Chapter | Thirteen

Equilibrium and Efficiency with Private Values

The previous chapter outlined the workings of some auction formats for the sale of multiple units of the same good. These formats differed in the manner in which prices at which the units were sold were determined, and we now examine how the different pricing rules affect bidding behavior. As in the case of a single object, we begin by considering the different auctions in a model where bidders' values are private and independently distributed, and again as a first step, we assume that bidders are *ex ante* symmetric.

13.1 THE BASIC MODEL

There are K identical objects for sale and N potential buyers are bidding for these. Bidder i's valuation for the objects is given by a private *value vector* $\mathbf{X}^i = \left(X_1^i, X_2^i, \ldots, X_K^i\right)$, where X_k^i represents the *marginal* value of obtaining the kth object. The total value to the bidder of obtaining exactly $k \leq K$ objects is then the sum of the first k marginal values: $\sum_{l=1}^{k} X_l^i$. It is assumed that the marginal values are declining in the number of units obtained so that $X_1^i \geq X_2^i \geq \cdots \geq X_K^i$. Bidders are assumed to be risk neutral.

Bidders are symmetric—each $\mathbf{X}_i$ is independently and identically distributed on the set

$$\mathcal{X} = \left\{\mathbf{x} \in [0, \omega]^K : \forall k, x_k \geq x_{k+1}\right\} \tag{13.1}$$

according to the density function f.

In the previous chapter we inverted the vector of bids $\mathbf{b}^i$ submitted by a bidder to obtain the demand function d^i submitted by him (see (12.1)). We can similarly invert each bidder's valuation vector $\mathbf{x}^i$ to obtain his "true" demand function δ^i defined by

$$\delta^i(p) \equiv \max\left\{k : p \leq x_k^i\right\} \tag{13.2}$$

Sometimes it will be useful to think of each bidder as drawing a demand function at random.

Some special cases of the preceding model—involving restrictions on the probability distribution that values are drawn from—are of interest.

LIMITED DEMAND MODEL

It may be that even though K units are being sold, each bidder has use for at most $L < K$ units. In that case, the support of f is the set

$$\mathcal{X}(L) = \{\mathbf{x} \in \mathcal{X} : \forall k > L, x_k = 0\}$$

so there is no value derived from obtaining more than L units. Value vectors are of the form $(x_1, x_2, \ldots, x_L, 0, 0, \ldots, 0)$, and we then suppose that each bidder submits a bid $\mathbf{b}^i$, which is also an L vector. In an extreme instance, each bidder has use for only one unit ($L = 1$), and we will refer to this case as one of *single-unit demand*. If bidders value more than one unit with positive probability, then we will refer to that as the case of *multiunit demand*. The single-unit demand model is of interest because equilibrium behavior there is analogous to equilibrium behavior in auctions where only a single object is sold and demanded. As we will see, this is not true with multiunit demand.

MULTIUSE MODEL

A second, analytically useful, restriction on the form of the density function f occurs if the value vector $\mathbf{X}$ consists of order statistics of independent draws from some underlying distribution. Specifically, suppose that each bidder draws $L \leq K$ values $Z_1, Z_2, \ldots, Z_L$ independently from some distribution F, and it is useful to think of these as the values derived from the object in different uses. If he obtains only one unit, then it is used in the best way possible, so his value for the first unit is $X_1 = \max\{Z_1, Z_2, \ldots, Z_L\}$. If he obtains a second unit, it is put to the second-best use possible, so the marginal value of the second unit, X_2, is the second-highest of $\{Z_1, Z_2, \ldots, Z_L\}$. The marginal value of the third unit, X_3, is the third-highest of $\{Z_1, Z_2, \ldots, Z_L\}$, and so on.

We now turn to an examination of equilibrium bidding behavior in the three sealed-bid auction formats outlined in the previous chapter: the discriminatory, uniform-price, and Vickrey auctions. Since we consider a private values environment, it is the case that any equilibrium in the sealed-bid environment is outcome equivalent to an equilibrium of the corresponding open auction. Thus, any equilibrium in the discriminatory auction is equivalent to an equilibrium in the multiunit Dutch auction, any equilibrium in the uniform-price auction to one in the multiunit English auction, and any equilibrium in the Vickrey auction to one in the Ausubel auction.

The Vickrey auction is the simplest from a strategic standpoint, so we begin our analysis there. Next we turn to the uniform-price and discriminatory formats.

13.2 VICKREY AUCTIONS

Recall that in a *Vickrey* auction each bidder submits a K-vector of bids $\mathbf{b}^i$, and the K highest bids are awarded units. The total amount paid by a bidder who is awarded k^i units is

$$\sum_{k=1}^{k^i} c^{-i}_{K-k^i+k}, \tag{13.3}$$

where $\mathbf{c}^{-i}$ is the K-vector of competing bids obtained by rearranging in decreasing order the $(N-1)K$ bids b^j_k, of bidders j other than i, and selecting the first K of these. Recall that the residual supply function facing bidder i, denoted by s^{-i}, can be obtained from $\mathbf{c}^{-i}$ as in (12.3). It is useful to think of

$$p^i_k \equiv c^{-i}_{K-k^i+k}$$

as the price bidder i pays for the kth unit. Notice that, by definition, $p^i_1 \leq p^i_2 \leq \cdots \leq p^i_{k^i}$.

Just as it is a weakly dominant strategy to bid one's value in a second-price auction of a single object, it is a weakly dominant strategy to "bid one's true demand function" in a multiunit Vickrey auction. Figure 13.1 illustrates why, for instance, it does not pay bidder 1 to submit a demand function d^1 that lies below his true demand function δ^1. When he reports truthfully, bidder 1 obtains

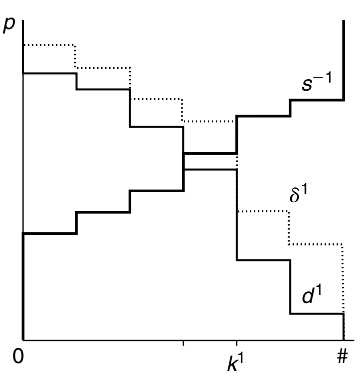

FIGURE 13.1 Dominant strategy property of Vickrey auction.

