MATH 425a MIDTERM EXAM 1 SOLUTIONS
Fall 2015
Prof. Alexander

(1)(a) (See text)

(b) No. Every open interval in $\mathbb{R}$ contains rationals, which are not in I, in particular this is true for every neighborhood $(\sqrt{2}-r, \sqrt{2}+r)$.

(2)(a) An open cover of E is a collection $\{G_\alpha, \alpha \in A\}$ of open sets such that $E \subset \cup_{\alpha \in A} G_\alpha$.

(b) $\{N_{1/2}(x) : x \in \mathbb{Z}\}$ is one example. Each $N_{1/2}(x)$ contains only one integer (x itself) so a finite subcollection of some size n can only cover n integers, so it can't cover all of $\mathbb{Z}$.

(c) SOLUTION 1: $\{N_x : x \in F\}$ is an open cover of F since each $x \in N_x$. If $\{N_{x_1}, \dots, N_{x_m}\}$ is any finite subcollection then the only points of F in $N_{x_1} \cup \cdots \cup N_{x_m}$ are $x_1, \dots, x_m$, which is not all of F (since F is infinite.) Thus $\{N_{x_1}, \dots, N_{x_m}\}$ is not a finite subcover. Since no finite subcover exists, F is not compact.

SOLUTION 2: No point x of F is a limit point of F, since the neighborhood N_x contains no other point of F besides x. Therefore F is an infinite subset of itself, which has no limit point in F. By Theorem 2.37, F is not compact.

(3)(a) Each point x is in either E or E^c.

If $x \in E$, then $x \in \overline{E}$. Also $x \notin E^c$, so by the assumption, every neighborhood of x contains a point of E^c other than x, which means $x \in (E^c)'$ so $x \in \overline{E^c}$. Thus $x \in \overline{E} \cap \overline{E^c} = \partial E$.

If instead $x \in E^c$ then the same proof with E and E^c switched shows that $x \in \overline{E} \cap \overline{E^c} = \partial E$.

(b) If $x \in \partial E$ then every neighborhood of x contains a point of E and a point of E^c.

(c) Let $x \in \partial E$ and let $N_r(x)$ be a neighborhood of x.

If $x \in E$, then we have $x \in \overline{E^c}$ but $x \notin E^c$, so x must be a limit point of E^c. Therefore $N_r(x)$ contains a point of E^c, and it also contains the point $x \in E$.

If instead $x \in E^c$ then the same proof with E and E^c switched shows that $N_r(x)$ contains a point of E and a point of E^c.

(4)(a) For $a \in A$ let $N(a)$ be the first index n with $\alpha_n = a$. If $a \neq a' \in A$ and $N(a) = n$, then $\alpha_n = a \neq a'$ so $N(a') \neq n$. This shows $N(.)$ is one-to-one on A, so it is a bijection with its range $N(A) \subset \mathbb{N}$.

(b) Since $N(A) \subset \mathbb{N}$, it is at most countable. There is a bijection from A to $N(A)$, so A is at most countable. Since A is infinite, it must be countable.