Note 7 and HW 7

English Auctions

It is a dynamic ascending auction in which bidders bid up the price of the item. It is an open rather than a sealed bid auction. You see all the previous bids, and decide whether you want to bid higher. There is a deadline, and the bidder with the highest bid before the deadline wins the object. The price you pay is the highest bid. It is also called an open-outcry ascending-price auction. Although it looks like a first-price auction (paying what you bid), it is actually strategically similar to a second-price auction when all bidders have independent values. The reason is that when you bid up the price, you don't actually bid your true value. Rather you bid something that is higher than the price other bidders are willing to pay, and raise the bid unless it is higher than what you are willing to pay. In other words, all bidders who have lower value stop bidding at the prices they are willing to pay. This means that the bidding war will stop at the second-highest price. So you actually pay the second-highest bid, rather than the highest bid.

In online auctions, there is an alternative format. You don't need to be at the computer screen to watch the bidding all the time, and decide whether you raise the bid and at what time. You can just submit a proxy bid instruction. You instruct the computer to slightly raise the bid for you when there is a bid that is lower than your willingness to pay. You tell the software what is your maximum willingness to pay. Then the software will bid on your behalf when it appears that someone is going to win the auction at a price below what you are willing to pay. The software will bid on your behalf, but not to exceed your maximum willingness to pay. The proxy bid is kept secret from others. So it is like a sealed bid in this sense even though the actual bids are observed. The optimal proxy bid is your willingness to pay. It is like a sealed bid second-price auction in which you bid your true value in sealed envelop. Most of the online auctions use proxy bids, and strategically it is like a second-price auction.

Dutch Auctions

Another well-known auction is called Dutch auction. It is also called an open-outcry descending-price auction. It is used often to sell the goods quickly as it requires only one bid to win the good. The auctioneer begins with a high asking price which is lowered until some bidder is willing to accept the auctioneer's price, or a predetermined reserve price (the seller's minimum acceptable price) is reached. The winning participant pays the last announced price. This is also known as a (descending) clock auction. The Dutch auction is strategically equivalent to a first-price auction.

Multi-unit version of the Dutch auction: you name the quantities you want. After you win the auction, if there is any item left, the auction continues until it is all sold out. Multi-unit auctions can be either uniform price auctions or pay-what-you-bid auctions (or discriminatory auctions).

The United States Department of the Treasury, through the Federal Reserve Bank of New York (FRBNY), raises funds for the U.S. Government using a Dutch auction. The FRBNY interacts with primary dealers, including large banks and broker-dealers who submit bids on behalf of themselves and their clients using the Trading Room Automated Processing System (TRAPS), and are generally told of winning bids within fifteen minutes.

For example, suppose the sponsor of the issuance is seeking to raise $10 billion in ten-year notes with a 5.125% coupon and in aggregate the bids are as follows:

$1.00 billion at 5.115% (highest bid)
$2.50 billion at 5.120%
$3.50 billion at 5.125%
$4.50 billion at 5.130%
$3.75 billion at 5.135%
$2.75 billion at 5.140%
$1.50 billion at 5.145% (lowest bid)

In this example the % at high is 66.66%, meaning only $3 billion of the $4.5 billion at 5.130% will get bonds. Bids will be filled from the lowest interest or yield (highest price) until the entire $10 billion has been raised. This auction will clear at a yield of 5.130%, and all bidders will pay the same amount (uniform price). In theory, the uniform-pricing feature of the Dutch auction format leads to more aggressive bidding as those who in this case bid 5.115% will receive the bonds at the higher yield (lower price) of 5.130%.

A variation on the Dutch auction, OpenIPO, was developed by WR Hambrecht and has been used for 19 IPOs in the US. Auctions have been used for hundreds of IPOs in more than two dozen countries, but have not been popular with issuers and thus were replaced by other methods. One of the largest uniform price "Dutch" auction IPOs was for Singapore Telecom in 1994. The 1994 auction IPO of Japan Tobacco was substantially larger (with proceeds more than double those of Singapore Telecom and triple those of Google), but this auction was discriminatory or pay-what-you-bid, not uniform price or "Dutch"

Share repurchase Dutch auctions

The introduction of the Dutch auction share repurchase in 1981 allows firms an alternative to the fixed price tender offer when executing a tender offer share repurchase. The first firm to utilize the Dutch auction was Todd Shipyards. A Dutch auction offer specifies a price range within which the shares will ultimately be purchased. Shareholders are invited to tender their stock, if they desire, at any price within the stated range. The firm then compiles these responses, creating a supply curve for the stock. The purchase price is the lowest price that allows the firm to buy the number of shares sought in the offer, and the firm pays that price to all investors who tendered at or below that price. If the number of shares tendered exceeds the number sought, then the company purchases less than all shares tendered at or below the purchase price on a pro rata basis to all who tendered at or below the purchase price. If too few shares are tendered, then the firm either cancels the offer (provided it had been made conditional on a minimum acceptance), or it buys back all tendered shares at the maximum price.

