Note 8 and HW 8

This note considers the issue of optimal contract under uncertain demand

or supply. We illustrate with a simple example of a monopolist selling to two consumers with different demands for the good for sale. Behind the discussions here are the issues of optimal auction design, and optimal way to minimize costs or maximize revenue$s$ under uncertainty.

Monopolist facing uncertain demand:

We take the contractual approach in a well-known monopoly problem. The monopoly firm has no private information, and makes offers to a consumer. The consumer has private information regarding his or her willingness to pay.

The consumer has the utility function

$$\theta v(q) - T$$

where θ can be high θ_H or low θ_L, where T is the total payment in the purchase of the quantity q. Think of $\theta v(q)$ as consumer willingness to pay the amount q. The monopoly firm does not know θ. Let c be the unit cost of production of the firm. Let β be the probability that $\theta = \theta_L$. We allow the firm to practice many different forms of pricing policy: for example non-linear pricing (including quantity discount), two-part pricing, menu of packages, and possible price discrimination. We can even let the buyers bid for the good. The idea is that in the pricing problems facing uncertainty, we want to be more flexible in designing the allocation and pricing problem.

More generally, the seller doesn't have to determine the pricing problem alone. We can also let the seller and buyer bargain over the pricing problem. The employment contract can be seen in this manner. The worker sells the service to the employer, with the two sides bargain over the wage policy and employment contract. This becomes a bargaining problem.

In the following we only consider the following possibilities:

Pricing with complete information: (a) linear pricing; (b) two-part pricing

Pricing with incomplete Information: (c) single linear pricing; (d) single two-parts tariff; (e) menu of pricing packages to be chosen by the consumer.

(a) standard linear pricing

If the firm can only use a linear price system (the standard pricing model), then the demand of the consumer facing a linear price system p is derived as follows: The consumer chooses quantity q to maximize

$$\theta v(q) - pq$$

with the first order condition for maximization

$$\theta v'(q) = p \tag{1}$$

The solution is denoted by $q(p)$. This means that the consumer has the demand function $q(p)$ and the inverse demand is given by $p(q) = \theta v'(q)$. Thus if the firm knows the identity of the consumer, the optimal linear price (monopoly pricing) is to choose p to maximize the profit of the firm:

$$(p - c)q(p)$$

and the first order condition for this maximization problem is

$$q(p) + (p - c)q'(p) = 0 \tag{2}$$

The solution of this equation gives us the optimal monopoly price $p^*(\theta)$. The monopoly quantity sold is $q^*(\theta) = q(p^*(\theta))$. The monopoly expected profit is given by

$$\beta(p^*(\theta_L) - c)q^*(\theta_L) + (1 - \beta)(p^*(\theta_H) - c)q^*(\theta_H).$$

As an example, let $\theta_L = 3, \theta_H = 4, \beta = 0.5$. Assume that $c = 1$. Let $v(q) = q - \frac{1}{2}q^2$. Then equation (1) is

$$\theta(1 - q) = p$$

and the demand function of the consumer is

$$q(p) = 1 - \frac{1}{\theta}p$$

Equation (2) is

$$(1 - \frac{p}{\theta}) - (p - c)\frac{1}{\theta} = 0$$

with the optimal monopoly price given by the solution of this equation in p. Hence

$$p^*(\theta) = \frac{\theta + c}{2}$$

is optimal monopoly price. This pricing solution is only possible if the firm knows θ. The profit of the monopolist is

$$\pi(\theta) = (\frac{\theta + c}{2} - c)(1 - \frac{\theta + c}{2\theta}) = \frac{(\theta - c)^2}{4\theta}$$

if the firm knows θ. If $\beta = 0.5$, the expected profit of the firm is

$$0.5\pi(\theta_H) + 0.5\pi(\theta_L) = 0.5(\frac{(4-1)^2}{4*4} + \frac{(3-1)^2}{4*3}) = 0.447\,92$$

This is the optimal profit if the firm knows the identity of the consumer, and uses a linear price system. However, the firm can do better by going beyond the linear pricing. In particular, the following will show that the firm can use a two-part pricing scheme to improve the profit.

(b) Two-parts pricing

The firm can do better by using a two-part pricing system. The optimal pricing scheme is to use a two-part tariffs: charge c per unit, and a fixed fee equal to the consumer surplus at the price c.

