Jonathan Levin
Stanford University

Game Theory Syllabus

Overview. This is an advanced course in game theory, intended for students who are interested in pursuing micro theory research or who want a good theory background to do applied work. The course will cover a combination of standard results and current research topics. The prerequisite is familiarity with the basic ideas of game theory — Nash equilibrium, subgame perfection, incomplete information — as introduced in Economics 203. Please talk to me if you haven't taken Economics 203 or 160 or another equivalent.

Logistics. The class meets MW 9:00–10:50 in Econ 218. I plan to have office hours M 3:15–5 in my Economics Department office (Landau 240). You can also contact me to make an appointment for some other time. My email address is jdlevin@stanford.edu. I'll post assignments, lecture notes, etc. on the Stanford Coursework class web page.

Assignments and Grading. There will be four problem sets and a final exam. Grades are based on a weighted average of assignments (40%) and the exam (60%). The assignments will be spaced evenly over the quarter. I encourage you to collaborate so long as solutions are written up individually. I'll post solutions on the course web page. The grader for the class is Xiaochen Fan (xfan@stanford.edu) — please don't harass her unnecessarily. The exam will have a 24 hour take-home format, with a several day window for taking it.

Reading. I will hand out notes for all the lectures. The lectures notes will have detailed references; virtually all the papers can be downloaded easily. There are also several books you may find useful.

Fudenberg, D. and J. Tirole, *Game Theory*, MIT Press, 1992.

Osborne, M. and A. Rubinstein, *A Course in Game Theory*, MIT Press, 1994.

Mailath, G. and L. Samuelson, *Repeated Games and Reputations: Long-Run Relationships*, Oxford, 2006.

The first two are general texts and have substantial overlap. Both are excellent, but close to fifteen years old. The Mailath-Samuelson book is not-yet released; you can find selected chapters on George Mailath's webpage at Penn.

Outline of Topics

1. Solution Concepts (2 lectures)

   FT, chapter 2; OR, chapters 2–4.

   Bernheim, D. (1984) "Rationalizable Strategic Behavior," *Econometrica*, 52, 1007-1028.

   Pearce. D. (1984) "Rationalizable Strategic Behavior and the Problem of Perfection," *Econometrica*, 52, 1029-1050.

   Aumann, R. (1987) "Correlated Equilibrium as an Extension of Bayesian Rationality," *Econometrica*, 55, 1-18.

   Brandenburger, A. and E. Dekel (1987): "Rationalizability and Correlated Equilibria," *Econometrica*, 55, 1391–1402.

   Gul, F. (1998) "A Comment on Aumann's Bayesian View," *Econometrica*, 66.

   Fudenberg, D. and D. Levine (1993) "Self-Confirming Equilibrium," *Econometrica*, 61, 523- 546.

   Dekel, E., D. Fudenberg and D. Levine (2004) "Learning to Play Bayesian Games," *Games and Econ. Behav.*, 46, 282-303.

2. Common Knowledge and Common Priors (2 lectures)

   OR, chapter 5.

   Samuelson, L. (2004) "Modeling Knowledge in Economic Analysis," *J. Econ. Lit.*, 62, 367-403.

   Aumann, R. and A. Brandenburger (1995) "Epistemic Conditions for Nash Equilibrium," *Econometrica*, 63, 1161-1180.

   Aumann, R. (1976) "Agreeing to Disagree," *Ann. of Stat.*

   Milgrom, P. and N. Stokey (1982) "Information, Trade and Common Knowledge," *J. Econ. Theory*, 26, 177-227.

   Harrison, M. and D. Kreps (1978): "Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations," *Quart. J. Econ*, 92, 323-336.

3. Supermodular Games and Global Games (2 lectures)

   Milgrom, P. and J. Roberts (1990) "Rationalizability and Learning in Games with Strategic Complementarities," *Econometrica*, 58, 1255-1277.

   Carlsson, H. and E. van Damme (1993) "Global Games and Equilibrium Selection," *Econometrica*, 61, 989-1018.

Morris, S. and H. Shin (1998) “Unique Equilibrium in a Model of Self-Fulfilling Attacks,” *Amer. Econ. Rev.*, 89, 587–597.

Morris, S. and H. Shin (2002) “The Social Value of Public Information,” *Amer. Econ. Rev.*, 92, 1521-1534.

Morris, S. and H. Shin (2003) “Global Games: Theory and Applications,” in *Advances in Economics and Econometrics*, Cambridge University Press.

Abreu, D. and M. Brunnermeier (2003) “Bubbles and Crashes,” *Econometrica*, 71, 173–204.

4. Repeated Games (4 lectures)

Fudenberg and Tirole, chapter 5.

Osborne and Rubinstein, chapter 8.

Abreu, D. “On the Theory of Infinitely Repeated Games with Discounting,” *Econometrica.*

Abreu, D. D. Pearce and E. Stacchetti (1990) “Toward a Theory of Discounted Repeated Games with Imperfect Monitoring,” *Econometrica*, 60, 1041-1063.

Fudenberg, D. and E. Maskin (1986) “The Folk Theorem for Repeated Games with Discounting or with Incomplete Information,” *Econometrica.*

Fudenberg, D. D. Levine and E. Maskin (1994) “The Folk Theorem with Imperfect Public Information,” *Econometrica*, 64, 997-1039.

Levin, J. (2003) “Relational Incentive Contracts,” *Amer. Econ. Rev.*

Sekiguchi, T. (1997) “Efficiency in Repeated Prisoners’ Dilemma with Private Monitoring,” *J. Econ. Theory*, 76, 345–361.

Ely, J. and J. Valimaki (2002) “A Robust Folk Theorem for the Prisoners’ Dilemma,” *J. Econ. Theory.*

Mailath, G. and S. Morris (2002) “Repeated Games with Almost-Public Monitoring,” *J. Econ. Theory.*

Matsushima, H. (2004) “Repeated Games with Private Monitoring: Two Players,” *Econometrica.*

Ely, J., J. Horner, and W. Olszewski (2005) “Belief-free Equilibria in Repeated Games,” *Econometrica.*

Horner, J. and W. Olszewski (2006) “The Folk Theorem for Games with Private Almost-Perfect Monitoring,” *Econometrica.*

5. Reputation (2-3 lectures)

Kreps, D. and R. Wilson (1982) “Reputation and Imperfect Information,” *J. Econ. Theory*, 27, 253-179.

Milgrom, P. and J. Roberts (1982) "Predation, Reputation, and Entry Deterrence," *J. Econ. Theory*, 27, 280-312.

Fudenberg, D. and D. Levine (1989) "Reputation and Equilibrium Selection in Ganes with a Patient Player," *Econometrica*, 57, 759-778.

Schmidt, K. (1993) "Reputation and Equilibrium Characterization in Repeated Games with Conflicting Interests," *Econometrica*, 61, 325-351.

Tiroloe, J. (1996) "A Theory of Collective Reputations," *Rev. Econ. Studies.*

Ely, J. and J. Valimaki (2002) "Bad Reputation," *Quarterly J. Econ.*

Tadelis, S. (2003) "The Market for Reputations as an Incentive Mechanism," *J. Pol. Econ.*

Mailath, G. and L. Samuelson (2001) "Who Wants a Good Reputation," *Rev. Econ. Studies.*

Abreu, D. and F. Gul (2001) "Bargaining and Reputation," *Econometrica.*

Cripps, M., G. Mailath and L. Samuelson (2004) "Imperfect Monitoring and Impermanent Reputations," *Econometrica.*

Abreu, D. and D. Pearce (2005) "Reputational Wars of Attrition with Complex Bargaining Postures," Working Paper.

6. Learning in Games (2 lectures)

Fudenberg, D. and D. Levine, *Theory of Learning in Games*, MIT Press, 1998.

Kalai, E. and E. Lehrer (1993) "Rational Learning Leads to Nash Equilibrium, *Econometrica*, 61, 1019–1045.

Nachbar, J. (1997) "Prediction, Optimization, and Learning in Repeated Games," *Econometrica*, 65, 275-309.

Erev, I. and A. Roth (1998) "Predicting how People Play Games: Reinforcement Learning in Experimental Games with Unique Mixed Strategy Equilibria," *Amer. Econ. Rev.*, 85, 848–881.

Hart, S. and A. Mas-Colell (2003) "Uncoupled Dynamics Do Not Lead to Nash Equilibrium," *Amer. Econ. Rev.*, 93, 1830-1836.

Bereby-Meyer, Y. and A. Roth (2006) "Learning in Noisy Games: Partial Reinforcement and the Sustainability of Cooperation," *Amer. Econ. Rev.*

7. Behavioral Game Theory: Experiments (1 lecture)

Camerer, C. (2003) *Behavioral Game Theory: Experiments on Strategic Interaction*, Princeton: Princeton University Press, 2003.

Crawford, V. (1997) "Theory and Interaction in the Analysis of Strategic Interaction," *Advances in Economics and Econometrics*, Seventh World Congress of the Econometric Society, Volume 1.

Nagel, R. (1995) "Unraveling in Guessing Games: An Experimental Study," *Amer. Econ. Rev.*, 85, 1313–1326.

Costa-Gomes, M., V. Crawford and B. Boseta (2001) "Cognition and Behavior in Normal-Form Games: An Experimental Study," *Econometrica.*

Costa-Gomes, M. and V. Crawford, "Cognition and Behavior in Two-Person Guessing Games: An Experimental Study," Working Paper.

8. Behavioral Game Theory: Models (2-3 lectures)

Geanakoplos, J., D. Pearce and E. Stacchetti (1989) "Psychological Games and Sequential Rationality," *Games Econ. Behav.*, 1, 60-79.

Rabin, M. (1993) "Incorporating Fairness into Game Theory and Economics," *Amer. Econ. Rev.*, 85, 1281-1302.

Fehr, E. and K. Schmidt (1999) "A Theory of Fairness, Competition and Cooperation," *Quart. J. Econ.* 114, 817-868.

Bernheim, D. and A. Rangel (2004) "Addiction and Cue-Triggered Decision Processes," *Amer. Econ. Rev.*

Fudenberg, D. and D. Levine (2005) "A Dual Self Model of Impulse Control," Working Paper.

Koszegi, B. and M. Rabin (2006) "A Model of Reference Dependent Preferences," *Quart. J. Econ.*

Heidhues, P. and B. Koszegi (2006) "The Impact of Consumer Loss Aversion on Pricing," Working Paper.

Dewatripont, M. and J. Tirole (2005) "Modes of Communication," *J. Pol. Econ.*

Eyster, E. and M. Rabin (2005) "Cursed Equilibrium," *Econometrica.*

Esponda, I. (2006) "Behavioral Equilibrium in Economies with Adverse Selection," Working Paper.

Solution Concepts

Jonathan Levin

April 2006

These notes discuss some of the central solution concepts for normal-form games: Nash and correlated equilibrium, iterated deletion of strictly dominated strategies, rationalizability, and self-confirming equilibrium.

1 Nash Equilibrium

Nash equilibrium captures the idea that players ought to do as well as they can given the strategies chosen by the other players.

Example 1 Prisoners' Dilemma

	C	D
C	$1,1$	$-1,2$
D	$2,-1$	$0,0$

The unique Nash Equilibrium is (D, D).

Example 2 Battle of the Sexes

	B	F
B	$2,1$	$0,0$
F	$0,0$	$1,2$

There are two pure Nash equilibria (B, B) and (F, F) and a mixed strategy equilibrium where Row plays $\frac{2}{3}B + \frac{1}{3}F$ and Column plays $\frac{1}{3}B + \frac{2}{3}F$.

Definition 1 *A normal form game G consists of*

1. *A set of players $i = 1, 2, ..., I$.*

2. *Strategy sets $S_1, ..., S_I$; let $S = S_1 \times ... \times S_I$.*

3. *Payoff functions: for each* $i = 1, ..., I$, $u_i : S \to \mathbb{R}$

A (mixed) strategy for player i, $\sigma_i \in \Delta(S_i)$, is a probability distribution on S_i. A pure strategy places all probability weight on a single action.

Definition 2 *A strategy profile* $(\sigma_1, ..., \sigma_I)$ *is a* ***Nash equilibrium*** *of* G *if for every* i, *and every* $s_i \in S_i$,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i}).$$

And recall Nash's famous result:

Proposition 1 *Nash equilibria exist in finite games.*

A natural question, given the wide use of Nash equilibrium, is whether or why one should expect Nash behavior. One justification is that rational players ought somehow to reason their way to Nash strategies. That is, Nash equilibrium might arrive through introspection. A second justification is that Nash equilibria are self-enforcing. If players agree on a strategy profile before independently choosing their actions, then no player will have reason to deviate if the agreed profile is a Nash equilibrium. On the other hand, if the agreed profile is not a Nash equilibrium, some player can do better by breaking the agreement. A third, and final, justification is that Nash behavior might result from learning or evolution. In what follows, we take up these three ideas in turn.

2 Correlated Equilibrium

2.1 Equilibria as a Self-Enforcing Agreements

Let's start with the account of Nash equilibrium as a self-enforcing agreement. Consider Battle of the Sexes (BOS). Here, it's easy to imagine the players jointly deciding to attend the Ballet, then playing (B, B) since neither wants to unilaterally head off to Football. However, a little imagination suggests that Nash equilibrium might not allow the players sufficient freedom to communicate.

Example 2, cont. Suppose in BOS, the players flip a coin and go to the Ballet if the coin is Heads, the Football game if Tails. That is, they just randomize between two different Nash equilibria. This coin flip allows a payoff $(\frac{3}{2}, \frac{3}{2})$ that is *not* a Nash equilibrium payoff.

So at the very least, one might want to allow for randomizations between Nash equilibria under the self-enforcing agreement account of play. Moreover, the coin flip is only a primitive way to communicate prior to play. A more general form of communication is to find a mediator who can perform clever randomizations, as in the next example.

Example 3 This game has three Nash equilibria (U, L), (D, R) and $(\frac{1}{2}U + \frac{1}{2}D, \frac{1}{2}L + \frac{1}{2}R)$ with payoffs $(5, 1)$, $(1, 5)$ and $(\frac{5}{2}, \frac{5}{2})$.

	L	R
U	$5, 1$	$0, 0$
D	$4, 4$	$1, 5$

Suppose the players find a mediator who chooses $x \in \{1, 2, 3\}$ with equal probability $\frac{1}{3}$. She then sends the following messages:

- If $x = 1 \Rightarrow$ tells Row to play U, Column to play L.
- If $x = 2 \Rightarrow$ tells Row to play D, Column to play L.
- If $x = 3 \Rightarrow$ tells Row to play D, Column to play R.

Claim. It is a Perfect Bayesian Equilibrium for the players to follow the mediator's advice.

Proof. We need to check the incentives of each player.

- If Row hears U, believes Column will play $L \Rightarrow$ play U.
- If Row hears D, believes Column will play L, R with $\frac{1}{2}, \frac{1}{2}$ probability $\Rightarrow$ play D.
- If Column hears L, believes Row will play U, D with $\frac{1}{2}, \frac{1}{2}$ probability $\Rightarrow$ play L.
- If Column hears R, believes Row will play $D \Rightarrow$ play R.

Thus the players will follow the mediator's suggestion. With the mediator in place, expected payoffs are $(\frac{10}{3}, \frac{10}{3})$, strictly higher than the players could get by randomizing between Nash equilibria.

2.2 Correlated Equilibrium

The notion of correlated equilibrium builds on the mediator story.

Definition 3 *A* ***correlating mechanism*** $(\Omega, \{H_i\}, p)$ *consists of:*

- *A finite set of states* Ω
- *A probability distribution* p *on* Ω.
- *For each player* i, *a partition of* Ω, *denoted* $\{H_i\}$. *Let* $h_i(\omega)$ *be a function that assigns to each state* $\omega \in \Omega$ *the element of* i*'s partition to which it belongs.*

Example 2, cont. In the BOS example with the coin flip, the states are $\Omega = \{\text{Heads, Tails}\}$, the probability measure is uniform on Ω, and Row and Column have the same partition, $\{\{\text{Heads}\}, \{\text{Tails}\}\}$.

Example 3, cont. In this example, the set of states is $\Omega = \{1, 2, 3\}$, the probability measure is again uniform on Ω, Row's partition is $\{\{1\}, \{2, 3\}\}$, and Column's partition is $\{\{1, 2\}, \{3\}\}$.

Definition 4 *A* ***correlated strategy*** *for* i *is a function* $f_i : \Omega \to S_i$ *that is measurable with respect to* i*'s information partition. That is, if* $h_i(\omega) = h_i(\omega')$ *then* $f_i(\omega) = f_i(\omega')$.

Definition 5 *A strategy profile* $(f_1, ..., f_I)$ *is a* ***correlated equilibrium*** *relative to the mechanism* $(\Omega, \{H_i\}, p)$ *if for every* i *and every correlated strategy* $\tilde{f}_i$:

$$\sum_{\omega \in \Omega} u_i\left(f_i(\omega), f_{-i}(\omega)\right) p(\omega) \geq \sum_{\omega \in \Omega} u_i\left(\tilde{f}(\omega), f_{-i}(\omega)\right) p(\omega) \tag{1}$$

This definition requires that f_i maximize i's *ex ante* payoff. That is, it treats the strategy as a contingent plan to be implemented after learning the partition element. Note that this is equivalent to f_i maximizing i's *interim* payoff for each H_i that occurs with positive probability — that is, for all i,ω, and every $s_i' \in S_i$,

$$\sum_{\omega' \in h_i(\omega)} u_i(f_i(\omega), f_{-i}(\omega'))p(\omega'|h_i(\omega)) \geq \sum_{\omega' \in h_i(\omega)} u_i(s_i', f_{-i}(\omega'))p(\omega'|h_i(\omega))$$

Here, $p(\omega'|h_i(\omega))$ is the conditional probability on ω' given that the true state is in $h_i(\omega)$. By Bayes' Rule,

$$p(\omega'|h_i(\omega)) = \frac{\Pr\left(h_i(\omega)|\omega'\right) p(\omega')}{\sum_{\omega'' \in h_i(\omega)} \Pr\left(h_i(\omega)|\omega''\right) p(\omega'')} = \frac{p(\omega')}{p(h_i(\omega))}$$

The definition of CE corresponds to the mediator story, but it's not very convenient. To search for all the correlated equilibria, one needs to consider millions of mechanisms. Fortunately, it turns out that we can focus on a special kind of correlating mechanism, callled a *direct mechanism.* We will show that for any correlated equilibrium arising from some correlating mechanism, there is a correlated equilibrium arising from the direct mechanism that is precisely equivalent in terms of behavioral outcomes. Thus by focusing on one special class of mechanism, we can capture all possible correlated equilibria.

Definition 6 *A* ***direct mechanism*** *has* $\Omega = S$*,* $h_i(s) = \{s' \in S : s'_i = s_i\}$*, and some probability distribution* q *over pure strategy profiles.*

Proposition 2 *Suppose* f *is a correlated equilibrium relative to* $(\Omega, \{H_i\}, p)$*. Define* $q(s) \equiv \Pr(f(\omega) = s)$*. Then the strategy profile* $\tilde{f}$ *with* $\tilde{f}_i(s) = s_i$ *for all* $i, s \in S$ *is a correlated equilibrium relative to the direct mechanism* $(S, \{\tilde{H}_i\}, q)$*.*

Proof. Suppose that s_i is recommended to i with positive probability, so $p(s_i, s_{-i}) > 0$ for some s_{-i}. We check that under the direct mechanism $(S, \{\tilde{H}_i\}, q)$, player i cannot benefit from choosing another strategy s'_i when s_i is suggested. If s_i is recommended, then i's expected payoff from playing s'_i is:

$$\sum_{s_{-i} \in S_{-i}} u_i(s'_i, s_{-i}) q(s_{-i}|s_i).$$

The result is trivial if there is only one information set H_i in the original mechanism for which $f_i(H_i) = s_i$. In this case, conditioning on s_i is the same as conditioning on H_i in the original. More generally, we substitute for q to obtain:

$$\frac{1}{\Pr(f_i(\omega) = s_i)} \cdot \sum_{\omega | f_i(\omega) = s_i} u_i(s'_i, f_{-i}(\omega)) p(\omega).$$

Re-arranging to separate each H_i at which s_i is optimal:

$$\frac{1}{\Pr(f_i(\omega) = s_i)} \cdot \sum_{H_i | f_i(H_i) = s_i} \Pr(H_i) \left[\sum_{\omega \in H_i} u_i(s'_i, f_{-i}(\omega)) p(\omega | H_i) \right]$$

Since $(\Omega, \{H_i\}, p, f)$ is a correlated equilibrium, each bracketed term for which $\Pr(H_i) > 0$ is maximized at $f_i(H_i) = s_i$. So s_i is optimal given recommendation s_i. *Q.E.D.*

Thus what really matters in correlated equilibrium is the probability distribution over strategy profiles. We refer to any probability distribution q over strategy profiles that arises as the result of a correlated equilibrium as a *correlated equilibrium distribution (c.e.d.).*

Example 2, cont. In the BOS example, the c.e.d. is $\frac{1}{2}(B, B)$, $\frac{1}{2}(F, F)$.

Example 3, cont. In this example, the c.e.d is $\frac{1}{3}(U, L)$, $\frac{1}{3}(D, L)$, $\frac{1}{3}(D, R)$.

The next result characterizes correlated equilibrium distributions.

