where λ_1, ..., λ_I are the Lagrange multipliers associated with $x^*(\widehat{\theta})$.

To grasp the intuition for the ET, think about a simple one-dimensional optimization problem with no constraints:

$$V(\theta) = \max_{x \in \mathbb{R}} f(x, \theta),$$

where $\theta \in [0, 1]$. If the solution $x^*(\theta)$ is differentiable, then $V(\theta) = f(x^*(\theta), \theta)$ is differentiable. Applying the chain rule, we get:

$$V'(\theta) = \underbrace{\frac{\partial f(x^*(\theta), \theta)}{\partial x}}_{=0 \text{ (at an optimum)}} \times \frac{\partial x^*(\theta)}{\partial \theta} + \frac{\partial f(x^*(\theta), \theta)}{\partial \theta} = \frac{\partial f(x^*(\theta), \theta)}{\partial \theta}.$$

A change in θ has two effects on the value function: (i) a direct effect $f_\theta(x^*(\theta), \theta)$, and (ii) an indirect effect $f_x(x^*(\theta), \theta)\frac{\partial x^*(\theta)}{\partial \theta}$. The ET tells us that under certain conditions, we can ignore the indirect effect and focus on the direct effect. In problems with constraints, there is also a third effect — the change in the constraint set. If constraints are binding (some λ's are positive), this effect is accounted for by the ET above.

A nice implication of the ET is that it provides some meaning for the mysterious Lagrange multipliers. To see this, think of b_k as a parameter of the problem, and consider $\partial V / \partial b_k$ — the marginal value of relaxing the kth constraint ($g_k(x, \theta) \leq b_k$). The ET tells us that (think about why this is true):

$$\frac{\partial V(\theta; b)}{\partial b_k} = \lambda_k.$$

Thus, λ_k is precisely the marginal value of relaxing the kth constraint.

One drawback to the ET stated above is that it requires $x^*(\theta)$ to be (at least locally) a differentiable function. In many cases (for instance in many unconstrained problems), this is a much stronger requirement than is necessary. The next result (due to Milgrom and Segal, 2001) provides an alternative ET that seems quite useful.

Theorem 11 *Consider the maximization problem*

$$\max_{x \in R^n} f(x, \theta) \text{subject to } g_k(x, \theta) \leq b_k \text{for all } k = 1, ..., K.$$

and suppose that (i) $f, g_1, ..., g_K$ are continuous and concave in x; (ii) $\partial f / \partial \theta$, $\partial g_1 / \partial \theta, ..., \partial g_K / \partial \theta$ are continuous in (x, θ); and (iii) there is some $\widehat{x} \in \mathbb{R}^n$ such that $g_k(\widehat{x}, \theta) > 0$ for all $k = 1, ..., K$ and all $\theta \in \Theta$. Then at any point $\hat{\theta}$ where $V(\theta)$ is differentiable:

$$\frac{\partial V(\widehat{\theta})}{\partial \theta_i} = \frac{\partial f(x^*(\widehat{\theta}), \widehat{\theta})}{\partial \theta_i} - \sum_{k=1}^{K} \lambda_k \frac{\partial g_k(x^*(\widehat{\theta}), \widehat{\theta})}{\partial \theta_i},$$

where λ_1, ..., λ_I are the Lagrange multipliers associated associated with $x^*(\widehat{\theta})$.

5.3 Monotone Comparative Statics

The theory of monotone comparative statics (MCS) provides a very general answer to the question: when is the solution set $x^*(\cdot)$ nondecreasing (or strictly increasing) in θ?

You may wonder if the answer is already given by the Implicit Function Theorem. Indeed, when $x^*(\cdot)$ is a differentiable function we can apply the IFT to compute $D_\theta x^*(\theta)$ and then proceed to sign it. But this approach has limitations. First, $x^*(\cdot)$ might not be differentiable, or might not even be a function. Second, even when $x^*(\theta)$ is a differentiable function, it might not be possible to sign $D_\theta x^*(\theta)$.

Consider a concrete example:

$$\max_{x \in \mathbb{R}} f(x, \theta),$$

where $\Theta = \mathbb{R}$, f is twice continuously differentiable, and strictly concave (so that $f_{xx} < 0$.)[10] Under these conditions the problem has a unique solution for every parameter θ that is characterized by the first order condition

$$f_x(x^*(\theta), \theta) = 0.$$

Since f is strictly concave, the conditions of the IFT hold (you can check this) and we get:

$$\frac{\partial x^*(\theta)}{\partial \theta} = -\frac{f_{xt}(x^*(\theta), \theta)}{f_{xx}(x^*(\theta), \theta)}.$$

[10] In this section we will use a lot of partial derivatives. Let f_{xy} denote $\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$.

And so

$$sign(\frac{\partial x^*(\theta)}{\partial \theta}) = sign(f_{xt}(x^*(\theta), \theta)).$$

In other words, the solution is increasing if and only if $f_{xt}(x^*(\theta), \theta) > 0$.

The example illustrates the strong requirements of the IFT: it only works if f is smooth, if the maximization problem has a unique solution for every parameter θ, and we can sign the derivative over the entire range of parameters only when f is strictly concave.

Also, from the example one might conclude (mistakenly) that smoothness or concavity were important to identify the effect of θ on x^*. To see why this would be the wrong conclusion, consider the class of problems:

$$x^*(\theta) = \arg \max_{x \in D(\theta)} f(x, \theta) \text{ for } \theta \in \Theta.$$

Now if $\varphi : \mathbb{R} \to \mathbb{R}$ is any increasing function, we have that

$$x^*(\theta) = \arg \max_{x \in D(\theta)} \varphi(f(x, \theta)) \text{ for } \theta \in \Theta.$$

In other words, applying an increasing transformation to the objective function changes the value function, but not the solution set. But this tells us that continuity, differentiability, and concavity of f have little to do with whether $x^*(\cdot)$ is increasing in θ. Even if f has all these nice properties, the objective function in the second problem won't be continuous if φ has jumps, won't be differentiable if φ has kinks, and won't be concave if φ is very convex. And yet any comparative statics conclusions that apply to f will also apply to $g = \varphi \circ f$!

We now develop a powerful series of results that allow us to make comparative statics conclusions *without* assumptions about continuity, differentiability, or concavity. A bonus is that the conditions required are often much easier to check than those required by the IFT.

5.3.1 MCS with One Choice Variable

Consider the case in which there is one variable, one parameter, and a fixed constraint set. In other words, let $f : S \times T \to \mathbb{R}$, with $S, T \subset \mathbb{R}$, and consider the class of problems

$$x^*(\theta) = \arg \max_{x \in D} f(x, \theta) \text{ for } \theta \in T.$$

We assume that the maximization problem has a solution ($x^*(\theta)$ always exists), but not that the solution is unique (so $x^*(\theta)$ may be a set).

The fact that $x^*(\theta)$ may be a set introduces a complication. How can a set be nondecreasing? For two sets of real numbers A and B, we say that $A \leq_s B$ in the *strong set order* if for any $a \in A$ and $b \in B$, $\min\{a, b\} \in A$ and $\max\{a, b\} \in B$. Note that if $A = \{a\}$ and $B = \{b\}$, then $A \leq_s B$ just means that $a \leq b$.

The strong set order is illustrated in Figure 17. In the top figure, $A \leq_s B$, but this is not true in the bottom figure. You should try using the definition to see exactly why this is true (hint: in the bottom figure, pick the two middle points as a and b).

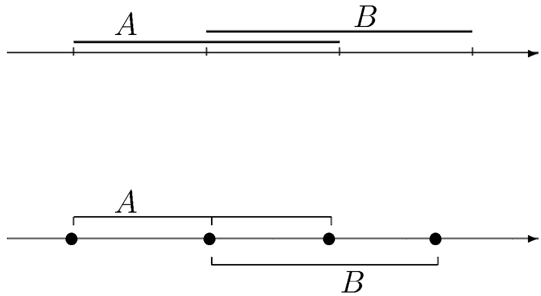

Figure 17: The Strong Set Order.

We will use the strong set order to make comparative statics conclusions: $x^*(\cdot)$ is nondecreasing in θ if and only if $\theta < \theta'$ implies that $x^*(\theta) \leq_s x^*(\theta')$. If $x^*(\cdot)$ is a function, this has the standard meaning that the function is nondecreasing. If $x^*(\cdot)$ is a compact-valued correspondence, then the functions $\overline{x}^*(\theta) = \max_{x \in x^*(\theta)} x$ and $\underline{x}^*(\theta) = \min_{x \in x^*(\theta)} x$ (i.e. the largest and smallest maximizers) are nondecreasing.

Definition 8 *The function $f : \mathbb{R}^2 \to \mathbb{R}$ is **supermodular** in (x, θ) if for all $x' > x$, $f(x', \theta) - f(x, \theta)$ is nondecreasing in θ.*[11]

[11]If f is supermodular in (x, θ), we sometimes say that f has **increasing differences** in (x, θ).

What exactly does this mean? If f is supermodular in (x, θ), then the incremental gain to choosing a higher x (i.e. x' rather than x) is greater when θ is higher. You can check that this is equivalent to the property that if $\theta' > \theta$, $f(x, \theta') - f(x, \theta)$ is nondecreasing in x.

The definition of supermodularity does not require f to be "nice". If f happens to be differentiable, there is a useful alternative characterization.

Lemma 12 *A twice continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ is supermodular in (x, θ) if and only if $f_{x\theta}(x, \theta) \geq 0$ for all (x, θ).*

The next result, *Topkis' Monotonicity Theorem*, says that supermodularity is sufficient to draw comparative statics conclusions in optimization problems.

Theorem 13 (Topkis' Monotonicity Theorem) *If f is supermodular in (x, θ), then $x^*(\theta) = \arg\max_{x \in D} f(x, \theta)$ is nondecreasing.*

Proof. Suppose $\theta' > \theta$, and that $x \in x^*(\theta)$ and $x' \in x^*(\theta')$. We first show that $\max\{x, x'\} \in x^*(\theta')$. Since $x \in x^*(\theta)$, then $f(x, \theta) - f(\min\{x, x'\}, \theta) \geq 0$. This implies (you should check this) that $f(\max\{x, x'\}, \theta) - f(x', \theta) \geq 0$. So by supermodularity, $f(\max\{x, x'\}, \theta') - f(x', \theta) \geq 0$, which implies that $\max\{x, x'\} \in x^*(\theta')$.

We now show that $\min\{x, x'\} \in x^*(\theta)$. Since $x' \in x^*(\theta')$, then $f(x', \theta') - f(\max\{x, x'\}, \theta') \geq 0$, or equivalently $f(\max\{x, x'\}, \theta') - f(x', \theta') \leq 0$. By supermodularity, $f(\max\{x, x'\}, \theta) - f(x', \theta) \leq 0$. This implies (again you should verify) that $f(x, \theta) - f(\min\{x, x'\}, \theta) \leq 0$, so $\min\{x, x'\} \in x^*(\theta)$. *Q.E.D.*

Topkis' Theorem is handy in many situations. It allows one to prove that the solution to an optimization problem is nondecreasing simply by showing that the objective function is supermodular in the choice variable and the parameter.

Example Consider the following example in the theory of the firm. A monopolist faces the following problem:

$$\max_{q \geq 0} p(q)q - c(q, \theta)$$

where q is the quantity produced by the monopolist, $p(q)$ denotes the market price if q units are produced, and $c(\cdot)$ is the cost function. The parameter θ affects the monopolist's costs (for instance, it might be the price of a key input). Let $q^*(\theta)$ be the monopolist's optimal quantity choice. The objective function is supermodular in (x, θ) as long as $c_{q\theta}(q, \theta) \leq 0$. Thus, $q^*(\theta)$ is nondecreasing as long as θ decreases the marginal cost of production. Note that we can draw this conclusion with *no assumptions* on the demand function p, or the concavity of the cost function!

Remark 1 *In the example, we found conditions under which $q^*(\theta)$ would be nondecreasing in θ. But what if we wanted to show that $q^*(\theta)$ was nonincreasing in θ? We could have done this by showing that the objective function was supermodular in $(x, -\theta)$, rather than supermodular in (x, θ). The former means that q^* will be nondecreasing in $-\theta$, or nonincreasing in θ.*

5.3.2 Useful Tricks for Applications

There are several useful tricks when it comes to applying Topkis' Theorem.

1. **Parameterization.** In many economics models, we end up wanting to compare the solution to two distinct maximization problems, $\max_{x \in D} g(x)$ and $\max_{x \in D} h(x)$. (For instance, we might want to show that a profit-maximizing monopolist will set a higher price that a benevolent social-surplus maximizing firm.) It turns out we can do this using Topkis' Theorem, if we introduce a parameter $\theta \in \{0, 1\}$ and construct a "dummy objective function" f as follows:

$$f(x, \theta) = \begin{cases} g(x) \text{ when } \theta = 0 \\ h(x) \text{ when } \theta = 1 \end{cases}.$$

   If the function f is supermodular (i.e. if $h(x) - g(x)$ is nondecreasing in x), then $x^*(1) \geq x^*(0)$. Or in other words, the solution to the second problem is greater than the solution to the first. This trick is *very* useful — there is an exercise at the end that lets you try it.

2. **Aggregation.** Sometimes, an optimization problem will have many choice variables, but we want to draw a conclusion about only one of them. In these cases, it may be possible to apply Topkis' Theorem by "aggregating" the choice variables we care less about.

Consider the problem

$$\max_{\substack{x\in\mathbb{R},y\in\mathbb{R}^k\\(x,y)\in S}} f(x,y,\theta).$$

If we are only interested in the behavior of x we can rewrite this problem as

$$\max_{x\in\mathbb{R}} g(x,\theta) \text{ where } g(x,\theta) = \max_{\substack{y\in\mathbb{R}^k\\(x,y)\in S}} f(x,y,\theta).$$

In other words, we can decompose the problem in two parts: first find the maximum value $g(x,\theta)$ that can be achieved for any x, then maximize over the "value function" $g(x,\theta)$. If g is supermodular in (x,θ), Topkis' Theorem says that in the original problem, x^* will be nondecreasing in θ.

Example Here's an example to see how the aggregation method works. Consider the profit maximization problem of a competitive firm that produces an output x, using k inputs z_1,..., z_k,

$$\max_{\substack{x\in\mathbb{R},z\in\mathbb{R}^k\\x\leq F(z)}} px - w\cdot z.$$

p is the price of output, w the vector of input prices, and $F(\cdot)$ the firm's production function. Suppose that we are only interested on how p affects output $x^*(p)$. Then we can rewrite the problem as

$$\max_{x\in\mathbb{R}} g(x,p) = px - c(x) \text{ where } c(x) = \min_{\substack{z\in\mathbb{R}^k\\x\leq F(z)}} w\cdot z,$$

where $c(\cdot)$ is called the cost function. Then, since $g(x,p)$ has increasing differences in (x,p), we get that the firm's supply curve is nondecreasing, regardless of the shape of the production function.

5.3.3 Some Extensions

Topkis' Theorem says that f being supermodular in (x,θ) is a sufficient condition for $x^*(\theta)$ to be nondecreasing. It turns out supermodularity is a stronger assumption than what one really requires.

Definition 9 *A function $f:\mathbb{R}^2\to\mathbb{R}$ is* ***single-crossing*** *in (x,θ) if for all $x'>x$ and $\theta'>\theta$, (i) $f(x',\theta)-f(x,\theta)\geq 0$ implies that $f(x',\theta')-f(x,\theta')\geq 0$ and also (ii) $f(x',\theta)-f(x,\theta)>0$ implies that $f(x',\theta')-f(x,\theta')>0$.*

It's easy to show that if f is supermodular in (x, θ), then it is also single crossing in (x, θ). However, as illustrated in Figure 18, the opposite is not true. The figure plots an objective function for a problem with a discrete choice set $X = \{0, 1, 2\}$ and a continuous parameter set. You should verify that f satisfies the single crossing property, but it does not have increasing differences.

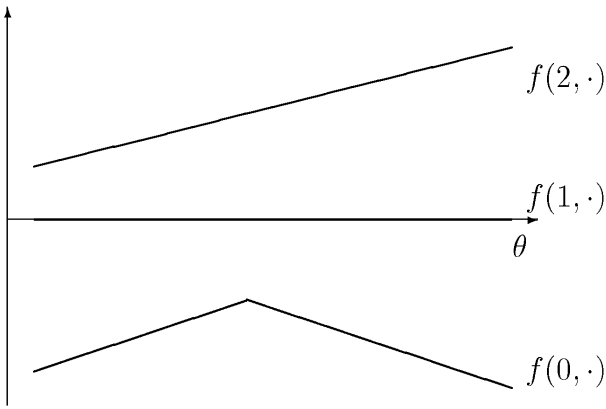

Figure 18: A function that is single crossing, but not supermodular.

Theorem 14 (Milgrom-Shannon Monotonicity Theorem) *If f is single-crossing in (x, θ), then $x^*(\theta) = \arg\max_{x \in D(\theta)} f(x, \theta)$ is nondecreasing. Moreover, there is a converse: if $x^*(\theta)$ is nondecreasing in θ for all choice sets D, then f is single-crossing in (x, θ).*

The first part of the Milgrom-Shannon Theorem is a bit stronger than Topkis' result. The proof is quite similar (in fact, if you look at the proof above, you will see that an identical proof suffices for Milgrom-Shannon). The second part says that single crossing is actually a *necessary* condition for monotonicity conclusions. What you should take away from this is the following: if you ever want to show that some variable is nondecreasing in a parameter, at some level you will necessarily be trying to verify the single-crossing condition.

There are times when the conclusion of the Milgrom-Shannon or Topkis Theorems is not exactly what we want. Sometimes we would like to show that a variable is strictly increasing in a parameter, and not just nondecreasing.

It turns out that the stronger assumption that $f_x(x,\theta)$ is *strictly increasing* in θ is almost enough to guarantee this, as the following result (due to Edlin and Shannon, 1998) shows.

Theorem 15 *Suppose that f is continuously differentiable in x and that $f_x(x,\theta)$ is increasing in θ for all x. Then for all $\theta < \theta'$, $x \in x^*(\theta)$, and $x' \in x^*(\theta')$ we get that:*

$$x \text{ is in the interior of the constraint set } \Rightarrow x < x'.$$

Proof. Since f is supermodular in (x,θ), Topkis' Theorem implies that $x \leq x'$. To see that the inequality is strict, note that because $x \in int(D)$ and $x \in x^*(\theta)$, we must have $f_x(x,\theta) = 0$. But then $f(x,\theta') > 0$. Since $x \in int(D)$, there must be some $\widehat{x} > x$ with $f(\widehat{x},\theta') > f(x,\theta')$. So $x \notin x^*(\theta')$. *Q.E.D.*

Unlike the above results, we can apply this result *only* if f is differentiable in x. However, its assumptions are still significantly weaker than the IFT — for instance, it applies equally well when Θ is discrete. Note that the theorem does require x to be an interior solution. Why? If x was at the upper boundary, there would be nowhere to go, so we could only have the weak conclusion $x \leq x'$.

6 Duality Theory

This final section looks at special class of PCOP that arise in consumer and producer theory. These problems have the following form:

$$V(\theta) = \max_{x \in K} \; \theta \cdot x,$$

where K is a convex subset of $\mathbb{R}^n$, and $\theta \in \mathbb{R}^n$. The essential feature of this problem is that the objective function is linear. This will allow us to say a lot of useful things about the solution and the value functions.

To do this we need to develop some new concepts.

Definition 10 *A **half-space** is a set of the form*

$$H_s(\theta, c) = \{x \in \mathbb{R}^n \mid \theta \cdot x \geq c\}, \; \theta \neq 0.$$

Definition 11 *A **hyperplane** is a set of the form*

$$H(\theta, c) = \{x \in \mathbb{R}^n \mid \theta \cdot x = c\}, \theta \neq 0.$$

These two concepts are illustrated in figure 19. In the case of $n = 2$, a hyperplane is a line, and a half-space is the set to one side of that line. In the case of $n = 3$, a hyperplane is a plane in three dimensional space.

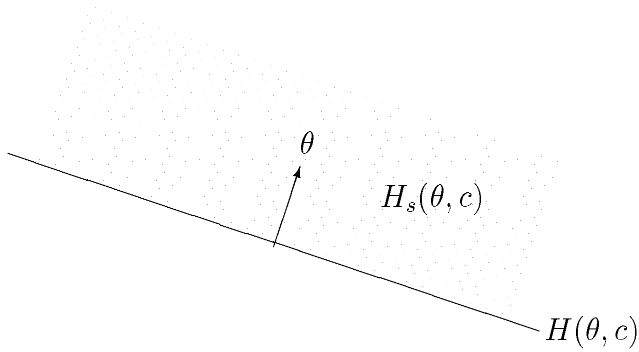

Figure 19: A Hyperplane and a Half-Space.

Theorem 16 (Separating Hyperplane Theorem) *Suppose that S and T are two convex, closed, and disjoint ($S \cap T = \emptyset$) subsets or $\mathbb{R}^n$. Then there exists $\theta \in \mathbb{R}^n$, and $c \in \mathbb{R}$ such that*

$$\theta \cdot x \geq c \textit{ for all } x \in S \textit{ and } \theta \cdot y < c \textit{ for all } y \in T.$$

This result is illustrated in figure 20. If the sets S and T are not intersecting and they are closed, then we can draw a hyperplane between them such that S lies strictly to one side of the hyperplane and T lies to the other. Note that the vector θ points towards the set S and that it need not be uniquely defined. (Can you think of an example when it is uniquely defined?) The figure on the bottom also shows that separation by a hyperplane may not be possible if one of the sets is not convex.

A consequence of the Separating Hyperplane Theorem is that *any* closed and convex set S can be described as the intersection of *all* half-spaces that contain it:

$$S = \bigcap_{(\theta,c) \text{ s.t. } S \subset H_s(\theta,c)} H_s(\theta, c).$$

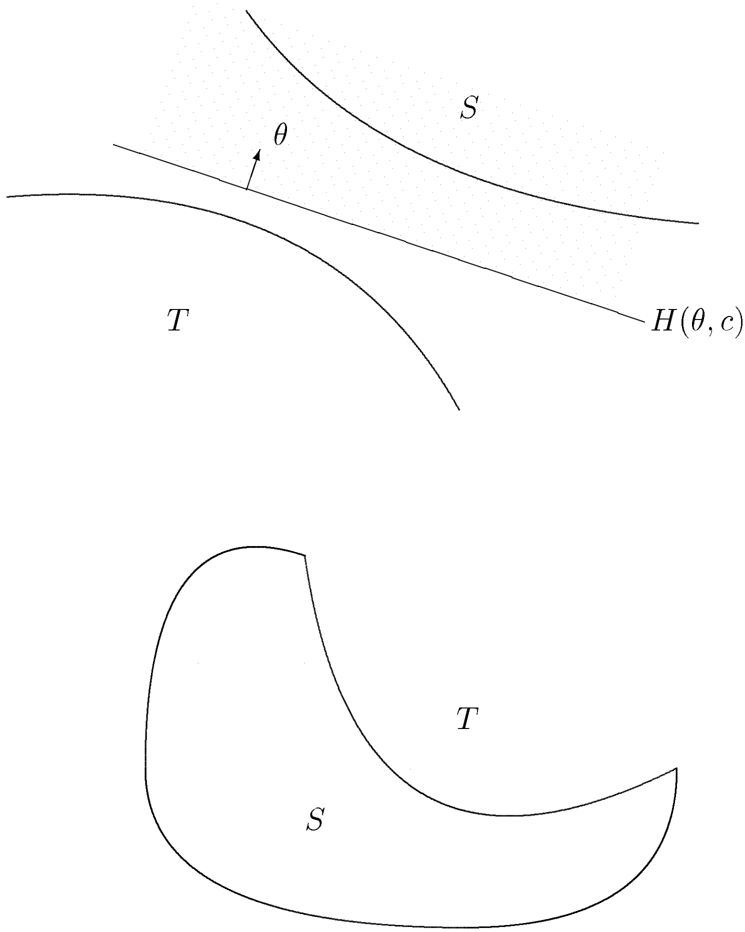

Figure 20: Separating Convex Sets.

This is easily seen in figure 21. If follows directly from the fact that for any point $x \notin S$ there is a half-space that contains S but not x. As simple as it is, this is the key idea in duality theory, and we will get a lot of mileage out of it.

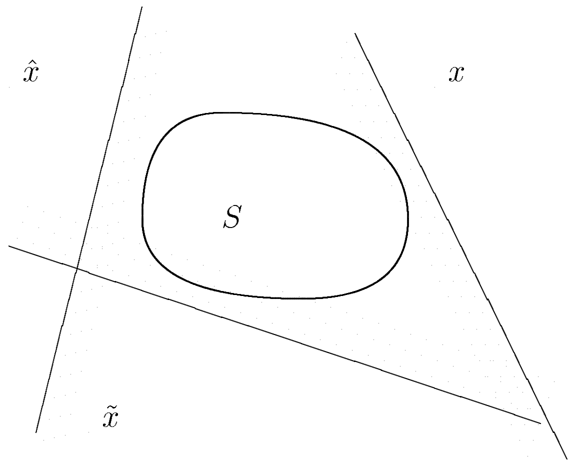

Figure 21: A convex set is the intersection of all half-spaces that contain it.

The same idea can be expressed using the notion of support functions. Any set S has two support functions:

$$M_S(\theta) = \sup\{\theta \cdot x \mid x \in S\} \qquad \text{and} \qquad m_S(\theta) = \inf\{\theta \cdot x \mid x \in S\}.$$

To understand these definitions you need to know that "sup" and "inf" are extensions of the ideas of "max" and "min". In particular:

Definition 12 *Let A be a non-empty set of real numbers.*

1. *The* ***infimum*** *of A, $\inf A$, is given by $x \in \mathbb{R} \cup \{-\infty, \infty\}$ such that: (i) $x \leq y$ for all $y \in A$, and (ii) there is no $\widehat{x}$ such that $x < \widehat{x} \leq y$ for all $y \in A$.*

2. *The* ***supremum*** *of A, $\sup A$, is given by $x \in \mathbb{R} \cup \{-\infty, \infty\}$ such that: (i) $x \geq y$ for all $y \in A$, and (ii) there is no $\widehat{x}$ such that $x > \widehat{x} \geq y$ for all $y \in A$.*

The following examples emphasize the distinction between inf and min, and sup and max.

1. $\inf [0,1] = 0$, and $\min [0,1] = 0$
2. $\sup [0,1) = 1$, but $\max [0,1)$ does not exist.
3. $\inf (0,\infty) = 0$ and $\sup (0,\infty) = \infty$, but neither $\min (0,\infty)$ nor $\max (0,\infty)$ exist.

If $\min S$ exists, then $\min S = \inf S$. The difference between inf and min arises when the set does not have a min. In that case, if S is bounded below then $\inf S$ is the greatest lower bound, if not it is $-\infty$. In other words, $\inf S = -\infty$ is an equivalent way of saying that the set S is not bounded below.

Let's try an example using support functions.

Example Figure 22 depicts the set $S = \{x \in \mathbb{R}^2 \mid x_1 \leq 0 \text{ and } x_2 \leq 0\}$. Let's characterize $m_S(\theta)$. First, consider any vector θ that points to a direction outside of S. If we choose a series of x's in the direction of $-\theta$, this will decrease $\theta \cdot x$ without bound. Hence $m_S(\theta) = \inf \{\theta \cdot x \mid x \in S\} = -\infty$. By contrast, consider a vector $\theta^{'}$ on the axis, or a vector such as $\widehat{\theta}$ that points towards the interior of S. In these cases, the value of $\theta \cdot x$ is minimized by taking $x = 0$, and $m_S(\theta) = 0$.

