1 Cournot Oligopoly with n firms

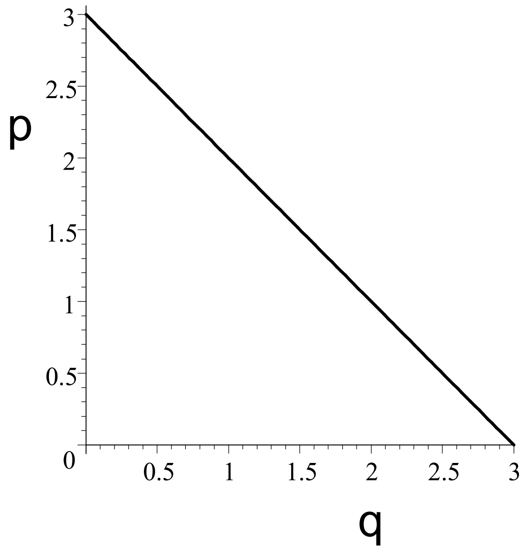
firm i's output: q_i
total output: $q = q_1 + q_2 + \cdots + q_n$
opponent's output: $q_{-i} = q - q_i = \Sigma_{j \neq i} q_i$
constant marginal costs of firm i: c_i
inverse demand function: $p(q)$
firm i's profit:

$$\Pi_i (q_{-i}, q_i) = p(q) \times q_i - c_i \times q_i = (p(q_{-i} + q_i) - c_i) q_i$$

FOC for profit maximum given q_{-i}:

$$\frac{\partial \Pi_i}{\partial q_i} = \frac{\partial p}{\partial q_i} \times q_i + p - c_i = 0$$

Solution defines *reaction curve* $q_i = r_i (q_{-i})$ which is often *decreasing* in q_{-i}.
Linear case: $p = A - Bq = A - B(q_{-i} + q_i)$



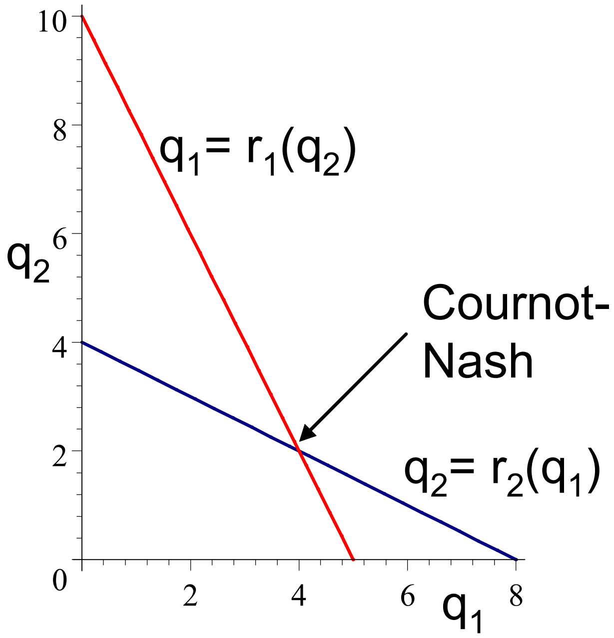
$$\frac{\partial p}{\partial q_i} = -B$$

FOC:

$$\begin{aligned} -Bq_i + (A - B(q_{-i} + q_i)) - c_i &= 0 \\ 2Bq_i &= A - c_i - Bq_{-i} \end{aligned}$$

Reactionfunction

$$q_i = r_i (q_{-i}) = \frac{A - c}{2B} - \frac{1}{2} q_{-i}$$



Cournot-Nash equilibrium:

1. Every firm maximizes profit given her expectation of q_{-i}.
2. The expectation is correct.

This yields the simultaneous system of equations

$$q_i = r_i\left(q_{-i}\right)$$

for all $i = 1, \ldots, n$. In the linear case the FOC yields, since $q_i + q_{-i} = q$

$$\begin{aligned} -Bq_1 + (A - Bq) - c_1 &= 0 \\ -Bq_2 + (A - Bq) - c_2 &= 0 \\ &\vdots \\ -Bq_n + (A - Bq) - c_n &= 0 \end{aligned}$$