Proposition 3 *The distribution $q \in \Delta(S)$ is a correlated equilibrium distribution if and only if for all i, every s_i with $q(s_i) > 0$ and every $s_i' \in S_i$,*

$$\sum_{s_{-i} \in S_{-i}} u_i\left(s_i, s_{-i}\right) q\left(s_{-i}|s_i\right) \geq \sum_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i}) q(s_{-i}|s_i). \tag{2}$$

Proof. ($\Leftarrow$) Suppose q satisfies (2). Then the "obedient" profile f with $f_i(s) = s_i$ is a correlated equilibrium given the direct mechanism $(S, \{H_i\}, q)$ since (2) says precisely that with this mechanism s_i is optimal for i given recommendation s_i. ($\Rightarrow$) Conversely, if q arises from a correlated equilibrium, the previous result says that the obedient profile must be a correlated equilibrium relative to the direct mechanism $(S, \{H_i\}, q)$. Thus for all i and all recommendations s_i occuring with positive probability, s_i must be optimal — i.e. (2) must hold. *Q.E.D.*

Consider a few properties of correlated equilibrium.

Property 1 Any Nash equilibrium is a correlated equilibrium

Proof. Need to ask if (2) holds for the probability distribution q over outcomes induced by the NE. For a pure equilibrium s^*, we have $q(s^*_{-i}|s^*_i) = 1$ and $q(s_{-i}|s^*_i) = 0$ for any $s_{-i} \neq s^*_{-i}$. Therefore (2) requires for all i, s_i :

$$u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i}).$$

This is precisely the definition of NE. For a mixed equilibrium, σ^*, we have that for any s^*_i in the support of σ^*_i, $q(s_{-i}|s^*_i) = \sigma_{-i}(s_{-i})$. This follows from the fact that in a mixed NE, the players mix independently. Therefore (2) requires that for all i, s^*_i in the support of σ^*_i, and s_i,

$$\sum_{s_{-i} \in S_{-i}} u_i\left(s^*_i, s_{-i}\right) \sigma_{-i}\left(s_{-i}\right) \geq \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \sigma_{-i}(s_{-i}),$$

again, the definition of a mixed NE. *Q.E.D.*

Property 2 Correlated equilibria exist in finite games.

Proof. Any NE is a CE, and NE exists. Hart and Schmeidler (1989) show the existence of CE directly, exploiting the fact that a CE is just a probability distribution q satisfying a system of linear inequalities. Their proof does not appeal to fixed point results! *Q.E.D.*

Property 3 The sets of correlated equilibrium distributions and payoffs are convex.

Proof. Left as an exercise.

2.3 Subjective Correlated Equilibrium

The definition of correlated equilibrium assumes the players share a common prior p over the set of states (or equivalently share the same probability distribution over equilibrium play). A significantly weaker notion of equilibrium obtains if this is relaxed. For this, let $p_1, p_2, ..., p_I$ be *distinct* probability measures on Ω.

Definition 7 *The profile f is a* ***subjective correlated equilibrium*** *relative to the mechanism $(\Omega, \{H_i\}, p_1, ..., p_I)$ if for every i, ω and every alternative strategy $\tilde{f}_i$,*

$$\sum_{\omega \in \Omega} u_i(f_i(\omega), f_{-i}(\omega))p_i(\omega) \geq \sum_{\omega \in \Omega} u_i(\tilde{f}(\omega), f_{-i}(\omega))p_i(\omega)$$

Example 3, cont. Returning to our example from above,

	L	R
U	$5,1$	$0,0$
D	$4,4$	$1,5$

Here, $(4,4)$ can be obtained as a SCE payoff. Simply consider the direct mechanism with $p_1 = p_2 = \frac{1}{3}(U,L) + \frac{1}{3}(D,L) + \frac{1}{3}(D,R)$. This is a SCE, and since there is no requirement that the players have *objectively correct* beliefs about play, it may be that (D,L) is played with probability one!

2.4 Comments

1. The difference between mixed strategy Nash equilibria and correlated equilibria is that mixing is *independent* in NE. With more than two players, it may be important in CE that one player believes others are correlating their strategies. Consider the following example from Aumann (1987) with three players: Row, Column and Matrix.

$0,0,3$	$0,0,0$
$1,0,0$	$0,0,0$

$2,2,2$	$0,0,0$
$0,0,0$	$2,2,2$

$0,0,0$	$0,0,0$
$0,1,0$	$0,0,3$

   No NE gives any player more than 1, but there is a CE that gives everyone 2. Matrix picks middle, and Row and Column pick (Up,Left) and (Down,Right) each with probability $\frac{1}{2}$. The key here is that Matrix must expect Row to pick Up precisely when Column picks Left.

2. Note that in CE, however, each agent uses a pure strategy — he just is uncertain about others' strategies. So this seems a bit different than mixed NE if one views a mixed strategy as an explicit randomization in behavior by each agent i. However, another view of mixed NE if that it's not i's actual choice that matters, but j's beliefs about i's choice. On this account, we view σ_i as what others expect of i, and i as simply doing some (pure strategy) best response to σ_{-i}. This view, which is consistent with CE, was developed by Harsanyi (1973), who introduced small privately observed payoff perturbations so that in pure strategy BNE, players would be uncertain about others behavior. His "purification theorem" showed that these pure strategy BNE are observably equivalent to mixed NE of the unperturbed game if the perturbations are small and independent.

3. Returning to our pre-play communication account, one might ask if a mediator is actually needed, or if the players could just communicate by flipping coins and talking. With two players, it should be clear from the example above that the mediator is crucial in allowing for messages that are not common knowledge. However, Barany (1992) shows that if $I \geq 4$, then any correlated equilibrium payoff (with rational numbers) can be achieved as the Nash equilibrium of an extended game where prior to play the players communicate through cheap talk. Girardi (2001) shows the same can be done as a sequential equilibrium provided $I \geq 5$. For the case of two players, Aumann and Hart (2003) characterize the set of attainable payoffs if players can communicate freely, but without a mediator, prior to playing the game.

3 Rationalizability and Iterated Dominance

Bernheim (1984) and Pearce (1984) investigated the question of whether one should expect rational players to introspect their way to Nash equilibrium play. They argued that even if rationality was common knowledge, this should not generally be expected. Their account takes a view of strategic behavior that is deeply rooted in single-agent decision theory.

To discuss these ideas, it's useful to explicitly define rationality.

Definition 8 *A player is rational if he chooses a strategy that maximizes his expected payoff given his belief about opponents' strategies.*

Note that assessing rationality requires defining beliefs, something that the formal definition of Nash equilibrium does not require. Therefore, as a matter of interpretation, if we're talking about economic agents playing a game, we might say that Nash equilibrium arises when each player is rational and know his opponents' action profile. But we could also talk about Nash equilibrium in an evolutionary model of fish populations without ever mentioning rationality.

3.1 (Correlated) Rationalizability

Rationalizability imposes two requirements on strategic behavior.

1. Players maximize with respect to their beliefs about what opponents will do (i.e. are rational).
2. Beliefs cannot conflict with other players being rational, and being aware of each other's rationality, and so on (but they need not be correct).

Example 4 In this game (from Bernheim, 1984), there is a unique Nash equilibrium (a_2, b_2). Nevertheless a_1, a_3, b_1, b_3 can all be rationalized.

	b_1	b_2	b_3	b_4
a_1	$0,7$	$2,5$	$7,0$	$0,1$
a_2	$5,2$	$3,3$	$5,2$	$0,1$
a_3	$7,0$	$2,5$	$0,7$	$0,1$
a_4	$0,0$	$0,-2$	$0,0$	$10,-1$

- Row will play a_1 if Column plays b_3
- Column will play b_3 if Row plays a_3

- Row will play a_3 if Column plays b_1
- Column will play b_1 if Row plays a_1

This "chain of justification" rationalizes a_1, a_3, b_1, b_3. Of course a_2 and b_2 rationalize each other. However, b_4 cannot be rationalized, and since no rational player would play b_4, a_4 can't be rationalized.

Definition 9 *A subset $B_1 \times ... \times B_I \subset S$ is a* ***best reply set*** *if for all i and all $s_i \in B_i$, there exists $\sigma_{-i} \in \Delta(B_{-i})$ to which s_i is a best reply.*

- In the definition, note that σ_{-i} can reflect correlation — it need not be a mixed strategy profile for the opponents. This allows for more "rationalizing" than if opponents mix independently. More on this later.

Definition 10 *The set of* ***correlated rationalizable strategies*** *is the component by component union of all best reply sets:*

$$R = R_1 \times ... \times R_I = \bigcup_{\alpha} B_1^{\alpha} \times ... \times B_I^{\alpha}$$

where each $B^{\alpha} = B_1^{\alpha} \times ... \times B_I^{\alpha}$ is a best reply set.

Proposition 4 *R is the maximal best reply set.*

Proof. Suppose $s_i \in R_i$. Then $s_i \in B_i^{\alpha}$ for some α. So s_i is a best reply to some $\sigma_{-i} \in \Delta(B_{-i}^{\alpha}) \subset \Delta(R_{-i})$. So R_i is a best reply set. Since it contains all others, it is maximal. *Q.E.D.*

3.2 Iterated Strict Dominance

In contrast to asking what players might do, iterated strict dominance asks what players *won't* do, and what they won't do conditional on other players not doing certain things, and so on. Recall that a strategy s_i is *strictly dominated* if there is some mixed strategy σ_i such that $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $s_i \in S_{-i}$, and that iterated dominance applies this definition repeatedly.

- Let $S_i^0 = S_i$
- Let $S_i^k = \left\{ \begin{array}{l} s_i \in S_i^{k-1} : \text{There is no } \sigma_i \in \Delta(S_i^{k-1}) \text{ s.t.} \\ u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}^{k-1} \end{array} \right\}$
- Let $S_i^{\infty} = \bigcap_{k=1}^{\infty} S_i^k$.

Iterated strict dominance never eliminates Nash equilibrium strategies, or any strategy played with positive probability in a correlated equilibrium (proof left as an exercise!). Indeed it is often quite weak. Most games, including many games with a unique Nash equilibrium, are not dominance solvable.

Example 4, cont. In this example, b_4 is strictly dominated. Eliminating b_4 means that a_4 is also strictly dominated. But no other strategy can be eliminated.

Proposition 5 *In finite games, iterated strict dominance and correlated rationalizability give the same solution set, i.e.* $S_i^\infty = R_i$.

This result is suggested by the following Lemma (proved in the last section of these notes).

Lemma 1 *A pure strategy in a finite game is a best response to some beliefs about opponent play if and only if it is not strictly dominated.*

Proof of Proposition.

$R \subset S^\infty$. If $s_i \in R_i$, then s_i is a best response to some belief over R_{-i}. Since $R_{-i} \subset S_{-i}$, Lemma 1 implies that s_i is not strictly dominated. Thus $R_i \subset S_i^2$ for all i. Iterating this argument implies that $R_i \subset S_i^k$ for all i, k, so $R_i \subset S_i^\infty$.

$S^\infty \subset R$. It suffices to show that S^∞ is a best-reply set. By definition, no strategy in S^∞ is strictly dominated in the game in which the set of actions is S^∞. Thus, any $s_i \in S_i^\infty$ must be a best response to some beliefs over S_{-i}^∞. *Q.E.D.*

3.3 Comments

1. Bernheim (1984) and Pearce (1984) originally defined rationalizability assuming that players would expect opponents to mix independently. So B is a best reply set if $\forall s_i \in B_i$, there is some $\sigma_{-i} \in \times_{j\neq i}\Delta(S_j)$ to which s_i is a best reply. For $I = 2$, this makes no difference, but when $I \geq 3$, their concept refines ISD (it rules out more strategies).

2. Brandenburger and Dekel (1987) relate correlated rationalizability to subjective correlated equilibrium. While SCE is more permissive than rationalizability, requiring players to have well defined conditional beliefs (and maximize accordingly) even for states $\omega \in \Omega$ to which they assign zero probability leads to a refinement of SCE that is the same as correlated rationalizability.

3. One way to categorize the different solution concepts is to note that if one starts with rationality, and common knowledge of rationality, the concepts differ precisely in how they further restrict the beliefs of the players about the distribution of play. Doug Bernheim suggests the following table:

	Different Priors	Common Prior
Correlation	Corr. Rationalizability/ ISD/Refined Subj CE	Correlated Equilibrium
Independence	Rationalizability	Nash Equilibrium

3.4 Appendix: Omitted Proof

For completeness, this section provides a proof of the Lemma equating dominated strategies with those that are never a best response. The proof requires a separation argument, so let's first recall the Duality Theorem for linear programming. To do this, start with the following problem:

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} \sum_{j=1}^{n} c_j x_j \\ &\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad \forall i = 1, ..., m \end{aligned} \tag{3}$$

This problem has the same solution as

$$\max_{y \in \mathbb{R}^m_+} \left(\min_{x \in \mathbb{R}^n} \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{n} y_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) \right). \tag{4}$$

Rearranging terms, we obtain

$$\max_{y \in \mathbb{R}^m_+} \left(\min_{x \in \mathbb{R}^n} \sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} y_i a_{ij} \right) x_j + \sum_{i=1}^{m} y_i b_i \right). \tag{5}$$

Swapping the order of optimization give us

$$\min_{x \in \mathbb{R}^n} \left(\max_{y \in \mathbb{R}^m_+} \sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} y_i a_{ij} \right) x_j + \sum_{i=1}^{m} y_i b_i \right), \tag{6}$$

which can be related to the following "dual" problem:

$$\max_{y \in \mathbb{R}_+^m} \sum_{i=1}^{m} y_i b_i \tag{7}$$
$$\text{s.t.} \quad \sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} y_i a_{ij} \right) = 0 \qquad \forall j = 1, ..., n.$$

Theorem 1 *Suppose problems (3) and (7) are feasible (i.e. have non-empty constraint sets). Then their solutions are the same.*

We use the duality theorem to prove the desired Lemma.

Lemma 2 *A pure strategy in a finite game is a best response to some beliefs about opponent play if and only if it is not strictly dominated.*

Proof (Myerson, 1991). Let $s_i \in S_i$ be given. Our proof will be based on a comparison of two linear programming problems.

Problem I:

$$\min_{\sigma_{-i}, \delta} \delta$$
$$\text{s.t.} \quad \sigma_{-i}(s_{-i}) \geq 0 \qquad \forall s_{-i} \in S_{-i}$$
$$\sum_{s_{-i}} \sigma_{-i}(s_{-i}) \geq 1 \qquad \text{and} \qquad -\sum_{s_{-i}} \sigma_{-i}(s_{-i}) \geq -1$$
$$\delta + \sum_{s_{-i}} \sigma_{-i}(s_{-i}) \left[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) \right] \geq 0 \qquad \forall s_i' \in S_i$$

Observe that s_i is a best response to some beliefs over opponent play if any only if the solution to this problem is less than or equal to zero.

Problem II:

$$\max_{\eta, \varepsilon_1, \varepsilon_2, \sigma_i} \varepsilon_1 - \varepsilon_2$$
$$\text{s.t.} \quad \varepsilon_1, \varepsilon_2 \in \mathbb{R}_+, \ \sigma_i \in \mathbb{R}_+^{|S_i|}, \ \eta \in \mathbb{R}_+^{|S_{-i}|}$$
$$\sum_{s_i'} \sigma_i(s_i') \geq 1$$
$$\eta(s_{-i}) + \varepsilon_1 - \varepsilon_2 + \sum_{s_i'} \sigma_i(s_i') \left[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) \right] = 0 \qquad \forall s_{-i} \in S_{-i}$$

Observe that s_i is strictly dominated if and only if the solution to this problem is strictly greater than zero — i.e. s_i is *not* strictly dominated if and only if the solution to this problem is less than or equal to zero.

Finally, the Duality Theorem for linear programming says that so long as these two problems are feasible (have non-empty constraint sets), their solutions must be the same, establishing the result. *Q.E.D.*

4 Self-Confirming Equilibria

The third possible foundation for equilibrium is learning. We'll look at explicit learning processes later; for now, we ask what might happen as the end result of a learning processes. For instance, if a learning process settles down into steady-state play, will this be a Nash Equilibrium? Fudenberg and Levine (1993) suggest that a natural end-result of learning is what they call *self-confirming equilibria.* In a self-confirming equilibrium:

1. Players maximize with respect to their beliefs about what opponents will do (i.e. are rational).

2. Beliefs cannot conflict with the empirical evidence (i.e. must match the empirical distribution of play).

The difference with Nash equilibrium and rationalizability lies in the restriction on beliefs. In a Nash equilibrium, players hold correct beliefs about opponents' strategies and hence about their behavior. By contrast, with rationalizability, beliefs need not be correct, they just can't conflict with rationality. With SCE, beliefs need to be consistent with available data.

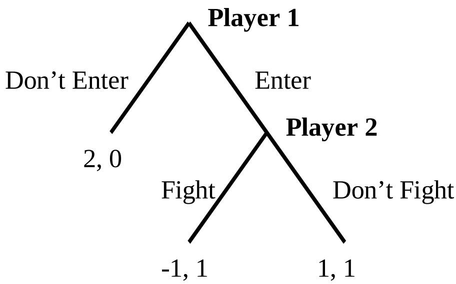
4.1 Examples of Self-Confirming Equilibria

In a simultaneous game, assuming actions are observed after every period, every Nash equilibrium is self-confirming. Moroeover, any self-confirming equilibrium is Nash.

Example 1 Consider matching pennies. It is an SCE for both players to mix 50/50 and to both believe the other is mixing 50/50. On the other hand, if player i believes anything else, he must play a pure strategy. But then player j must believe i will play this strategy or else he would eventually be proved wrong. So the only SCE is the same as the NE.

In extensive form games, the situation is different, as the next example shows.

Example 2 In the entry game below, the only Nash equilibria are (Enter, Don't Fight) and (Don't Enter, Fight). These equilibria, with correct beliefs, are also self-confirming equilibria.

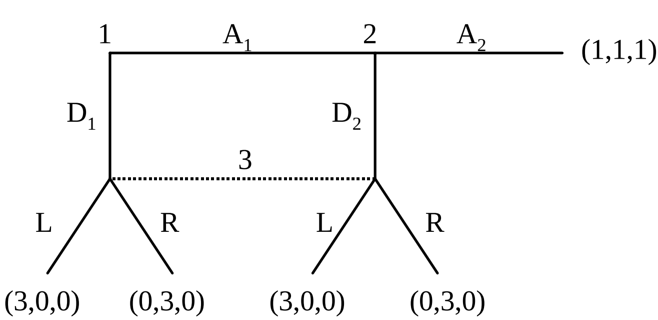


An Entry Game

There is also another SCE, however, where Player 1 plays Don't Enter and believes Fight, while Player 2 player plays Don't Fight and believes Don't Enter. In this SCE, player 1 has the wrong beliefs, but since he never enters, they're never contradicted by the data!

In the entry game, the non-Nash SCE is indistinguishable from a Nash equilibrium in terms of observed behavior. But even that need not be the case, as the next example shows.

Example 3 (Fudenberg-Kreps, 1993) Consider the three player game below.



SCE are different than NE

In this game, there is a self-confirming equilibrium where (A_1, A_2) is played. In this equilibrium, player 1 expects player 3 to play R, while player 2 expects 3 to play L. Given these beliefs, the optimal strategy for player 1 is to play A_1, while the optimal strategy or player 2 is to play A_2. Player 3's beliefs and strategy can be arbitrary so long as the strategy is optimal given beliefs.

The key point here is that there is *no* Nash equilibrium where (A_1, A_2) is played. The reason is that in any Nash equilibrium, players 1 and 2 *must* have the *same* (correct) beliefs about player 3's strategy. But if they have the same beliefs, then at least one of them must want to play D.

The distinction between Nash and self-confirming equilibria in extensive form games arises because players do not get to observe all the relevant information about their opponents' behavior. The same issue can arise in simultaneous-move games as well if the information feedback that players' receive is limited.

Example 4 Consider the following two-player game. Suppose that after the game is played, the column player observes the row player's action, but the row player observes only whether or not the column player chose R, and gets no information about her payoff.

	L	M	R
U	$2, 0$	$0, 2$	$0, 0$
D	$0, 0$	$2, 0$	$3, 3$

This game has a unique Nash equilibrium, (D, R); indeed the game is dominance-solvable. The profile (D, R) is self-confirming too; however that is not the only self-confirming profile. The profile (U, M) is also self-confirming. In this SCE, column has correct beliefs, but row believes that column will play L. This mistaken belief isn't refuted by the evidence because all row observes is that column does not play R. There are also SCE where row mixes, and where both players mix.

Note that in the (U, M) SCE, row's beliefs do not respect column's rationality. This suggests that one might refine SCE by further restricting beliefs to respect rationality or common knowledge of rationality — Dekel, Fudenberg and Levine (1998) and Esponda (2006) explore this possibility.

The above example is somewhat contrived, but the idea that players might get only partial feedback, and this might affect the outcome of learning, is natural. For instance, in sealed-bid auctions it is relatively common to announce only the winner and possibility not even the winning price, so the information available to form beliefs is rather limited.

4.2 Formal Definition of SCE

To define self-confirming equilibrium in extensive form games, let s_i denote a strategy for player i, and σ_i a mixture over such strategies. Let H_i denote the set of information sets at which i moves, and $H(s_i, \sigma_{-i})$ denote the set of information sets that can be reached if player i plays s_i and opponents play σ_{-i}. Let $\pi_i(h_i|\sigma_i)$ denote the mixture over actions that results at information set h_i, if player i is using the strategy σ_i (i.e. π_i is the behavior strategy induced by the mixed strategy σ_i). Let μ_i denote a belief over $\Pi_{-i} = \times_{j \neq i} \Pi_j$ the product set of other players' behavior strategies.

Definition 11 *A profile σ is a Nash equilibrium if for each $s_i \in$support(σ_i), there exists a belief μ_i such that (i) s_i maximizes i's expected payoff given beliefs μ_i, and (ii) player i's beliefs are correct, for all $h_j \in H_{-i}$*

$$\mu_i \left[\{ \pi_{-i} \mid \pi_j(h_j) = \pi_j(h_j|\sigma_j) \} \right] = 1.$$

The way to read this is that for each information set at which some player $j \neq i$ moves, player i's belief puts a point mass on the probability distribution that exactly coincides with the distribution induced by j's strategy. Thus i has correct beliefs at all opponent information sets.

