Chapter | Fifteen

Sequential Sales

In all of the multiunit auctions considered thus far—the discriminatory, uniform-price, and Vickrey formats—all the units are sold at one go. In this chapter we examine situations in which the units are sold one at a time in separate auctions that are conducted sequentially.

15.1 SEQUENTIAL FIRST-PRICE AUCTIONS

Consider a situation in which the K identical items are sold to $N > K$ bidders using a series of first-price sealed-bid auctions. Specifically, one of the items is auctioned using the first-price format, and the price at which it is sold—the winning bid—is announced. The second item is then sold and again the price at which it is sold—the winning bid in the second auction—is announced. The third item is then sold, and so on.

We restrict attention to situations in which each bidder has use for at most one unit—the case of *single-unit demand*. We also suppose that bidders have private values and that each bidder's value X_i is drawn independently from the same distribution F on $[0, \omega]$. It is then natural to look for a symmetric equilibrium.

A bidding strategy for a bidder consists of K functions $\beta_1^{\mathrm{I}}, \beta_2^{\mathrm{I}}, \ldots, \beta_K^{\mathrm{I}}$ where $\beta_k^{\mathrm{I}}\left(x, p_1, p_2, \ldots, p_{k-1}\right)$ denotes the bid in the kth auction given that the bidder's value is x and the prices in the $k-1$ previous auctions were $p_1, p_2, \ldots, p_{k-1}$, respectively. All this assumes, of course, that the particular bidder has not already won an object and so is still active in the kth auction. (From now on, if there is no ambiguity, the superscript "I" identifying the first-price format will be omitted.)

Notice that if the equilibrium strategies β_k are increasing functions of the value x, then the items will be sold in order of decreasing values. The first item will go to the bidder with the highest value, the second to the bidder with the second-highest value, and so on. In that case the K units will be allocated efficiently.

In what follows it will be convenient to think of the auctions as being held in different periods. Moreover, the auctions are assumed to be held in a short enough time—say, the same day—so bidders do not discount payoffs from later periods.

15.1.1 Two Units

We begin by looking at a situation in which only two units are sold ($K = 2$), so a symmetric equilibrium consists of two functions (β_1, β_2), denoting the bidding strategies in the first and second periods, respectively. We conjecture that these are increasing and differentiable. The first-period bidding strategy is a function $\beta_1 : [0, \omega] \to \mathbb{R}_+$ that depends only on the bidder's value. The bid in the second period may depend on both the bidder's value and the price paid in the first auction, p_1. As usual, denote by $Y_1 \equiv Y_1^{(N-1)}$ the highest of $N - 1$ values, by $Y_2 \equiv Y_2^{(N-1)}$ the second-highest, and so on. Let F_1 and F_2 be the distributions of Y_1 and Y_2, respectively, and let f_1 and f_2 be the corresponding densities.

Since the first-period strategy β_1 is assumed to be invertible, the value of the winning bidder in the first period is commonly known; it is just $y_1 = \beta_1^{-1}(p_1)$. Thus, the second-period strategy can be thought of as a function $\beta_2 : [0, \omega] \times [0, \omega] \to \mathbb{R}_+$ so a bidder with value x bids an amount $\beta_2(x, y_1)$ if $Y_1 = y_1$.

We are interested in equilibria that are sequentially rational—that is, equilibria with the property that following any outcome of the first-period auction, the strategies in the second period form an equilibrium. We begin with the second period.

SECOND-PERIOD STRATEGY

Consider the second-period auction and the decision problem facing a particular bidder—say, 1—whose value is x. Suppose all other bidders follow the equilibrium strategy $\beta_2(\cdot, y_1)$ and bidder 1 bids $\beta_2(z, y_1)$ in the second auction. Since the bidders competing against bidder 1 in the second auction have values $Y_2, Y_3, \ldots, Y_{N-1}$ and in equilibrium $Y_2 < y_1$, it makes no sense for bidder 1 to bid an amount greater than $\beta_2(y_1, y_1)$. His expected payoff in the second auction if he bids $\beta_2(z, y_1)$ for some $z \le y_1$ is

$$\Pi(z, x; y_1) = F_2(z \mid Y_1 = y_1) \times [x - \beta_2(z, y_1)]$$

Differentiating $\Pi(z, x; y_1)$ with respect to z we obtain the first-order condition that in equilibrium, for all x,

$$f_2(x \mid Y_1 = y_1)[x - \beta_2(x, y_1)] - F_2(x \mid Y_1 = y_1)\beta_2'(x, y_1) = 0,$$

where β_2' is the derivative of β_2 with respect to its first argument. Rearranging this results in the differential equation

$$\beta_2'(x, y_1) = \frac{f_2(x \mid Y_1 = y_1)}{F_2(x \mid Y_1 = y_1)}[x - \beta_2(x, y_1)] \tag{15.1}$$

together with the boundary condition $\beta_2(0, y_1) = 0$.

