MATH 425b SAMPLE MIDTERM EXAM 2 SOLUTIONS
Spring 2016
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(1) Let $\mathbf{q}$ be any value taken by f, that is, $f(\mathbf{x}) = \mathbf{q}$ for some $\mathbf{x} \in E$. Since f is differentiable, it is continuous, so $f^{-1}(\mathbf{q})$ is closed in E. By the Corollary to 9.19, if $\mathbf{x} \in f^{-1}(\mathbf{q})$ then there is a ball $B_{\mathbf{x}} \subset E$ centered at $\mathbf{x}$ and f is constant on this ball: $f \equiv \mathbf{q}$ on $B_{\mathbf{x}}$, that is, $B_x \subset f^{-1}(\mathbf{q})$. This shows that $f^{-1}(\mathbf{q})$ is also open in E. Since E is connected, and $f^{-1}(\mathbf{q})$ is open, closed, and nonempty, it must be all of E, that is, $f \equiv \mathbf{q}$ is constant on E.

(2) We first show $g(x) = d(x, \varphi(x))$ is continuous. (As noted in lecture, if this were an actual exam problem, this fact would be given!) For $x, y \in X$ we have

$$d(x, \varphi(x)) \leq d(x, y) + d(y, \varphi(y)) + d(\varphi(x), \varphi(y))$$

so

$$d(x, \varphi(x)) - d(y, \varphi(y)) \leq d(x, y) + d(\varphi(x), \varphi(y)) \leq 2d(x, y),$$

and similarly

$$d(y, \varphi(y)) - d(x, \varphi(x)) \leq 2d(x, y),$$

so $|g(x) - g(y)| = |d(y, \varphi(y)) - d(x, \varphi(x))| \leq 2d(x, y)$ which shows g is indeed continuous.

Therefore, since X is compact, the inf of g is achieved at some $x_0 \in X$. We claim the infimum is 0, that is $g(x_0) = 0$, which means x_0 is a fixed point of φ. If instead the infimum is some $a > 0$, then by the assumed property of φ, we have $g(\varphi(x_0)) = d(\varphi(x_0), \varphi(\varphi(x_0))) < d(x_0, \varphi(x_0)) = a$ which contradicts a being the infimum. Thus the infimum must be $g(x_0) = 0$, meaning x_0 is a fixed point. To show uniqueness, if there were another fixed point $x_1 \neq x_0$, then $d(x_0, x_1) = d(\varphi(x_0), \varphi(x_1)) < d(x_0, x_1)$, a contradiction. Thus x_0 is the unique fixed point.

(3)(a) Since $|g(\mathbf{x})| \leq c|\mathbf{x}|^\alpha$, we must have $|g(0)| = 0$ so $g(0) = 0$. Therefore

$$|g(\mathbf{h}) - g(0)| = |g(\mathbf{h})| \leq c|\mathbf{h}|^\alpha = o(|\mathbf{h}|),$$

which shows $g'(0) = 0$. Since the derivative of $h(\mathbf{x}) = T\mathbf{x}$ is $h'(\mathbf{x}) = T$, this means $f'(0) = g'(0) + h'(0) = T$.

(b) Let $f(\mathbf{x}) = |\mathbf{x}|$ for $\mathbf{x} \in \mathbb{R}^n$. Then $f(0 + he_1) = |h|$ so $(D_1 f)(0) = \frac{d}{dh} f(0 + he_1)$ does not exist. This means $f'(0)$ does not exist, either, by 9.21.

(4)(a) Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x, y, z) = (-4x^2 + y^2 + z^2, yz - 2xy)$. Then the matrix of $f'(x, y, z)$ is

$$\begin{bmatrix} -8x & 2y & 2z \\ -2y & z - 2x & y \end{bmatrix},$$

and in particular, for $(1, 2, 2)$ it is

$$A = \begin{bmatrix} -8 & 4 & 4 \\ -4 & 0 & 2 \end{bmatrix}.$$

Since $A_{(x,y)} = \begin{bmatrix} -8 & 4 \\ -4 & 0 \end{bmatrix}$ is invertible, by the Implicit Function Theorem we can solve for x, y uniquely in terms of z in a neighborhood of $(1, 2, 2)$, say $(x, y) = (h_1(z), h_2(z))$. This is the desired parametrization.

(b) We have $A_{(y,z)} = \begin{bmatrix} 4 & 4 \\ 0 & 2 \end{bmatrix}$ which is also invertible, so as in (a) we can solve for (y, z) in terms of x. But $A_{(x,z)} = \begin{bmatrix} -8 & 4 \\ -4 & 2 \end{bmatrix}$ is not invertible so we cannot necessarily solve for (x, z) in terms of y.