Definition 12 *A profile σ is a Self-Confirming equilibrium if for each $s_i \in$support$(\sigma_i$ there exists a belief μ_i such that (i) s_i maximizes i's expected payoffs given beliefs μ_i, and (ii) player i's beliefs are empirically correct, for all histories $h_j \in H(s_i, \sigma_{-i})$ and all $j \neq i$*

$$\mu_i \left[\{ \pi_{-i} \mid \pi_j(h_j) = \pi_j(h_j|\sigma_j) \} \right] = 1.$$

The difference is that in self-confirming equilibrium, beliefs must be correct only for reachable histories. This definition assumes that the players observe their opponents' actions perfectly, in contrast to the last example above; it's not hard to generalize the definition. Note that this definition formally encompasses games of incomplete information (where Nature moves first); Dekel et. al (2004) study SCE in these games.

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Knowledge and Equilibrium

Jonathan Levin

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These notes develop a formal model of knowledge. We use this model to prove the "Agreement" and "No Trade" Theorems and to investigate the reasoning requirements implicit in different solution concepts. The notes follow Osborne and Rubinstein (1994, ch. 5). For an outstanding survey that covers much the same material, see Samuelson (2004).

1 A Model of Knowledge

The basic model we consider works as follows. There are a set of states Ω, one of which is true. For each state $\omega \in \Omega$, and a given agent, there is a set of states $h(\omega)$ that the agent considers possible when the actual state is ω. Finally, we will say that an agent *knows* E if E obtains at all the states that the agent believes are possible. These basic definitions will provide a language for talking about reasoning and the implications of reasoning.

States. The starting point is a set of states Ω. We can vary our interpretation of this set depending on the problem. In standard decision theory, a state describes contingencies that are relevant for a particular decision. In game theory, a state is sometimes viewed as a *complete* description of the world, including not only an agent's information and beliefs, but also his behavior.

Information. We describe an agent's knowledge in each state using an *information function.*

Definition 1 *An **information function** for Ω is a function h that associates with each state $\omega \in \Omega$ a nonempty subset $h(\omega)$ of Ω.*

Thus $h(\omega)$ is the set of states the agent believes to be possible at ω.

Definition 2 *An information function is* ***partitional*** *if there is some partition of* Ω *such that for any* $\omega \in \Omega$*,* $h(\omega)$ *is the element of the partition that contains* ω*.*

It is straightforward to see that an information function is partitional if and only if it satisfies the following two properties:

P1 $\omega \in h(\omega)$ for every $\omega \in \Omega$.

P2 If $\omega' \in h(\omega)$, then $h(\omega') = h(\omega)$.

Given some state ω, Property P1 says that the agent is not convinced that the state is not ω. Property P2 says that if ω' is also deemed possible, then the set of states that would be deemed possible were the state actually ω' must be the same as those currently deemed possible at ω.

Example 1 Suppose $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and that the agent's partition is $\{\{\omega_1, \omega_2\}, \{\omega_3\}\{\omega_4\}\}$. Then $h(\omega_3) = \{\omega_3\}$, while $h(\omega_1) = \{\omega_1, \omega_2\}$.

Example 2 Suppose $\Omega = \{\omega_1, \omega_2\}$, $h(\omega_1) = \{\omega_1\}$ but $h(\omega_2) = \{\omega_1, \omega_2\}$. Here h is not partitional.

Knowledge. We refer to a set of states $E \subset \Omega$ as an *event*. If $h(\omega) \subset E$, then in state ω, the agent views $\neg E$ as impossible. Hence we say that the agent *knows* E. We define the agent's *knowledge function* K by:

$$K(E) = \{\omega \in \Omega : h(\omega) \subset E\}.$$

Thus, $K(E)$ is the set of states at which the agent knows E.

Example 1, cont. In our first example, suppose $E = \{\omega_3\}$. Then $K(E) = \{\omega_3\}$. Similarly, $K(\{\omega_3, \omega_4\}) = \{\omega_3, \omega_4\}$ and $K(\{\omega_1, \omega_3\}) = \{\omega_3\}$.

Example 2, cont. In our second example, $K(\{\omega_1\}) = \{\omega_1\}$, $K(\{\omega_2\}) = \emptyset$ and $K(\{\omega_1, \omega_2\}) = \{\omega_1, \omega_2\}$.

Now, notice that for every state $\omega \in \Omega$, we have $h(\omega) \subset \Omega$. Therefore it follows that:

K1 (Axiom of Awareness) $K(\Omega) = \Omega$.

That is, regardless of the actual state, the agent knows that he is in some state. Equivalently, the agent can identify (is aware of) the set of possible states.

A second property of knowledge functions derived from information functions is that:

K2 $K(E) \cap K(F) = K(E \cap F)$.

Property K2 says that if the agent knows E and knows F, then he knows $E \cap F$. An implication of this property is that

$$E \subset F \Rightarrow K(E) \subset K(F),$$

or in other words, if F occurs whenever E occurs, then knowing F means knowing E as well. To see why this additional property holds, suppose $E \subset F$. Then $E = E \cap F$, so $K(E) = K(E \cap F)$. Applying K2 implies that $K(E) = K(E) \cap K(F)$, from which the result follows.

If the information function satisfies P1, the knowledge function also satisfies a third property:

K3 (Axiom of Knowledge) $K(E) \subset E$.

This says that if the agent knows E, then E must have occurred — the agent cannot know something that is false. This is a fairly strong property if you stop to think about it.

Finally, if the information function is partitional (i.e. satisfies both P1 and P2), the knowledge function satisfies two further properties:

K4 (Axiom of Transparency) $K(E) \subset K(K(E))$

Property K4 says that if the agent knows E, then he knows that he knows E. To see this, note that if h is partitional, then $K(E)$ is the union of all partition elements that are subsets of E. Moreover, if F is any union of partition elements $K(F) = F$ (so actually $K(E) = K(K(E))$).

K5 (Axiom of Wisdom) $\Omega \backslash K(E) \subset K(\Omega \backslash K(E))$.

Property K5 states the opposite: if the agent does not know E, then he knows that he does not know E. This brings home the strong assumption in the model that the agent understands and is aware of all possible states and can reason based on states that *might have* occurred, not just those that actually do occur (see Geanakoplos, 1994 for discussion).

We started with a definition of knowledge functions based on information functions, and derived K1-K5 as implications of the definition and the properties of partitional information. As it happens, it is also possible to go the other way because K1-K5 completely characterize knowledge functions. That is, Bacharach (1985) has shown that if we start with a set of states Ω and a function $K : \Omega \rightarrow \Omega$, then if K satisfies K1-K5 it is possible to place a partition on Ω to characterize the agent's information.

2 Common Knowledge

Suppose there are I agents with partitional information functions $h_1, ..., h_I$ and associated knowledge functions $K_1, ..., K_I$. We say that an event $E \subset \Omega$ is *mutual knowledge* in state ω if it is known to all agents, i.e. if $\omega \in K_1(E) \cap K_2(E) \cap ... \cap K_I(E) \equiv K^1(E)$. An event E is *common knowledge* in state ω if it is know to everyone, everyone knows this, and so on.

Definition 3 *An event $E \subset \Omega$ is* ***common knowledge in state*** *ω if $\omega \in K^1(E) \cap K^1K^1(E) \cap .....$*

A definition of common knowledge also can be stated in terms of information functions. Let us say that an event F is *self-evident* if for all $\omega \in F$ and $i = 1, ..., I$, we have $h_i(\omega) \subset F$.

Definition 4 *An event $E \subset \Omega$ is* ***common knowledge*** *in state $\omega \in \Omega$ if there is a self-evident event F for which $\omega \in F \subset E$.*

Example 3 Suppose $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and there are two individuals with information partitions:

$$\begin{aligned} \mathcal{H}_1 &= \{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}\} \\ \mathcal{H}_2 &= \{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\} \end{aligned}$$

- Is the event $E = \{\omega_1, \omega_2\}$ ever common knowledge? Not according to the second definition, since E does not contain any self-evident event. Moreover, not according to the first either since, $K_1(E) = \{\omega_1\}$ implies that $K_2K_1(E) = \emptyset$. Note, however, that $K^1(E) = \{\omega_1\}$ so E is mutual knowledge at ω_1.

- Is $F = \{\omega_1, \omega_2, \omega_3\}$ ever common knowledge? Apparently F is common knowledge at any $\omega \in F$ since F is self-evident. Moreover, since $K_1(F) = F$ and $K_2(F) = F$, it easy to check the first definition as well.

Lemma 1 *The following are equivalent: (i) $K_i(E) = E$ for all i, (ii) E is self-evident, (iii) E is a union of members of the partition induced by h_i for all i.*

Proof. To see *(i)* and *(ii)* are equivalent, note that F is self-evident iff $F \subset K_i(F)$ for all i. By K4, $K_i(F) \subset F$ always, so F is self-evident iff $K_i(F) = F$ for all i. To see *(ii)* implies *(iii)*, note that if E is self-evident, then $\omega \in E$ implies $h_i(\omega) \subset E$, so $E = \cup_{\omega \in E} h_i(\omega)$ for all i. Finally *(iii)* implies *(i)* immediately. *Q.E.D.*

Proposition 1 *The two definitions of common knowledge are equivalent.*

Proof. Assume E is common knowledge at ω according to the first definition. Then $E \supset K^1(E) \supset K^1K^1(E) \supset ...$ and ω is a member of each of these sets, which are thus non-empty. Since Ω is finite, there is some set $F = K^1 \cdots K^1(E)$ for which $K_i(F) = F$ for all i. So this set F, with $\omega \in F \subset E$ is self-evident and E is ck by the second definition.

Assume E is common knowledge at ω according to the second definition. There is a self-evident event F with $\omega \in F \subset E$. Then $K_i(F) = F$ for all i. So $K^1(F) = F$, and $K^1(F)$ is self-evident. Iterating this argument, $K^1 \cdots K^1(F) = F$ and each is self-evident. Now $F \subset E$, so by K2, $F \subset K^1 \cdots K^1(E)$. Since $\omega \in F$, E is ck by the first definition. *Q.E.D.*

3 The Agreement Theorem

Aumann (1976) posed the following question: could two individuals who share the same prior ever agree to disagree? That is, if i and j share a common prior over states, could a state arise at which it was commonly known that i assigned probability η_i to some event, j assigned probability η_j to that same event and $\eta_i \neq \eta_j$. Aumann concluded that this sort of disagreement is impossible.

Formally, let p be a probability measure on Ω — interpreted as the agents' prior belief. For any state ω and event E, let $p(E|h_i(\omega))$ denote i's posterior belief, so $p(E|h_i(\omega))$ is obtained by Bayes' rule. The event that "i assigns probability η_i to E" is $\{\omega \in \Omega : p(E|h_i(\omega) = \eta_i\}$.

Proposition 2 *Suppose two agents have the same prior belief over a finite set of states Ω. If each agent's infomation function is partitional and it is common knowledge in some state $\omega \in \Omega$ that agent 1 assigns probability η_1 to some event E and agent 2 assigns probability η_2 to E, then $\eta_1 = \eta_2$.*

Proof. If the assumptions are satisfied, then there is some self-evident event F with $\omega \in F$ such that:

$$F \subset \{\omega' \in \Omega : p(E|h_1(\omega') = \eta_1\} \cap \{\omega' \in \Omega : p(E|h_2(\omega') = \eta_2\}$$

Moreover, F is a union of members of i's information partition. Since Ω is finite, so is the number of sets in each union — let $F = \cup_k A_k = \cup_k B_k$. Now, for any nonempty disjoint sets C, D with $p(E|C) = \eta_i$ and $p(E|D) = \eta_i$, we have $p(E|C \cup D) = \eta_i$. Since for each k, $p(E|A_k) = \eta_1$, then $p(E|F) = \eta_1$ and similarly $p(E|F) = p(E|B_k) = \eta_2$. *Q.E.D.*

4 The No-Trade Theorem

The agreement theorem underlies an important set of results that place limits on the trades that can occur in differential information models under the common prior assumption (cf. Kreps, 1977; Milgrom and Stokey, 1982; Tirole, 1982; Osborne and Wolinsky, 1990). These "no-trade" theorems state, in various ways, that rational risk-averse traders cannot take opposite sides on a purely speculative bet. To see the basic idea, note that for some agent 1 to make an even money bet that a coin will come up Heads, he must believe that $\Pr(Heads) > 1/2$. For some agent 2 to take the other side of this bet, he must believe that $\Pr(Heads) < 1/2$. Aumann's theorem says the bet cannot happen since these opposing beliefs would then be common knowledge!

To formalize this idea, suppose there are two agents. Let Ω be a set of states and X a set of consequences (trading outcomes). A contingent contract is a function mapping Ω into X. Let A be the space of contracts. Each agent has a utility function $u_i : X \times \Omega \to \mathbb{R}$. Let $U_i(a) = u_i(a(\omega), \omega)$ denote i's utility from contract a — $U_i(a)$ is a random variable that depends on the realization of ω. Let $\mathbb{E}[U_i(a)|H_i]$ denote i's expectation of $U_i(a)$ conditional on his information H_i.

The following result is close cousin to the agreement theorem.

Proposition 3 *Let ϕ be a random variable on Ω. If i and j have a common prior on Ω, it cannot be common knowledge between them that i's expectation of ϕ is strictly greater than j's expectation of ϕ.*

Proof. This is on the problem set. *Q.E.D.*

Now, let us say that a contingent contract b is *ex ante efficient* if there is no contract a such that, for both i, $\mathbb{E}[U_i(a)] > \mathbb{E}[U_i(b)]$. We now state Milgrom and Stokey's (1982) no trade theorem.

Proposition 4 *If a contingent contract b is ex ante efficient, then it cannot be common knowledge between the agents that every agent prefers contract a to contract b.*

The proof will follow the same lines as the proof of the Agreement Theorem.

Proof. The claim is that there cannot be a state ω, which occurs with positive probability, at which the set

$$E = \{\omega : \mathbb{E}[U_i(a)|h_i(\omega)] > \mathbb{E}[U_i(b)|h_i(\omega)] \text{ for all } i\}$$

if common knowledge. Suppose to the contrary that there was such a state ω and hence a self-evident set F such that $\omega \in F \subset E$. By the definition of a self-evident set, for all $\omega' \in F$ and all i, $h_i(\omega') \in F$. So for all $\omega' \in F$ and all i:

$$\mathbb{E}[U_i(a) - U_i(b)|h_i(\omega')] > 0.$$

Now, using the fact that i's information is partitional, we know that $F = h(\omega^1) \cup h(\omega^2) \cup ... \cup h(\omega^n)$ for some set of states $\omega^1, ..., \omega^n \in F$ (in fact, we can choose these states so that $h(\omega^k) \cap h(\omega^l) = \varnothing$). It follows that for all i:

$$\mathbb{E}[U_i(a) - U_i(b)|F] > 0.$$

In other words, contract a strictly Pareto dominates contract b conditional on the event F. But now we have a contradiction to the assumption that b is ex ante efficient, because it is possible to construct a better contract c by defining c to be equal to a for all states $\omega \in F$ and equal to b for all states $\omega \notin F$. *Q.E.D.*

This theorem has remarkable implications because it says that under conditions that are quite standard in economic modelling — a common prior, Bayesian rationality and common knowledge being reached of any trades — there cannot be purely speculative transactions. Trades can only occur when the allocation of resources is not Pareto optimal and they must have an efficiency rationale.

This result seems nearly impossible to square with the enormous volume of trade in real-world financial markets. This suggests weakening one or more of the assumptions.

1. A route often followed in finance is to introduce "behavioral" or "liquidity" traders who are not rational but trade anyway. This can do

two things. First, to the extent that these traders lose money, they create gains from trade for rational traders. Second the presence of such traders may mean that two rational traders can trade without reaching common knowledge that they are trading – because one or both may think her trading partner is non-rational.

2. A second route is to relax the rationality assumption more broadly, for instance positing that agents use simple rules of thumb to process information or are systematically optimistic or pessimistic in interpreting information.

3. A third possibility is to weaken the common prior assumption so that agents "agree to disagree". In fact, Feinberg (2000) has shown that the common prior assumption is equivalent to the no-trade theorem in the sense that if it fails, agents can find some purely speculative bet that they would all want to take. Morris (1994) characterizes the set of trades that can occur with heteregenous priors.

5 Speculative Trade with Heterogeneous Priors

When agents have differing prior beliefs, the no-trade theorem fails. Here we'll cover an interesting case of trade with differing priors, taken from Harrison and Kreps (1978). To do this, we'll depart from the formal notation and just look at a simple example with numbers.

Consider a market where a single asset is traded by two classes of investors with infinite wealth, and identical discount factor $\delta = 3/4$. In every period $t = 1, 2, ...$, the asset pays a dividend d_t of either zero or one, and then can be traded. Both types of investors perceive the stream of dividends to be a stationary markov process with state space $\{0, 1\}$. But they disagree over the transition probabilities. Specifically, investors of type $i = 1, 2$ belive the transition probabilies are (here the ij entry reflects the probability of transiting from state i to state j):

$$Q_1 = \begin{matrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{matrix} \qquad\qquad Q_2 = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}.$$

The first type of investor believes the returns are independent over time; the second type believes the returns are perfectly positively correlated over time. Suppose we compute the value to each type of investor of holding the asset for the entire future, as a function of today's dividend:

$$\begin{array}{cc} v^1(0) = 3/2 & v^1(1) = 3/2 \\ v^2(0) = 0 & v^2(1) = 3 \end{array}.$$

Thus in periods when the dividend is low, the first type of investor, who believes returns are iid, has a higher belief about the asset's fundamental value. But in periods when the dividend is high, the second investor, who believes returns are persistent, believes the fundamental value is higher. This suggests the asset will be traded back and forth depending on the current period dividend. But at what price?

Harrison and Kreps show that for every state s, and time t, prices must satisfy:

$$p_t(s_t) = \delta \bullet \max_k \sum_{s_{t+1}} [d_{t+1}(s_{t+1}) + p_t(s_{t+1})]\, Q_k(s_t, s_{t+1}).$$

That is, given a common expectation about the price process $p_t(\bullet)$, the price in state s must equal the maximum expected return across investors of buying the asset and holding it for a single period. (Why just a single period? Think about the optimality principle of dynamic programming.)

Given the stationarity of our example, equilibrium asset prices will also be stationary — that is, the price will be the same in every period the asset returns zero and in every period it returns one. Combining this observation with the Harrison and Kreps characterization of prices, we have:

$$\begin{aligned} p(0) &= \frac{3}{4}\left[\frac{1}{2}p(0) + \frac{1}{2}(1 + p(1))\right] \\ p(1) &= \frac{3}{4}[1 + p(1)]. \end{aligned}$$

Solving these equations, we arrive at:

$$p(0) = 12/5 \qquad\qquad p(1) = 3.$$

In the high dividend state, type two investors buy the asset and are willing to hold it forever, so the price equals their fundamental valuation. In the low dividend, state, however, the price exceeds the fundamental valuation of both types of investors. Why? Because the price is set by type one investors who buy the asset for *speculative* purposes — that is, with the intention of holding the asset only until a dividend is paid at which point they intend to immediately unload the asset onto a type two investor.

This example seems quite special, but it can be generalized considerably (see Harrison and Kreps, 1978 and Scheinkman and Xiong, 2003). Nevertheless, this line of research has languished until recently despite many interesting questions one might ask. For instance, one might wonder how investor learning affects speculative trade. Morris (1996) shows that there can still be a speculative premium as learning occurs even if ultimately everyone agrees. Fudenberg and Levine (2005) argue that for natural learning processes, the end-result will be a situation where there is no trade (roughly they argue that a no-trade result obtains for a particular form of self-confirming equilibrium). Finally, you might notice that in the Harrison and Kreps model, investors agree on the state-price density (i.e. the price process) despite not agreeing on the transition probabilities over states. Mordecai Kurz has written a series of papers, using a quite different formalization, that attempt to relax this by allowing for heteregeneous beliefs directly about prices.

6 Epistemic Conditions for Equilibrium

Fix a game $G = (I, \{S_i\}, \{u_i\})$. Let Ω be a set of states. Each state is a complete description of each player's knowledge, action and belief. Formally, each state $\omega \in \Omega$ specifies for each i,

- $h_i(\omega) \subset \Omega$, i's knowledge in state ω.
- $s_i(\omega) \in S_i$, i's pure strategy in state ω.
- $\mu_i(\omega) \in \Delta(S_{-i})$, i's belief about the actions of others (note that i may believe other players actions are correlated).

We assume that among the players, it is common knowledge that the game being played is G. Thus, we assume each player knows his strategy set and the strategy sets and payoffs of the other players. Nevertheless, the model can be extended to games of incomplete information, where the game (e.g. payoff functions) may be uncertain. We will also maintain the assumption that each agent's information function is partitional.

Our first result is that if, in some state, each player is rational, knows the other players' strategies, and has a belief consistent with his knowledge, then the strategy profile chosen in that state is a Nash equilibrium of G.

Proposition 5 *Suppose that in state $\omega \in \Omega$, each player i:*

(i) knows the others' actions: $h_i(\omega) \subset \{\omega' \in \Omega : s_{-i}(\omega') = s_{-i}(\omega)\}$.

