ECON 504
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CHAPTER 12 PROBLEMS

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12.2 Consider the strategies for player 1 $L \to q_L^1$, $H \to q_H^1$ & player 2 $L \to q_L^2$, $H \to q_H^2$

for player 1's L type; q_L^1 maximizes:

$$\pi_1^L = \mu\left[q_L^1\left(a - b\left(q_L^1 + q_L^2\right) - q_L^1 c_L\right] + (1-\mu)\left[q_L^1\left(a - b\left(q_L^1 + q_H^2\right) - q_L^1 c_L\right]$$

$$= q_L^1\left[a - b\left(q_L^1 + \left(\mu q_L^2 + (1-\mu) q_H^2\right) - q_L^1 c_L\right] \Rightarrow \text{F.O.C } \frac{\partial \pi_L^1}{\partial q_L^1} = 0$$

$$\Rightarrow \quad a - b\left(q_L^1 + \left(\mu q_L^2 + (1-\mu) q_H^2\right) - c_L + q_L^1(-b) = 0$$

$$q_L^1 = \frac{a - c_L}{2b} - \frac{1}{2}\left(\mu q_L^2 + (1-\mu) q_H^2\right)$$ and similarly for player 1 H;

$$q_H^1 = \frac{a - c_H}{2b} - \frac{1}{2}\left(\mu q_L^2 + (1-\mu) q_H^2\right)$$ and similarly for player 2's L & H types.

4 linear eqns in 4 unknowns should be generically solvable (uniquely).
For a symmetric BNE, $q_L^1 = q_L^2$ & $q_H^1 = q_H^2$ and solve 2 eqn in 2 unknowns.

12.3 The strategy of a player is $a(\theta)$ $\theta \in \{S, W\}$ and $a(\theta) \in \{A, N\}$
Hence there are 4 pure strategies; AA, AN, NA, NN
when Strong, when Weak

For example consider $EU_1(AN, AA)$

$$= Pr(SS) \cdot v_1(A, A, S, S) + Pr(S, W)\, v_1(A, A, S, W) + Pr(W, S)\, v_1(N, A, W, S) + Pr(W, W)\, v_1(N, A, W, W)$$

Player 1's type, Player 2's type, pl.1 action when S, pl.2 action when S, pl.1 type, pl.2's type.

$$= \frac{1}{4}(-s) + \frac{1}{4}(m-s) + \frac{1}{4}0 + \frac{1}{4}0 = \frac{m-2s}{4}$$

NOTE THAT THIS IS A GAME of COMMON VALUES; suppose I am strong and I attack & my opponent attacks: whether I get m depends on my opponent's type!

	AA	AN	NA	NN
AA	$\frac{m}{4}-\frac{s}{2}-\frac{w}{2}$	$\frac{m}{2}-\frac{w}{4}-\frac{s}{4}$	$\frac{3m}{4}-\frac{s}{4}-\frac{w}{4}$	m
AN	$\frac{m}{4}-\frac{2s}{4}$	$\frac{m}{4}-\frac{s}{4}$	$\frac{2m}{4}-\frac{s}{4}$	$\frac{m}{2}$
NA	$-\frac{2w}{4}$	$\frac{m}{4}-\frac{w}{4}$	$\frac{m}{4}-\frac{w}{4}$	$\frac{m}{2}$
NN	0	0	0	0

For another box;

$$U_1(AN, NA) = \frac{1}{4}\overset{ss}{\overline{m}} + \frac{1}{4}\overset{sw}{\overline{(m-s)}} + \frac{1}{4}\overset{ws}{\overline{0}} + \frac{1}{4}\overset{ww}{\overline{0}}$$

$$= \frac{2m}{4} - \frac{s}{4}$$

Player 2's payoffs are mirror images with respect to the diagonal from symmetry; for e.g. $u_1(AN, AA) = u_2(AA, AN)$.

Substituting $m=3$ $w=2$ $s=1$ and multiplying each box values by 4 (scaling up by 4) to get rid of fractions;

	AA	AN	NA	NN
AA	-3,-3	3,1	6,-4	12,0
AN	1,3	2,2	5,1	6,0
NA	-4,6	1,5	1,1	6,0
NN	0,12	0,6	0,6	0,0

(AN, AA) and (AA, AN) are pure BNE.
One always attacks & the other attacks only when strong.

Substituting $m=3$ $w=4$ $s=2$

	AA	AN	NA	NN
AA	-9,-9	0,-1	3,-8	12,0
AN	-1,0	1,1	4,-1	6,0
NA	-8,3	-1,4	-1,-1	6,0
NN	0,12	0,6	0,6	0,0

3 pure BNE.

- One always attacks & the other never attacks.
- Each attacks only when strong.