The probability that bidder 1 will win the second auction is the probability that Y_2, the second-highest of $N - 1$ values, is less than z, conditional on the event that Y_1, the highest of $N-1$ values, equals y_1. But because the different values are drawn independently, this is the same as the probability that $Y_1^{(N-2)}$, the highest of $N-2$ values, is less than z, conditional on the event that $Y_1^{(N-2)} < y_1$. (See (C.6) in Appendix C for a formal demonstration.) This implies that the probability that bidder 1 will win the second auction is

$$\begin{aligned} F_2(z \mid Y_1 = y_1) &= F_1^{(N-2)}\left(z \mid Y_1^{(N-2)} < y_1\right) \\ &= \frac{F(z)^{N-2}}{F(y_1)^{N-2}} \end{aligned} \tag{15.2}$$

Now using (15.2) the differential equation (15.1) can be written as

$$\beta_2'(x, y_1) = \frac{(N-2)f(x)}{F(x)}\left[x - \beta_2(x, y_1)\right]$$

or equivalently,

$$\frac{\partial}{\partial x}\left(F(x)^{N-2}\beta_2(x, y_1)\right) = (N-2)\,F(x)^{N-3}f(x)x$$

Notice that the right-hand side of the preceding equation is independent of y_1 and hence of the price announcement. This means that β_2 is, in fact, independent of y_1; in other words, the *price announcements have no effect* on the equilibrium bids in the second period. The solution to the differential equation is

$$\begin{aligned} \beta_2(x) &= \frac{1}{F(x)^{N-2}}\int_0^x y d\left(F(y)^{N-2}\right) \\ &= E\left[Y_1^{(N-2)} \mid Y_1^{(N-2)} < x\right] \\ &= E\left[Y_2 \mid Y_2 < x < Y_1\right], \end{aligned} \tag{15.3}$$

where the last equality follows from (15.2). Thus, the complete bidding strategy for the second period is to bid $\beta_2(x)$ if $x \le y_1$ and to bid $\beta_2(y_1)$ if $x > y_1$. The latter may occur if bidder 1 himself underbid in the first auction—say, by mistake—causing someone else, with a lower value, to win. Even though this represents "off-equilibrium" behavior on the part of bidder 1 himself, recall that a strategy must prescribe actions in all contingencies.

The derivation of the preceding strategy is virtually the same as the derivation of the equilibrium bidding strategy in a first-price auction with symmetric private values (see Proposition 2.2 on page 15) and has been carried out using the necessary first-order conditions. Verifying the optimality of $\beta_2(x)$, and thus that β_2 constitutes a symmetric equilibrium in the second auction, is almost the same as in the case of a single-object first-price auction.

FIRST-PERIOD STRATEGY

In the first-period auction the decision problem facing a bidder is slightly more complex. Again let us take the perspective of bidder 1 with value x and suppose that all other bidders are following the first-period strategy β_1. Further, suppose that all bidders, including bidder 1, will follow β_2 in the second period, regardless of what happens in the first period.

The equilibrium calls on bidder 1 to bid $\beta_1(x)$ in the first stage, but consider what happens if he decides to bid $\beta_1(z)$ instead. If $z \geq x$, his payoff is

$$\begin{aligned}\Pi(z,x) &= F_1(z)[x-\beta_1(z)] \\ &\quad + (N-1)(1-F(z))F(x)^{N-2}[x-\beta_2(x)],\end{aligned}$$

where the first term results from the event $Y_1 < z$, so that he wins the first auction with a bid of $\beta_1(z)$. The second term results from the event $Y_2 < x \leq z \leq Y_1$, so he loses the first auction but wins the second. On the other hand, if $z < x$, his payoff is

$$\begin{aligned}\Pi(z,x) &= F_1(z)[x-\beta_1(z)] + [F_2(x)-F_1(x)][x-\beta_2(x)] \\ &\quad + \int_z^x [x-\beta_2(y_1)] f_1(y_1)\,dy_1,\end{aligned}$$

where the first term again results from the event $Y_1 < z$. The second term results from the event $Y_2 < x < Y_1$, so he loses the first auction but wins the second with a bid of $\beta_2(x)$. The third term results from the event $z < Y_1 < x$ so he loses the first auction but wins the second with a bid of $\beta_2(Y_1)$.

