MATH 425b ASSIGNMENT 8
SPRING 2016
Prof. Alexander
Due Wednesday April 13.

Rudin Chapter 9 #30ab and:

(I) Let $f : E \to \mathbb{R}$ be a $\mathcal{C}'$ function on an open set $E \subset \mathbb{R}^3$, and define the surface $S = \{(x, y, z) : f(x, y, z) = 0\}$. Suppose $(x_0, y_0, z_0) \in S$ and $D_1 f(x_0, y_0, z_0) \neq 0$. By the Implicit Function Theorem, in a neighborhood W of (x_0, y_0, z_0) one can express the surface as $x = u(y, z)$ for some $\mathcal{C}'$ function u. We then define two curves in the surface S passing through (x_0, y_0, z_0): $\gamma(z) = (u(y_0, z), y_0, z)$ and $\tilde{\gamma}(y) = (u(y, z_0), y, z_0)$. The derivatives $\gamma'(z)$ and $\tilde{\gamma}'(y)$ are then vectors tangent to S and we define the *translated tangent plane* at (x_0, y_0, z_0) to be the span of $\gamma'(z_0)$ and $\tilde{\gamma}'(y_0)$. ("Translated" means it passes through the origin, not necessarily through the point (x_0, y_0, z_0).)

(a) Express the tangent vectors $\gamma'(z_0)$ and $\tilde{\gamma}'(y_0)$ in terms of the partial derivatives $D_1 u$ and $D_2 u$. (Some entries may just be numbers.)

(b) Let $\varphi : E \to \mathbb{R}$ be another $\mathcal{C}'$ function. We say φ has a local maximum on S at (x_0, y_0, z_0) if there is a neighborhood Y of (x_0, y_0, z_0) such that $\varphi(x_0, y_0, z_0) = \sup\{\varphi(x, y, z) : (x, y, z) \in Y \cap S\}$. Show that in this case, $\nabla\varphi$ is perpendicular to the translated tangent plane at (x_0, y_0, z_0), and $\nabla\varphi$ is a scalar multiple of ∇f at (x_0, y_0, z_0).

As an observation, part (b) is what underlies the method of Lagrange multipliers—to find local maxima/minima of φ on a surface $f =$ contstant, you look for places where $\nabla\varphi$ is a scalar multiple of ∇f.

(II) You may take the following as given. A 2×2 symmetric matrix H is always diagonalizable to $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, that is, $H = U^t D U$ for some unitary matrix U and some λ_i's which are eigenvalues of H. (*Unitary* means $|U\mathbf{x}| = |\mathbf{x}|$ for all $\mathbf{x}$, i.e. U preserves length, and U^t means the transpose.) The sum at the top of p. 244 is called the *Taylor polynomial of f of order $m-1$ at a.*

(a) Let $f : E \to \mathbb{R}$ be a $\mathcal{C}^{(3)}$ function on an open subset $E \subset \mathbb{R}^2$. The *quadratic part* of the Taylor polynomial at a is the sum of the 4 terms which are constant multiples of $x_1^2, x_1 x_2$ and x_2^2. Show that there is a 2×2 symmetric matrix H such that the quadratic part can be expressed as $\mathbf{x}^t H \mathbf{x}$ for the column vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The entries of H are given by some derivatives of f.

(b) Show that if both eigenvalues of H are positive, $\lambda_1 \geq \lambda_2 > 0$, then the quadratic part satisfies $\mathbf{x}^t H\mathbf{x} \geq \lambda_2|\mathbf{x}|^2$ for all $\mathbf{x}$.

(c) Show that if the total derivative $Df(a) = \nabla f(a) = 0$ and both eigenvalues of H are positive, then f has a local minimum at a.

HINTS:

(30)(a) The hardest part of this is "bookkeeping," meaning using notation correctly and precisely enough to formalize what's not a complicated idea. "Repeated application" means induction on k: assume you have shown $h^{(k-1)}(t) = \sum(D_{i_2\cdots i_k}f)(p(t))x_{i_2}\cdots x_{i_k}$ and prove the statement for $h^{(k)}$.

(I)(b) To get perpendicularity, consider the function $\varphi(u(y,z),y,z)$ and its partial derivatives in y and z, at (y_0,z_0). Can you apply this result to $\varphi = f$?

(II)(b) Use $H = U^t DU$ and recall that for matrices A, B, $(AB)^t = B^t A^t$.

(c) If you didn't have the remainder term $r(\mathbf{x})$ at the top of p. 244, this would be easy. So you need to show that the remainder term is too small to alter the existence of a minimum.