Appendix | B

Stochastic Orders

There are numerous senses in which one distribution F may "dominate" or may be "greater than" another distribution G and each induces a partial order over the space of all distributions.[1] In this appendix we discuss some of these notions, called *stochastic orders*, that play an important role in auction theory.

For the sake of convenience, we assume that all distributions being compared have the same support, say $[0,\omega]$. As usual, we allow for the possibility that the support is the nonnegative portion of the real line.

FIRST-ORDER STOCHASTIC DOMINANCE

Given two distribution functions F and G, we say that F (first-order) *stochastically dominates* G if for all $z \in [0,\omega]$,

$$F(z) \leq G(z) \tag{B.1}$$

If the random variables X and Y are distributed according to F and G, respectively, and (B.1) holds, then we will also say the X stochastically dominates Y.

Now suppose $\gamma : [0,\omega] \to \mathbb{R}$ is an increasing and differentiable function. If X stochastically dominates Y, and these have distribution functions F and G, respectively, then

$$E[\gamma(X)] - E[\gamma(Y)] = \int_0^\omega \gamma(z)[f(z) - g(z)]dz$$

and integrating by parts, we obtain

$$E[\gamma(X)] - E[\gamma(Y)] = -\int_0^\omega \gamma'(z)[F(z) - G(z)]dz \geq 0$$

since $\gamma' > 0$ and $F \leq G$. In particular, $E[X] \geq E[Y]$.

[1]A partial order is reflexive and transitive. It need not be complete.

HAZARD RATE DOMINANCE

Given two distribution functions F and G, we say that F *dominates G in terms of the hazard rate* if for all $z \in [0, \omega]$,

$$\lambda_F(z) = \frac{f(z)}{1 - F(z)} \leq \frac{g(z)}{1 - G(z)} = \lambda_G(z)$$

This order is also referred in short as hazard rate dominance.

If F dominates G in terms of the hazard rate, then (A.3) immediately implies that

$$F(x) = 1 - \exp\left(-\int_0^x \lambda_F(t)\, dt\right) \leq 1 - \exp\left(-\int_0^x \lambda_G(t)\, dt\right) = G(x)$$

and thus F stochastically dominates G. Thus, hazard rate dominance implies first-order stochastic dominance.

REVERSE HAZARD RATE DOMINANCE

Given two distribution functions F and G, we say that F *dominates G in terms of the reverse hazard rate* if for all $z \in [0, \omega]$,

$$\sigma_F(z) = \frac{f(z)}{F(z)} \geq \frac{g(z)}{G(z)} = \sigma_G(x)$$

This order is also referred to as reverse hazard rate dominance, in short.

If F dominates G in terms of the reverse hazard rate, then (A.5) immediately implies that

$$F(x) = \exp\left(-\int_x^\omega \sigma_F(t)\, dt\right) \leq \exp\left(-\int_x^\omega \sigma_G(t)\, dt\right) = G(x)$$

and so, again, F stochastically dominates G. Thus, reverse hazard rate dominance also implies first-order stochastic dominance.

LIKELIHOOD RATIO DOMINANCE

The distribution function F *dominates G in terms of the likelihood ratio* if for all $x < y$,

$$\frac{f(x)}{g(x)} \leq \frac{f(y)}{g(y)} \tag{B.2}$$

or equivalently, if the ratio f/g is nondecreasing in x. We will refer to this order as likelihood ratio dominance. See Figure B.1 for an illustration. As shown, if F dominates G in terms of the likelihood ratio, then f and g can "cross" only once.

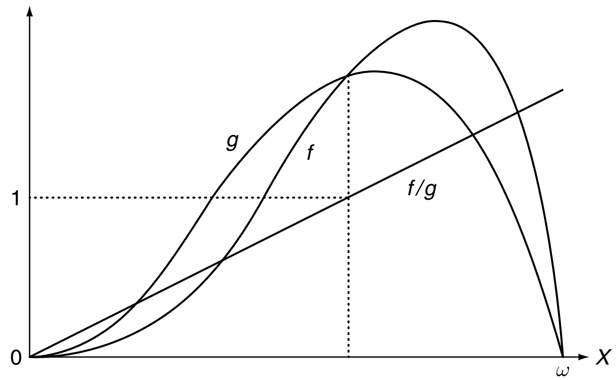

FIGURE B.1 Likelihood ratio dominance.

