MATH 425a ASSIGNMENT 10 SOLUTIONS
FALL 2015 Prof. Alexander

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Chapter 5:

(15) Fix $x \in (a, \infty)$. Taylor's Theorem 5.15 says that for $h > 0$,

$$f(x + 2h) = f(x) + 2hf'(x) + \frac{1}{2}f''(\xi)(2h)^2$$

for some $\xi \in (x, x + 2h)$, or after solving for $f'(x)$,

$$f'(x) = \frac{1}{2h}[f(x + 2h) - f(x)] - hf''(\xi).$$

Taking the magnitude gives

$$|f'(x)| \le \frac{1}{2h} \cdot 2M_0 + hM_2,$$

and this is valid for all $x \in (a, \infty)$. Therefore we can take the sup over x and conclude that

$$(*) \qquad M_1 \le \frac{M_0}{h} + hM_2.$$

This is valid for all $h > 0$ so we can take the minimum over h on the right side. This is just a calculus problem: the minimum of the right side is achieved where the derivative $M_2 - M_0/h^2 = 0$, that is at $h = \sqrt{M_0/M_2}$. Plugging this value of h into $(*)$ we get $M_1 \le 2\sqrt{M_0 M_2}$, or equivalently $M_1^2 \le 4M_0M_2$.

Chapter 6:

(1) Suppose α is increasing on $[a, b]$, $a \le x_0 \le b$, α is continuous at x_0. By 6.7a we need only consider partitions $P = \{t_0, \dots, t_n\}$ containing the point x_0, so $t_j = x_0$ for some j. Then $M_i = m_i = 0$ for all i except $i = j, j + 1$, and $M_j = M_{j+1} = 1$, $m_j = m_{j+1} = 0$. Therefore $L(P, f, \alpha) = 0$ and

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i = 1 \cdot \Delta\alpha_j + 1 \cdot \Delta\alpha_{j+1} = \alpha(t_{j+1}) - \alpha(t_{j-1}).$$

Since α is continuous at x_0, by taking t_{j-1} and t_{j+1} close enough to $t_j = x_0$ we can make $\alpha(t_{j+1}) - \alpha(t_{j-1})$ as close to 0 as we like. Therefore $\inf_P U(P, f, \alpha) = 0 = \sup_P L(P, f, \alpha)$, which means $f \in \mathcal{R}(\alpha)$ with $\int f \, d\alpha = 0$.

(2) Prove the contrapositive: suppose $f \in \mathcal{R}$ and $f(x_0) > 0$ for some $x_0 \in [a,b]$. Taking $\epsilon = f(x_0)/2$ we see that there exists $\delta > 0$ such that

$$x \in [a,b], |x - x_0| \leq \delta \implies |f(x) - f(x_0)| < \frac{f(x_0)}{2} \implies f(x) > \frac{f(x_0)}{2}.$$

Let $[c,d]$ be an interval contained in $[a,b] \cap (x_0 - \delta, x_0 + \delta)$, so that for all $x \in [c,d]$ we have $f(x) > \frac{f(x_0)}{2}$. Let $P = \{t_0, \ldots, t_n\}$ be any partition containing the points c and d, say $t_j = c, t_k = d$. For all $j+1 \leq i \leq k$ we have $m_i \geq \frac{f(x_0)}{2}$, so

$$\int_a^b f\ dx \geq L(P,f) \geq \sum_{i=j+1}^{k} m_i \Delta x_i \geq \frac{f(x_0)}{2} \sum_{i=j+1}^{k} \Delta x_i = \frac{f(x_0)}{2}(d-c) > 0.$$

Therefore $\int_a^b f\ dx = 0$ implies $f(x) = 0$ for all x.

(3) By Theorem 6.4 we need only consider partitions P containing the point 0. Let $P = \{x_0, \ldots, x_n\}$ with $x_j = 0$. Then $\Delta(\beta_1)_i = 0$ for all $i \neq j+1$, and $\Delta(\beta_1)_{j+1} = 1$, so

$$(*) \quad U(P,f,\beta_1) - L(P,f,\beta_1) = M_{j+1}\Delta(\beta_1)_{j+1} - m_{j+1}\Delta(\beta_1)_{j+1} = M_{j+1} - m_{j+1}.$$

(a) Suppose first that $f(0+) = f(0)$ and let $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$(**) \quad x \in [0,\delta] \implies |f(x) - f(0)| < \frac{\epsilon}{2} \implies f(0) - \frac{\epsilon}{2} < f(x) < f(0) + \frac{\epsilon}{2}.$$

