Chapter | Seven

The Revenue Ranking ("Linkage") Principle

In the previous chapter we compared the three common auction formats by directly computing the expected revenues in the respective symmetric equilibria. In particular, we showed that the revenue in a second-price auction exceeded that in its first-price counterpart. Just as the revenue equivalence principle isolates the reasons underlying the equality of revenues between the second- and first-price auctions, the main result of this chapter, the *linkage* principle, isolates the reasons underlying the revenue rankings of the previous chapter.

7.1 THE MAIN RESULT

As in the derivation of the revenue equivalence principle, it is convenient to abstract away from the specific rules of a particular format and concentrate on its essential mechanics. Consider the symmetric setting of the previous chapter. Suppose A is a standard auction in which the highest bid wins the object and that it has a symmetric equilibrium, β^A. Consider bidder 1, say, and suppose that all other bidders follow the symmetric equilibrium strategy. Let $W^A(z,x)$ denote the expected price paid by bidder 1 if he is the *winning* bidder when he receives a signal x but bids as if his signal were z (that is, he bids $\beta^A(z)$).

In a first-price auction, the winning bidder pays exactly what he bid, so

$$W^{\mathrm{I}}(z,x) = \beta^{\mathrm{I}}(z),$$

where β^{I} is the symmetric equilibrium strategy in the auction. In a second-price sealed-bid auction, however, the winning bidder pays the second-highest bid. From his perspective, the amount he will have to pay is uncertain, so the expected payment upon winning is

$$W^{\mathrm{II}}(z,x) = E\left[\beta^{\mathrm{II}}(Y_1) \mid X_1 = x, Y_1 < z\right],$$

where β^{II} is the symmetric equilibrium strategy in the second-price auction.

Let $W_2^A(z,x)$ denote the partial derivative of the function $W^A(\cdot,\cdot)$ with respect to its second argument, evaluated at the point (z,x). The following result is called the revenue ranking or linkage principle.

Proposition 7.1. *Let A and B be two auctions in which the highest bidder wins and only he pays a positive amount. Suppose that each has a symmetric and increasing equilibrium such that (1) for all x, $W_2^A(x,x) \geq W_2^B(x,x)$; (2) $W^A(0,0) = 0 = W^B(0,0)$. Then the expected revenue in A is at least as large as the expected revenue in B.*

Proof. Consider auction A and suppose that all bidders $j \neq 1$ follow the symmetric equilibrium strategy β^A. The probability that bidder 1 with signal x who bids $\beta^A(z)$ will win is just $G(z\,|\,x) \equiv \text{Prob}\,[Y_1 < z\,|\,X_1 = x]$. Thus, each bidder in auction A maximizes

$$\int_0^z v(x,y)\,g(y\,|\,x)\,dy - G(z\,|\,x)\,W^A(z,x)$$

In equilibrium it is optimal to choose $z = x$, so the relevant first-order condition is

$$g(x\,|\,x)\,v(x,x) - g(x\,|\,x)\,W^A(x,x) - G(x\,|\,x)\,W_1^A(x,x) = 0,$$

where W_1^A denotes the partial derivative of W^A with respect to its first argument. This can be rearranged so that

$$W_1^A(x,x) = \frac{g(x\,|\,x)}{G(x\,|\,x)}v(x,x) - \frac{g(x\,|\,x)}{G(x\,|\,x)}W^A(x,x)$$

Similarly,

$$W_1^B(x,x) = \frac{g(x\,|\,x)}{G(x\,|\,x)}v(x,x) - \frac{g(x\,|\,x)}{G(x\,|\,x)}W^B(x,x)$$

and thus

$$W_1^A(x,x) - W_1^B(x,x) = -\frac{g(x\,|\,x)}{G(x\,|\,x)}\left[W^A(x,x) - W^B(x,x)\right] \tag{7.1}$$

Now define

$$\Delta(x) = W^A(x,x) - W^B(x,x)$$

so

$$\Delta'(x) = \left[W_1^A(x,x) - W_1^B(x,x)\right] + \left[W_2^A(x,x) - W_2^B(x,x)\right] \tag{7.2}$$