(ii) has a belief consistent with this knowledge: supp$(\mu_i(\omega)) \subset \{s_{-i}(\omega') \in S_{-i} : \omega' \in h_i(\omega)\}$

(iii) is rational: $s_i(\omega)$ *is a best response to* $\mu_i(\omega)$*,*

Then $s(\omega)$ *is a pure strategy Nash equilibrium of* G*.*

Proof. By *(iii)*, $s_i(\omega)$ is a best response for i to his belief, which by *(ii)* and *(i)* assigns probability one to the profile $s_{-i}(\omega)$. *Q.E.D.*

Clearly the assumption that each player knows the strategy choices of others is quite strong. What if we relax the assumption that players actually know the strategy choices of others, and replace it with an assumption that players know the *beliefs* of others and also that the others are rational? With two players, these conditions imply a Nash equilibrium in beliefs.

Proposition 6 *Suppose that* $I = 2$ *and that in state* $\omega \in \Omega$*, each player* i*:*

(i) knows the other's belief: $h_i(\omega) \subset \{\omega' \in \Omega : \mu_j(\omega') = \mu_j(\omega)\}$*,*

(ii) has a belief consistent with this knowledge,

(iii) knows that the other is rational: for any $\omega' \in h_i(\omega)$*, the action* $s_j(\omega')$ *is a best response of* j *to* $\mu_j(\omega')$ *for* $j \neq i$*.*

Then the profile $\sigma = (\mu_2(\omega), \mu_1(\omega))$ *is a Nash equilibrium of* G*.*

Proof. Let $s_i \in S_i$ be in the support of $\mu_j(\omega)$. By (ii) there is a state $\omega' \in h_j(\omega)$ such that $s_i(\omega') = s_i$. By (iii), s_i must be a best response to $\mu_i(\omega')$, which by (i) is equal to $\mu_i(\omega)$. *Q.E.D.*

Interestingly, neither of these results requires that beliefs be derived from a common prior on Ω. Indeed, beliefs need only be consistent with a player's knowledge. It is also not crucial that the game be common knowledge. For the first result, each player need only know his own preferences and strategy set. For the second, the game needs to be mutual knowledge.

Aumann and Brandenburger (1995) show that with three or more players, stronger assumptions are needed to justify mixed strategy Nash equilibrium. The main issue is to ensure that players i and j have the *same* beliefs about k. To ensure this, Aumann and Brandenburger assume a common prior, mutual knowledge of rationality and *common knowledge* of beliefs —

common knowledge of beliefs and a common prior imply identical beliefs by the Agreement Theorem.

Finally, we show that common knowledge of rationality implies rationalizability as we earlier suggested.

Proposition 7 *Suppose that in state $\omega \in \Omega$ it is common knowledge that each player's belief is consistent with his knowledge and that each player is rational. Then the profile $s(\omega)$ is rationalizable.*

Proof (for $I = 2$). Let $F \ni \omega$ be a self-evident event such that for every $\omega' \in F$ and each i, $s_i(\omega')$ is a best response to $\mu_i(\omega')$ and $\mu_i(\omega')$ is consistent with i's knowledge at ω'. For each i, let $B_i = \{s_i(\omega') \in S_i : \omega' \in F\}$. If $\omega' \in F$, then $s_i(\omega')$ is a best response to $\mu_i(\omega')$, whose support is a subset of $\{s_j(\omega'') \in S_j : \omega'' \in h_i(\omega')\}$. Since F is self-evident, $h_i(\omega') \subset F$, so $\{s_j(\omega'') \in S_j : \omega'' \in h_i(\omega')\} \subset B_j$. Thus $B_1 \times B_2$ is a best-reply set containing $s(\omega)$. *Q.E.D.*

It is also possible to provide an epistemic characterization of correlated equilibrium. In contrast to the other characterizations, which are *local* in nature (i.e. they refer to stategies, beliefs and knowledge at a particular state), correlated equilibrium is justified by assuming that players have a common prior and that *at every state* players are rational (this implies ck of rationality at every state). To state the result, due to Aumann (1987), let $\{H_i\}$ denote the partition induced by h_i for each i.

Proposition 8 *Suppose that for all $\omega \in \Omega$, (i) all players are rational, (ii) each player's belief is derived from a common prior p on Ω such that $p(h_i(\omega)) > 0$ for all i, ω, (iii) for all i, $s_i : \Omega \to S_i$ is measurable with respect to i's information, then $(\Omega, \{H_i\}, p, s)$ is a correlated equilibrium.*

Proof. Follows immediately from the definition of CE. *Q.E.D.*

Note that while we earlier motivated CE with an explicit communication story, here the information structure is just there. Gul (1998) argues that this leads to a conceptual problem — if the information structure is "just there," it's hard to know what to make of the common prior assumption.

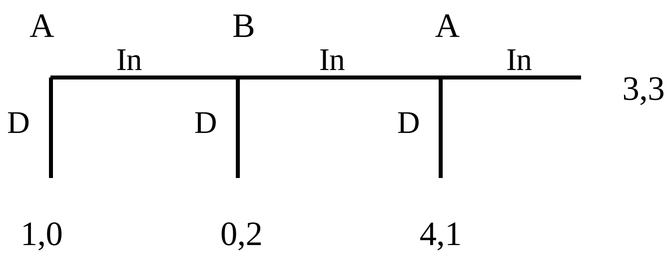
7 Comments

1. An idea we will return to throughout the term is that small departures from common knowledge can have a dramatic effect on the set of equilibria. In particular, even if each player is quite certain about the game

being player (i.e. the payoffs structure), even small uncertainty about other's information can eliminate equilibria that exist when payoffs are common knowledge. For one striking example of this, see Rubinstein (1989). Formally, the fact that small perturbations of the information structure can eliminate Nash equilibria occurs because the Nash equilibrium correspondence (mapping from the parameters of the game to the set of equilibrium strategies) is not lower semi-continuous.

2. Monderer and Samet (1989) define a version of "almost common knowledge" that has proved useful in some applications. There basic idea is that an event E is *common p-belief* if everyone assigns probability at least p to E, everyone assigns probability at least p to everyone assigning probability at least p to E and so on. Belief with probability 1 is then very close to knowledge (though not strictly speaking the same without some further assumptions), and common 1-belief the analogue of common knowledge.

3. Recent work has attempted to provide epistemic characterizations of dynamic equilibrium concepts such as subgame perfection, sequential equilibrium and so on. The difficulty is that rationality is a much more subtle issue in the extensive form. For example, consider the centipede game:



Centipede Game

Consider the following argument for why common knowledge of rationality should imply the backward induction solution that A play D immediately:

If A is rational, A will play D at the last node; if B knows A is rational, then B knows this; if B herself is rational, she must then play D at the second to last node; if A knows B is rational, and that B knows that A is rational, then A knows that B will play D at the second to last node; thus, if A is rational, A must play D immediately.

The question, however, is *what will happen if* A plays In? At that point, how will B assess A's rationality? This puzzle has inspired much recent research (see Brandenburger, 2002, for a survey).

4. One interesting approach to these problems is Feinberg (2004), who proposes to treat each agent at each information set as essentially a separate individual (though all A's have the same preferences), with each individual having a frame of reference that assumes his node has been reached. Then B might still be confident of A's future rationality even if A's past rationality has been refuted. In Feinberg's account, common knowledge of rationality may actually contradict the logical structure of the game (the centipede game is such a game).

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Supermodular Games

Jonathan Levin

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These notes develop the theory of supermodular games. Supermodular games are those characterized by "strategic complementarities" — roughly, this means that when one player takes a higher action, the others want to do the same. Supermodular games are interesting for several reasons. First, they encompass many applied models. Second, they have the remarkable property that many solution concepts yield the same predictions. Finally, they tend to be analytically appealing — they have nice comparative statics properties and behave well under various learning rules. Much of the theory is due to Topkis (1979), Vives (1990) and Milgrom and Roberts (1990).

1 Monotone Comparative Statics

We take a brief detour to review monotone comparative statics, starting with the property of increasing differences (or supermodularity). For this, suppose $X \subset \mathbb{R}$ and T some partially ordered set.

Definition 1 *A function $f : X \times T \to \mathbb{R}$ has* ***increasing differences*** *in (x, t) if for all $x' \geq x$ and $t' \geq t$,*

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t).$$

What does this mean? If f has increasing differences in (x, t), then the incremental gain to choosing a higher x (i.e. x' rather than x) is greater when t is higher. That is, $f(x', t) - f(x, t)$ is nondecreasing in t. You can check that increasing differences is symmetric — an equivalent statement is that if $t' > t$, then $f(x, t') - f(x, t)$ is nondecreasing in x.

Note that f need not be nicely behaved, nor do X and T need to be intervals. For instance, we could have $X = \{0, 1\}$ and just a few parameter values $T = \{0, 1, 2\}$. If, however, f *is* nicely behaved, we can re-write increasing differences in terms of derivatives.

Lemma 1 *If f is twice continuously differentiable, then then f has increasing differences in (x,t) if and only if $t' \geq t$ implies that $f_x(x,t') \geq f_x(x,t)$ for all x, or alternatively that $f_{xt}(x,t) \geq 0$ for all x,t.*

A central question in monotone comparative statics is to identify when:

$$x(t) = \arg\max_{x \in X} f(x,t)$$

will be non-decreasing (or increasing) in t. The main result we will use is due to Topkis (1968).

Theorem 1 *Let $X \subset \mathbb{R}$ be compact and T a partially ordered set. Suppose $f : X \times T \rightarrow \mathbb{R}$ has increasing differences in (x,t), and is upper semi-continuous in x.[1] Then (i) for all t, $x(t)$ exists and has a greatest and least element $\overline{x}(t)$ and $\underline{x}(t)$; and (ii) if $t' \geq t$, then $x(t') \geq x(t)$ in the sense that $\overline{x}(t') \geq \overline{x}(t)$ and $\underline{x}(t') \geq \underline{x}(t)$.*

Proof. *(i)* Fix t, and pick $x^1 \leq x^2 \leq ...$, with each $x^k \in x(t)$, and let $\overline{x} = \lim_{k \rightarrow \infty} x^k$. Then for all $x \in X$,

$$f(x^k,t) \geq f(x,t) \qquad \Rightarrow \qquad f(\overline{x},t) \geq f(x,t)$$

by continuity. Thus, $\overline{x} \in x(t)$. It follows that $x(t)$ must have a largest (and by the same argument, smallest) element.

(ii) Fix t and t', and let $x \in x(t)$ and $x' \in x(t')$ to be two maximizers. By the fact that x maximizes $f(x,t)$,

$$f(x,t) - f(\min(x,x'),t) \geq 0.$$

This implies (check the two cases that $x \geq x'$ and $x \leq x'$) that:

$$f(\max(x,x'),t) - f(x',t) \geq 0,$$

so by supermodularity

$$f(\max(x,x'),t') - f(x',t') \geq 0.$$

Thus, $\max(x,x')$ maximizes $f(\cdot,t')$. Now if we pick $x = \overline{x}(t)$ and $x' = \overline{x}(t')$, an immediate implication is that $x' \geq x$. A similar argument applies to the lowest maximizers. *Q.E.D.*

Topkis' Theorem says that if f has increasing differences, then the set of maximizers $x(t)$ is increasing in t in the sense that both the highest and lowest maximizers will not decrease if t increases.

[1] Recall that a function $f : X \rightarrow \mathbb{R}$ is upper semi-continuous at x_0 if for any ε, there exists a neighborhood $U(x_0)$ such that $x \in U(x_0)$ implies that $f(x) < f(x_0) + \varepsilon$. The function f is upper semi-continuous if it is upper semi-continuous at each $x_0 \in X$.

2 Supermodular Games

We now introduce the notion of a supermodular game, or game with strategic complementarities.

Definition 2 *The game* $(S_1, ..., S_I; u_1, .., u_I)$ *is a **supermodular game** if for all* i*:*

- S_i *is a compact subset of* $\mathbb{R}$*;*
- u_i *is upper semi-continuous in* s_i, s_{-i}*.*
- u_i *has increasing differences in* (s_i, s_{-i})*.*

Applying Topkis' Theorem in this context shows immediately that each player's best response function is increasing in the actions of other players.

Corollary 1 *Suppose* (S, u) *is a supermodular game, and let*

$$BR_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$$

Then

(i) $BR_i(s_{-i})$ *has a greatest and least element* $\overline{BR}_i(s_{-i})$, $\underline{BR}_i(s_{-i})$*.*

(ii) If $s'_{-i} \geq s_{-i}$*, then* $\overline{BR}_i(s'_{-i}) \geq \overline{BR}_i(s_{-i})$ *and* $\underline{BR}_i(s'_{-i}) \geq \underline{BR}_i(s_{-i})$*.*

2.1 Examples

1. (Investment Game) Suppose firms $1, ..., I$ simultaneously make investments $s_i \in \{0, 1\}$ and payoffs are:

$$u_i(s_i, s_{-i}) = \begin{cases} \pi\left(\sum_{j=1}^{I} s_j\right) - k & \text{if } s_i = 1 \\ 0 & \text{if } s_i = 0 \end{cases}$$

   where π is increasing in *aggregate* investment.

2. (Bertrand Competition) Suppose firms $1, ..., I$ simultaneously choose prices, and that

$$D_i(p_i, p_{-i}) = a_i - b_i p_i + \sum_{j \neq i} d_{ij} p_j$$

   where $b_i, d_{ij} \geq 0$. Then $S_i = \mathbb{R}^+$ and $\pi(p_i, p_{-i}) = (p_i - c_i) D_i(p_i, p_{-i})$ has $\left(\partial^2 \pi_i\right) / (\partial p_i \partial p_j) = d_{ij} \geq 0$. So the game is supermodular.

3. (Cournot Competition) Cournot duopoly is supermodular if we take

$$\begin{aligned} s_1 &= \text{Firm 1's quantity} \\ s_2 &= \text{Negative of Firm 2's quantity} \end{aligned}$$

4. (Diamond Search Model) Consider a simplified version of Diamond's (1982) search model (suggested by Milgrom and Roberts, 1990). There are I agents who exert effort looking for trading partners. Let e_i denote the effort of agent i, and $c(e_i)$ the cost of this effort, where c is increasing and continuous. The probability of finding a partner is $e_i \cdot \sum_{j \neq i} e_j$ and the cost is $c(e_i)$. Then:

$$u_i(e_i, e_{-i}) = e_i \cdot \sum_{j \neq i} e_j - c(e_i)$$

has increasing differences in e_i, e_{-i} so the game is supermodular.

2.2 Solving the Bertrand Game

Consider the Bertrand game from above, where firms 1 and 2 choose prices p_1, p_2. Suppose they have zero marginal costs, and that $D_i(p_i, p_j) = 1 - 2p_i + p_j$. Then

$$\Pi_i(p_i, p_j) = p_i \left[1 - 2p_i + p_j\right].$$

Note that

$$\frac{\partial \Pi_i}{\partial p_i}(p_i, p_j) = 1 - 4p_i + p_j$$

Let's apply iterated strict dominance to this game.

Set $S_i^0 = [0, 1]$.

- If $p_i < \frac{1}{4}$, then $\frac{\partial \Pi_i}{\partial p_i} > 1 - 4\frac{1}{4} + p_j \geq 0 \Rightarrow$ any $p_i < \frac{1}{4}$ is strictly dominated.
- If $p_i > \frac{1}{2}$, then $\frac{\partial \Pi_i}{\partial p_i} < 1 - 4\frac{1}{2} + p_j \leq 0 \Rightarrow$ any $p_i > \frac{1}{2}$ is strictly dominated.

So $S_i^1 = [\frac{1}{4}, \frac{1}{2}]$. Note that this is the interval of best-responses: $BR_i(p_j) \in [\frac{1}{4}, \frac{1}{2}]$.

Let $S_i^k = [\underline{s}^k, \overline{s}^k]$, where

$$\underline{s}^k = \frac{1}{4} + \frac{\underline{s}^{k-1}}{4} = \frac{1}{4} + \frac{1}{16} + \frac{\underline{s}^{k-2}}{16} = \dots = \frac{1}{4} + \frac{1}{16} + \dots + \frac{1}{4^k} + \frac{\underline{s}^0}{4^k}$$

$$\overline{s}^k = \frac{1}{4} + \frac{\overline{s}^{k-1}}{4} = \frac{1}{4} + \frac{1}{16} + \frac{\overline{s}^{k-2}}{16} = ... = \frac{1}{4} + \frac{1}{16} + ... + \frac{1}{4^k} + \frac{\overline{s}^k}{4^k}$$

So

$$\lim_{k\to\infty} \underline{s}^k = \lim_{k\to\infty} \overline{s}^k = \frac{1}{3}.$$

So $(\frac{1}{3}, \frac{1}{3})$ is the only Nash equilibrium, and the unique rationalizable profile.

3 Main Result

We now use the properties of supermodular games to show that the correspondence between rationalizable and Nash strategies in the Bertrand example is significantly more general than might appear at first glance.

Theorem 2 *Let (S, u) be a supermodular game. Then the set of strategies surviving iterated strict dominance has greatest and least elements $\underline{s}, \overline{s}$ and $\underline{s}, \overline{s}$ are both Nash equilibria.*

Corollary 2 *This implies the following.*

1. *Pure strategy NE exist in supermodular games*
2. *The largest and smallest strategies compatible with ISD, rationalizability, correlated equilibrium and Nash equilibrium are the same.*
3. *If a supermodular game has a unique NE, then it is dominance solvable (& lots of learning or adjustment rules will converge to it (e.g. best-response dynamics)).*

Proof. As in the example, we iterate the best response mapping. Let $S^0 = S$, and let $s^0 = (s_1^0, ..., s_I^0)$ be the largest element of S. Let $s_i^1 = \overline{BR}_i(s_{-i}^0)$, and $S_i^1 = \{s_i \in S_i^0 : s_i \leq s_i^1\}$. If $s_i \notin S_i^1$, i.e. $s_i > s_i^1$, then it is *dominated* by s_i^1 when $s_{-i} \in S_{-i}^0$ because (by increasing differences and the fact that s_i' is the biggest maximizer)

$$u_i(s_i, s_{-i}) - u_i(s_i^1, s_{-i}) \leq u_i(s_i, s_{-i}^0) - u_i(s_i^1, s_{-i}^0) < 0$$

Note that $s_i^1 = \overline{BR}_i(s_{-i}^0)$ and $s_i^1 \leq s_i^0$.

Iterating this argument, define

$$s_i^{k+1} = \overline{BR}_i(s_{-i}^k) \qquad \text{and} \qquad S_i^{k+1} = \left\{s_i \in S_i : s_i \leq s_i^{k+1}\right\}$$

Now, if $s^k \le s^{k-1}$, this implies that $s_i^{k+1} = \overline{BR}_i(s_{-i}^k) \ge \overline{BR}_i(s_{-i}^{k-1}) = s_i^k$. So by induction, s_i^k is a decreasing sequence for each i. Define:

$$\overline{s}_i = \lim_{k \to \infty} s_i^k$$

This limit exists and only strategies $s_i \le \overline{s}_i$ are undominated.

Similarly, we can start with $s^0 = (s_1^0, ..., s_I^0)$ the smallest elements in S and identify $\underline{s}$.

To complete the proof, we need to show that $\overline{s} = (\overline{s}_1, ..., \overline{s}_I)$ is a Nash equilibrium. Then for all i, s_i,

$$u_i(s_i^{k+1}, s_{-i}^k) \ge u_i(s_i, s_{-i}^k)$$

Taking limits as $k \to \infty$,

$$u_i(\overline{s}_i, \overline{s}_{-i}) \ge u_i(s_i, \overline{s}_{-i}).$$

Q.E.D.

4 Properties of Supermodular Games

A useful propery of supermodular games is that we can use monotonicity to prove comparative statics results. Our first result shows how changes in parameters that affect the marginal returns to action shift the equilibria of a supermodular game.

- A supermodular game (S, u) is indexed by t if each players payoff function is indexed by $t \in T$, some ordered set, and for all i, $u_i(s_i, s_{-i}, t)$ has increasing differences in (s_i, t).

Proposition 1 *Suppose (S, u) is a supermodular game indexed by t. The largest and smallest Nash equilibria are increasing in t.*

Proof. Let $\overline{BR}(s, t) : S \times T \to S$ be the largest best response function as defined above for the game with parameter t. Then $\overline{BR}_i(s, t)$ is i's best response to s_{-i} given parameter value t and is nondecreasing in s and t by Topkis' Theorem. Thus $\overline{BR}(s, t)$ is nondecreasing. Every Nash equilibrium satisfies $\overline{BR}(s, t) \ge s$, and moreover $\overline{s}(t) = \sup\{s : \overline{BR}(s, t) \ge s\}$ is the largest first point of $\overline{BR}(s, t)$ and hence the largest Nash equilibrium (formally this follows from Tarski's Fixed Point Theorem). Since $\overline{BR}(s, \cdot)$ is nondecreasing, $\overline{s}$ is nondecreasing. A similar argument proves the result for the smallest Nash equilibrium. *Q.E.D.*

Because there is a positive feedback between the strategic choices of different players in a supermodular game, there are often multiple equilibria. The second property we consider a welfare theorem that is particularly useful when considering such games.

- A supermodular game (S, u) has *positive spillovers* if for all i, $u_i(s_i, s_{-i})$ is increasing in s_{-i}.

Proposition 2 *Suppose (S, u) is a supermodular game with positive spillovers. Then the Nash equilibrium are ordered in accordance with Pareto preference.*

This result implies that the largest Nash equilibrium is Pareto-preferred among the set of all Nash equilibria. Nevertheless, it need not be Pareto optimal among the set of all strategy profiles.

We have now seen that the greatest and least equilibria in a supermodular game are pure strategy nash equilibria and that it is possible to obtain nice comparative statics results for these equilibria. But what about mixed strategy equilibria? Echenique and Edlin (2003) show that when a supermodular game has mixed strategy equilibria, these equilibria are always "unstable" under a variety of dynamic adjustment processes, thus justifing a focus on pure strategy equilibria.