In the case of linear demand above, the price axis intercept of the linear inverse demand computed above is θ, we have $T_H = \frac{1}{2}(\theta_H - c)q_H$ for the high valuation buyer, and $T_L = \frac{1}{2}(\theta_L - c)q_L$ for the low valuation buyer, where $q_i, i = H, L$ are the optimal demand quantities in maximizing

$$\theta_i v(q) - cq$$

with q_i satisfying the first order condition

$$\theta_i v'(q_i) = c \tag{3}$$

This is a first-best solution for the monopolist. The highest profit is realized. The idea is that it is better to earn no profit on the per unit price, and charge the maximum possible fixed fee (equal to the consumer surplus). We can refer to this as the Disney pricing scheme.

Note that the above formula for computing surplus is specific to the linear demand case. A more general way to compute T_H, T_L at $p = c$ is as follows:

$$T_i = \theta_i v(q_i) - cq_i, i = H, L \tag{4}$$

which is the total willingness to pay minus the amount paid. More generally, the surplus of consumer i at p is given by $T_i = \theta_i v(q_i(p)) - pq_i(p)$.

To illustrate with the linear deamnd example, When $i = H$, the solution for (3) is

$$4(1 - q_H) = 1$$

or $q_H = 0.75$. Similarly $q_L = \frac{2}{3}$. Hence the fixed fees are

$$T_H = \frac{1}{2}(\theta_H - 1)(1 - \frac{1}{\theta_H}) = \frac{1}{2} * 3 * \frac{3}{4} = \frac{9}{8}$$

$$T_L = \frac{1}{2}(3 - 1)(1 - \frac{1}{3}) = \frac{2}{3}$$

The expected profit of the firm is

$$0.5T_H + 0.5T_L = 0.5(\frac{9}{8} + \frac{2}{3}) = 0.895\,83$$

which is much better than the monopoly pricing earlier.

Note that if the same price and fixed fee has to be charged for both consumers (no discrimination), it is treated below under incomplete information (as if the firm does not have the needed information to discriminate). The effect of legal requirement of no discrimination is the same as incomplete information.

(c) Optimal linear pricing with incomplete information

(for comparison purposes with earlier materials on wage offers, note that the consumer is assumed to have 0 option in this problem. Also the uncertainty is discrete here. We also assume that monopoly does not want to price any consumers out of the market, and the trade-off is between higher price or higher quantity.)

If the monopolist does not know θ, and offer a single price P per unit for both types of buyers. A consumer of type i has demand $D_i(P)$ satisfying

$$\theta_i v'(D_i(P)) = P$$

Let the total demand be

$$D\,(P) = \beta D_L(P) + (1 - \beta) D_H(P)$$

then the monopolist maximizes the expected profit

$$(P - c)D(P)$$

with the optimal monopoly price determined by the following first-order condition

$$P = c - \frac{D(P)}{D'(P)}$$

For the numeric example earlier, we have $D(P) = 0.5(1 - \frac{P}{3}) + 0.5(1 - \frac{P}{4}) = 1 - \frac{7}{24}P$. Hence the optimal price is given by

$$P = 1 + \frac{1 - \frac{7}{24}P}{\frac{7}{24}} = \frac{31}{7} - P,$$

or $P = \frac{31}{14}$. The maximum profit is

$$(P - c)D(P) = (\frac{31}{14} - 1)(1 - \frac{7}{24}\frac{31}{14}) = \frac{289}{672} = 0.430\,06.$$

As we would expect, this profit is lower than the case of complete information case (a).

(d) Optimal single two-parts tariff with incomplete information

If the monopolist can use a two-part tariff: (Z, P), where Z is the fixed fee and P is the per unit price, in general the firm profit will also be higher. Let

$$S_L(P) = \theta_L v(D_L(P)) - PD_L(P)$$

be the total surplus for the low demander. Assume that the monopolist serves both types of buyers. The fixed fee will be $Z = S_L(P)$. This fixed fee is paid by both types of consumers. The total expected demand at P is $D(P)$, and the expected profit for the firm is

$$S_L(P) + (P - c)D(P)$$

The first-order condition of this maximization problem gives us the optimal P^* satisfying

$$P = c - \frac{D(P) + S_L'(P)}{D'(P)}$$

The maximum expected profit for the monopolist firm is

$$S_L(P^*) + (P^* - c)D(P^*)$$

Using the same numeric example, we can compute the optimal price. First we compute

$$S_L(P) = \theta_L v(D_L(P)) - PD_L(P) = 3((1-\frac{P}{3}) - \frac{1}{2}(1-\frac{P}{3})^2) - P(1-\frac{P}{3}) = \frac{1}{6}P^2 - P + \frac{3}{2}.$$