How about $M_S(\theta)$? Once more, there are three cases. If θ points to a direction outside of S, then $\sup \{\theta \cdot x \mid x \in S\} = 0$. The same is true in the direction θ' on the axis. However, for any vector $\widehat{\theta}$ that points towards the interior of S, we have that $M_S(\theta) = \infty$. (To see this notice that as we move in the direction of $\widehat{\theta}$ farther and farther down we increase the value of $\widehat{\theta}.x$ while staying in the set S. Thus, $\{\theta \cdot x \mid x \in S\}$ is not bounded above).

We can use the support functions to give a characterization of any closed, convex set S:

$$S = \bigcap_{\theta} \{x \mid \theta \cdot x \leq M_S(\theta)\} \qquad \text{or} \qquad S = \bigcap_{\theta} \{x \mid \theta \cdot x \geq m_S(\theta)\}\,.$$

The intuition for this is straightforward. We have seen before that any convex and closed set is equal to the intersection of all the half-spaces that contain it. The expression $S = \cap_\theta \{x|\theta.x \geq m_S(\theta)\} = \cap_\theta H_S(\theta, m_S(\theta))$ is almost identical except that it takes the intersection only over some half-spaces, not all half-spaces. But by construction of the support function $m_S(\theta)$, we have that if $m > m_S(\theta)$, then S is not contained in $H_S(\theta, m)$, whereas if $m < m_S(\theta)$,

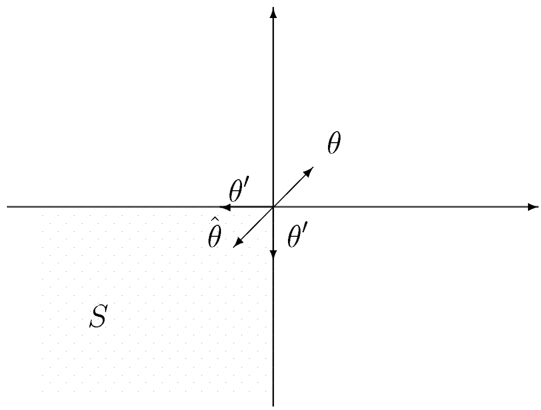

Figure 22: Support functions for a convex set.

then $H_S(\theta, m) \subset H_S(\theta, m_S(\theta))$. This suggests that for each direction θ there is a unique hyperplane that is relevant for characterizing the set S — the one given by the support function.

Support functions have some very useful properties. To state them, we need one definition.

Definition 13 *A function $h : \mathbb{R}^n \to \mathbb{R}$ is* ***homogeneous (of degree one)*** *if for all $t > 0$, $h(tx) = th(x)$.*

Theorem 17 *1. $M_S(\cdot)$ is homogeneous and convex in θ.*

2. $m_S(\cdot)$ is homogeneous and concave in θ.

Proof. (1) Homogeneity follows immediately from the fact that if $t > 0$:

$$M_S(t\theta) = \sup\{t\theta \cdot x \mid x \in S\} = t \cdot \sup\{\theta \cdot x \mid x \in S\} = tM_S(\theta).$$

Now consider convexity. We will need two facts (proving them is an exercise). Fact 1: For all non-empty sets of real numbers A and B, $\sup(A + B) \leq \sup A + \sup B$. Fact 2: If $A \subset B \subset \mathbb{R}$, then $\sup A \leq \sup B$.

We need to show that for all $t \in [0, 1]$, and θ, θ',

$$M_S(\theta^t) \leq tM_S(\theta) + (1 - t)M_S(\theta').$$

Note that

$$tM_S(\theta) = \sup \underbrace{\{t\theta \cdot x \mid x \in S\}}_{A},$$

and

$$(1-t)M_S(\theta') = \sup \underbrace{\{(1-t)\theta \cdot x \mid x \in S\}}_{B}.$$

Then since $M_S(\theta^t) = \{t\theta \cdot x + (1-t)\theta \cdot x \mid x \in S\} \subset A+B$, we can use Facts 1 and 2 to conclude:

$$M_S(\theta^t) \leq \sup A + B \leq \sup A + \sup B = tM_S(\theta) + (1-t)M_S(\theta').$$

(2) The proof is essentially the same. *Q.E.D.*

We are now ready to prove one last big result.

Theorem 18 (Duality Theorem) *Let S be any non-empty and closed subset of $\mathbb{R}^n$. Then*

1. *$M_S(\cdot)$ is differentiable at $\widehat{\theta}$ if and only if there exists a unique $\widehat{x} \in S$ such that $\widehat{\theta} \cdot \widehat{x} = M_S(\widehat{\theta})$. Furthermore, in that case $D_\theta M_S(\widehat{\theta}) = \widehat{x}$.*

2. *$m_S(.)$ is differentiable at $\widehat{\theta}$ if, and only if, there exists a unique $\widehat{x} \in S$ such that $\widehat{\theta} \cdot \widehat{x} = m_S(\widehat{\theta})$. Furthermore, in that case $D_\theta m_S(\widehat{\theta}) = \widehat{x}$.*

The Duality Theorem establishes necessary *and* sufficient conditions for the support function to be differentiable. The support function $M_S(\cdot)$ is differentiable if and only if the maximization problem $\max\{\widehat{\theta} \cdot x | x \in S\}$ has a unique solution $\widehat{x}$. Furthermore, in that case the "envelope" derivative $D_\theta M_S(\widehat{\theta})$ is equal to $\widehat{x}$. Similarly, the support function $m_S(\cdot)$ is differentiable if and only if the minimization problem $\min\{\widehat{\theta} \cdot x | x \in S\}$ has a unique solution $\tilde{x}$, in which case the envelope derivative $D_\theta M_S(\widehat{\theta})$ is equal to $\tilde{x}$.

Okay, so we have worked very hard to define these support functions. But why do we care? We care because the value function for the class of problems that we are interested in,

$$V(\theta) = \max_{x \in K} \theta \cdot x,$$

looks like a support function. In fact, as long as the problem has a unique solution we know that $V(\theta) = M_K(\theta)$. Thus, we have learned that in this maximization problem:

1. The value function is homogeneous and convex; and

2. As long as the maximization problem has a unique solution $x^*(\theta)$, the value function is differentiable and $D_\theta V(\theta) = x^*(\theta)$.

Analogous statements also hold for minimization problems.

It is difficult to overstate the usefulness of these results: if you keep them in mind, you will be *amazed* at how many results in price theory follow as a direct consequence!

7 Exercises

1. Are the following functions continuous? Provide a careful proof.

   (a) $X = Y = \mathbb{R}$, and $h(x) = x^2 + 5$

   (b) $X = (-\infty, 0] \cup \{1\} \cup [2, \infty)$, $Y = \mathbb{R}$, and $h(x) = \begin{cases} x & \text{if } x \leq 0 \\ 2x & \text{if } x = 1 \\ 10x & \text{otherwise} \end{cases}$

   (c) $X = \mathbb{R}^{2n}$, $Y = \mathbb{R}$, and $h(x, y) = \|x - y\|$.

2. Prove that the correspondence in Figure 4 is usc but not lsc, and that the correspondence in Figure 5 is lsc but not usc. (Note: we proved the negative claims in the text, but in a "chatty" way; try writing down a careful proof).

3. Consider the correspondence given by

$$\phi(x) = \begin{cases} \{0, 1/x\} & \text{if } x > 0 \\ \{0\} & \text{if } x = 0 \end{cases}.$$

   Show that ϕ is lower semi-continuous, but not upper semi-continuous.

4. A Walrasian budget set at prices p and wealth w is a subset of $\mathbb{R}^n_+$ given by

$$B(p, w) = \{x \in \mathbb{R}^n_+ | p.x \leq w\},$$

   for $p \in R^n_{++}$ and w.

   (a) Prove that the correspondence $B : \mathbb{R}^n_{++} \times \mathbb{R} \Rightarrow \mathbb{R}^n_+$ is usc and lsc.

   (b) Does our definition of usc applies if we extend the domain of the correspondence to include zero prices? Why?

5. Consider the function $f : [0, 1] \to \mathbb{R}$, given by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1 \end{cases}.$$

Does the problem $\max_{x \in [0,1]} f(x)$ have a solution? Why doesn't the Theorem of the Maximum apply?

6. Prove the version of the Kuhn-Tucker Theorem that includes non-negativity constraints. (Hint: Apply the original Kuhn-Tucker Theorem to the optimization problem that includes the additional n non-negative constraints.)

7. Solve the following optimization problem using the Kuhn Tucker Theorem.

$$\min_{x \in \mathbb{R}^2_+} w_1 x_1 + w_2 x_2 \text{ subject to } y \geq (x_1^\rho + x_2^\rho)^{1/\rho},$$

assuming that $y, w, \rho > 0$.

8. Consider the following optimization problem

$$\max_{x \in \mathbb{R}^n_+} \prod_{i=1}^{n} x_i^{\alpha_i} \text{ subject to } p.x \leq w$$

where $\alpha_i, p_i, w > 0$. Compute the solution and value function to this problem for any parameters (p, w) using Lagrange's Theorem. (You need to show first that this Theorem characterizes the solution to the problem even though it has inequality constraints.

9. Let x^* and V denote the solution set and value function for the previous problem. The goal of this problems is to familiarize you with the mechanics of the IFT and the ET. Thus, for the moment ignore the explicit solutions that you obtained in problem 6.

   (a) Look at the FOCs for the problem. Are the conditions of the IFT satisfied?

   (b) Suppose that $n = 2$. Compute the $D_{(p,w)}x^*$ using the IFT.

   (c) Now compute $D_{(p,w)}V$ using the Envelope Theorem.

   (d) Now compute $D_{(p,w)}x^*$ using the IFT.

(e) Now compute $D_{(p,w)}V$ and $D_{(p,w)}x^*$ directly by taking the derivatives of x^* and V. Do you get the same answer?

10. Show that for any two sets of real number A, B: $A \leq_s B$ if and only if (1) $A \backslash B$ lies below $A \cap B$ and (2) $B \backslash A$ lies above $A \cap B$.

11. (due to Segal) A function $f : \mathbb{R}^2 \to \mathbb{R}$ has strict increasing differences in (x, θ) if $x' > x$, $\theta > \theta'$

$$f(x', \theta') - f(x, \theta') > f(x', \theta) - f(x, \theta).$$

(a) Show that if f has strict increasing preferences, then for *any selection* of $x^*(\theta) \in Arg\max_{x \in X} f(x, \theta)$ and $x^*(\theta') \in \arg\max_{x \in X} f(x, \theta')$ with $\theta' > \theta'$ we must have $x^*(\theta') \geq x^*(\theta)$.

(b) Show by means of an example that property does not guarantee that $x^*(\theta') > x^*(\theta)$. (Hint: the easiest examples involve a discrete choice space.)

12. Consider the problem of a profit maximizing firm discussed in Section 5.3

$$\max_{\substack{x \in \mathbb{R}, z \in \mathbb{R}^k \\ x \leq F(z)}} \; px - wz.$$

Show that for any input k, $z_k^*(p, w)$ is non-increasing in w_k. (Hint: Use the aggregation method.)

13. Use the aggregation method to show that Hicksian demand curves are downward-sloping. The consumer expenditure problem is:

$$\min_{x \in \mathbb{R}^n} p \cdot x \text{ subject to } u(x) \geq \overline{u},$$

where $p = (p_1, ..., p_n)$ is the price of each good and $x_1, ..., x_n$ the quantities consumed. Your goal is to show that $x_1^*(p_1, ..., p_n)$ is nonincreasing in p_1.

14. Consider the following problem that arises in monopoly theory. A monopolist solves the problem

$$\max_{x \geq 0} \pi(x) = P(x)x - c(x),$$

where $P(x)$ is the inverse demand function. (It provides the price at which the market demands exactly x units of the good.) The following problem characterizes the maximization of total social surplus:

$$\max_{x \geq 0} W(x) = \pi(x) + CS(P(x)),$$

where $CS(p) = \int_p^\infty x(p)dp$ is consumer surplus, and $x(p)$ is the demand function (i.e., assume that $P(x(p)) = p$ for all p and $x(P(x)) = x$ for all x.).

The goal of the exercise is to compare the monopoly output to the surplus-maximizing output level. Let $\theta = 0$ correspond to the firm's profit-maximization program, $\theta = 1$ correspond to the total surplus maximization program, and define

$$\max_{x \geq 0} f(x, \theta) = \pi(x) + \theta \cdot CS(P(x)).$$

Suppose that the inverse demand function $P(x)$ and consumer surplus $CS(p)$ are non-increasing functions. Is this enough to show that the monopoly output is always (weakly) less than the surplus maximizing level?

15. Let $S = \{x \in \mathbb{R}^2 | \, \|x - 0\| \leq r^2\}$. Compute the support functions $M_S(\theta)$ and $m_S(\theta)$ for this set.

16. Prove that for every set convex and closed subset S of $\mathbb{R}^n$, $S = \cap_\theta \{x | \theta.x \geq m_S(\theta)\}$. Show also that the result is not true if S is open or non-convex.

17. Prove that for all non-empty sets of real numbers A and B, $\sup(A + B) \leq \sup A + \sup B$.

Normal-Form Games

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1 Introduction

Game theory is the study of strategic behavior. The key feature of game-theoretic settings is that they involve multiple agents who are strategically interdependent. In contrast, many of the problems we studied last quarter involved purely individual decisions about consumption or production. What is the difference? Imagine that two people walk into a restaurant and look at the menu. We can model this as two individual decision problems. But what if these people decide to each order an entree and share them? Now we have a game.

Since its introduction in the 1940s, game theory has become a central tool in many branches of economcs. To whet your appetite, consider just a few examples.

- Political Economy: How legislators vote depends on how others are voting, and how they expect their constituents and interest groups to react.

- Public Finance: The benefits from contributing to a public good depend on what everyone else contributes.

- Industrial Organization: Firm's profits depend on the prices they set and products they offer, but also on the prices and products offered by other firms.

- Labor Economics: Firms try to design incentive schemes and structure compensation to modify behavior.

- International Trade: Levels of imports and exports, and prices, depend on your tariffs, but also on the tariffs of others.

- Urban Economics: What time should a commuter leave home, knowing the traffic will depend on when others leave?

2 Normal Form Games

We will start by considering one representation of strategic environments — games in normal (or strategic) form.

Definition 1 *A normal form game G consists of:*

- *A finite set of agents $\mathcal{I} = \{1, 2, ..., I\}$*
- *Strategy sets $S_1, ..., S_I$.*
- *Payoff functions $u_i : S_1 \times ... \times S_I \to \mathbb{R}$ for each $i = 1, ..., I$.*

A *strategy profile* $s = (s_1, ..., s_I)$ specifies a strategy for each player. We'll write $S = S_1 \times S_2 \times ... \times S_I$ for the space of strategy profiles. It will also be useful to let s_{-i} denote the vector of strategies chosen by everyone except i, $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_I)$.

3 Examples

Let's look at some examples. First, we consider two simple games.

Prisoners' Dilemma Two prisoners are taken into separate rooms to be interrogated. Each is told that they can either plead innocence or implicate the other prisoner. If just one prisoner implicates the other, the one who talks will be immediately released, while the other will hang. If both implicate the other, they will end up in jail. Finally, if both plead innocence, they will eventually be released, but not for a while.

We can represent this strategic situation as follows. Both players must choose independently from the strategy set $\{C, D\}$, where C represents the strategy of holding out and pleading innocent, and D represents the strategy of finking (or "defecting") on the other prisoner. The payoffs are as follows: if both hold out, then each get 1, so $u_i(C, C) = 1$. If i holds out, but j finks, then i gets -1, while j gets 2, so $u_i(C, D) = -1$ and $u_i(D, C) = 2$. Finally, if both fink, they each get 0, so $u_i(D, D) = 0$.

It is often convenient to summarize simple games like this in (bi-) matrix form.

	C	D
C	$1, 1$	$-1, 2$
D	$2, -1$	$0, 0$

Note that the key feature of the prisoners' dilemma is that each player benefits by doing something bad to the other (playing D instead of C). This is true in many strategic interactions — for example arms' races (where building up a nuclear arsenal offers protection, but threatens other countries).

Battle of the Sexes On New Year's Eve, a couple finds themselves separated at a party as midnight approaches. There are two natural spots to meet — the bar, or the dance floor. All things equal, one prefers the bar, the other the dance floor. But most of all, they want to coordinate and end up in the same place at midnight.

	B	F
B	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

We capture this as follows. Both must choose independently from the strategy set $\{B, F\}$. Both have a payoff 0 from miscoordinating, i.e. $u_i(B, F) = u_i(F, B) = 0$. If they meet at the bar, one gets a payoff of 2, the other a payoff of 1. Payoffs are reversed if they meet on the dance floor.

Now consider two models of duopolistic competition.

Cournot Competition In Cournot's (1838) model of imperfect competition, firms 1 and 2 choose quantities q_1, q_2. Firm i has a cost function $c_i(\cdot)$, and the market inverse demand curve is $P(Q)$, where $Q = q_1 + q_2$. The normal form of this game has $s_i = q_i$, $S_i = [0, \infty)$.

$$u_i(s_i, s_{-i}) = P(s_1 + s_2)\, s_i - c_i(s_i).$$

Bertrand Competition In Bertrand's model of competition, firms set prices rather than quantities. If there are two firms, they set prices p_1, p_2. Consumers all choose the firm with the lowest price, or, if the prices are the same split their purchases equally. Assume there is a mass Q of consumers, and firm i has per-unit costs $c_i(\cdot)$. To represent this as a normal form game, we take $s_i = p_i$, so that $S_i = [0, \infty)$ and define:

$$u_i(s_i, s_{-i}) = \begin{cases} 0 & \text{if } s_i > s_{-i} \\ s_i Q - c_i(Q) & \text{if } s_i < s_{-i} \\ s_i \frac{Q}{2} - c_i\left(\frac{Q}{2}\right) & \text{if } s_i = s_{-i} \end{cases}.$$

4 Dominant and Dominated Strategies

We now start to investigate the following question: how will people behave in strategic environments? We start by exploring the most basic implications of rationality for strategic play.

Definition 2 *A strategy s_i is* ***strictly dominated*** *if there exists some $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,*

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}).$$

If a given strategy is strictly dominated, there is some other strategy that does strictly better *regardless of opponents' play.* This suggests that a rational agent will not use a dominated strategy. To formalize this idea, we need to have a definition of what it means to behave rationally in a strategic environment.

Definition 3 *Player i is rational with beliefs μ_i if:*

$$s_i \in \arg\max_{s_i \in S_i} \mathbb{E}_{\mu_i(s_{-i})} u_i\left(s_i, s_{-i}\right),$$

that is, if s_i maximizes i's expected payoff $\sum_{s_{-i}} u_i(s_i, s_{-i}) \mu_i(s_{-i})$.

Proposition 1 *If player i is rational, then she wil not play a strictly dominated strategy.*

Proof. Suppose that s_i is strictly dominated by s_i'. Then for any beliefs μ_i,

$$\begin{aligned} \mathbb{E}_{\mu_i(s_{-i})} u_i\left(s_i, s_{-i}\right) &= \sum_{s_{-i}} u_i(s_i, s_{-i}) \mu_i(s_{-i}) \\ &< \sum_{s_{-i}} u_i(s_i', s_{-i}) \mu_i(s_{-i}) = \mathbb{E}_{\mu_i(s_{-i})} u_i\left(s_i', s_{-i}\right). \end{aligned}$$

The key inequality holds because $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for *all* s_{-i}, and $\mu_i(s_{-i})$ must be strictly positive for some s_{-i}. It follows that s_i cannot be justified for any beliefs. *Q.E.D.*

Prisoners' Dilemma In the prisoners' dilemma, the strategy C is strictly dominated by the strategy D.

	C	D
C	$1, 1$	$-1, 2$
D	$2, -1$	$0, 0$

If a player has one single strategy that strictly dominates all others, we say this strategy is *strictly dominant* (or sometimes that the player has a dominant strategy).

Battle of the Sexes In the battle of the sexes, neither strategy is strictly dominated. Playing B is best if you believe your opponent will play B; playing F is best if you believe your opponent will play F.

5 Iterated Strict Dominance

The line of argument we are following can be pushed further. Suppose that player j *knows* that player i is rational. If j himself is rational, and understands the above proposition, he then *knows* that i will never use a dominated strategy. Thus, when j forms his beliefs about i's play, he should put probability zero on i playing a dominated strategy. If follows that if j is rational, then not only will j not want to play a dominated strategy, j will not want to play a strategy that is dominated *conditional on i not playing a dominated strategy.*

Example Consider the following game:

	L	M	R
U	$2, 2$	$1, 1$	$4, 0$
D	$1, 2$	$4, 1$	$3, 5$

In this game, neither of Row's strategies are dominated. But M is dominated for Column. Once M is eliminated, then D is dominated for Row. But once both D is eliminated, then R is dominated for Column. We are left with the strategy profile (U, L).

To formalize this idea, we define the process of iterated strict deletion.

- Define $S_i^0 = S_i$.

- For each $k = 0, 1, 2....$, define:

$$S_i^{k+1} = \left\{ \begin{array}{c} s_i \in S_i^k \mid \nexists s_i' \in S_i^k \text{ with} \\ u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}^k \end{array} \right\}.$$

- Define $S_i^\infty = \cap_{k=1}^\infty S_i^k$.

To understand this process, observe that initially all strategies are allowed. At the first round, strictly dominated strategies are eliminated to obtain the restricted strategy sets S_i^1. At the second round, we eliminate those strategies that are strictly dominated conditional on opponents using strategies in S_{-i}^1. Eventually this process must converge since each set S_i^0, S_i^1, etc.. is getting progressively smaller.

Let $S^\infty = S_1^\infty \times ... \times S_I^\infty$ denote the set of strategy profiles that survive the entire process of iterated strict deletion. If S^∞ contains a single strategy profile, we say the game is *dominance solvable.* Some games (such as the prisoners' dilemma) are dominance solvable, but most games are not. For example, in battle of the sexes, no strategies can be eliminated, so $S^\infty = S$.

Note that each level of iterated dominance invokes one further implication of rationality. First, that each i is rational. Second, that each i is rational and knows everyone else is rational. And so on.... The full restriction to strategies in S^∞ depends on *common knowledge of rationality* (everyone is rational, knows everyone is rational, and so on).

Proposition 2 *If players $1, ..., I$ are rational, and this fact is common knowledge, then no player i will play a strategy that is eliminated by iterated strict deletion of dominated strategies.*

6 Rationalizability

In their respective dissertations, Bernheim (1984) and Pearce (1984) investigated the implications of rationality from an alternative perspective. Instead of asking what players won't do, they asked what sort of behavior could be *directly* justified in an environment where rationality was common knowledge. Their intuitive idea is that a strategy should be *rationalizable* if it is a best response to beliefs that are themselves consistent with the rationality of others.

Example Consider the following game.

	a	b	c	d
U	$4, 10$	$3, 0$	$1, 3$	$2, 6$
D	$0, 0$	$2, 10$	$10, 3$	$3, 6$

- U is rationalizable. It is a best response to a, which is itself a best response to U.
- D is also rationalizable. It is a best response to d. In turn, d is a best response if Column believes that U, D are equally likely (and we already know that U is rationalizable).

This logic also tells us that a, d are rationalizable. Similarly, b is rationalizable because it is a best-response to D (which we just showed was rationalizable). However, c is not rationalizable because it is not a best response to anything.

Now consider a formal definition.

Definition 4 *A subset $B_1 \times ... \times B_I \subset S$ is a* ***best reply set*** *if for all i and all $s_i \in B_i$, there exists a probability distribution over opponent strategies $\mu_i \in \Delta(B_{-i})$ such that s_i is a best reply to beliefs μ_i.*[1]

Definition 5 *The set of* ***rationalizable strategies*** *is the component by component union of all best reply sets:*

$$R = R_1 \times ... \times R_I = \bigcup_{\alpha} B_1^{\alpha} \times ... \times B_I^{\alpha}$$

where each $B^{\alpha} = B_1^{\alpha} \times ... \times B_I^{\alpha}$ is a best reply set.

It is not hard to show that R is itself a best reply set, and of course contains all the others. Thus R is the *maximal* best reply set.

In thinking about solution concepts, it is often useful to think about what these solution concepts imply about (i) how players form beliefs about opponents' behavior, and (ii) how players act given their beliefs. In these terms, rationalizability makes two essential assumptions about strategic behavior.

1. Each player is rational, and maximizes his payoff given his beliefs about his opponents' play.

2. Each player's beliefs do not conflict with others being rational, or being aware of others' rationality, and so on. However, beliefs need not be correct.

[1] For the cognoscenti, note that I have allowed beliefs μ_i to reflect correlation between opponent choices, i.e. $\mu_i \in \Delta(B_{-i})$ rather than $\mu_i \in \times_{j \neq i} \Delta(B_j)$. This leads to a definition of rationalizability that coincides exactly with ISD (as commented on below). The original papers by Bernheim and Pearce assume independence, which (in games of three or more players) leads to a slightly less permissive solution concept.

You might be wondering about the relationship between strategies that are rationalizable and those that survive iterated strict deletion. As we have defined ISD, these sets are not exactly the same. However, it turns out that once we allow for mixed strategies, rationalizability and iterated strict deletion yield identical solutions — that is, $R = S^\infty$!

7 Nash Equilibrium

So far, we have considered the implications of rationality (and common knowledge of rationality) for predicting how people will play games. However, while iterated strict dominance is an attractive solution concept that makes relatively weak assumptions about behavior, it does not have much bite in many games. Relatively few games are actually dominance solvable, and often virtually every strategy is rationalizable.

This leads us to the notion of Nash equilibrium, the most important concept for solving games. Nash equilibrium captures the idea that players ought to do as well as they can given the strategies actually chosen by the other players.

Definition 6 *A strategy profile $(s_1, ..., s_I)$ is a* ***(pure strategy) Nash equilibrium*** *of G if for every i, and every $s_i' \in S_i$,*

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}).$$

A Nash equilibrium profile is an "equilibrium" in the following sense: given that everyone knows what everyone else is doing, no one wants to change their behavior. It is also useful to define the notion of a strict Nash equilibrium.

Definition 7 *A strategy profile $(s_1, ..., s_I)$ is a* ***strict Nash equilibrium*** *of G if for every i, and every $s_i' \in S_i$,*

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}).$$

Consider two simple examples.

Prisoners' Dilemma

	C	D
C	$1, 1$	$-1, 2$
D	$2, -1$	$0, 0$

The unique (strict) Nash Equilibrium is (D, D).

Battle of the Sexes

	B	F
B	$2,1$	$0,0$
F	$0,0$	$1,2$

There are two (strict) pure Nash equilibria (B, B) and (F, F).

Nash equilibrium has an appealing internal consistency in that no player would unilaterally wish to deviate from a Nash profile. An important issue, however, is how the equilibrium arises in the first place. To see what is required, it is useful again to think of Nash equilibrium as imposing a requirement both on how players form beliefs about their opponents' play, and about how they act given their beliefs. Nash equilibrium supposes that:

1. Each player is rational and maximizes his payoff with respect to his beliefs about opponent play.

2. Each player's beliefs about opponent play are *correct*.

The second assumption is quite strong. How exactly could a player come to *know* what his opponents will play? Depending on the situation being studied, several possible interpretations of Nash equilibrium might be offered.

1. Introspection. In some games (for example, those that are dominance solvable, or those with a "focal" equilibrium), introspection may lead players to make correct conjectures about their opponents' behavior. But not all games (e.g. Battle of the Sexes) will have an obvious play.

2. Communication/Self-enforcing agreement. If players can communicate prior to play, and agree on a strategy profile, any Nash equilibrium profile will be *self-enforcing* in the sense that no one will unilaterally want to deviate if it is agreed on.

3. Result of Learning/Convention. If the strategic situation arises repeatedly in a given society, people over time may learn what typical behavior is. A Nash equilibrium would then constitute a stable (non-changing) social arrangement.