$k^1 = 4$ units. If he were to report d^1, he would win only three units forgoing some surplus on the fourth unit. More generally, we have

Proposition 13.1. *In a Vickrey auction, it is a weakly dominant strategy to bid according to $\boldsymbol{\beta}^{\mathrm{V}}(\mathbf{x}) = \mathbf{x}$.*

Proof. Consider bidder i and the bids $\mathbf{b}^{-i}$ submitted by the other bidders. As before, let $\mathbf{c}^{-i}$ be a vector consisting of the K highest bids of the other bidders. Suppose further that when bidder i submits a bid $\mathbf{b}^i = \mathbf{x}^i$, he is awarded k^i units. According to the Vickrey pricing rule, his payment is given by (13.3) and it is the case that for all $k \leq k^i$, $x^i_k \geq c^{-i}_{K-k^i+k} = p^i_k$, whereas for all $k > k^i$, $x^i_k \leq c^{-i}_{K-k^i+k} = p^i_k$.

Now suppose bidder i were to submit a bid vector $\mathbf{b}^i \neq \mathbf{x}^i$ such that he is awarded the same number of units as when he submitted his true value vector $\mathbf{x}^i$; then the prices he pays for these units would be unaffected, as would his overall surplus—the total value less the sum of the prices paid.

If bidder i were to submit a $\mathbf{b}^i \neq \mathbf{x}^i$ such that he is awarded a greater number of units—say, $l^i > k^i$—than if he were to submit his true value vector $\mathbf{x}^i$, then the prices he would pay for the first k^i units would be unchanged and, therefore, so would the surplus derived from these. For any unit $k > k^i$, however, the price p^i_k exceeds (or, at best equals) the kth marginal value x^i_k, so the surplus from these $l^i - k^i$ units would be negative (or at best, zero). As a result, the overall surplus would be lower (or at best, the same) than that if he were to bid truthfully.

Finally, if bidder i were to submit a $\mathbf{b}^i \neq \mathbf{x}^i$ such that he is awarded a smaller number of units, say $l^i < k^i$, as when he submitted his true value vector $\mathbf{x}^i$, then the prices he would pay for the first l^i units would be unchanged and therefore so would the surplus derived from these. But the surplus from any unit $k < k^i$ was positive and is now forgone. Thus, by winning fewer units bidder 1's overall surplus would be lower than if he were to bid truthfully. ■

It is important to observe that nowhere in the proof did we make use of the assumption that bidders were symmetric—Proposition 13.1 continues to hold even if bidders are asymmetric. An immediate consequence of the dominant strategy nature of the Vickrey auction is that the K objects are awarded in an efficient manner—they are awarded to the K highest values x^i_k. For future reference we record this observation as follows:

Proposition 13.2. *The Vickrey auction allocates the objects efficiently.*

The efficiency property of the Vickrey auction also extends to its open ascending-price counterpart, the Ausubel auction: It is an equilibrium strategy for a bidder to reduce his demand according to his *true demand function* δ^i, obtained by inverting his value vector $\mathbf{x}^i$. The resulting allocation is always efficient.

In the private value context, Vickrey auctions thus inherit the most important property of second-price auctions: It is a dominant strategy to bid truthfully and as a result, the allocations are efficient. While Vickrey auctions are always efficient, in some circumstances the outcome of a Vickrey auction may be deemed to be unfair. This is seen most easily in the context of a simple example. Suppose that there are two bidders with values $\mathbf{x}^1 = (10, 6)$ and $\mathbf{x}^2 = (9, 2)$. In a Vickrey auction each bidder bids his value vector so that each wins one unit. But notice that bidder 1 pays only $x_2^2 = 2$ for the unit he wins, whereas bidder 2 pays $x_2^1 = 6$. Thus, while bidder 1 attaches higher values to both units than does bidder 2, and indicates this by bidding truthfully, he ends up paying less than what bidder 2 pays. More generally, suppose there are two bidders i and j such that $\mathbf{x}^i \geq \mathbf{x}^j$ and the bids are such that i and j win the same number of units, say k. The vector of competing bids $\mathbf{c}^{-j}$ that j faces is at least as large as the vector of competing bids $\mathbf{c}^{-i}$ that i faces, so the amount that j pays for the k units that he wins is at least as large as the amount that i pays. Thus, in a Vickrey auction, if two bidders win the same number of units, then the one who indicates a willingness to pay more than the other will actually pay less.

What can be said about equilibrium behavior in the uniform-price and discriminatory auctions? Before addressing this question, we take a slight detour to examine the question of efficiency in general.

13.3 EFFICIENCY IN MULTIUNIT AUCTIONS

The requirement of efficiency restricts the form that equilibrium strategies can take. Consider any multiunit auction format in which bidders submit bid vectors $\mathbf{b}^i$ and the objects are awarded to the K highest bids—that is, a *standard auction*. As we have seen, the discriminatory, uniform-price, and Vickrey auctions fall under this label.

Suppose that all bidders' value vectors $\mathbf{X}^i$ lie in the set $\mathcal{X}$ defined in (13.1) and that the density of $\mathbf{X}^i$, denoted by f, has full support. In any standard auction, the bidding strategy of a bidder, say i, is a function of the form $\boldsymbol{\beta}^i : \mathcal{X} \to \mathbb{R}_+^K$ satisfying, for all k, $\beta_k^i(\mathbf{x}^i) \geq \beta_{k+1}^i(\mathbf{x}^i)$.

