MATH 425b ASSIGNMENT 10
SPRING 2016
Prof. Alexander
Due Wednesday May 4.

Rudin Chapter 10 #16($k = 2, 3$ cases only), 20, 24 and:

(I) A differential equation of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0, \qquad (x,y) \in \mathbb{R}^2, \quad M, N \in \mathcal{C}''$$

is called *exact* if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Show that there then exists an $F(x,y)$ such that $\frac{\partial N}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$, so the equation can be rewritten

$$\frac{d}{dx}F(x,y) = 0.$$

Notice this is not a partial derivative—instead think of $y = y(x)$ as a function of x and use the chain rule.

(II) Let S be the portion of the surface $z = x^4 + y^2$ (in $\mathbb{R}^3$) which lies above $[0,1]^2$ and let

$$\omega = x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy.$$

Find a natural parametrization Φ of S and calculate $\int_\Phi \omega$.

(III) Let S be the positively oriented boundary of the set $E \subset \mathbb{R}^3$ enclosed by $y = x^2, y = 4, z = 0$ and $z = 1$. Let

$$\omega = xyz\ dy \wedge dz + (x^2 + y^2 + z^2)\ dz \wedge dx + (x + y + z)\ dx \wedge dy.$$

Calculate $\int_S \omega$.

(IV) Define the 1-form

$$\eta = \frac{x\ dy - y\ dx}{x^2 + y^2}$$

in $\mathbb{R}^2 \backslash \{0\}$.

(a) Show that η is closed, that is, $d\eta = 0$.

(b) Show that $\eta = d(\arctan \frac{y}{x})$ in any convex open set not intersecting the y-axis (where $x = 0$), and $\eta = d(-\arctan \frac{x}{y})$ in any convex open set not intersecting the x-axis (where

$y=0$). (In this sense we have $\eta = d\theta$, where θ is the usual angle in the plane, but note that θ cannot be defined as a $\mathcal{C}'$ function on *all* of in $\mathbb{R}^2\backslash\{0\}$, it is only true locally. If you give θ values in $[0, 2\pi)$, for example, then it has a discontinuity along the positive x-axis, and if you use $(-\pi, \pi]$ then it's the negative x-axis.)

(c) Let $\gamma(t) = (\cos t, \sin t), t \in [0, 2\pi]$. Show that $\int_\gamma \eta = 2\pi$.

(d) Show that despite part (b), η is not exact in $\mathbb{R}^2\backslash\{0\}$, that is, there is no one $\mathcal{C}'$ function f defined on all of in $\mathbb{R}^2\backslash\{0\}$ such that $df = \eta$.

Thus closed $\not\Rightarrow$ exact in in $\mathbb{R}^2\backslash\{0\}$. But closed $\Rightarrow$ exact in $\mathbb{R}^2$, by Theorem 10.39! In general, the relation between closed and exact forms depends on topological properties of the domain, such as whether it has "holes" in it. This goes under the heading "de Rham cohomology."

HINTS:

(16) When you write out an expression for $\partial^2\sigma$, what is the coefficient of σ_{kl} for a fixed k, l?

(I) Consider the form $M\ dx + N\ dy$. You may use Theorem 10.39, though we may not get to discuss it.

(IV)(d) If η were exact, what would happen in part (c)?