8 Applications & Examples

8.1 Cournot Competition

Let's consider a particular case of Cournot duopoly, with linear demand and linear costs. Assume that:

$$P(q_1 + q_2) = 1 - (q_1 + q_2),$$

and for some $1 > c > 0$,

$$C(q_i) = cq_i.$$

Firm i's payoff function is:

$$u_i(q_i, q_j) = q_i \left[1 - (q_i + q_j) - c\right]$$

Consider the problem facing player i given that he expects his opponent to produce q_j. He solves:

$$\max_{q_i} q_i \left[1 - (q_i + q_j) - c\right].$$

His marginal returns to higher quantity are:

$$\frac{\partial u_i(q_i, q_j)}{\partial q_i} = 1 - 2q_i - q_j - c$$

so using the first order condition, the "best response" to opponent quantity q_j is

$$BR(q_j) = \max\left\{\frac{1-c}{2} - \frac{q_j}{2}, 0\right\}.$$

To find a Nash equilibrium, we look for a pair (q_i, q_j) with the property that:

$$q_i \in BR_i(q_j) \qquad \text{and} \qquad q_j \in BR_j(q_i).$$

It is easy to see that there will be a unique Nash equilibrium.

Proposition 3 *The unique Nash equilibrium is for each firm to choose quantity* $q = (1-c)/3$.

Interestingly, it turns out that the Cournot duopoly model is also dominance solvable. To show this, we need one preliminary result.

Lemma 1 *If* $q_{-i} \in [\underline{q}, \overline{q}]$, *then any quantity* $q_i > BR(\underline{q})$ *or* $q_i < BR(\overline{q})$ *is strictly dominated.*

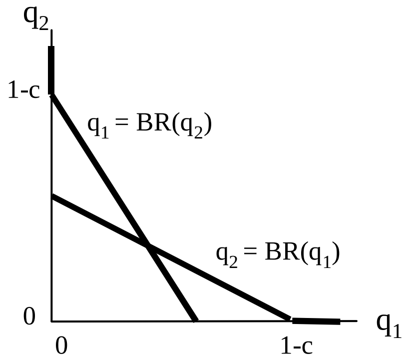

Figure 1: Cournot Duopoly

Proof. Note that the marginal returns to higher quantity are decreasing in both q_i and q_j. Thus, for all $q_j \leq \overline{q}$, and $q_i < BR(\overline{q})$, we have

$$\frac{\partial u_i(q_i, q_j)}{\partial q_i} > 0.$$

So $q_i < BR(\overline{q})$ will be strictly dominated by $BR(\overline{q})$, so long as $q_j \leq \overline{q}$. Similarly, we can show that $q_i > BR(\underline{q})$ will be strictly dominated by $BR(\underline{q})$ so long as $q_j \geq \underline{q}$. *Q.E.D.*

Now, let's use this claim to apply ISD.

- $S_i^0 = [0, \infty)$
- $S_i^1 = [0, BR(0)]$, where $BR(0)$ is the monopoly quantity $(1-c)/2$.
- $S_i^2 = [BR^2(0), BR(0)]$, where $BR^2(0) = (1-c)/2 - (1-c)/4$.

Continuing, we get a sequence of intervals $S_i^k = [\underline{q}^k, \overline{q}^k]$, where $\underline{q}^k = BR(\overline{q}^{k-1})$ and $\overline{q}^k = BR(\underline{q}^{k-1})$. Clearly, if either the sequence of upper or lower bounds converge, they both will converge to the same single point. Consider the lower bound:

$$\underline{q}^k = \frac{1-c}{2} - \frac{\overline{q}^{k-1}}{2} = \frac{1-c}{4} + \frac{\underline{q}^{k-2}}{4}.$$

It is easy to check that this sequence with initial value $q^0 = 0$, will converge to:

$$\underline{q} = \overline{q} = \frac{1-c}{3}.$$

Thus, $S^\infty = \left\{\frac{1-c}{3}, \frac{1-c}{3}\right\}$, and the game is dominance solvable.

8.2 Bertrand Competition

Consider the Bertrand model of competition with linear costs, $C_i(q_i) = cq_i$, with $1 > c > 0$, and a total market demand of $Q \geq 1$.

Proposition 4 *The unique Nash equilibrium is for both firms to set a price $p = c$.*

Proof. We first show that (c, c) is a Nash equilibrium, i.e. that $p_i = c$ is a best response to $p_j = c$. Note that if $p_j = c$, then i has three options. He can set $p_i < p_j$ and make Q sales, each at a loss. He can set $p_i < p_j$ and make no sales, or he can set $p_i = c$ and make $Q/2$ sales, each a zero profit. So $p_i = c$ is among his best responses, and it follows that (c, c) is a NE.

To see that (c, c) is the unique equilibrium, we have to consider different cases.

- Suppose $p_i \leq p_j < c$. Firm i makes positive sales, each at a loss, and would prefer to raise his price above p_j. So this cannot be a NE.
- Suppose $p_i < c \leq p_j$. Firm i again makes positive sales, each at a loss, and would prefer to raise his price above c. So this cannot be a NE.
- Suppose $c \leq p_i < p_j$. Firm i makes positive sales, but can always find a price $p \in (p_i, p_j)$ at which he would make the same number of sales, but at a higher price. Also, Firm j makes no sales, so if $c < p_i$, he would like to undercut Firm i. For both reasons, this cannot be a NE. *Q.E.D.*

Unlike Cournot duopoly, Bertrand duopoly is not dominance solvable. It is fairly easy to see setting $p_i = 0$ is strictly dominated by $p_i = c_i$ (since the former results in a demand of at least $Q/2$ and negative profits, while the latter always gives zero profits). However, no other strategies (not even prices below marginal cost) can be elimated by ISD! To see why, consider two prices $p_i, p' > 0$. Neither is dominated by the other because there will always be some opponent price $p_j < p_i, p'$ against which p_i, p' yield identical (zero) profits.

8.3 Bertrand Competition with Differentiated Products

The Bertrand model of competition has the appealing feature that firms choose prices, but the unappealing feature that firm's end up pricing at marginal cost even if there are only two of them. A more realistic model emerges if we assume that consumers have preferences for one firm or the other, so that an ε-price difference does not lead to wild swings in demand. In particular, suppose that firms have constant marginal costs c, and choose prices $p_1, p_2 \in [0, \infty)$, and that the demand for Firm i's product is:

$$D_i(p_i, p_j) = 1 - p_i + \lambda p_j.$$

Then Firm i's chooses his price to maximize:

$$\max_{p_i} (p_i - c) \left[1 - p_i + \lambda p_j\right].$$

The first order condition for this problem is:

$$1 - 2p_i + \lambda p_j + c = 0$$

leading to a best response function:

$$BR_i(p_j) = \frac{1+c}{2} + \frac{\lambda p_j}{2}.$$

Proposition 5 *The unique Nash equilibrium is for both firms to set a price* $p = \frac{1+c}{2-\lambda}$.

Again, a graphical derivation is illuminating.

8.4 Games with Multiple Equilibria

Even if one believes strongly in the Nash equilibrium concept, in some games, there may be a great many such equilibria. This raises the difficult problem of *equilibrium selection*: which Nash equilibrium is most likely to be played?

Bargaining: Nash Demand Game Imagine two parties who must bargain over a pie of size 1. Each player can make a demand x_i. If $x_1 + x_2 > 1$, then both parties get zero. If $x_1 + x_2 \leq 1$, the demands are compatible, and player i can take home a share x_i of the pie.

Claim. Any split of the pie $(x, 1-x)$ with $x \in [0, 1]$ is a Nash equilibrium outcome.

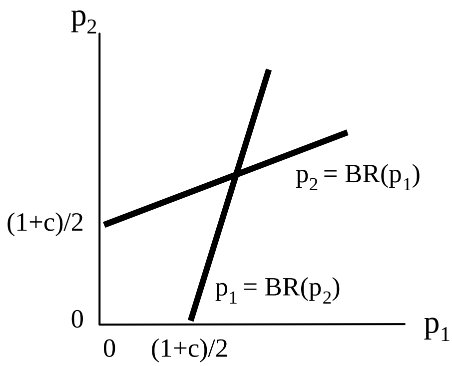

Figure 2: Bertrand-Nash Duopoly

Proof. Suppose that j demands x_j. Then i's best-response is to demand $x_i = 1 - x_j$. Thus, it must be that in any Nash equilibrium, $x_i + x_j = 1$, but there are no further constraints on the values of x_i, x_j. *Q.E.D.*

Stag Hunt This game has two pure strategy Nash equilibria, (A, A) and (B, B). Both are strict.

	A	B
A	$9, 9$	$-5, 8$
B	$8, -5$	$7, 7$

It is natural to ask which of these two equilibria is more likely. One view is that (A, A) should be played, since it is the Pareto-preferred equilibrium. For instance, shouldn't the players meet before the game and agree to play (A, A)? There is a question, however, of whether this sort of communication would be credible. Even if Row was planning to play B, he would want to convince Column to play A by claiming that he himself would play A. In fact, Harsanyi and Selten (1988) argue that (B, B) is in fact more likely, because it is *risk-dominant* (i.e. B is a best reply if you are completely uncertain and assign probability $1/2$ to each opponent strategy).

9 Mixed Strategies

While pure strategy Nash equilibrium is a natural solution concept, there are some situations where a pure strategy Nash equilibrium profile does not exist.

Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

This game has *no* pure strategy Nash equilibrium. At a matched profile (H, H) or (T, T), Column wants to switch strategies, while at a mismatched profile (H, T) or (T, H), Row wants to switch strategies.

An important feature of matching pennies is that it is a game of *pure conflict*, or a zero-sum game. Any gain for i is a loss for j. Zero-sum games were the original class of games studied by game theorists. Other examples include parlor games such as poker, sporting events such as soccer or football, and military conflicts.

In games of pure conflict, if player i knows his opponent's strategy, he can exploit this to his advantage. Fearing this, his opponent may prefer to be unpredictable and to choose a *randomized* or *mixed strategy*. For example, in matching pennies, Column could decide to play H with probability $1/2$ and T with probability $1/2$.

Definition 8 *A* ***mixed strategy*** σ_i *for player* i *is a probability distribution on* S_i*, i.e. for* S_i *finite, a mixed strategy is a function* $\sigma_i : S_i \rightarrow \mathbb{R}_+$ *such that*

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1.$$

We will use several different notations for mixed strategies. For example, the strategy of mixing equally between Heads and Tails could be represented alternatively as: $\sigma_1(H) = \frac{1}{2}$, $\sigma_1(T) = \frac{1}{2}$; or as $(\sigma_1(H), \sigma_1(T)) = (\frac{1}{2}, \frac{1}{2})$, or finally as $\sigma_1 = \frac{1}{2}H + \frac{1}{2}T$.

We use $\Sigma_i = \Delta(S_i)$ to represent the set of mixed strategies available to player i, and Σ to represent the set of mixed strategy profiles. And we write $u_i(\sigma_i, \sigma_{-i})$ to represent i's payoff given a profile σ:

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i, s_{-i}} u_i(s_i, s_{-i})\sigma_i(s_i)\sigma_{-i}(s_{-i}).$$

Definition 9 *A mixed strategy profile $\sigma = (\sigma_1, ..., \sigma_I)$ is a Nash equilibrium of G if for all $\sigma_i' \in \Sigma_i$,*

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}).$$

A somewhat thorny question is what is means exactly for a player to play a mixed strategy. The most obvious interpretation is that players *explicitly* randomize, perhaps by flipping a coin. However, this is somewhat unsatisfactory, given that we don't often see people going around flipping precisely weighted coins to make decisions. Another interpretation, due to Harsanyi, is that mixed strategies reflect small uncertainties in payoffs, and the probability of playing a given strategy is the probability that this small uncertainty will resolve in favor of that strategy. We will return to this idea in a few weeks. A third interpretation, which we will return to sooner is that mixed strategy equilibria describe a stable state for a *population* of players, each of whom uses a pure strategy. Finally, the interpretation favored by some current theorists (e.g. Aumann and Brandenburger, 1995) is that people don't actually randomize, but that mixed strategy equilibria can be thought of as equilibria in *beliefs*. That is, given a mixed strategy equilibrium (σ_1, σ_2) we think of σ_i simply as j's belief about what i will do, but *not necessarily* as the analyst's prediction about what i will do, or even as i's belief about what i will do.[2]

9.1 Characterizing Mixed Strategy Equilibria

An important feature of mixed strategy equilibria is that if (σ_i, σ_{-i}) is a mixed Nash equilibrium profile, then i must be indifferent between every strategy s_i in the support of σ_i.

Definition 10 *In a finite game, the support of a mixed strategy σ_i is the set of pure strategies to which σ_i assigns positive probability.*

$$supp(\sigma_i) = \{s_i \in S_i \mid \sigma_i(s_i) > 0\}.$$

Proposition 6 *If σ is a mixed strategy Nash equilibrium, and $s_i, s_i' \in supp(\sigma_i)$, then:*

$$u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i}).$$

[2] If this seems confusing, don't worry.

This suggest a method for identifying mixed strategy equilibria in a given game. First identify the support of each player's mixed strategy profile. Second, identify the mixture that j must be using to keep i indifferent between all pure strategies in the support of his mixed strategy.

Example Consider the following game:

	L	R
U	$1,2$	$3,1$
D	$2,1$	$0,2$

This game has no pure strategy Nash equilibrium. We look for a mixed equilibrium where Row randomizes between U, D. For this to happen, it must be that Row is indifferent between U, D. But this can only happen if Column is playing the mixed strategy $\frac{3}{4}L + \frac{1}{4}R$. But for Column to be mixing, it must be that Column is indifferent between L, R. This can only happen if Row is playing the mixed strategy $\frac{1}{2}U + \frac{1}{2}D$. It follows that the unique NE is $(\frac{1}{2}U + \frac{1}{2}D, \frac{3}{4}L + \frac{1}{4}R)$.

There is also a nice graphical way to identify this same equilibrium. Let $pU+(1-p)D$ denote an arbitrary strategy for Row (with $p \in [0,1]$) and $qL + (1-q)R$ denote an arbitrary strategy for Column. Then

$$BR_R(qL + (1-q)R) = \begin{cases} U & \text{if } q < 3/4 \\ pU + (1-p)D & \text{if } q = 3/4 \\ D & \text{if } q > 3/4 \end{cases}$$

and

$$BR_C(pU + (1-p)D) = \begin{cases} L & \text{if } p > 1/2 \\ qL + (1-q)R & \text{if } p = 1/2 \\ R & \text{if } p < 1/2 \end{cases}$$

We find a Nash equilibrium, we need a profile (σ_R, σ_C) with the property that:

$$\sigma_R \in BR_R(\sigma_C) \qquad \text{and} \qquad \sigma_C \in BR_C(\sigma_R)$$

This is shown in Figure 3.

Note that if σ is a mixed equilibrium, then σ cannnot be a strict NE. As we just noted, if σ_i is mixed, then i must be indifferent between every strategy in the support of σ_i. But why should a player play precisely his mixed

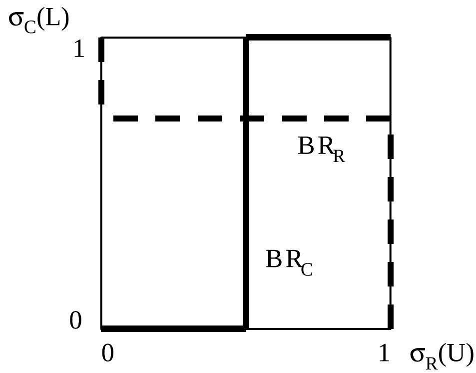

Figure 3: Graphical Derivation of Mixed NE

strategy, given that he is indifferent between that an many other strategies. For instance, in our example, Row is indifferent between playing U, playing D and playing any randomization. Why should Row play precisely $\frac{1}{2}U + \frac{1}{2}D$ rather than some other strategy? How problematic this is depends on how one wishes to interpret or justify Nash equilibrium. It does not seem terribly problematic if we view Nash equilibrium as a self-enforcing agreement, or as a stable social configuration. It seems more problematic if we want to think about Nash equilibrium as arising just from introspection.

9.2 Iterated Dominance Revisited

Now that we have defined mixed strategies, it is possible to re-define iterated strict dominance in a way that more completely exploits rationality.

Example Consider the following game:

	L	M	R
U	$4, 10$	$3, 0$	$1, 3$
D	$0, 0$	$2, 10$	$10, 3$

In this game, no strategy is strictly dominated by another pure strategy. However, for Column, R is strictly dominated by the mixed strategy $(1/2, 1/2, 0)$ that assigns equal probability to L, M. Once R is deleted, then U strictly dominates D. It follows that the game is dominance solvable giving the profile (U, L).

Formally, we can expand our definition of ISD as follows:

- $S_i^0 = S_i$
- $S_i^{k+1} = \left\{ \begin{array}{c} s_i \in S_i^k \mid \nexists \sigma_i \in \Delta(S_i^k) \text{ such that} \\ u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}^k \end{array} \right\}.$
- $S_i^\infty = \cap_{k=0}^\infty S_i^k$.

Proposition 7 *Suppose σ is a Nash Equilibrium of G, and that $\sigma_i^*(s_i) > 0$. Then s_i is not removed by ISD, $s_i \in S_i^\infty$.*

Proof. Let σ be a Nash equilibrium of G, and suppose (by way of contradiction) that not every pure strategy in its support survives ISD. Let s_i be he first pure strategy with $\sigma_i(s_i) > 0$ that is removed during ISD, and suppose this occurs at round k. This means that there exists some $\sigma_i' \in \Delta(S_i)$ with the property that:

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i}) \qquad \text{for all } s_{-i} \in S_{-i}^k.$$

However, since s_i is the *first* strategy in the support of the NE σ to be removed, it must be that for any $s_{-i} \in$supp(σ_{-i}), it is also the case that $s_{-i} \in S_{-i}^k$. Thus,

$$u_i(\sigma_i', \sigma_{-i}) > u_i(s_i, \sigma_{-i})$$

which contradicts the fact that σ is a Nash Equilibrium. *Q.E.D.*

9.3 Application: Equilibrium Price Dispersion

Some compelling applications of mixed strategy Nash equilibrium come in games with a large population of players. Consider the following model of price competition. There are a continuum of firms, $i \in [0, 1]$, each of whom posts a price p_i. Once firms have posted prices, consumers make purchase decisions. There is a mass $m > 1$ of consumers, each of whom has willingness to pay v. A fraction μ of these consumers sample a single firm's price and buy if the price is below v, while the remainder $1 - \mu$ sample two prices and purchase from the firm that sets the lower price, conditional on this price being below v. All firms have constant marginal cost equal to $c < v$.

Consider the problem facing firm i. Suppose that i sets a price p_i. A fraction μ of consumers sample only one price, meaning that Firm i can expect a mass μm of these consumers to arrive and purchase if and only if $p_i < v$. Firm i can also expect a mass $(1 - \mu)m$ of consumers to arrive, who also will see a second price. Let $F_i(p_i)$ denote the probability that such a consumer will see a price below p_i as her second price. Then Firm i's demand is:

$$D_i(p_i) = \begin{cases} \mu m + (1-\mu)m\left[1 - F_i(p_i)\right] & \text{if } p_i \leq v \\ 0 & \text{if } p_i > v \end{cases}$$

Let's look for a Nash equilibrium of this model. First, note that so long as $\mu \in (0,1)$, there cannot be an equilibrium where all firms set the same price p. If all firms set the same price $p > c$, then by dropping its price by ε, firm i can gain $(1-\mu)m/2$ new sales for new profit of $(p-c-\varepsilon)(1-\mu)m/2$, and incur a loss of only $\varepsilon\mu m$. On the other hand, if all firms set $p \leq c$, then by charging $p_i = v$ a deviating firm can go from at best zero profits to $\mu m(v-c)$ in profits.

Instead, we look for a mixed equilibrium, where each firm uses all prices between $\underline{p}$ and v, for some $\underline{p}$, and where the probability of setting a price below p is $F(p)$. For this to be an equilibrium, we need firms to realize the same profits for any price in use. Since

$$u_i(p_i = v, F) = (v-c)\,\mu m$$

we have that for all $p \in [\underline{p}, v]$,[3]

$$u_i(p,F) = (p-c)\left[\mu m + (1-\mu)m\,(1-F(p))\right] = (v-c)\mu m.$$

This allows us to solve for $F(\cdot)$:

$$F(p) = 1 - \frac{\mu}{1-\mu}\frac{v-p}{p-c},$$

in turn, we solve $F(\underline{p}) = 0$ to find that:

$$\underline{p} = \mu v + (1-\mu)c.$$

Note that if $\mu = 1$, then all consumers look at only a single price, and hence there is a pure strategy equilibrium where all firms set the monopoly price $p = v$. If $\mu = 0$, then all consumers look at two prices, so there is effectively Bertrand competition and there is a pure strategy equilibrium in which all

[3] You might be wondering why v is the highest price used. Clearly it makes no sense to set a price above v, since no consumer will purchase. On the other hand, if the highest price used was $\overline{p} < v$, then a firm that priced at $\overline{p}$ would sell only to consumers who looked at only one price. Given that it sells only to these consumers, it does better to price at v than $\overline{p} < v$.

firms price a marginal cost $p = c$. Finally, for $\mu \in (0,1)$, we have equilibrium price dispersion.

Observe that this price dispersion can be interpreted in two ways. One is that each firm literally plays a mixed strategy, randomizing in its choice of price. Alternatively, we can think of each firm as playing a different pure strategy, but as the population distribution of prices being $F(\cdot)$. Either way, we have an equilibrium!

10 Properties of Nash Equilibrium

We now take up a few of the important properties of the Nash equilibrium solution concept.

10.1 Existence of Nash Equilibrium

The first is the celebrated existence theorem from Nash's Ph.D. thesis (1951).

Proposition 8 *(Nash) At least one Nash equilibrium exists in any finite game.*

To prove this, we will rely on a characterization of Nash equilibrium that we have already been using informally. Define the *best response correspondence* of player i as follows:

$$BR_i(\sigma_{-i}) = \arg\max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_{-i}).$$

Lemma 2 *σ is a Nash equilibrium if and only if for all i, $\sigma_i \in BR_i(\sigma_{-i})$.*

This observation permits us to identify Nash equilibria as *fixed points* of the correspondence $BR : \Sigma \rightrightarrows \Sigma$ given by

$$BR(\sigma) = \left(BR_1(\sigma_{-1}), ..., BR_I(\sigma_{-I})\right).$$

To proceed further, we need to take a short mathematical detour to discuss fixed points of functions and correspondences.[4]

Let $X, Y \subset \mathbb{R}^n$, and let $r : X \rightrightarrows Y$ be a correspondence. Recall that X is *convex* if and only if $x, x' \in X$ implies that $\lambda x + (1-\lambda)x' \in X$ for all $\lambda \in [0,1]$. Recall also that X is *compact* if and only if it is closed and bounded.

[4] For those of you who have seen the proof of the existence of Walrasian equilibria in exchange economies, much of this will look familiar.

The correspondence r is *non-empty valued* if for all $x \in X$, $r(x) \neq \emptyset$. The correspondence r is *convex-valued* if for all $x \in X$, $r(x)$ is a convex set. Finally, we can define the *graph* of r as follows:

$$Graph(r) = \{(x, y) \in X \times Y \mid y \in r(x)\}.$$

The correspondence r has a *closed graph* if $Graph(r)$ is a closed subset of $X \times Y$.

Lemma 3 *(Kakutani's Fixed Point Theorem) Suppose that X is a non-empty, compact, convex subset of $\mathbb{R}^n$ and that $r : X \rightrightarrows X$ is a non-empty and convex-valued correspondence with a closed graph. Then r has a fixed point, i.e. there exists some $x \in X$ with the property that $x \in r(x)$.*

To prove Nash's Theorem, we apply Kakutani's Fixed Point Theorem to the best response correspondence $BR : \Sigma \rightrightarrows \Sigma$.

Proof of Nash's Theorem. Consider the best response correspondence $BR : \Sigma \rightrightarrows \Sigma$. We proceed to verify the conditions for Kakutani's Theorem.

1. Σ is a compact, convex subset of $\mathbb{R}^I$.

2. BR is non-empty valued. To see this, recall that

$$BR_i(\sigma_{-i}) = \arg\max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_{-i})$$

   Since $u_i(\sigma_i, \sigma_{-i})$ is continuous in σ_i, and Σ_i is compact, a maximizer exists, so $BR_i(\sigma_{-i})$ exists for all σ_{-i} and hence $BR(\sigma)$ exists for all σ.

3. BR is convex-valued. To see this, suppose $\sigma_i, \sigma_i' \in BR_i(\sigma_{-i})$. Then, any s_i in the support of σ_i gives the same payoff, and this payoff is the same as that given by any s_i in the support of σ_i'. It follows that i is indifferent to any randomization amongst these pure strategies. So $BR_i(\sigma_{-i})$ must be convex, and hence so is $BR(\sigma)$.

4. BR has a closed graph. Suppose that $(\sigma^k, \hat{\sigma}^k) \in Graph(BR)$, and that $\sigma^k \to \sigma$ and $\hat{\sigma}^k \to \hat{\sigma}$. We want to show that $\hat{\sigma} \in BR(\sigma)$, that is, for all i that $\hat{\sigma}_i \in BR_i(\sigma_{-i})$. To see this, note that for all s_i, and all k,

$$u_i(\hat{\sigma}_i^k, \sigma_i^k) \geq u_i(s_i, \sigma_i^k).$$

   Furthermore, u_i is continuous in all arguments. So by taking limits:

$$u_i(\hat{\sigma}_i, \sigma_i) \geq u_i(s_i, \sigma_i).$$

Applying Kakutani's Theorem, there exists some profile σ such that $\sigma \in BR(\sigma)$, and hence σ is a Nash Equilibrium. *Q.E.D.*

Nash's Theorem establishes the existence of a *mixed strategy equilibrium* in *finite games.* It is also interesting to inquire whether there are conditions under which *pure strategy equilibrium* are known to exist. This is clearly not the case in arbitrary finite games, as we have already seen examples (such as matching pennies) where there is no pure strategy Nash equilibrium. However, pure strategy equilibrium will certainly exist in certain well–behaved games with continuous strategy spaces.

Proposition 9 *(Debreu-Fan-Glicksberg) Let G be a normal form game. Suppose that each strategy set S_i is a non-empty, compact, convex subset of $\mathbb{R}^n$ and that each payoff function u_i is continuous in s_i, s_{-i} and quasi-concave in s_i. Then G has a pure strategy Nash equilibrium.*

There are also mixed strategy existence theorems for games with continuous (and infinite) strategy spaces. Problems with existence typically arise because the payoff functions are discontinuous. In this case, best responses may not necessarily exist. A recent and beautiful paper on the existence of equilibrium in games with discontinuous payoff functions is by Reny (2000).

10.2 Other Properties of Nash Equilibrium

A useful property of Nash equilibrium is that as the payoff functions change, the set of Nash equilibria changes in an upper hemi-continuous way.