Using (7.1) in (7.2) yields

$$\Delta'(x) = -\frac{g(x\,|\,x)}{G(x\,|\,x)}\Delta(x) + \left[W_2^A(x,x) - W_2^B(x,x)\right] \tag{7.3}$$

By hypothesis, the second term in (7.3) is nonnegative. Thus, if $\Delta(x) \le 0$, then $\Delta'(x) \ge 0$. Furthermore, by assumption $\Delta(0) = 0$. Thus, we must have that for all x, $\Delta(x) \ge 0$. ■

Proposition 7.1 leads to a ranking of alternative auction forms by comparing the statistical linkages between a bidder's own signal and the price he would pay upon winning. The greater the linkage between a bidder's own information and how he perceives others will bid, the greater the expected price paid upon winning.

As such, Proposition 7.1 does not make any assumptions regarding the distribution of bidders' signals—whether they are affiliated or not. It relies only on the some properties of the maximization problem that bidders face. In applying it, however, we will make use of the assumption that bidders' signals are affiliated.

FIRST-PRICE VERSUS SECOND-PRICE AUCTIONS

The revenue ranking principle sheds useful light on why the second-price auction outperforms the first-price auction in terms of revenue (Proposition 6.4). This is because for the first-price auction,

$$W^{\mathrm{I}}(z,x) = \beta^{\mathrm{I}}(z),$$

where β^{I} is the symmetric equilibrium strategy, so $W_2^{\mathrm{I}}(x,x) = 0$ for all x.

For a second-price auction, on the other hand,

$$W^{\mathrm{II}}(z,x) = E\left[\beta^{\mathrm{II}}(Y_1)\,|\,X_1 = x, Y_1 < z\right],$$

where β^{II} is the symmetric equilibrium strategy in a second-price auction. Since β^{II} is increasing, affiliation now implies that for all x, $W_2^{\mathrm{II}}(x,x) \ge 0$. Thus, from Proposition 7.1 we conclude that the revenue from the latter is no less than that from the former.

If signals are independently distributed, then in any auction A satisfying the preceding hypotheses, $W^A(z,x)$ does not depend on x. So $W_2^A(z,x) = 0 = W_2^B(z,x)$ for any two auctions A and B. In that case, $W^A(x,x) = W^B(x,x)$, so the revenues in the two auctions are the same. Proposition 7.1 thus implies the revenue equivalence principle in Proposition 3.1. This argument also highlights the fact that the assumption of private values is unimportant for revenue equivalence—as long as the bidders' signals are independently distributed, revenue equivalence obtains even with interdependent values.

7.2 PUBLIC INFORMATION

In many instances the seller may have information that is potentially useful to the bidders. What should the seller do with this information? Should the seller keep it hidden or should she reveal it publicly? Should she be strategic, revealing the information only when it is favorable?

To address these and related questions, the symmetric model considered in Chapter 6 needs to be slightly amended. Specifically, let S be a random variable that denotes the information available to the seller. This information, if known, would affect the valuations of the bidders, so we now write these as a function of the $N+1$ signals

$$V_i = v_i(S, X_1, X_2, \ldots, X_N)$$

and we assume, as before, that $v_i(\mathbf{0}) = 0$. In the symmetric case, which is our focus here, we write

$$v_i(S, \mathbf{X}) = u\big(S, X_i, \mathbf{X}_{-i}\big),$$

where u is, as before, a symmetric function of its last $N-1$ arguments. The variables $S, X_1, X_2, \ldots, X_N$ are assumed to be affiliated and distributed according to a joint density function f, which is a symmetric function of its last N arguments, the bidders' signals.

When public information is *not* available, the bidders do not know the realization of S before bidding, so we can, as before, define

$$v(x, y) = E\,[V_1 \mid X_1 = x, Y_1 = y], \tag{7.4}$$

where now the unknown public information S has also been integrated out.