Consider an equilibrium of some standard auction $(\boldsymbol{\beta}^1, \boldsymbol{\beta}^2, \ldots, \boldsymbol{\beta}^N)$ and a particular realization of bidders' values $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N$. In a standard auction, the K units will be awarded to the K highest of the $N \times K$ bids $\beta_k^i(\mathbf{x}^i)$, whereas efficiency demands that the K units be awarded to the K highest of the $N \times K$ marginal values x_k^i. For the equilibrium to allocate efficiently for *every* realization of the values, the ranking of the $N \times K$ bids $\beta_k^i(\mathbf{x}^i)$ must agree with the ranking of the $N \times K$ values x_k^i. In other words, efficiency requires that for all i,j and k,l,

$$x_k^i > x_l^j \text{ if and only if } \beta_k^i(\mathbf{x}^i) > \beta_l^j(\mathbf{x}^j) \tag{13.4}$$

The requirement in (13.4) has two implications. First, it must be that bidder i's bid on the kth object $\beta_k^i(\mathbf{x}^i)$ cannot depend on the value of, say, the lth object, x_l^i where $l \neq k$. Otherwise, with positive probability there are situations in which $\beta_k^i(\mathbf{x}^i)$ is the Kth highest of all the bids and some other bidder, say j, submits a bid

$\beta^j_{k'}(\mathbf{x}^j)$, which is the $K+1$st highest bid and this is just below $\beta^i_k(\mathbf{x}^i)$. Now there exists a change in x^i_l to $x^i_l - \varepsilon$ that affects i's bid so $\beta^i_k\left(\mathbf{x}^i_{-l}, x^i_l - \varepsilon\right) < \beta^j_{k'}(\mathbf{x}^j)$, but ε is small enough so that the efficient allocation is unaffected. For such a change, the equilibrium allocation would be inefficient. Thus, we have argued that an implication of efficiency is that bidding strategies must be *separable*—the bid on the kth object can only depend on the kth marginal valuation.

A second implication is that the different components of the bidding strategy must be *symmetric* across both bidders and objects—that is, for all i,j and k,l, $\beta^i_k(\cdot) = \beta^j_l(\cdot)$. Otherwise, with positive probability there are situations in which the allocation will be inefficient; and this is for much the same reason why a first-price auction may not allocate a single object efficiently when there are asymmetries (see Chapter 4). In particular, if $i \neq j$, then there will be situations in which $x^i_k > x^j_l$, but $\beta^i_k\left(x^i_k\right) < \beta^j_l\left(x^j_l\right)$ and bidder i does not win an object he should have won on grounds of efficiency.

Together these imply that if an equilibrium is efficient then the bidding strategies must map values into bids using a single increasing function. The converse is also true—if values are mapped into bids using a single increasing function, then from (13.4) the equilibrium must be efficient. Finally, note that in a Vickrey auction, bids equal values, so values are mapped into bids using the identity function.

We summarize our findings as follows:

Proposition 13.3. *An equilibrium of a standard auction is efficient if and only if the bidding strategies are separable and symmetric across both bidders and objects—that is, there exists an increasing function β such that for all i and k,*

$$\beta^i_k\left(\mathbf{x}^i\right) = \beta\left(x^i_k\right)$$

13.4 UNIFORM-PRICE AUCTIONS

We noted in the previous chapter that both the Vickrey and the uniform-price auctions were multiunit extensions of the single-unit second-price auction. As we have seen, the Vickrey auction inherits the dominant strategy property from the second-price auction and delivers efficient allocations. What of the uniform-price auction? This section explores the strategic properties of the uniform-price auction and finds that, in general, the uniform-price auction does not inherit the dominant strategy property of the second-price auction. In fact, the conditions required by Proposition 13.3 fail; as a result, the uniform-price auction is generally inefficient.

We begin by noting that in the independent private values setting studied here the uniform-price auction is known to have a pure strategy equilibrium. But a closed form expression for the strategies is not available, so we proceed indirectly. Rather than explicitly calculating equilibrium strategies—a difficult task even in specific examples—we will instead deduce some structural features that any equilibrium must have. These will then allow us to infer some important economic properties of the auction.

Recall that in a uniform-price auction, bidder i wins exactly $k^i > 0$ units if and only if

$$b^i_{k^i} > c^{-i}_{K-k^i+1} \quad \text{and} \quad b^i_{k^i+1} < c^{-i}_{K-k^i}$$

where $\mathbf{c}^{-i}$ is the vector of competing bids facing bidder i. The highest losing bid—the price at which all units are sold—is then just

$$p = \max\left\{b^i_{k^i+1}, c^{-i}_{K-k^i+1}\right\}$$

Figure 13.2 illustrates the workings of the uniform-price auction when there are only two units for sale and bidder 1 submits a bid vector $\mathbf{b} = (b_1, b_2)$ (superscripts are omitted). If $b_2 > c_1$, and this occurs in the small triangular region to the left, bidder 1 wins both units and the price is c_1. If $b_1 < c_2$, and this occurs in the small triangular region to the right, bidder 1 does not win any units. In the remaining region he wins one unit and the price paid is $\max\{b_2, c_2\}$.

We begin with some simple observations regarding equilibrium bidding behavior. First, the bids cannot exceed marginal values—that is, for all i and k, $b^i_k \le x^i_k$. To see this, suppose that some bidder i bids an amount $b^i_k > x^i_k$. We claim that this is weakly dominated by the strategy of bidding $b^i_k = x^i_k$ (and if there is another bid, say b^i_{k+1}, such that $b^i_k \ge b^i_{k+1} > x^i_k$, then this is also reduced to x^i_k). If $b^i_k = p$, the price at which the units are sold, then bidder i is winning exactly $k-1$ units and reducing his bid to x^i_k can only improve his profits by possibly decreasing the price. If $b^i_k < p$, then reducing this bid to x^i_k makes no

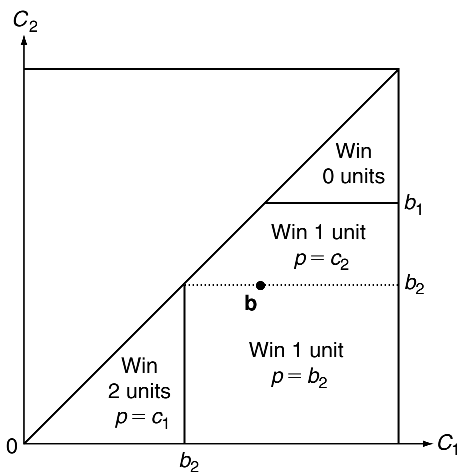

FIGURE 13.2 Outcomes in a uniform-price auction with two units for sale.

difference. If $b_k^i > p > x_k^i$, then bidder i is making a loss on at least one unit and decreasing b_k^i to x_k^i will reduce his loss; he will no longer win the units for which the price exceeded the marginal value. If $b_k^i > x_k^i > p$, then reducing his bid to x_k^i again makes no difference since he would still win the kth unit at the same price.