Proposition 10 *Consider a sequence of games G^k with finite strategy space S, but differing in the payoff functions u^k. Suppose that for all i, $u_i^k \to u_i$ as $k \to \infty$, and let G denote the game (S, u). If $\sigma^k \in NE(G^k)$ for all k, and $\sigma^k \to \sigma$, then $\sigma \in NE(G)$.*

Proof. This is just like the proof that BR has a closed graph, and it left as an exercise. *Q.E.D.*

Another useful result is that finite games typically have only a finite (and odd) number of equilibria. To state this result, we need a notion of genericity. First, letting G and G' be two games with finite strategy space S and payoff functions u, u', we define the *distance* between G, G' as:

$$\|G' - G\| = \left(\sum_{s \in S} \sum_{i \in I} \left(u_i'(s) - u_i(s) \right)^2 \right)^{1/2}.$$

Definition 11 *A property holds generically for finite normal form games if it holds for a set X such that: (i) if $G \in X$, then there is some $\varepsilon > 0$ such that if $\|G' - G\| < \varepsilon$, then $G' \in X$, and (ii) if $G \notin X$, then for any $\varepsilon > 0$, there exists some $G' \in X$ with $\|G' - G\| < \varepsilon$.*

Proposition 11 *(Wilson) Finite normal form games generically have a finite and odd number of Nash equilibria.*

11 Evolution and Nash Equilibrium

We now briefly take up the idea that Nash equilibrium might arise as the end result of "evolutionary pressure" or adaptive behavior. To do this, we imagine a population of N players who are randomly matched against each other over time to play a given game. It is useful to think of N as very large (or even infinite). When N is large, players are unlikely to ever meet their opponent again, so each interaction is truly a one-time event. We focus on symmetric $n \times n$ games, meaning that each player $i = 1, 2$ has the same strategy set and that $u_i(s_i, s_j) = u_j(s_j, s_i)$ for all $s_i, s_j \in S = \{s_1, ..., s_n\}$.

To specify an evolutionary model, we specify a rule about how players change their behavior over time, and then track the fraction of players who are using each strategy at each point in time.[5] Let $x_1^t, ..., x_n^t$ denote the fraction of players who are using strategy $s_1, ..., s_n$ at time t. It will always be the case that $x_i^t \geq 0$ for all i and that $\sum_i x_i^t = 1$. Let $x^t = (x_1^t, ..., x_n^t)$ denote the vector of population shares. Note that we can easily interpret x as a *mixed strategy* of the underlying game (denoted G).

Evolutionary models typically have two forces: *selection* and *mutation.* Selection means that strategies that perform well will tend to spread. Mutation captures that idea that there may be some noise in this process — players may randomly try new strategies for no particular reason. We start by looking at one of the simplest versions of evolution, the *replicator dynamics.*

Under replicator dynamics, the fraction of players playing a given strategy s_i evolves according to:

$$x_i^{t+1} - x_i^t = x_i^t \left[u_i\left(s_i, x^t\right) - u_i(x^t, x^t)\right],$$

where $u_i(s_i, x)$ denotes the expected payoff of a player using strategy s_i given that the population is playing x, and $u(x, x) = \sum u_i(s_i, x)x_i$ denotes

[5] In evolutionary biology, it is sometimes useful to think about a strategy as being expressed by a gene. Then a given player never changes her strategy, but strategies are passed down from generation to generation.

the average population payoff given population play x. A starting point for this dynamic process is a vector $x^0 = (x_1^0, ..., x_n^0)$.

Lemma 4 *The population shares always sum to one, i.e. for all* t, $\sum_i x_i^t = 1$.

Proof. To see this, note that:

$$\begin{aligned}\sum_i x_i^{t+1} &= \sum_i x_i^t + \sum_i x_i^t \left[u_i\left(s_i, x^t\right) - u_i(x^t, x^t)\right] \\ &= \sum_i x_i^t + \left[u_i\left(x^t, x^t\right) - u_i(x^t, x^t)\right] = \sum_i x_i^t.\end{aligned}$$

So long as $\sum_i x_i^0 = 1$, we are ok. *Q.E.D.*

There are several important features to note about the replicator dynamics. First, they are deterministic – there is selection, but no mutations. Second, they are monotonic — strategies that do better (have a higher expected payoff) realize a larger percentage gain in their population share. Finally, growth is proportionate to the present share — so if $x_i^0 = 0$ for some strategy s_i, then $x_i^t = 0$ for all t.

We are interested in looking at social configurations, or population shares, that might be *stable* under these dynamics. We first give a definition of what this means.

Definition 12 *A population distribution x is a* ***steady-state*** *under the replicator dynamics if $x^t = x$ for some t implies that $x^T = x$ for all $T > t$.*

Proposition 12 *If a strategy profile σ is a symmetric Nash equilibrium of G, then the population distribution $x = \sigma$ is a steady-state under the replicator dynamics.*

Proof. If σ is a symmetric Nash equilibrium, then if $\sigma(s_i) > 0$, $u_i(s_i, \sigma) = u_i(\sigma, \sigma)$, so σ will also be a steady-state under the replicator dynamics. *Q.E.D.*

Note that the converse is not true. For example, it is a steady-state for all players to cooperate in the prisoner's dilemma.

Example In matching pennies, the *only* steady-state is the mixed Nash equilibrium, $x = (1/2, 1/2)$.

A stronger concept is the idea of stability under replicator dynamics.

Definition 13 *A steady-state population distribution x is* ***stable*** *if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x^t - x| < \delta$, then $|x^T - x| < \varepsilon$ for all $T > t$.*

We say that x is *asymptotically stable* if there is some $\delta > 0$ such that whenever $|x^t - x| < \delta$ for some t, then $x^T \to x$ as $T \to \infty$.

Proposition 13 *If a steady-state population distribution x is stable, then the profile $\sigma = x$ is a Nash equilibrium.*

Proof. Assume σ is stable, but not a Nash equilibrium. Then there exists some s_i such that $u_i(s_i, \sigma) - u_i(\sigma, \sigma) = a > 0$. By continuity, it follows that there is some $\varepsilon > 0$ such that if $|x^t - \sigma| < \varepsilon$, then $u_i(s_i, x^t) - u_i(x^t, x^t) > a/3$. Now, because σ is stable, we can find some δ such that if $|x^t - \sigma| < \delta$, then $|x^T - \sigma| < \varepsilon$ for all $T > t$. Fix $x^t = (1 - \frac{\delta}{2})\sigma + \frac{\delta}{2} s_i$. Then for all $T > t$, $x_i^{T+1} - x_i^T > x_i^T \frac{a}{3}$, or $x_i^{T+1} > x_i^T(1 + \frac{a}{3})$. But then

$$x_i^T > \frac{\delta}{2}\left(1 + \frac{a}{3}\right)^{T-t} \qquad \Rightarrow \qquad x_i^T \to \infty \text{ as } T \to \infty,$$

which yields a contradiction. *Q.E.D.*

Stability focuses attention on the selection aspect of evolution. The idea of an *evolutionary steady state* focuses more on the role of mutations.

Definition 14 *A population distribution x is an* ***evolutionary steady state (ESS)*** *if, for all $y \neq x$, there exists some $\overline{\varepsilon}$ such that $u(x, (1-\varepsilon)x + \varepsilon y) > u(y, (1-\varepsilon)x + \varepsilon y)$ for all $0 < \varepsilon < \overline{\varepsilon}$.*

Suppose the population starts at state x, and a fraction ε mutate to play y. The original state x is an ESS if, whatever the nature of this mutation, the average payoff of the non-mutators is higher than the payoff of the mutators. Note that this definition is independent of the specific dynamics through which the population distribution changes. It is defined directly on the underlying game G.

Proposition 14 *A population distribution x is an ESS if and only if for all $y \neq x$ either (a) $u(x,x) > u(y,x)$ or (b) $u(x,x) = u(y,x)$ and $u(x,y) > u(y,y)$.*

There is a close link between Nash equilibria and evolutionary steady-states.

Proposition 15 *If x is an ESS, then it is also a Nash equilibrium. Conversely, if x is a strict Nash equillibrium, then it is an ESS.*

Our final result establishes a link between ESS and stability under the replicator dynamics.

Proposition 16 *If x is an ESS, then it is asymptotically stable under the replicator dynamics.*

You will be asked to work out one or two examples on the homework.

Extensive Form Games

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1 Introduction

The models we have described so far can capture a wide range of strategic environments, but they have an important limitation. In each game we have looked at, each player moves once and strategies are chosen simultaneously. This misses a common feature of many economic settings (and many classic "games" such as chess or poker). Consider just a few examples of economic environments where the timing of strategic decisions is important:

- Compensation and Incentives. Firms first sign workers to a contract, then workers decide how to behave, and frequently firms later decide what bonuses to pay and/or which employees to promote.

- Research and Development. Firms make choices about what technologies to invest in given the set of available technologies and their forecasts about the future market.

- Monetary Policy. A central bank makes decisions about the money supply in response to inflation and growth, which are themselves determined by individuals acting in response to monetary policy, and so on.

- Entry into Markets. Prior to competing in a product market, firms must make decisions about whether to enter markets and what products to introduce. These decisions may strongly influence later competition.

These notes take up the problem of representing and analyzing these dynamic strategic environments.

2 The Extensive Form

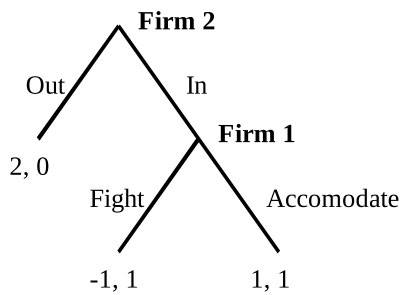
The extensive form of a game is a complete description of:

1. The set of players
2. Who moves when and what their choices are
3. What players know when they move
4. The players' payoffs as a function of the choices that are made.

2.1 Examples

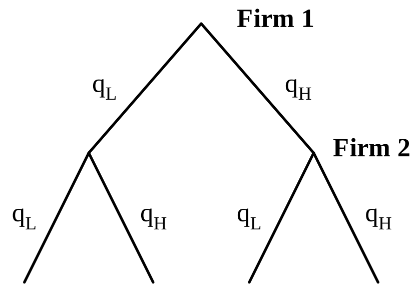
We start with a few examples.

An Entry Model Firm 1 is an incumbent monopolist. A second firm, Firm 2, has the opportunity to enter. After Firm 2 enters, Firm 1 will have to choose how to compete: either aggressively (Fight), or by ceding some market share (Accomodate). The strategic situation can be represented as follows.

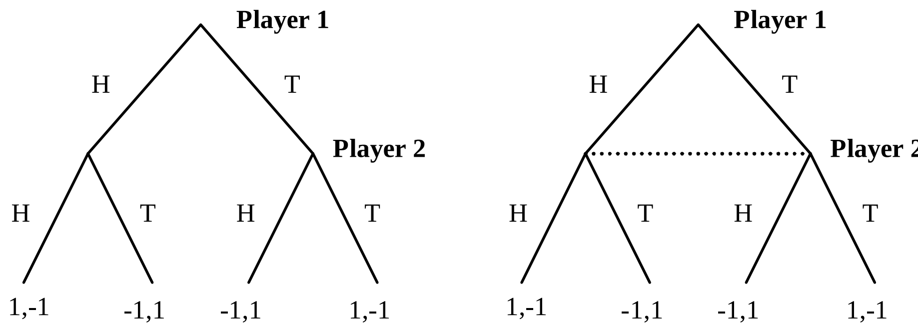


Stackleberg Competition An alternative to the Bertrand or Cournot models of imperfect competition is to assume that one firm is the market *leader*, while the other firm (or firms) are *followers*. In the Stackleberg model, we think of Firm 1 as moving first, and setting a quantity q_1, and Firm 2 as moving second, and setting a quantity q_2, *after having observed* q_1. The price is then determined by

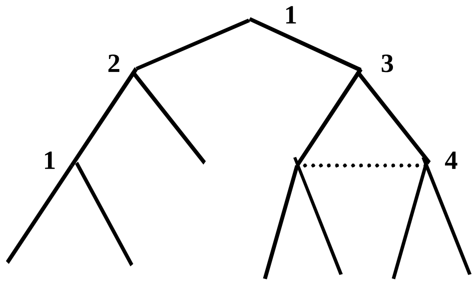
$P(q_1 + q_2) = 1 - (q_1 + q_2)$. Let's assume that the two firms have constant marginal costs, $c = 0$. To keep the picture simple, we think of q_i as taking only two values, Low and High.



Matching Pennies Or consider two variants of the matching pennies game. In the first variant, player one moves first, and then player two moves second *after having observed player one's action.* In the second, player two does not observe player one's action.



Example We can also have more complicated games, where players move multiple times, or select which player will move.



2.2 Formal Definitions

Formally, a finite extensive form game consists of:

- A finite set of players $i = 1, ..., I$.
- A finite set X of nodes that form a tree, with $Z \subset X$ being the terminal nodes.
- A set of functions that describe for each $x \notin Z$,
  - The player $i(x)$ who moves at x.
  - The set $A(x)$ of possible actions at x.
  - The successor node $n(x, a)$ resulting from action a.
- Payoff functions $u_i : Z \to \mathbb{R}$ assigning payoffs to players as a function of the terminal node reached.
- An information partition: for each x, let $h(x)$ denote the set of nodes that are possible given what player $i(x)$ knows. Thus, if $x' \in h(x)$, then $i(x') = i(x)$, $A(x') = A(x)$ and $h(x') = h(x)$.

We will sometimes use the notation $i(h)$ or $A(h)$ to denote the player who moves at information set h and his set of possible actions.

Matching Pennies, cont. Let's revisit the two versions of matching pennies above. In both, we have seven nodes, three of which are non-terminal. The key difference is the information partition. In the first version, each $h(x) = \{x\}$ for each x. In the second, for the two middle nodes we have $h(x) = h(x') = \{x, x'\}$.

In an extensive form game, write H_i for the set of information sets at which player i moves.

$$H_i = \{S \subset X : S = h(x) \text{ for some } x \in X \text{ with } i(x) = i\}$$

Write A_i for the set of actions available to i at any of his information sets.

2.3 Strategies

Definition 1 *A pure strategy for player i in an extensive form game is a function* $s_i : H_i \to A_i$ *such that* $s_i(h) \in A(h)$ *for each* $h \in H_i$.

A strategy is a complete contingent plan explaining what a player will do in every situation. Let S_i denote the set of pure strategies available to player i, and $S = S_1 \times ... \times S_I$ denote the set of pure strategy profiles. As before, we will let $s = (s_1, ..., s_I)$ denote a strategy profile, and s_{-i} the strategies of i's opponents.

Matching Pennies, cont. In the first version of matching pennies, $S_1 = \{H, T\}$ and $S_2 = \{HH, HT, TH, TT\}$. In the second version, $S_1 = S_2 = \{H, T\}$.

There are two ways to represent mixed strategies in extensive form games.

Definition 2 *A mixed strategy for player i in an extensive form game is a probability distribution over pure strategies, i.e. some* $\sigma_i \in \Delta(S_i)$.

Definition 3 *A behavioral strategy for player i in an extensive form game is a function* $\sigma_i : H_i \to \Delta(A_i)$ *such that support*$(\sigma_i(h)) \subset A(h)$ *for all* $h \in H_i$.

A famous theorem in game theory, *Kuhn's Theorem*, says that in games of *perfect recall* (these are games where *(i)* a player never forgets a decision he or she took in the past, and *(ii)* never forgets information she had when making a past decision — see Kreps, p. 374 for formalities), mixed and behavioral strategies are equivalent, in the sense that for any mixed strategy there is an equivalent behavioral strategy and vice versa. Since essentially all the games we will consider have perfect recall, we will use mixed and behavioral strategies interchangeably.

3 The Normal Form and Nash Equilibrium

Any extensive form game can also be represented in the normal form. If we adopt a normal form representation, we can solve for the Nash equilibrium.

Matching Pennies, cont. For our two versions of Matching Pennies, the normal forms are:

	HH	HT	TH	TT
H	$1,-1$	$1,-1$	$-1,1$	$-1,1$
T	$-1,1$	$1,-1$	$-1,1$	$1,-1$

	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

In the first version, Player two has a winning strategy in the sense that she can always create a mismatch if she adopts the strategy TH. Any strategy for player one, coupled with this strategy for player two is a Nash equilibrium. In the second version, the Nash equilibrium is for both players to mix $\frac{1}{2}H + \frac{1}{2}T$.

Entry Game, cont. For the entry game above, the normal form is:

	Out	In
F	$2,0$	$-1,1$
A	$2,0$	$1,1$

There are several Nash equilibria: (A, In), (F, Out) and $(\alpha F + (1-\alpha)A, Out)$ for any $\alpha \geq 1/2$.

Note that in the entry game, some of the Nash equilibria seem distinctly less intuitive that others. For instance, in the (F, Out) equilibrium, it is the threat of $Fight$ that keeps Firm 2 from entering. However, if Firm 2 *were* to enter, is it reasonable to think that Firm 1 will *actually* fight? At this point, it is not in Firm 1's interest to fight, since it does better by accomodating.

Consider another example, where this problem of incredible threats arises in way that is potentially even more objectionable.

Stackleberg Competition In the Stackleberg model, for any $q_1' \in [0,1]$, there is a Nash equilibrium in which Firm 1 chooses quantity q_1'. To see this, consider the strategies:

$$s_1 = q_1'$$

and

$$s_2 = \begin{cases} \frac{1-q_1'}{2} & \text{if } q_1 = q_1' \\ 1 - q_1' & \text{if } q_1 \neq q_1' \end{cases}.$$

Let's check that these are all equilibria. First, given Firm 2's strategy, Firm 1 can either set $q_1 \neq q_1'$, or $q_1 = q_1'$. If it does the former, the eventual price will be zero, and Firm 1 will make zero profits. If it does the latter, then Firm 1 will make profits:

$$q_1'\left(1 - q_1' - \frac{1 - q_1'}{2}\right) = \frac{1}{2}q_1'\left(1 - q_1'\right) \geq 0.$$

Now, consider Firm 2. Does it have a strategy that yields strictly higher payoff. Clearly, changing its strategy in response to $s_1 \neq q_1'$ will have no effect on its payoff given Firm 1's strategy. On the other hand, in response to $s_1 = q_1'$, its best response solves:

$$\max_{q_2} q_2\left(1 - q_1' - q_2\right).$$

The solution to this problem is $\left(1 - q_1'\right)/2$, so Firm 2 is playing a best response.

As in the previous case, many of the Nash equilibria in the Stackelberg model seem unreasonable. If Firm 1 sets $q_1 \neq q_1'$, then Firm 2 typically has options that give a positive payoff. However, it chooses to flood the market and drive the price to zero. Thus, off the equilibrium path, unreasonable things are happening. And not only is Firm 2 being allowed to make incredible threats, we have a huge multiplicity of equilibria.

4 Subgame Perfect Equilibrium

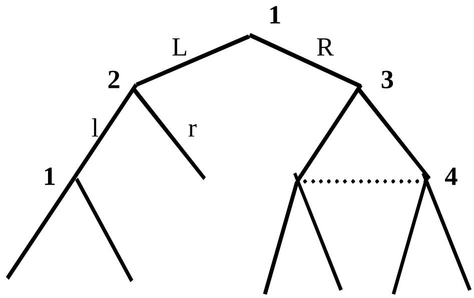
In response to the problems of credibility we have seen in the last two examples, we now introduce the idea of a *subgame perfect equilibrium*. Subgame perfection tries to rule out incredible threats by assuming that once something has happened, players will always optimize going forward.

4.1 Subgame Perfection

Definition 4 *Let G be an extensive form game, a subgame G' of G consists of (i) a subset Y of the nodes X consisting of a single non-terminal node x and all of its successors, which has the property that if $y \in Y$, $y' \in h(y)$ then $y' \in Y$, and (ii) information sets, feasible moves, and payofs at terminal nodes as in G.*

Entry Game, cont. In the entry game, there are two subgames. The entire game (which is always a subgame) and the subgame after Firm 2 has entered the market.

Example In game below, there are four subgames: (1) The entire game, (2) the game after player one chooses R, (3) the game after player one choose L, and (4) the game after Player 1 chooses L and player 2 chooses l.



Definition 5 *A strategy profile s is a **subgame perfect equilibrium** of G if it induces a Nash equilibrium in every subgame of G.*

Note that since the entire game is always a subgame, any SPE must also be a NE.

Entry Game, cont. In the entry game, only (A, In) is subgame perfect.

4.2 Application: Stackleberg Competition

Consider the model of Stackleberg Competition where Firm 1 moves first and chooses quantity q_1, and then Firm 2 moves second and chooses quantity q_2. Once both firms have chosen quantities, the price is determined by: $P(Q) = 1 - Q$, where $Q = q_1 + q_2$. So that we can compare this model to Bertrand and Cournot competition, let's assume that both firms have constant marginal cost equal to $0 \leq c < 1$.

To solve for the subgame perfect equilibrium, we work backward. Suppose that Firm 1 has set some quantity q_1. Then Firm 2's best response solves:

$$\max_{q_2} q_2 (1 - q_1 - q_2 - c)$$

The first-order condition for this problem is:

$$0 = 1 - q_1 - c - 2q_2,$$

which gives a best response:

$$q_2^*(q_1) = \max\left\{0, \frac{1 - q_1 - c}{2}\right\}.$$

Now consider the problem facing Firm 1, knowing that if it chooses q_1, Firm 2 will respond with a quantity $q_2^*(q_1)$. It solves

$$\max_{q_1} q_1 \left(1 - q_1 - q_2^*(q_1) - c\right)$$

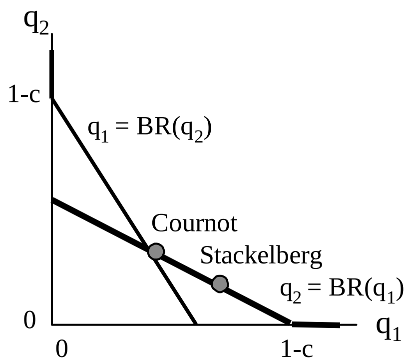
The first-order condition for this problem is:

$$0 = 1 - q_2 - c - 2q_1 - q_1 \frac{dq_2^*}{dq_1}.$$

Solving this out yields:

$$q_1 = \frac{1-c}{2} \qquad \text{and} \qquad q_2 = \frac{1-c}{4}.$$

The total quantity is $\frac{3}{4}(1-c)$ and the price is $p^S = (1 + 3c)/4$. In comparison, under Cournot competition, both firms set identical quantity $\frac{1-c}{3}$, so total quantity is $\frac{2}{3}(1-c)$ and the price is $p^C = (1 + 2c)/3$.



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Stackelberg Competition

Relative to Cournot, in Stackelberg Competition, the Leader (Firm 1) can choose any point on the Follower's (Firm 2's) best-response curve. Note that this game has a first-mover advantage in the sense that there is an advantage to being the leader.

4.3 Backward Induction

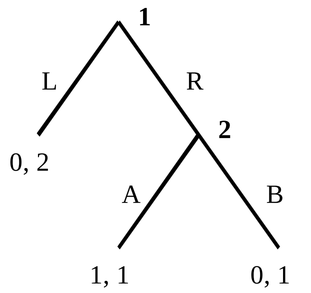
As the previous examples illustrate, a common technique for identifying subgame perfect equilibria is to start at the end of the game and work back to the front. This process is called *backward induction.*

Definition 6 *An extensive form game is said to have* ***perfect information*** *if each information set contains a single node.*

Proposition 7 *(Zermelo's Theorem) Any finite game of perfect information has a pure strategy subgame perfect equilibrium. For generic payoffs, there is a unique SPE.*

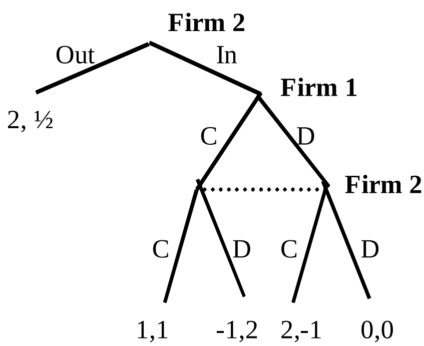
Proof. (very informal) In a finite game of perfect information, we can label each node as belonging to stage 1, stage 2, ..., stage K. To find a pure strategy SPE (and for generic payoffs, the unique SPE), we first consider all nodes x in stage K. Each node starts a subgame, so by NE the player who moves must maximize his or her expected payoff. Identify the optimal choice (generically, it will be unique). Now move back to stage $K-1$, and consider the problem facing a player who moves here. This player can assume that at stage K, play will follow the choices just identified. Since each node starts a subgame, we look for the payoff-maximizing choice facing a player who gets to move. Once these choices have been identified, we move back to stage $K-2$. This process of continues until we reach the beginning of the game, at which point we will have at least one (and typically no more than one). *Q.E.D.*

Example Here is a non-generic game where backward induction reveals three pure strategy subgame perfect equilibria: (R, A), (R, B) and (L, B).



We can use backward induction even in games without perfect information as the next example demonstrates.

Example To use backward induction in the game below, we first solve for the subgame perfect equilibrium after Firm 2 has entered. We see that the unique equilibrium in this subgame is for both Firm 1 and Firm 2 to play D. (Note that this subgame is a prisoners' dilemma. Hence Firm 2 will choose not to enter at the first node.

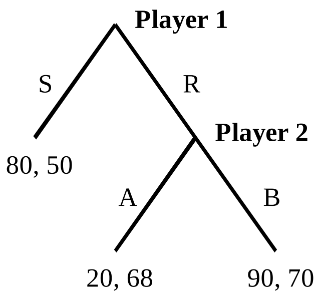


Another Entry Game

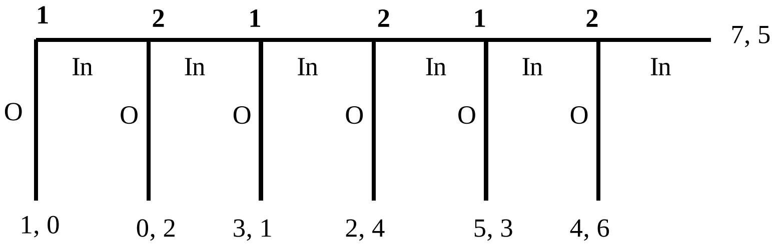
4.4 Criticisms of Subgame Perfection

We motivated Subgame Perfection as an attempt to eliminate equilibria that involved incredible threats. As we go on to consider applications, we will use SPE regularly as a solution concept. Before we do this, however, it is worth pausing momentarily to ask whether SPE might be *over*-zealous in eliminating equilibria.

Example: Trusting Someone to be Rational Here the unique SPE is for Player 1 to choose R and Player 2 to choose B. However, Goeree and Holt (2001) report an experiment in which more than 50% of player ones play S.



The Centipede Game In games with many stages, backward induction *greatly* stresses the assumption of rationality (and common knowledge of rationality). A famous example due to Rosenthal (1981) is the centipede game. The unique SPE is for Player 1 to start by moving Out, but in practice people do not seem to play the game this way.



5 Stackelberg (Leader-Follower) Games

Above, we considered Stackelberg competition in quantities. We now consider Stackelberg competition in prices, then fit both models into a more general framework.

5.1 Stackelberg Price Competition

In the Stackelberg version of price competition, Firm 1 moves first and commits to a price p_1. Firm 2 observes p_1 and responds with a price p_2. Sales for Firm i are then given by $Q_i(p_1, p_2)$. Supposing that the two firms have constant marginal costs equal to c, firm i's profits can be written as:

$$\pi_i(p_1, p_2) = (p_i - c) Q_i(p_i, p_j).$$

Homogeneous Products. If the firms' products are heteregenous, then firm that sets a lower price gets demand $Q(p)$ (where $p = \min\{p_1, p_2\}$), and the firm that sets a higher price gets no demand. Suppose that if $p_1 = p_2$ the consumers split equally between the two firms. In this case, if Firm 1 chooses $p_1 > c$, Firm 2 would like to choose the highest price less than p_1. Unfortunately, such a price does not exist! So there are subgames in which Nash equilibria do not exist and no SPE.

