Appendix | F

Games of Incomplete Information

Auction theory models the decision problems collectively facing bidders in an auction as a game of incomplete information. This appendix contains definitions of some conceptual tools that are used throughout the book. It is intended only as a sketch of the relevant material. The reader should consult one of the references mentioned in the notes at the end of this appendix for a more complete treatment.

A *game* G consists of (1) a set $\mathcal{N}$ of players; (2) for each player $i \in \mathcal{N}$ a nonempty set $\mathcal{A}_i$ of actions; and (3) for each player $i \in \mathcal{N}$ a payoff function $u_i : \times_j \mathcal{A}_j \to \mathbb{R}$. A *Nash equilibrium* of a game G is a vector $\mathbf{a}^* \in \times_j \mathcal{A}_j$ of actions such that for all i and $a_i \in \mathcal{A}_i$,

$$u_i\left(\mathbf{a}^*\right) \geq u_i\left(a_i, \mathbf{a}^*_{-i}\right)$$

In a game of incomplete information, a player's payoff depends not only on the actions of other players but also on information that is only partly known to the player. Because of this each player evaluates strategies on the basis of expected payoffs conditional on the information available to the player.

Formally, a *game of incomplete information* Γ consists of (i) a set $\mathcal{N}$ of players; (ii) for each player $i \in \mathcal{N}$ a nonempty set $\mathcal{A}_i$ of actions; (iii) for each player $i \in \mathcal{N}$ a set of signals $\mathcal{X}_i$; (iv) for each player $i \in \mathcal{N}$ a payoff function $u_i : \times_j \mathcal{A}_j \times_j \mathcal{X}_j \to \mathbb{R}$; (v) a probability distribution f over the product set of signals $\times_j \mathcal{X}_j$. A (pure) *strategy* for player i is a function $\alpha_i : \mathcal{X}_i \to \mathcal{A}_i$ mapping signals into actions.

This formulation postulates the following timing of events. First, the signals $\mathbf{X}$ are drawn according to f and player i is told the realization $X_i = x_i$ of his signal. Second, armed with the knowledge that $X_i = x_i$ each player chooses an action a_i. Finally, based on the signals $\mathbf{x}$ of all the players and their actions $\mathbf{a}$, payoffs are realized.

In the context of auctions the set of players is, of course, the set of bidders. An action corresponds to a bid. The signals encode the information available to bidders prior to the auction. A bidding strategy maps this information into bids.

DOMINANT STRATEGIES

A strategy α_i is said to (weakly) *dominate* α_i' if for all $\mathbf{x} \in \mathcal{X}$ and all $\mathbf{a}_{-i}$,

$$u_i\left(\alpha_i(x_i), \mathbf{a}_{-i}, \mathbf{x}\right) \geq u_i\left(\alpha_i'(x_i), \mathbf{a}_{-i}, \mathbf{x}\right)$$

with a strict inequality for some $\mathbf{x}$ and $\mathbf{a}_{-i}$.

The strategy α_i is *dominant* if it (weakly) dominates every other strategy α_i'. If every player has a dominant strategy α_i^*, then we will refer to α^* as a *dominant strategy equilibrium.*

A strategy α_i is *undominated* if there does not exist another strategy α_i' that dominates it.

BAYESIAN-NASH EQUILIBRIA

A (pure strategy) *Bayesian-Nash equilibrium* of a game of incomplete information Γ is a vector of strategies $\boldsymbol{\alpha}^*$ such that for all i, for all $x_i \in \mathcal{X}_i$ and for all $a_i \in \mathcal{A}_i$,

$$E\left[u_i\left(\boldsymbol{\alpha}^*(\mathbf{X}), \mathbf{X}\right) \mid X_i = x_i\right] \geq E\left[u_i\left(a_i, \boldsymbol{\alpha}_{-i}^*(\mathbf{X}_{-i}), \mathbf{X}\right) \mid X_i = x_i\right], \qquad \text{(F.1)}$$

where $\boldsymbol{\alpha}^*(\mathbf{x}) = \left(\alpha_i^*(x_i)\right)_{i \in \mathcal{N}}$ denotes the vector of actions of all players and $\boldsymbol{\alpha}_{-i}^*(\mathbf{x}_{-i}) = \left(\alpha_j^*(x_j)\right)_{j \neq i}$ denotes the vector of actions of the other players. Suppose α_i is some alternative strategy. Then for all i and x_i (F.1) holds for $a_i = \alpha_i(x_i)$. Taking the expectation of both sides with respect to x_i implies that if $\boldsymbol{\alpha}^*$ is a Bayesian-Nash equilibrium, then for all i and all strategies α_i,

$$E\left[u_i\left(\boldsymbol{\alpha}^*(\mathbf{X}), \mathbf{X}\right)\right] \geq E\left[u_i\left(\alpha_i(X_i), \boldsymbol{\alpha}_{-i}^*(\mathbf{X}_{-i}), \mathbf{X}\right)\right] \qquad \text{(F.2)}$$

The inequality in (F.1) says that for all i, the strategy α_i^* is optimal against $\boldsymbol{\alpha}_{-i}^*$ when evaluated at the *interim* stage—that is, when players know their own signals, whereas the inequality in (F.2) says that α_i^* is optimal against $\boldsymbol{\alpha}_{-i}^*$ when evaluated at the *ex ante* stage—that is, before players know their own signals. As we have argued, interim optimality implies *ex ante* optimality. Conversely, *ex ante* optimality implies that interim optimality holds almost surely—that is, if (F.2) holds, then (F.1) can fail only for x_i in a set whose probability is zero.

