Answers to HW 7

1. The buyer value distribution is given by $F(v) = v^{0.5}$. There are two buyers in a first-price auction.

(a) Compute the expected revenue of the auction at reservation price 0.

(b) Compute the expected revenue of the auction at any reservation price r.

(c) From (b), find the optimal reservation price r. Verify the second-order condition for all the solutions you get from the first-order condition.

(d) Find the optimal reservation price from the solution of the following equation

$$x = \frac{1 - F(x)}{f(x)}.$$

2. The buyer value distribution is given by $F(v) = \frac{1+0.5v}{2}, v \in [0, 2]$. There are two buyers in a first-price auction.

(a) Compute the expected revenue of the auction at reservation price 0.

(b) Compute the expected revenue of the auction at any reservation price r.

(c) From (b), find the optimal reservation price r. Verify the second-order condition for all the solutions you get from the first-order condition.

(d) Find the optimal reservation price from the solution of the following equation

$$x = \frac{1 - F(x)}{f(x)}.$$

3. The buyer value distribution is given by $F(v) = e^{v-1}, v \in [0, 1]$. There are two buyers in a first-price auction.

(a) Compute the expected revenue of the auction at reservation price 0.

(b) Compute the expected revenue of the auction at any reservation price r.

(c) From (b), find the optimal reservation price r. Verify the second-order condition for all the solutions you get from the first-order condition.

(d) Find the optimal reservation price from the solution of the following equation

$$x = \frac{1 - F(x)}{f(x)}.$$

4. The buyer value distribution is given by $F(v) = v, v \in [0, 1]$. There are N buyers in a first-price auction.

(a) Compute the expected revenue of the auction at reservation price 0.

(b) Compute the expected revenue of the auction at any reservation price r.

(c) From (b), find the optimal reservation price r. Verify the second-order condition for all the solutions you get from the first-order condition.

(d) Find the optimal reservation price from the solution of the following equation

$$x = \frac{1 - F(x)}{f(x)}.$$

(e) If $N \to \infty$, what is the limit of the seller's revenue. How do you interpret the result?

Answers:

Let $F(v)$ $v \in [0, \beta]$ be the value distribution of the buyers. After double-checking, it is ok to use the formula

$$R = \int_r^\beta b(v) dF^2(v)$$

to compute the revenue of the first-price auction with two buyers and the reservation price r. This should give you the same revenue computed from the alternative formula

$$R = \int_r^\beta J(v) dF^2(v) = 2\int_r^\beta (vf(v) + F(v) - 1)F(v)dv.$$

Both formulas are used in the following.

1. (a) The equilibrium bidding strategy is

$$b(v) = v - \frac{1}{v^{0.5}} \int_0^v x^{0.5} dx = \frac{1}{1.5v^{0.5}} v^{1.5} = v - \frac{2}{3}v = \frac{1}{3}v.$$

Hence the revenue is given by

$$\int_0^1 \frac{1}{3} v dF^2(v) = \int_0^1 \frac{1}{3} v dv = \frac{1}{6}.$$

(b) For any reservation price r, we have

$$b(v) = v - \frac{1}{v^{0.5}} \int_r^v x^{0.5} dx = \frac{1}{3}v + \frac{2r^{1.5}}{3v^{0.5}}.$$

Hence the revenue is

$$2\int_r^1 (xf(x) + F(x) - 1)F(x)dx = 2\int_r^1 (\frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}} - 1)x^{\frac{1}{2}} dx$$

$$= \frac{4}{3}r^{\frac{3}{2}} - \frac{3}{2}r^2 + \frac{1}{6}.$$

This is the same as computing it from

$$\int_r^1 b(v) dF^2(v) = \int_r^1 (\frac{1}{3}v + \frac{2r^{\frac{3}{2}}}{3v^{\frac{1}{2}}})dv = \frac{4}{3}r^{\frac{3}{2}} - \frac{3}{2}r^2 + \frac{1}{6}.$$

(c) Taking the derivative of the revenue with respect r, we get

$$2r^{0.5} - 3r = 0,$$

and we get the optimal $r^* = \frac{4}{9}$. The second-order condition is easily checked.

(d) We can also find the optimal reservation price from the equation

$$r - \frac{1 - F(r)}{f(r)} = 0,$$

or

$$v - \frac{1 - v^{0.5}}{0.5v^{-0.5}} = 0,$$
$$1.5v^{0.5} - 1 = 0,$$

and we get the same answer.