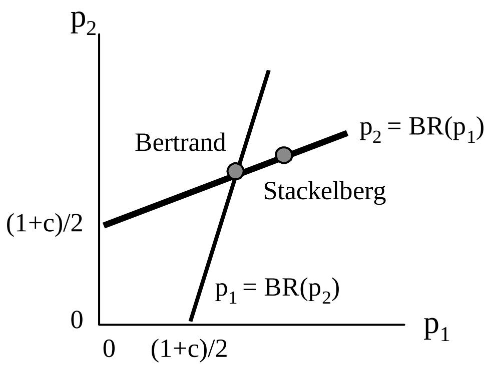
To resolve this, assume that if $p_1 = p_2$, then all consumers purchase from Firm 2. Then after observing $p_1 \geq c$, Firm 2 responds by taking the entire market — either by choosing $p_2 = p_1$ if $p_1 \leq p^m$ (the monopoly price), or by choosing $p_2 = p^m$ if $p_1 > p^m$. It follows that Firm 2's best response function is given by:

$$p_2^*(p_1) = \begin{cases} p^m & \text{if } p_1 > p^m \\ p_1 & \text{if } p_1 \in [c, p^m] \\ c & \text{if } p_1 < c \end{cases}.$$

Thus, for any $p_1 < c$, Firm 1 gets the entire market but loses money, which for any $p_1 \geq c$, Firm 1 gets no demand. It follows that any pair $(p_1, p_2^*(p_1))$ with $p_1 \geq c$ is an SPE.

Note that compared to the Bertrand (simultaneous move) equilibrium, the price may be higher. In particular, both Firms 1 and 2 set (weakly) higher prices. Moreover, Firm 2 (the follower) does better than in the simultaneous game, while Firm 1 does the same. Thus we say the game has a *second mover advantage.*

Heterogeneous Products. With heterogeneous products, the situation is similar to Stackelberg quantity competition, except with Bertrand best-responses rather than Cournot!



Stackelberg Price Competition

5.2 General Leader-Follower Games

The Stackelberg models of imperfect competition are examples of what Gibbons calls "Leader-Follower" games. These games have the following structure:

1. Player 1 moves first and chooses an action $a_1 \in A_1$.
2. Player 2 observes a_1 and chooses an action $a_2 \in A_2$.

There is a simple algorithm to identify the subgame perfect equilibria of this sort of game. We just apply backwards induction. We first define player 2's best response to any action by Player 1:

$$a_2^*(a_1) = \arg\max_{a_2 \in A_2} \pi_2(a_1, a_2).$$

We then identity what player one should do, assuming that player two will best respond. To do this, define:

$$a_1^* = \arg\max_{a_1 \in A_1} \pi_1\left(a_1, a_2^*(a_1)\right).$$

A subgame perfect equilibrium is a pair $(a_1^*, a_2^*(a_1^*))$.

From our examples, we can make several observations about leader-follower games.

1. The Leader always does (weakly) better than in a simultaneous move pure strategy equilibrium setting (note that this is not true for mixed strategy equilibria — think about matching pennies).

2. The Leader tends to distort his behavior relative to the simultaneous move game (how he does so depends on Firm 2's best response function and on what sort of action he prefers Firm 2 to choose).

3. Whether the Follower does better than in the simultaneous game depends on both Firm 2's best response function and the interdependence in payoffs (how i's action affects j's payoff and vice-versa).

6 Strategic Pre-Commitment

We now turn to a class of problems that arise frequently in industrial organization. These problems have the following structure. First, one player (Firm 1) has the opportunity to take some sort of action or investment — for instance, installing capacity, investing in a cost-saving technology or in product development, signing an exclusive contract with a key upstream suppliers, building a relationship with customers, or so on. Then Firm 2 decides whether or not to enter the market. Finally, firms compete with Firm 1 either operating as a monopoly or the two firms competing as a duopoly.

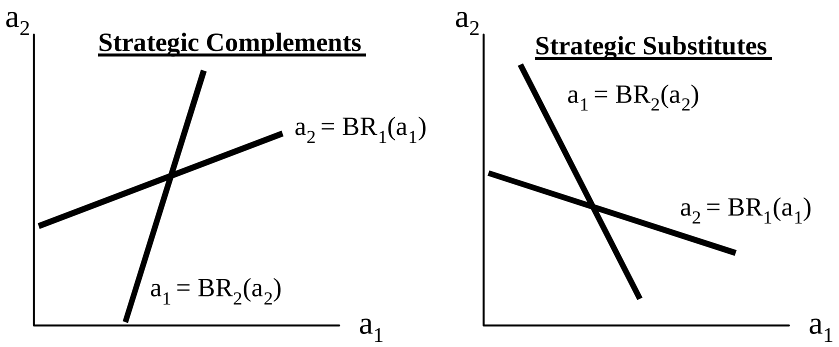
By taking an action in advance of competition, Firm 1 has the opportunity to *strategically pre-commit* (just as the Stackelberg leader pre-commits to a price or quantity). It turns out that it is possible to give a very intuitive analysis of the economics of pre-commitment by using the idea of strategic complements and substitutes.[1] This analysis can then be used to shed light on a range of problems in industrial organization and other fields.

6.1 Strategic Complements and Substitutes

Let G be a simultaneous move game in which the players 1 and 2 takes actions $a_1, a_2 \in \mathbb{R}$, and have payoffs $\pi_1(a_1, a_2)$, $\pi_2(a_1, a_2)$. Let $BR_1(a_2)$ and $BR_2(a_1)$ denote the best-response functions (assume best responses are unique).

Definition 8 *The players' actions are* ***strategic complements*** *if* $BR_i'(\cdot) \geq 0$. *The actions are* ***strategic substitutes*** *if* $BR_i'(\cdot) \leq 0$.

[1] This section is drawn from Tirole (1988, Chapter 8). See Fudenberg and Tirole (1984, AER) or Bulow, Geanokoplos and Klemperer (1985, JPE) for the original analyses.



We have already seen that in Cournot Competition, quantities are strategic substitutes, while in differentiated products Bertrand Competition, prices are strategic complements.

Proposition 9 *Suppose for* $i = 1, 2,.$ π_i *is twice continuously differentiable. Then* G *has strategic complements if* $\frac{\partial^2 \pi_i}{\partial a_i \partial a_j} \geq 0$ *and strategic substitutes if* $\frac{\partial^2 \pi_i}{\partial a_i \partial a_j} \leq 0$.

For those of you who took Econ 202, these conditions should look very familiar from our study of Monotone Comparative Statics. A good exercise is to try to prove this result using Topkis' Theorem.

6.2 Strategic Pre-Commitment

We consider the following model:

- Firm 1 moves first and chooses and investment k.
- Firm 2 observes k and decides whether or not to compete (enter).
- Firms 1 and 2 choose actions $a_1 \in A_1$, $a_2 \in A_2$.
- Payoffs are given by $\pi_1(k, a_1, a_2)$ and $\pi_2(k, a_1, a_2)$.[2]

[2]Note that Leader-Follower games are a special case of this setting. To get the Leader-Follower case, let A_1 be a singleton, and think of k as Firm 1's action and a_2 as Firm 2's action.

We assume that for any choice k, the competition subgame has a unique Nash Equilibrium, which we can denote $a_1^c(k), a_2^c(k)$. Payoffs in this game are given by:

$$\pi_i\left(k, a_1^c\left(k\right), a_2^c\left(k\right)\right) \text{ for } i = 1, 2.$$

Thus given a choice of k, Firm 2 will choose to enter if

$$\pi_2\left(k, a_1^c(k), a_2^c(k)\right) > 0.$$

If Firm 2 does not enter, then Firm 1 sets chooses the monopoly strategy a_1^m. Payoffs are

$$\pi_1^m\left(k, a_1^m(k)\right)$$

for Firm 1 and zero for Firm 2. Let k^* denote the subgame perfect level of investment.

We say that:

- Entry is *deterred* if $\pi_2\left(k^*, a_1^c(k^*), a_2^c(k^*)\right) \leq 0$.
- Entry is *accomodated* if $\pi_2\left(k^*, a_1^c(k^*), a_2^c(k^*)\right) > 0$,

If entry is deterred, then the SPE involves Firm 1 choosing $a_1^m(k^*)$, and achieves profits $\pi_1^m\left(k^*, a_1^m(k^*)\right)$. If entry is accomodated, Firms choose $a_1^c(k^*), a_2^c(k^*)$.

An alternative to this model would be a case without pre-commitment where Firm 1 chooses k at the same time as a_1, a_2 (or chooses k in advance but without it being observed). Let's assume that this game also has a unique Nash Equilibrium, denoted $(k^{nc}, a_1^{nc}, a_2^{nc})$. We will say that:

- Entry is *blockaded* if $\pi_2\left(k^{nc}, a_1^{nc}, a_2^{nc}\right) \leq 0$.

In what follows, we assume that entry is not blockaded. In addition, we will assume that π_1^m and π_1 are both concave in k.

6.3 Entry Deterrence

Let's first consider SPE in which entry is deterred. In this case, Firm 1 need to choose a level of k that makes Firm 2 less profitable. Indeed, it will choose an investment k^* such that:[3]

$$\pi_2\left(k^*, a_1^c(k^*), a_2^c(k^*)\right) = 0.$$

[3] The fact that Firm 1 will choose k^* to make Firm 2's competition profits *exactly* zero follows from the assumption that entry is not blockaded and that payoffs functions are concave.

Let's consider what sort of strategy by Firm 1 works to make Firm 2 unprofitable and hence deter entry. From second period optimization:

$$\frac{\partial \pi_2}{\partial a_2}\left(k^*, a_1^c(k^*), a_2^c(k^*)\right) = 0$$

Hence:

$$\frac{d\pi_2}{dk} = \underbrace{\frac{\partial \pi_2}{\partial k}}_{\text{direct effect}} + \underbrace{\frac{\partial \pi_2}{\partial a_1}\frac{\partial a_1}{\partial k}}_{\text{strategic effect}}.$$

To deter entry, Firm 1 wants to choose an investment that will make Firm 2 *less profitable*. It has two ways to do this. It may be able to invest in a way that makes Firm 2 *directly less profitable*. It make also be able to change the nature of competition — for example, if k is investment in capacity, k has no direct effect, but only a *strategic effect*.

To classify Firm 1's strategies, we adopt the terminology of Fudenberg and Tirole (1984).

Definition 10 *Investment makes Firm 1* ***tough*** *if* $\frac{d\pi_2}{dk} < 0$. *Investment makes Firm 1* ***soft*** *if* $\frac{d\pi_2}{dk} > 0$.

Fudenberg and Tirole suggest the following typology of strategies for investment.

- *Top Dog*: Be big (invest a lot) to look tough.
- *Puppy Dog:* Be small (invest only a little) to look soft.
- *Lean and Hungry Look:* Be small to look tough.
- *Fat Cat:* Be big to look soft.

Example: Reducing Own Costs Suppose that Firm 1 has the opportunity to invest to lower its marginal costs. If Firm 2 enters, competition will be Cournot. There is no direct on Firm 2. But there is a strategic effect. Investment will shift out Firm 1's best-response function, and lead to a competitive outcome where Firm 1 produces higher quantity and Firm 2 lower quantity. This *top dog* strategy may deter entry.

Example: Building a Customer Base Suppose that Firm 1 has the opportunity to invest in customer relations, building up a loyal customer base. The direct effect of this is to limit the potential market for Firm

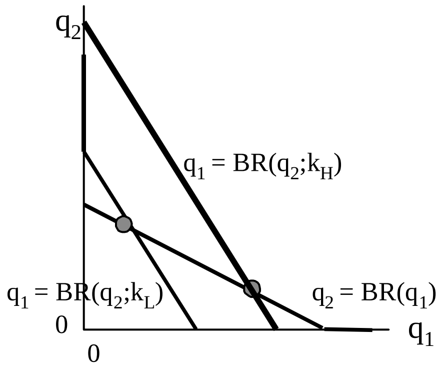


Figure 1:

2, making Firm 1 tough. However, there may be a second effect. If Firm 1 cannot price discriminate, it may be tempted to set a high price to take advantage of its locked in customers. This strategic effect can work to make Firm 1 soft. The overall effect is ambiguous. Thus either a "top dog" strategy or a "lean and hungry look" might work to deter entry depending on the specifics.

6.4 Accomodation: Changing the Nature of Competition

Suppose now that Firm 1 finds it too costly to deter entry. How should it behave in order to make more profits once Firm 2 enters? In this case, Firm 1 is interested in choosing its investment to maximize:

$$\pi_1\left(k, a_1^c\left(k\right), a_2^c\left(k\right)\right).$$

The total derivative is given by (using the envelope theorem):

$$\frac{d\pi_1}{dk} = \underbrace{\frac{\partial\pi_1}{\partial k}}_{\text{direct effect}} + \underbrace{\frac{\partial\pi_1}{\partial a_2}\frac{da_2}{dk}}_{\text{strategic effect}}.$$

That is, the first term is the *direct* effect. This would exist even if investment was not observed by Firm 2. The *strategic* effect results from the fact that Firm 1's investment can change the way Firm 2 will compete.

Let's investigate the strategic effect further. To do this, let's assume that the second-period actions of the two firms have the same nature in the sense that $\partial\pi_1/\partial a_2$ has the same sign as $\partial\pi_2/\partial a_1$. For instance, either both firms choose quantities or both choose prices. Then:

$$\frac{da_2^c}{dk} = \frac{da_2^c}{da_1}\frac{da_1}{dk} = \frac{dBR_2(a_1)}{da_1}\frac{da_1}{dk}$$

Thus,

$$\text{sign}\left(\frac{\partial\pi_1}{\partial a_2}\frac{da_2}{dk}\right) = \text{sign}\left(\frac{\partial\pi_1}{\partial a_2}\frac{da_2}{dk}\right)\cdot\text{sign}\left(\frac{dBR_2(a_1)}{da_1}\right)$$

Assuming that $\partial\pi_2/\partial k = 0$, we can identify the sign of the strategic effect with two things: (1) whether investment makes Firm 1 Tough or Soft (the first term) and (2) whether there are strategic substitutes or complements (the second term). We have four cases:

- If investment makes Firm 1 tough and reaction curve slope down, investment by Firm 1 softens Firm 2's action — thus Firm 1 should overinvest (top dog).
- If investment makes Firm 1 tough and reaction curves slope up, Firm 1 should overinvest so as not to trigger an aggressive response by Firm 2 (puppy dog).
- If invesmtnent makes Firm 1 soft and reaction curves slope down, Firm 1 should stay lean and hungry.
- If invesment makes Firm 1 soft and reaction curves slope up, Firm 1 should overinvest to become a fat cat.

To summarize,

	Investment makes Firm 1	
	Tough	Soft
Strategic Complements	Puppy Dog	Fat Cat
Strategic Substitutes	Top Dog	Lean and Hungry

Example: Reducing Costs Suppose Firm 1 can invest to reduce its costs before competing. With Cournot competition, the strategic effect to make Firm 1 more aggressive. The equilibrium changes so that Firm 2 ends up producing less. Thus Firm 1 wants to be a Top Dog and invest heavily. On the other hand, if Firm 1 can reduce its costs before price competition, the strategic effect is to make Firm 1 more aggressive so that in equilibrium Firm 2 ends up pricing more aggressively as well. Thus, Firm 2 might want to be a Puppy Dog to soften price competition.

6.5 Applications in Industrial Organization

1. **Product Differentiation**. Suppose Firm 1 can choose its product's "location" prior to competing in price with Firm 2. Producing a product that is "close" to Firm 2's will tend to make price competition more intense, lowering prices and profits. To deter entry or to drive Firm 2 from the market, Firm 1 might want to adopt a "Top Dog" strategy. But to change the nature of competition in a favorable way, Firm 1 might want to adopt a "Puppy Dog" ploy and differentiate its offering from Firm 2's.

2. **Most-Favored Customer Clause**. With price competition, if Firm 1 wants to accomodate Firm 2, it wants to look inoffensive so as to keep Firm 2 from cutting price. In particular, it would like to commit itself to charging high prices (a "Puppy Dog" ploy). One way to do this is most-favored customer clauses. Firm 1 can write contracts with its customers promising that if it ever offers a low price to another customer, the original customer can get the new low price. This makes it *very* costly for Firm 1 to drop prices, effectively committing itself to be unaggressive in competing with Firm 2.

3. **Advertising**. Suppose Firm 1 can invest in advertising that makes customer more excited not just about its own product, but about the whole market. This kind of advertising makes Firm 1 soft. To deter entry, Firm 1 should not do much of this sort of advertising — rather it should run advertisements that increase the demand for its own product and decrease the demand for other firms' product. But what if Firm 2 surely plans to enter. If competition is in prices, then Firm 1 will want to advertise in a way that increases demand (direct effect) and in a way that softens price competition (for example by establishing separate niches for different products in the market). This is a fat cat approach to advertising.

4. **Leverage and Tying**. An old story in IO is that a firm with monopoly power in one market can leverage this power to monopolize a second. (Think Microsoft and browsers.) One way to do this is to tie the two products together. Suppose that there are two markets, that Firm 1 has a monopoly in the first, and that Firms 1 and 2 may compete in prices in the second. Firm 1 must decide whether to bundle or tie its two products. The question is how this action will effect its pricing behavior. This depends on how we model demand, but in many cases,

bundling will make demand more elastic. This leads Firm 1 to price *more aggressively* in response to Firm 2's prices. It follows that from a strategic point of view, bundling is a top dog strategy that works to deter entry. But if entry is to be accomodated, it may be better to use the puppy dog ploy of not bundling.

Bargaining and Repeated Games

Jonathan Levin

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1 Sequential Bargaining

A classic economic question is how people will bargain over a pie of a certain size. One approach, associated with Nash (1950), is to specify a set of axioms that a "reasonable" or "fair" division should satisfy, and identify the division with these properties. For example, if two identical agents need to divide a pie of size one, one might argue that a reasonable division would be $(1/2, 1/2)$.

The other approach, associated with Nash's 1951 paper, is to write down an explicit game involving offers and counter-offers and look for the equilibrium. Even a few minutes' reflection suggests this will be difficult. Bargaining can involve bluffing and posturing, and there is no certainty that an agreement will be reached. The Nash demand game demonstrates that a sensible bargaining protocol might have many equilibria. A remarkable paper by Rubinstein (1982), however, showed that there was a fairly reasonable dynamic specification of bargaining that yielded a *unique* subgame perfect equilibrium. It is this model of sequential bargaining that we now consider.

1.1 The Model

Imagine two players, one and two, who takes turns making offers about how to divide a pie of size one. Time runs from $t = 0, 1, 2, ...$ At time 0, player one can propose a split $(x_0, 1 - x_0)$ (with $x_0 \in [0, 1]$), which player 2 can accept or reject. If player 2 accepts, the game ends and the pie is consumed. If player two rejects, the game continues to time $t = 1$, when she gets to propose a split $(y_1, 1 - y_1)$. Once player two makes a proposal, player one can accept or reject, and so on ad infinitum.

We assume that both players want a larger slice, and also that they both dislike delay. Thus, if agreement to split the pie $(x, 1-x)$ is reached at time

t, the payoff for player one is $\delta_1^t x$ and the payoff for player two is $\delta_2^t (1-x)$, for some $\delta_1, \delta_2 \in (0,1)$.

1.2 A Finite-Horizon Version

To get a flavor for this sort of sequential-offer bargaining, consider a variant where there is some *finite* number of offers N that can be made. This model was first studied by Stahl (1972). To solve for the subgame perfect equilibrium, we can use backward induction, starting from the final offer.

For concreteness, assume $N = 2$. At date 1, player two will be able to make a final take-it-or-leave-it offer. Given that the game is about to end, player one will accept any split, so player two can offer $y = 0$.

What does this imply for date zero? Player two anticipates that if she rejects player one's offer, she can get the whole pie in the next period, for a total payoff of δ_2. Thus, to get her offer accepted, player one must offer player two at least δ_2. It follows that player one will offer a split $(1-\delta_2, \delta_2)$, and player two will accept.

Proposition 1 *In the $N = 2$ offer sequential bargaining game, the unique SPE involves an immediate $(1-\delta_2, \delta_2)$ split.*

1.3 Solving the Rubinstein Model

It is fairly easy to see how a general N-offer bargaining game can be solved by backward induction to yield a unique SPE. But the infinite-horizon version is not so obvious. Suppose player one makes an offer at a given date t. Player two's decision about whether to accept will depend on her belief about what she will get if she rejects. This in turn depends on what sort of offer player one will accept in the next period, and so on. Nevertheless, we will show:

Proposition 2 *There is a unique subgame perfect equilibrium in the sequential bargaining game described as follows. Whenever player one proposes, she suggests a split $(x, 1-x)$ with $x = (1-\delta_2)/(1-\delta_1\delta_2)$. Player two accepts any division giving her at least $1-x$. Whenever player two proposes, she suggests a split $(y, 1-y)$ with $y = \delta_1(1-\delta_2)/(1-\delta_1\delta_2)$. Player one accepts any division giving her at least y. Thus, bargaining ends immediately with a split $(x, 1-x)$.*

Proof. We first show that the proposed equilibrium is actually an SPE. By a classic dynamic programming argument, it suffices to check that no player can make a profitable deviation from her equilibrium strategy in one single period. (This is known as the *one-step deviation principle* — see e.g. Fudenberg and Tirole's book for details.)

Consider a period when player one offers. Player one has no profitable deviation. She cannot make an acceptable offer that will get her more than x. And if makes an offer that will be rejected, she will get $y = \delta_1 x$ the next period, or $\delta_1^2 x$ in present terms, which is worse than x. Player two also has no profitable deviation. If she accepts, she gets $1 - x$. If she rejects, she will get $1 - y$ the next period, or in present terms $\delta_2 (1 - x) = \delta_2(1 - \delta_1 x)$. It is easy to check that $1 - x = \delta_2 - \delta_1\delta_2 x$. A similar argument applies to periods when player two offers.

We now show that the equilibrium is unique. To do this, let $\underline{v}_1, \overline{v}_1$ denote the lowest and highest payoffs that player one could conceivably get in any subgame perfect equilibrium starting at a date where he gets to make an offer.

To begin, consider a date where player *two* makes an offer. Player one will certainly accept any offer greater than $\delta_1\overline{v}_1$ and reject any offer less than $\delta_1\underline{v}_1$. Thus, starting from a period in which she offers, player two can secure *at least* $1 - \delta_1\overline{v}_1$ by proposing a split $(\delta_1\overline{v}_1, 1 - \delta_1\overline{v}_1)$. On the other hand, she can secure *at most* $1 - \delta_1\underline{v}_1$.

Now, consider a period when player one makes an offer. To get player two to accept, he must offer her *at least* $\delta_2 (1 - \delta_1\overline{v}_1)$ to get agreement. Thus:

$$\overline{v}_1 \leq 1 - \delta_2 (1 - \delta_1\overline{v}_1) .$$

At the same time, player two will certainly accept if offered more than $\delta_2(1 - \delta_1\underline{v}_1)$. Thus:

$$\underline{v}_1 \geq 1 - \delta_2 (1 - \delta_1\underline{v}_1) .$$

It follows that:

$$\underline{v}_1 \geq \frac{1 - \delta_2}{1 - \delta_1\delta_2} \geq \overline{v}_1 .$$

Since $\overline{v}_1 \geq \underline{v}_1$ by definition, we know that *in any subgame perfect equilibrium*, player one receives $v_1 = (1 - \delta_2) / (1 - \delta_1\delta_2)$. Making the same argument for player two completes the proof. *Q.E.D.*

A few comments on the Rubinstein model of bargaining.

1. It helps to be patient. Note that player one's payoff, $(1 - \delta_2) / (1 - \delta_1\delta_2)$, is increasing in δ_1 and decreasing in δ_2. The reason is that if you are more patient, you can afford to wait until you have the bargaining power (i.e. get to make the offer).

2. The first player to make an offer has an advantage. With identical discount factors δ, the model predicts a split

$$\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$$

which is better for player one. However, as $\delta \to 1$, this first mover advantage goes away. The limiting split is $(1/2, 1/2)$.

3. There is no delay. Player two accepts player one's first offer.

4. The details of the model depend a lot on there being *no immediate counter-offers*. With immediate counter-offers, it turns out that there are many many equilibria!

2 Finitely Repeated Games

So far, one might have a somewhat misleading impression about subgame perfect equilibria, namely that they do such a good job of eliminating unreasonable equilibria that they typically make a unique prediction. However, in many dynamic games, we still have a very large number of SPE.

2.1 A Simple Example

Consider the following game G:

	A	B	C
A	$4,4$	$0,0$	$0,5$
B	$0,0$	$1,1$	$0,0$
C	$5,0$	$0,0$	$3,3$

This game has three Nash equilibria, (B, B), (C, C) and $(\frac{3}{4}B + \frac{1}{4}C, \frac{3}{4}B + \frac{1}{4}C)$. Note that A is strictly dominated by C.

Suppose that the players play G twice, and observe the first period actions before choosing their second period actions. Suppose that both players discount the future by δ. Thus player i's total payoff is $u_i\left(a^1\right) + \delta u_i\left(a^2\right)$, where $a^t = \left(a_1^t, a_2^t\right)$ is the time t action profile.

In this setting, note that repeating any one of the Nash equilibria twice is an SPE of the two period game. So is playing one of the NE in the first period and another in the seond. Moreover, by making the choice of *which* Nash Equilibrium to play in period two *contingent* on play in period one, we can construct SPE where play in the first period does not correspond to Nash play.

Proposition 3 *If $\delta \geq 1/2$ then there exists an SPE of the two-period game in which (A, A) is played in the first period.*

Proof. Consider the following strategies.

- Play (A, A) in the first period.
- If first period play is (A, A), play (C, C) in the second period. Otherwise, play (B, B) in the second period.

To see that this is an SPE, we again look for profitable one-time deviations. Consider the second period. Following (A, A), playing (C, C) is a NE so neither player benefits from deviating. Similarly, following something else, (B, B) is a NE, so there is no gain to deviating.

Now consider the first period. By following the strategy, a player gets:

$$\begin{aligned} \text{Payoff to Strategy} &: \quad 4 + \delta \cdot 3 \\ \text{Best Deviation Payoff} &: \quad 5 + \delta \cdot 1 \end{aligned}$$

So long as $2\delta \geq 1$, there is no gain to deviating. *Q.E.D.*

2.2 Finitely Repeated Prisoners' Dilemma

The construction we just used requires that the stage game have at least two Nash equilibria, with different payoffs. Of course, not every game has this property. For instance, consider the prisoners' dilemma:

	C	D
C	$1, 1$	$-1, 2$
D	$2, -1$	$0, 0$

Proposition 4 *In a T-period repetition of the prisoners' dilemma, the unique SPE is for both players to defect in every period.*

Proof. We use backward induction. Consider period T. In this period, play must be a Nash equilibrium. But the only equilibrium is (D, D). Now back up to period $T - 1$. In that period, players know that *no matter what happens, they will play (D, D) at date T.* Thus D is strictly dominant at date $T - 1$, and the equilibrium must involve (D, D). Continuing this for periods $T - 2, T - 3$, etc... completes the proof. *Q.E.D.*

3 Infinitely Repeated Games

We now consider infinitely repeated games. In the general formulation, we have I players and a stage game G which is repeated in periods $t = 0, 1, 2, ...$ If a^t is the action profile played at t, player i's payoff for the infinite horizon game is:

$$u_i\left(a^0\right) + \delta u_i\left(a^1\right) + \delta^2 u_i\left(a^2\right) + ... = \sum_{t=0}^{\infty} \delta^t u_i\left(a^t\right).$$

To avoid infinite sums, we assume $\delta < 1$. Sometimes, it is useful to look at *average payoffs*:

$$(1-\delta)\sum_{t=0}^{\infty} \delta^t u_i\left(a^t\right)$$

Note that for player i, maximizing total payoff is the same as maximizing average payoff.

Before we go on to consider some examples, it's worth commenting one two interpretations of the discount factor.

- Time preference. One interpretation of δ is that player's prefer money today to money tomorrow. That is, if the economy has an interest rate r, then $\delta = 1/(1+r)$. If there is a longer gap between stage games, we would think of δ as smaller.

- Uncertain end date. Another interpretation of δ is that it corresponds to the probability that the interaction will continue until the next date. That is, after each stage game, there is a $1-\delta$ probability that the game will end. This leads to the same repeated game situation, but with the advantage that the game will actually end in finite time — just randomly.