Hence

$$S'_L(P) = \frac{1}{3}P - 1.$$

The expected demand is

$$D(P) = 1 - \frac{7}{24}P,$$

and the optimal price is the solution of the equation

$$P = 1 + \frac{1 - \frac{7}{24}P + \frac{1}{3}P - 1}{\frac{7}{24}} = 1 + \frac{1}{7}P.$$

We have $P = \frac{7}{6}$. The fixed fee is

$$S_L(P) = \frac{1}{6}P^2 - P + \frac{3}{2} = \frac{1}{6}(\frac{7}{6})^2 - \frac{7}{6} + \frac{3}{2} = \frac{121}{216} = 0.560\,19.$$

The equilibrium expected profit of the monopolist is

$$S_L(P^*) + (P^* - c)D(P^*) = 0.56019 + \frac{1}{6}(1 - \frac{7}{24}\frac{7}{6}) = 0.670\,14.$$

which is higher than what the monopolist can achieve with a linear price.

Note that the firm may want to exclude low demand consumers from participation (by setting the fixed fee high enough). This in some cases can increase the profit of the firm further depending on the parameters.

(e) Optimal menu of contracts (second best) with incomplete information

By allowing a menu of choices T_i, P_i, where T_i is the total payment of the consumer type i, and P_i is the per unit pricing. The interpretation of T_i is now slightly different here. This is the total payment to the monopolist by consumer of type i (including fixed fees and consumption purchases). This is a more conveni

Here we assume that the monopolist wants to sell to both types of consumers. The firm can exclude certain consumers from purchase and in some situations it is better to practice exclusive pricing and serve only certain types of consumers. This is ruled out now by assumption.

In choosing different pricing plans, the consumers reveal their information, and self-select. The firm designs the contracts keeping in mind the types of consumers who will choose a particular contract. The maximization problem of the monopolist now is done subject to these constraints: participation constraint (meaning that a particular contract is attractive enough for the type of consumers it is designed for to accept it), and incentive constraint (the contract should be the best possible for the type of consumers it is designed for among all the contracts they can choose).

The monopolist firm now offer two contracts $(q_H, T_H), (q_L, T_L)$ and maximizes

$$\beta(T_L - cq_L) + (1 - \beta)(T_H - cq_H)$$

subject to the participation constraints

$$\theta_i v(q_i) - T_i \geq 0 \text{ for } i = H, L$$

and the incentive constraints

$$\theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L$$

$$\theta_L v(q_L) - T_L \geq \theta_L v(q_H) - T_H$$

The only binding constraints are the incentive constraint for the high type and the participation constraint for the low type (The Spence-Mirrlees single crossing condition— indifference curves in the q, T space can only cross once—is satisfied). This means that instead of four constraints, only two of them will be used in the optimization problem. Binding means that the constraints are in equalities rather than inequalities.

We can rewrite the maximization problem as

$$\max \beta(T_L - cq_L) + (1 - \beta)(T_H - cq_H) \tag{5}$$

subject to

$$\theta_H v(q_H) - T_H = \theta_H v(q_L) - T_L$$

$$\theta_L v(q_L) - T_L = 0$$

The two equations allow us to substitute q_L, q_H for T_L, T_H and transform the constrained maximization problem into an unconstrained maximization problem with two variables q_L, q_H. We can then use the first-order conditions to determine the optimal solution.

Taking the derivative with respect to each, we have two equations from which the optimal q_L, q_H can be solved. We have

$$\theta_H v'(q_H^*) = c \tag{6}$$

$$\theta_L v'(q_L^*) = \frac{c}{1 - (\frac{1-\beta}{\beta}\frac{\theta_H - \theta_L}{\theta_L})} > c$$

In the optimal contract, the high type consumption is the first-best amount (marginal benefit equals marginal cost) , and enjoys positive surplus, while the low type consumption is lower than the first-best amount, and has zero rent. There is efficiency in allocation for the high type, and inefficiency of allocation for the low type. The low-type would like to consumer more (and the firm would like to supply it because marginal cost is lower than the price the consumer is willing to pay for it. However, the firm refrains from doing that), but is prevented from higher consumption, because the package $(q_{H,}T_H)$ is too expensive for them.