12.4 a)- I'm trying to maximize my "expected grade". If I chose:

X ⟶ my opponent always exchanges, hence expected grade (of his) is

$\frac{4+3+2+1+0}{5} = 2$, and after bumping +1; I end up with an expected grade = 3.

Hence my BR is: X if $g_i \leq 3$ H iff $g_i > 3$

(the equality doesn't matter; I am indifferent between X & H when my grade $g_i = 3$; hence X iff $g < 3$ H iff $g_i \geq 3$ is also best response)

b)- A player's type $g \in G = \{0,1,2,3,4\}$ and actions $A_i = \{H, X\}$

Hence a (pure) strategy is $s_i : G \to A_i$; (equivalently to choose a subset of G for which I exchange (chose X) my grade.)

Given a strategy for my opponent $a_2(g_2)$ and my type g_1 ;

if I chose H ⟶ $EU_1(H, s_2(g_2), g_1) = g_1$

if I chose X ⟶ $EU_1(X, s_2(g_2), g_1) = \sum_{g_2=0}^{4} \frac{1}{5} \left\{ \begin{matrix} g_2+1 & \text{if } s_2(g_2) = X \\ g_1 & \text{if } s_2(g_2) = H \end{matrix} \right\}$

$= \underbrace{pr(s_2(g_2) = H)\, g_1 + pr(s_2(g_2) = X)}_{\text{sum to } 1\,!} \underbrace{E}_{\text{expected}}(g_2 + 1 \mid s_2(g_2) = X)$

Hence I chose X iff $E(g_2 \mid s_2(g_2) = X) + 1 \geq g_1$

Notice that independent of whatever strategy my opponent has, my best response is of "cutoff" or "threshold" type; I chose to exchange (X) if g_1 is small enough; and Hold (H) if large enough.

This logic of course applies to my opponent, hence any BNE (pure) is of the form $a_1(g_1) = \left\{ \begin{matrix} X & g_1 \leq k_1 \\ H & g_1 > k_1 \end{matrix} \right\}$ and $a_2(g_2) = \left\{ \begin{matrix} X & g_2 \leq k_2 \\ H & g_2 > k_2 \end{matrix} \right\}$

for some cutoff grades k_1 & k_2.

We are looking for a symmetric BNE (pure) ; $k_1 = k_2 = k$

Hence if I get a grade $g_1 \le k$ I'd like to exchange X & hold on to it, H, if $g_1 > k$;

opponent's expected grade given he is willing to exchange

bump up

$$EU_1\left(X, s_2(g_2), g_1\right) = \left(1 - \frac{k+1}{5}\right) g_1 + \left(\frac{k+1}{5}\right)\left(\frac{k}{2} + 1\right)$$

prob. opponent gets $g_2 > k$ prob. opponent gets $g_2 \le k$

$$EU_1\left(H, s_2(g_2), g_1\right) = g_1$$

We need to have

if $g_1 > k \Rightarrow g_1 \ge \left(1 - \frac{k+1}{5}\right) g_1 + \left(\frac{k+1}{5}\right)\left(\frac{k}{2}+1\right)$ $\quad g_1 \ge \frac{k}{2} + 1$

$g_1 \le k \rightarrow g_1 \le \left(1 - \frac{k+1}{5}\right) g_1 + \left(\frac{k+1}{5}\right)\left(\frac{k}{2}+1\right)$ $\quad g_1 \le \frac{k}{2} + 1$

$\Rightarrow k+1 \ge \frac{k}{2} + 1$ and $k \le \frac{k}{2} + 1$

$k \ge 0 \qquad k \le 2$

So we have $\underline{3}$ pure BNE corresponding to the (symmetric) thresholds $k = 0, 1, 2$

In the eqm with $k=0$; $EU_1^{k=0} = 2 + \frac{1}{25}\cdot 1 = EU_2 = 2.04$

Both players get the same exp. utility from symmetry. The expected grade is $\frac{4+3+2+1+0}{5} = 2$ and only when both parties get $g_i = 0$ simultaneously, they swap & grade is up by 1.

In the eqm with $k=1$, the ex-ante (before types are chosen; over the whole type space weighted with the prior) utility is;

$EU_1^{k=1} = 2 + \frac{4}{25}\cdot 1 = \underline{2.16}$ and similarly for $k=2$;

$EU_1^{k=2} = 2 + \frac{9}{25}\cdot 1 = \boxed{2.36}$

Pareto dominates

note the eqm payoffs are pareto ranked: $\underset{k=2}{(2.36, 2.36)} > \underset{k=1}{(2.16, 2.16)} > \underset{k=0}{(2.04, 2.04)}$

NOTE THAT THIS IS A GAME OF COMMON VALUES! My partner's grade matters to me, of course!!

