MATH 425a ASSIGNMENT 9
FALL 2015 Prof. Alexander
Due Wednesday November 18.

Rudin Chapter 5 #1, 6, 7, 13ab plus the problems (A)–(E) below:

(A) A function f on $\mathbb{R}$ is called *even* if $f(x) = f(-x)$ for all x. Show that if f is even, and differentiable at $x = 0$, then $f'(0) = 0$.

(B) Suppose f, g are differentiable on $[a, b]$, $f(a) = g(a)$, and $f' \leq g'$ on $[a, b]$. Show that $f(x) \leq g(x)$ for all $x \in [a, b]$.

(C) Suppose $f : [a, b] \to \mathbb{R}$ is differentiable at x, and let $g(t) = f(t)^3$. Show *directly from the definition of derivative* (i.e. *not* using the product rule, chain rule, etc.) that $g'(x) = 3f(x)^2 f'(x)$.

(D) Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable on $\mathbb{R}$ with $f''(x) > 0$ for all x. Show that for each $c \in \mathbb{R}$ there are at most two points where $f(x) = c$.

(E) Suppose f is differentiable in (a, b). State hypotheses under which $|f|$ is differentiable in (a, b), and prove it. Your hypotheses should not prohibit $f(x)$ from ever being 0.

(F) In class we proved the following corollary to Theorem 5.8: if $f : [a, b] \to \mathbb{R}$ is differentiable and $f(a) = f(b)$, then there exists $x \in (a, b)$ with $f'(x) = 0$. Prove the following analog for a possibly infinite interval: suppose $f : (a, b) \to \mathbb{R}$ is differentiable, with $-\infty \leq a \leq b \leq \infty$, and suppose $\lim_{x\to a} f(x) = \lim_{x\to b} f(x)$ (a finite value.) Show that there exists $x \in (a, b)$ with $f'(x) = 0$.

(G) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

Find $f'(x)$ for all $x \in \mathbb{R}$, and show that $f''(0)$ does not exist.

HINTS:

(2) Use 4.17, and use the idea of substitution in limits.

(6) Use Theorem 5.3c. To help you see what's happening, draw a picture for some example like $f(x) = x^2$.

(7) The derivatives are only assumed to exist at one point, so you can't use L'Hospital. Instead, what quantity do you know (essentially by definition) converges to $f'(x)/g'(x)$? Relate $f(t)/g(t)$ to this quantity.

(13b) Justify carefully—at $x = 0$ you can't just differentiate the formula, calculus-style.

(A) Use the definition of derivative.

(B) Apply the Mean Value Theorem (but not on the whole interval $[a, b]$) to one of the following functions: $h = f + g, h = f - g$ or $h = fg$.

(C) You can factor the difference of two cubes.

(D) Suppose there are three points, and get a contradiction. A picture may be helpful to let you see what is going on here, but a picture is not a proof.