A *history* in a repeated game is a list $h^t = \left(a^0, a^1, ..., a^{t-1}\right)$ of what has previously occurred. Let H^t be the set of t-period histories. A *strategy* for player i is a sequence of maps $s_i^t : H^t \to A_i$. A mixed strategy σ_i is a sequence of maps $\sigma_i^t : H^t \to \Delta(A_i)$. A strategy profile is $\sigma = (\sigma_1, ..., \sigma_I)$.

3.1 The Prisoners' Dilemma

Proposition 5 *In the infinitely repeated prisoners' dilemma, if $\delta \geq 1/2$ there is an equilibrium in which (C, C) is played in every period.*

Proof. Consider the following symmetric strategies — called "grim trigger" strategies:

- Play C in the first period and in every following period so long as no one ever plays D.
- Play D if either player has ever played D.

To see that there is no profitable single deviation, suppose that D has already been played. At this point, player i has two choices:

- Play C for a payoff $-1 + \delta \cdot 0 + \delta^2 \cdot 0 + ... = -1$
- Play D for a payoff $0 + \delta \cdot 0 + \delta^2 \cdot 0 + ... = 0$.

So player i should certainly play D. On the other hand, suppose D has not been played. At this point i has two choices:

- Play C for a payoff $1 + \delta + \delta^2 + ... = 1/(1-\delta)$.
- Play D for a payoff $2 + \delta \cdot 0 + \delta^2 \cdot 0 + ... = 2$.

If $\delta \geq 1/2$ it is better to play C so we have an SPE. *Q.E.D.*

It is tempting to think of this proposition as saying that if people interact repeatedly then they **will** cooperate. However, it does not say this. What it says is that cooperation is one **possible** SPE outcome. However, there are many others.

- For any δ, there is a SPE in which players play D in every period.
- For $\delta \geq 1/2$, there is a SPE in which the players play D in the first period and C in every following period.
- For $\delta \geq 1/\sqrt{2}$, there is a SPE in which the players alternate between (C, C) and (D, D).
- For $\delta \geq 1/2$, there is a SPE in which the players alternate between (C, D) and (D, C).

A good exercise is to try to construct these SPE.

3.2 The Folk Theorem

Perhaps the most famous result in the theory of repeated games is the folk theorem. It says that if players are *really* patient and far-sighted (i.e. if $\delta \to 1$), then not only can repeated interaction allow many SPE outcomes, but actually SPE can allow virtually *any* outcome in the sense of average payoffs.

Let G be a simultaneous move game with action sets $A_1, ..., A_I$, and mixed strategy sets $\Sigma_1, ..., \Sigma_I$, and payoff functions $u_1, ..., u_I$.

Definition 6 *A payoff vector $v = (v_1, ..., v_I) \subset \mathbb{R}^I$ is* ***feasible*** *if there exist actions profiles $a^1, ..., a^K \in A$ and non-negative weights $\lambda^1, ..., \lambda^K$ with $\sum_k \lambda^k = 1$ such that for each i,*

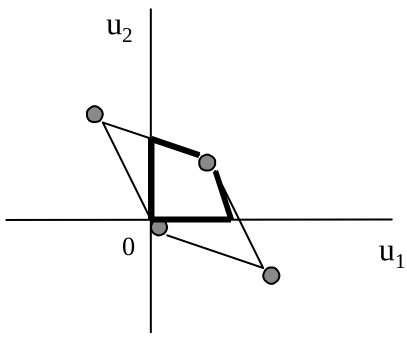
$$v_i = \lambda^1 u_i\left(a^1\right) + ... + \lambda^K u_i\left(a^K\right).$$

Definition 7 *A payoff vector v is* ***strictly individually rational*** *if for all i,*

$$v_i > \min_{\sigma_{-i} \in \Sigma_{-i}} \max_{\sigma_i \in \Sigma_i} u_i\left(\sigma_i, \sigma_{-i}\right) = \underline{v}_i.$$

We can think $\underline{v}_i$ as the lowest payoff a rational player could ever get in equilibrium if he anticipates his opponents' (possibly non-rational) play. We refer to $\underline{v}_i$ as i's *min-max payoff*.

Example In the prisoners' dilemma, the figure below outlines the set of feasible and individually rational payoffs.



Theorem 8 *(Folk Theorem) Suppose that the set of feasible payoffs of G is I-dimensional. Then for any feasible and strictly individually rational payoff vector v, there exists $\underline{\delta} < 1$ such that for any $\delta \geq \underline{\delta}$ there is a SPE σ^* of G such that the average payoff to σ^* is v_i for each player i.*

The Folk Theorem says that *anything* that is feasible and individually rational is possible.

Sketch of Proof. The proof is pretty involved, but here is the outline:

1. Have players on the equilibrium path play an action profile with payoff v (or alternate if necessary).
2. If some player deviates, *punish him* by having the other players for T periods play some σ_{-i} against which i can get at most $\underline{v}_i$.
3. After the punishment period, *reward* all players other than i for carrying out the punishment. To do this, switch to an action profile that gives each player $j \neq i$ some payoff $v_j > \underline{v}_j$.

Q.E.D.

Note that the Folk Theorem does not call for the players to revert to the static Nash equilibrium as a punishment. Instead, they do someting potentially worse — they min-max the deviating player. Of course, in some games (i.e. the prisoners' dilemma), the static Nash equilibrium *is* the min-max point. But in other games (e.g. Cournot), it is not.

4 Applications of Repeated Games

Repeated game models are perhaps the simplest way to capture the idea of ongoing interaction between parties. In particular, they allow us to capture in a fairly simple way the idea that actions taken today will have consequences for tomorrow, and that there is no "last" date from which to unfold the strategic environment via backward induction. In these notes, we will give only a small flavor of the many applications of repeated game models.

4.1 Employment and Efficiency Wages

We consider a simple model of employment. There is a firm and a worker. The firm makes a wage offer w to the worker. The worker then chooses whether to accept, and if he accepts whether to "Work" or "Shirk". Thus, the firms action space is $[0,\infty)$, and the worker chooses $a \in \{\emptyset, W, S\}$.

Payoffs are as follows. If the worker is not employed, he gets $\overline{u} > 0$. If he is employed and works, he gets $w - c$ (here c is the disutility of effort). If he is employed and shirks, he gets w. The firm gets nothing if the worker turns down the offer, $v - w$ if the worker accepts and works, and $-w$ if the worker accepts and shirks. Assume that $v > \overline{u} + c$.

In this environment, it's efficient for the firm to hire the worker if the worker will actually work. However, once hired, the worker would like to slack off. In the one-shot game, if the firm offers a wage $w \geq \overline{u}$ the worker will accept the job and shirk. The worker will reject the offer if $w < \overline{u}$. The firm thus gets $-w$ if the worker accepts and 0 otherwise. So the firm will offer $w < \overline{u}$ and there will be no employment. Is there any way for the parties to strike and efficient deal?

One solution is an incentive contract. Suppose that the worker's behavior is *contractible* in the sense that the firm can pay a contingent wage $w(a)$. If the firm offers a contract $w : A \to \mathbb{R}$ given by $w(W) = \overline{u} + c$ and $w(S) = \overline{u}$, the worket will accept and work. However, this solution requires in essence that a court could come in and verify whether or not a worker performed as mandated — and the employment contract would have to specificy the worker's responsibilities very carefully.

Another possibility arises if the employment relationship is ongoing. Suppose the stage game is repeated at dates $t = 0, 1, \ldots.$

Proposition 9 *If $v \geq \overline{u} + c/\delta$, there is a subgame perfect equilibrium in the firm offers a wage $w \in [\overline{u} + c/\delta, v]$ and the worker works in every period.*

Proof. Consider strategies as follows. In every period, the firm offers a wage $w \in [\overline{u} + c/\delta, v]$ and the worker works. If the worker ever deviates by turning down the offer or shirking, the firm forever makes an offer of $w = 0$ and the worker never accepts (i.e. they repeat the stage game equilibria for every following period. If the firm ever deviates by offering some other w, the worker either rejects the offer — if $w < \overline{u}$ — or accepts and shirks. From then on they repeat the stage game equilibrium.

The firm's best deviation payoff is zero, so it has no incentive to deviate so long as $v \geq w$. Now consider the worker. He has no incentive to reject the offer so long as $w \geq \overline{u} + c$. He has an incentive to work rather than shirk if:

$$\underbrace{w - c}_{\text{Payoff today if } W} + \underbrace{\frac{\delta}{1-\delta}(w - c)}_{\text{Future payoff if employed}} \geq \underbrace{w}_{\text{Payoff today if } S} + \underbrace{\frac{\delta}{1-\delta}\overline{u}}_{\text{Future payoff if fired}}$$

Or in other words if $w \geq \overline{u} + c/\delta$. *Q.E.D.*

The relationship described in the equilibrium is sometimes referred to as an *efficiency wage contract* since the worker receives more than his opportunity cost. That is, since $w - c > \overline{u}$, the worker makes a pure rent due to the incentive problem. However, by paying more, the firm gets more!

4.2 Collusion on Prices

Consider a repeated version of Bertrand competition. The stage game has $N \geq 2$ firms, who each select a price. Customers purchase from the least expensive firm, dividing equally in the case of ties. Quantity purchased is given by $Q(P)$, which is continuous and downward-sloping. Firms have constant marginal cost c. Let

$$\pi(p) = (p - c)\, Q(p)$$

and assume that π is increasing in p on $[c, p^m]$ — where p^m is the monopoly price. This game is repeated at each date $t = 0, 1, 2, ...$ Firms discount at δ.

Note that the static Nash equilibrium involves each firm pricing at marginal cost. The question is whether there is a repeated game equilibrium in which the firms sustain a price above marginal cost.

Proposition 10 *If $\delta < N/(N-1)$ all SPE involve pricing at marginal cost in every period. If $\delta \geq N/(N-1)$, there is a SPE in which the firms all price at $p^* \in [c, p^m]$ in every period.*

Proof. We look for a collusive equilibrium. Note that since the stage nash equilibrium gives each firm it's min-max payoff zero, we can focus on Nash reversion punishments. Suppose the firms try to collude on a price p^*. A firm will prefer not to deviate if and only if:

$$\frac{1}{N}\pi(p^*) \cdot \frac{1}{1-\delta} \geq \pi(p^*).$$

This is equivalent to

$$\delta \geq \frac{N}{N-1}.$$

Thus patient firms can sustain a collusive price. *Q.E.D.*

Note that cooperation becomes more difficult with more firms. For $N = 2$, the critical discount factor is $1/2$, but this increases with N. Note also that the equilibrium condition is independent of π — thus either monopoly collusion is possible ($p = p^m$) or nothing at all.

4.3 Collusion on Quantities

Consider a repreated version of the Cournot model. The stage game has $N = 2$ firms, who each select a quantity. The market price is then $P(Q) = 1 - Q$. Firms produce with constant marginal cost c.

Let $Q^m = (1-c)/2$ denote the monopoly quantity, and π^m the monopoly profit. Let $Q^c = (1-c)/3$ denote the Cournot quantities and π^c the Cournot profit for each firm.

Note that since the static Cournot model has a unique equilibrium, we must focus on the infinitely repeated game to get collusion. We consider SPE with Nash reversion.

Proposition 11 *Firms can collude on the monopoly price as part of a SPE if the discount factor is sufficiently high.*

Proof. Let's look for an equilibrium in which the firms set quantities q^m in each period and play static Cournot forever should a firm ever deviate. Provided no one deviates, each firm expects a payoff

$$\text{Collusive Payoff} = \frac{1}{1-\delta}\frac{1}{2}\pi^m$$

On the other hand, by deviating a firm can get:

$$\text{Deviation Payoff} = \max_{q_i} \left(P\left(q^m + q_i\right) - c\right) q_i + \frac{\delta}{1-\delta}\pi^c.$$

The best deviation is to set $q_i = (3/4 - c)/2$. For the case of $c = 0$, there is no incentive to deviate provided that:

$$\frac{1}{1-\delta}\frac{1}{2}\pi^m = \frac{1}{1-\delta}\frac{1}{8} \geq \frac{9}{64} + \frac{\delta}{1-\delta}\frac{1}{9} = \left(1 - \frac{1}{4} - q^{dev}\right) q^{dev} + \frac{\delta}{1-\delta}\pi^c.$$

That is if $\delta \geq 9/17$. *Q.E.D.*

Several comments are in order.

1. First, note that as N increases the monopoly share of each firm becomes smaller. However, the static Cournot profits also become smaller. As an exercise see if you can figure out whether collusion becomes harder or easier to sustain.

2. In the Cournot model, the static Nash equilibrium is not the min-max point. Thus, there are typically "worse" SPE punishments than Nash reversion. In turn, this means that collusion can be sustained for lower discount factors than with Nash reversion. Finding the "optimal" collusive agreement for a given discount factor turns out to be a difficult problem. See Abreu (1986, JET and 1988 EMA) for details.

4.4 Multimarket Contact

A common wisdom in Industrial Organization is that collusion will be easier to sustain if firms compete simultaneously in many markets. The classic intuition for this is that opportunism is likely to be met with retaliation in many markets. This may limit the temptation to compete agressively. We now explore this idea in a framework pioneered by Bernheim and Whinston (1990).

Consider two firms that simultaneously compete in two markets.

- Let G_{ik} denote the gain to firm i from deviating in market k for the current period, for a particular equilibrium.
- Let π_{ik}^c denote the discounted payoff from continuation (next period forward) for firm i in market k, assuming no deviation from the equilibrium in the current period.
- Let π_{ik}^p denote the discounted "punishment" payoff from continuation (next period forward) for firm i in market k, assuming that i deviates from the equilibrium in the current period.

Separate Markets. If the markets are separate, the equilibrium condition is that for each i and each k,

$$\delta\pi_{ik}^c \geq G_{ik} + \delta\pi_{ik}^p.$$

There are $N \times K$ such constraints.

Linked Markets. Now suppose that a deviation in *any* market is met by punishment in all markets. The equilibrium condition is that for each i,

$$\sum_k \delta\pi_{ik}^c \geq \sum_k G_{ik} + \sum_k \delta\pi_{ik}^p.$$

There are now N constraints.

We can make the immediate observation that *multimarket contact pools constraints across markets.* Clearly, if the separate constraints are satisfied, the linked constraints will be as well. The interesting question is whether linking strictly expands the set of collusive outcomes. The answer is typically yes *so long as there is enforcement slack in some market.* If there is slack in one of the individual constraints, with pooling there will be slack in the aggregate constraint — potentially allowing for a more collusive equilibrium.

Games of Incomplete Information

Jonathan Levin

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1 Introduction

We now start to explore models of incomplete information. Informally, a game of *incomplete information* is a game where the players do not have common knowledge of the game being played. This idea is tremendously important in capturing many economic situations, where a variety of features of the environment may not be commonly known. Among the aspects of the game that the players might not have common knowledge of are:

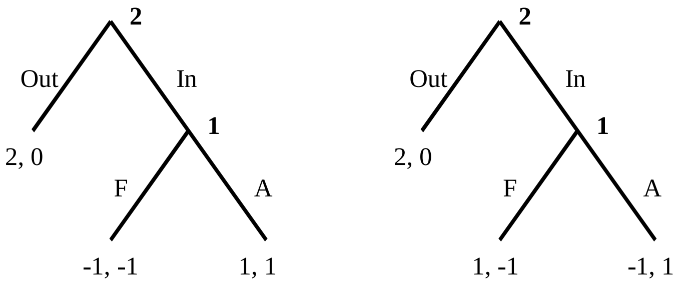
- Payoffs
- Who the other players are
- What moves are possible
- How outcome depends on the action.
- What opponent knows, and what he knows I know....

To take a couple of simple examples: (1) in price or quantity competition, firms might know their own costs, but not the costs of their rivals; (2) firms investing in R&D might know how their project is coming along, but have no idea who else is working on the same problem; (3) the government may design the tax code not envisioning what ploys people will come up with to avoid taxs; (4) countries may negotiate climate change agreements having different beliefs about the costs and benefits of global climate change; (5) plaintiffs may offer settlements to defendants not knowing what sort of case the defendant will be able to bring to court, or what sort of case the defendant thinks the plaintiff will be able to bring.

2 Examples

2.1 Entry with a "Tough" Incumbent

Recall our canonical entry model where Firm 2 (the entrant) must decide whether or not to enter a market, and Firm 1 (the incumbent) must decide whether to Fight or Accomodate entry. Let's modify this game by assuming that with probability $1-p$, the incumbent is "tough," while with probability p, the incumbent is normal. The payoffs in the game depend on whether the incumbent is tough or normal:



Entry w/ Normal Incumbent **Entry w/ Tough Incumbent**

2.2 Sealed Bid Auction

Two bidders are trying to purchase the same item in a sealed bid auction. The bidders simultaneously submit bids b_1 and b_2 and the auction is sold to the highest bidder at his bid price (this is called a "first price" auction). If there is a tie, there is a coin flip to determine the winner. Suppose the players utilities are:

$$u_i(b_i, b_{-i}) = \begin{cases} v_i - b_i & \text{if } b_i > b_{-i} \\ \frac{1}{2}(v_i - b_i) & \text{if } b_i = b_{-i} \\ 0 & \text{if } b_i < b_{-i} \end{cases}$$

The key informational feature is the each player knows his own value for the item (i.e. bidder i knows v_i), but does not know the valuation of his rival. Instead, we assume that each bidder had a prior belief that his rival's valuation is a draw from a uniform distribution on $[0, 1]$, and that these prior beliefs are common knowledge.

2.3 Public Good Provision

Two faculty members in the economics department both want to recruit a top graduate student to their department. Either faculty member can ensure the student will accept the offer by getting on the phone and shamelessly promoting the graduate program. However, there is some cost to making this call. Assume the payoffs can be represented as follows:

	Call	Don't
Call	$1-c_1, 1-c_2$	$1-c_2, 1$
Don't	$1, 1-c_1$	$0, 0$

Assume the faculty members choose their actions simultaneously (or at least without learning the other's action) and the faculty have private information about their costs of making the call. That is, faculty member i knows c_i, and believes that c_j is a random draw from a uniform distribution on $[\underline{c}, \overline{c}]$. Faculty member i's belief about c_j is commonly known.

An alternative formulation of this problem is that faculty member one tells everyone everything, so that everyone knows his cost $c_1 = 1/2$. However, $c_2 \in \{\underline{c}, \overline{c}\}$ is known only to player two.

Or, we could assume that player one is a senior faculty member who knows from long experience that $c_1 = 1/2$ and $c_2 = 2/3$. However, player two is a new assistant professor whose prior belief is that $c_1, c_2 \sim U[0, 2]$ and are independent.

3 Definitions

Definition 1 *A game with incomplete information $G = (\Theta, S, P, u)$ consists of:*

1. *A set $\Theta = \Theta_1 \times ... \times \Theta_I$, where Θ_i is the (finite) set of possible types for player i.*

2. *A set $S = S_1 \times ... \times S_I$, where S_i is the set of possible strategies for player i.*

3. *A joint probability distribution $p(\theta_1, ..., \theta_I)$ over types. For finite type space, assume that $p(\theta_i) > 0$ for all $\theta_i \in \Theta_i$.*

4. *Payoff functions $u_i : S \times \Theta \to \mathbb{R}$.*

Consider how this definition relates to each of our examples.

1. Entry: $\Theta_1 = \{\text{tough, normal}\}$; $\Theta_2 = \{\text{normal}\}$.

2. Auction: $\Theta_1 = \Theta_2 = [0, 1]$.

3. Public Good: $\Theta_1 = \Theta_2 = [\underline{c}, \overline{c}]$.

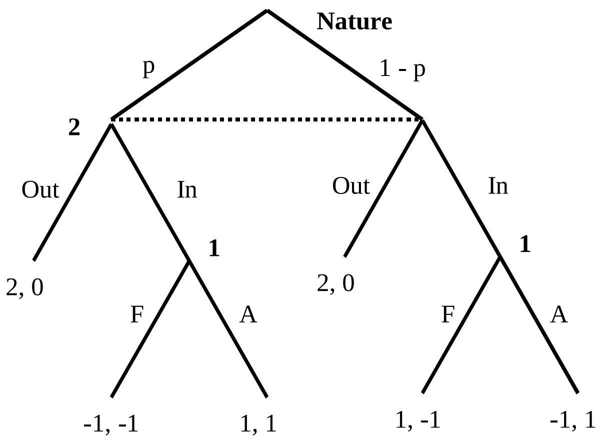
We assume that players know their own types, but do not know the types of other players.

Remark 1 *Note that payoffs can depend not only on your own type, but on your rivals' types. If u_i depends on θ_i, but not on θ_{-i}, we sometimes say the game has private values.*

In order to analyze these types of games, we rely on a fundamental (and Nobel-prize winning) observation by Harsanyi (1968):

> Games of incomplete information can be thought of as games of complete but imperfect information where nature makes the first move (selecting $\theta_1, ..., \theta_I$), but not everyone observes nature's move (i.e. player i learns θ_i but not θ_{-i}).

Consider formulating the entry model in exactly this way.



In analyzing this sort of game, we cam think of nature simply as another player. The only difference is that rather than maximizing a payoff, nature just uses a fixed mixed strategy.

This observation should make the following definitions look obvious. They just say that to analyze a game of incomplete information, we can look at the Nash Equilibrium of the game where nature is a player.

Definition 2 *A Bayesian pure strategy for player i in G is a function $f_i : \Theta_i \to \Sigma_i$. Write S^{Θ_i} for the set of Bayesian pure strategies.*

Definition 3 *A Bayesian strategy profile $(f_1, ..., f_I)$ is a* ***Bayesian Nash Equilibrium*** *if for all i,*

$$f_i \in \arg \max_{f_i' \in S_i^{\Theta_i}} \sum_{\theta \in \Theta} u_i \left(f_i' (\theta_i), f_{-i} (\theta_{-i}), \theta_i, \theta_{-i} \right) p (\theta_i, \theta_{-i})$$

or alternatively, for all i, θ_i and s_i :

$$\begin{aligned} & \sum_{\theta_{-i} \in \Theta_{-i}} u_i (f_i (\theta_i), f_{-i} (\theta_{-i}), \theta_i, \theta_{-i}) p (\theta_{-i} | \theta_i) \\ \geq & \sum_{\theta_{-i} \in \Theta_{-i}} u_i (s_i, f_{-i} (\theta_{-i}), \theta_i, \theta_{-i}) p (\theta_{-i} | \theta_i) \end{aligned}$$

The second part of the definition just says that in order to maximize your expected payoff given that you know your types, then the strategy you choose for each type should maximize your payoff conditional on your having that type.

Remark 2 *A Bayesian Nash Equilibrium is simply a Nash Equilibrium of the game where Nature moves first, chooses $\theta \in \Theta$ from a distribution with probability $p(\theta)$ and reveals θ_i to player i.*

4 Solving Bayesian Games

4.1 Public Good: version A

Consider a version of the public good game where

- Player 1 has a *known* cost $c_1 < 1/2$;
- Player 2 has cost $\underline{c}$ with probability p and $\overline{c}$ with probability $1 - p$.

Assume that $0 < \underline{c} < 1 < \overline{c}$ and that $p < 1/2$.

Proposition 1 *The unique Bayesian Nash Equilibrium is $f_1 = Call$ and $f_2(c) = Don't$ for all c.*

To prove this, keep in mind that each type of player must play a best response. When player 2 has type $\overline{c}$, then calling is strictly dominated:

$$u_2(s_1, Call; \overline{c}) < u_2(s_1, Don't; \overline{c}),$$

for all s_1. Thus, $f_2(\overline{c}) = Don't$.

Now, for player 1,

$$\begin{aligned} u_1(Call, f_2; c_1) &= 1 - c_1 \\ u_1(Don't, f_2; c_1) &= pu_1(Don't, f_2(\underline{c}); c_1) \\ &\quad + (1-p) u_1(Don't, f_2(\overline{c}); c_1) \\ &\leq p \cdot 1 + (1-p) \cdot 0 = p. \end{aligned}$$

Since $1 - c_1 > p$, then $f_1(c_1) = Call$.

But then when player 2 has type $\underline{c}$:

$$\begin{aligned} u_2(f_1, Call; \underline{c}) &= 1 - \underline{c} \\ u_2(f_1, Don't; \underline{c}) &= 1 \end{aligned}$$

so $f_2(\underline{c}) = Don't$.

Note that this process works a bit like iterated dominance.

4.2 Public Good: version B

Now imagine that c_1 and c_2 are independent random draws from a uniform distribution on $[0, 2]$.

Proposition 2 *The (essentially) unique Bayesian Nash Equilibrium is*

$$f_i(c_i) = \begin{cases} Call & \text{if } c_i \leq 2/3 \\ Don't & \text{if } c_i > 2/3 \end{cases}$$

To prove that this is actually a BNE is easy. We can just check that each player's conjectured strategy is a best response to the other's — in particular, that each player's strategy is a best response given that his opponent will call with probability $1/3$ and won't call with probability $2/3$.

To illustrate uniqueness, let's work through how to derive the equilibrium. The first observation is the following: *if* $f_i(c_i) = Call$ *then* $f_i(c_i') = Call$ *for all* $c_i' < c_i$. To see this, note that if $f_i(c_i) = Call$, then:

$$\mathbb{E}_{c_{-i}} u_i(Call, f_2(c_{-i}); c_i) \geq \mathbb{E}_{c_{-i}} u_i(Don't, f_2(c_{-i}); c_i)$$

which implies that:

$$1 - c_i \geq z_{-i}$$

where we let z_{-i} denote the payoff to not calling. This implies that for all $c_i' < c_i$:

$$1 - c_i' > z_{-i}$$

or equivalently

$$\mathbb{E}_{c_{-i}} u_i \left(Call, f_2 \left(c_{-i} \right) ; c_i' \right) \geq \mathbb{E}_{c_{-i}} u_i \left(Don't, f_2 \left(c_{-i} \right) ; c_i' \right).$$

There is a simple intuition: namely that calling is more attractive if the costs are lower.

In light of Observation 1, a Bayesian Nash Equilibrium must be of the form:

$$f_1 \left(c_1 \right) = \begin{cases} Call & \text{if } c_1 \leq c_1^* \\ Don't & \text{if } c_1 > c_1^* \end{cases}$$

$$f_2 \left(c_2 \right) = \begin{cases} Call & \text{if } c_2 \leq c_2^* \\ Don't & \text{if } c_2 > c_2^* \end{cases}$$

for some "cut-off" costs c_1^*, c_2^*. (Note: it will turn out that when c_i is *exactly* equal to c_i^*, then agent i is indifferent to calling or not. This is why the equilibrium is "essentially" unique.)

Let

$$z_j = \Pr \left[i \text{ will call given cut-off } c_j^* \right] = \Pr \left[c_j \leq c_j^* \right] = \frac{1}{2} c_j^*.$$

For these strategies to be a BNE, we need:

$$\begin{aligned} 1 - c_i &\geq z_{-i} && \text{for all } c_i \leq c_i^* \\ 1 - c_i &< z_{-i} && \text{for all } c_i > c_i^* \end{aligned}$$

Or equivalently that $1 - c_i^* = z_{-i}$. Thus, for $i = 1, 2$,

$$1 - c_i^* = \frac{1}{2} c_i^*$$

and hence the unique equilibrium is to call whenever $c_i < 2/3$.

Remark 3 *Note that the equilibrium outcome is inefficient in several ways. First, there is "under-investment" in the public good — it is* ***always*** *efficient for someone to call, and yet with probability* $4/9$*, no one calls. Second, there is "miscoordination" — with probability* $1/9$ *both parties call even though this is inefficient.*

4.3 Sealed Bid Auction

Proposition 3 *In the first price sealed bid auction with valuations uniformly distributed on* $[0,1]$, *the unique BNE is* $f_i(v_i) = v_i/2$ *for* $i = 1, 2$.