Their idea can be seen using Battle of the Sexes as an example.

	B	F
B	$2,1$	$0,0$
F	$0,0$	$1,2$

Recall that Battle of the Sexes has two pure nash equilibria (B, B) and (F, F) and a mixed equilibrium $(\frac{2}{3}B+\frac{1}{3}F, \frac{1}{3}B+\frac{2}{3}F)$. To make this a supermodular game, we need to define an order on the strategy sets. Let $F >_i B$ for both players. Then $u_i(s_i, s_{-i})$ has increasing differences in (s_i, s_{-i}).

In the mixed equilibrium it is crucial that player 1 believes that player 2 is playing exactly $\frac{1}{3}B + \frac{2}{3}F$. If player 1 believes player 2 will play F with probability $2/3 + \varepsilon$, even for $\varepsilon > 0$ small, then player 1 will strictly prefer F. Similarly, if player 2 believes 1 will play F with any probability above $1/3$, player 2 will strictly prefer F.

Now, imagine the players play repeatedly, with player 1 initially believing 2 will play F with probability $2/3 + \varepsilon$ and player 2 initially believing 1 will play F with probability $1/3 + \eta$. Both will play F. If they then adjust their beliefs so they put more weight on their opponent's playing F (I'm purposely being a little loose about the dynamic adjustment process here), they will play F again in the next period, and so on until they always play F and have moved away from mixed strategy beliefs.

This situation is not contrived. The more general point is that so long as player i adjusts his beliefs toward j playing F when j does play F, and so long as i's response to this change is to herself play F more often, then any move toward (F, F) (or toward (B, B)) and away from the mixed equilibrium is self-reinforcing, and many reasonable dynamic processes will move away from the mixed equilibrium toward a pure equilibrium.[2]

5 Comments

1. (Extensions) These results extend to games where players have multi-dimensional strategy spaces. If $S_i \subset \mathbb{R}^n$, we need two further assumptions. First, for all i, S_i must be a *complete sublattice*; second, for all i, u_i must be supermodular in s_i as well as having increasing differences in (s_i, s_{-i}). For precise definitions, see that Monotone Comparative Statics handout. The results also extend to the case where u_i satisfies the single crossing property in (s_i, s_{-i}) as opposed to the stronger assumption of increasing differences (see Milgrom and Shannon, 1994).

2. (Comparing Fixed Points) Milgrom and Roberts (1994) use similar arguments to derive comparative statics for models where equilibria are the solutions to some equation $f(x, t) = 0$. Roughly, they show that if f is increasing in t and continuous (in a weak sense) in x, then the largest fixed point of $f(x, t) = 0$ is increasing in t. Thus their results provide analogues of Proposition 1 for another class of models.

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Coordination and Higher Order Uncertainty

Jonathan Levin

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In these notes, we discuss work on coordination in situations of uncertainty and investigate the importance of "higher order beliefs" — that is, players' beliefs about their opponents beliefs about their beliefs about.

We start with a particular class of "global" coordination games of incomplete information where players' beliefs are highly, but not perfectly, correlated. There games are interesting for several reasons. First, they capture in simple form the idea that in strategic settings where actions are conditioned on beliefs, in particular settings where coordination is important, players need to be concerned with what their opponents believe, what their opponents believe about their beliefs, and so on. Second, global games can allow us to refine equilibria in coordination games in a very strong way. In some models, we can use global games analysis to show that even if common knowledge of payoffs gives rise to multiple equilibria, there will be a unique equilibrium if the players' information is perturbed in even a "small" way. A recent applied literature has arisen using these techniques particularly in finance and macroeconomic (Morris and Shin, 2002, is a nice survey).

We then use related ideas to look at the difference between public and private information in situations where higher order beliefs matter greatly. Finally, we discuss the broader relevance of higher order beliefs in "typical" games. We discuss the "types" approach of Harsanyi, the modeling of higher-order uncertainty, and some further applications.

1 Global Games

We work with an example drawn from Morris and Shin (2002) and based on Carlsson and van Damme (1993). There are two players $i = 1, 2$ who choose one of two actions, "Invest" or "Not Invest". The payoffs are as follows:

	Invest	Not Invest
Invest	θ, θ	$\theta - 1, 0$
Not Invest	$0, \theta - 1$	$0, 0$

Not Invest is a safe action that yields zero, while Invest yields $\theta - 1$ if the opponent doesn't invest and θ if she does. If θ is known to the players, there are three possibilities:

- $\theta > 1$, Invest is a dominant strategy
- $\theta \in [0, 1]$, (Invest, Invest) and (NI,NI) are both NE.
- $\theta < 0$, Not invest is a dominant strategy

Suppose we introduce incomplete information by assuming that each player does not observe θ, but rather a private signal $x_i = \theta + \varepsilon_i$, where $\varepsilon_i \sim N(0, \sigma^2)$, each ε_i is independent. Suppose also that θ is drawn from a uniform distribution over the entire real line (i.e. players have an uninformative or improper prior on θ).

Proposition 1 *In the incomplete information game, there is a unique equilibrium in which both players invest if and only if* $x_i > x^* = \frac{1}{2}$.

Proof. Let's first verify that the stated equilibrium is in fact an equilibrium. First, observe that if player j uses the stated strategy, then i's payoff from investing conditional on having signal x_i is

$$\mathbb{E}[\theta \mid x_i] - \Pr\left[x_j \leq \frac{1}{2} \mid x_i\right] = x_i - \Phi\left(\frac{\frac{1}{2} - x_i}{\sqrt{2}\sigma}\right),$$

where we use the fact that $x_j | x_i \sim N(x_i, 2\sigma^2)$. This payoff increasing in x_i, and equal to zero when $x_i = 1/2$. So i's best response is to invest if and only if $x_i \geq 1/2$, verifying the equilibrium.

To prove uniqueness, we'll show that the stated profile is the only one that survives iterated deletion of dominated strategies. As a preliminary, consider i's payoff from investing conditional on having signal x_i and conditional on j playing the strategy "invest if and only if $x_j \geq k$" for some "switching point" k. The payoff is

$$\mathbb{E}[\theta \mid x_i] - \Pr[x_j \leq k \mid x_i] = x_i - \Phi\left(\frac{k - x_i}{\sqrt{2}\sigma}\right),$$

which is again increasing in x_i, and also decreasing in k. Let $b(k)$ be the unique value of x_i at which the payoff to investing is zero. Observe that $b(0) > 0$, $b(1) < 1$, that $b(\cdot)$ is strictly increasing in k, and that there is a unique value k^* that solves $b(k) = k$, namely $k^* = 1/2$.

We now show that if a strategy s_i survives n rounds of iterated deletion of strictly dominated strategies, then:

$$s_i(x) = \begin{cases} \text{Invest} & \text{if } x_i > b^{n-1}(1) \\ \text{Not Invest} & \text{if } x_i \leq b^{n-1}(0) \end{cases}.$$

Note that the value of $s_i(x)$ for values of x between $b^{n-1}(0)$ and $b^{n-1}(1)$ is not pinned down here.

This claim follows from an induction argument. Suppose $n = 1$. The worst case for i investing is that j never invests (i.e. uses a switching strategy with $k = \infty$). If j never invests, then i should invest if and only if $\mathbb{E}[\theta|x_i] > 1$, or in other words if $x_i \geq 1$ (recall that $\mathbb{E}[\theta|x_i] = x_i$). This means that Not Invest is dominated by Invest if and only if $x_i > 1$. Conversely, Invest is dominated by Not Invest if and only if $x_i < 0$. This verifies the claim for $n = 1$.

Now suppose the claim is true for n. Now the worst case for i investing is that j invests only if $x_j > b^{n-1}(1)$, i.e. plays a switching strategy with cut-off $k = b^{n-1}(1)$. In this case, i should invest if and only if $x_i > b(b^{n-1}(1)) = b^n(1)$, meaning that Not Invest is in fact dominated if $x_i > b^n(1)$. Conversely, Invest is dominated by Not Invest at this round if and only if $x_i < b(b^{n-1}(0)) = b^n(0)$. So we have proved the claim.

To complete the result, we observe that:

$$\lim_{n\to\infty} b^n(0) = \lim_{n\to\infty} b^n(1) = \frac{1}{2}.$$

Therefore iterated deletion of dominated strategies yields a unique profile in which both players invest if and only if their respective signal exceeds one-half. *Q.E.D.*

1.1 Generalizing the Model

The logic of this example can be generalized without much trouble to other two-action two-player supermodular games. In particular, suppose we have two players, each of who chooses an action $a \in \{0, 1\}$. For simplicity, they symmetric payoffs $u : \{0, 1\} \times \{0, 1\} \to \mathbb{R}$, where $u(a_i, a_j, x_i)$ is i's payoff from choosing a_i, given that j chooses a_j, and that i's private signal is x_i.

Signals are generated in the following way. First, nature selects a state $\theta \in \mathbb{R}$ drawn from the (improper) uniform distribution on $\mathbb{R}$. Player i then observes a signal $x_i = \theta + \sigma\varepsilon_i$, where ε_i has a continuous density $f(\cdot)$ with support $\mathbb{R}$. Call this game $G(\sigma)$.

Define the incremental returns to choosing $a = 1$ as:

$$\Delta(a_j, x) = u(1, a_j, x) - u(0, a_j, x).$$

We impose the following assumptions.

1. **Action Monotonicity**. Δ is increasing in a_j.

2. **State Monotonicity**. Δ is strictly increasing in x.

3. **Continuity.** Δ is continuous in x.

4. **Limit Dominance**. There exist $\underline{\theta}, \overline{\theta} \in \mathbb{R}$ such that $\Delta(a_j, x) < 0$ whenever $x < \underline{\theta}$, and $\Delta(a_j, x) > 0$ whenever $x \geq \overline{\theta}$.

Proposition 2 *Under (A1)–(A4), the essentially unique strategy profile that survives iterated deletion of strictly dominated strategies in $G(\sigma)$ satisfies $s(x) = 0$ for all $x < \theta^*$ and $s(x) = 1$ for all $x > \theta^*$, where θ^* is the unique solution to:*

$$\left(\Delta(1, \theta^*) + \Delta(0, \theta^*)\right)/2 = 0.$$

Proof. Nearly identical to the one above. *Q.E.D.*

In the unique equilibrium, each player uses a cut-off strategy (we say essentially unique because the strategy is indeterminate at $x = \theta^*$. Moreover, the cut-off θ^* is such that if i receives the signal θ^*, and believes that j is equally likely to play 0 or 1, then i will be just indifferent between playing 0 and 1.

Note one difference between this game and the first one is that this has "private values" while the other game had "common values". This distinction is not so important, however. Suppose payoffs in the more general case were $u(a, \theta)$, rather than $u(a, x)$. We could simply define $\Delta(a_j, x)$ as the *expected* return to playing 1 rather than 0 after observing a signal x, given that one's opponent was playing a_j. Assuming j was using a cut-off strategy, Δ would satisfy all the same properties, so everything will still go through.

1.2 Discussion

We now discuss several features and extensions of the model.

1. (Inefficiency of the Unique Equilibrium). If we consider a sequence of incomplete information games with $\sigma \to 0$, then, since in equilibrium each player invests if and only if $x = \theta + \varepsilon \geq \frac{1}{2}$, in the limit players coordinate on (Invest,Invest) whenever $\theta > \frac{1}{2}$, and on (Not Invest,Not Invest) whenever $\theta \leq \frac{1}{2}$. Coordination on $(Invest,\ Invest)$, however, is *efficient* whenever $\theta > 0$. So the fact that the players act in a decentralized fashion means that they generally won't coordinate efficiently.

2. (Risk Dominance) In 2×2 symmetric games, an action is *risk-dominant* if it is a best-response given that one's opponent is mixing uniformly. In the underlying symmetric information game, investing is risk-dominant if $\theta \geq \frac{1}{2}$ and not investing is risk-dominant if $\theta \leq \frac{1}{2}$. Hence as $\sigma \to 0$, and we converge to the complete information game, the players play the risk dominant action.

3. (Equilibrium Refinements) More generally, note that if $0 < \theta < \frac{1}{2}$, (Invest, Invest) is an equilibrium of the complete information game, but not of the closely related incomplete information game (with σ small but positive). This can be related to the general problem of selecting more or less plausible equilibria in a given game — or *refining* the set of equilibria. A common approach in this regard is to consider a family of nearby games and ask if these games have equilibria that are "close" to a given equilibrium of the original game. Kajii and Morris (1997) say that an equilibrium of a given game is *robust* if it is an equilibrium of all nearby games of incomplete information. Some games have no robust equilibria, but Kajii and Morris show that some interesting classes of games do have robust equilibria.

4. (Generalizations) It is quite easy to duplicate the above analysis to more general 2×2 games with strategic complementarities, provided some technical conditions are satisfied (see Morris and Shin, 2002). Frankel, Morris and Pauzner (2002) extend the above result to asymmetric n-player many action games with strategic complementarities. They provide conditions under which, if there is only a small amount of noise, equilibrium will be unique. The selected equilibrium, however, may depend on the fine structure of the noise. Interestingly, however, if a game has a unique robust equilibrium, this equilibrium will be

selected regardless of the noise structure. Frankel and Pauzner (2000) and Levin (2000) use global game arguments to identify unique equilibria in dynamic games with strategic complementarities. One such problem is on your homework.

5. (Higher Order Beliefs) One way to understand why the incomplete information game has a unique equilibrium when nature selects $\theta \in (0, 1)$ despite the complete information game having multiple strict Nash equilibria, is that there is a failure of *common knowledge* in the incomplete information game. We return to this point below.

6. (Applications) There has been a lot of interest in the application of global games to macroeconomics, in particular currency crises — the view being that currency pegs tend to fall when there is an attack by many investors, a situation that naturally gives rise to a coordination game. A twist in such settings is that the ability to observe prices may restore common knowledge (one analysis of this is by Angeletos and Werning, 2005).

2 Public and Private Information

In the coordination game above, players care about fundamentals (the value of θ) and also about the actions of their fellow player (the value of a_{-i}). Each player's signal is informative about both variables of interest. Of course, in the context of the above example, information is private. An interesting question that arises in a coordination setting concerns the role of *public* information. Intuitively, public information about fundamentals should be valuable in a coordination setting because (barring unfortunate coordination failure) it will allow for coordination on the more appropriate action. Morris and Shin (2003), however, show that this intuition fails in settings where players have private information as well as public information. The basic idea, as we will see shortly, is that players are "too sensitive" to the public information because even if it is not that informative about fundamentals, it tends to be quite informative about other players' beliefs and hence about their actions.

2.1 The "Beauty Contest" Model

Morris and Shin's model is based on a famous "beauty contest" parable in Keynes' *General Theory.*[1] There are a continuum of agents, indexed by $i \in [0,1]$. Agent i chooses a_i, and receives a payoff:

$$u_i(a_i, a_{-i}, \theta) = -(1-r)(a_i - \theta)^2 - r(L_i - \overline{L}),$$

where $0 < r < 1$ and:

$$L_i = \int_0^1 (a_j - a_i)^2 dj \qquad \text{and} \qquad \overline{L} = \int_0^1 L_j dj \ .$$

Agent i wants to minimize the distance between his action and the true state θ and also minimize the distance between his action and the actions of the other agents. The parameter r weights these two parts of the objective function.

The beauty contest part of the game has a zero-sum flavor. If we define social welfare as the average of individual utilities:

$$W(a,\theta) = \int_0^1 u_i(a,\theta)di = -(1-r)\int_0^1 (a_i - \theta)^2 di.$$

From a social point of view, what matters is how close the individual actions are to θ, not to each other.

Each agent will maximize his utility by choosing a_i to minimize his expected loss:

$$a_i = (1-r)\mathbb{E}_i[\theta] + r\mathbb{E}_i[\overline{a}],$$

where $\overline{a} = \int_0^1 a_j dj$ is the population average utility, and $\mathbb{E}_i$ is the expectation operator conditioning on i's information.

2.2 Public Information Benchmark

As a benchmark, consider the case where the agent's only have public information. Suppose everyone shares an improper uniform prior on θ and then observes a public signal:

$$y = \theta + \eta,$$

[1]Keynes drew an analogy between the stock market and a certain beauty contest in a London newspaper. The paper printed pictures of young women. Keynes said that making money in the stock market was like trying to pick the girl who the most people would vote for as most beatiful — what mattered was not the true beauty of the girls, but whether or not people would vote for them.

where $\eta \sim N(0, \sigma_\eta^2)$. Then $\mathbb{E}_i[\theta|y] = y$ by Bayesian updating, and the unique equilibrium has each agent choose:

$$a_i(y) = y.$$

In this equilibrium, the expected welfare is:

$$\mathbb{E}(W|\theta) = -(1-r)\mathbb{E}[(y-\theta)^2|\theta] = -(1-r)\sigma_\eta^2.$$

So clearly better public information unambiguously improves social welfare.

2.3 Public and Private Information

In contrast, now suppose that in addition to y, each agent observes a *private* signal:

$$x_i = \theta + \varepsilon_i,$$

where $\varepsilon_i \eta \sim N(0, \sigma_\varepsilon^2)$, and moreover, ε_i and ε_j are independent for all $i \neq j$. In this set-up, agent i's information is the pair (x_i, y). She needs to use this information to forecast both the true state θ and the average action in the population.

By the wonderful properties of Bayes updating with normal random variables, we have:

$$\mathbb{E}_i[\theta|x_i, y] = \frac{h_\eta y + h_\varepsilon x_i}{h_\eta + h_\varepsilon},$$

where $h_\varepsilon = 1/\sigma_\varepsilon^2$ and $h_\eta = 1/\sigma_\eta^2$ are the precisions of ε and η.

We look for a linear equilibrium of the form:

$$a_i(x_i, y) = \kappa x_i + (1-\kappa)y.$$

If the equilibrium has this linear form, then:

$$\begin{aligned}
\mathbb{E}_i[\overline{a}|x_i, y] &= \kappa\left(\frac{h_\eta y + h_\varepsilon x_i}{h_\eta + h_\varepsilon}\right) + (1-\kappa)y \\
&= \left(\frac{\kappa h_\varepsilon}{h_\eta + h_\varepsilon}\right) x_i + \left(1 - \frac{\kappa h_\varepsilon}{h_\eta + h_\varepsilon}\right) y.
\end{aligned}$$

Agent i's optimal action is:

$$\begin{aligned}
a_i(x_i, y) &= (1-r)\mathbb{E}_i[\theta] + r\mathbb{E}_i[\overline{a}] \\
&= \left(\frac{h_\varepsilon(r\kappa + 1 - r)}{h_\eta + h_\varepsilon}\right) x_i + \left(1 - \frac{h_\varepsilon(r\kappa + 1 - r)}{h_\eta + h_\varepsilon}\right) y,
\end{aligned}$$

meaning we have a linear equilibrium with:

$$\kappa = \frac{h_\varepsilon(1-r)}{h_\eta + h_\varepsilon(1-r)}.$$

In this equilibrium, agent i's action is:

$$a_i(x_i, y) = \frac{h_\eta y + h_\varepsilon(1-r)x_i}{h_\eta + h_\varepsilon(1-r)}.$$

Morris and Shin (2003) verify that this is the unique equilibrium in their model. You can consult their paper for what they describe as a "brute force" proof.

2.4 Discussion and Welfare Properties

The key point to notice about the equilibrium behavior in the beauty contest model is that agents actions *over-weight* public information relative to its informativeness about economic fundamentals. In particular, both $\mathbb{E}_i[\theta|x_i, y]$ and $a_i(x_i, y)$ are linear combinations of x_i and y. In forming her expectation of θ, agent i puts a weight $h_\eta/(h_\eta + h_\varepsilon)$ on y. But in choosing her action, agent i puts a weight $h_\eta/(h_\eta + (1-r)h_\varepsilon)$ on y. Why the larger weight? Because even if x_i and y were to be equally informative about θ ($h_\eta = h_\varepsilon$), the public signal y would be *more informative* about other player's beliefs, and hence about their actions. An early version of Morris and Shin's paper referred to this as the "publicity multiplier".

Because agent's are forecasting other agents' actions — and hence other agents' beliefs and beliefs about beliefs and so on — public information is given disproportionate weight relative to its true informativeness about fundamentals. This can give rise to surprising welfare effects.

In particular, suppose we re-write the equilibrium strategy of each agent i as:

$$a_i = \theta + \frac{h_\eta \eta + h_\varepsilon(1-r)\varepsilon_i}{h_\eta + h_\varepsilon(1-r)}.$$

Then expected welfare is given by:

$$\mathbb{E}[W|\theta] = -(1-r)\frac{h_\eta + h_\varepsilon(1-r)^2}{(h_\eta + h_\varepsilon(1-r))^2}.$$

An increase in h_ε, the informativeness of the private signals, has an unambiguously positive effect on social welfare. On the other hand, an increase in h_η, the informativeness of the public signal, has an effect:

$$\frac{\partial \mathbb{E}[W|\theta]}{\partial h_\eta} \overset{sign}{=} h_\eta - (2r-1)(1-r)h_\varepsilon.$$

Better public information is always beneficial is $r < 1/2$, so that the "beauty contest" incentive is relatively small. If the beauty contest component of payoffs is large, however, so that $r > 1/2$ and there is a significant element of coordination involved in the equilibrium, better public information improves welfare only if the public information is reasonably good relative to the quality of private information — if h_η is small relative to h_ε, an increase in h_η will cause the agents' too substitute toward y in choosing their actions and increase the variance in the population action around θ.

