Chapter | Three

The Revenue Equivalence Principle

In the previous chapter we saw that regardless of the distribution of values, the expected selling price in a symmetric first-price auction is the same as that in a second-price auction. As a result a risk-neutral seller is indifferent between the two formats. The fact that the expected selling prices in the two auctions are equal is quite remarkable. The two auctions are not strategically equivalent as defined in Chapter 1, and in particular instances, the price in one or the other auction may be higher. This chapter explores the reasons underlying the equality of expected revenues in Proposition 2.3. In the process, we will discover that this equality extends beyond first- and second-price auctions to a whole class of auction forms.

3.1 MAIN RESULT

The auction forms we consider all have the feature that buyers are asked to submit *bids*—amounts of money they are willing to pay. These bids alone determine who wins the object and how much the winner pays. We will say that an auction is *standard* if the rules of the auction dictate that the person who bids the highest amount is awarded the object. Both first- and second-price auctions are, of course, standard in this sense, but so, for instance, is a *third-price auction*, discussed later in this chapter, in which the winner is the person bidding the highest amount but pays the third-highest bid. An example of a nonstandard method is a *lottery* in which the chances that a particular bidder wins is the ratio of his or her bid to the total amount bid by all. Such a lottery is nonstandard, since the person who bids the most is not necessarily the one who is awarded the object.

Given a standard auction form, A, and a symmetric equilibrium β^A of the auction, let $m^A(x)$ be the equilibrium *expected payment* by a bidder with value x. It turns out, quite remarkably, that provided that the expected payment of a bidder with value 0 is 0, the expected payment function $m^A(\cdot)$ does not depend on the particular auction form A. As a result, the expected revenue in any standard auction is the same, a fact known as the revenue equivalence principle.

Proposition 3.1. *Suppose that values are independently and identically distributed and all bidders are risk neutral. Then any symmetric and increasing equilibrium of any standard auction, such that the expected payment of a bidder with value zero is zero, yields the same expected revenue to the seller.*

Proof. Consider a standard auction form, A, and fix a symmetric equilibrium β of A. Let $m^A(x)$ be the equilibrium expected payment in auction A by a bidder with value x. Suppose that β is such that $m^A(0) = 0$.

Consider a particular bidder—say, 1—and suppose other bidders are following the equilibrium strategy β. It is useful to abstract away from the details of the auction and consider the expected payoff of bidder 1 with value x and when he bids $\beta(z)$ instead of the equilibrium bid $\beta(x)$. Bidder 1 wins when his bid $\beta(z)$ exceeds the highest competing bid $\beta(Y_1)$, or equivalently, when $z > Y_1$. His expected payoff is

$$\Pi^A(z,x) = G(z)x - m^A(z)$$

where as before $G(z) \equiv F(z)^{N-1}$ is the distribution of Y_1. The key point is that $m^A(z)$ depends on the other players' strategy β and z but is independent of the true value, x.

Maximization results in the first-order condition,

$$\frac{\partial}{\partial z}\Pi^A(z,x) = g(z)x - \frac{d}{dz}m^A(z) = 0$$

At an equilibrium it is optimal to report $z = x$, so we obtain that for all y,

$$\frac{d}{dy}m^A(y) = g(y)y \tag{3.1}$$

Thus,

$$\begin{aligned} m^A(x) &= m^A(0) + \int_0^x y g(y)\,dy \\ &= \int_0^x y g(y)\,dy \\ &= G(x) \times E[Y_1 \mid Y_1 < x] \end{aligned} \tag{3.2}$$

since, by assumption, $m^A(0) = 0$. Since the right-hand side does not depend on the particular auction form A, this completes the proof. ■

For the specification in Example 2.1, the expected payment function can be easily calculated.

Example 3.1. *Values are uniformly distributed on* $[0,1]$.

If $F(x)=x$, then $G(x)=x^{N-1}$ and for any standard auction satisfying $m^A(0)=0$, (3.2) implies that

$$m^A(x)=\frac{N-1}{N}x^N$$

and

$$E\left[m^A(X)\right]=\frac{N-1}{N(N+1)}$$

while the expected revenue is

$$E\left[R^A\right]=N\times E\left[m^A(X)\right]=\frac{N-1}{N+1}$$

▲

3.2 SOME APPLICATIONS OF THE REVENUE EQUIVALENCE PRINCIPLE

The revenue equivalence principle is a powerful and useful tool. In this section we show how, with judicious use, it can be used to derive equilibrium bidding strategies in alternative, unusual auction forms. We then show how it can be extended and applied to situations in which bidders are unsure as to how many other, rival bidders they face.