History of the Dutch auctions

It's the early 1600's in Holland, and the tulip bulb has recently been introduced to the country by traders from the Ottoman empire. The flower is unique and unlike any other flower in Europe at the time. The demand for tulip started to grow quickly.

However, the botanical limitations of the tulip serve to constrain the supply curve. It takes 7-12 years to grow a tulip from a seed to its tradable asset, the bulb. And the most sought after bulbs, the multi-colored bulbs, would take even longer to grow. So with a quickly growing demand curve, and a relatively fixed supply curve with an extremely long production cycle, the possibility for a spike in prices is extremely high. To compare it to modern products, imagine if the demand for a time-intensive alcoholic beverage suddenly exploded in popularity. Take something like Johnny Walker Black and assume that its popularity explodes 3x over the next year. Because Johnny Walker takes 12 years to make, there's no possible way to meet the demand, and as a result the price starts to grow extremely quickly.

The flower only blooms during April and May. During the Summer months, the flower returns into its bulb form, which means it can be dug up and moved around. The rest of the year, the tulip must be in dirt and doesn't show a flower. Essentially, the tulip can only be traded for 3 months a year, but the market and demand exist all year round. What resulted was the first modern futures contract - at any time during the year, tulip sellers and tulip buyers would sign a contract to deliver X amount of tulips on a given date in the Summer at price Y. Further, these contracts could be traded amongst other buyers and sellers to create a functional futures market. So, the futures market as we know it today is a direct result of the Dutch traders wanting to buy and sell tulips year-round despite being botanically limited to physically trading them and delivering them for only 3 months a year.

The beginning of the option market

Finally, during the height of the Panic, the Dutch Parliament passed a law in an effort to curb the panic that stated that all futures contracts could be canceled at any point if the buyer paid the seller 3.5% of the contract price. Yes, even the primitive option market was created during the Dutch Tulip Panic. These new option contracts allowed traders to increase their bets through massive leverage.

You can imagine traders milling around talking loudly, the noise, the hubbub, the contracts flying around. Total commotion, very noisy, and very hard to keep everything organized. The Exchange figured, correctly, that the best solution to selling the bulbs was to do it quickly, to do it with as few bids as possible, while at the same time getting as high a price as possible. The solution, of course, was a Dutch Auction.

Tulip mania

The fame of this beautiful bloom spread rapidly and the flower was immediately popular with the upper classes. As the tulip gained ground, competition amongst growers started to produce the most beautiful specimens. Most admired were tulips in vivid colors, which were multicolored or had lines, stripes and flames on the petals – ironically we now know that these variations come from a tulip-specific virus and are actually imperfections.

Tulips became a luxury item and a status symbol. People were willing to pay vast sums of money for a single bulb and the prices rose constantly. Soon tulip mania was gripping the country. The speculation in bulbs increased as people saw this as a quick and easy route to making their fortune. In 1636, stock exchanges were established to trade in bulbs and their future options. Despite attempts by the authorities to limit the craze, trade blossomed and people sold land, houses and valuable objects to invest in tulip bulbs. The biggest sellers were the Semper Augustus and Viceroy bulbs and by 1635, a sale of 100,000 florins for 40 bulbs was recorded. This was ten times the average salary of a skilled labourer!

Of course this hype had to end sometime and in 1637 the market crashed, leaving most traders with not much more than a bunch of flowers. Today, tulip mania, Tulipomania or tulip madness is a term used to refer to any large economic bubble that cannot last. The Dutch do still love their tulips. Dutch tulip growers still dominate the world tulip bulb industry. There are 12,000 hectares of tulip bulbs. The Dutch are serious about their love for tulips, and put these special flowers in the spotlight on National Tulip Day. The celebration traditionally takes place in January each year on Amsterdam's Dam Square, inviting tulip-lovers to pick their own bloom for free from a specially constructed 'picking garden'. Later in the year, the Amsterdam Tulip Days allow visitors to marvel at a variety of colorful (and occasionally rare) tulips in the gardens of museums, private homes and institutions throughout the city.

All pay auctions: auctions in which a loser may still need to pay. Many sports competition can be modeled as an all pay auction. Instead of competing with money, you compete by spending time effort and skill. The more you pay or spend, the more you are likely to win. Even if you lose, you still pay in your effort and time.

Ideas of Revenue Equivalence Theorem in auctions

Envelop theorem in economics
Assume $F(.)$ is defined over $[0, b]$.
Suppose you have a maximization (or minimization) problem:

$$Max_x u(c, x)$$

the maximum solution $x(c)$ depends on c. We want to know how $u^*(c) = \max u(c, x) = u(c, x(c))$ changes with c. The Envelop Theorem says that $\frac{d}{dc} u^*(c) = \frac{\partial}{\partial c} u(c, x)|_{x=x(c)}$.