The first-order conditions in the two cases are

$$\begin{aligned}0 &= f_1(z)[x-\beta_1(z)] - F_1(z)\beta_1'(z) \\ &\quad - (N-1)f(z)F(x)^{N-2}[x-\beta_2(x)]\end{aligned}$$

and

$$\begin{aligned}0 &= f_1(z)[x-\beta_1(z)] - F_1(z)\beta_1'(z) \\ &\quad - f_1(z)[x-\beta_2(z)]\end{aligned}$$

respectively. In equilibrium it is optimal to bid $\beta_1(x)$ and setting $z = x$ in *either* first-order condition results in the differential equation

$$\beta_1'(x) = \frac{f_1(x)}{F_1(x)}[\beta_2(x) - \beta_1(x)] \tag{15.4}$$

together with the boundary condition $\beta_1(0) = 0$. This can be rearranged so that

$$\frac{d}{dx}[F_1(x)\beta_1(x)] = f_1(x)\beta_2(x)$$

which has a solution

$$
\begin{aligned}
\beta_1(x) &= \frac{1}{F_1(x)} \int_0^x \beta_2(y) f_1(y)\, dy \\
&= E[\beta_2(Y_1) \mid Y_1 < x] \\
&= E[E[Y_2 \mid Y_2 < Y_1] \mid Y_1 < x] \\
&= E[Y_2 \mid Y_1 < x]
\end{aligned}
\tag{15.5}
$$

using (15.3).

Thus, we obtain the following:

Proposition 15.1. *Suppose bidders have single-unit demand and two units are sold by means of sequential first-price auctions. Symmetric equilibrium strategies are*

$$
\begin{aligned}
\beta_1^{\mathrm{I}}(x) &= E[Y_2 \mid Y_1 < x] \\
\beta_2^{\mathrm{I}}(x) &= E[Y_2 \mid Y_2 < x < Y_1],
\end{aligned}
$$

where $Y_1 \equiv Y_1^{(N-1)}$ is the highest, and $Y_2 \equiv Y_2^{(N-1)}$ is the second highest, of $N-1$ independently drawn values.

15.1.2 More than Two Units

The construction underlying the equilibrium strategies in the two-unit case readily generalizes. So suppose that K units are sold in a sequence of K first-price auctions. The prices $p_1, p_2, \ldots, p_{k-1}$ in the first $k-1$ auctions are commonly known to the bidders in auction $k > 1$. In what follows, $Y_k \equiv Y_k^{(N-1)}$ denotes the kth highest of $N-1$ values, F_k denotes the distribution of Y_k, and f_k denotes the corresponding density.

As in the previous subsection, we will derive symmetric bidding strategies $(\beta_1, \beta_2, \ldots, \beta_K)$ by working backward from the last auction. So first consider the Kth auction—conducted in the last period. Following exactly the same reasoning as in the derivation of (15.3) leads to the conclusion that the bidding strategy in the last period is

$$
\beta_K(x) = E[Y_K \mid Y_K < x < Y_{K-1}] \tag{15.6}
$$

Once again, the strategy in the last period does not depend on the price announcements.

Now consider the kth auction for some $k < K$. Given the detailed analysis of the two-unit case, we proceed somewhat heuristically. Again let us take the perspective of bidder 1 with value x and suppose that all other bidders are following the kth period strategy β_k. Further, suppose that all bidders, including bidder 1, will follow the strategies $\beta_{k+1}, \beta_{k+2}, \ldots, \beta_K$ in the subsequent auctions.