3.2.1 Unusual Auctions

We consider two unusual formats: an all-pay auction and a third-price auction. Although neither is used as a real-world auction to sell objects, the former is a useful model of other auction-like contests (some examples are offered next), while the latter is a useful theoretical construct.

EQUILIBRIUM OF ALL-PAY AUCTIONS

Consider an *all-pay* auction with the following rules. Each bidder submits a bid, and, as in the standard auctions discussed earlier, the highest bidder wins the object. The unusual aspect of an all-pay auction is that all bidders pay what they bid. The all-pay auction is a useful model of lobbying activity. In such models, different interest groups spend money—their "bids"—in order to influence government policy and the group spending the most—the highest "bidder"—is able to tilt policy in its favored direction, thereby "winning the auction." Since money spent on lobbying is a sunk cost borne by all groups regardless of which group is successful in obtaining its preferred policy, such situations have a natural all-pay aspect. We are interested in symmetric equilibrium strategies in an all-pay auction with symmetric, independent private values.

Suppose for the moment that there is a symmetric, increasing equilibrium of the all-pay auction such that the expected payment of a bidder with value 0 is 0. In other words, the assumptions of Proposition 3.1 are satisfied. Then we know that the expected payment in such an equilibrium must be the same as in (3.2). Now in an all-pay auction, the expected payment of a bidder with value x is the *same* as his bid—he forfeits his bid regardless of whether he wins or not—and so if there is a symmetric, increasing equilibrium of the all-pay auction β^{AP}, it must be that

$$\begin{aligned}\beta^{\mathrm{AP}}(x) &= m^{A}(x) \\ &= \int_0^x y g(y)\, dy\end{aligned}$$

To verify that this indeed constitutes an equilibrium of the all-pay auction, suppose that all bidders except one are following the strategy $\beta \equiv \beta^{\mathrm{AP}}$. If he bids an amount $\beta(z)$, the expected payoff of a bidder with value x is

$$G(z)x - \beta(z) = G(z)x - \int_0^z y g(y)\, dy$$

and integrating the second term by parts, this becomes

$$G(z)(x - z) + \int_0^z G(y)\, dy$$

which is the same as the payoff obtained in a first-price auction by bidding $\beta^{\mathrm{I}}(z)$ against other bidders who are following β^{I}. For the same reasons as in Proposition 2.2, this is maximized by choosing $z = x$. Thus, β^{AP} is a symmetric equilibrium.

EQUILIBRIUM OF THIRD-PRICE AUCTIONS

Suppose that there are at least three bidders. Consider a sealed-bid auction in which the highest bidder wins the object but pays a price equal to the third-highest bid. A *third*-price auction, as it is called, is a purely theoretical construct: There is no known instance of such a mechanism actually being used. It is an interesting construct nevertheless; equilibria of such an auction display some unusual properties, and it leads to a better understanding of the workings of the standard auction forms. Here we show how the revenue equivalence principle can once again be used to derive equilibrium bidding strategies.

Again, suppose for the moment that there is a symmetric, increasing equilibrium of the third-price auction—say, β^{III}—such that the expected payment of a bidder with value 0 is 0. Once again, since the assumptions of Proposition 3.1 are satisfied, we must have that for all x, the expected payment in a third-price auction is

$$m^{\mathrm{III}}(x) = \int_0^x y g(y)\, dy \tag{3.3}$$

On the other hand, consider bidder 1, and suppose that he wins in equilibrium when his value is x. Winning implies, of course, that his value x exceeds the highest of the other $N-1$ values—that is, $Y_1 < x$. The price bidder 1 pays is the random variable $\beta^{\mathrm{III}}(Y_2)$, where Y_2 is the second highest of the $N-1$ other values. The density of Y_2, conditional on the event that $Y_1 < x$, can be written as

$$f_2^{(N-1)}(y \mid Y_1 < x) = \frac{1}{F_1^{(N-1)}(x)}(N-1)(F(x)-F(y))f_1^{(N-2)}(y),$$

where $(N-1)(F(x)-F(y))$ is the probability that Y_1 exceeds $Y_2 = y$ but is less than x and $f_1^{(N-2)}(y)$ is the density of the highest of $N-2$ values. The expected payment in a third-price auction can then be written as

$$\begin{aligned} m^{\mathrm{III}}(x) &= F_1^{(N-1)}(x)E\left[\beta^{\mathrm{III}}(Y_2) \mid Y_1 < x\right] \\ &= \int_0^x \beta^{\mathrm{III}}(y)(N-1)(F(x)-F(y))f_1^{(N-2)}(y)dy \end{aligned} \tag{3.4}$$

