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CHAPTER 13 PROBLEM SOLUTIONS

(13.2) a). YES. The logic behind "bidding own value" still applies; consider bidding own value, θ_i, versus bidding sth. else; say $b_i < \theta_i$. Consider b^* the highest bid among the rest of the players. If $b^* > \theta_i$ or $b^* < b_i$; you get the same whether you bid θ_i or b_i. ($b^* > \theta_i > b_i \rightarrow$ you lose in both cases, $b^* < b_i < \theta_i \Rightarrow$ you win in both cases and pay the same, b^*.) When $\theta_i > b^* > b_i$ however you win and net utility is $\theta_i - b^* > 0$ when you bid θ_i; however you lose if you bid b_i. Hence θ_i weakly dominates b_i.

b). Assume $v_1 > v_2 > \ldots > v_n$, values commonly known, and $b_1, b_2, \ldots, b_n$ are the bids in a pure NE. As anybody guarantees 0 utility by bidding 0; the winner should be making ≥ 0 utility from winning; If i is the winner, b_i highest bid, b^* second highest bid;

$b_i \geq b^*$ and $v_i \geq b^*$. For others not to have an incentive to deviate; $b_i \geq v_j$ $\forall j \neq i$. Otherwise they would have bid above b_i & get the good at price b_i. (now b_i would be the second highest bid) Hence for any i, $b_i \geq \max_{j \neq i} v_j$ and $b_i, v_i \geq b^*$ would be a NE.

For example n bidding very high ($b_n > v_1$, say), and everybody else bidding 0; n (the lowest valuation player) getting the good for free, $b^* = 0$, a NE.

13.3 a)- YES. Think of the reservation price r as the seller's "bid", as any other player's bid. If r is the highest "bid", then the seller retains his good.
For the other players, this is just another player with a known bid.
The logic behind bidding own value still applies as in 13.2.a

b)- Seller's revenue $= E(\text{second highest bid}) = 2 \cdot pr(\text{second highest bid} = 2) + 1 \cdot pr(\text{second highest bid} = 1) + 0 \cdot pr(\text{second highest bid} = 0)$

$Pr(\text{second highest bid} = 2) = Pr(\theta_1 = \theta_2 = 2) = \frac{1}{9}$

$Pr(\quad " \quad = 1) = Pr(\theta_1 = 2\ \theta_2 = 1) + Pr(\theta_1 = 1\ \theta_2 = 2) + Pr(\theta_1 = \theta_2 = 1) = \frac{3}{9}$

$Pr(\quad = 0) = 1 - \left(\frac{1}{9} + \frac{3}{9}\right) = \frac{5}{9}$

Seller rev $= 2 \cdot \frac{1}{9} + 1 \cdot \frac{3}{9} + 0 \cdot \frac{5}{9} = \frac{5}{9}$

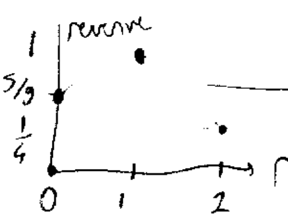
c)- Seller's revenue $= Pr(\text{highest bid} \geq r) \cdot E(\max(r, \text{second highest bid}) \mid \text{highest bid} \geq r)$

$= \underbrace{Pr\begin{pmatrix}(2,2)\ (2,1)\ (2,0) \\ (1,2), (1,1), (1,0) \\ (0,2), (0,1)\end{pmatrix}}_{\frac{8}{9}} \cdot \frac{2+1+1+1+1+1+1+1}{8} = \frac{8}{9} \cdot \frac{9}{8} = \boxed{1}$

all cases except $(0,0)$

d)- In (c), seller screens out low types from having the good, by charging a minimum price (r) and the higher minimum price elicits a higher payment from the high types; more than offsetting the loss due to elimination from auction low type customers.
Think of the 2^{nd} highest type (bid) distribution as the "demand" for the monopolist seller. By charging "above MC", he excludes low willingness to pay types but overcharges the high willingness to pay types, making more profits overall.

e)- Only consider $r = 0, 1, 2$ as any other reserve price ($r = 0.7$, say) is the same as putting the next integer up as the reserve price ($r = 0.7$ is the same as $r = 1$, given types in the problem). For $r = 2$ Seller rev $= \underbrace{Pr\begin{pmatrix}2,0 & 2,1 & 2,2 \\ 0,2 & 1,2 & \end{pmatrix}}_{\frac{5}{8}} \cdot \frac{2}{5} = \frac{1}{4}$



$\longrightarrow$ revenue is inverse U shaped!