The equilibrium calls on bidder 1 to bid $\beta_k(x)$ in the kth stage but consider what happens if he decides to bid slightly higher—say, $\beta_k(x+\Delta)$. Now if $Y_k < x$, he would have won with his equilibrium bid $\beta_k(x)$ anyway, so the only consequence of his bidding higher is that he pays more than he would have. His expected payment increases by

$$F_k\left(x \mid Y_{k-1} = y_{k-1}\right) \times [\beta_k(x+\Delta) - \beta_k(x)] \tag{15.7}$$

The expression in (15.7) thus represents the expected loss from bidding $\beta_k(x+\Delta)$ as opposed to $\beta_k(x)$. On the other hand, if $x < Y_k < x+\Delta$, he would have lost the kth auction with his equilibrium bid, whereas bidding higher results in his winning. Now there are two subcases. In the event that $Y_{k+1} < x < Y_k < x+\Delta$, he would have lost the kth auction but won the $k+1$st. In the event that $x < Y_{k+1} < Y_k < x+\Delta$, however, he would have lost both the kth and the $k+1$st auctions, and possibly won a later auction, say the lth for some $l > k+1$. When Δ is small, however, the probability that $x < Y_{k+1} < Y_k < x+\Delta$ is very small—it is of second order in magnitude. Thus, the contribution to the expected gain from all events in which the bidder loses both the kth and the $k+1$st auctions can be safely neglected when Δ is small. The overall expected gain from bidding $\beta_k(x+\Delta)$ is approximately

$$\begin{aligned}&\left[F_k\left(x+\Delta \mid Y_{k-1} = y_{k-1}\right) - F_k\left(x \mid Y_{k-1} = y_{k-1}\right)\right] \\ &\quad \times \left[(x - \beta_k(x)) - \left(x - \beta_{k+1}(x)\right)\right]\end{aligned} \tag{15.8}$$

that is, it is the probability that $x < Y_k < x+\Delta$ times the difference in the equilibrium price paid tomorrow and the price paid today. Equating (15.7) and (15.8), dividing by Δ, and taking the limit as $\Delta \to 0$, we obtain the differential equation

$$\beta_k'(x) = \frac{f_k\left(x \mid Y_{k-1} = y_{k-1}\right)}{F_k\left(x \mid Y_{k-1} = y_{k-1}\right)}[\beta_{k+1}(x) - \beta_k(x)]$$

together with the boundary condition $\beta_k(0) = 0$. Now notice that as in (15.2),

$$\begin{aligned}F_k\left(x \mid Y_{k-1} = y_{k-1}\right) &= F_1^{(N-k)}\left(x \mid Y_1^{(N-k)} < y_{k-1}\right) \\ &= \frac{F(z)^{N-k}}{F\left(y_{k-1}\right)^{N-k}}\end{aligned}$$

As before, the solution is independent of y_{k-1} and hence of the price announcements. The solution is

$$\begin{aligned}\beta_k(x) &= \frac{1}{F(x)^{N-k}} \int_0^x \beta_{k+1}(y)\, d\left(F(y)^{N-k}\right) \\ &= E\left[\beta_{k+1}\left(Y_1^{(N-k)}\right) \mid Y_1^{(N-k)} < x\right] \\ &= E[\beta_{k+1}(Y_k) \mid Y_k < x < Y_{k-1}]\end{aligned} \tag{15.9}$$

An explicit solution to (15.9) can be obtained by working backward from the last period. Using (15.6), we deduce that

$$\begin{aligned}\beta_{K-1}(x) &= E\left[\beta_K\left(Y_{K-1}\right) \mid Y_{K-1} < x < Y_{K-2}\right] \\ &= E\left[E\left[Y_K \mid Y_K < Y_{K-1}\right] \mid Y_{K-1} < x < Y_{K-2}\right] \\ &= E\left[Y_K \mid Y_{K-1} < x < Y_{K-2}\right]\end{aligned}$$

and proceeding inductively in this fashion results in the solution for all k,

$$\begin{aligned}\beta_k(x) &= E\left[\beta_{k+1}(Y_k) \mid Y_k < x < Y_{k-1}\right] \\ &= E\left[E\left[Y_K \mid Y_{k+1} < Y_k\right] \mid Y_k < x < Y_{k-1}\right] \\ &= E\left[Y_K \mid Y_k < x < Y_{k-1}\right]\end{aligned} \tag{15.10}$$

Thus, we obtain the following generalization of Proposition 15.1.

Proposition 15.2. *Suppose bidders have single-unit demand and K units are sold by means of sequential first-price auctions. Symmetric equilibrium strategies are given by*

$$\beta_k^{\mathrm{I}}(x) = E[Y_K \mid Y_k < x < Y_{k-1}],$$

where β_k^{I} denotes the bidding strategy in the kth auction and $Y_k \equiv Y_k^{(N-1)}$ is the kth highest of $N-1$ independently drawn values.