Now suppose that the seller reveals the information in a nonstrategic manner—it is made public in all circumstances. As a result, the bidders know the realization of S before bidding and we define in an analogous manner

$$\widehat{v}(s, x, y) = E\,[V_1 \mid S = s, X_1 = x, Y_1 = y] \tag{7.5}$$

to be the expectation of the value to bidder 1 when the public signal is s, the signal the bidder receives is x, and the highest signal among the other bidders is y. Because of symmetry, this function is the same for all bidders, and because of affiliation, $\widehat{v}$ is an increasing function of its arguments. Moreover, $\widehat{v}(0,0,0) = 0$. With publicly available information, the function $\widehat{v}$ will play the same role as that played by v in the previous chapter. Clearly,

$$v(x, y) = E\,[\widehat{v}(S, X_1, Y_1) \mid X_1 = x, Y_1 = y] \tag{7.6}$$

What effect does a policy of revealing information in all circumstances have on the seller's expected revenue? We begin by looking at first-price auctions.

PUBLIC INFORMATION IN A FIRST-PRICE AUCTION

To derive the effects of public information it is useful to think of the two situations—with and without the information—as two different "auctions." Then we can use the machinery of Proposition 7.1 to compare the two.

When public information is available, a bidder's strategy is a function of both the public information S and his own signal X_i. Temporarily, suppose that there exists a symmetric equilibrium strategy of the form $\widehat{\beta}(S, X_i)$, which is increasing in both variables. The expected payment of a winning bidder when he receives a signal x but bids as if his signal were z (that is, for all $S = s$, he bids $\widehat{\beta}(s, z)$) is

$$\widehat{W}^{\mathrm{I}}(z, x) = E\left[\widehat{\beta}(S, z) \mid X_1 = x\right]$$

so $\widehat{W}_2^{\mathrm{I}}(z, x) \geq 0$, because S and X_1 are affiliated.

When public information is not available, then, as before, we have that if $\beta \equiv \beta^{\mathrm{I}}$ is the equilibrium strategy in a first-price auction,

$$W^{\mathrm{I}}(z, x) = \beta(z)$$

so $W_2^{\mathrm{I}}(z, x) = 0$.

Thus,

$$\widehat{W}_2^{\mathrm{I}}(z, x) \geq W_2^{\mathrm{I}}(z, x)$$

We can now apply the linkage principle: Proposition 7.1 implies that the expected revenue in a first-price auction is *higher* when public information is made available than when it is not.

In arguing that publicly available information enhances the revenue from a first-price auction, we temporarily supposed that there exists a symmetric and increasing equilibrium in the case when public information is made available. This may be verified in a manner analogous to Proposition 6.3. By mimicking the arguments there, it can be shown that the strategy

$$\widehat{\beta}^{\mathrm{I}}(s, x) = \int_0^x \widehat{v}(s, y, y)\, d\widehat{L}(y \mid s, x),$$

where

$$\widehat{L}(y \mid s, x) = \exp\left(-\int_y^x \frac{g(t \mid s, t)}{G(t \mid s, t)}\, dt\right)$$

constitutes an equilibrium of the first-price auction with publicly available information.

PUBLIC INFORMATION IN SECOND-PRICE AND ENGLISH AUCTIONS

The release of public information also raises revenues in both second-price and English auctions. The arguments are almost the same as in Proposition 6.4, so they are omitted.

7.3 AN ALTERNATIVE LINKAGE PRINCIPLE

Proposition 7.1 applies to auctions in which only the winner pays a positive amount; it does not apply, for instance, to all-pay auctions. A similar result that does apply to such situations is available.

Let $M^A(z,x)$ be the expected payment by a bidder with signal x who bids as if his or her signal were z in an auction mechanism A. The advantage of this specification is that we do not assume that only the winner pays a positive amount.

For instance, for an all-pay auction, $M^{\mathrm{AP}}(z,x) = \beta^{\mathrm{AP}}(z)$, where β^{AP} is a symmetric and increasing equilibrium strategy if one exists. For auctions in which only the winner pays, $M^A(z,x) = F_{Y_1}(z\,|\,x)\,W^A(z,x)$. So in a first-price auction, $M^{\mathrm{I}}(z,x) = F_{Y_1}(z\,|\,x)\,\beta^{\mathrm{I}}(z)$.