(13.4) a). Intuitively, when the seller sets $r = \varepsilon > 0$ rather than $r = 0$, he loses any revenues when <u>both</u> buyers have valuations less than ε; but charges them approximately ε higher when one buyer has less than ε and the other more than ε. The latter case happens with $\varepsilon(1-\varepsilon)$ probability; on the order of ε whereas the former case happens with ε^2. Thus the overcharging more than compensates for the lost sales. Mathematically;

$$\text{Loss} = \varepsilon \cdot \varepsilon \cdot \left(\tfrac{1}{3}\varepsilon\right)$$ expected second highest bid when bids are $U[0,\varepsilon]$

$$\text{Gain} = 2\,\varepsilon \cdot (1-\varepsilon) \cdot \left(\varepsilon - \frac{\varepsilon}{2}\right)$$

two ways it can happen; one is below ε; other is above ε; now the winner is charged ε rather than the expected second highest bid; which is $\frac{\varepsilon}{2}$ on average.

$$\text{Gain} - \text{Loss} = \varepsilon^2(1-\varepsilon) - \frac{\varepsilon^3}{3} = \varepsilon^2 - \frac{4}{3}\varepsilon^3 > 0 \text{ for small } \varepsilon > 0.$$

Notice that if both have valuations over ε, reserve price of ε has no effect

b) $$\text{Revenue}(\text{reserve price} = r) = \text{Rev}(r) = 2\,r(1-r)\,r + (1-r)^2\left(r + (1-r)\frac{1}{3}\right)$$

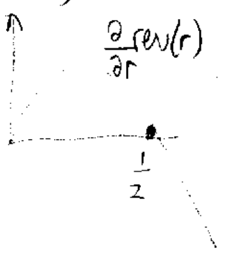
one valuation below r, other above r, winner pays r. both above r. expected 2nd highest bid given both valuations above r

Bids in this case are Uniform on $[r,1]$; hence $r + (1-r)\,U[0,1]$ distributed.

(From the textbook we know second highest (among 2) bid (eqv. valuation) from $U[0,1] = \frac{1}{3}$)

$$\frac{\partial}{\partial r}\text{Rev}(r) = 4r(1-r) + 2r^2(-1) + 2(1-r)(-1)\left(\frac{2r+1}{3}\right) + (1-r)^2\,\frac{2}{3} = -4r^2 + 2r$$

$$\frac{\partial\,\text{Rev}(r)}{\partial r} = 2r(1-2r)$$ notice that $\frac{\partial \text{Rev}(r)}{\partial r}$ is positive upto $r = \frac{1}{2}$, then negative, hence $\text{Rev}(r)$ increases upto $r = \frac{1}{2}$, then decreases.



$$r^* = \frac{1}{2} \qquad \text{Revenue}^* = 2 \cdot \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2} + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\,\frac{1}{3}\right) = \frac{5}{12}$$

(13.5) IGNORE THIS PROBLEM. There is an error in part (c).

13.6 a)- Suppose a bidder i has valuation θ_i and bids b_i.

$$EU_i = \underbrace{pr\,(b_i \text{ is the highest bid})}_{pr(\text{win})} \cdot E\left(\theta_i - 3^{rd}\text{ highest bid} \;\middle|\; b_i \text{ is the highest bid}\right)$$

Denote $s_j(\theta_j)$ for the strategy of player j; assume it is strictly increasing.

$$EU_i = pr\left(s_j(\theta_j) \le b_i \;\; \forall j \neq i\right) \cdot E_{\theta_{-i}}\left(\theta_i - \left(3^{rd}\text{ highest bid}\right) \;\middle|\; s_j(\theta_j) < b_i \;\; \forall j \neq i\right)$$

b)- For player i type θ_i; $s_i(\theta_i) = b_i$ maximizes

$$EU_i = pr\left(\frac{n-1}{n-2}\theta_j \le b_i \;\; \forall j \neq i\right) \cdot \left(\theta_i - \underbrace{b_i \cdot \frac{n-2}{n}}\right)$$

$$= \left(\frac{b_i}{\frac{n-1}{n-2}}\right)^{n-1} \cdot \left(\theta_i - b_i\frac{n-2}{n}\right)$$

expected second highest bid among the $n-1$ opponent bids, given all are below b_i.