Apply this idea to the case of first-price auctions. Assume that the Bayesian Nash equilibrium is symmetric.

Bidders choose b maximize Maximize $u(v,b) = F(\phi(b))(v-b)$. Let $b(v)$ be the optimal bidding strategy. We have $\frac{\partial}{\partial v}u(v,b) = F(\phi(b))$. Hence $\frac{d}{dv}u(v,b(v)) = F(\phi(b(v))) = F(v)$. In other words, the derivative of the equilibrium payoff $u(v,b(v))$ with respect to v is $F(v)$. Assume that bidders with 0 value has 0 payoff in equilibrium. Then the fundamental theorem of calculus says that

$$u(v,b(v)) = \int_0^v F(x)dx.$$

But we also know that

$$u(v,b(v)) = F(\phi(b(v)))(v - b(v)) = F(v)(v - b(v)).$$

Hence we have

$$F(v)(v - b(v)) = \int_0^v F(x)dx.$$

and we have the formula for the equilibrium bidding strategy

$$b(v) = v - \frac{1}{F(v)}\int_0^v F(x)dx.$$

This is the formula that I gave you before.

What about second price auctions?
We have

$$u(v,b) = \int_0^{\phi(b)} (v-x)dF(x) = vF(\phi(b)) - \int_0^{\phi(b)} xdF(x).$$

$$\frac{\partial}{\partial v}u(v,b)|_{b=b(v)} = F(v).$$

Hence, for second-price auctions, we also have

$$u(v,b(v)) = \int_0^v F(x)dx.$$

For any payment rule which can be any function of the submitted bids, we have

$$u(v,b) = vF(\phi(b)) - p(b,\phi),$$

where $p(b,\phi)$ is the expected payment amount when the winning bid is b. When you take the partial derivative with respect to v, the term $p(b,\phi)$ disappear, and we have

$$\frac{\partial}{\partial v}u(v,b)|_{b=b(v)} = F(v).$$

For any auction yielding 0 payoff to the bidder with 0 payoff, and highest bidder wins, we have the same formula. The bidders equilibrium payoff is the same. This means that each bidder's contribution to the seller revenue is the same. Hence we have the revenue equivalence theorem.

Optimal reservation price:

Because of revenue equivalence, we can use any particular auction format to obtain the optimal reservation price. Let $F(v) = v$. There are two buyers. The seller has 0 value for the object. We assume private value assumptions. We try the first-price auctions. Given the reservation price r, the equilibrium bidding strategy is given by

$$b_r(v) = v - \frac{1}{F(v)} \int_r^v F(x)dx,$$

or

$$b_r(v) = v - \frac{1}{v} \int_r^v xdx = v - \frac{1}{2v}(v^2 - r^2) = \frac{1}{2}v + \frac{r^2}{2v}.$$

When r is higher, you lose the revenue from selling to buyers with value below r. However, the bidding amount for each buyer with value above r is higher. So we have a trade-off. The former is loss from higher reservation price, while the latter is gain from higher reservation price. The optimal reservation price is chosen so that the gain and loss are balanced in a optimal way. The revenue is given by

$$\int_r^1 (\frac{1}{2}v + \frac{r^2}{2v})dv^2 = 2\int_r^1 (\frac{1}{2}v + \frac{r^2}{2v})vdv = -\frac{4}{3}r^3 + r^2 + \frac{1}{3}.$$

Taking derivative with respect to r, we have

$$-4r^2 + 2r = 0,$$

hence $r = 0.5$ is the optimal reservation. In the case of N buyers, we have the following equilibrium bidding strategy

$$b_r(v) = v - \frac{1}{v^{N-1}} \int_r^v x^{N-1}dx = v - \frac{1}{Nv^{N-1}}(v^N - r^N) = \frac{N-1}{N}v + \frac{r^N}{Nv^{N-1}}.$$

The revenue is

$$N\int_r^1 (\frac{N-1}{N}v + \frac{r^N}{Nv^{N-1}})v^{N-1}dv = \int_r^1 ((N-1)v^N + r^N)dv$$

$$= \frac{N-1}{N+1}(1 - r^{N+1}) + r^N(1-r)$$

$$= \frac{N-1}{N+1} - \frac{2N}{N+1}r^{N+1} + r^N.$$

Taking the derivative, we get

$$-2Nr^N + Nr^{N-1} = 0$$

or

$$Nr^{N-1}(-2r + 1) = 0,$$

or $r = 0.5$ is the optimal reservation price. This reservation price is independent of the number of buyers. The solution $r = 0$ is not the optimal one because of the second order condition fails. The revenue is increasing at 0, as for r close to 0, the derivative is positive.