A numeric example: Let $\theta_L = 3, \theta_H = 4, \beta = 0.5$. Assume that $c = 1$. Let $v(q) = q - \frac{1}{2}q^2$. The binding participation constraint of the low type is

$$3v(q_L) = T_L$$

and the binding incentive constraint for the high type is

$$T_H = 4v(q_H) - 4v(q_L) + T_L = 4v(q_H) - v(q_L)$$

Substitute into the following objective function

$$0.5(T_L - q_L) + 0.5(T_H - q_H)$$

we get the new objective function

$$0.5(q_L - q_L^2) + 1.5q_H - q_H^2$$

This is an unconstrained problem. Take the derivatives, and we have

$$q_L^* = \frac{1}{2}, q_H^* = \frac{3}{4}$$

You can also apply the equations in (6):

$$4(1-q_H^*)=1$$

$$3(1-q_L^*)=\frac{1}{1-\frac{1}{3}}$$

and get the same solution. From these quantities, we can compute the payments

$$T_L=\theta_L v(q_L)=3(\frac{1}{2}-\frac{1}{8})=\frac{9}{8}$$

and

$$T_H=4(\frac{3}{4}-\frac{1}{2}(\frac{3}{4})^2)-(\frac{1}{2}-\frac{1}{8})=\frac{3}{2}$$

The quantities imply the per unit prices

$$p_H=4(1-q_H)=1, p_L=3(1-q_L)=\frac{3}{2}$$

The contracts are often stated in prices rather than quantitates. Given the prices, quantities are determined, and vice versa. Thus the optimal contract for the high type is $p_H=1, T_H=\frac{3}{2}$, and for the low type is $p_L=\frac{3}{2}, T_L=\frac{9}{8}$. This means that the fixed fee for the high demander is $T_H-p_Hq_H=\frac{3}{2}-\frac{3}{4}=\frac{3}{4}$. The fixed fee for the low demander is $T_L-p_Lq_L=\frac{9}{8}-\frac{3}{2}\frac{1}{2}=\frac{3}{8}$. Substitute these values into the formula (5), we get the optimal second-best profit of the firm. The optimal profit is

$$0.5T_H+0.5T_L-0.5c(q_H+q_L)=\frac{1}{2}(\frac{9}{8}+\frac{3}{2}-\frac{3}{4}-\frac{1}{2})=\frac{7}{8}=\frac{11}{16}=0.687\,5$$

This is second-best because it is smaller than 0.895 83 (the best you can get with two-part pricing and complete information), and larger than 0.670 14 (the best you can get with one single two-part pricing, incomplete information, and no choice of menus of contracts).

Homework 8

Due March 27

1. Assume that a monopolist faces an a consumer with uncertain preferences defined by

$$\theta(4q - q^2)$$

where θ is either 5 or 10 with equal probability. Assume that cost of production is $c = 1$ per unit.

(a) If the monopolist knows θ when the consumer arrives and can practice price discrimination, but only linear price system is allowed, what is the maximum expected profit of the monopolist?

(b) If the monopolist cannot practice price discrimination, and only linear price system is allowed, what is the maximum expected profit of the monopolist? (Use the corresponding part with asymmetric information).

(c) Compare the profits in (a) and (b). You should get higher profit in (a)? Why?

2. The monopolist faces the same market as described in problem 1. Here we consider two-part pricing problems.

(a) If the monopolist knows θ when the consumer arrives and can practice price discrimination using a two-part pricing system, what is the maximum expected profit of the monopolist? (Be careful with the computations for T_H, T_L. They are given by slightly different formulas because price axis intercept is different here.)

(b) If the monopolist does not practice price discrimination in a two-part pricing system, what is the maximum expected profit of the monopolist?

(c) Compare the profits in (a) and (b). You should get higher profit in (a)? Why?

3. In case (d), we have noted that the firm may want to exclude low demand consumers from participation (by setting the fixed fee high enough). This in some cases can increase the profit of the firm further depending on the parameters. Determine whether this is true in the example considered in case (d). In other words, determine the optimal profit by allowing the firm to consider the possibility of excluding some consumers from participation, and then compare it to optimal profit in case (d).

4. The monopolist faces the same market as described in problem 1, except that we now assume that $\theta_H = 6, \theta_L = 5$. Here we assume that the monopolist cannot observe the parameter θ, but knows the function $v(q) = 4q - q^2$.

(a) Find out the optimal menu of contracts for each type of consumer.

(b) Compute the expected profit of the monopolist from the optimal menu of contracts.

(c) Translate that optimal contracts into a description of fixed fees and per unit charged of each type of the consumers. Note that according to our notations, T_H, T_L represent total payments (rather just fixed fee).

5. Assume that the value distribution of each bidder is $F(v) = v^2$ over $[0, 1]$. In the first-price with two bidders,

(a) Use the formula to compute the equilibrium bidding strategy of each bidder.

(b) Compute the revenue of the seller.

(c) In the second-price auction with truthful bidding, compute the seller's revenue, and see whether you get the same revenue in (b).