Equating (3.3) and (3.4), we obtain that

$$\int_0^x \beta^{\mathrm{III}}(y)(N-1)(F(x)-F(y))f_1^{(N-2)}(y)dy = \int_0^x yg(y)\,dy$$

and differentiating with respect to x, this implies that

$$\begin{aligned} (N-1)f(x)\int_0^x \beta^{\mathrm{III}}(y)f_1^{(N-2)}(y)dy &= xg(x) \\ &= x \times (N-1)f(x)F(x)^{N-2} \end{aligned}$$

since $G(x) \equiv F(x)^{N-1}$. This can be rewritten as

$$\int_0^x \beta^{\mathrm{III}}(y)f_1^{(N-2)}(y)dy = xF_1^{(N-2)}(x)$$

since, $F_1^{(N-2)}(x) \equiv F(x)^{N-2}$. Differentiating once more with respect to x,

$$\beta^{\mathrm{III}}(x)f_1^{(N-2)}(x) = xf_1^{(N-2)}(x) + F_1^{(N-2)}(x)$$

and rearranging this we get

$$\begin{aligned} \beta^{\mathrm{III}}(x) &= x + \frac{F_1^{(N-2)}(x)}{f_1^{(N-2)}(x)} \\ &= x + \frac{F(x)}{(N-2)f(x)} \end{aligned}$$

This derivation, however, is valid only if β^{III} is increasing, and from the preceding equation it is clear that a sufficient condition for this is that the ratio F/f is increasing. This condition is the same as requiring that $\ln F$ is a concave function or equivalently that F is *log-concave*.

Proposition 3.2. *Suppose that there are at least three bidders and F is log-concave. Symmetric equilibrium strategies in a third-price auction are given by*

$$\beta^{\text{III}}(x) = x + \frac{F(x)}{(N-2)f(x)} \tag{3.5}$$

An important feature of the equilibrium in a third-price auction is worth noting: The equilibrium bid *exceeds* the value. To better understand this phenomenon, first notice that for much the same reason as in a second-price auction, it is dominated for a bidder to bid below his value in a third-price auction. Unlike in a second-price auction, however, it is not dominated for a bidder to bid above his value. Fix some equilibrium bidding strategies of the third-price auction—say, β—and suppose that all bidders except 1 follow β. Suppose bidder 1 with value x bids an amount $b > x$. If $\beta(Y_2) < x < \beta(Y_1) < b$, this is better than bidding x, since it results in a profit, whereas bidding x would not. If, however, $x < \beta(Y_2) < \beta(Y_1) < b$, then bidding b results in a loss. When $b - x \equiv \varepsilon$ is small, the gain in the first case is of order ε^2, whereas the loss in the second case is of order ε^3. Thus, it is optimal to bid higher than one's value in a third-price auction.

Comparing equilibrium bids in first-, second-, and third-price auctions in case of symmetric private values, we have seen that

$$\beta^{\text{I}}(x) < \beta^{\text{II}}(x) = x < \beta^{\text{III}}(x)$$

(assuming, of course, that the distribution of values is log-concave).

3.2.2 Uncertain Number of Bidders

In our analysis so far, each bidder knows his or her own value but is uncertain about the values of others. All other aspects of the situation—the number of bidders, the distribution from which they draw their values—are assumed to be common knowledge. In many auctions—particularly in those of the sealed-bid variety—a bidder may be uncertain about how many other interested bidders there are. In this section we show how the standard model may be amended to include this additional uncertainty.

Let $\mathcal{N} = \{1, 2, \ldots, N\}$ denote the set of *potential* bidders and let $\mathcal{A} \subseteq \mathcal{N}$ be the set of *actual* bidders—that is, those that participate in the auction. All potential bidders draw their values independently from the same distribution F.

Consider an actual bidder $i \in \mathcal{A}$ and let p_n denote the probability that any participating bidder assigns to the event that he is facing n other bidders. Thus, bidder i assigns the probability p_n that the number of actual bidders is $n+1$. The exact process by which the set of actual bidders is determined from the set of potential bidders is not important. What is important is that the process be symmetric so every actual bidder holds the *same* beliefs about how many other bidders he faces; the probabilities p_n do not depend on the identity of the bidder nor on his value. It is also important that the set of actual bidders does not depend on the realized values.