Consider P with $x_{j+1} < \delta$; for such P, by $(**)$, M_{j+1} and m_{j+1} are both between $f(0) - \frac{\epsilon}{2}$ and $f(0) + \frac{\epsilon}{2}$, so by $(*)$, $U(P,f,\beta_1) - L(P,f,\beta_1) \leq \epsilon$. By 6.6 this shows $f \in \mathcal{R}(\beta_1)$. To evaluate $\int f\ d\beta_1$, observe that by the above, for all P containing 0, $U(P,f,\beta_1) = M_{j+1}\Delta(\beta_1)_{j+1} = M_{j+1}$, which is between $f(0)$ and $f(0) + \frac{\epsilon}{2}$. Since ϵ is arbitrary, we must have $\int f\ d\beta_1 = \inf_P U(P,f,\beta_1) = f(0)$.

Conversely suppose $f \in \mathcal{R}(\beta_1)$ and let $\epsilon > 0$. Then by 6.6 there exists a partition P with $U(P,f,\beta_1) - L(P,f,\beta_1) < \epsilon$, and as noted above, we can take $x_j = 0$ for some j. By $(*)$ this means $M_{j+1} - m_{j+1} < \epsilon$, which says that for all $0 \leq t \leq x_{j+1}$ we have $f(t), f(0) \in [m_{j+1}, M_{j+1}]$ and hence $|f(t) - f(0)| < \epsilon$. Since ϵ is arbitrary this shows that $f(0+) = f(0)$.

(8) Let $f \geq 0$ be monotone decreasing on $[1,\infty)$. Define g, h on $[1,\infty)$ by

$$g(x) = f(n) \quad \text{for } x \in [n, n+1), \qquad h(x) = f(n+1) \quad \text{for } x \in [n, n+1).$$

Then $h \leq f \leq g$. Let

$$F(b) = \int_1^b f(x)\ dx, \qquad G(b) = \int_1^b g(x)\ dx, \qquad H(b) = \int_1^b h(x)\ dx.$$

Then $H \leq F \leq G$. Also F, G, H are nondecreasing, so each has a limit as $b \to \infty$ if and only if it is bounded. For integers n we have

$$H(n) = \sum_{i=2}^{n} f(i), \qquad G(n) = \sum_{i=1}^{n} f(i),$$

so H and G are bounded if and only if $\sum_{i=1}^{\infty} f(i)$ converges. Thus

$$\sum_{i=1}^{\infty} f(i) \text{ converges} \implies G \text{ bounded} \implies F \text{ bounded} \implies \lim_{b\to\infty} F(b) \text{ exists}$$

$$\implies \int_1^{\infty} f(x)\, dx \text{ converges,}$$

and

$$\int_1^{\infty} f(x)\, dx \text{ converges} \implies \lim_{b\to\infty} F(b) \text{ exists} \implies F \text{ bounded} \implies H \text{ bounded}$$

$$\implies \sum_{i=1}^{\infty} f(i) \text{ converges.}$$

Handout:

(I) The idea is to take a function with a standard local maximum at some x_0 (we will use $g(x) = -2x^2$, which has a maximum at $x_0 = 0$) and add something to it so that the resulting $f(x)$ oscillates between increasing and decreasing, on both sides of the maximum. But what we add must be small enough that it does alter the fact we have a local maximum at x_0.

One example that works: let $f(x) = -x^2(2 + \sin\frac{1}{x})$; this is obtained by starting with $-2x^2$ and adding the oscillating function $-x^2 \sin\frac{1}{x}$. Then $f(x) \le 0$ for all x and $f(0) = 0$, so f has a local maximum at $x = 0$. Note that $f(x)$ oscillates between $-x^2$ and $-3x^2$. Then

$$f'(x) = \cos\frac{1}{x} - 4x - 2x\sin\frac{1}{x},$$

so $f'(x) - \cos\frac{1}{x} \to 0$ as $x \to 0$. Since $\cos\frac{1}{x}$ oscillates infinitely many times between 1 and -1 as $x \to 0$, this means that in any neighborhood of 0, there are intervals where f' is positive (so f is increasing) and intervals where f' is negative (so f is decreasing.) Thus there is no interval $(-\delta, 0]$ where f is increasing, nor an interval $[0, \delta)$ where f is decreasing.

(II) Let $g(x) = f(x) - 3x$. Then $g'(x) < 0$ for all $x < 0$, and $g'(x) > 0$ for all $x > 0$. Since g is differentiable it is continuous, so given $x < 0$ we can apply the Mean Value Theorem to say that there exists $t \in (x, 0)$ with $g(0) - g(x) = g'(t)(0 - x) < 0$ so $g(x) > g(0)$. Similarly if $x > 0$ then $g(x) > g(0)$. It follows that g has a local minimum at $x = 0$, so $g'(0) = 0$, which is the same as $f'(0) = 3$.