3 Modeling Beliefs and Higher Order Beliefs

In this section, we take a short detour to discuss the modeling of beliefs and higher order beliefs. A running theme in the coordination models we have looked at is the importance not only of the players' beliefs about underlying fundamentals (i.e. about the true payoffs), but also about the beliefs of others. As emphasized in the last model, information about the beliefs of others (so as to ascertain their actions) can easily be more important than beliefs about fundamentals.[2]

We have modeled beliefs, and beliefs about beliefs, and so on, as arising from a model where players have a common prior belief about some objective payoff uncertainty, and update based on a private signal about payoffs. A player's signal conveys information about fundamentals and about the signals of others. Indeed, the one signal determines a player's "first-order" beliefs about fundamentals, his second-order beliefs about the first-order beliefs of others, his third-order beliefs about the second-order beliefs others have about his first-order beliefs, and so on.

[2]Note that higher-order beliefs can also be relevant in *complete information* games, in the sense that player i has to form a belief about player j's action, j has to form a belief about i's belief about j's action, and so on. Nash equilibrium "cuts through" this infinite regress of beliefs by assuming the first order beliefs are correct and player are rational with respect to these correct beliefs. Rationalizability, on the other hand, makes use of higher order beliefs to eliminate strategies that only iteratively dominated. With incomplete information, there is a sense in which beliefs about beliefs must be tackled head on — because to the extent that j has private information about i's payoff, i cares *directly* about j's belief, as well as about j's action.

This formulation, while powerful, is also limiting in the sense that a player's beliefs about fundamentals uniquely tie down his higher-order beliefs. A natural question is whether it is possible to develop models with a greater diversity of higher-order uncertainty. It certainly seems possible to posit a model where we specify first the beliefs of the players about fundamentals, then their beliefs about each other's beliefs, and so on. Harsanyi (1968) argued that rather than write down an entire hierarchy of beliefs, it would be possible to capture even higher order uncertainty in a model similar to the ones with which you are familiar. That is, Harsanyi argued that one could consider a model where each player initially drew a random "type" that described *all* of his beliefs.

Mertens and Zamir (1985) and Brandenburger and Dekel (1993) made this idea precise. In doing so, they show that it is possible to define a "universal" type space that would allow for all possible hierarchies of belief. These papers are quite technical, particularly Mertens and Zamir's, but it is worth sketching the ideas. I'll follow Brandenburger and Dekel, glossing over virtually all of the technical issues.

The starting point is a space of possible fundamentals S (e.g. payoffs). Each player will have a first-order belief about fundamentals, that is some $t_1 \in T_1 = \Delta(S)$. Each player will then have a second-order belief over fundamentals and the other players first order belief, that is some $t_2 \in T_2 = \Delta(S \times \Delta(S))$. Formally, define the spaces:

$$
\begin{aligned}
X_0 &= S \\
X_1 &= X_0 \times \Delta(X_0) \\
&\vdots \\
X_n &= X_{n-1} \times \Delta(X_{n-1})
\end{aligned}
$$

The space of each player's nth order beliefs is then $T_n = \Delta(X_{n-1})$. A *type* for player i is a hierarchy of beliefs $t^i = (t_1^i, t_2^i, ...) \in \times_{n=1}^{\infty} T_n$. Let $T = \times_{n=1}^{\infty} T_n$ denote the space of all possible types for a given player.

Under this formulation, i knows his own type but not the type of his opponent j. So perhaps we will need to specify a belief for i about j's type, a belief for j about i's belief and so forth. Brandenburger and Dekel show that one additional assumption, however, pins down i's belief about j's type.

Definition 1 *A type $t = (t_1, t_2, ...) \in T$ is **coherent** if for every $n \geq 2$, $marg_{X_{n-2}} t_n = t_{n-1}$, where $marg_{X_{n-2}}$ denotes the marginal probability distribution on the space X_{n-2}.*

This simply says that i's beliefs do not contradict one another. This leads to the following result, for which you will need to know that a *homeomorphism* is a 1-1 map that is continuous and has a continuous inverse. For this result, let T' denote the set of all coherent types.

Proposition 3 *There is a homeomorphism $f : T' \to \Delta(S \times T)$.*

This result says that i's hierarchy of beliefs (his "type") also determines i's belief about j's type. Brandenburger and Dekel go on to show that if there is common knowledge of coherency (where i "knows" something if he assigns probability 1 to it), then i's type will determine not only his belief about j's type, but his belief about j's belief about his type, and so on. To state this formally, let $T'' \times T''$ denotes the set of types for which there is common knowledge of coherency.

Proposition 4 *There is a homeomorphism $g : T'' \to \Delta(S \times T'')$.*

So assuming common knowledge of coherency, it is possible to have a *universal type space* (the space $T'' \times T''$) for which each player's type will specifies *all* his beliefs. In this sense, it is possible to have a "closed" model of incomplete information.

As noted above, few applied economic models explicitly model higher order uncertainty. Rather, types are drawn from a small subset of the universal type sapce. There is, however, a recent literature that asks whether standard models that have only first-order uncertainty lead to predictions that are *robust* to perturbations of higher order beliefs (see, for instance, Rubinstein, 1989). Indeed, this is one way to view the global games analysis — as questioning the robustness of coordination equilibria that are not risk-dominant — even though the global games model we have considered retained the standard modeling approach.

Bergemann and Morris (2004) pursue the robustness line of inquiry in the context of mechanism design. They ask whether when a mechanism that implements some outcome in a standard model with first-order uncertainty will implement the same outcome if types are drawn from the universal type space constructed above. Their answer is that for a mechanism to be robust in this sense, it must implement the desired outcome as an *ex post* equilibrium, which in the case of private values is equivalent to *dominant strategy implementation.*

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Bubbles and Crashes

Jonathan Levin

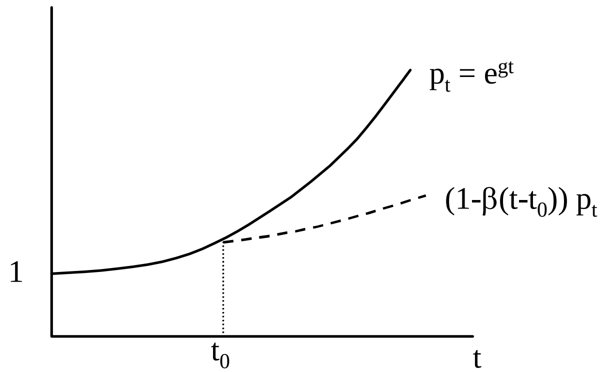
May 2006

These notes consider Abreu and Brunnermeier's (2003) paper on the failure of rational arbitrage in asset markets. Recall that the "no-trade" theorem states that speculative bubbles cannot exist in a world with only rational traders even if there is asymmetric information, so long as these traders share a common prior. Believers in the efficient market hypothesis argue that even if there are also behavioral or boundedly rational traders in the market, the presence of rational arbitrageurs will still push asset prices to fundamental values. Various models have been put forward as to why this might not be the case (e.g. Delong et al. 1990; Shleifer and Vishny, 1997). The idea of AB's paper is that even when rational arbitrageurs are aware of mispricing, a lack of common knowledge may prevent them from coordinating their attacks on a buble. As a result, persistent mispricing may occur even in the presence of rational traders.

1 Model

The market is composed of behavioral traders, whose behavior we will not directly model, and rational traders. The idea is that the behavioral traders cause a bubble in asset prices, which can be punctured only if there is sufficient selling pressure from the rational traders.

The price process for stocks works as follows. Starting from time $t = 0$, stock prices rise exponentially with $p_t = e^{gt}$. We assume that $g > r$, the risk-free interest rate. At the outset, these prices coincide with fundamental values. However at some random time t_0, stock prices and fundamental values diverge. At any time $t > t_0$, fundamental value is given by $(1 - \beta(t - t_0))p_t$, where $\beta : [0, \overline{\tau}] \to [0, \overline{\beta}]$ is increasing. Thus, after t_0, only a fraction $1 - \beta(\cdot)$ of the price is due to fundamentals and a fraction $\beta(\cdot)$ is due to the bubble. The time t_0 at which prices depart from fundamentals is distributed exponentially on $[0, \infty)$ so its cdf is $\Phi(t_0) = 1 - e^{-\lambda t_0}$.

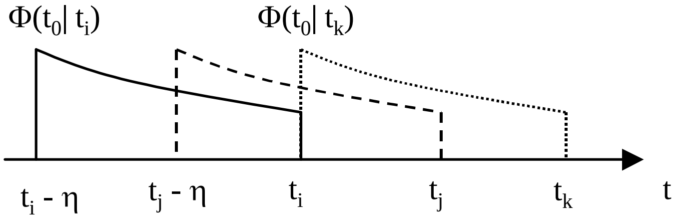


The Price Process

Stock prices are kept on the price path p_t by behavioral traders. To alter the price path, a significant number of rational traders must sell. Specifically, if a fraction κ of rational traders sell, prices drop to fundamentals. If any fraction less than κ sells, prices continue to grow at rate g. However, we assume that the bubble cannot grow to size greater than $\overline{\beta}$ — this occurs after $\beta(\overline{\tau})$ periods, so if at time $t_0 + \overline{\tau}$ the bubble has not burst, it will burst exogenously with prices returning to fundamentals.

With perfect information among rational traders as to when prices depart from fundamentals (i.e. time t_0 is common knowledge), it is clear that the bubble cannot grow at all. Since each rational trader would want to sell just before the bubble collapsed, this "pre-emption" motive will unravel their sales from the final date $t_0 + \overline{\tau}$ all the way back to t_0.

The key ingredient in the model is that rational traders do not all share the same information. Instead, once stock prices depart from fundamentals, rational arbitrageurs figure this out sequentially. Starting at t_0, a cohort of mass $1/\eta$ becomes aware at each moment. By time $t_0+\eta$, all arbitrageurs are aware that prices are above fundamental value. However, since t_0 is random, arbitrageurs do not know how many other arbitrageurs became aware of the bubble before them. In particular, if a trader wakes up at t, he learns only that $t_0 \in [t - \eta, t]$, and hence that all traders will become informed at some point between t and $t + \eta$. Traders' posterior beliefs on t_0 are shown in the next figure.



Posterior Beliefs

After time t_0, the price p_t exceeds fundamentals, but only a few traders realize this. After time $t_0 + \eta\kappa$, however, enough traders are aware of the mispricing to burst the bubble. Thus, AB say that there is a true "bubble" if mispricing exists beyond $t_0 + \eta\kappa$.

1.1 Strategies and Equilibrium

We can identify each rational trader with the time $t_i \in [t_0, t_0 + \eta]$ at which he becomes aware of the bubble. A strategy for trader t_i is a function $\sigma(\cdot, t_i) : [0, \infty) \to \{0, 1\}$, where $\sigma(t, 1) = 0$ means "Hold at t" and 1 means "Sell".

AB show, and we will assume, that each trader uses a simple "cut-off" strategy, so that:

$$\sigma(t, t_i) = \begin{cases} 0 & \text{for all } t < T(t_i) \\ 1 & \text{for all } t \geq T(t_t) \end{cases}.$$

Thus, the strategy for trader t_i is summarized by the time $T(t_i)$ at which he sells. We also follow AB in restricting attention to equilibria that satisfy a monotonicity property whereby traders who become aware of mispricing earlier also sell out earlier in equilibrium.

Definition 1 *A **trading equilibrium** $\{T(t_i)\}$ is a perfect Bayesian equilibrium with the property that if $t_i < t_j$, then $T(t_i) \leq T(t_j)$.*

1.2 Optimal Trading Strategies

Given selling times $\{T(t_i)\}$ satisfying the monotonicity property, a bubble that starts at time t_0 will burst at time:

$$T^*(t_0) = \min\{T(t_0 + \eta\kappa), t_0 + \overline{\tau}\}.$$

AB establish that in any trading equilibrium, the function $T^*(\cdot)$ is continuous and strictly increasing, and so is its inverse $T^{*-1}(\cdot)$.

Now, define $\Pi(t|t_i)$ to be trader t_i's belief that the bubble will burst by time t, given the strategies of the other traders. Then:

$$\Pi(t|t_i) \equiv \Phi\left(T^{*-1}(t)\right),$$

and let $\pi(t|t_i)$ denote the corresponding density.

Thus the expected payoff to t_i if he sells out at t is:

$$\int_{t_i}^{t} e^{-rs}(1 - \beta(s - T^{*-1}(s)))p(s)\pi(s|t_i)ds + e^{-rt}p(t)(1 - \Pi(t|t_i)).$$

Differentiating with respect to t yields the optimal sell-out time for t_i.

Lemma 1 *Given random bursting time $T^*(t_0)$, trader t_i optimally sells time t such that:*

$$h(t|t_i) = \frac{\pi(t|t_i)}{1 - \Pi(t|t_i)} = \frac{g - r}{\beta(t - T^{*-1}(t))}.$$

To see the intuition, consider the benefits of attacking the buble at time t rather than time $t + \Delta$. These benefits are equal to the probability the bubble will burst at t times the profits from selling before the burst:

$$h(t|t_i) \cdot p(t) \cdot \beta(t - T^{*-1}(t))$$

Note that the bubble size at t is $\beta(t - t_0)$ and if the bubble does burst at t, then $t_0 = T^{*-1}(t)$. On the other hand, the benefit of waiting a bit longer is $(g - r) \cdot p(t)$. Combining these terms yields the result.

1.3 Exogenous and Endogenous Crashes

We now have the optimal selling time for each trader in response to the random bursting time $T^*(t_0)$. Since the random bursting time is determined by these trading strategies, we are in position to characterize trading equilibria.

Suppose each trader believes the bubble will burst ξ units of time after the bubble begins, i.e. at time $t_0+\xi$. Each trader has a different belief about the burst time, because each has a different belief about the start time.

Let's consider i's belief about the start time. Let S be the random start date for the bubble, and S_i the random wake up time for agent i. Then S has a cdf $\Phi(t) = 1 - e^{-\lambda t}$, and if $S = t_0$, then S_i is uniformly distributed on $[t_0, t_0 + \eta]$. So

$$
\begin{aligned}
\Pr\left(S \leq t_0 | S_i = t_i\right) &= \frac{\Pr\left(S_i = t_i \text{ and } S \leq t_0\right)}{\Pr(S_i = t_i)} \\
&= \frac{\int_0^{t_0} \Pr(S_i = t_i | S = s) \Pr(S = s) ds}{\int_0^{\infty} \Pr(S_i = t_i | S = s) \Pr(S = s) ds}
\end{aligned}
$$

Now, note that

$$
\Pr(S_i = t_i | S = s) = \begin{cases} \frac{1}{\eta} & \text{if } s \in [t_i - \eta, t_i]. \\ 0 & \text{otherwise} \end{cases}
$$

and the density of S is $\lambda e^{-\lambda t}$ so loosely, $\Pr(S = s) = \lambda e^{-\lambda s}$. So we have

$$
\begin{aligned}
\Pr\left(S \leq t_0 | S_i = t_i\right) &= \frac{\int_{t_i-\eta}^{t_0} \frac{1}{\eta} \lambda e^{-\lambda s} ds}{\int_{t_i-\eta}^{t_i} \frac{1}{\eta} \lambda e^{-\lambda s} ds} \\
&= \frac{e^{-\lambda(t_i-\eta)} - e^{-\lambda t_0}}{e^{-\lambda(t_i-\eta)} - e^{-\lambda t_i}} \\
&= \frac{e^{\lambda\eta} - e^{-\lambda(t_i-t_0)}}{e^{\lambda\eta} - 1}.
\end{aligned}
$$

Therefore trader i believes the start time is randomly randomly distributed on $[t_i - \eta, t_i]$ with distribution

$$
\Phi(t_0 | t_i) = \frac{e^{\lambda\eta} - e^{\lambda(t_i-t_0)}}{e^{\lambda\eta} - 1}.
$$

Thus, she believes that the bursting date $t_i + \tau$ has distribution:

$$
\Pi(t_i + \tau | t_i) = \frac{e^{\lambda\eta} - e^{\lambda(\xi-\tau)}}{e^{\lambda\eta} - 1},
$$

and hazard rate:

$$
h(t_i + \tau | t_i) = \frac{\lambda}{1 - e^{-\lambda(\xi-\tau)}}.
$$

The optimal selling Lemma says that t_i should sell at some time $t_i + \tau$ such that:

$$h(t_i + \tau | t_i) = \frac{g - r}{\beta(t - T^{*-1}(t))} = \frac{g - r}{\beta(\xi)}.$$

Note that if $\xi = \overline{\tau}$, so the trader believes the bubble will burst exogenously, then $\beta(\xi) = \overline{\beta}$.

After re-arranging these two equalities involving the hazard rate, we find that if all traders expect the bubble to burst ξ periods after t_0, they will sell τ periods after becoming aware of the bubble, where:

$$\tau = \xi - \frac{1}{\lambda} \ln \left(\frac{g - r}{g - r - \lambda \beta(\xi)} \right). \tag{1}$$

Moreover, if all traders sell τ periods after becoming aware of the bubble, the bubble will burst at:

$$\xi = \min \left\{ \overline{\tau}, \eta\kappa + \tau \right\}. \tag{2}$$

A trading equilibrium is thus a pair (τ, ξ) satisfying these two equations. We thus two possibilities covering the cases where the bubble bursts exogenously and endogenously.

Proposition 1 *There is a unique trading equilibrium.*

1. *If* $\frac{\lambda}{1-e^{-\lambda\eta\kappa}} \leq \frac{(g-r)}{\overline{\beta}}$, *then each trader sells out* $\tau^* = \overline{\tau} - \frac{1}{\lambda} \ln \left(\frac{g-r}{g-r-\lambda\overline{\beta}} \right)$ *periods after becoming aware of the bubble, and for all* t_0, *the bubble bursts exogenously at time* $t_0 + \overline{\tau}$.
2. *If* $\frac{\lambda}{1-e^{-\lambda\eta\kappa}} > \frac{(g-r)}{\overline{\beta}}$, *then each trader* t_i *with* $t_i \geq \eta\kappa$ *sells out* $\tau^* = \beta^{-1}\left(\frac{g-r}{\lambda/(1-e^{-\lambda\eta\kappa})}\right) - \eta\kappa$ *periods after becoming aware of the bubble, and all traders* t_i *with* $t_i < \eta\kappa$ *sells out* $\tau^* + \eta\kappa$ *periods after becoming aware of the bubble. Thus the bubble bursts endogenously at time:*

$$t_0 + \xi^* = t_0 + \beta^{-1} \left(\frac{1 - e^{-\lambda\eta\kappa}}{\lambda} \cdot (g - r) \right) < t_0 + \overline{\tau}.$$

Thus, if traders' prior belief is that fundamentals will justify prices for a relatively long period of time, and the bubble will not grow too quickly, the bubble will last for the maximum length of time. On the other hand, if the bubble is expected to start quickly or grow rapidly, it will burst endogenously. However, even if it bursts endogeously, arbitrage trades are *delayed* by τ^* periods in equilibrium, so the bubble still grows significantly above fundamentals.

2 Comments

1. The uniqueness aspect of the equilibrium is similar to global games analysis, but there is a difference in that this game combines aspects of coordination and competition. In particular, traders want to attack just before other traders. Thus, with perfect information, a backward induction argument yields immediate attack, rather than multiple coordination equilibria.

2. Backward induction fails to bite because although the existence of the bubble eventually becomes mutual knowledge, it does not become common knowledge. To see this, note that at time $t_0 + \eta$, all traders are aware, so the bubble is mutual knowledge. However, not until time $t_0 + 2\eta$ is the mutual knowlege of the bubble itself mutual knowledge. The bubble is mth order mutual knowledge at time $t_0+m\eta$, but clearly never common knowledge.

3. An interesting point here is that due to the behavioral traders, rational traders actually benefit from their failure to coordinate and burst the bubble.

4. In the last section of their paper, AB show that public "news" can have a disproportionate effect on prices. The idea is that even though public news announcements may not reveal any new information about fundamentals, they can create common knowledge that allows traders to coordinate. In particular, traders may learn from the news not about fundamentals, but about what other traders know about fundamentals and thus about how they are likely to trade.

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Wars of Attrition

Jonathan Levin*

October 2004

These notes discuss "war of attrition" models. These models are used to study industry shakeouts in industrial organization, rent-seeking and lobbying in political economy. The first war of attrition models are actually from evolutionary biology, where they were used to study conflict. As you saw from the last problem set, war of attrition models are closely related to auction models, in that they feature several players competing for a prize or set of prizes by expending resources.

We'll start with a quick overview of a war of attrition between two players with known values. Then we'll talk about the case where they either have a known value or, with small probability, are a type that never quits. Then we'll talk about games with more than two players, all of whose values are known. Then two players whose values are drawn from a distribution. Then more than two players with values drawn from a distribution.

1 Two Players, Known Values

Suppose there are two players competing for a single object. Both players start out competing and can drop out at any time. The game ends when one of the players drops out. Assume player 1's value is known to be v_1 and player 2's value is known to be v_2. Each player incurs a cost of fighting equal to c per unit of time. A strategy for player i species a drop-out time t_i, or alternatively a distribution $G_i(\cdot)$ over drop out times. The former is a pure strategy; the latter is a mixed strategy.

One kind of equilibrium in this game is for one player to exit immediately, while the other never exits. These asymmetric pure strategy equilibria are efficient in the sense that no resources are "wasted" fighting for the prize.

*These notes were originally written by Jeremy Bulow, and revised by me in 2003 and 2004.