For more general $F(.)$ over $[0, \beta]$, we can follow the derivation of Myerson (1981). We have the equilibrium bidding strategy:

$$b(v) = v - \frac{1}{F^{N-1}(v)} \int_r^v F^{N-1}(x)dx.$$

Hence the expected revenue of the auction with the reservation price r is given by

$$\begin{aligned}
& N \int_r^\beta \left(F^{N-1}(v)v - \int_r^v F^{N-1}(x)dx \right) dF(v) \\
&= \int_r^\beta vdF^N(v) - N \int_r^\beta \left(\int_r^v F^{N-1}(x)dx \right) dF(v) \\
&= \int_r^\beta t(v)vdF^N(v) - N \int_r^\beta F^{N-1}(x) \left(\int_x^\beta dF(v) \right) dx \\
&= \int_r^\beta vdF^N(v) - N \int_r^\beta (1 - F(x))F^{N-1}(x)dx \\
&= \int_r^\beta vdF^N(v) - \int_r^\beta \frac{1 - F(v)}{f(v)} dF^N(v) \\
&= \int_r^\beta (v - \frac{1 - F(v)}{f(v)})dF^N(x)
\end{aligned}$$

Let the virtual value be defined by

$$J(v) = v - \frac{1 - F(v)}{f(v)},$$

we have

$$R = \int_r^\beta J(v)dF^N(v).$$

For the uniform distribution $F(v) = v$, we have

$$J(v) = 2v - 1.$$

The highest revenue is obtained when $2v - 1 = 0$, or $v = 0.5$. More generally, if $J(v)$ is an increasing function, the optimal reservation price is is the solution of the equation

$$v - \frac{1 - F(v)}{f(v)} = 0.$$

Note that this solution is independent of the number of buyers N.

We can apply this revenue formula to the auction with two buyers and $F(v) = v$. We get revenue with no reserve price:

$$R = \int_0^1 (2v - 1) dv^2 = \int_0^1 2v(2v - 1) dv = \frac{1}{3}.$$

This is the same amount of revenue computed earlier. We now compute it with a very different formula, and we get the same number. For another example, let $F(v) = v^2$, and $N = 2, r = 0$. We have

$$J(v) = v - \frac{1 - v^2}{2v},$$

hence the revenue is given by

$$R = \int_0^1 (v - \frac{1 - v^2}{2v}) dv^4 = \int_0^1 4(v - \frac{1 - v^2}{2v}) v^3 dv$$
$$= \int_0^1 (6v^4 - 2v^2) dv = \frac{8}{15}.$$

You can try to get the revenue with an earlier formula, and verify whether you get the same revenue. For $r > 0$, this new formula may be easier to compute than the earlier one.

Homework 7
Due March 20

1. The buyer value distribution is given by $F(v) = v^{0.5}$. There are two buyers in a first-price auction.

(a) Compute the expected revenue of the auction at reservation price 0.

(b) Compute the expected revenue of the auction at any reservation price r.

(c) From (b), find the optimal reservation price r. Verify the second-order condition for all the solutions you get from the first-order condition.

(d) Find the optimal reservation price from the solution of the following equation

$$x = \frac{1 - F(x)}{f(x)}.$$

2. The buyer value distribution is given by $F(v) = \frac{1+0.5v}{2}, v \in [0, 2]$. There are two buyers in a first-price auction.

(a) Compute the expected revenue of the auction at reservation price 0.

(b) Compute the expected revenue of the auction at any reservation price r.

(c) From (b), find the optimal reservation price r. Verify the second-order condition for all the solutions you get from the first-order condition.

(d) Find the optimal reservation price from the solution of the following equation

$$x = \frac{1 - F(x)}{f(x)}.$$

3. The buyer value distribution is given by $F(v) = e^{v-1}, v \in [0, 1]$. There are two buyers in a first-price auction.

(a) Compute the expected revenue of the auction at reservation price 0.

(b) Compute the expected revenue of the auction at any reservation price r.

(c) From (b), find the optimal reservation price r. Verify the second-order condition for all the solutions you get from the first-order condition.

(d) Find the optimal reservation price from the solution of the following equation

$$x = \frac{1 - F(x)}{f(x)}.$$

4. The buyer value distribution is given by $F(v) = v, v \in [0, 1]$. There are N buyers in a first-price auction.

(a) Compute the expected revenue of the auction at reservation price 0.

(b) Compute the expected revenue of the auction at any reservation price r.

(c) From (b), find the optimal reservation price r. Verify the second-order condition for all the solutions you get from the first-order condition.

(d) Find the optimal reservation price from the solution of the following equation
$$x = \frac{1 - F(x)}{f(x)}.$$

(e) If $N \to \infty$, what is the limit of the seller's revenue. How do you interpret the result?