MATH 425b ASSIGNMENT 9
SPRING 2009
Prof. Alexander
Due Friday April 22.

Rudin Chapter 10 #2, 3b, 8 and:

(I) Let $\Phi : D \to \mathbb{R}^3$ be a 2-surface and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be $\mathcal{C}'$. (We can think of $f$ as a vector field.) Define a 2-form

$$\omega_f = f_1 \; dx_2 \wedge dx_3 + f_2 \; dx_1 \wedge dx_3 + f_3 \; dx_1 \wedge dx_2.$$

Assume $\Phi$ is one-to-one; for $p \in D$ the *tangent space* to $\Phi$ at $\Phi(p)$ is range$(\Phi'(p))$, which is the linear span of $\Phi'(p)e_1$ and $\Phi'(p)e_2$. The *normal vector* at $\Phi(p)$ is the cross product $N(p) = \Phi'(p)e_1 \times \Phi'(p)e_2$. Show that

$$\int_\Phi \omega_f = \int_D f(\Phi(p)) \cdot N(p) \; dp.$$

What happens if the vector field "flows along the surface", that is, $f(\Phi(p))$ is in the tangent space at $\Phi(p)$ for all $p$?

(II) Let

$$\omega = (x_1x_2 + x_3^2) \; dx_4 \wedge dx_1 \wedge dx_2 - 3x_4 \; dx_4 \wedge dx_2 \wedge dx_1, \quad \omega' = x_3 \; dx_2 \wedge dx_1 + x_4^2 \; dx_3 \wedge dx_5$$

Calculate $\omega \wedge \omega'$ and $d\omega$ and give the standard presentation of each.

(III) Let $\gamma : [c, d] \to \mathbb{R}^n$ be a 1-surface (i.e. a curve) and let $\omega$ be a 1-form in $\mathbb{R}^n$. Let $\varphi$ be a $\mathcal{C}'$ map of $[a, b]$ into $[c, d]$ with $\varphi' > 0$, so that $\alpha = \gamma \circ \varphi$ defines a parametrization of $\gamma$. Show that $\int_\gamma \omega = \int_\alpha \omega$.

(IV) Suppose the trace of $\gamma$ is the intersection of the unit circle and the upper half plane, and $\omega = y \; dx$. Find a natural parametrization of $\gamma$ and calculate $\int_\gamma y \; dx$.

(V) Let $Q$ be the unit square $[0, 1] \times [0, 1]$, define $T$ on $\mathbb{R}^2$ by $T(u, v) = (u - v^2, u^2 + v)$, and let $A = T(Q)$.
(a) Sketch $A$, by computing the image under $T$ of each of the 4 sides of $Q$.
(b) Show that $T$ is 1-1 on $Q$.
(c) Calculate $\int_A x \; dx \; dy$.

(VI) For $\mathbf{x} \in \mathbb{R}^3$, let $R(\mathbf{x})$ denote the distance from $\mathbf{x} = (x, y, z)$ to the $x$-axis. The moment of inertia of a body $A \subset \mathbb{R}^3$ about the $x$-axis is given by $\int_A R(\mathbf{x})^2 \; d\mathbf{x}$. Suppose $f$ is a positive continuous function on an interval $[a, b]$ and $A$ is the body obtained by rotating the region $0 \leq y \leq f(x)$ around the $x$-axis. Find the moment of inertia of such an $A$. Express your answer as a one-dimensional integral, where the integrand depends on $f(x)$, like for example $\int \cos(f(x))\, dx$ or $\int f(x)^2\, dx$, with appropriate limits on the integral.

**HINTS:**

(3)(b) This is similar to the proof done in lecture for 90-degree rotation.

(8) We know $T\mathbf{x} = \mathbf{b} + A\mathbf{x}$ for some $\mathbf{b}, A$. $T(0, 0) = (1, 1)$ tells you $\mathbf{b}$; then you must find $Ae_1$ and $Ae_2$ to determine $A$.

(V)(b) This is "brute force" calculation—no real tricks or short cuts. Suppose $T(s, t) = T(u, v)$ with$(s, t), (u, v) \in Q$. Use the fact you can factor the difference of two squares, and the fact that all of $s, t, u, v$ are nonnegative, to help you show that $s = u, t = v$.

(c) Change of variable.

(VI) Figure out what change of variable makes this easier.