$$\frac{\partial EU}{\partial b_i} = 0 = \left(b_i^{n-1}\left(-\frac{n-2}{n}\right) + (n-1)b_i^{n-2}\left(\theta_i - b_i\frac{n-2}{n}\right)\right) \Big/ \left(\frac{n-1}{n-2}\right)^{n-1}$$

$$\frac{n-2}{n} b_i^{\cancel{n-1}} = (n-1)\cancel{b_i^{n-2}}\left(\theta_i - b_i\frac{n-2}{n}\right)$$

$$(n-2)b_i = (n-1)n\,\theta_i - (n-1)(n-2)b_i \qquad b_i(\cancel{n})(n-2) = (n-1)\cancel{n}\,\theta_i \qquad \boxed{b_i = \frac{n-1}{n-2}\theta_i}$$

hence for any θ_i, $b_i = \frac{n-1}{n-2}\theta_i$ maximizes EU_i, hence a BNE.

c)- For the second prize auction it is optimal to bid own value; when you're supposed to pay the 3^{rd} highest bid rather than 2^{nd} highest you can "inflate" your bids as you're going to pay much less conditional on winning with a given bid.

d)-
$$rev_3 = E\Big(\frac{n-1}{n-2}\underbrace{\theta^{[3]}}_{3^{rd}\text{ highest value among all players.}}\Big) = \frac{n-1}{n-2}\;\frac{n-2}{n+1} = \frac{n-1}{n+1} = rev_1 = rev_2$$

Revenue Equivalence theorem applies; they all deliver the same revenue.

13.7 a) Given i has valuation θ_i; for each auction;

$$EU_i^{first}(b_i, s_{-i}(\theta_{-i}), \theta_i) = \int_{\theta_{-i}} \begin{cases} \sqrt{\theta_i - b_i} & \text{if } b_i > s_j(\theta_j) \ \forall j \neq i \\ 0 & \text{otherwise} \end{cases} \prod_{j\neq i} f_j(\theta_j) d\theta_j$$

$$EU_i^{second}(b_i, s_{-i}(\theta_{-i}), \theta_i) = \int_{\theta_{-i}} \begin{cases} \sqrt{\theta_i - b^\star(\theta_{-i})} & \text{if } b_i > s_j(\theta_j) \ \forall j \neq i \\ 0 & \text{otherwise} \end{cases} \cdot \prod_{j\neq i} f_j(\theta_j) d\theta_j$$

second highest bid.

$$EU_i^{first}(\ldots) = \sqrt{\theta_i - b_i} \cdot Pr(b_i > s_j(\theta_j) \ \forall j \neq i) = \sqrt{\theta_i - b_i} \prod_{j\neq i} F(s_j^{-1}(b_i))$$

$$EU_i^{second}(\ldots) = E_{\theta_{-i}}\left(\sqrt{\theta_i - b^\star(\theta_{-i})} \,\middle|\, b_i \text{ is the highest bid}\right) \cdot Pr(b_i > s_j(\theta_j) \ \forall j \neq i)$$

(second highest bid, given b^i highest bid) $\quad$ Pr b_i is the highest bid $= \prod_{j\neq i} F(S_j^{-1}(b_i))$

$$= E_{\theta_{-i}}\left(\sqrt{\theta_i - \max_{j\neq i} s_j(\theta_j)} \,\middle|\, b_i > \max_{j\neq i} s_j(\theta_j)\right) \cdot \prod_{j \neq i} F(s_j^{-1}(b_i))$$. Equivalently;

$$EU_i^{second}(\ldots) = \int_{\theta_{-i} \text{ s.t. } s_j(\theta_j) < b_i \ \forall j \neq i} \sqrt{\theta_i - \max_{j\neq i} s_j(\theta_j)} \cdot \prod_{j\neq i} f_j(\theta_j) d\theta_j$$

b) Typo in the book: the payoff to winning is $(\theta_i - p)^{\frac{1}{m}}$; in particular for $m=2$ $= \sqrt{\theta_i - p}$. Now, assume all other players are using $b_j = s_j(\theta_j) = \frac{m(n-1)}{m(n-1)+1}\theta_j$, and player i with valuation θ_i chooses $s_i(\theta_i) = b_i$ to maximize

$$EU^{first} = (\theta_i - b_i)^{\frac{1}{m}} \cdot Pr\left(b_i > \frac{m(n-1)}{m(n-1)+1}\theta_j \ \forall j\right) = (\theta_i - b_i)^{\frac{1}{m}} \left(\frac{b_i}{\frac{m(n-1)}{m(n-1)+1}}\right)^{n-1}$$

$$\frac{\partial EU}{\partial b_i} = 0 = \left(\frac{1}{m}(\theta_i - b_i)^{\frac{1}{m}-1}(-1)(b_i)^{n-1} + (\theta_i - b_i)^{\frac{1}{m}}(n-1)\, b_i^{n-2}\right) \Big/ \left(\frac{m(n-1)}{m(n-1)+1}\right)^{n-1}$$

$$\frac{1}{m}(\theta_i - b_i)^{\frac{1}{m}-1} b_i^{\cancel{n-1}} = (\theta_i - b_i)^{\cancel{\frac{1}{m}}}(n-1)\, \cancel{b_i^{n-2}} \qquad b_i = m(\theta_i - b_i)(n-1)$$

$$b_i = \frac{m(n-1)}{m(n-1)+1}\theta_i$$. θ_i was arbitrary; hence $s_i(\theta_i) = \frac{m(n-1)}{m(n-1)+1}\theta_i$ indeed maximizes EU_i.