Example 15.1. *Values are uniformly distributed on* $[0,1]$.

In the last period, the equilibrium bidding strategy is

$$\beta_K(x) = \frac{N-K}{N-K+1}x$$

Proceeding inductively, it may be verified that the bidding strategy in the kth first-price auction is

$$\beta_k(x) = \frac{N-K}{N-k+1}x$$

Figure 15.1 depicts the bidding strategies for the case of three objects ($K=3$) and five bidders ($N=5$). ▲

15.1.3 Equilibrium Bids and Prices

Some features of the equilibrium strategies are worth noting. First, for all k, $\beta_{k+1}(x) > \beta_k(x)$—that is, a bidder with value x who is active in the kth auction

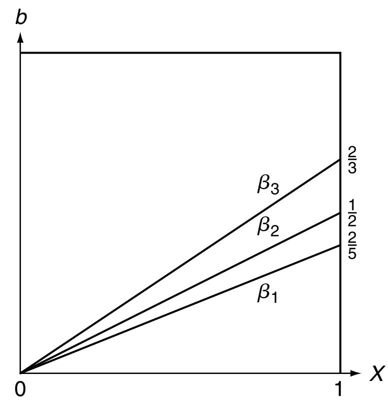

FIGURE 15.1 Equilibrium of sequential first-price auction.

and fails to win bids higher in the $k+1$st auction. Informally, this is due to the deterioration of available supply relative to current demand. The higher bids from those who did not win in the previous period is, however, mitigated by the fact that there is one fewer bidder in this period. Indeed, the remaining bidders have smaller values than the winner of the previous round. Remarkably, the equilibrium is such that the two effects exactly offset each other and the prices in successive auctions show no trend. Precisely, the equilibrium price path is a *martingale*—at the end of the kth auction, the expected price in the $k+1$st auction is the same as the realized price in the kth auction.

To see this, suppose that in equilibrium, bidder 1 with value x wins the kth auction. Then, absent any ties, it must be that

$$Y_{K-1} < \ldots < Y_k < x < Y_{k-1} < \ldots < Y_1$$

Now let the random variables P_k and P_{k+1} denote the prices in periods k and $k+1$, respectively. We know that the realized price in period k, $p_k = \beta_k(x)$. Moreover, the price in the $k+1$st period is the random variable $P_{k+1} = \beta_{k+1}(Y_k)$ and

$$\begin{aligned} E[P_{k+1} \mid P_k = p_k] &= E[\beta_{k+1}(Y_k) \mid Y_k < x < Y_{k-1}] \\ &= \beta_k(x) \\ &= p_k \end{aligned}$$

from (15.9). This establishes that the price path is a martingale. An implication of this property of the price path is that there are no opportunities for intertemporal arbitrage. For instance, if $E[P_{k+1} \mid P_k = p_k] < p_k$, then it would benefit bidders to decrease their bids on the current item with a view to waiting for the next item.

15.2 SEQUENTIAL SECOND-PRICE AUCTIONS

In this section we amend the model of the previous section so that the K items are sold using a series of second-price sealed-bid auctions. In all other respects, the model is the same. In particular, we suppose that, prior to bidding in a particular auction, the prices in all previous auctions are announced to all. And once again, we are interested in symmetric equilibrium strategies that are increasing. Instead of finding the equilibrium strategies directly, however, we will make use of the revenue equivalence principle.

15.2.1 Revenue Equivalence

If $\beta_1^{\mathrm{II}}, \beta_2^{\mathrm{II}}, \ldots, \beta_K^{\mathrm{II}}$ is a symmetric increasing equilibrium, then, as in the case of sequential first-price auctions, the K units will be allocated efficiently. Indeed, the first unit will go to the bidder with the highest value, the second to the bidder with the second-highest value, and so on. This means that the two mechanisms—the sequential first- and second-price formats—are revenue equivalent (see Proposition 14.1 on page 206). Specifically, if $m^{\mathrm{I}}(x)$ and $m^{\mathrm{II}}(x)$ denote the expected payment by a bidder with value x in K sequential first- and second-price formats, respectively, then for all x,

$$m^{\mathrm{I}}(x) = m^{\mathrm{II}}(x)$$

Now define $m_k^{\mathrm{I}}(x)$ to be the expected payment made in the kth auction by a bidder with value x when the items are sold by means of K first-price auctions. Define $m_k^{\mathrm{II}}(x)$ in analogous fashion for the sequential second-price format. Clearly,

$$m^{\mathrm{I}}(x) = \sum_{k=1}^{K} m_k^{\mathrm{I}}(x) \text{ and } m^{\mathrm{II}}(x) = \sum_{k=1}^{K} m_k^{\mathrm{II}}(x)$$