Again, to verify that this is a BNE is relatively easy. We just show that each type of each player is using a best response. Note that:

$$\begin{aligned}\mathbb{E}_{v_2} u_1(b_1, f_2; v_1, v_2) &= (v_1 - b_1)\Pr[f_2(v_2) < b_1] \\ &\quad + \frac{1}{2}(v_1 - b_1)\Pr[f_2(v_2) = b_1].\end{aligned}$$

We assume $b_1 \in (0, 1/2]$. No large bid makes sense given f_2. Then:

$$\mathbb{E}_{v_2} u_1(b_1, f_2; v_1, v_2) = (v_1 - b_1)\, 2b_1$$

Maximizing this by choice of b_1, we obtain the first order condition:

$$0 = 2v_1 - 4b_1 \qquad \Rightarrow \qquad b_1 = v_1/2.$$

To show uniqueness (or to find the equilibrium if you didn't already know it) is harder. To do it, let's consider bidder one's optimization problem given f_2. If we *assume* that f_2 is strictly increasing, then ties occur with probability zero, so bidder one's problem is:

$$\max_{b_1} (v_1 - b_1)\Pr[f_2(v_2) < b_1]$$

If we further assume that f_2 is increasing, then we have

$$\Pr[f_2(v_2) < b_1] = \Pr\left[v_2 < f_2^{-1}(b_1)\right] = f_2^{-1}(b_1).$$

Finally if f_2 is differentiable, we can solve:

$$\max_{b_1} (v_1 - b_1)\, f_2^{-1}(b_1)$$

to obtain the first order condition:

$$0 = -f_2^{-1}(b_1) + (v_1 - b_1)\frac{1}{f_2'\left(f_2^{-1}(b_1)\right)}.$$

For f_1, f_2 to be an equilibrium, it must be that:

$$b_1 = f_1(v_1)$$

so the first order condition is

$$0 = -f_2^{-1}\left(f_1\left(v_1\right)\right) + \left(v_1 - f_1\left(v_1\right)\right)\frac{1}{f_2'\left(f_2^{-1}\left(f_1\left(v_1\right)\right)\right)}.$$

Finally, assuming we can look for a symmetric equilibrium with $f_1 = f_2 = f$, the first order condition for optimality becomes:

$$0 = -v_1 + \left(v_1 - f_1\left(v_1\right)\right)\frac{1}{f'\left(v_1\right)}.$$

Solving this differential equation gives the equilibrium.

4.4 A Lemons Problem

Consider a seller of a used car and a potential buyer of that car. Suppose that quality of the car, θ, is a uniform draw from $[0,1]$. This quality is known to the seller, but not to the buyer. Suppose that the buyer can make an offer $p \in [0,1]$ to the seller, and the seller can then decide whether to accept or reject the buyer's offer. (Note: This sounds like a dynamic game, but we can think of it as simultaneous-move if we think of the seller as announcing the set of all prices she will accept and all those she will reject.)

Payoffs are as follows:

$$u_S = \begin{cases} p & \text{if offer accepted} \\ \theta & \text{if offer rejected} \end{cases}$$

$$u_B = \begin{cases} a + b\theta - p & \text{if offer accepted} \\ 0 & \text{if offer rejected} \end{cases}$$

Assume that $a \in [0,1)$, that $b \in (0,2)$, and that $a + b > 1$. These assumptions imply that for all θ, it is more efficient for the buyer to own the car.

Proposition 4 *The (essentially) unique BNE is for the buyer to offer $p = a/(2-b)$ and the seller to accept if and only if $p \geq \theta$.*

To prove this, we first consider the seller. It is easy to see that the strategy of accepting if and only if $p \geq \theta$ is weakly dominant for the seller. Now consider the buyer's problem.

$$\begin{aligned} \mathbb{E}_\theta u_B\left(p; S \text{ accepts if } p \geq \theta\right) &= \int_0^1 \mathbf{1}_{\{\theta<p\}}\left(a + b\theta - p\right) d\theta \\ &= \int_0^p \left(a + b\theta - p\right) d\theta \\ &= p\left(a + \frac{1}{2}bp - p\right) \end{aligned}$$

Choosing p to maximize this expression gives the first order condition

$$0 = a + (b - 2)\, p.$$

Note that the ex ante quality of the car is $1/2$. However, given an offer p, the expected value of the car *conditional on the seller accepting the offer* (i.e. conditional on $\theta \leq p$) is $p/2 \leq 1/2$. This is sometime's called the *winner's curse (*here it's really a *buyer's curse*). Note that what gives rise to this effect is that the seller's information is directly payoff-relevant to the buyer. Unlike our previous examples, we do not have private values.

Remark 4 *If $a = 0$, then the buyer's curse is so strong that the unique equilibrium is for the buyer to offer a price $p = 0$. Trade never occurs despite it the fact that there are always gains from trading.*

Dynamic Games with Incomplete Information

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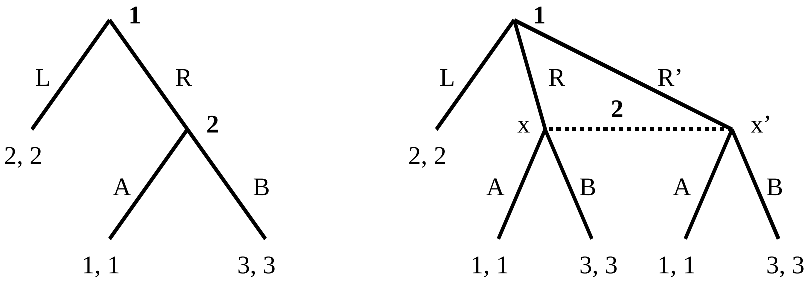
Our final topic of the quarter is dynamic games with incomplete information. This class of games encompasses many interesting economic models — market signalling, cheap talk, and reputation, among others. To study these problems, we start by investigating a new set of solution concepts, then move on to applications.

1 Perfect Bayesian Equilibrium

1.1 Problems with Subgame Perfection

In extensive form games with incomplete information, the requirement of subgame perfection does not work well. A first issue is that subgame perfection may fail to rule out actions that are sub-optimal given any "beliefs" about uncertainty.

Example 1 Consider the following games:



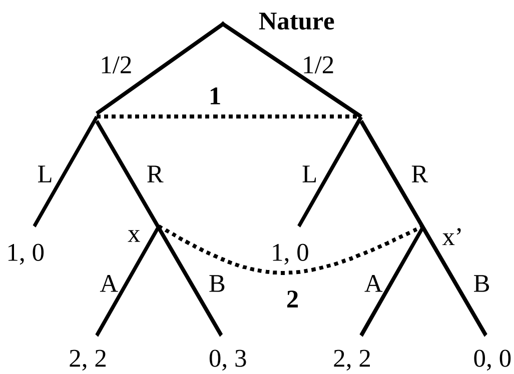
These two games are very similar. However, in the game on the left, (R, B) is the only SPE. In the game on the right, there are other SPE: $(pR + (1-p)\,R', B)$ for any p, and $(L, qA + (1-q)\,B)$ for any $q \geq 1/2$.

The problem here is that the game on the right has no subgames other than the game itself. So SPE has no bite. What can be done about this? One solution is to require players to choose optimally at all information sets. To make sense of this we need to introduce the idea of beliefs.

Example 1, cont. Suppose at the information set $h = \{x, x'\}$, we require player two to choose the action that maximizes his *expected* payoff given some belief assigning probability $\mu(x)$ to being at x and $\mu(x')$ to being at x', with $\mu(x) + \mu(x') = 1$. Then for *any belief* player two might have, choosing B is optimal.

A second issue is that subgame perfection may allow actions that are possible only with beliefs that are "unreasonable".

Example 2 Consider the following game:



In this game, (L, B) is a subgame perfect equilibrium.

As in the previous example, (L, B) is an SPE in this example because there are no subgames. Note, however, that there are in fact beliefs for which B is an optimal choice for player 2. If player 2 places probability at least $2/3$ on being at x given that he is at $h = \{x, x'\}$, then B is an optimal choice. However, these beliefs seem quite unreasonable: if player 1 chooses R, then player 2 should place equal probability on being at either x or x'.

1.2 Perfect Bayesian Equilibrium

Let G be an extensive form game. Let H_i be the set of information sets at which player i moves. Recall that:

Definition 1 *A behavioral strategy for player i is a function $\sigma_i : H_i \to \Delta(A_i)$ such that for any $h_i \in H_i$, the support of $\sigma_i(h_i)$ is contained in the set of actions available at h_i.*

We now augment a player's strategy to explicitly account for his beliefs.

Definition 2 *An* ***assessment*** *(σ_i, μ_i) for player i is a strategy σ_i and belief function μ_i that assigns to each $h_i \in H_i$ a probability distribution over nodes in h_i. Write $\mu_i(x|h)$ as the probability assigned to node x given information set h.*

Example 1, cont. Player two's belief function μ_2 must satisfy $\mu_2(x) + \mu_2(x') = 1$. If player 2 uses Bayesian updating, and $\sigma_1(R) + \sigma_1(R') > 0$, then $\mu_2(x) = \sigma_1(R) / (\sigma_1(R) + \sigma_1(R'))$.

Example 2, cont. Player two's belief function μ_2 again sets $\mu_2(x) + \mu_2(x') = 1$. If player 2 uses Bayesian updating, and $\sigma_1(R) > 0$, then $\mu_2(x) = \mu_2(x') = 1/2$.

Definition 3 *A profile of assessments (σ, μ) is a* ***perfect bayesian equilibrium*** *if*

1. *For all i and all $h \in H_i$, σ_i is sequentially rational, i.e. it maximizes i's expected payoff conditional on having reached h given μ_i and σ_{-i}.*
2. *Beliefs μ_i are updated using Bayes' rule whenever it applies (i.e. at any information set on the equilibrium path).*

Example 1, cont. Sequential rationality implies that player 2 must play B, so the unique PBE is (R, B).

1.3 Refinements of PBE

While PBE is a bread and butter solution concept for dynamic games with incomplete information, there are many examples where PBE arguably allows for equilibria that seem quite unreasonable. Problems typically arise because PBE places no restrictions on beliefs in situations that occur with probability zero — i.e. "off-the-equilibrium-path".

Example 2, cont. Bayes' rule implies that if $\sigma_1(R) > 0$, then $\mu_2(x) = \mu_2(x') = 1/2$, so player 2 must play A. Therefore (R, A) is a PBE. However, (L, B) is also a PBE! If $\sigma_1(R) = 0$, then Bayes' Rule does not apply when player 2 forms beliefs. So μ_2 is arbitrary. If $\mu_2(x) > 2/3$, then it is sequentially rational for 2 to play B. Hence (L, B) is a PBE.

This example may seem pathological, but it turns out to be not all that uncommon. In response, game theorists have introduced a large number of "equilibrium refinements" to try to rule out unreasonable PBE of this sort. These refinements restrict the sorts of beliefs that players can hold in situations that occur with probability zero.

One example is Kreps and Wilson's (1982) notion of *consistent beliefs.*

Definition 4 *An assessment (σ, μ) is* ***consistent*** *if $(\sigma, \mu) = \lim_{n\to\infty}(\sigma^n, \mu^n)$ for some sequence of assessments (σ^n, μ^n) such that σ^n is totally mixed and μ^n is derived from σ^n by Bayes' Rule.*

Example 2, cont. In this game, the only consistent beliefs for player two are $\mu_2(x) = \mu_2(x') = 1/2$. To see why, note that for any player 1 strategy with $\sigma_1^n(L), \sigma_1^n(R) > 0$, it must be that $\mu_2^n(x) = \frac{1}{2}$. But then $\lim_{n\to\infty}\mu_2^n(x) = 1/2$, so $\mu_2(x) = 1/2 = \mu_2(x')$ is the only consistent belief for player 2.

Definition 5 *An assessment (σ, μ) is a* ***sequential equilibrium*** *if (σ, μ) is both consistent and a PBE.*

Sequential equilibrium is a bit harder to apply than PBE in practice, so we will typically work with PBE. It also turns out that there are games where sequential equilibrium still seems to allow unreasonable outcomes, and one might want a stronger refinement. We will mention one such refinement — the "intuitive criterion" for signalling games — below.

2 Signalling

We now consider an important class of dynamic models with incomplete information. These *signalling* models were introduced by Spence (1974) in his Ph.D. thesis. There are two players and two periods. The timing is as follows.

Stage 0 Nature chooses type $\theta \in \Theta$ of player 1 from a distribution p.

Stage 1 Player 1 observes θ and chooses $a_1 \in A_1$.

Stage 2 Player 2 observes m and chooses $a_2 \in A_2$.

Payoffs $u_1(a_1, a_2, \theta)$ and $u_2(a_1, a_2, \theta)$.

Many important economic models take this form.

Example: Job Market Signalling In Spence's original example, player 1 is a student or worker. Player 2 is the "competitive" labor market. The worker's "type" is his ability and his action is the level of education he chooses. The labor market observes the worker's education (but not his ability) and offers a competitive wage equal to his expected ability conditional on education. The worker would like to use his education choice to "signal" that he is of high ability.

Example: Initial Public Offerings Player 1 is the owner of a private firm, while Player 2 is the set of potential investors. The entrepreneur's "type" is the future profitability of his company. He has to decide what fraction of the company to sell to outside investors and the price at which to offer the shares (so a_1 is both a quantity and a price). The investors respond by choosing whether to accept or reject the entrepeneur's offer. Here, the entrepeneur would like to signal that the company is likely to be profitable.

Example: Monetary Policy Player 1 is the Federal Reserve. It's type is its preferences for inflation versus unemployment. In the first period, it chooses an inflation level $a_1 \in A_1$. Player 2 is the firms in the economy. They observe first period inflation and form expectations about second period inflation, denoted $a_2 \in A_2$. Here, the Fed wants to signal a distaste for inflation so that firms will expect prices not to rise too much.

Example: Pretrial Negotiation Player 1 is the Defendant in a civil lawsuit, while Player 2 is the Plaintiff. The Defendant has private information about his liability (this is θ). He makes a settlement offer $a_1 \in A_1$. The Plaintiff then accepts or rejects this offer, so $A_2 = \{A, R\}$. If the Plaintiff rejects, the parties go to trial. Here, the Defendant wants to signal that he has a strong case.

2.1 Equilibrium in Signalling Models

We start by considering Perfect Bayesian equilibrium in the signalling model.

Definition 6 *A perfect bayesian equilibrium in the signalling model is a strategy profile $s_1(\theta)$, $s_2(a_1)$ together with beliefs $\mu_2(\theta|a_1)$ for player two such that:*

1. *Player one's strategy is optimal given player two's strategy:*

$$s_1(\theta) \text{ solves } \max_{a_1 \in A_1} u_1(a_1, s_2(a_1), \theta) \text{ for all } \theta \in \Theta$$

2. *Player two's beliefs are compatible with Bayes' rule, i.e. if any type of player one plays a_1 with positive probability then*

$$\mu_2(\theta|a_1) = \frac{\Pr(s_1(\theta) = a_1)\, p(\theta)}{\sum_{\theta' \in \Theta} \Pr(s_1(\theta') = a_1)\, p(\theta')};$$

   if player one never uses a_1, then $\mu_2(\theta|a_1)$ is arbitrary.

3. *Player two's strategy is optimal given his beliefs and given player one's action:*

$$s_2(a_1) \text{ solves } \max_{a_2 \in A_2} \sum_{\theta \in \Theta} u_2(a_1, a_2, \theta)\, \mu_2(\theta|a_1) \text{ for all } a_1 \in A_1.$$

It is not hard to allow for mixed strategies, in which case player i's strategy is denoted σ_i.

As we will see in the examples to follow, it helps to think of PBE as falling into different categories:

1. Separating: Different types of player one use different actions, so player two perfectly learns player one's type in equilibrium.

2. Pooling: All types of player one use the same action, so no information is transmitted in player one's action.

3. Semi-Separating: Some actions of player one are chosen by several types of player one, other actions are chosen by a single type. Thus, there is some learning, but not perfect learning.

With this general framework established, we move on to applications.

2.2 Job Market Signalling

Consider a single worker, whose ability (productivity) is given by $\theta \in \{\theta_L, \theta_H\}$ with $\theta_H > \theta_L > 0$. The worker knows his own ability, and the labor market assigns prior probability λ to him having type θ_H. The worker first chooses his level of education e. Education is costly, and the cost $c(e, \theta)$ depends on the worker's ability.

Assumption Suppose that $c_e > 0$ (education is costly on the margin), and that $c_{e\theta} < 0$ (education is *less* costly on the margin for more able workers).

Once the worker chooses his education level, firms make wage offers. Suppose that workers all have a reservation wage 0 regardless of their ability. Assume also that the labor market is competitive.

The key idea we are headed for is that high ability workers may be able to communicate their ability by getting a lot of costly education. Indeed they may choose to become educated despite the fact that education has no direct effect on productivity!

To solve the model, we work backward. Let $\mu(e)$ denote the labor market's belief that a worker who has chosen education e is of high ability. Then the market assesses that the worker will produce θ_H with probability $\mu(e)$ and otherwise θ_L. Since the labor market is competitive, the wage for a worker with education e is given by:

$$w(e) = \mu(e)\,\theta_H + (1 - \mu(e))\,\theta_L.$$

Now consider the problem facing the worker, given beliefs $\mu(e)$ and the resulting wage $w(e)$. A worker with ability θ must solve:

$$\max_e w(e) - c(e, \theta).$$

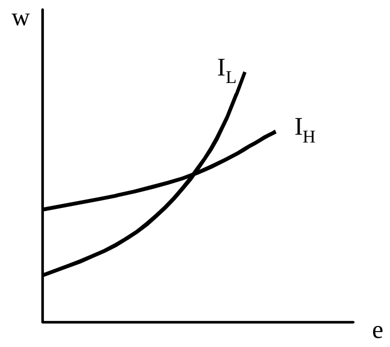
To solve this problem, consider the worker's indifference curves in (e, w) space. Implicitly differentiating $u(w, e, \theta) = U$, we obtain:

$$\left.\frac{dw}{de}\right|_{u=U} = c_e(e, \theta) > 0$$

so indifference curves slope up. Also:

$$\frac{d}{d\theta}\left.\frac{dw}{de}\right|_{u=U} = c_{e\theta}(e, \theta) < 0$$

so indifference curves are flatter for high productivity workers. Graphically, this means that indifference curves exhibit the *Spence-Mirrlees Single Crossing Property.*



For any given $w(e)$, we can find the optimal choice of each worker type by selecting the point of tangency with an indifference curve.

Remark 1 *If we apply Topkis' Theorem to the worker's problem* $\max_e w(e) - c(e, \theta)$*, we see immediately that because* $c_{e\theta} < 0$*, then for any wage function* $w(e)$ *it must be the case that a worker of type* θ_H *selects a (weakly) higher education level than a worker of type* θ_L*. We could also derive this result in the indifference curve picture.*

The key question is where $w(e)$ (or equivalently $\mu(e)$) comes from. On the equilibrium path, it is implied by theworker's choice. However, for levels of education that are not chosen in equilibrium, it can be anything between θ_L and θ_H since PBE imposes no restriction on beliefs other than that $\mu(e) \in [0, 1]$.

This flexibility in setting $w(e)$ gives rise to many possible equilibria.

Separating Equilibria. We first look for separating equilibria where $e(\theta_H) \neq e(\theta_L)$.

Claim 1 In a separating equilibrium, $w(e(\theta)) = \theta$ for $\theta \in \{\theta_L, \theta_H\}$. (I.e. workers are paid their marginal product).

Proof. In a PBE, beliefs are derived from Bayes rule when possible. Type θ_L workers always choose $e(\theta_L)$, while type θ_H workers always choose θ_H. Thus, if $e(\theta)$ is observed, the market must believe the worker is type θ for sure. Thus, $w(e(\theta)) = \theta$. *Q.E.D.*

Claim 2 In a separating equilibrium, $e(\theta_L) = 0$.

Proof. Suppose not. By the previous claim, type θ_L workers are paid θ_L if they choose $e(\theta_L)$. Suppose instead they choose $e = 0$. Then $\mu(e = 0) \geq 0$, so $w(0) \geq \theta_L$. This deviation reduces education costs, but not wages, so it is beneficial. *Q.E.D.*

We are now ready to derive separating equilibria in the job market model. We need to find an education level $e(\theta_H)$ for high ability workers such that:

- High ability workers prefer to select $e(\theta_H)$ and get a wage θ_H rather than selecting $e(\theta_L) = 0$ and getting a wage $w(0) = \theta_L$. Thus:

$$\theta_H - c(e(\theta_H), \theta_H) \geq \theta_L - c(0, \theta_H) \tag{1}$$

- Low ability workers must prefer the opposite:

$$\theta_L - c(0, \theta_L) \geq \theta_H - c(e(\theta_H), \theta_L). \tag{2}$$

Claim 3 For any $e(\theta_H)$ that satisfies (1)–(2), there is a separating equilibrium in which high ability workers choose $e(\theta_H)$.

Proof. If (1),(2) hold then no worker will want to mimic a worker of the other ability. The only thing to worry about is that a worker might want to choose some $e \notin \{e(\theta_L), e(\theta_H)\}$. To ensure against this, we need to choose $\mu(e)$ sufficiently low so that for all $e \notin \{e(\theta_L), e(\theta_H)\}$

$$w(e) - c(e, \theta_H) \leq \theta_H - c(e(\theta_H), \theta_H)$$

and

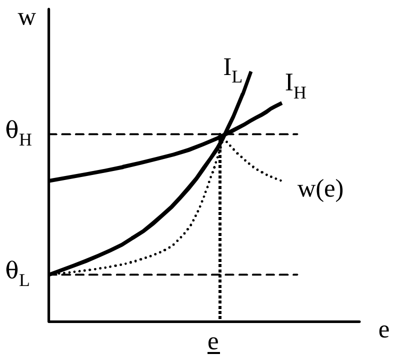
$$w(e) - c(e, \theta_L) \leq \theta_L - c(0, \theta_L).$$

One choice that will definitely work is $\mu(e) = 0$, so that $w(e) = \theta_L$ for all $e \notin \{e(\theta_L), e(\theta_H)\}$. *Q.E.D.*

What is the range of values of $e(\theta_H)$ for which as separating equilibrium (i.e. the range of values of $e(\theta_H)$ that satisfy (1) and (2)? Note that by the single crossing property, if (2) is satisfied with equality, then (1) will be satisfied strictly. So let $\underline{e}$ be defined to solve:

$$\theta_L - c(0, \theta_L) = \theta_H - c(\underline{e}, \theta_L).$$

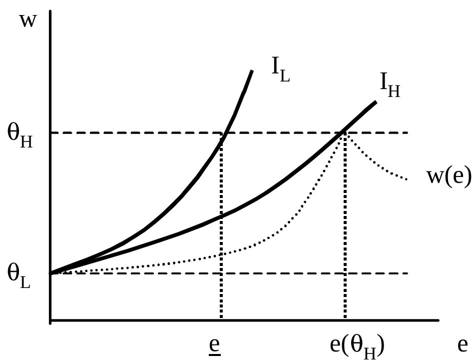
The separating equilibrium with $e(\theta_H) = \underline{e}$ is depicted below.



The single-crossing property also implies that if (1) is satisfied with equality, then (2) will be satisfied strictly. Define $\overline{e}$ to solve:

$$\theta_H - c(\overline{e}, \theta_H) = \theta_L - c(0, \theta_H).$$

The separating equilibrium with $e(\theta_H) = \overline{e}$ is depicted below.



In fact, there is separating equilibrium for any $e \in [\underline{e}, \overline{e}]$, but not for any e outside of this interval.

Remark 2 *Note the important role played by the single crossing property. This is what allows high ability workers to choose a positive education level that is costly, but less costly for them than it would be for low ability workers. It is this **differential** cost that allows separation.*

Pooling Equilibria. In a pooling equilibrium, every worker chooses the same education level e^P with probability one. Therefore, the labor market must have beliefs $\mu(e^P) = \lambda$ in equilibrium, and thus:

$$w\left(e^P\right) = \lambda\theta_H + (1-\lambda)\,\theta_L = \theta^E.$$

Now, let $\hat{e}$ be defined so that:

$$\theta^E - c\left(\hat{e}, \theta_L\right) = \theta_L - c\left(0, \theta_L\right).$$

That is, $\hat{e}$ is chosen so that a low ability worker is just indifferent to acquiring education $\hat{e}$ and being paid θ^E, and acquiring no education and being paid θ_L.

Proposition 1 *For any $e^P \in [0, \hat{e}]$, there is a pooling equilibrium in which all workers choose e^P for certain.*

Proof. Let $e^P \in [0, \hat{e}]$ be given. Suppose that $\mu(e^P) = \lambda$ and that $\mu(e) = 0$ for all $e \neq e^P$, with wages given accordingly. Then $w(e) < w(e^P)$, so clearly no worker wants to deviate to $e > e^P$. Moreover, θ_L workers prefer e^P to any $e < e^P$ by definition of $\hat{e}$. And since θ_L workers prefer e^P to any lower e, so must θ_H workers (by the single crossing property).

There are a few general points to make about the signalling model. First, in the separating equilibrium, education does not increase productivity, but it does reveal information. Thus it is correlated with wages in equilibrium. In the pooling equilibrium, education neither increases productivity nor reveals information. Nevertheless, workers may incur education costs in equilibrium because there is a wage penalty for doing something unexpected (i.e. not becoming educated). Thus pooling pooling equilibria with positive education levels are very inefficient.

A special feature of the model is that education is not productive at all. We could, however, generalize the model to let the worker's productivity be a function of both ability and education: $\theta + e$, for example. Things would work out in a similar manner — with the key idea being that signalling would lead workers to invest more in education than is efficient.

2.3 The Intuitive Criterion

What can be said about the vast multiplicity of equilibria in the signalling model? Cho and Kreps (1987) argue that some equilibria are less appealing than others. We now consider their "intuitive criterion" for eliminating signalling equilibria.

Definition 7 *Let $BR(T, a_1)$ denote the set of player 2's best responses if player 1 has chosen a_1 and player 2's beliefs have support in $T \subset \Theta$:*

$$BR(T, a_1) = \cup_{\mu \in \Delta(T)} \arg \max_{a_2 \in A_2} \sum u_2(a_1, a_2, \theta) \mu(\theta)$$

Definition 8 *A PBE s^* of G* ***fails the intuitive criterion*** *if there exists $a_1 \in A_1$, $\theta' \in \Theta$ and $J \subset \Theta$ such that*

1. $u_1(s^*, \theta) > \max_{a_2 \in BR(\Theta, a_1)} u_1(a_1, a_2, \theta)$ *for all $\theta \in J$*
2. $u_1(s^*, \theta) < \min_{a_2 \in BR(\Theta/J, a_1)} u_2(a_1, a_2, \theta')$

Condition (1) implies that types in J would never try to play a_1 since even if they could convince player 2 that they were a particular type, they would do worse. Condition (2) implies that type θ' definitely does better by playing a_1 rather than the equilibrium so long as she can convince player 2 that her type is not in J.

Proposition 2 *In the job market signalling model, any pooling equilibrium fails the intuitive criterion.*

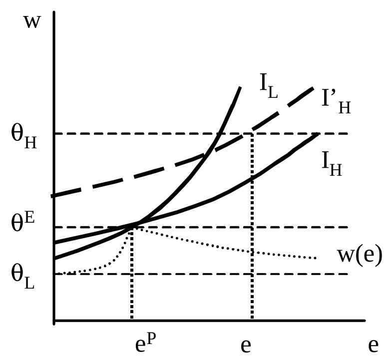
Proof. Consider a pooling equilibrium with some education level e^P. Then we look for a education level e that satisfies:

$$\theta^E - c\left(e^P, \theta_L\right) > \theta_H - c(e, \theta_L)$$

and

$$\theta^E - c\left(e^P, \theta_H\right) < \theta_H - c(e, \theta_H).$$

The Figure below show how to identify e.



