MATH 425a ASSIGNMENT 10
FALL 2015 Prof. Alexander
Due Wednesday December 2.

Rudin Chapter 5 #15, Chapter 6 #1, 2, 3a, 8, plus the problems (I)-(V) below. Problems 1, 2, and (VI) should be relatively "quick" ones, the type that most often appears on exams.

(I) Typically, if a continuous function $f : [a,b] \to \mathbb{R}$ has a local maximum at some $c \in [a,b]$, then f is increasing in $(c-\delta, c]$ and decreasing in $[c, c+\delta)$ for some $\delta > 0$. Find an example, though, in which this is false.

(II) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere, with $f'(x) < 3$ for all $x < 0$ and $f'(x) > 3$ for all $x > 0$. Show that $f'(0) = 3$.

(III) Let $f(x) = 2 + 3x$ and

$$\alpha(x) = \begin{cases} x^2, & 0 \le x < 1 \\ 2, & 1 \le x < 2 \\ 2x, & x \ge 2. \end{cases}$$

Calculate $\int_0^3 f \, d\alpha$. (Just a calculation, you don't have to prove the steps.)

(IV) Suppose f, α, β are real-valued functions on $[a,b]$ with α and β nondecreasing. Define $I_\alpha = \inf_P U(P, f, \alpha)$, which is just a shorter notation for $\overline{\int_a^b} f \, d\alpha$, and define I_β and $I_{\alpha+\beta}$ similarly.

(a) Show that for fixed P we have $U(P, f, \alpha+\beta) = U(P, f, \alpha) + U(P, f, \beta)$.
(b) Show that $I_{\alpha+\beta} \ge I_\alpha + I_\beta$.
(c) Let $\epsilon > 0$. Show that $I_{\alpha+\beta} \le I_\alpha + I_\beta + 2\epsilon$.
(d) Show that $I_{\alpha+\beta} = I_\alpha + I_\beta$.

(V) For $x \in [1,4]$ let $f(x) = 2x$ and

$$\alpha(x) = \begin{cases} 0 & \text{if } x < 2, \\ \frac{1}{2} & \text{if } 2 \le x \le 3, \\ \frac{3}{2} & \text{if } x > 3. \end{cases}$$

(a) Let P be a partition of $[1,4]$ such that some $x_i = 2$ and some $x_j = 3$. Find $U(P, f, \alpha)$. The only unspecified quantities in your answer should be some or all of the points $x_0, \dots, x_n$.
(b) Find $\overline{\int_1^4} f \, d\alpha$.
(c) Similarly to (a) and (b), find $L(P, f, \alpha)$ and $\underline{\int_1^4} f \, d\alpha$.
(d) Is $f \in \mathcal{R}(\alpha)$? How can you tell from (a)–(c) alone?

(VI) Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable and bounded, say $|f(t)| \le M$ for all t. Let

$$F(x) = \int_a^x f(t) \, dt, \qquad x \in [a,b].$$

Show that F is uniformly continuous.

HINTS:

(15) You can omit the vector-valued part. The bound on $|f'(x)|$ is valid for *all* $h > 0$. How can you take advantage of this?

This isn't really a hint, just an informal explanation of what problem 15 shows. If M_1 is large, this means $|f'(x)|$ is large for some x. This means one of two things must happen. One possibility is that $|f'(x)|$ remains large for points in the vicinity of x, in which case $f(x)$ must climb to a very large value, meaning M_0 is large. The other possibility is that $|f'(x)|$ does not remain large for points in the vicinity of x, in which case $f'(x)$ must change rapidly, forcing M_2 to be large. Either way, if M_1 is large then the product $M_0 M_2$ must be large. The bound in the problem quantifies this exactly.

(1) For a partition $a = t_0 < \cdots < t_n = b$, focus on the interval $[t_{i-1}, t_i]$ containing x_0.

(2) If f is continuous and $f(x) > 0$ for some x, what must be true for values close to x?

(3) Part (c) was done in lecture; (a) and (b) are similar.

(8) Compare $f(n), f(n+1)$ and $\int_n^{n+1} f(x)\, dx$.

(I) Take a function which has a known local maximum, for example $g(x) = -x^2$ at $x = 0$. Add something to it so that it's no longer monotone on either side of $x = 0$, but it still has a local maximum at $x = 0$.

(II) The problem would be easier if 3 were replaced everywhere by 0. Add something to f to make a new function g, so that the problem converts to this easier question, for g.

(IV)(b) $I_{\alpha+\beta}$ is by definition the *greatest* lower bound, so it's enough to show $I_\alpha + I_\beta$ is a lower bound. (Lower bound for what, exactly? Specify!)

(c) There exist P_1 with $U(P_1, f, \alpha) \le I_\alpha + \epsilon$ and P_2 with $U(P_2, f, \beta) \le I_\beta + \epsilon$. Why is there then a P with $U(P, f, \alpha + \beta) \le I_\alpha + I_\beta + \epsilon$?