While the revenue equivalence principle as such only guarantees that the overall expected payments in the two formats are the same, we claim that, in fact, for all k,

$$m_k^{\mathrm{I}}(x) = m_k^{\mathrm{II}}(x)$$

that is, the expected payment in the kth first-price auction is the same as the expected payment in the kth second-price auction. In other words, we claim that the two auctions are payment equivalent period by period. We argue by induction, starting with the Kth auction. Prior to the last auction, the information available to the remaining $N-K+1$ bidders in either format is the same. For instance, bidder 1 knows his own value x, that his competitors have values $Y_{K+1}, Y_{K+2}, \ldots, Y_N$, and that $Y_K = y_K$. The revenue equivalence principle implies that $m_K^{\mathrm{I}}(x) = m_K^{\mathrm{II}}(x)$. Now consider the start of auction $K-1$ and think of the remaining two formats as mechanisms for allocating *two* units. Once again, the information available to the remaining $N-K+2$ bidders is the same. For instance, bidder 1 knows his own value x, that his competitors have values $Y_K, Y_K, \ldots, Y_N$ and that $Y_{K-1} = y_{K-1}$. Once again, the revenue equivalence principle implies that

$$m_{K-1}^{\mathrm{I}}(x) + m_K^{\mathrm{I}}(x) = m_{K-1}^{\mathrm{II}}(x) + m_K^{\mathrm{II}}(x)$$

and since $m_K^{\mathrm{I}}(x) = m_K^{\mathrm{II}}(x)$, we have $m_{K-1}^{\mathrm{I}}(x) = m_{K-1}^{\mathrm{II}}(x)$. Proceeding inductively in this way establishes that for all k, $m_k^{\mathrm{I}}(x) = m_k^{\mathrm{II}}(x)$.

15.2.2 Equilibrium Bids

With the equality of the per-period expected payments in the sequential first- and second-price auctions in hand, we are ready to find the equilibrium bidding strategies in the latter. Clearly, in the last period it is a dominant strategy to bid one's value—that is,

$$\beta_K^{\mathrm{II}}(x) = x$$

and this is for the same reason that it is a dominant strategy in a single-unit second-price auction. Now notice that for any $k < K$, if bidder 1 with value x wins the kth auction, then it must be that

$$Y_{K-1} < \ldots < Y_k < x < Y_{k-1} < \ldots < Y_1$$

and the price he pays—the highest competing bid—is $\beta_k^{\mathrm{II}}(Y_k)$. Thus,

$$m_k^{\mathrm{II}}(x) = \mathrm{Prob}\left[Y_k < x < Y_{k-1}\right] \times E\left[\beta_k^{\mathrm{II}}(Y_k) \mid Y_k < x < Y_{k-1}\right]$$

On the other hand, in the first-price format a winning bidder pays his own bid, so

$$m_k^{\mathrm{I}}(x) = \mathrm{Prob}\left[Y_k < x < Y_{k-1}\right] \times \beta_k^{\mathrm{I}}(x)$$

But since $\beta_k^{\mathrm{I}}(x) = E\left[\beta_{k+1}^{\mathrm{I}}(Y_k) \mid Y_k < x < Y_{k-1}\right]$, from (15.10), the fact that the kth period expected payments are equal implies that

$$\begin{aligned}
&\mathrm{Prob}\left[Y_k < x < Y_{k-1}\right] \times E\left[\beta_k^{\mathrm{II}}(Y_k) \mid Y_k < x < Y_{k-1}\right] \\
&= \mathrm{Prob}\left[Y_k < x < Y_{k-1}\right] \times E\left[\beta_{k+1}^{\mathrm{I}}(Y_k) \mid Y_k < x < Y_{k-1}\right]
\end{aligned}$$