Second, the bid on the *first* unit must be the same as its value. Precisely, any strategy that calls upon a bidder to submit a bid $b_1^i < x_1^i$ is weakly dominated by a strategy in which $b_1^i = x_1^i$. To see this, suppose that $b_1^i < x_1^i$. If $p \geq x_1^i > b_1^i$, then bidder i is not winning any objects—all his bids are below the market-clearing price—and this would not change if he were to raise b_1^i to x_1^i. If $x_1^i > p \geq b_1^i$, then again bidder i is not winning any objects, but if he were to raise b_1^i to x_1^i, then he may win a unit and at a price that would be profitable. Finally, if $x_1^i > b_1^i > p$, then raising his bid to x_1^i makes no difference. Thus, we have argued that in a uniform-price auction it is a weakly dominant strategy for a bidder to bid truthfully for the first unit. Put another way, bidders do not have any incentive to shade their bids b_1^i for the first unit.

Bidders do have the incentive to shade their bids $b_2^i, b_3^i, \ldots, b_K^i$ for additional units, however, and this feature distinguishes the uniform-price auction from the Vickrey auction; in the latter, there is no incentive to shade bids for *any* of the units. Submitting a vector $\mathbf{b}^i \leq \mathbf{x}^i$, $\mathbf{b}^i \neq \mathbf{x}^i$ is equivalent to submitting a demand function d^i such that for some prices p the amount demanded $d^i(p)$ is lower than the true demand $\delta^i(p)$ (see (12.1) and (13.2)). Bid shading is thus sometimes referred to as *demand reduction*.

13.4.1 Demand Reduction

Let us look more closely at the case when there are only two units for sale—that is, $K = 2$. Further, suppose that the density of values f has full support on the set $\mathcal{X}$ defined in (13.1). Fix a symmetric equilibrium of the uniform-price auction $\boldsymbol{\beta} = (\beta_1, \beta_2)$ that satisfies $\boldsymbol{\beta}(\mathbf{0}) = \mathbf{0}$. Suppose that all bidders other than bidder 1, say, follow $\boldsymbol{\beta}$. Suppose further that the marginal values bidder 1 assigns to the two units are given by $\mathbf{x} = (x_1, x_2)$ and bidder 1 bids $\mathbf{b} = (b_1, b_2)$. (Bidder indices are omitted so that we write b_k instead of b_k^1, and so on.) Let $\mathbf{c} = (c_1, c_2)$ be the competing bids facing bidder 1 and suppose that the distribution of the random variable $\mathbf{C}$ has a density on the set $\mathcal{X}$ given by $h(\cdot)$. Bidder 1's expected payoff is then given by

$$\begin{aligned} \Pi(\mathbf{b}, \mathbf{x}) = & \int_{\{\mathbf{c}: c_1 < b_2\}} (x_1 + x_2 - 2c_1)\, h(\mathbf{c})\, \mathbf{dc} \\ & + \int_{\{\mathbf{c}: c_2 < b_1 \text{ and } c_1 > b_2\}} (x_1 - \max\{b_2, c_2\})\, h(\mathbf{c})\, \mathbf{dc} \end{aligned}$$

The first term is bidder 1's payoff when he wins both units and the second term is his payoff when he wins only one unit. (See Figure 13.2.)

Let H_1 denote the marginal distribution of the higher competing bid C_1 and H_2 that of the lower competing bid C_2 with densities h_1 and h_2, respectively. Thus, $H_1(b_2) = \text{Prob}[C_1 < b_2]$ is the probability that bidder 1 will defeat both competing bids and win two units. Similarly, $H_2(b_1) = \text{Prob}[C_2 < b_1]$ is the probability that he will defeat the lower competing bid, so win *at least* one unit. The probability that he will win *exactly* one unit is then the difference $H_2(b_1) - H_1(b_2)$. Also, $H_2(b_2) - H_1(b_2) = \text{Prob}[C_2 < b_2 < C_1]$ is the probability that the highest losing bid—the price at which the units are sold—is b_2. Using these facts, bidder 1's expected payoff can be rewritten as

$$\begin{aligned}\Pi(\mathbf{b},\mathbf{x}) &= H_1(b_2)(x_1+x_2) - 2\int_0^{b_2} c_1 h_1(c_1)\,dc_1 \\ &\quad + [H_2(b_1) - H_1(b_2)]x_1 \\ &\quad - [H_2(b_2) - H_1(b_2)]b_2 - \int_{b_2}^{b_1} c_2 h_2(c_2)\,dc_2\end{aligned}$$

(Again, it may help to refer to Figure 13.2.)

Differentiating with respect to b_2 results in

$$\frac{\partial \Pi}{\partial b_2} = h_1(b_2)(x_2 - b_2) - [H_2(b_2) - H_1(b_2)] \tag{13.5}$$

and when $b_2 = x_2$, we determine that

$$\left.\frac{\partial \Pi}{\partial b_2}\right|_{b_2 = x_2} = -[H_2(x_2) - H_1(x_2)] < 0$$

since H_1 stochastically dominates H_2.

We have thus argued that a bidder can increase his payoff by shading his bid for the second unit—that is, the equilibrium bid for the second unit must be such that $b_2 < x_2$.