There's one more eqm. If we had defined threshold str. as X if $g_i < k$ rather than $\le k$ we'd also have the pure BNE = never exchange!! ($k=0$ in this formulation; or, $k=-1$ in the original formulation.

In this case $EU^{k=-1} = 2$ and $(2,2)$ is Pareto dominated by all of the above.

12.4 . c). Now there is no Bayesian game, as players will decide before learning their types. $A_i = \{X, H\}$, 2x2 normal form game;

	X	H
X	3,3	2,2
H	2,2	2,2

Unless both choose X; they get their own grade; average is 2.
When they both choose X; grade is bumped by $1 \Rightarrow 2+1=3$.

X is weakly dominant for both players and a pure NE.

It is Better than all BNE in part (b), for both players.

Think of this as "COMMITTING" to exchange grades, whatever your grade turns out to be. It kills the adverse selection aspect; and both students are better off.

d). Alternative to the "commitment to eXchange" interpretation, you can also interpret this as you don't want to know your grade as this would tempt you to exchange only when you get bad grades, inducing the other student to do the same, hence getting to a worse equilibrium. More info. hurts the students in this strategic situation.

12.5 a). For price $p \geq 0$ if $3x \leq p$ prospector wishes to sell, if $3x > p$ he wants to keep the mine; $x \leq \frac{p}{3} \leftrightarrow x(p) = \frac{p}{3}$ is the "critical type".

b). Given (a), prospector's strategy in a BNE should be;
sell if $p \geq 3x$, keep if $p < 3x$ (indifferences are not important)

Given this, the owner's utility would be, (owner has no types)

$$EU_{owner}(A, p) = E(4x - p \mid x \text{ accepts to sell at } p)$$

$$= 4 \cdot E(x \mid 3x \leq p) - p = 4 \cdot \frac{p}{6} - p = -\frac{p}{3} \leq 0 = EU_{owner}(R, p)$$

(reject)

unless $p = 0$. Hence owner only accepts $p=0$, knowing that this is the type $x=0$.
Hence there is no trade (probability $(x=0) = 0$!)

$EU_{prospector} = E(3x) = 3 \cdot \frac{1}{2} = \frac{3}{2}$ $\quad EU_{owner} = 0$ (no trade)

(12.6) a)- Farmer strategy is to ask wage; $w(\theta)$ when his type is θ.

Similar to 12.5; farmer should ask a wage $w(\theta) \geq \theta \quad \forall \theta$.

owner (man. plant's) utility of accepting the wage w is;

$$U_{owner}(w, A) = E\left(\frac{3}{2}\theta - w \mid \theta \text{ accepts wage } w\right)$$

$$= \frac{3}{2} E(\theta \mid \theta \text{ accepts } w) - w = \frac{3}{2} \cdot \frac{w}{2} - w = -\frac{w}{4} \leq 0 = U(w, R)$$

expected type of farmer given he accepts w. (reject)

Hence, as in 12.5. each type of farmer offers $w(\theta) = \theta$, owner accepts only $w = 0$ (knowing the farmer has $w=0$); so no trade.

b)- $EU_{farmer} = E(\theta) = \frac{1}{2}$ $\quad EU_{owner} = 0 \quad$ Social Surplus $= \frac{1}{2} + 0 = \frac{1}{2}$

c)- Revising (a) $w(\theta) \geq \frac{\theta}{2}$ should be offered by the farmer, $\forall \theta$.

$$U_{owner}(w, A) = E\left(\frac{3}{2}\theta - w \mid \theta \text{ accepts } w\right)$$

$$= \frac{3}{2} E(\theta \mid \theta \text{ accepts } w) - w = \frac{3}{2} \cdot w - w = \frac{w}{2} \geq 0 = U(w, R)$$

Hence "each type farmer offering $\frac{\theta}{2} \leq w$ $w(\theta) = 1/2$, owner accepting $w = 1/2$" is a BNE.

$EU_{farmer} = 1/2$

$$EU_{owner} = E\left(\frac{3}{2}\theta - 1/2 \mid \theta \text{ accepts } w = 1/2\right)$$

Every body accepts $w = 1/2$!

$$= \frac{3}{2} \cdot \frac{1}{2} - \frac{1}{2} = \frac{1}{4}$$

surplus $= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ $\qquad \frac{3}{4} > \frac{1}{2}$ in (b) hence society is better off.

Notice that as the "outside option" for farmer type θ is decreased from θ to $\frac{\theta}{2}$, even higher types are available for work, which overcomes the adverse selection problem.

(12.7) a)- Suppose $\left\{\begin{array}{c}\text{Player 1 strategy.} \\ \text{exchange if } v_1 < \bar{v}_1 \\ \text{don't exchange otherwise}\end{array}\right\}$ and $\left\{\begin{array}{c}\text{Player 2 strategy} \\ \text{exchange if } v_2 < \bar{v}_2 \\ \text{don't, otherwise}\end{array}\right\}$ is a pure BNE.