Tracing the indifference curves through $\left(w = \theta^E, e^P\right)$, we look for the education level that puts the low ability type on the same indifference curve when $w = \theta_H$. A slight increase to e will then suffice — at $(w = \theta_H, e)$, the low ability type will be on a lower indifference curve than at $\left(\theta^E, e^P\right)$, while the high ability type is on a higher indifference curve. *Q.E.D.*

It is also possible to show that of the separating equilibria derived above, only the most efficient — the separating equilibrium where the high ability cost type gets education $\underline{e}$ survives the intuitive criterion. This is typical of the intuitive criterion — in models with separating equilibria, it selects the most efficient. You are asked to work out another example on the problem set.

3 Cheap Talk

In signalling models, a player uses *costly action* to communicate information about his type. In some situations, however, effective costly signals may not be available. This raises the question of whether a player could credibly communicate information about his type simply through a costless process of communication. It is this possibility that we now explore.

3.1 The Model

The basic model of cheap talk is exactly like the signalling model, only player 1's action is a message that has no direct effect on payoffs.

Stage 0 Nature chooses type $\theta \in \Theta$ of player 1 from a distribution p.

Stage 1 Player 1 observes θ and chooses $m \in M$.

Stage 2 Player 2 observes m and chooses $a \in A$.

Payoffs $u_1(a, \theta)$ and $u_2(a, \theta)$.

We sometimes call player 1 the "sender" and player 2 the "receiver". We are interested in studying perfect bayesian equilibria of this game. The basic question is whether player 1's message will ever convey meaning in equilibrium.

3.2 Basic Observations

1. *Cheap talk doesn't work if different types of senders have the same preferences over actions.*

Example Consider a job market game where instead of going to school players just say "I'm high θ" or "I'm low θ". This game has payoffs:

	a_L	a_M	a_H
$\theta = L$	$1, 1$	$2, 0$	$3, -2$
$\theta = H$	$1, -2$	$2, 0$	$3, 1$

Claim. In this game there are no separating PBE.

Proof. Consider a candidate PBE where $s_1(\theta = L) = m$ and $s_1(\theta = H) = m'$ with $m \neq m'$. By Bayes Rule':

$$\mu_2(L|m) = 1 \qquad \Rightarrow \qquad a(m) = a_L$$

and so:

$$\mu_2(L|m') = 0 \qquad \Rightarrow \qquad a(m') = a_H$$

But then when $\theta = L$, player 1 would like to deviate since

$$u_1(a(m), L) = 1 < 3 = u_2(a(m'), L)$$

so this cannot be a PBE. *Q.E.D.*

2. *Cheap talk doesn't work if different types of senders have completely opposed preferences over actions.*

Example Suppose that Vince McMahon (player 2) is thinking about hiring Mike Tyson (player 1) as a WWF wrestler. In the Mike Tyson game, Mike can be either Normal or Crazy.

$$\Theta = \{\text{Normal, Crazy}\}$$

Mike wants the job if he is normal, but not is he is crazy. On the other hand, Vince wants to hire Mike if and only if Mike is crazy. We write the payoffs for Mike and Vince as follows:

	Hire	Don't Hire
$\theta = N$	$2, -2$	$0, 0$
$\theta = C$	$-2, 2$	$0, 0$

Prior to Vince's decision, Mike holds a press conference to announce his state of mind.

Claim. In this game, there are no separating PBE.

Proof. Suppose that $s_1(N) = m$, $s_2(C) = m'$ with $m \neq m'$. By Bayes' Rule:

$$\begin{aligned} \mu_2(N|m) &= 1 \quad \Rightarrow \quad a(m) = D \\ \mu_2\left(N|m'\right) &= 0 \quad \Rightarrow \quad a\left(m'\right) = H \end{aligned}$$

It follows that *both types* of Tyson want to switch messages:

$$\begin{aligned} u_1(a(m), N) &= 0 < 2 = u_1\left(a\left(m'\right), N\right) \\ u_1(a(m), C) &= 0 > -2 = u_1\left(a\left(m'\right), C\right). \end{aligned}$$

which means this cannot be an equilibrium. *Q.E.D.*

3. *Cheap talk can work great in coordination games.*

Example Consider a version of the "meeting place in New York" game where player 1 is already there and is not allowed to move. Suppose that player 1's "type" is his location in New York.

$$\Theta = \{\text{Grand Central Station, Empire State Building}\}.$$

Player 1 knows θ and calls player 2 on the phone. Player 2 listens and then chooses $\{G, E\}$. Payoffs are given by:

	G	E
$\theta = G$	$1,1$	$0,0$
$\theta = E$	$0,0$	$1,1$

This game has a separating equilibrium:

$$\begin{aligned} s_1(G) &= m & s_1(E) &= m' \\ \mu_2(\theta = G|m) &= 1 & \mu_2\left(\theta = G|m'\right) &= 0 \\ a_2(m) &= G & a_2\left(m'\right) &= E \end{aligned}$$

Remark 3 *Note that this last example also has a pooling or "babbling" equilibrium, where player 1 always says m (or m' or randomizes over all elements of M), and player 2 does not update her beliefs at all.*

3.3 A Richer Model

We now tackle a richer model of cheap talk where the possibilities for communication are more interesting. We assume that θ is uniformly distributed on the interval $[0, 1]$. Let player 2's action be denoted $a \in \mathbb{R}$. We assume that player 2 (the receiver) has payoffs $u_2(a, \theta) = -(a - \theta)^2$, while player 1 (the sender) has payoffs $u_1(a, \theta) = -(a - \theta - c)^2$.

Player 2's optimal action is to choose $a = \theta$, while player 1's is to choose $a = \theta + c$. Importantly, preferences are congruent in the sense that both like to take higher actions when the state is higher. However, player 1 is systematically biased by an amount c.

We investigate different types of perfect bayesian equilibria.

Babbling Equilibrium. Player 1 uses each message $m \in M$ with equal probability regardless of θ. Player 2 assigns a uniform belief to all values $\theta \in [0, 1]$ regardless of the message. She then chooses that action $a = 1/2$.

Clearly player 1 cannot deviate profitably given Player 2's beliefs. Similarly, Player 2's action is optimal given her beliefs. Finally, Player 2's beliefs are consistent with Bayes' Rule.

A Two Message Equilibrium. Let's try to construct an equilibrium where Player 1 uses two messages, m_1 and m_2. Let $a(m)$ denote player 2's action in response to message m. If the equilibrium conveys information, it must be the case that $a(m_1) \neq a(m_2)$. Assume without loss of generality that $a(m_1) < a(m_2)$.

Claim 1 In a two message equilibrium, player 1 uses a threshold strategy, choosing the message m_1 whenever $\theta \in [0, \theta^*)$ and m_2 whenever $\theta \in (\theta^*, 1]$, with θ^* indifferent (and choosing either).

Proof. To prove this note that player 1's payoff are as follows:

$$\begin{aligned} \text{Payoff to } m_1 &: \quad -(a(m_1) - \theta - c)^2 \\ \text{Payoff to } m_2 &: \quad -(a(m_2) - \theta - c)^2 \end{aligned}$$

So the incremental gain to choosing message m_2 versus m_1 is:

$$\Delta(\theta) = -(a(m_2) - \theta - c)^2 + (a(m_1) - \theta - c)^2$$

This is increasing in θ. Thus, it follows that if type θ prefers m_2 to m_1, so will every type $\theta' > \theta$. This gives a threshold characterization with θ^* being the type that is just indifferent. *Q.E.D.*

Claim 2 In equilibrium:

$$a(m_1) = \frac{\theta^*}{2} \qquad\qquad a(m_2) = \frac{1+\theta^*}{2}$$

Proof. This follows from Bayes' Rule and optimization by player 2. In equilibrium, player 2 must set $a(m) = \mathbb{E}[\theta|m]$. If all types between $[0,\theta^*)$ send message m_1, and all types between $(\theta^*,1]$ send message m_2, then she must behave as stated. *Q.E.D.*

We are now ready to solve for the equilibrium. We use the fact that θ^* must be just indifferent between the two messages: $u_1(a(m_1),\theta^*) = u_1(a(m_2),\theta^*)$. This is equivalent to:

$$\theta^* + c - \frac{\theta^*}{2} = \frac{1+\theta^*}{2} - (\theta^* + c)$$

which simplifies to $\theta^* = \frac{1}{2} - 2c$. This equilibrium "works" so long as $\theta^* \in (0,1)$, i.e. so long as $c < 1/4$.

A few comments on the two message equilibrium.

1. Even if there are more than two messages available, this two message equilibrium can still be made into a PBE. For example, we can group all of the messages into two "classes" or message, with types in $[0,\theta^*)$ choosing from the first message class randomly and those in $(\theta^*,1]$ choosing from the second.

2. Note also that the equilibrium is asymmetric — $\theta^* < 1/2$. This means that m_1 is a more informative message than m_2 in the sense that it narrows down the possibilities more.

A Three Message Equilibrium. What about an equilibrium with more communication? Let's try to construct an equilibrium with three messages m_1, m_2, m_3. Using precisely the same arguments, we can show that:

1. The equilibrium must divide types into three segments, with those in $[0,\theta_1)$ choosing m_1, those in (θ_1,θ_2) choosing m_2 and those in $(\theta_2,1]$ choosing m_3, with θ_1,θ_2 indifferent between m_1, m_2 and m_2, m_3 respectively.

2. In equilibrium, player two must use:

$$a(m_1) = \frac{\theta_1}{2} \qquad a(m_2) = \frac{\theta_1+\theta_2}{2} \qquad a(m_3) = \frac{1+\theta_2}{2},$$

i.e. she must set $a(m) = \mathbb{E}[\theta|m]$ for $m = m_1, m_2, m_3$.

3. Finally, we have two indifference conditions. Type θ_1 must be indifferent between m_1 and m_2 :

$$(\theta_1 + c) - \frac{\theta_1}{2} = \frac{\theta_1+\theta_2}{2} - (\theta_1 + c)$$

and type θ_2 must be indifferent between m_2 and m_3 :

$$(\theta_2 + c) - \frac{\theta_1+\theta_2}{2} = \frac{1+\theta_2}{2} - (\theta_2 + c).$$

Solving these two equations implies that $\theta_1 = \frac{1}{3} - 4c$ and $\theta_2 = \frac{2}{3} - 4c$.

This "works" as an equilibrium provided that $\theta_1 > 0$, or in other words if $c < \frac{1}{12}$. So if there is a three message equilibrium, there is also a two message equilibrium and a babbling equilibrium. Note also that the degree of bias must be smaller to have more communication (three messages in equilibrium as opposed to just two).

Arbitrary Message Equilibrium. It is possible to show that there are also 4,5,6,...,T message equilibria for some T which depends on c. In general, however, there is no fully separating (infinite message) equilibrium. The reason is that if messages were fully informative, then type θ would send the report corresponding to type $\theta + c$ and achieve a better outcome (i.e. the action $a = \theta + c$ rather than the action θ). Thus in any equilibrium, there is some loss of information. And moreover, the loss of information is directly related to the bias of the informed party. These results were originally worked out in a paper by Crawford and Sobel (1982).

4 Models of Reputation

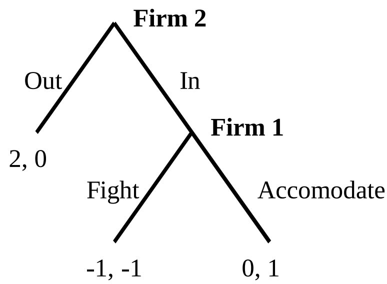
Some of the most striking applications of incomplete information are to models of reputation. We will consider two kinds of reputation models.

- In the first model, we consider a long-run player who plays against a sequence of short-run playerss We find is that if short-run players entertain the possibility that the long-run player is irrational in a particular way, the long-run player may be able to build a reputation that works to his advantage.

- In the second model, we consider two long-run players. We find that again the slight possibility of irrationality changes the equilibrium greatly, for instance, allowing cooperation in the finitely repeated prisoners' dilemma.

4.1 Building a Reputation for Toughness

We start by considering the following entry game.



We consider a situation where this game is played T times. In each case, Player 1 is the same, but he faces a succession of Player 2s. Thus Player 1's objective is to maximize the undiscounted sum of his payoffs for all T periods, while each Player 2 wants to maximize her payoffs in the present period.

As an example of this type of situation, think of Player 1 as Microsoft, fending off a succession of lawsuits accusing it of anti-competitive behavior. There are a succession of small firms who feel they have a case against Microsoft and have to decide whether to launch a court battle or not. Microsoft then decides whether to settle (accomodate) or to fight tooth and nail (fight). The question is whether Microsoft might be able to fight some early lawsuits, and thus "build a reputation" for fighting so that later firms will not challenge it.

Proposition 3 *In the T-period complete information game, there is a unique SPE in which Player 1 accomodates in all periods and all entrants enter.*

Proof. Use backward induction. *Q.E.D.*

This result points to the strength of subgame perfection in complete information games. (Recall the centipede game or the Enron speculation game.)

Even if Microsoft is going to face a large number of challengers, it cannot build a reputation for being a fighter in this setting.

To incorporate reputation in this model, we relax the assumption that Player 1 is *necessarily* rational, and suppose instead that with small probability Player 1 actually enjoys fighting. To do this, suppose that Player 1 has two possible types:

$$\theta_1 \in \{\text{Crazy, Normal}\}$$

Player 1 is crazy with probability q. For a crazy type, we assume that Fight is a dominant strategy — a Crazy type of player 1 will fight in every period that an opponent enters.

Even in the $T = 1$ period game, this aspect of craziness may be enough to deter entry.

Proposition 4 *In the $T = 1$ period game with incomplete information:*

1. *If $q > 1/2$, the unique PBE has player 2 play Out.*
2. *If $q < 1/2$, the unique PBE has player 2 play In.*

Proof. This also can be solved by backward induction. *Q.E.D.*

However, the Normal player 1 behaves normally by accomodating the unique PBE. Things get more interesting when we go to the $T = 2$ period game.

Proposition 5 *In the $T = 2$ period game with incomplete information:*

1. *If $q > 1/2$, the unique PBE has the form:*

   Period 1: Player 2 stays Out
   Player 1 Fights
   Period 2: Player 2 stays Out (unless he has seen Player 1 Accomodate)
   Player 1 Accomodates

2. *If $q < 1/2$, the unique PBE has the form:*

   Period 1: Player 2 stays Out if $q > 1/4$ and Enters if $q < 1/4$.
   Player 1 Fights with Prob $q/(1-q)$ if 2 enters
   Period 2: Player 2 enters (unless he has seen Player 1 Fight in which case he randomizes $\frac{1}{2}In + \frac{1}{2}Out$.
   Player 1 Accomodates

Remark 4 *The strategies for Player 1 are for the case when he is Normal. If he is Crazy, he always Fights.*

We consider the two cases in turn.
Proof of (a): $q > 1/2$.

1. Because a Crazy Player 1 always plays Fight, Player 2 will not enter if she believes the probability that Player 1 is Crazy is greater than 1/2. Consequently, Player 2 certainly will not enter in the first period. Moreoever, if there is no first period entry, Player 2 will continue to believe in the second period that Player 1 is Crazy with probability $q > 1/2$. Consequently, she will not enter in the second period either.

We conclude that Player 2's equilibrium strategy is not to enter in either the first or the second period. However, we still need to describe what happens *off the equilibrium path* — i.e. when Player 2 *does enter* in the first period.

2. If Player 2 enters in the first period and Player 1 responds by Accomodating, this reveals that Player 1 is Normal. So in the second period, Player 2 will find it optimal to enter, and Player 1 will find it optimal to respond by Accomodating.

3. If Player 2 enters in the first period and Player 1 responds by Fighting, then regardless of the Normal Player 1's strategy, Player 2 must believe at the beginning of the second period that Player 1 is crazy with probability at least q. From Bayes' rule:

$$\Pr\left[\text{Crazy}|a_1 = \textit{Fight}\right] = \frac{q}{q + (1-q)\cdot \Pr\left[\text{Normal P1 } \textit{Fights}\right]} \geq q.$$

   It follows that in the second period, Player 2 will find it optimal not to enter, even though in the event of entry a Normal Player 1 would Accomodate.

We conclude that a Normal Player 1 would always Accomodate in the second period, but that following entry in the first period, Player 2 would enter in the second if and only if she saw Player 1 Accomodate in the first. Finally, we need to consider what a Normal Player 1 would do in the first period should Player 2 enter.

4. If Player 2 enters in the first period, then by playing Fight a Normal Player 1 can deter second-period entry. In particular, playing Fight

gives a payoff of $-1+2=1$. On the other hand, Accomodating gives a payoff of 0 today and encourages entry leading to 0 tomorrow. So Player 1 will optimally Fight should Player 2 enter in period 1.

This completely describes the perfect bayesian equilibrium if $q > 1/2$. *Q.E.D.*

Proof of (a): $q < 1/2$.

We start by considering what happens if Player 2 does not enter in the first period.

1. If Player 2 does not enter in the first period, then there is no learning about player 1's type. In the second period, we have a one-shot game with probability $q < 1/2$ that Player 1 is Crazy. Consequently, Player 2 will enter in the second period and a Normal Player 1 will accomodate.

Now consider what happens if Player 2 enters in the first period.

2. It *cannot* be a PBE for Player 1 to fight with probability 1 in the first period. If it was, then Player 2 would not update when she saw Player 1 fight and thus would enter in the second period regardless of whether or nor Player 1 fought or accomodated in the first period. Consequently, a Normal Player 1 would do better to accomodate in the first period.

3. It also *cannot* be a PBE for Player 1 to accomodate with probability 1 in the first period. If it was, then fighting would perfectly signal craziness. After seeing Fight, Player 2 would not enter the second period. But then the payoff to Fighting in the first period for Normal Player 1 would be $-1+2=1$, greater than the payoff 0 to accomodating.

We conclude that the PBE must have Player 1 randomize in the first period. For Player 1 to be willing to randomize, he must be indifferent to playing Fight and Accomodate in the first period. This means that:

$$\begin{aligned} 0 &= -1 + 2 \cdot \Pr\left[2 \text{ Out in period } 2 | a_1 = \textit{Fight}\right] \\ &\quad +0 \cdot \Pr\left[2 \text{ Enters in period } 2 | a_1 = \textit{Fight}\right]. \end{aligned}$$

For this to happen, it must be that:

$$\Pr\left[2 \text{ Out in period } 2 |\ a_1 = \textit{Fight}\right] = \frac{1}{2}.$$

We conclude from this that the PBE must have Player 2 randomize in the second period after seeing Fight in the first.

For Player 2 to be willing to randomize in the second period after seeing Fight in the first, it must be that:

$$\Pr\left[\text{Crazy}|a_1 = \textit{Fight}\right] = \frac{1}{2}.$$

In equilibrium, this belief will be obtained through Bayes' updating:

$$\Pr\left[\text{Crazy}|\ a_1 = \textit{Fight}\right] = \frac{q}{q + (1-q) \cdot \Pr\left[\text{Normal P1 } \textit{Fights} \text{ in Period 1}\right]}.$$

Combining these requirement, we find that the Normal Player 1 must fight entry in period 1 with probability $q/(1-q)$.

At this point, we have determined (a) what Normal Player 1 must do in period 1 if Player 2 enters, and (b) what Player 2 will do after (i) not entering in period 1, (ii) entering in period 1 and seeing Fight, and (iii) entering in Period 1 and seeing Accomodate.

The remaining question is whether Player 2 will enter in the first period. Her payoff to entering is:

$$-\left(q + (1-q) \cdot \frac{q}{1-q}\right) + (1-q) \cdot \frac{1-2q}{1-q} = 1 - 4q.$$

while staying out gives zero We conclude that Player 2 will optimally enter in the first period if $q < 1/4$, and will optimally stay out in the first period if $q > 1/4$. *Q.E.D.*

The general T-period version of this model has also been studied. It turns out that if T is large, then even if q is quite small, the Normal type of Player 1 will be able to mimic the Crazy type and obtain a large period period payoff. Remarkably, as $T \to \infty$, the Normal player's average payoff per-period approaches 2 (which is the highest payoff that player 1 could get by committing to a strategy in advance — i.e. his "Stackelberg" payoff). In this sense, the slight chance that a player is crazy may allow that player to do very well in equilibrium.

4.2 Cooperation in the Finitely Repeated Prisoners' Dilemma

In the above example, the long-run Player 1 was able to build a reputation to his own advantage. Reputation can also play a powerful role in situations where players may be able to *constructively* mislead others about their objectives. To see how this works, we consider Kreps, Milgrom, Roberts, and Wilson's (1982) treatment of the finitely repeated prisoners' dilemma.

The stage game is the standard prisoners' dilemma.

	C	D
C	$1, 1$	$-1, 2$
D	$2, -1$	$0, 0$

We assume the game is repeated T times with no discounting, with both player 1 and 2 being long-run players. Recall that if both players are maximizing their payoffs and there is complete information, then backward induction tells us that (D, D) will be played in every period of the unique subgame perfect equilibrium.

To add reputation, we imagine that player 2 is either "crazy" with probability q or "normal" with probability $1 - q$. A normal player maximizes his expected payoff in the standard way, but a crazy player mechanically plays the Grim Trigger strategy. That is a crazy Player 2 plays C so long as Player 1 has never played D, and plays D forever once Player 1 has played D for the first time.

If this this incomplete information game is played once, both Player 1 and the Normal Player 2 have a dominant strategy, which is to play D. However, if the game is played more than once, the possibility that Player 2 might be crazy gives Player 1 some incentive to play C in the early rounds. More subtle is the fact that it also gives Player 2 an incentive to play C — by doing so Player 2 can potentially convey the impression that he really is crazy.

Proposition 6 *In the $T = 2$ period model, the normal Player 2 always plays D, while Player 1 plays C in the first period if $q > 1/2$.*

Proof. In the second period, both the Normal Player 2 and Player 1 will clearly want to choose D regardless of what happened in the first period and regardless of Player 1's beliefs about Player 2's type. The Crazy Player 2 will play C if Player 1 played C in the first period and D if Player 1 played D.

In the first period, the Normal Player 2 will certainly play D since regardless of what she plays she expects Player 1 to play D in the last peiriod. The Crazy Player 2 will play C. Now consider Player 1. She has payoffs:

$$\begin{aligned} \text{Payoff to } C &: \ q - (1 - q) + q \cdot 2 = 4q - 1 \\ \text{Payoff to } D &: \ 2q \end{aligned}$$

Thus Player 1 optimally plays C in the first period if $q > 1/2$ and optimally plays D if $q < 1/2$. *Q.E.D.*

Proposition 7 *In the $T = 3$ period model, if $q > 1/2$, then both players cooperate in the first round.*

Proof. To analyze the three period game, we start by considering continuation play from date $t = 2$.

- If player 1 played D in the first period, then both players will play D thereafter.
- If player 1 played C in the first period, then let μ denote player 1's belief that player 2 is crazy. Equilibrium play in the last two periods will be just as in the 2-period model. In particular, player 1 will optimally play C in the second period if $\mu > 1/2$ and D in the second period if $\mu < 1/2$, and be indifferent if $\mu = 1/2$.

Now let's back up and try to construct an equilibrium in which both players cooperate in round one. Clearly if this happens, then no information will be revealed as to whether player 2 is truly crazy, so $\mu = q$.

- Supposing that $q > 1/2$, and that player 1 plays C in the first period, what should a normal player 2 do?

$$\begin{aligned} \text{Payoff to } C &: \quad 1 + 2 + 0 \\ \text{Payoff to } D &: \quad 2 + 0 + 0 \end{aligned}$$

  Playing D results in D always being played in the last two periods, while playing C results in player 1 playing C in the next period. Thus a normal player 2 should play C in the first period so as not to reveal that he is normal.

- Supposing that $q > 1/2$, and that player 2 plays C in the first period, what should player 1 do?

$$\begin{aligned} \text{Payoff to } C &: \quad 1 + q \cdot (1 + 2) + (1 - q) \cdot (-1) = 4q \\ \text{Payoff to } D &: \quad 2 + 0 + 0 \end{aligned}$$

So player 1 should also play C in the first period. *Q.E.D.*

Note that the equilibrium involves full cooperation in the first period, partial cooperation in the second period (only player 1 cooperates) and no cooperation in the third round. This is consistent with the way people play the finitely repeated prisoners' dilemma in experiments: some cooperation at the start, but hten tailing off.

A second point is that this is only one equilibrium, and we need parametric assumptions ($q > 1/2$) to get it. So cooperation is possible, but not necessary. The remarkable fact is that if the game is repeated enough, then cooperation is not only possible, but *necessary* in the sense that players cooperate in *almost every period.*

Proposition 8 *In any sequential equilibrium of the T-period game where T is "large," the number of periods where one player or the other plays D is bounded above by a constant that depends on q, but is independent of T.*

Proof (sketch)**.** The first point to note is that if it ever becomes known that player 2 is normal prior to some round t, then both players must select D in every following period. This follows from backward induction. Thus, if player 1 has never played D and player 2 plays D, D will be played in all following periods.

Similarly, if player 1 has never played D and player 2 plays D, then D will be played in all following periods. This happens because either player 2 is crazy, in which case he is triggered and will play D forever, or else he is normal. If he is normal and ever plays C, then he reveals normality and ends up playing (D, D) forever anyway. So player 2 will certainly play D forever following a D by player 1. And if player 2 is playing D forever, so should player 1.

We conclude that once D is played once, it will be played forever by player 1 and both types of player 2 in any sequential equilibrium.

Next, define $M = \frac{3-q}{q}$. We claim that if neither player has yet played D in any round up to and including t where $t < T - M$, then player 1 must select C in round $t + 1$. To see this note that once player 1 plays D, (D, D) will be played forever, so

$$\text{Player 1's payoff to } D: \text{ at most } 2 + 0 + ... + 0 = 2$$

On the other hand, Player 1 has the option of playing C and continuing to play C until either the last period, or until player 2 plays D for the first time. This strategy will give a payoff of *at least:*

$$q \cdot \left(\underbrace{1 + 1 + ... + 1}_{= T - t - 2} + 2 \right) + (1 - q) \cdot (-1) = (M + 1)\, q - 1$$

So playing C is certainly optimal if:

$$(T - t - 1)\, q - 1 \geq 2 \qquad \Leftrightarrow \qquad T - t \geq \frac{3 - q}{q},$$

which is the case since $T - t > M$.

Finally, we claim that if no player has yet played D in any round up to and including t where $t < T - M - 1$, then player 2 must select C in period $t + 1$. To see this, note that the crazy player 2 will certainly select C. If the normal player 2 selects D, then (D, D) will be played forever, so

$$\text{Player 1's payoff to } D: \text{ at most } 2 + 0 + ... + 0 = 2.$$

On the other hand, Player 2 has the option of playing C once and then D in all following rounds. This will give a payoff of *at least*

$$1 + 2 + 0 + ... + 0 = 3.$$

So player 2 certainly will play C in this case.

We conclude that in any period $t < T - M$, neither player will select D. Note that M is independent of T, so as $T \to \infty$, players cooperate for almost every period. *Q.E.D.*

We have not shown that an equilibrium actually exists, but there is a general existence result for sequential equilibrium, which tells us that there is an equilibrium in which (C, C) is played for a significant amount of time in equilibrium. More generally, equilibrium will have the following structure. Players start out by cooperating. As the end draws near, one or both players may start to mix between C and D with the probability that D is played in any given period increasing over time. Once a D has been played, (D, D) is played forever.

1. This pattern of cooperation matches up reasonably well with lab experiments: players start out cooperating and cooperation tails off as the end of the game approaches.

2. Interestingly, our last result did not require specific assumptions on q. Even if q is small, for large T there will be a lot of cooperation.

3. We relied heavily on a particular form of craziness to get this result. It turns out that if one allows for all conceivable forms of craziness, and if both players may be crazy, there is a folk theorem type result — anything can happen.

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