MATH 425a ASSIGNMENT 11 SOLUTIONS
FALL 2011 Prof. Alexander

These solutions are for the individual use of Math 425a students and are not to be distributed outside that group.

Rudin Chapter 7:

(1) Suppose $f_n \to f$ uniformly and each f_n is bounded, say $|f_n(x)| \le M_n$ for all x. Taking $\epsilon = 1$, there exists N such that $|f_N(x) - f(x)| \le 1$ for all $n \ge N$ and all x. Then for all $n \ge N$ and all x we have

$$\begin{aligned}|f_n(x)| &\le |f_n(x) - f_N(x)| + |f_N(x)| \\ &\le |f_n(x) - f(x)| + |f(x) - f_N(x)| + |f_N(x)| \\ &\le 1 + 1 + M_N = M_N + 2.\end{aligned}$$

Let $K = \max\{M_1, \dots, M_{N-1}\}$; then $|f_n(x)| \le K$ for all $n < N$ and all x, and hence for all n and x we have $|f_n(x)| \le \max\{M_N + 2, K\}$. Thus $\{f_n\}$ is uniformly bounded.

(2) Suppose $f_n \to f, g_n \to g$, both uniformly, and let $\epsilon > 0$. There exist N_1, N_2 such that

$$n \ge N_1, x \in X \implies |f_n(x) - f(x)| < \frac{\epsilon}{2},$$

$$n \ge N_2, x \in X \implies |g_n(x) - g(x)| < \frac{\epsilon}{2}.$$

Then

$$n \ge \max(N_1, N_2), x \in X \implies |(f_n+g_n)(x)-(f+g)(x)| \le |f_n(x)-f(x)|+|g_n(x)-g(x)| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

This shows $f_n + g_n \to f + g$ uniformly.

Now suppose also that each sequence is uniformly bounded, that is, there exist M_1, M_2 such that

$$|f_n(x)| \le M_1, \quad |g_n(x)| \le M_2 \qquad \text{for all } n \text{ and all } x.$$

Since $g_n(x) \to g(x)$ for all x, this means $|g(x)| \le M_2$ for all x, as well. Let $\epsilon > 0$. There exists N such that

$$n \ge N, x \in X \implies |f_n(x) - f(x)| \le \epsilon \text{ and } |g_n(x) - g(x)| \le \epsilon.$$

Then for $n \ge N$ and $x \in X$ we have

$$\begin{aligned}|f_n(x)g_n(x) - f(x)g(x)| &\le |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)| \\ &= |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\ &\le M_1\epsilon + M_2\epsilon = (M_1 + M_2)\epsilon.\end{aligned}$$

Since ϵ is arbitrary, this shows $f_n g_n \to fg$ uniformly.

Handout:

(A) Let $\epsilon > 0$. Since f is continuous at 0, there exists $\delta > 0$ such that

$$x \in [0, \delta] \implies |h(x)| \le \epsilon \implies |g_n(x)| = |h(x)|e^{-nx} \le \epsilon \quad \text{for all } n.$$

Since h is continuous on the compact set $[0, 1]$, it is bounded, that is, there exists M such that $|h(x)| \le M$ for all x. Choose N so $e^{-N\delta} < \epsilon/M$. Then

$$x \in [\delta, 1], n \ge N \implies |g_n(x)| = |h(x)|e^{-nx} \le Me^{-N\delta} < M \cdot \frac{\epsilon}{M} = \epsilon.$$

Combining these we get

$$x \in [0, 1], n \ge N \implies |g_n(x)| \le \epsilon,$$

and since ϵ is arbitrary this shows $g_n \to 0$ uniformly.

(B) Let $\epsilon > 0$. Since $f_n \to f$ uniformly, there exists N such that $|f_n(x) - f(x)| < \epsilon/3$ for all $n \ge N$ and all x. Since f_N is uniformly continuous, there exists $\delta > 0$ such that

$$x, y \in \mathbb{R}, |y - x| < \delta \implies |f_N(y) - f_N(x)| < \frac{\epsilon}{3}.$$

Then

$$\begin{aligned} x, y \in \mathbb{R}, |y - x| < \delta \implies |f(y) - f(x)| &\le |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This shows f is uniformly continuous.

(C)(a) Let $x \in [a, b]$. If there are infinitely many values of n for which f_n is continuous at x, then for the subsequence $\{f_{n_k}\}$ consisting of these values, we have $f_{n_k} \to f$ uniformly and f_{n_k} continuous at x, so by 7.11, f is continuous at x. This shows that if f is discontinuous at x, then there can be at most finitely many values of n for which f_n is continuous at x. In other words, there exists $N(x)$ such that f_n is discontinuous at x for all $n \ge N(x)$.

If f were discontinuous at 3 points, say x_1, x_2, x_3, then for $n \ge \max\{N(x_1), N(x_2), N(x_3)\}$, we would then have f_n discontinuous at all three points x_i. Equivalently, if each f_n has at most 2 discontinuities, then f has at most 2 discontinuities.

(b) Define $f_n, f : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in [0, 1], \\ 0 & \text{otherwise,} \end{cases} \qquad f(x) \equiv 0.$$

Then $f_n \to 0$ uniformly, and each f_n has 2 discontinuities, but f has no discontinuities.

(D) Suppose E is a finite set, and f_n, f are functions from E to $\mathbb{R}$ with $f_n \to f$ pointwise. This means that given $\epsilon > 0$, for each $x \in E$ there exist N_x such that $n \ge N_x \implies |f_n(x) - f(x)| < \epsilon$. Since E is finite, $N = \max_{x \in E} N_x$ is finite, and $n \ge N \implies |f_n(x) - f(x)| < \epsilon \, \forall x \in E$. This shows $f_n \to f$ uniformly on E.