The following is an alternative version of the linkage principle. While it reaches the same conclusion as does Proposition 7.1, its hypotheses concern the relative responsiveness of the unconditional expected payment, $M^A(z,x)$, to a change in a bidder's own signal rather than the relative responsiveness of the expected payment conditional on winning, $W^A(z,x)$, to the same change.

Proposition 7.2. *Let A and B be two auctions in which the highest bidder wins. Suppose that each has a symmetric and increasing equilibrium such that (1) for all x, $M_2^A(x,x) \geq M_2^B(x,x)$; and (2) $M^A(0,0) = 0 = M^B(0,0)$. Then the expected revenue in A is at least as large as the expected revenue in B.*

Proof. The expected payoff of a bidder with signal x who bids $\beta^A(z)$ is

$$\int_{-\infty}^{z} v(x,y)\,g(y\,|\,x)\,dy - M^A(z,x)$$

In equilibrium, it is optimal to choose $z = x$ and the resulting first-order conditions imply that

$$M_1^A(x,x) = v(x,x)\,g(x\,|\,x) \tag{7.7}$$

Now writing $\Delta(x) = M^A(x,x) - M^B(x,x)$ and using (7.7) we deduce that

$$\Delta'(x) = M_2^A(x,x) - M_2^B(x,x) \geq 0$$

by assumption. Since $\Delta(0) = 0$, for all x, $\Delta(x) \geq 0$. ■

RANKING ALL-PAY AUCTIONS

To see how Proposition 7.2 can be used to rank other auction forms, consider an all-pay auction in an environment with interdependent values and affiliated signals.

First, as noted earlier, in a first-price auction,

$$M^{\mathrm{I}}(z,x) = G(z \mid x)\,\beta^{\mathrm{I}}(z)$$

so

$$M_2^{\mathrm{I}}(z,x) = \frac{\partial}{\partial x}\left[G(z \mid x)\,\beta^{\mathrm{I}}(z)\right] < 0$$

since affiliation implies that $G(z \mid \cdot)$ is decreasing.

Suppose that there is a symmetric, increasing equilibrium in the all-pay auction, say β^{AP}. Then, by definition,

$$M^{\mathrm{AP}}(z,x) = \beta^{\mathrm{AP}}(z)$$

so

$$M_2^{\mathrm{AP}}(z,x) = \frac{\partial}{\partial x}\beta^{\mathrm{AP}}(z) = 0$$

Since, $M_2^{\mathrm{AP}}(x,x) > M_2^{\mathrm{I}}(x,x)$, an application of Proposition 7.2 implies that the expected revenue from an all-pay auction, provided it has an increasing equilibrium, is greater than that from a first-price auction.

It can be argued that, provided that the function $v(\cdot,y)g(y \mid \cdot)$ is increasing, the following is a symmetric increasing equilibrium of the all-pay auction:

$$\beta^{\mathrm{AP}}(x) = \int_0^x v(y,y)g(y \mid y)\,dy$$

CHAPTER NOTES

The revenue ranking (or linkage) principle, Proposition 7.1, was first set forth and used by Milgrom and Weber (1982). The results on public information are also from this paper.

The alternative form of the linkage principle, Proposition 7.2, and its application to ranking the all-pay auction relative to the first-price auction is due to Krishna and Morgan (1997). This paper also derives some sufficient conditions for the existence of a symmetric, increasing equilibrium in the all-pay auction. Amann and Leininger (1995) also compare the revenues from all-pay auctions to those from a first-price auction.

Biologists have studied the *war of attrition*, in which two players engage in a struggle during which both expend resources. The game ends when one of the players gives up so that the winning player expends the same amount as the losing player. This is essentially a *second-price all-pay auction*, and it can be shown that if it has a symmetric, increasing equilibrium, then it is revenue superior to the ordinary second-price auction (see Krishna and Morgan, 1997).