There is also a mixed strategy equilibrium in which both player fight for a positive expected time. In a mixed equilibrium, j must be just indifferent between dropping out at t and waiting to drop out at $t + dt$. This means that:

$$c \cdot dt = v_j \cdot \frac{g_i(t)}{1 - G_i(t)} \cdot dt = v_j \cdot \lambda_i(t) \cdot dt,$$

where $c \cdot dt$ is the flow cost to staying in, and $v_j \cdot \lambda_i(t) \cdot dt$ is the flow benefit, equal to the probability i drops out in the interval between t and $t+dt$ times the value to winning. Here $g_i(t)/(1-G_i(t)) = \lambda_i(t)$ is player i's "hazard rate" of quitting or her "exit rate." In equilibrium, player 1 exits with probability c/v_2 per unit time, while player 2 exists with probability c/v_1.

Proposition 1 *In the unique symmetric mixed strategy equilibrium, player i uses the strategy: $G_i(t) = 1 - e^{-\lambda_i t} = 1 - e^{-c/v_j}$.*

Note that the key in the mixed equilibrium is the ratio of benefit to cost. If player i's value of winning doubles, but so does her cost of fighting, she is no stronger or weaker (assuming risk-neutrality), so neither player's equilibrium mixed strategy will change.

Something is a bit odd about this mixed equilibrium. For example, if $c = 1$, $v_1 = 5$ and $v_2 = 10$ then the implication is that player 1 exits at half the rate of player 2. So player 1 wins the war with probability 2/3. This is strange because player 2 has higher value. So maybe a more realistic equilibrium is for player 1 to drop out immediately.

Are there other equilibria? For instance, can we have an equilibrium where the players fight but if time T is ever reached one player drops out for sure (for example because he will run out of resources)? No. If player i necessarily had to drop out at T, he would lose for sure in equilibrium, so he would do better to just drop out at the beginning.

2 Many Players, Known Values

Let's generalize to the case where there are N identical prizes and $N + K$ contestants in the war of attrition. Again all contestants start out competing, and each contestant can drop out at any time. We assume that contestants observe the exit behavior of their fellow contestants. The game ends as soon as only N contestants remain. Assume the contestants incur a common flow cost c to competing and have known values $v_1, ..., v_{N+K}$.

It is easy to see that there are lots of equilibria, both pure and mixed. One set of (pure) strategy equilibria is for all exit to take place immediately.

For example, with five players competing for two prizes, it can easily happen that three players exit immediately, with the other two not intending to exit for a long time.

It is also possible for there to be a mixed strategy equilibrium where some deterministic exit takes place at either the beginning or the end of the game. For instance, with five players and two prizes, it is possible that two players could exit for sure at the beginning and three could continue on fighting for the two prizes. We can also have more than one person exit at the end. If five players A, B, C, D and E are competing for two prizes, there could be an equilibrium where everyone starts off by staying in, but then coordinates on the two winners being B and C if A is the first to drop out, C and D if B is the first to drop out, and so on. In this equilibrium, all five would randomize at first; then after one person exited two others would follow immediately.

What we cannot have in equilibrium is for some player to exit for sure in the middle of the game. Why not? Consider someone who has a strategy that specifies mixing behavior contingent on who has already dropped out, and suppose the player's strategy involves immediate exit after some intermediate number of drop outs. For instance, in the five player, two prize case, imagine that all five start by mixing and A's strategy (but no one else's) specifies dropping out immediately after the first guy drops out. Then A will earn negative profits from this strategy, because she will stay in for a while and pay costs and then lose for sure. She would do better to drop out right at the beginning, so such a strategy cannot be part of the equilibrium.

3 Two players, Two Types

In the two player case, we observed that an odd phenomenon was that players who appeared strong (i.e. had high values) might have to drop out faster in the mixed strategy equilibrium, and might win less than half the time. A slight change in the model that pushes in the direction of getting the "sensible" equilibrium of the weaker player quitting at the beginning of the game is one in which there is a small possibility that each player has negative costs of continuing to play and so will never drop out.

Suppose that if j wins the game, he receives a payoff of W_j, while his payoff is L_j if he concedes first. As before, the game ends immediately once one player concedes. Therefore the "prize that j gets from winning is $p_j = W_j - L_j$. Let the "interest rate" for player j be r_j. Suppose that j also has cash out-of-pocket costs per unit of time of b_j until the game ends (i.e.

costs decline by that much after resolution of the war). We can thus write the total cost per unit time for j of delaying resolution as $c_j = r_j L_j + b_j$.

Define z_j as the probability that j is a type who "irrationally" will never exit. For convenience define $\lambda_j = c_j/p_j$ as the probability that a rational j must have of winning per unit of time to be indifferent to be exiting at different times along the path.

A strategy for j, $\hat{G}_j(\cdot)$ specifies the distribution of j's quitting time *conditional* on j being rational. Given $\hat{G}_j(\cdot)$, the probability that j exits by time t, assessed from i's perspective, will therefore be $G_j(\cdot) = (1 - z_j)\hat{G}_j(\cdot)$. We will look for a perfect bayesian equilibrium. To do this, it should be clear that we could in principle work with either $\hat{G}_j(\cdot)$ or $G_j(\cdot)$; we'll work with the latter. Our approach to solving for the equilibrium, as it was for auction models, will be to identify necessary conditions for equilibrium, then show that the behavior we identify is in fact an equilibrium.

First, a few basic facts. First, it cannot be the case for two times $t' > t$, we have G_j constant on $[t, t']$ and increasing above t'. The reason is that i would then strictly prefer to quit at $t+\varepsilon$ rather than in the interval $[t', t'+\varepsilon]$. But if i never exited in the interval $[t', t'+\varepsilon]$ it could not be optimal for j to exit in this interval rather than at t'. So therefore G_i and G_j must be strictly increasing until they equal $1 - z_i$ and $1 - z_j$.

A second point is that there cannot be a time $t > 0$ where i exits with discrete probability. If there was, j would strictly prefer to exit at $t+\varepsilon$ rather than at any time in some interval before t, which would contradict the claim above. A third point, is that although at time $t = 0$ one player could exit with discrete probability, both cannot do so in equilibrium, because then each would have an incentive to wait a second and win the game with positive probability at very low cost.

Third, because G_i and G_j must be strictly increasing up to $1 - z_i$ and $1 - z_j$, but cannot have atoms, the rational i and j types must be mixing after time zero. For "rational" j's to be indifferent between exiting and staying in, i must exit at a constant rate λ_j. This implies that at some point during the game, a time T_i will be reached at which point all rational i's will have exited. After T_i, i will only be in the game if she is a type who never quits.

Fourth, both sides must reach the point where they will never again exit at the same time — that is, it must be the case that $T_i = T_j$ in equilibrium. The reason is that once all the rational i types have exited, a rational j has no reason to stay in the game. He would do better to quit immediately.

This implies in equilibrium:

$$\frac{g_i(t)}{1 - G_i(t)} = \lambda_j,$$

and by integration:

$$G_i(t) = 1 - (1 - G_i(0))e^{-\lambda_j t}.$$

Moreover, there is some time $T = T_i = T_j$ where

$$G_i(T) = 1 - z_i \qquad \text{and} \qquad G_j(T) = 1 - z_j.$$

Beyond T, neither player ever exits.

What remains is to specify behavior at the beginning of the game, i.e. $G_i(0)$ and $G_j(0)$. There are three possibilities: either player 1 or player 2 drop out with some discrete probability at time 0, or neither player does so. As noted above, it cannot be that both players drop out with discrete probability at time 0.

To figure out which case we have, observe that:

$$G_i(T_i) = 1 - (1 - G_i(0))e^{-\lambda_j T_i} = 1 - z_i.$$

Solving out for T_i, we get

$$T_i = -\frac{1}{\lambda_j} \ln \frac{z_i}{1 - G_i(0)}.$$

We know that either $G_i(0) = 0$ or $G_j(0) = 0$ or both and also that $T_i = T_j$.

Therefore, we want to compare:

$$-\frac{1}{\lambda_j} \ln z_i \qquad \text{vs.} \qquad -\frac{1}{\lambda_i} \ln z_j$$

If these two expression are equal, we can have $G_i(0) = G_j(0) = 0$ and $T_i = T_j$. On the other hand, if the first expression is smaller, then we must have $G_i(0) = 0$ and $G_j(0) > 0$ chosen just large enough that $T_i = T_j$. Specifically, we have:

$$T_i = -\frac{1}{\lambda_j} \ln z_i = -\frac{1}{\lambda_i} \ln \frac{z_j}{1 - G_j(0)}.$$

So after some algebra we have

$$G_j(0) = 1 - z_j z_i^{-\lambda_i/\lambda_j}.$$

Of course, the reverse logic applies in the case where $G_j(0) = 0$ and $G_i(0) > 0$.

The preceding argument identifies a unique strategy profile that is potentially consistent with perfect bayesian equilibrium. It is not hard to show that the profile is in fact an equilibrium, giving the following result.

Proposition 2 *There is a unique perfect bayesian equilibrium in which, assuming* $\lambda_i \ln z_i \geq \lambda_j \ln z_j$*,*

$$\begin{aligned} G_i(t) &= 1 - e^{-\lambda_j t} \\ G_j(t) &= 1 - z_j z_i^{-\lambda_i/\lambda_j} e^{-\lambda_i t} \end{aligned}$$

Relative to the complete information war of attrition, this model has at least two nice features. First, the equilibrium is unique. Second, the model has nice comparative statics. To expand on the latter point, suppose for instance that everything is equal for the two players except that $W_i > W_j$. In this case: $p_i > p_j$, so $\lambda_i < \lambda_j$, so in equilibrium the "weaker" player j must drop out immediately with some positive probability. Similarly appealing comparative statics can be obtained with respect to z_i and z_j and the other parameters.

It is interesting to note that Abreu and Gul (2000) use this model to provide a war of attrition theory of bargaining. In their formulation, each player demands a share of a size one pie, so that W_i is i demand and $L_i = 1 - W_j$. They drop the time cost of bargaining b_i and have just the opportunity cost of not settling $r_i L_i$ balanced against the flow probability of the other player conceding. There's more to the paper and I recommend it to anyone interesting in bargaining. Shin Kambe, a former GSB student, also has a nice paper in this vein (Kambe, 1998), as do Abreu and Pearce (2002).

4 Two players, Many Types

We now consider a model with two symmetric players with values drawn i.i.d. from a distribution $F(v)$, with a minimum value of 0 and positive density everywhere up to the maximum value. As before, players can stop fighting at any time and the game ends immediately once there is only a single player left. Each player has costs of 1 per unit of time. The bounded value of winning and positive cost of fighting will be enough to rule out the "commitment to fighting" types we saw in the previous model.

We are interested in finding a symmetric equilibrium for this game. Suppose such an equilibrium exists, where both players follows the strategy of exiting at time $T(v)$. Clearly to have an equilibrium, we must have $T(0) = 0$ because a type zero player will never win in equilibrium and hence could not expend positive resources fighting. It is also not hard to show that $T(v)$ must be continuous and strictly increasing.

Now, for $T(v)$ to be an equilibrium, a type v player must be just indifferent to exiting at time $T(v)$ and exiting slightly earlier. Given that the opponent uses the strategy $T(v)$ the benefit from waiting the last dt up to $T(v)$ is equal to v times the probability the opponent will exit in this interval, which $(f(v)/(1 - F(v)) \cdot (1/T'(v)) \cdot dt$. The cost is dt.

Therefore a necessary condition for $T(v)$ to be a symmetric equilibrium is that

$$T'(v) = vf(v)/[1 - F(v)] = vh(v).$$

Here $h(v)$ is the hazard rate of the value distribution.

If we have $N+1$ players competing for N prizes (e.g. a group of penguins standing around until one jumps in the water to see if there are any sharks), each with a value of winning drawn i.i.d. from the distribution $F(\cdot)$, then the logic easily generalizes. A symmetric equilibrium $T(v)$ must satisfy $T(0) = 0$ and

$$T'(v) = Nvh(v).$$

An intuition for both these equations is that $T'(v)$ is the cost that must be paid to beat players of type v while the benefit for someone of type v (who must be indifferent to doing this in the symmetric equilibrium) is $vh(v)$ if there is one opponent and $Nvh(v)$ if there are N opponents, the exit of any one of whom would make everyone else a winner.

Starting with the differential equation above and integrating up, we obtain the equilibrium concession times:

$$T(v) = T(0) + \int_0^v T'(x)dx = \int_0^v xh(x)dx.$$

As in the case of the first price auction, this is a necessary condition for a symmetric equilibrium. It is not hard to verify that $T(v)$ is a best-response to $T(v)$.

Also, just as we were able to solve the first price auction via a differential equation and via the envelope theorem, we can also derive this equilibrium using revenue equivalence, or at least the payoff equivalence that we obtain using the envelope theorem.

How does this work? In a second price auction between two players with value distributed i.i.d. from F, a player with a value of v will have an expected payments of $\int_0^v xf(x)dx$. This must be equal her payment in the war of attrition. If the player wins the war of attrition he pays based on the exit time of the opponent; if he loses he pays based on his own exit time. So expected payments are $\int_0^v T(x)f(x)dx + T(v)[1-F(v)]$. Equating these two expected payments:

$$\int_0^v xf(x)dx = \int_0^v T(x)f(x)dx + T(v)[1-F(v)].$$

Differentiating both sides with respect to v reduces to

$$vf(v) = T(v)f(v) + T'(v)[1-F(v)] - T(v)f(v)$$

and then to

$$T'(v) = vf(v)/[1-F(v)] = vh(v)$$

as before.

5 Many Players, Prizes, and Types

Okay, we're now ready to take a shot at solving for the symmetric equilibrium of a war of attrition with lots of players competing for lots of prizes, given that there is asymmetric information about values.

Suppose there are $N+K$ players competing for N prizes. Suppose the prizes are identical, and each player i has a value of winning drawn iid from $F(\cdot)$. Suppose that F has support $[0, \overline{v}]$. Let's assume that each player has a flow cost 1 per unit of time while she is still competing. In contrast to the earlier models, let's also assume that if you drop out before the game ends, then in the interval between when you drop out and when the game ends you pay a cost c per unit of time. We'll call the model with $c > 0$ the "generalized" war of attrition.

Why change the game? Well, the tricky part here is going to be getting people to play symmetrically and drop out as the game goes along. Remember that in the N prize, $N+K$ player game all the exits would have to come at the beginning or the end. Here are a couple of interpretations of the c cost:

1. Tim Bresnahan insists that everyone stay at the faculty meeting and pay attention until at least five people have agreed to be on the committee to review the econometrics comprehensive exam. Even if you "drop out" and agree to be on the committee, you're still stuck as long as everyone else (so $c = 1$).

2. Tim Bresnahan insists that everyone stay until he's got his five volunteers, but once you "drop out" and agree to be on the committee, you can tune out and write referee reports while sitting there (so $0 < c < 1$).

3. Tim Bresnahan lets the volunteers leave immediately (so $c = 0$).

The last case is the exact extension of the models we've been looking at, as you can drop out immediately and still ensure a surplus of zero. This case turns out to be hard to solve (technically it won't have a symmetric PBE), but by solving the case for $c > 0$, we can take the limit of games as $c \to 0$ and basically recover what must happen when c is very, very small. One reason to be interested in the $c = 0$ case is that it will be revenue equivalent to a Vickrey auction where $N + K$ bidders compete for N prizes.

5.1 The Symmetric Equilibrium

We're going to look, as we said, for a symmetric PBE. To introduce a little notation, let $T(v, \underline{v}; k)$ denote the amount of time that type v will wait to drop out in a subgame where *(i)* there are $N + k$ firms left and *(ii)* the lowest remaining type is $\underline{v}$.

To solve the model and characterize the symmetric PBE, let's start with the subgame where $K - 1$ guys have dropped out and we have $N + 1$ guys competing for N prizes. In this case the c is irrelevant because any drop-out will end the game immediately. In fact, we saw above that:

$$T'(v; \underline{v}, 1) = Nvh(v),$$

where $h(v) = f(v)/(1 - F(v))$ is the hazard rate of F. Remember the intuition is that at each moment the marginal firm with type v faces the prospect of paying an extra $T'(v; \underline{v}, 1)$ to outlast any types between v and $v + dv$, and equates these costs with the probability $Nh(v)$ of being a winner times the value v of actually winning.

Of course, now we have to start with $\underline{v}$ dropping out immediately rather than type 0, so $T(\underline{v}; \underline{v}, 1) = 0$. Integrating up, we have:

Lemma 3 *The unique symmetric perfect bayesian equilibrium of the subgame in which just one more exit is required to end the game is defined by:*

$$T(v; \underline{v}, 1) = \int_{\underline{v}}^{v} Nxh(x)dx.$$

Let's now state the main result before we give the intuition.

Proposition 4 *The unique symmetric perfect bayesian equilibrium of the generalized war of attrition is defined by:*

$$T(v;\underline{v},k) = c^{k-1} \int_{\underline{v}}^{v} Nxh(x)dx.$$

Let's start by talking about case (1), where $c = 1$. In this case, the Proposition states that people won't condition their strategies on k at all! Instead they'll just have in mind some stopping time $T(v;0;K)$ and stick with it regardless of who drops out when beforehand.

How can this be? The key insight is to realize that if you are not among the final $N+1$ players there will come a time shortly before you exit where you know with probability 1 you will not win (because there will not be 2 or more other exits in the short time before it is your equilibrium turn to leave). Therefore once we get near v's drop-out time, her marginal decision about whether to stay in a bit longer or just drop out is not about whether she has a chance to win or lose but rather about whether she can shorten the game by dropping out more quickly. For v to be willing to exit at the "right" time it must be that dropping out slightly earlier won't shorten the game. But how can the game's length be independent of when you exit (so that if you are "supposed" to leave at some time and there are 3 more people who need to drop out you don't do better by just waiting for three more to leave and then leaving instantly, and you also cannot gain by leaving as soon as it becomes apparent that you will not be a winner even though it is still a bit before your turn to leave)?

The answer is, there can be an equilibrium in which everyone's strategy is independent of how many other people have dropped out. What must be the strategies in such a game? We know that once there are $N+1$ players competing for N prizes that each player will use the strategy $T'(v) = Nvh(v)$, and since their strategy is the same regardless of what others do, it must be that everyone plays according to that strategy from the very beginning of the game.

Now consider what happens in case (2), where costs drop from 1 to c per unit of time after you exit and until the end of the game. Again, when it becomes apparent that you will lose your real option is to lose on time, or a moment earlier or later. But doing so cannot effect your total costs, otherwise you would go in the cheaper direction. So it must be that the game slows down by a factor $1/c$ after each player leaves. We know that the strategies followed in the last round when there are $N+1$ players left fighting for N prizes (the prize being to stay off the committee in the example) are $T'(v) = Nvh(v)$, so the strategies in the next to last round must be $T'(v) = cNvh(v)$, while in the round before that the strategies must be $T'(v) = c^2Nvh(v)$ etcetera, or generally $T'(v) = c^{k-1}Nvh(v)$ where k is the number of excess players in the game at the time.

This means, a bit more formally, that:

$$T'(v; \underline{v}, k) = c^{k-1}Nvh(v),$$

and we have the initial conditions $T(\underline{v}; \underline{v}, k) = 0$, so combining this together gives the equilibrium.

Finally, what about the case where $c \to 0$. In this case, even the round before the last becomes very fast and the round before that faster still. So basically people exit really quickly but efficiently in a very short time until the last $N + 1$ players. These guys then fight it out.

5.2 An Example with Numbers

Let's try a numerical example. Suppose there are three guys competing for one prize and that $c = 1/2$. Values are independent and uniform on [0,1]. Let's try to figure out the total expected length and cost of the game in a couple of different ways.

First, note that in the first round everyone is using the strategy

$$T'(v) = \frac{1}{2}vh(v).$$

Note that with 3 guys competing for two prizes the strategies would be:

$$T'(v) = 2vh(v),$$

which would be exactly four times as slow.

In this "3 for 2" case, the c is irrelevant for the reasons we mentioned before (the game will end as soon as one guy drops out). So we're in revenue equivalence world — the "3 for 2" war of attrition is revenue equivalent to a "3 for 2" Vickrey auction. In such an auction, the two winners would expect to pay the value of the lowest guys, which is $1/4$. Therefore the total expected cost is $\frac{1}{4} \cdot 2 = \frac{1}{2}$.

Taking this observation to the war of attrition, note that the first round of the "3 for 1" generalized war of attrition goes four times as fast as the "3 for 2" war of attrition. Therefore, the expected cost in the first round must be $\frac{1}{8}$. The costs are paid by three players, so the expected length of the first stage is $\frac{1}{8}/3 = \frac{1}{24}$.

The second stage is easy because we have a "2 for 1" game. Strategies will be

$$T'(v) = vh(v).$$

Moreover, this subgame is revenue equivalent to a standard "2 for 1" Vickrey auction, so the winner must expect to pay the second highest value, i.e. the winner expects to pay 1/2. The expected costs are spread across two players, so the expected length of the subgame is therefore $\frac{1}{4}$. Of course, the third guy who dropped out in the first stage also has a cost from this subgame, equal to $\frac{1}{4}c = \frac{1}{8}$. Therefore total costs in the game are $\frac{1}{8}$ in the first stage and $\frac{1}{2} + \frac{1}{8} = \frac{5}{8}$ in the second stage for a total of $\frac{3}{4}$.

Note that if we just had a regular Vickrey auction with three people competing for one prize the expected cost would be $\frac{1}{2}$, the expected value of the second highest bidder. Where does the extra $\frac{1}{4}$ come from? The answer is that it comes from the positive $c > 0$ cost that must be incurred even by losers!

That is, in a Vickrey auction a bidder with value 0 would simply get zero surplus. In the generalized war of attrition, however, a guy with a value of 0 drops out right away, but still has expected costs of 1/12 (since the lower of the remaining two bidders would have a value averaging 1/3, implying a length of the last stage of 1/6 and a cost to the bottom bidder of 1/12 if $c = 1/2$. This makes the expected surplus of the bottom type $S(0) = -1/12$. If we did the expected surplus for each bidder in this game using the envelope condition $S(v) = S(0) + \int_0^v p(x)dx$ we would see that the expected surplus of each type is therefore 1/12 lower than in a second price auction. Adding up across three bidders, expected surplus is reduced by $3/12 = 1/4$ and total costs are that much higher than in the second price auction.