(III) We have

$$\alpha'(x) = \begin{cases} 2x, & 0 \le x < 1 \\ 0, & 1 < x < 2 \\ 2, & x > 2. \end{cases}$$

Also, α has a jump of size 1 at $x = 1$, and size 2 at $x = 2$. Therefore

$$\begin{aligned}
\int_0^3 f(x)\ d\alpha(x) &= \int_0^3 f(x)\alpha'(x)\ dx + 1 \cdot f(1) + 2 \cdot f(2) \\
&= \int_0^1 (2x)(2+3x)\ dx + \int_2^3 2(2+3x)\ dx + 5 + 16 \\
&= 4 + 19 + 5 + 16 = 44.
\end{aligned}$$

(IV)(a) Let $P = \{x_0, x_1, \ldots, x_n\}$. Then

$$\begin{aligned}
U(P, f, \alpha + \beta) &= \sum_{i=1}^{n} M_i[\alpha(x_i) + \beta(x_i) - \alpha(x_{i-1}) - \beta(x_{i-1})] \\
&= \sum_{i=1}^{n} M_i \Delta\alpha_i + \sum_{i=1}^{n} M_i \Delta\beta_i \\
&= U(P, f, \alpha) + U(P, f, \beta).
\end{aligned}$$

(b) From (a), for all P we have $U(P, f, \alpha + \beta) \leq I_\alpha + I_\beta$, meaning $I_\alpha + I_\beta$ is an upper bound for $\{U(P, f, \alpha + \beta) : P \text{ a partition}\}$. Therefore by the definition of sup we have $I_{\alpha+\beta} \leq I_\alpha + I_\beta$.

(c) Let $\epsilon > 0$ and let P_1 and P_2 be partitions satisfying $U(P_1, f, \alpha) < I_\alpha + \epsilon$ and $U(P_2, f, \beta) < I_\beta + \epsilon$. Let $P = P_1 \cup P_2$ be the common refinement. Then by (a),

$$I_{\alpha+\beta} \leq U(P, f, \alpha + \beta) = U(P, f, \alpha) + U(P, f, \beta) \leq U(P_1, f, \alpha) + U(P_2, f, \beta) < I_\alpha + I_\beta + 2\epsilon.$$

(d) Since ϵ is arbitrary in (c), we have $I_{\alpha+\beta} \leq I_\alpha + I_\beta$. This and (b) show that $I_{\alpha+\beta} = I_\alpha + I_\beta$.

(V)(a) We have $\Delta\alpha_i = \frac{1}{2}, \Delta\alpha_{j+1} = 1$ and $\Delta\alpha_k = 0$ for all $k \neq i, j+1$. Also since f is increasing, $M_i = f(2) = 4$ and $M_j = f(x_{j+1}) = 2x_{j+1}$. Therefore

$$U(P, f, \alpha) = \sum_{k=1}^{n} M_k \Delta\alpha_k = 4 \cdot \frac{1}{2} + 2x_{j+1} = 2 + 2x_{j+1}.$$

(b) Since adding points to P can only decrease U, we need only consider P containing 2 and 3, as in (a). The infimum of all possible values of x_{j+1} in (a) is $x_j = 3$, so $\overline{\int_1^4} f\ d\alpha = \inf_P U(P, f, \alpha) = 2 + 2 \cdot 3 = 8$.

(c) As in (b) we need only consider P containing 2 and 3, as in (a). We have $m_i = f(x_{i-1}) = 2x_{i-1}$ and $m_j = f(3) = 6$. Therefore

$$L(P, f, \alpha) = \sum_{k=1}^{n} m_k \Delta\alpha_k = 2x_{i-1} \cdot \frac{1}{2} + 6 \cdot 1 = x_{i-1} + 6.$$

The supremum of all possible values of x_{i-1} in (a) is $x_i = 2$, so $\underline{\int_1^4} f\ d\alpha = 2 + 6 = 8$.

(d) Yes, answers in (b) and (c) are equal.

(VI) Let $\epsilon > 0$ and suppose $x < y$ with $|y - x| < \epsilon/M$. Then by 6.12d,

$$|F(y) - F(x)| = \left| \int_x^y f(t)\ dt \right| \leq M|y - x| < \epsilon.$$

This shows that $\delta = \epsilon/M$ "works" so f is uniformly continuous.