Differentiating both sides of the equality with respect to x results in the identity

$$\beta_k^{\mathrm{II}}(x) = \beta_{k+1}^{\mathrm{I}}(x) \tag{15.11}$$

We have thus used the revenue equivalence principle to establish the following:

Proposition 15.3. *Suppose bidders have single-unit demand and K units are sold by means of sequential second-price auctions. Symmetric equilibrium strategies are given by*

$$\beta_K^{\mathrm{II}}(x) = x$$

and for all $k < K$,

$$\beta_k^{\mathrm{II}}(x) = \beta_{k+1}^{\mathrm{I}}(x)$$

where β_{k+1}^{I} *is the* $k+1$*st period equilibrium bidding strategy in the sequential first-price auction format, derived in Proposition 15.2.*

Example 15.2. *Values are uniformly distributed on* $[0, 1]$.

In the last period of a sequential second-price auction, it is a weakly dominant strategy to bid one's value, so

$$\beta_K(x) = x$$

Bidding strategies in earlier periods can be found using the strategies for the sequential first-price auction derived in Example 15.1 and applying the characterization obtained in Proposition 15.3. This results in

$$\beta_k(x) = \frac{N-K}{N-k}x$$

The bidding strategies are portrayed in Figure 15.2 for the case of three objects ($K = 3$) and five bidders ($N = 5$) and should be compared with the strategies for the sequential first-price auction in Figure 15.1. ▲

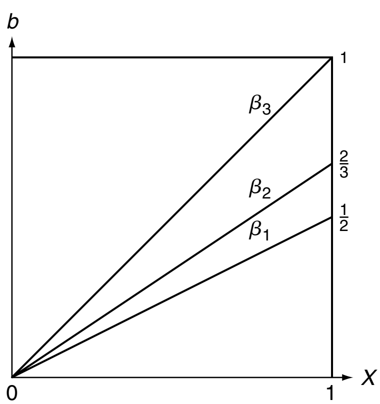

FIGURE 15.2 Equilibrium of sequential second-price auction.

Again, some properties of the equilibrium bidding strategies are worth noting. First, for all k, $\beta_k^{\mathrm{II}}(x) > \beta_k^{\mathrm{I}}(x)$—that is, every bidder bids more in a second-price sequential auction than in its first-price counterpart. Second, while it is a dominant strategy to bid one's value in the last period, this is not the case in earlier periods. This is because in any period $k < K$, there is an "option value" associated with not winning the current auction—the expected payoff arising from the possibility of winning an auction in some later period. In contrast to the case of a single object, the strategies in a sequential second-price auction are optimal only if other bidders also adopt them. Third, the equilibrium price process in a sequential second-price auction is also a martingale. Suppose bidder 1 with value x wins in period $K-1$. Then $Y_{K-1} < x < Y_{K-2}$ and the price in period $K-1$ is the realization of the random variable $P_{K-1} = \beta_{K-1}^{\mathrm{II}}(Y_{K-1})$. Let the realized price be $p_{K-1} = \beta_{K-1}^{\mathrm{II}}(y_{K-1})$, where y_{K-1} is the realized value of Y_{K-1}. In the last period the bidder with value $Y_{K-1} = y_{K-1}$ will win and the price in the last period will be $P_K = \beta_K^{\mathrm{II}}(Y_K) \equiv Y_K$, since it is weakly dominant to bid one's value in the last auction. Now

$$
\begin{aligned}
E[P_K \mid P_{K-1} = p_{K-1}] &= E[Y_K \mid Y_K < y_{K-1}] \\
&= \beta_{K-1}^{\mathrm{II}}(y_{K-1}) \\
&= p_{K-1}
\end{aligned}
$$

In earlier periods, the martingale property of prices in a sequential second-price auction is a consequence of the corresponding property in a sequential first-price auction and the relationship between the two equilibrium strategies in Proposition 15.3.

PROBLEMS

15.1. (Power distribution) Consider a situation in which two identical objects are to be sold to three interested bidders in two auctions conducted sequentially. Each bidder has use for at most one item—there is single-unit demand. Bidders' private values are identically and independently distributed according to the distribution $F(x) = x^2$ on $[0, 1]$.

a. Find a symmetric equilibrium bidding strategy if a sequential first-price format is used.

b. Find a symmetric equilibrium bidding strategy if a sequential second-price format is used.

c. Compare the distribution of prices in the two auctions under the first- and second-price formats.

