MATH 425b MIDTERM EXAM 2 SOLUTIONS
Spring 2016
Prof. Alexander

(1) Since $\varphi(x^*) = x^*$ and $\varphi(x_n) = x_{n+1}$, we have

$$d(x_{n+1}, x^*) = d(\varphi(x_n), \varphi(x^*)) \leq d(x_n, x^*).$$

Thus $\{d(x_n, x^*)\}$ is a decreasing sequence, bounded below by 0, so this sequence has a limit. Since a subsequence converges to 0, 0 must be the limit of the full sequence, meaning $x_n \to x^*$.

(2)(a) Since all partial derivatives are continuous functions, $f = (f_1, f_2)$ is $\mathcal{C}'$, and those partial derivatives give the matrix of f':

$$f'(x, y, z) = \begin{bmatrix} 2x & 2y & 2z \\ 2(x-2) & 2y & 2z \end{bmatrix},$$

so in particular

$$f'(x, y, z) = A = \begin{bmatrix} 2 & 2\sqrt{2} & 2 \\ -2 & 2\sqrt{2} & 2 \end{bmatrix}.$$

Let A_{xy} denote the left two columns, and A_z the right column. Then $\det(A_{xy}) = 8\sqrt{2} \neq 0$ so A_{xy} is invertible. The existence of the desired neighborhood therefore follows from the Implicit Function Theorem. Also from that theorem,

$$u'(\mathbf{x}) = -A_{xy}^{-1} A_z = -\frac{1}{8\sqrt{2}} \begin{bmatrix} 2\sqrt{2} & -2\sqrt{2} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

(b) When $f_1(x, y, x) = f_2(x, y, z) = 0$, subtracting f_2 from f_1 shows that $x^2 = (x-2)^2$, so $x = 1$. We cannot express (y, z) as a function of x when $x = 1$ is the only possible value consistent with $f(x, y, z) = (0, 0)$. This does not contradict the Implicit Function Theorem because the matrix $A_{yz} = \begin{bmatrix} 2\sqrt{2} & 2 \\ 2\sqrt{2} & 2 \end{bmatrix}$ consisiting of the y, z columns of A is not invertible.

(Note that to show there is no expression $(y, z) = v(x)$ it is not enough to show A_{yz} is not invertible, because the Implicit Function Theorem is not "if and only if".)

(3)(a)

$$\begin{aligned}
\big|(f+g)(\mathbf{x}+\mathbf{h}) - (f+g)(\mathbf{x}) &- (A_f + A_g)\mathbf{h}\big| \\
&\leq \big|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - A_f\mathbf{h}\big| + \big|g(\mathbf{x}+\mathbf{h}) - g(\mathbf{x}) - A_g\mathbf{h}\big| \\
&= o(|\mathbf{h}|) + o(|\mathbf{h}|) \\
&= o(|\mathbf{h}|),
\end{aligned}$$

which shows that $(f+g)'(\mathbf{x}) = A_f + A_g$.

(b*) Since f is continuous, there exist ϵ and M such that $|\mathbf{h}| < \epsilon \implies |f(\mathbf{x}+\mathbf{h})| \le M$. Therefore using the Schwartz inequality,

$$
\begin{aligned}
&\big|(f\cdot g)(\mathbf{x}+\mathbf{h}) - (f\cdot g)(\mathbf{x}) - f(\mathbf{x})A_g\mathbf{h} - g(\mathbf{x})A_f\mathbf{h}\big| \\
&\quad = \big|f(\mathbf{x}+\mathbf{h})\cdot g(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}+\mathbf{h})\cdot g(\mathbf{x}) + f(\mathbf{x}+\mathbf{h})\cdot g(\mathbf{x}) - f(\mathbf{x})\cdot g(\mathbf{x}) - f(\mathbf{x})A_g\mathbf{h} - g(\mathbf{x})A_f\mathbf{h}\big| \\
&\quad = \big|f(\mathbf{x}+\mathbf{h})\cdot\big[g(\mathbf{x}+\mathbf{h}) - g(\mathbf{x}) - A_g\mathbf{h}\big] + (f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}))\cdot A_g\mathbf{h} \\
&\qquad\qquad + \big[f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - A_f\mathbf{h}\big]\cdot g(x)\big| \\
&\quad \le \big|f(\mathbf{x}+\mathbf{h})\big|\,\big|g(\mathbf{x}+\mathbf{h}) - g(\mathbf{x}) - A_g\mathbf{h}\big| + \big|(f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}))\big|\,\big|A_g\mathbf{h}\big| \\
&\qquad\qquad + \big|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - A_f\mathbf{h}\big|\,\big|g(x)\big| \\
&\quad \le Mo(|\mathbf{h}|) + \big|A_f\mathbf{h} + o(|\mathbf{h}|)\big|\,\|A_g\|\,\big|\mathbf{h}\big| + o(|\mathbf{h}|)\big|g(\mathbf{x})\big| \\
&\quad \le o(|\mathbf{h}|) + (\|A_f\|\,|\mathbf{h}| + o(|\mathbf{h}|))\,\|A_g\|\,\big|\mathbf{h}\big| + o(|\mathbf{h}|).
\end{aligned}
$$

Since $(\|A_f\|\,|\mathbf{h}| + o(|\mathbf{h}|))\,\|A_g\| \to 0$ as $\mathbf{h} \to 0$, the middle term in the bottom row is $o(|\mathbf{h}|)$, so the whole bottom row is $o(|\mathbf{h}|)$. This shows that $(f\cdot g)'(\mathbf{x}) = f(\mathbf{x})A_g + g(\mathbf{x})A_f$.

(4)(a) $S = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}$ is closed and bounded, hence compact. $A\mathbf{x}$ is a continuous function of $\mathbf{x}$, hence so is $|A\mathbf{x}|$, so its sup is achieved on S.

(b*) $D\mathbf{x} = (\lambda_1 x_1, \ldots, \lambda_n x_n)$ for all $\mathbf{x}$, so $|D\mathbf{x}|^2 = \sum_{j=1}^n \lambda_j^2 x_j^2$. Let λ_m be the largest diagonal entry (in magnitude): $|\lambda_m| = \max_{j\le n}|\lambda_j|$. Then for $|\mathbf{x}| = 1$,

$$
|D\mathbf{x}|^2 \le \sum_{j=1}^n \lambda_m^2 x_j^2 = \lambda_m^2 \sum_{j=1}^n x_j^2 = \lambda_m^2,
$$

with equality if $\mathbf{x} = e_m = (0, \ldots, 1, \ldots, 0)$ (1 is the mth coordinate.) It follows that $\sup\{|D\mathbf{x}|^2 : |\mathbf{x}| = 1\}$ is acheived at $\mathbf{x} = e_m$, with value λ_m^2, meaning $\|D\| = \sup\{|D\mathbf{x}| : |\mathbf{x}| = 1\} = |\lambda_m| = \max_{j\le n}|\lambda_j|$.