It is instructive to scrutinize the incentive to shade the bid for the second unit more closely. An increase in b_2 has two effects on a bidder's payoff. First, it increases the likelihood that the bidder will win the second object. To do this, b_2 must exceed the highest competing bid, c_1, and a small increase in b_2 raises the likelihood by $h_1(b_2)$. The gain from winning the second unit is just $(x_2 - b_2)$, so the first term in (13.5) represents the gain in expected payoff from raising the bid on the second unit slightly. The second effect is that an increase in b_2 raises the expected payment on the *first* unit (even though it does not affect the chances of winning it). This is because, with probability $H_2(b_2) - H_1(b_2)$, the amount bid on the second unit is the highest losing bid and hence determines the price paid for the first unit. The second term in (13.5) represents the resulting loss. When b_2 is close to x_2, the second effect dominates, so the bidder has an incentive to shade his bid.

In a uniform-price auction, the shading of bids for units other than the first—demand reduction—results from the fact that, with positive probability, every bid other than that for the first unit may determine the price paid on all units. In other words, a bidder's own bids influence the price he pays. By contrast, in a Vickrey auction, a bidder's own bids determine how many units he wins but have no influence on the prices paid—each unit is purchased at a competing bid.

This situation occurs in the example presented in the previous chapter and is depicted in the middle panel of Figure 12.3 on page 177. The highest losing bid, and thus the market-clearing price, is bidder 1's bid for the fourth unit, b_4^1, and thus his own bid determines the amount he pays for the three units that he wins.

Two aspects of our analysis were special. First, we examined only the case when the number of units was two. The argument for demand reduction is quite general—considering more units only adds notational complexity—and applies no matter how many units are sold. Thus, no matter what K is, $\beta_1^i\left(x_1^i\right) = x_1^i$ and for all $k > 1$, $\beta_k^i\left(x_k^i\right) < x_k^i$. Figure 13.3 is a schematic portrayal of demand reduction when the number of units for sale is greater than two. Second, we assumed that the distribution of the competing bids facing a bidder admitted a density $h(\mathbf{c})$, so the distribution of bids did not have any mass points. This is not entirely innocuous since it rules out the possibility that for some open set of value vectors, the bids on some units are constant. It turns out, however, that the demand reduction occurs even if the bidding strategies have mass points.

Proposition 13.4. *Every undominated equilibrium of the uniform-price auction has the property that the bid on the first unit is equal to the value of the first unit. Bids on other units are lower than the respective marginal values.*

Since the bidding strategies associated with different units are different—there is no shading on the first unit but there is on other units—by applying Proposition 13.3 we obtain the following:

Proposition 13.5. *Every undominated equilibrium of the uniform-price auction is inefficient.*

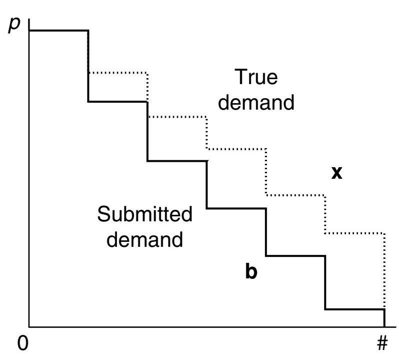

FIGURE 13.3 Demand reduction in a uniform-price auction.

Demand reduction (or bid shading) can be severe when the number of bidders is small relative to the number of units for sale. An extreme case of this phenomenon occurs in the following example.

Example 13.1. *There are two units for sale and two bidders with value vectors* $\mathbf{X}$ *that are identically and independently distributed according to the density function* $f(x) = 2$ *on* $\mathcal{X} = \{\mathbf{x} \in [0,1]^2 : x_1 \geq x_2\}$.

With this distribution of values, a symmetric equilibrium of the uniform-price auction is $\beta_1(x_1, x_2) = x_1$ and $\beta_2(x_1, x_2) = 0$. In every realization, each bidder wins one unit and the price is zero!

To see this, suppose that bidder 2 is following the strategy β as specified and bidder 1 with value vector $\mathbf{x}$ submits a bid vector $\mathbf{b} = (b_1, b_2) \gg \mathbf{0}$. As usual, let $\mathbf{c}$ denote the competing bids facing bidder 1—in this case, these are just the bids submitted by bidder 2—and we know that $\mathbf{c} = (y_1, 0)$ where $\mathbf{y}$ is bidder 2's value vector. Since $c_2 = 0$, bidder 1 is sure to win at least one unit. He wins only one unit if his low bid is a losing bid—that is, if $y_1 > b_2$ and in that case he pays b_2. He wins both units if his low bid of b_2 exceeds the high bid of bidder 2, that is, if $y_1 < b_2$, and in that case the price he pays for each unit is y_1.

Let F_1 be the marginal distribution of X_1 with corresponding density f_1. Since $f(\mathbf{x}) = 2$ on $\mathcal{X}$, we know that $F_1(x_1) = (x_1)^2$ and $f_1(x_1) = 2x_1$ on the interval $[0,1]$.

Bidder 1's expected payoff from bidding $\mathbf{b}$ when his values are $\mathbf{x}$ is simply

$$\begin{aligned} \Pi(\mathbf{b}, \mathbf{x}) &= F_1(b_2)(x_1 + x_2) - 2\int_0^{b_2} y_1 f_1(y_1)\, dy_1 \\ &\quad + (1 - F_1(b_2))(x_1 - b_2), \end{aligned}$$

where the first term is the payoff from winning both units and the second from winning only one unit.

Differentiating with respect to b_2 we obtain

$$\begin{aligned} \frac{\partial \Pi}{\partial b_2} &= f_1(b_2)(x_2 - b_2) - 1 + F_1(b_2) \\ &= 2b_2(x_2 - b_2) - 1 + (b_2)^2 \\ &\leq -(x_2 - b_2)^2 \\ &\leq 0 \end{aligned}$$

and this is strictly negative whenever $b_2 < x_2$. So it is optimal to set $b_2 = 0$ whatever the value of x_2.