15.2. (Multiunit demand) Consider a situation in which two identical objects are to be sold to two interested bidders in two second-price auctions conducted sequentially. Bidders have multiunit demand with values determined as follows. Each bidder draws two values Z_1 and Z_2 from the uniform distribution $F(z) = z$ on $[0,1]$. The bidder's value for the first unit is $X_1 = \max\{Z_1, Z_2\}$ and his marginal value for the second unit is $X_2 = \min\{Z_1, Z_2\}$. (This is just an instance the multiuse model discussed in Chapter 13.)

a. Show that the following strategy constitutes a symmetric equilibrium of the sequential second-price format:

i. in the first auction, bid $\beta_1\left(x_1^i, x_2^i\right) = \frac{1}{2}x_1^i$; and

ii. in the second auction, bid truthfully—that is, bidder i bids x_2^i if he won the first auction; otherwise he bids x_1^i.

b. Show that the sequence of equilibrium prices (P_1, P_2) is a *submartingale*—that is, $E[P_2 \mid P_1 = p_1] \geq p_1$ and with positive probability, the inequality is strict.

15.3. (Multiunit demand) Consider the same environment as in the previous problem.

a. Show that the following strategy *also* constitutes a symmetric equilibrium of the second-price format:

i. in the first auction, bid $\beta_1\left(x_1^i, x_2^i\right) = x_2^i$; and

ii. in the second auction, bid truthfully.

b. What can you say about the resulting sequence of equilibrium prices (P_1, P_2)?

CHAPTER NOTES

Equilibria of sequential auctions were first derived by Milgrom and Weber (1999) in a paper written in 1982, but published only recently. The martingale property of prices with symmetric independent private values was derived there. They also studied sequential auctions in the interdependent value model with affiliated signals, extending their work in the single-object case (see Chapter 6). With the latter specifications, the price process is no longer a martingale. Prices have a tendency to drift upward—that is, $E[P_{k+1} \mid P_k = p_k] > p_k$—and the reason is that now price announcements carry valuable information. These results have also been reported by Weber (1983).

There is some evidence that in real-world sequential auctions—art and wine auctions, in particular—the prices tend to drift downward (Ashenfelter, 1989). Because the theoretical models predict either a (stochastically) constant or increasing price path, this fact has been dubbed the "declining price anomaly." McAfee and Vincent (1993) explore the theoretical implications of risk aversion on the part of bidders in sequential first- and second-price auctions. They find that symmetric equilibria exist and prices decline over time, but only if bidders' risk aversion *increases* with wealth. Increasing risk aversion leads to declining prices because the bidder who wins in the first period, say, has a higher value than those of the remaining bidders, so this bidder is more risk averse. Even though prices are expected to decline in the future, his greater aversion to risk offsets the incentive to wait for a random future price, which is lower on average. But since we typically expect risk aversion to decrease with wealth, this does not seem like a particularly compelling explanation of declining prices in sequential auctions.

Jeitschko (1999) offers a somewhat different explanation. He considers situations in which the number of units to be sold is not known to the bidders beforehand. For instance, suppose that the number of units to be sold is at most two but at the time of the first auction bidders are unsure whether or not a second unit will be sold at all. This means that the expected profits from the now uncertain second auction are lower than they would be if a second item were to be sold for sure. Arbitrage now drives up the price in the first auction, so prices decline. But this explanation, while interesting, is only partial at best. Declining prices are observed even when the number of units to be sold is announced beforehand, so there is no uncertainty in this regard. See, for example, the account of the auction for 24 satellite transponder leases reported in Weber (1983).

The results of this chapter were all derived under the restrictive assumption that all buyers have *single-unit demand*. A full treatment of sequential auctions with multiunit demands is problematic for reasons that are by now familiar—once a particular bidder has won the first unit his behavior and interests are different from those of the other bidders. In other words, even if bidders are symmetric *ex ante* multiunit demands introduce asymmetries in later auctions. There is one tractable case, however, as pointed out by Katzman (1999). Suppose only *two* units are for sale and these are sold by means of two second-price auctions. In the first stage, bidders are symmetric. In the second and last stage, even though they are no longer symmetric, each bidder has a dominant strategy to bid his or her value since no considerations regarding the future are necessary. Katzman (1999) is thus able to find a symmetric equilibrium of the two-stage second-price auction when bidders have multi- (that is, two-) unit demand. Problems 15.2 and 15.3 are based on his paper.