Likelihood ratio dominance is, of course, equivalent to the following: for all $x < y$,

$$\frac{f(y)}{f(x)} \geq \frac{g(y)}{g(x)}$$

This implies that for all x,

$$\int_x^{\omega} \frac{f(y)}{f(x)}\, dy \geq \int_x^{\omega} \frac{g(y)}{g(x)}\, dy$$

so

$$\frac{1-F(x)}{f(x)} \geq \frac{1-G(x)}{g(x)}$$

which is the same as

$$\lambda_F(x) \leq \lambda_G(x)$$

Thus, likelihood ratio dominance implies hazard rate dominance.

Similarly, by writing (B.2) as follows: for all $x < y$,

$$\frac{f(x)}{f(y)} \leq \frac{g(x)}{g(y)}$$

and integrating, we obtain

$$\int_0^{y} \frac{f(x)}{f(y)}\, dx \leq \int_0^{y} \frac{g(x)}{g(y)}\, dx$$

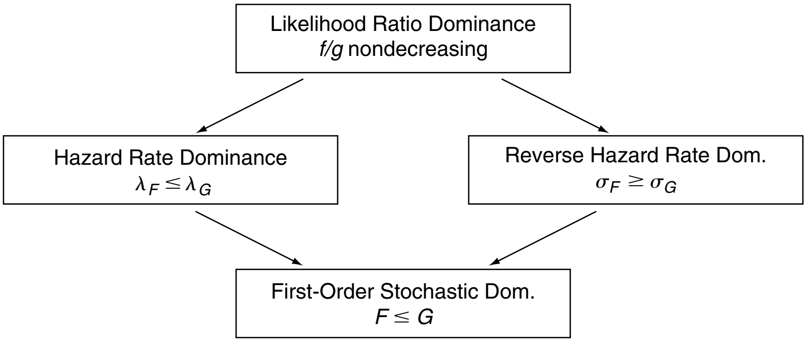

FIGURE B.2 Different notions of stochastic ordering.

so for all y,

$$\frac{F(y)}{f(y)} \leq \frac{G(y)}{g(y)}$$

which is the same as: for all y,

$$\sigma_F(y) \geq \sigma_G(y)$$

Thus, likelihood ratio dominance also implies reverse hazard rate dominance.

We have already shown that hazard rate dominance and reverse hazard rate dominance both imply first-order stochastic dominance. Figure B.2 summarizes the relationships among the four notions of stochastic orders considered so far.

MEAN-PRESERVING SPREADS

A fifth notion of stochastic order is useful in comparing distributions with the same mean.

Suppose X is a random variable with distribution function F. Let Z be a random variable whose distribution conditional on $X = x$, $H(\cdot \mid X = x)$ is such that for all x, $E[Z \mid X = x] = 0$. Suppose $Y = X + Z$ is the random variable obtained from first drawing X from F and then for each realization $X = x$, drawing a Z from the conditional distribution $H(\cdot \mid X = x)$ and adding it to X. Let G be the distribution of Y so defined. We will then say that G is a *mean-preserving spread* of F.

As the name suggests, while the random variables X and Y have the same mean—that is, $E[X] = E[Y]$—the variable Y is “more spread out” than X since it is obtained by adding a “noise” variable Z to X.

Now suppose $U : [0, \omega] \to \mathbb{R}$ is a concave function. Then we have

$$\begin{aligned} E_Y[U(y)] &= E_X[E_Z[U(X+Z) \mid X = x]] \\ &\leq E_X[U(E_Z[X+Z \mid X = x])] \\ &= E_X[U(X)], \end{aligned}$$

where the inequality follows from the assumption that U is concave.

This fact is used to define another notion of order between distributions with the same mean. Given two distributions F and G with the same mean, we say that F *second-order stochastically dominates* G if for all concave functions $U : [0, \omega] \to \mathbb{R}$,

$$\int_0^\omega U(x)f(x)\,dx \geq \int_0^\omega U(y)g(y)\,dy$$

We have shown that if G is a mean-preserving spread of F, then F second-order stochastically dominates G. The converse is also true—although we omit a proof of this fact here—and so the two notions are equivalent.

Second-order stochastic dominance is also equivalent to the statement that for all x,

$$\int_0^x G(y)\,dy \geq \int_0^x F(x)\,dx$$

with an equality when $x = \omega$. Again, we omit a proof of this fact.

NOTES ON APPENDIX B

A comprehensive discussion of the various notions of stochastic orders and their relationships can be found in Shaked and Shanthikumar (1994). This book contains a very useful collection of results that had hitherto been scattered about in the literature and, as a result, had been somewhat inaccessible.