In the example, demand reduction is so extreme that the equilibrium price is always zero and the two units are always split between the two bidders regardless of their values. The resulting inefficiency is clear. Finally, note that low revenue equilibria of this sort cannot arise if the number of bidders exceeds the number of units for sale. ▲

13.4.2 Single-Unit Demand

The inefficiency of the uniform-price auction does not result from the fact that multiple units are sold *per se* but rather from the fact that multiple units are demanded. Consider a situation in which $K > 1$ units are up for sale but each bidder has use for at most one unit—the case of single-unit demand. This is equivalent to supposing that the value vectors are drawn from the set

$$\mathcal{X}(1) = \{\mathbf{x} \in [0, \omega]^K : \forall k > 1, x_k = 0\}$$

Thus, f no longer has full support on $\mathcal{X}$. We already know that in a uniform-price auction it is weakly dominant to bid $b_1 = x_1$, and since the value of all additional units is zero, each bidder is bidding truthfully. Put another way, if all bidders have unit demand, there is no possibility that a winning bidder will influence the price paid, and hence there is no incentive for demand reduction. The upshot of this is that with single-unit demand the uniform-price auction is efficient.

13.5 DISCRIMINATORY AUCTIONS

Recall that in a discriminatory or "pay-your-bid" auction, a bidder who is awarded $k^i \leq K$ units pays the sum of his first k^i bids, $b_1^i + b_2^i + \cdots + b_{k^i}^i$.

Once again, we note that pure strategy equilibria are known to exist in a discriminatory auction when bidders have independent private values. When bidders are symmetric, as assumed in this chapter, a symmetric equilibrium is known to exist. No explicit characterization of the strategies is available, however, so, as in the previous section, we proceed indirectly by deducing properties that any equilibrium must satisfy.

It is obvious that there will be demand reduction in a discriminatory auction—to bid $b_k^i = x_k^i$ would ensure only that there is no gain from winning the kth object. To further understand the nature of equilibrium bids, let us look more closely at the two unit and two bidder case. Fix a symmetric equilibrium (β_1, β_2) of the discriminatory auction. First, notice that if the highest amount ever bid on the second unit is $\overline{b} = \max \beta_2(\mathbf{x})$, then it makes no sense for a bidder to bid more than $\overline{b}$ on the first unit. This is because any bid b_1 on the first unit that is greater than $\overline{b}$ will win with probability 1 and the bidder could do better by reducing it slightly. Thus, we have that in equilibrium

$$\max_{\mathbf{x}} \beta_1(\mathbf{x}) = \overline{b} = \max_{\mathbf{x}} \beta_2(\mathbf{x}) \tag{13.6}$$

Second, consider a particular bidder and let the random variable $\mathbf{C} = (C_1, C_2)$ denote the competing bids—that is, the bids of the other bidder. Let H_1 denote the marginal distribution of a bidder's high bid C_1 and let H_2 denote the marginal distribution of the other bidder's low bid C_2. Thus,

$$H_k(c) = \text{Prob}[\beta_k(\mathbf{X}) \leq c]$$

Since for all $\mathbf{x}$, $\beta_1(\mathbf{x}) \geq \beta_2(\mathbf{x})$, it is clear that the distribution H_1 stochastically dominates the distribution H_2. As usual, let h_1 and h_2 denote the corresponding densities.

Suppose a bidder has values (x_1, x_2) and bids (b_1, b_2). He wins both units if $C_1 < b_2$ and the probability of this event is $H_1(b_2)$. He wins exactly one unit if $C_2 < b_1$ and $C_1 > b_2$ and the probability of this event is $H_2(b_1) - H_1(b_2)$. Thus, the expected payoff is

$$\begin{aligned}\Pi(\mathbf{b}, \mathbf{x}) &= H_1(b_2)(x_1 + x_2 - b_1 - b_2) \\ &\quad + [H_2(b_1) - H_1(b_2)](x_1 - b_1) \\ &= H_2(b_1)(x_1 - b_1) + H_1(b_2)(x_2 - b_2) \end{aligned} \tag{13.7}$$

The bidder's optimization problem is to choose $\mathbf{b}$ to maximize $\Pi(\mathbf{b}, \mathbf{x})$ subject to the constraint that $b_1 \geq b_2$.

When the constraint $b_1 \geq b_2$ does not bind at the optimum, so $b_1 > b_2$, the first-order conditions for an optimum are

$$h_2(b_1)(x_1 - b_1) = H_2(b_1) \tag{13.8}$$

$$h_1(b_2)(x_2 - b_2) = H_1(b_2) \tag{13.9}$$

Thus, we deduce that whenever $b_1 > b_2$, the bids are completely *separable* in the values—that is, β_1 does not depend on x_2 and β_2 does not depend on x_1.

When the constraint $b_1 \geq b_2$ binds at the optimum, so that $b_1 = b_2 \equiv b$, the first-order condition is

$$h_2(b)(x_1 - b) + h_1(b)(x_2 - b) = H_2(b) + H_1(b) \tag{13.10}$$

In this case, the bidder submits a "flat demand" function—bidding the same amount for the each of the two units. An examination of (13.7) reveals that if it is optimal to submit the flat demand bid b for the value vector $\mathbf{x} = (x_1, x_2)$ and also for the value vector $\mathbf{z} = (z_1, z_2)$, then it is optimal to submit the same flat demand bid b for any convex combination of the values $\lambda\mathbf{x} + (1 - \lambda)\mathbf{z}$, where $0 \leq \lambda \leq 1$.

13.5.1 Structure of Equilibria

Once again, we deduce some structural properties of equilibrium strategies without deriving the strategies explicitly. Indeed, unlike in the case of a single object, even if bidders are symmetric no closed form expression for the bidding strategies is available. This is because even symmetric bidders value different units of the same object differently. For example, when there are two units for sale to two bidders, bids on the first unit compete with rival bids on the second unit. Since the marginal distributions of the values are different, the gains and losses from, say, bidding higher on the first unit are different from those from bidding higher on the second unit.

Indeed, when the constraint $b_1 \geq b_2$ is not binding, the relevant first-order conditions, (13.8) and (13.9), that determine bidding strategies in a symmetric multiunit auction are the *same* as those that determine bidding strategies in an asymmetric single-object auction (see Chapter 4, especially (4.17) on page 46). The only difference is notational—the subscripts now index units—reflecting the fact that asymmetries across bidders have been replaced by asymmetries across different units of the same good. The two problems are not isomorphic, however. In the discriminatory auction, the constraint $b_1 \geq b_2$ binds some of the time—bidders submit flat demands with positive probability.