2. (a) We have the equilibrium bidding strategy

$$b(v) = v - \frac{2}{1 + \frac{1}{2}v} \int_0^v \frac{1 + \frac{1}{2}x}{2} dx = v - \frac{v\,(v+4)}{2(v+2)}$$
$$= v - \frac{v}{2}(1 + \frac{2}{v+2}) = \frac{v}{2} - \frac{v}{v+2}.$$

We have the revenue

$$R = \int_0^2 (\frac{v}{2} - \frac{v}{v+2}) d(\frac{1 + \frac{1}{2}x}{2})^2 = 2\int_0^2 (\frac{v}{2} - \frac{v}{v+2})(\frac{1 + \frac{1}{2}v}{2})\frac{1}{4}dv = \frac{1}{6}.$$

(b) We have the bidding strategy

$$b(v) = v - \frac{2}{1 + \frac{1}{2}v} \int_r^v \frac{1 + \frac{1}{2}x}{2} dx$$
$$= v + \frac{2}{\frac{1}{2}v + 1}\left(\frac{1}{8}r^2 + \frac{1}{2}r - \frac{1}{8}v^2 - \frac{1}{2}v\right)$$

The revenue is

$$R = 2\int_r^2 \left(v + \frac{2}{\frac{1}{2}v + 1}\left(\frac{1}{8}r^2 + \frac{1}{2}r - \frac{1}{8}v^2 - \frac{1}{2}v\right)\right)(\frac{1 + \frac{1}{2}v}{2})\frac{1}{4}dv$$
$$= -\frac{1}{12}r^3 - \frac{1}{8}r^2 + \frac{1}{2}r + \frac{1}{6}.$$

(c) Taking the derivative of the revenue with respect to r, we get the equation

$$-\frac{1}{4}r^2 - \frac{1}{4}r + \frac{1}{2} = 0,$$

and we get the optimal reservation price $r^* = 1$.

(d) We can also compute the optimal reservation price by the following equation

$$r - \frac{1 - \frac{1 + \frac{1}{2}v}{2}}{\frac{1}{4}} = 0,$$

and we get the same solution $r^* = 1$.

3.

(a) We have the bidding strategy

$$b(v) = v - \frac{1}{e^{v-1}} \int_0^v e^{x-1} dx = v - 1 + e^{-v}.$$

The revenue is

$$\int_0^1 (v-1+e^{-v}) de^{2(v-1)} = 2\int_0^1 (v-1+e^{-v}) e^{2(v-1)} dv = 2e^{-1} - \frac{1}{2}e^{-2} - \frac{1}{2} = 0.168\,09.$$

(b) We have the bidding strategy

$$b(v) = v - \frac{1}{e^{v-1}} \int_r^v e^{x-1} dx = v + \frac{1}{e^{v-1}} \left(e^{r-1} - e^{v-1}\right) = v - 1 + e^{r-v}.$$

The revenue is

$$R = \int_r^1 (v - 1 + e^{r-v}) de^{2(v-1)} = 2\int_r^1 (v - 1 + e^{r-v}) e^{2(v-1)} dv$$

$$= 2e^{r-1} - \frac{1}{2}e^{2r-2} - re^{2r-2} - \frac{1}{2}.$$

(c) Taking the derivative of the revenue with respect to r, we get the first-order condition

$$2e^{r-1} - 2e^{2r-2} - 2re^{2r-2} = 0,$$

or

$$2 - 2e^{r-1} - 2re^{r-1} = 0$$

We get the optimal reservation price $r^* = 0.557\,15$.

(d) We can also compute the optimal reservation price by the equation

$$r - \frac{1 - e^{r-1}}{e^{r-1}} = 0,$$

or

$$re^{r-1} + e^{r-1} - 1 = 0,$$

which is the same equation, with the same solution.

4.

(a) For N buyers, we use the formula

$$b(v) = v - \frac{1}{F(v)^{N-1}} \int_0^v F(x)^{N-1} dx$$

$$= v - \frac{1}{v^{N-1}} \int_0^v x^{N-1} dx = \frac{N-1}{N} v.$$

The revenue is

$$\int_0^1 \frac{N-1}{N} v dx^N = \frac{N-1}{N} \int_0^1 N v^N dx = \frac{N-1}{N+1}.$$

(b) When the reservation price is r, we have the formula

$$b(v) = v - \frac{1}{F(v)^{N-1}} \int_r^v F(x)^{N-1} dx$$

$$= v - \frac{1}{v^{N-1}} \int_r^v x^{N-1} dx = \frac{N-1}{N} v + \frac{r^N}{N v^{N-1}}. \tag{1}$$

The revenue is given by

$$N \int_r^1 \left(\frac{N-1}{N} v + \frac{r^N}{N v^{N-1}} \right) v^{N-1} dv$$

$$= \int_r^1 \left((N-1) v^N + r^N \right) dv = r^N (1-r) + \frac{N-1}{N+1} (1 - r^{N+1}).$$

The derivative with respect to r is given by

$$-r^N + N r^{N-1}(1-r) - (N-1) r^N = 0$$

$$r^{N-1}(1-r) - r^N = 0,$$

or

$$1 - r = r,$$

and we get the optimal reservation price $r^* = 0.5$.

(d) The optimal reservation price can be solved by the following equation

$$r - \frac{1-r}{1} = 2r - 1 = 0,$$

and we get $r^* = 0.5$.

(e) When $N \to \infty$, the seller revenue for $r = 0$ is

$$\frac{N-1}{N+1} \to 1,$$

This means that the seller gets the highest possible revenue.