In this book the term *equilibrium* of a game of incomplete information is always taken to mean "Bayesian-Nash equilibrium."

EX POST EQUILIBRIA

An *ex post equilibrium* is a Bayesian-Nash equilibrium $\boldsymbol{\alpha}^*$ with the property that for all i, for all $\mathbf{x} \in \mathcal{X}$ and all a_i,

$$u_i\left(\boldsymbol{\alpha}^*(\mathbf{x}), \mathbf{x}\right) \geq u_i\left(a_i, \boldsymbol{\alpha}^*_{-i}(\mathbf{x}_{-i}), \mathbf{x}\right)$$

In other words, an *ex post* equilibrium $\boldsymbol{\alpha}^*$ is a Bayesian-Nash equilibrium with the additional requirement that even if all players' signals $\mathbf{x}$ were known to a particular bidder i, it would still be optimal for him to choose $\alpha_i^*(x_i)$, that is, i would not suffer from any regret. Put another way, for all $\mathbf{x}$, the actions $\left(\alpha_i^*(x_i)\right)_{i\in\mathcal{N}}$ constitute a Nash equilibrium of the game $G(\mathbf{x})$ in which each player $i \in \mathcal{N}$ chooses an action a_i from $\mathcal{A}_i$ and has the payoff function $u_i(\cdot, \mathbf{x})$. Finally, an *ex post* equilibrium $\boldsymbol{\alpha}^*$ is robust in the sense that it is independent of the probability distribution f of signals. Precisely, if $\boldsymbol{\alpha}^*$ is an *ex post* equilibrium when the signals are distributed according to the distribution f, then it is also an *ex post* equilibrium when the signals are distributed according to some $g \neq f$.

Every dominant strategy equilibrium is an *ex post* equilibrium, and by definition, every *ex post* equilibrium is a Bayesian-Nash equilibrium. Thus, we have

$$\begin{array}{ccccc} \text{Set of Dominant} & & \text{Set of } \textit{Ex Post} & & \text{Set of Bayesian-} \\ & \subseteq & & \subseteq & \\ \text{Strategy Equilibria} & & \text{Equilibria} & & \text{Nash Equilibria} \end{array}$$

A Bayesian-Nash equilibrium $\boldsymbol{\alpha}^*$ is said to be an *undominated equilibrium* if for all i, α_i^* is undominated.

NOTES ON APPENDIX F

Strategic situations in which players are unsure about some aspect of the game—the set of available strategies, the payoffs, or what other players believe about them—are called games of incomplete information. The conceptual underpinnings for such games were developed by Harsanyi (1967/1968) who argued first that all uncertainty faced by a player can be summarized as a single variable, called his "type," and, second, that the prior distribution over the vector of types is common to all the players. According to Harsanyi (1967/1968), all differences in what players know and believe about each other should stem from differences in their private information—their types—alone and not from differences in their initial beliefs. Harsanyi's conception thus allows a game of incomplete information to be reformulated as a game of imperfect information—one in which only players' information differs—and these can be analyzed along conventional lines. The term *Bayesian-Nash* equilibrium of a game of incomplete information then refers to a *Nash* equilibrium of the resulting imperfect information game.

As mentioned in the preface, Vickrey's model (Vickrey, 1961) of auctions as games was already along these lines. In particular, Vickrey (1961) implicitly assumed that the joint distribution of values was commonly known to all bidders.

More detailed discussions of these issues may be found in the game theory textbooks by Fudenberg and Tirole (1991) and Osborne and Rubinstein (1994).

The notion of dominance used in this book involves *ex post* payoffs—a strategy dominates another only if it is never worse against any vector of actions of other players in any circumstances and sometimes strictly better. An alternative weaker notion involves *interim* payoffs—a strategy $\alpha_i(\cdot)$ dominates $\alpha_i'(\cdot)$ in the interim sense if, when evaluated according to expected payoffs conditional on a player's own signal, $\alpha_i(\cdot)$ is never worse than $\alpha_i'(\cdot)$ against any $\boldsymbol{\alpha}_{-i}$ and strictly better against some $\boldsymbol{\alpha}_{-i}$.

The notion of *ex post* equilibrium has been used in many contexts by different authors under different rubrics. It is the same as the notion of "uniform incentive compatibility" introduced by Holmström and Myerson (1983). In the auction context, its use appears to originate in the work of Crémer and McLean (1985). Maskin (1992) refers to an *ex post* equilibrium as a "robust" Bayesian-Nash equilibrium.