Given player 2's strategy, my strategy should be optimal;

if ① $v_1 \geq \bar{v}_1$ $\quad EU_1(Ex, s_2(\cdot)) \geq EU_1(No, s_2(\cdot), v_1) = v_1$ should hold.

and ② $v_1 \leq \bar{v}_1$ $\quad$ " $\quad <$ $\quad$ "

① $v_1 \geq \bar{v}_1 \implies \underbrace{(1-\bar{v}_2)}_{\text{prob. pl 2 has } v_2 > \bar{v}_2} \cdot v_1 + \underbrace{\bar{v}_2}_{\text{prob pl. 2 has } v_2 \leq \bar{v}_2} \cdot \frac{3}{2} \cdot \frac{\bar{v}_2}{2} \leq v_1$ → pl. 2's house value for me given he exchanges, ie, $v_2 < \bar{v}_2$

no exch happens

$\longrightarrow \frac{3}{2}\frac{\bar{v}_2}{2} \leq v_1$ $\qquad \frac{3}{2}\frac{\bar{v}_2}{2} \geq \bar{v}_1 \geq \frac{3}{2}\frac{\bar{v}_2}{2} \implies \boxed{\bar{v}_1 = \frac{3\bar{v}_2}{4}}$

② $v_1 \leq \bar{v}_1$ $\quad$ similarly implies $\quad \frac{3}{2}\frac{\bar{v}_2}{2} \geq v_1 \rightarrow$

Similarly, the condition from player 2's best response would yield $\bar{v}_2 = \frac{3\bar{v}_1}{4}$

$\implies \bar{v}_1 = \bar{v}_2 = 0$. Nobody exchanges!

b)- then the equation would be modified to: $\quad \bar{v}_1 = \frac{5\bar{v}_2}{4}$ $\left(\text{or } =1 \text{ if } \frac{5\bar{v}_2}{4} \geq 1\right)$

$\bar{v}_2 = \frac{5\bar{v}_1}{4}$ $\left(\text{or } =1 \text{ if } \frac{5\bar{v}_1}{4} \geq 1\right)$

$\implies \bar{v}_1 = \bar{v}_2 = 1$; everybody (each type of each player) exchanges!

c)- Just as in (12.4), given player 2's strategy and given your type (player 1);

$\underbrace{EU_1(Ex, s_2(\cdot))}_{\text{the cutoff}} > v_1 \longrightarrow$ you should exchange

$< v_1 \longrightarrow$ " not exchange.

Hence, even if player 2 doesn't use a threshold strategy; it is optimal for you to use one.

note that when you "Exchange" EU_1 doesn't depend on your type v_1.

Hence Left hand side is a number that you take as given (opponent sets it)

12.8 a)- Given player 2 is always convicting, you're always pivotal (it's up to you whether defendant gets convicted or not)

if you are type = G (got guilty signal), you know that pl 2 chose C (but as both types of pl 2 convicts, you cannot infer his signal/type). Your belief is the posterior belief after your own signal; hence you'd rather C.

if you're type I, as you don't get any info from player 2 in equilibrium and as prior + 1 I signal $\Rightarrow$ posterior that is still above $\frac{1}{2}$ (*), you still think guilty is more likely; hence you'd rather convict.

(*) Here we assume that $p < q$ so that after one innocent signal the posterior belief still is $\geq \frac{1}{2}$. If not; then (CC, CC) is NOT a BNE, hence the textbook has an error. In this case ($p \geq q$), (CC, CA) is a BNE, though.

$p < q$ unfortunately is the less interesting case, because then one player (one signal) does not overturn the prior weight on guilty, hence we don't expect following the signal as optimal in the one-player case.

The interesting case $p > q$ has (CC, CA) BNE, where effectively 1 player decides the fate of the defendant, the other always convicting (hence not transmitting his own piece of information to the decision process.)

b)- For $p > q$ (CC, CA) is BNE as described above corresponds to 1 player deciding the fate.

c)- For "following your signal" to be BNE; it should be optimal to C after θ_G signal and A after θ_I signal when the player is pivotal. Pivotal here means that your decision determines majority; that is the other two players have chosen C & A, resp. This means they have received θ_G & θ_I signals respectively.

Hence $pr(G \mid \underset{\text{your signal}}{\theta_G}, \underbrace{\theta_G, \theta_I}_{\text{other players' signal}}) \geq \frac{1}{2}$ and $pr(G \mid \theta_I, \theta_G, \theta_I) \leq \frac{1}{2}$ should hold.

1 Guilty signal cancels out 1 innocent signal and you're left with your own posterior after your own signal. If $p > q$ both eqns hold.