As long as bidders hold the same beliefs about the likelihood of meeting different numbers of rivals, the conclusion of Proposition 3.1 obtains in a straightforward manner. Consider a standard auction A and a symmetric and increasing equilibrium β of the auction. Note that since bidders are unsure about how many rivals they face, β does not depend on n. Consider the expected payoff of a bidder with value x who bids $\beta(z)$ instead of the equilibrium bid $\beta(x)$. The probability that he faces n other bidders is p_n. In that case, he wins if $Y_1^{(n)}$, the highest of n values drawn from F, is less than z and the probability of this event is $G^{(n)}(z) = F(z)^n$. The overall probability that he will win when he bids $\beta(z)$ is therefore

$$G(z) = \sum_{n=0}^{N-1} p_n G^{(n)}(z)$$

His expected payoff from bidding $\beta(z)$ when his value is x is then

$$\Pi^A(z,x) = G(z)x - m^A(z)$$

and the remainder of the argument is the same as in Proposition 3.1. Thus, we conclude that the revenue equivalence principle holds even if there is uncertainty about the number of bidders.

Suppose that the object is sold using a second-price auction. Even though the number of rival buyers that a particular bidder faces is uncertain, it is still a dominant strategy for him to bid his value. The expected payment in a second-price auction of an actual bidder with value x is therefore

$$m^{\mathrm{II}}(x) = \sum_{n=0}^{N-1} p_n G^{(n)}(x)\, E\left[Y_1^{(n)} \mid Y_1^{(n)} < x\right]$$

Now suppose that the object is sold using a first-price auction and that β is a symmetric and increasing equilibrium. The expected payment of an actual bidder with value x is

$$m^{\mathrm{I}}(x) = G(x)\,\beta(x)$$

where $G(x)$ is as defined earlier. The revenue equivalence principle implies that for all x, $m^{\mathrm{I}}(x) = m^{\mathrm{II}}(x)$, so

$$\begin{aligned}\beta(x) &= \sum_{n=0}^{N-1} \frac{p_n G^{(n)}(x)}{G(x)} E\left[Y_1^{(n)} \mid Y_1^{(n)} < x\right] \\ &= \sum_{n=0}^{N-1} \frac{p_n G^{(n)}(x)}{G(x)} \beta^{(n)}(x)\end{aligned}$$

where $\beta^{(n)}$ is the equilibrium bidding strategy in a first-price auction in which there are exactly $n+1$ bidders for sure (see Proposition 2.2 on page 15). Thus, the equilibrium bid for an actual bidder with value x when he is unsure about the number of rivals he faces is a weighted average of the equilibrium bids in auctions when the number of bidders is known to all.

PROBLEMS

3.1. (War of attrition) Consider a two-bidder *war of attrition* in which the bidder with the highest bid wins the object but both bidders pay the losing bid. Bidders' values independently and identically distributed according to F.

- **a.** Use the revenue equivalence principle to derive a symmetric equilibrium bidding strategy in the war of attrition.
- **b.** Directly compute the symmetric equilibrium bidding strategy and the seller's revenue when the bidders' values are uniformly distributed on $[0,1]$.

3.2. (Losers-pay auction) Consider a N-bidder *losers-pay auction* in which the bidder with the highest bid wins the object and pays nothing, while all losing bidders pays their own bids. Bidders' valuations independently and identically distributed according to F.

- **a.** Use the revenue equivalence principle to derive a symmetric equilibrium bidding strategy in the losers-pay auction.
- **b.** Directly compute the symmetric equilibrium bidding strategy for the case when the bidders' values are distributed according to $F(x) = 1 - e^{-ax}$ over $[0, \infty)$.

CHAPTER NOTES

The revenue equivalence principle was established by Riley and Samuelson (1981) and Myerson (1981), showing, in effect, that the phenomenon noticed by Vickrey (1961, 1962) was quite general.

For a model of interest group lobbying as an all-pay auction, albeit in a complete information setting, see Baye, Kovenock, and de Vries (1993). Third-price auctions were first analyzed by Kagel and Levin (1993) who pointed out the over-bidding phenomenon . The explicit derivation of equilibrium strategies is due to Wolfstetter (2001).

Auctions with an uncertain number of bidders have been considered by McAfee and McMillan (1987b), Matthews (1987), and Harstad, Kagel, and Levin (1990). The first two papers are particularly interested in how risk-averse bidders—considered in the next chapter—are affected by uncertainty regarding the number of competitors they face. Harstad, Kagel, and Levin (1990) derive equilibrium bidding strategies in different auctions under number uncertainty when bidders are risk neutral.