MATH 425b SAMPLE MIDTERM EXAM 1 SOLUTIONS
Spring 2016
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(1) Suppose first that f is real-valued. Then for $n \geq 1$,

$$c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx + i\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,$$

$$c_{n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx - i\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Since the two integrals with sine and cosine are real-valued, these two numbers are conjugates of each other, that is, $c_{-n} = \overline{c_n}$.

Conversely suppose $c_{-n} = \overline{c_n}$. Taking $n = 0$ shows that c_0 is its own conjugate, so c_0 is real. Since the series converges pointwise, regrouping terms gives

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty}(c_n e^{inx} + c_{-n}e^{-inx}) = c_0 + \sum_{n=1}^{\infty}(c_n e^{inx} + \overline{c_n e^{inx}}).$$

Any number added to its conjugate it real, so all terms of the last series are real, so $f(x)$ is real.

(2)(a) The formula shows that $f''(x) \to \infty$ as $x \to -1$, so f'' cannot be continuous at $x = -1$. This means the power series for f'' cannot converge in an interval containing -1, so the radius of convergence of the power series for f'' is at most 1. But this power series has the same radius of convergence as the one for f, so the power series for f has radius of convergence at most 1, also.

(b) Orthonormal means that if $u_{n+1} = \sum_{i=1}^{n} c_i u_i$, then

$$1 = \langle u_{n+1}, u_{n+1}\rangle = \langle u_{n+1}, \sum_{i=1}^{n} c_i u_i\rangle = \sum_{i=1}^{n} \overline{c_i}\langle u_{n+1}, u_i\rangle = \sum_{i=1}^{n} \overline{c_i}\cdot 0 = 0,$$

a contradiction. Thus u_{n+1} is not a linear combination of $u_1, \dots, u_n$.

(3) Let $\epsilon > 0$. By equicontinuity, there exists $\delta > 0$ such that $d(x,y) < \delta \implies |f_n(y) - f_n(x)| < \epsilon$ for all n. Since $f_n \to f$ pointwise, letting $n \to \infty$ we get $|f(y) - f(x)| \leq \epsilon$ also. Since D_δ is finite, there exists N such that

$$n \geq N, x \in D_\delta \implies |f_n(x) - f(x)| < \epsilon.$$

By defintion of δ-dense, given $y \in X$, there exists $x \in D_\delta$ with $d(x,y) < \delta$, so for $n \geq N$,

$$|f_n(y) - f(y)| \leq |f_n(y) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - f(y)| \leq \epsilon + \epsilon + \epsilon = 3\epsilon.$$

Since this is true for all $y \in X$ with the same N, it shows $f_n \to f$ uniformly on X.

(4)(a) Let $f \in C[0,1]$. Then $g(x) = f(x^{1/2})$ is also in $C[0,1]$, so by the Weierstrass Theorem there exists a polynomial Q for which $\|g - Q\|_\infty < \epsilon$. Letting $P(x) = Q(x^2)$ we have $P \in \mathcal{A}_1$, and

$$\|f - P\|_\infty = \sup_{y \in [0,1]} |f(y) - P(y)| = \sup_{y \in [0,1]} |g(y^2) - Q(y^2)| = \sup_{x \in [0,1]} |g(x) - Q(x)| = \|g - Q\|_\infty < \epsilon.$$

Here the third equality is because y^2 and x run over the same range, 0 to 1. Since ϵ is arbitrary, this shows $\mathcal{A}_1$ is dense in $C[0,1]$.

(b) Since all functions in $\mathcal{A}_2$ are even and continuous, so are the functions in the uniform closure $\overline{\mathcal{A}_2}$. Suppose f is an even continuous function on $[-1,1]$, and let $\epsilon > 0$. By part (a), there exists $P \in \mathcal{A}_1$ with $\sup_{x \in [0,1]} |f(x) - P(x)| < \epsilon$. Since f, P are even, we have $f(-x) - P(-x) = f(x) - P(x)$, so the sup is the same over $[-1,1]$ as it is over $[0,1]$, that is, $\sup_{x \in [-1,1]} |f(x) - P(x)| = \sup_{x \in [0,1]} |f(x) - P(x)| < \epsilon$. Since ϵ is arbitrary, this shows $f \in \overline{\mathcal{A}_2}$.