Summation yields

$$-Bq + n\left(A - Bq\right) - n\bar{c} = 0$$

where

$$\bar{c} = \frac{c_1 + c_2 + \cdots + c_n}{n}$$

is the average marginal cost in the market.
Thus we can deduce the total quantity produced and the price in the market

$$\begin{aligned} (n+1)\,Bq &= n\left(A - \bar{c}\right) \\ q &= \frac{n}{n+1}\frac{A - \bar{c}}{B} \\ p &= A - Bq = \frac{1}{n+1}A + \frac{n}{n+1}\bar{c} \to \bar{c} \text{ for } n \to \infty \end{aligned}$$

Each firm produces in the n-firm oligopoly

$$q_i^n = \frac{A - Bq - c_i}{B} = \frac{A - c_i}{B} - \frac{n}{n+1}\frac{A - \bar{c}}{B} = \frac{1}{n+1}\frac{A}{B} + \frac{n(\bar{c} - c_i) - c_i}{(n+1)B}.$$

Let us now, for simplicity, assume that firms have identical marginal costs $c_i = \bar{c} = c$. Then

$$\begin{aligned}
p &= \frac{1}{n+1}A + \frac{n}{n+1}c \to c \text{ as } n \to \infty \\
q_i^n &= \frac{1}{n+1}\frac{A-c}{B} \to 0 \text{ as } n \to \infty \\
\Pi_i^n &= (p - c)\, q_i^n = \left(\frac{1}{n+1}A + \frac{n}{n+1}c - c\right)\frac{1}{n+1}\frac{A-c}{B} = \frac{1}{(n+1)^2}\frac{(A-c)^2}{B} \\
n\Pi_i^n &= \frac{n}{(n+1)^2}\frac{(A-c)^2}{B} \to 0 \text{ as } n \to \infty
\end{aligned}$$

The total profit in the industry decreases with every additional firm entering the market since for all $n > 1$

$$\begin{aligned}
& (n-1)\,\Pi_i^{n-1} > n\Pi_i^n \\
\iff & \frac{n-1}{(n)^2} > \frac{n}{(n+1)^2} \\
\iff & (n-1)(n+1)^2 > n^3 \\
\iff & \left(n^2 - 1\right)(n+1) > n^3 \\
\iff & n^3 - n + n^2 - 1 > n^3 \\
\iff & n^2 > n - 1
\end{aligned}$$

which is true since $n^2 > n$ for all $n > 1$.

In particular, it always pays for the firms to form a cartel and share the monopolist profit since $n\Pi_i^n < \Pi_i^1$.

2 Stackelberg Equilibrium

Two firms with marginal costs 1. Different timing: Firm 1 moves first, firm 2 **observes** the move and then adapts.
If a rational firm 2 observes the quantity q_1 it will choose the quantity

$$q_2 = r_2(q_1) = \frac{A-c}{2B} - \frac{1}{2}q_1$$

Total output is

$$q_1 + q_2 = \frac{A-c}{2B} + \frac{1}{2}q_1$$

and the price will be

$$p = A - B(q_1 + q_2) = A - \frac{A-c}{2} - \frac{B}{2}q_1 = \frac{A + c - Bq_1}{2}$$

Anticipating this, firm 1 expects to make the profit

$$\Pi_1(q_1, r_1(q_2)) = \left(\frac{A + c - Bq_1}{2} - c\right) \times q_1 = \frac{A - c - Bq_1}{2} \times q_1$$

which is maximized for

$$q_1 = \frac{A-c}{2B}$$

yielding the price

$$p = \frac{A + c - B\frac{A-c}{2B}}{2} = \frac{A-c}{4}$$

and the profit

$$\Pi_1 = \frac{1}{8}\frac{(A-c)^2}{B}$$

Firm 2 produces

$$q_2 = \frac{A-c}{2B} - \frac{1}{2}q_1 = \frac{A-c}{4B}$$

and makes the profit

$$\Pi_2 = \frac{1}{2}\Pi_1 = \frac{1}{16}\frac{(A-c)^2}{B}$$

Notice that this would not be a Nash equilibrium if firm 2 could not observe the quantity choice because firm 2 reacts optimally while firm 1 should produce

$$q_1 = r_1(q_2) = \frac{A-c}{2B} - \frac{1}{2}q_2 = \frac{A-c}{2B} - \frac{A-c}{8B} = \frac{3}{8}\frac{A-c}{B}$$