Hence a BNE.

c). The logic in 13.2 & 13.3 still applies, with the utility now $(\theta_i - p)^{\frac{1}{m}}$ rather than $\theta_i - p$ ($m=1$ case).

d). The second price auction revenue is the expected second highest value; $\theta^{[2]}$

$$rev_2 = E\left(\theta^{[2]}\right) = \frac{n-1}{n+1} \quad \text{from pp 282 in the textbook.}$$

highest value among n valuations

For the first price auction revenue $rev_1 = E\left(\underbrace{\frac{m(n-1)}{m(n-1)+1}\theta^{[1]}}_{\text{highest bid}}\right) = \frac{m(n-1)}{m(n-1)+1}\frac{n}{n+1}$

Notice that $\frac{rev_1}{rev_2} = \frac{m(n-1)\, n}{(m(n-1)+1)\cancel{(n+1)}}\frac{\cancel{n+1}}{n-1} = \frac{n}{(n-1)+\frac{1}{m}}$ ↑ in m. ($=1$ for $m=1$, risk neutral case)

Notice that in first price auction, bids are <u>higher</u> than in the risk neutral case; $\frac{m(n-1)}{m(n-1)+1}$ ↑ in m. So risk averse bidders bid higher in first price auction hence the seller collects more revenue compared to the risk-neutral case.

(13.8) It's easier to consider first symmetric, pure BNE;

Suppose (b_L, b_H) for both players constitute an equilibrium, with $b_L < b_H$

Given opponent's strategy (b_L, b_H), your strategy when you are type L; b_L maximizes $EU_i(b, (b_L, b_H), L) = Pr(\text{win}) \cdot u(\text{owning oil field} \mid \text{win at price/bid } b_L)$

Remember that

	L	H
L	v=10	v=20
H	v=20	v=30

If $b_L > 10$, when you bid and win, it means opponent bid b_L too; hence $v = 10$;

$$EU_i = \frac{1}{2} \cdot \frac{1}{2} (10 - b_L) < 0$$

prob. opponent is L

in which case you share the good.

v=10 when types are (L, L) (bids are (b_L, b_L))

Similarly if $b_L < 10$, $EU_i > 0$ but if you bid $b_L' = b_L + \varepsilon$, you don't share the good anymore as you bid higher b_L and $EU_i = \frac{1}{2}(10 - (b_L + \varepsilon)) > \frac{1}{2} \cdot \frac{1}{2}(10 - b_L)$

Hence $b_L = 10$ should be the case.

For b_H; if $b_H > 30$ $\quad EU_i = \frac{1}{2}(20 - 10) + \frac{1}{2} \cdot \frac{1}{2}(30 - b_H) < 0$

$v(H,L) = 20$

types are (H, L) hence bids $(b_H, 10)$

you pay second highest bid; 10.

hence $b_H - \varepsilon$ would do better; $EU_i = \frac{1}{2}(20 - 10) + \frac{1}{2} \cdot 0$ you lose against b_H.

If $10 < b_H < 30$ $\quad EU_i = \frac{1}{2}(20 - 10) + \frac{1}{2} \cdot \frac{1}{2}(30 - b_H)$

> 0

Similar to above logic, $b_H + \varepsilon$ would get the good with pr. 1 in this case, increasing payoffs; $EU_i = \frac{1}{2}(20 - 10) + \frac{1}{2}(30 - b_H) > \frac{1}{2}(20 - 10) + \frac{1}{2} \frac{1}{2}(30 - b_H)$

Hence $b_H = 30$ should hold.

Indeed given opponent uses $(b_L, b_H) = (10, 30)$, when you are L type, you would want to bid $b_L = 10$ optimally (lower or higher gives you the same 0 utility)

If you're the H type, $b_H < 30$ or $b_H > 30$ similarly gets you the same utility as bidding $b_H = 30$; so it's a pure symmetric BNE.