To see why, suppose for the moment that F_1 dominates F_2 in terms of the reverse hazard rate—that is, for all x,

$$\frac{f_1(x)}{F_1(x)} > \frac{f_2(x)}{F_2(x)}$$

Suppose that (β_1, β_2) is a solution to the differential equations resulting from (13.8) and (13.9), that is, if we neglect the constraint that $b_1 \geq b_2$. Then we can deduce that for all $x \in (0, \omega)$, $\beta_2(x) > \beta_1(x)$, as shown in Figure 13.4. This is because Proposition 4.4 on page 47 applies to this situation unchanged—only the meaning of the subscripts is different—so the bids on the second unit will be more aggressive than bids on the first unit. But now for every $x_1 \in (0, \omega)$, if $x_2 > y_2 \equiv \phi_2(\beta_1(x_1))$, where ϕ_2 is the inverse of β_2, the constraint $b_1 \geq b_2$ surely binds and bidders must submit flat demands when that happens. Thus, we conclude that *bidders submit flat demands with positive probability*.

The assumption that F_1 dominates F_2 in terms of the reverse hazard rate is satisfied in the multi-use model outlined earlier. Suppose that each bidder first draws two values Z_1 and Z_2 independently from some distribution F—the

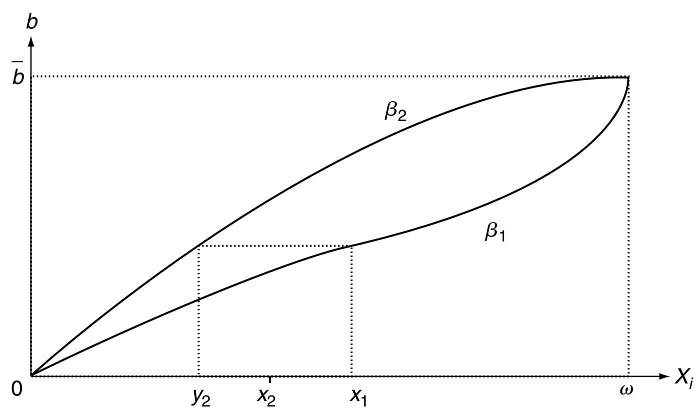

FIGURE 13.4 Illustration of the necessity of flat demands.

values of the object in two different uses. If he obtains only one unit, then he uses it in the best way possible, so his value for the first unit is $X_1 = \max\{Z_1, Z_2\}$. The marginal value of the second unit is then $X_2 = \min\{Z_1, Z_2\}$. With this specification, F_1 and F_2 are just distributions of the highest and second-highest order statistics, respectively, and it is routine to verify that F_1 dominates F_2 in terms of the reverse hazard rate.

At the same time, it cannot be optimal for bidders to *always* submit flat demands. In particular, when the values (x_1, x_2) are close to $(\omega, 0)$, then the only possible flat demand bid (b, b) is close to $(0, 0)$. But an examination of (13.10) reveals that this is impossible, so that we must have that $b_1 > b_2$, that is, the submitted demand function is "downward sloping." Thus, we also conclude that *bidders submit downward sloping demands with positive probability*.

Proposition 13.6. *Suppose that F_1 dominates F_2 in terms of the reverse hazard rate. In any equilibrium of the two-bidder, two-unit discriminatory auction, bidders submit flat demands if the difference in marginal values is small and submit downward sloping demands if the difference is large.*

Figure 13.5 illustrates Proposition 13.6 by delineating the set of values for which bidders submit flat demands (the region lying between the curve and the 45° line) and the set of values for which bidders submit downward sloping demands (the region lying below the curve). Points lying on the kinked solid lines are value pairs such that the bid on the first unit is a constant, and points on the kinked dotted lines are value pairs such that the bid on the second unit

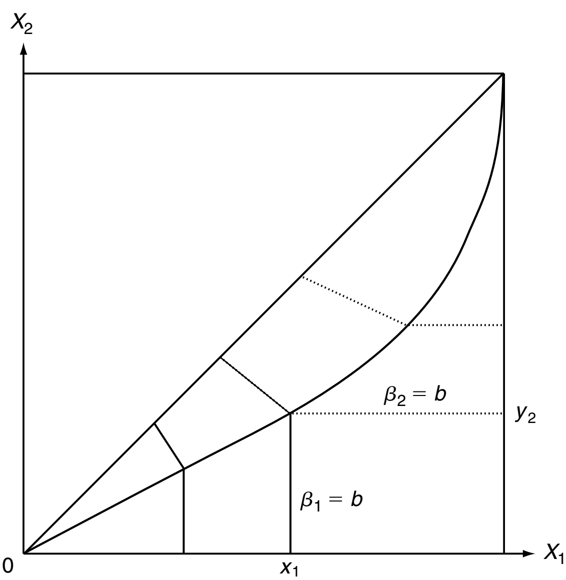

FIGURE 13.5 Equilibrium bids in a discriminatory auction.

is a constant. Below the curve, the bids on the two units are separable, so the "iso-bid" curves for the first unit are vertical lines and for the second unit are horizontal lines. Above the curve, the same amount b is bid for both units, so the "iso-bid" curves are common, downward sloping lines and, as indicated by (13.10), have slope

$$-\frac{h_1(b)}{h_2(b)}$$

The iso-bid lines when flat demands are submitted are thus are not parallel and typically get flatter as the amount bid b increases. Given a value x_1 of the first unit, the bid on the first unit b exceeds the amount bid on the second unit as long as $x_2 \leq y_2$ (compare with Figure 13.4). Once $x_2 > y_2$ the constraint $b_1 \geq b_2$ binds and flat demands are submitted.