Total quantity would be $\frac{5}{8}\frac{A-c}{B}$ and the the price would reduce to

$$p = A - \frac{5}{8}(A-c) = \frac{3A+5c}{8}$$

and yield the profit

$$\Pi_1 = \left(\frac{3A+5c}{8} - c\right)\left(\frac{3}{8}\frac{A-c}{4B}\right) = \frac{9}{8^2}\frac{(A-c)^2}{B} > \frac{1}{8}\frac{(A-c)^2}{B}$$

The leader produces in the Stackelberg equilibrium twice as much than the follower and makes twice the profit. In the Cournot duopoly the payoff $\Pi_i^2 = \frac{1}{9}\frac{(A-c)^2}{B}$ which is in between the profit of the leader and the follower.

3 Bertrand competition with differentiated products

The two firms have the demand functions

$$\begin{aligned} Q_1 &= 100 - 2P_1 + P_2 \\ Q_2 &= 100 - 2P_2 + P_1 \end{aligned}$$

and constant marginal costs $c = 5$. The profit function for firm i is

$$\Pi_i(p_1, p_2) = (P_i - c)\,Q_i = (P_i - 5)(100 - 2P_i + P_j)$$

where $j = 3 - i$. The first order condition for a profit optimum (taking the other firm's price as given) is

$$\frac{\partial \Pi_i}{\partial P_i} = (+1) \times (100 - 2P_i + P_j) + (P_i - 5) \times (-2) = 110 - 4P_i + P_j = 0,\ i = 1, 2$$

The solution to this system of equations is $P_1 = P_2 = \frac{110}{3} = 36\frac{2}{3}$. Each firm produces $\frac{2\times110}{3} = 73\frac{1}{3}$ units and makes the profit $73\frac{1}{3} \times 36\frac{2}{3} \approx 2688 \times 2$ is made. Together they make the profit 5376. If they would form a cartel they could make the profit $\Pi_1(p_1, p_2) + \Pi_2(p_1, p_2)$. Maximizing joint profit leads to the two first order conditions

$$\frac{\partial(\Pi_1 + \Pi_2)}{\partial P_i} = 110 - 4P_i + P_j + (P_i - 5) = 105 - 3P_i + P_j = 0,\ i = 1, 2$$

which have the solution $P_1 = P_2 = 52.5$. Of each commodity 57.5 units are produced and the total profit is $2 \times \left(47\frac{1}{2}\right) \times \left(57\frac{1}{2}\right) = 5462.5$, which is obviously higher than in competition.

4 Bertrand "competition" with perfect complements.

Two price-setting firms produce with constant marginal costs $c = 3$ produce goods which are perfect complements. Consumers therefore buy equal amounts from both firms. The total amount they by of each commodity is

$$Q = Q(P_1, P_2) = 15 - (P_1 + P_2)$$

The profit of firm $i = 1$ or $i = 2$ is

$$\Pi_i(P_1, P_2) = (P_i - 3)\,Q = (P_i - 3)(15 - (P_1 + P_2))$$

The first-order condition for a profit maximum is

$$\frac{\partial \Pi_i}{\partial P_i} = 15 - (P_1 + P_2) - (P_i - 3) = 18 - 2P_i - P_j = 0$$

where $j = 3 - i$. By symmetry, $P_1 = P_2$ in equilibrium, so $3P_i = 18$ or $P_1 = P_2 = 6$. It follows that $Q = 15 - 12 = 3$ pairs are sold at the price 6. Each firm makes the profit $(6-3) \times 3 = 9$ and the total profit in the industry is 18.

If a monopolist takes over both plants and takes the price $2P$ per pair his profit is

$$\Pi(P) = (2P - 6)(15 - 2P)$$

which is maximized for $2P = \frac{15+6}{2} = 10.5$ where $15 - 10.5 = 4.5$ pairs are demanded. Consumer surplus is up in the monopoly because they get more at a lower price. Producer surplus goes up because the monopolist's profit is $4.5^2 = 20.\,25 > 18$.