13.9 a)- Suppose player j is bidding $s_j(\theta_j) = a + b\theta_j$; player i type θ_i's bid b_i maximizes ; $EU_i(b_i, s_j(\theta_j), \theta_i) = pr(\text{win})(\text{Expected net utility} \mid \text{win})$

$$= pr\big(\underbrace{s_j(\theta_j)}_{a+b\theta_j} < b_i\big)\left(\theta_i + \frac{1}{2} - b_i\right)$$

$$= \frac{b_i - a}{b} \cdot \left(\theta_i + \frac{1}{2} - b_i\right)$$

$$= (b_i - a)(2\theta_i + 1 - 2b_i)/2b$$

$$\frac{\partial EU_i}{\partial b_i} = 0 = (2\theta_i + 1 - 2b_i) + (b_i - a)(-2)$$

$$2\theta_i + 1 - 2b_i - 2b_i + 2a = 0$$

$$4b_i = 2\theta_i + 1 + 2a$$

$$\boxed{b_i = \frac{1}{2}\theta_i + \frac{1+2a}{4}}$$

$$= b\theta_i + a \quad (\text{symmetric eqm})$$

$b = \frac{1}{2}$ $\quad a = \frac{1+2a}{4}$ $\quad a = \frac{1}{2}$

$$\boxed{s_i(\theta_i) = \frac{1}{2} + \frac{1}{2}\theta_i}$$

b)- $EU_i(\theta_i) = \theta_i \cdot \left(\theta_i - \frac{1}{2} - \left(\frac{1}{2} + \frac{1}{2}\theta_i\right)\right) = \theta_i\left(\frac{\theta_i}{2} - 1\right)$

13.10 a)- Suppose player j is using bidding strategy $s_j(\theta_j) = a + b\theta_j$.
In a BNE, player i type θ_i's bid $s_i(\theta_i) = b_i$ should maximize;

$$EU_i(b_i, s_j(\theta_j), \theta_i) = pr(\text{win}) \cdot (\text{Expected value} \mid \text{win})$$

$$= pr(b_i > a + b\theta_j) \cdot E(\theta_i + \theta_j - b_i \mid b_i > a + b\theta_j)$$

$$= \frac{b_i - a}{b} \cdot \left(\theta_i + E\left(\theta_j \,\middle|\, \theta_j < \frac{b_i - a}{b}\right) - b_i\right)$$

$$= \frac{b_i - a}{b} \cdot \left(\theta_i + \frac{b_i - a}{2b} - b_i\right) = (b_i - a)(2b\theta_i + b_i - a - 2bb_i) \Big/ 2b^2$$

$$\frac{\partial EU_i}{\partial b_i} = 0 = (b_i - a)(1 - 2b) + 1 \cdot (2b\,\theta_i + b_i - a - 2b \cdot b_i)$$

$$b_i(1 - 2b + 1 - 2b) = a(1 - 2b) + a - 2b\theta_i$$

$$b_i = a\,\frac{2-2b}{2-4b} + \frac{-2b}{2-4b}\,\theta_i$$

$$= a + b\,\theta_i$$

$$\Rightarrow \quad 2b - 4b^2 = -2b \qquad 4b = 4b^2 \qquad b = 1 \text{ or } 0$$

if $b = 1 \Rightarrow a = a \cdot \frac{2-2\cdot 1}{2-4\cdot 1} = 0$ $\quad a = 0 \qquad \boxed{s_i(\theta_i) = \theta_i}$ is a symmetric BNE.

(Note: If $b = 0$, then both bid $s_j(\theta_j) = a$ always, independent of their valuations.
$E(\theta_1 + \theta_2) = 1$ hence if $a < 1 \Rightarrow$ each player has an incentive to always bid $a + \varepsilon$
$a > 1 \Rightarrow$ " " $a - \varepsilon$
(so that he loses)
$\Rightarrow$ Only $a = 1$ is a candidate eqm.
However, then, I'd bid $b_i = 1 + \varepsilon$ if I have $\theta_i > 0.5$ and
$b_i = 1 - \varepsilon$ $\quad \theta_i < 0.5$. Hence not an eqm.)

b)- $EU_i(\theta_i) = \theta_i \left(\theta_i + \underbrace{\frac{\theta_i}{2}}_{E(\theta_j)} - \underbrace{\theta_i}_{\text{bid}}\right) = \frac{\theta_i^2}{2}$

c)- $\theta_i \leq \frac{1}{2} + \frac{1}{2}\theta_i \rightarrow$ from 13.9.a $\quad$ as $\frac{\theta_i}{2} \leq \frac{1}{2} \quad \theta_i \leq 1$