Given the features of equilibrium behavior highlighted above it is not surprising that the discriminatory auction is, in general, inefficient. The presence of flat demands and the fact that when downward sloping bidding on the second unit is more aggressive than on the first unit implies that the two units are not treated in symmetric fashion. Proposition 13.3 implies the following:

Proposition 13.7. *Every equilibrium of the discriminatory auction is inefficient.*

13.5.2 Single-Unit Demand

As in the case of the uniform-price auction, the general inefficiency of the discriminatory auction stems not from the multiplicity of units for sale but rather from the fact that bidders demand multiple units. Unlike the uniform-price auction, however, the discriminatory auction is efficient with single-unit demand only in the case where bidders are symmetric. This is not surprising since the single-unit first-price auction is also efficient only under the assumption of symmetry.

Suppose that there are N symmetric bidders, each of whom draws a single value independently from the same distribution F. A symmetric equilibrium of a discriminatory auction in which $K < N$ units are sold entails that each bidder follow the strategy

$$\beta(x) = E\left[Y_K^{(N-1)} \mid Y_K^{(N-1)} < x\right],$$

where $Y_K^{(N-1)}$ denotes the Kth-highest order statistic of $N-1$ draws from the distribution F.

The argument that this constitutes an equilibrium is the same as in Chapter 2, where $K = 1$. Since all bidders follow the same strategy, the K highest bids come from bidders with the K highest values; so with single-unit demand, the auction is efficient.

This chapter has focused on the efficiency of the various multiunit auctions. In the next chapter we turn to our other major concern—the revenue from the different auctions.

PROBLEMS

13.1. (Uniform-price auction) Consider a three-unit uniform-price auction with two bidders. Each bidder's value vector $\mathbf{X}^i = \left(X_1^i, X_2^i, X_3^i\right)$ is independently and identically distributed on the set $\mathcal{X} = \{\mathbf{x} \in [0,1]^3 : x_1 \geq x_2 \geq x_3\}$ according to a density function f such that the marginal distributions are:

$$\begin{aligned} F_1(x_1) &= (x_1)^2 \\ F_2(x_2) &= (2 - x_2)x_2 \end{aligned}$$

F_3 is left unspecified. Show that the bidding strategy $\boldsymbol{\beta}(x_1, x_2, x_3) = (x_1, (x_2)^2, 0)$ constitutes a symmetric equilibrium of the uniform-price auction.

13.2. (Uncertain supply) Consider a multiunit uniform-price auction with N bidders each of whom has use for one unit only. At the time of bidding, the actual number of units that will be available for sale is uncertain and could range anywhere between 1 and K, where $K < N$. Show that it is a weakly dominant strategy for each bidder to bid his or her value.

13.3. (Multiple equilibria) Consider a two-unit uniform-price auction with two bidders. Each bidder's value vector $\mathbf{X}^i$ is identically and independently distributed so that the marginal distributions of the values of both goods is uniform, that is, $F_1(x_1) = x_1$ and $F_2(x_2) = x_2$. Show that for *any* increasing function $\gamma(z)$ such that $0 \leq \gamma(z) \leq z$, the bidding strategy $\boldsymbol{\beta}(x_1, x_2) = (x_1, \gamma(x_1))$ constitutes a symmetric equilibrium.

CHAPTER NOTES

Vickrey (1961) first studied multiunit auctions and recognized that the discriminatory and uniform-pricing rules were inefficient. In fact, it is safe to say that this is the reason he proposed an entirely new pricing rule—which we now know by his name—that was efficient. He also recognized that the root cause of the inefficiency of the discriminatory and uniform-price auctions was not the multiplicity of units for sale but that bidders had multiunit demand.

There has been a lively debate regarding the merits of the various formats, mainly in the context of the auction of Treasury bills. The Treasury has traditionally used the discriminatory auction, but since the 1960s numerous economists have advocated a switch to the uniform-price format. The arguments in favor of the uniform-price auction have been made on many grounds—revenue, efficiency, strategic simplicity, and susceptibility to collusion. Back and Zender (1993) summarize the debate and highlight the fact that many of the arguments that have been made are based on extrapolating properties of the two auctions from situations where bidders have single-unit demand to more general situations. As we have seen, and as Vickrey (1961) himself realized, the properties of these auctions in the context of single-unit demand do not extend to situations with multiunit demand. As a case in point, the strategic simplicity of the uniform-price auction with single-unit demand—it is a dominant strategy to bid one's value—does not extend to situations with multiunit demand—it is now advantageous to engage in demand reduction. The policy debate has concerned the relative merits of the discriminatory and uniform-price auctions; the attractive properties of the Vickrey auction seem to have been largely overlooked. We hasten to add, however, that these properties hold in a model with private values and such a specification does not seem the most natural in the context of Treasury bills.

The derivation of equilibrium bidding behavior in private value uniform-price auctions follows the work of Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998a). The example of low revenue equilibria in the uniform-price auction is taken from the latter paper. The analysis of equilibrium strategies in the discriminatory auction is, to a large extent, based on Engelbrecht-Wiggans and Kahn (1998b). The issue of demand reduction has been studied in more detail by Ausubel and Cramton (2002).

In an important paper, Reny (1999) has shown that with independent private values, the discriminatory auction has a pure strategy equilibrium. In symmetric situations, there is a symmetric pure strategy equilibrium. Bresky (2000) extends Reny's work to show that a whole class of auctions—which includes both the uniform-price and discriminatory formats—has pure strategy equilibria, again in an independent private values setting that is possibly asymmetric.

We have argued that the discriminatory and uniform-price auctions are generally inefficient. Swinkels (1999) shows, however, that as the number of bidders gets arbitrarily large, the inefficiency in the discriminatory auction goes to zero. Essentially, in any equilibrium, once there are enough bidders, bidding behavior begins to resemble "price taking." Thus, the discriminatory auction is asymptotically efficient in the sense that the ratio of social surplus in equilibrium to the attainable social surplus approaches one. The rate at which the inefficiency goes to zero as the number of bidders increases may be slow, however, so the inefficiency may remain large even for a relatively large number of bidders. In a second paper, Swinkels (2001) studies a model in which the total supply is uncertain and derives results on the asymptotic efficiency of both the discriminatory and uniform-price auctions.