We can use revenue equivalence arguments of this sort more generally than in this numerical example. For instance, if there are $N + k$ bidders competing for $N + k - 1$ prizes everyone's strategy will be $T'(v) = Nvh(v)$ and the expected length of the game will be $(N + k - 1)E(v_{N+k})/(N + k)$ where v_{N+k} is the expected value of the $N + k^{th}$ highest (and therefore lowest) remaining bidder, times the number of winners. With that expected length expected payments per winner would be $E(v_{N+k})$,which must be so by the RET. In the war of attrition as we go from $N + k$ to $N + k - 1$ remaining players people use the strategy $T'(v) = c^{k-1}Nvh(v)$ so therefore the expected length of the stage must be $c^{k-1}(N + k - 1)E(v_{N+k})/(N + k)$. By plugging into this formula for $k = 1, 2, 3, ...K$ and summing we can figure out the expected length of any game and then, somewhat more tediously, figure out the expected costs.

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Auction Theory

Jonathan Levin

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Our next topic is auctions. Our objective will be to cover a few of the main ideas and highlights. Auction theory can be approached from different angles — from the perspective of game theory (auctions are bayesian games of incomplete information), contract or mechanism design theory (auctions are allocation mechanisms), market microstructure (auctions are models of price formation), as well as in the context of different applications (procurement, patent licensing, public finance, etc.). We're going to take a relatively game-theoretic approach, but some of this richness should be evident.

1 The Independent Private Value (IPV) Model

1.1 A Model

The basic auction environment consists of:

- Bidders $i = 1, ..., n$
- One object to be sold
- Bidder i observes a "signal" $S_i \sim F(\cdot)$, with typical realization $s_i \in [\underline{s}, \overline{s}]$, and assume F is continuous.
- Bidders' signals $S_1, ..., S_n$ are independent.
- Bidder i's value $v_i(s_i) = s_i$.

Given this basic set-up, specifying a set of auction rules will give rise to a game between the bidders. Before going on, observe two features of the model that turn out to be important. First, bidder i's information (her signal) is *independent* of bidder j's information. Second, bidder i's value is independent of bidder j's information — so bidder j's information is *private* in the sense that it doesn't affect anyone else's valuation.

1.2 Vickrey (Second-Price) Auction

In a Vickrey, or second price, auction, bidders are asked to submit sealed bids $b_1, ..., b_n$. The bidder who submits the highest bid is awarded the object, and pays the amount of the second highest bid.

Proposition 1 *In a second price auction, it is a weakly dominant strategy to bid one's value,* $b_i(s_i) = s_i$.

Proof. Suppose i's value is s_i, and she considers bidding $b_i > s_i$. Let $\hat{b}$ denote the highest bid of the other bidders $j \neq i$ (from i's perspective this is a random variable). There are three possible outcomes from i's perspective: (i) $\hat{b} > b_i, s_i$; (ii) $b_i > \hat{b} > s_i$; or (iii) $b_i, s_i > \hat{b}$. In the event of the first or third outcome, i would have done equally well to bid s_i rather than $b_i > s_i$. In (i) she won't win regardless, and in (ii) she will win, and will pay $\hat{b}$ regardless. However, in case (ii), i will win and pay more than her value if she bids $\hat{b}$, something that won't happen if she bids s_i. Thus, i does better to bid s_i than $b_i > s_i$. A similar argument shows that i also does better to bid s_i than to bid $b_i < s_i$. *Q.E.D.*

Since each bidder will bid their value, the seller's revenue (the amount paid in equilibrium) will be equal to the second highest value. Let $S^{i:n}$ denote the ith highest of n draws from distribution F (so $S^{i:n}$ is a random variable with typical realization $s^{i:n}$). Then the seller's expected revenue is $\mathbb{E}\left[S^{2:n}\right]$.

The truthful equilibrium described in Proposition 1 is the unique symmetric Bayesian Nash equilibrium of the second price auction. There are also asymmetric equilibria that involve players using weakly dominated strategies. One such equilibrium is for some player i to bid $b_i(s_i) = \overline{v}$ and all the other players to bid $b_j(s_j) = 0$.

While Vickrey auctions are not used very often in practice, open ascending (or English) auctions are used frequently. One way to model such auctions is to assume that the price rises continuously from zero and players each can push a button to "drop out" of the bidding. In an independent private values setting, the Nash equilibria of the English auction are the same as the Nash equilibria of the Vickrey auction. In particular, the unique symmetric equilibrium (or unique sequential equilibrium) of the English auction has each bidder drop out when the price reaches his value. In equilibrium, the auction ends when the bidder with the second-highest value drops out, so the winner pays an amount equal to the second highest value.

1.3 Sealed Bid (First-Price) Auction

In a sealed bid, or first price, auction, bidders submit sealed bids $b_1, ..., b_n$. The bidders who submits the highest bid is awarded the object, and pays his bid.

Under these rules, it should be clear that bidders will not want to bid their true values. By doing so, they would ensure a zero profit. By bidding somewhat below their values, they can potentially make a profit some of the time. We now consider two approaches to solving for symmetric equlibrium bidding strategies.

A. The "First Order Conditions" Approach

We will look for an equilibrium where each bidder uses a bid strategy that is a strictly increasing, continuous, and differentiable function of his value.[1] To do this, suppose that bidders $j \neq i$ use identical bidding strategies $b_j = b(s_j)$ with these properties and consider the problem facing bidder i.

Bidder i's expected payoff, as a function of his bid b_i and signal s_i is:

$$U(b_i, s_i) = (s_i - b_i) \cdot \Pr\left[b_j = b(S_j) \leq b_i,\ \forall j \neq i\right]$$

Thus, bidder i chooses b to solve:

$$\max_{b_i} (s_i - b_i)\, F^{n-1}\left(b^{-1}(b_i)\right).$$

The first order condition is:

$$(s_i - b_i)\,(n-1)F^{n-2}\left(b^{-1}(b_i)\right) f\left(b^{-1}(b_i)\right) \frac{1}{b'(b^{-1}(b_i))} - F^{n-1}\left(b^{-1}(b_i)\right) = 0$$

At a symmetric equilibrium, $b_i = b(s_i)$, so the first order condition reduces to a differential equation (here I'll drop the i subscript):

$$b'(s) = (s - b(s))\,(n-1)\,\frac{f(s)}{F(s)}.$$

This can be solved, using the boundary condition that $b(\underline{s}) = \underline{s}$, to obtain:

$$b(s) = s - \frac{\int_{\underline{s}}^{s_i} F^{n-1}(\tilde{s})d\tilde{s}}{F^{n-1}(s)}.$$

[1] In fact, it is possible to prove that in any symmetric equilibrium each bidder *must* use a continuous and strictly increasing strategy. To prove this, one shows that in equilbrium there cannot be a "gap" in the range of bids offered in equilibrium (because then it would be sub-optimal to offer the bid just above the gap) and there cannot be an "atom" in the equilibrium distribution of bids (because then no bidder would make an offer just below the atom, leading to a gap). I'll skip the details though.

It is easy to check that $b(s)$ is increasing and differentiable. So any symmetric equilibrium with these properties must involve bidders using the strategy $b(s)$.

B. The "Envelope Theorem" Approach

A closely related, and often convenient, approach to identify necessary conditions for a symmetric equilibrium is to exploit the envelope theorem.

To this end, suppose $b(s)$ is a symmetric equilibrium in increasing differentiable strategies. Then i's equilibrium payoff given signal s_i is

$$U(s_i) = (s_i - b(s_i))\, F^{n-1}(s_i). \tag{1}$$

Alternatively, because i is playing a best-response in equilibrium:

$$U(s_i) = \max_{b_i} (s_i - b_i)\, F^{n-1}(b^{-1}(b_i)).$$

Applying the envelope theorem (Milgrom and Segal, 2002), we have:

$$\left.\frac{d}{ds}U(s)\right|_{s=s_i} = F^{n-1}(b^{-1}(b(s_i)) = F^{n-1}(s_i)$$

and also,

$$U(s_i) = U(\underline{s}) + \int_{\underline{s}}^{s_i} F^{n-1}(\tilde{s})d\tilde{s}. \tag{2}$$

As $b(s)$ is increasing, a bidder with signal $\underline{s}$ will never win the auction — therefore, $U(\underline{s}) = 0$.

Combining (1) and (2), we solve for the equilibrium strategy (again dropping the i subscript):

$$b(s) = s - \frac{\int_{\underline{s}}^{s_i} F^{n-1}(\tilde{s})d\tilde{s}}{F^{n-1}(s)}.$$

Again, we have showed necessary conditions for an equilibrium (i.e. any increasing differentiable symmetric equilibrium must involve the strategy $b(s)$). To check sufficiency (that $b(s)$ actually is an equilibrium), we can exploit the fact that $b(s)$ is increasing and satisfies the envelope formula to show that it must be a selection from i's best response given the other bidder's use the strategy $b(s)$. (For details, see Milgrom 2004, Theorems 4.2 amd 4.6).

Remark 1 *In most auction models, both the first order conditions and the envelope approach can be used to characterize an equilibrium. The trick is to figure out which is more convenient.*

What is the revenue from the first price auction? It is the expected winning bid, or the expected bid of the bidder with the highest signal, $\mathbb{E}\left[b(S^{1:n})\right]$. To sharpen this, define $G(s) = F^{n-1}(s)$. Then G is the probability that if you take $n-1$ draws from F, all will be below s (i.e. it is the cdf of $S^{1:n-1}$). Then,

$$b(s) = s - \frac{\int_{\underline{s}}^{s} F^{n-1}(\tilde{s})d\tilde{s}}{F^{n-1}(s)} = \frac{1}{F^{n-1}(s)} \int_{\underline{s}}^{s} \tilde{s}dF^{n-1}(\tilde{s}) = \mathbb{E}\left[S^{1:n-1}|S^{1:n-1} \leq s\right].$$

That is, if a bidder has signal s, he sets his bid equal to the expectation of the highest of the other $n-1$ values, conditional on all those values being less than his own.

Using this fact, the expected revenue is:

$$\mathbb{E}\left[b\left(S^{1:n}\right)\right] = \mathbb{E}\left[S^{1:n-1}|S^{1:n-1} \leq S^{1:n}\right] = \mathbb{E}\left[S^{2:n}\right],$$

equal to the expectation of the second highest value. We have shown:

Proposition 2 *The first and second price auction yield the same revenue in expectation.*

1.4 Revenue Equivalence

The result above is a special case of the celebrated "revenue equivalence theorem" due to Vickrey (1961), Myerson (1981), Riley and Samuelson (1981) and Harris and Raviv (1981).

Theorem 1 *(Revenue Equivalence) Suppose n bidders have values $s_1, ..., s_n$ identically and independently distributed with cdf $F(\cdot)$. Then all auction mechanisms that (i) always award the object to the bidder with highest value in equilibrium, and (ii) give a bidder with valuation $\underline{s}$ zero profits, generates the same revenue in expectation.*

Proof. We consider the general class of auctions where bidders submit bids $b_1, ..., b_n$. An auction rule specifies for all i,

$$\begin{aligned} x_i &: B_1 \times ... \times B_n \to [0,1] \\ t_i &: B_1 \times ... \times B_n \to \mathbb{R}, \end{aligned}$$

where $x_i(\cdot)$ gives the probability i will get the object and $t_i(\cdot)$ gives i's required payment as a function of the bids $(b_1, ..., b_n)$.[2]

Given the auction rule, bidder i's expected payoff as a function of his signal and bid is:

$$U_i(s_i, b_i) = s_i \mathbb{E}_{b_{-i}} \left[x_i(b_i, b_{-i}) \right] - \mathbb{E}_{b_{-i}} \left[t_i(b_i, b_{-i}) \right].$$

Let $b_i(\cdot), b_{-i}(\cdot)$ denote an equilibrium of the auction game. Bidder i's *equilibrium* payoff is:

$$U_i(s_i) = U_i(s_i, b(s_i)) = s_i F^{n-1}(s_i) - \mathbb{E}_{s_{-i}} \left[t_i(b_i(s_i), b_{-i}(s_{-i})) \right],$$

where we use *(i)* to write $\mathbb{E}_{s_{-i}} \left[x_i(b(s_i), b(s_{-i})) \right] = F^{n-1}(s_i)$.

Using the fact that $b(s_i)$ must maximize i's payoff given s_i and opponent strategies $b_{-i}(\cdot)$, the envelope theorem implies that:

$$\left. \frac{d}{ds} U_i(s) \right|_{s=s_i} = \mathbb{E}_{b_{-i}} \left[x_i(b_i(s_i), b_{-i}(s_{-i})) \right] = F^{n-1}(s_i),$$

and also

$$U_i(s_i) = U_i(\underline{s}) + \int_{\underline{s}}^{s_i} F^{n-1}(\tilde{s}) d\tilde{s} = \int_{\underline{s}}^{s_i} F^{n-1}(\tilde{s}) d\tilde{s},$$

where we use *(ii)* to write $U_i(\underline{s}) = 0$.

Combining our expressions for $U_i(s_i)$, we get bidder i's expected payment given his signal:

$$\mathbb{E}_{s_{-i}} \left[t_i(b_i, b_{-i}) \right] = s_i F^{n-1}(s_i) - \int_{\underline{s}}^{s_i} F^{n-1}(\tilde{s}) d\tilde{s} = \int_{\underline{s}}^{s_i} \tilde{s} dF^{n-1}(\tilde{s}),$$

where the last equality is from integration by parts. Since $x_i(\cdot)$ does not enter into this expression, bidder i's expected equilibrium payment given his signal is the *same* under all auction rules that satisfy *(i)* and *(ii)*. Indeed, i's expected payment given s_i is equal to:

$$\mathbb{E} \left[S^{1:n-1} \mid S^{1:n-1} < s_i \right] = \mathbb{E} \left[S^{2:n} \mid S^{1:n} = s_i \right].$$

So the seller's revenue is:

$$\mathbb{E} \left[\text{Revenue} \right] = n \mathbb{E}_{s_i} \left[i\text{'s expected payment} \mid s_i \right] = \mathbb{E} \left[S^{2:n} \right],$$

[2] So in a first price auction, $x_1(b_1, ..., b_n)$ equals 1 if b_1 is the highest bid, and otherwise zero. Meanwhile $t_1(b_1, ..., b_n)$ equals zero unless b_1 is highest, in which case $t_1 = b_1$. In a second price auction, $x_1(\cdot)$ is the same, and $t_1(\cdot)$ is zero unless b_1 is highest, in which case t_1 equals the highest of $(b_2, ..., b_n)$.

a constant. *Q.E.D.*

The revenue equivalence theorem has many applications. One useful trick is that it allows us to solve for the equilibrium of different auctions, so long as we know that the auction will satisfiy *(i)* and *(ii)*. Here's an example.

Application: The All-Pay Auction. Consider the same set-up — bidders $1, .., n$, with values $s_1, ..., s_n$, identically and independently distributed with cdf F — and consider the following rules. Bidders submit bids $b_1, ..., b_n$ and the bidder who submits the highest bid gets the object. However, bidders must pay their bid regardless of whether they win the auction. (These rules might seem a little strange — the all-pay auction is commonly used as a model of lobbying or political influence).

Suppose this auction has a symmetric equilibrium with an increasing strategy $b^A(s)$ used by all players. Then, bidder i's expected payoff given value s_i will be (if everyone plays the equilibrium strategies):

$$U(s_i) = s_i F^{n-1}(s_i) - b^A(s_i) = \int_{\underline{s}}^{s_i} F^{n-1}(\tilde{s}) d\tilde{s}$$

So

$$b^A(s) = sF^{n-1}(s) - \int_{\underline{s}}^{s} F^{n-1}(\tilde{s}) d\tilde{s}$$

In addition to the all-pay auction, many other auction rules also satisfy the revenue equivalence assumptions when bidder values are independently and identically distributed. Two examples are:

1. The English (oral ascending) auction. All bidders start in the auction with a price of zero. The price rises continuously, and bidders may drop out at any point in time. Once they drop out, they cannot re-enter. The auction ends when only one bidder is left, and this bidder pays the price at which the second-to-last bidder dropped out.

2. The Dutch (descending price) auction. The price starts at a very high level and drops continuously. At any point in time, a bidder can stop the auction, and pay the current price. Then the auction ends.

2 Common Value Auctions

We would now like to generalize the model to allow for the possibility that *(i)* learning bidder j's information could cause bidder i to re-assess his estimate of how much he values the object, and *(ii)* the information of i and j is not independent (when j's estimate is high, i's is also likely to be high).

These features are natural to incorporate in many situations. For instance, consider an auction for a natural resource like a tract of timber. In such a setting, bidders are likely to have different costs of harvesting or processing the timber. These costs may be independent across bidders and private, much like in the above model. But at the same time, bidders are likely to be unsure exactly how much merchanteable timber is on the tract, and use some sort of statistical sampling to estimate the quantity. Because these estimates will be based on limited sampling, they will be imperfect — so if i learned that j had sampled a different area and got a low estimate, she would likely revise her opinion of the tract's value. In addition, if the areas sampled overlap, the estimates are unlikely to be independent.

2.1 A General Model

- Bidders $i = 1, ..., n$
- Signals $S_1, ..., S_n$ with joint density $f(\cdot)$
- Signals are *(i)* exchangeable, and *(ii)* affiliated.
  - Signals are *exchangeable* if s' is a permutation of $s \Rightarrow f(s) = f(s')$.
  - Signals are *affiliated* if $f(s \wedge s')f(s \vee s') \geq f(s')f(s)$ (i.e. $s_j|s_i$ has monotone likelihood ratio property).
- Value to bidder i is $v(s_i, s_{-i})$.

Example 1 *The independent private value model above is a special case: just let $v(s_i, s_{-i}) = s_i$, and suppose that $S_1, ..., S_n$ are independent.*

Example 2 *Another common special case is the pure common value model with conditionally independent signals. In this model, all bidders have the same value, given by some random variable V. The signals $S_1, .., S_n$ are each correlated with V, but independent conditional on it (so for instance, $S_i = V + \varepsilon_i$, where $\varepsilon_1, .., \varepsilon_n$ are independent. Then $v(s_i, s_{-i}) = \mathbb{E}[V|s_1, ..., s_n]$.*

Example 3 *A commonly used, but somewhat hard to motivate, variant of the general model is obtained by letting $v_i(s_i, s_{-i}) = s_i + \beta \sum_{j \neq i} s_j$, with $\beta \leq 1$, and assuming that $S_1, ..., S_n$ are independent. This model has the feature that bidder's have independent information (so a version of the RET applies), but interdependent valuations (so there are winner's curse effects).*

A new feature of the more general auction environment is that each bidder i will want to account for the fact that her opponents' bids reveal something about their signals, information that is relevant for i's own valuation. In particular, if i wins, then the mere fact of winning reveals that her opponent's values were not that high — hence winning is "bad news" about i's valuation. This feature is called the winner's curse.

2.2 Second Price Auction

Let's look for a symmetric increasing equilibrium bid strategy $b(s)$ in this more general environment. We start by considering the bidding problem facing i:

- Let s^i denote the highest signal of bidders $j \neq i$.
- Bidder i will win if she bids $b_i \geq b(s^i)$, in which case she pays $b(s^i)$

Bidder i's problem is then:

$$\max_{b_i} \int_{\underline{s}}^{\overline{s}} \left[\mathbb{E}_{S_{-i}} \left[v(s_i, S_{-i}) \mid s_i, S^i = s^i \right] - b(s^i) \right] \mathbf{1}_{\{b(s^i) \leq b_i\}} f(s^i | s_i) ds^i$$

or

$$\max_{b_i} \int_{\underline{s}}^{b^{-1}(b_i)} \left[\mathbb{E}_{S_{-i}} \left[v(s_i, S_{-i}) \mid s_i, S^i = s^i \right] - b(s^i) \right] dF(s^i | s_i)$$

The first order condition for this problem is:

$$0 = -\frac{1}{b'(b^{-1}(b_i))} \left[\mathbb{E}_{S_{-i}} \left[v(s_i, S_{-i}) \mid s_i, S^i = b^{-1}(b_i) \right] - b(b^{-1}(b_i)) \right] f(b^{-1}(b_i) | s_i)$$

or simplifying:

$$b_i = b(s_i) = \mathbb{E}_{S_{-i}} \left[v(s_i, S_{-i}) \mid s_i, \max_{j \neq i} s_j = s_i \right]$$

That is, in equilibrium, bidder i will bid her expected value conditional on her own signal *and* conditional on all other bidders having a signal less than hers (with the best of the rest equal to hers).

Remark 2 *While we will not purse it here, in this more general environment the revenue equivalence theorem fails. Milgrom and Weber (1982) prove a very general result called the "linkage principle" which basically states that in this general symmetric setting, the more information on which the winner's payment is based, the higher will be the expected revenue. Thus, the first price auction will have lower expected revenue than the second price auction because the winner's payment in the first price auction is based only on her own signal, while in the second price auction it is based on her own signal and the second-highest signal.*

3 Large Auctions & Information Aggregation

An interesting question that has been studied in the auction literature arises if we think about auction models as a story about how prices are determined in Walrasian markets. In this context, we might then ask to what extent prices will aggregate the information of market participants.

We now consider a series of results along these lines. For each result, the basic set-up is the same. There are a lot of bidders, and the auction has pure common values with conditionally independent signals. We consider second price auctions (or more generally, with k objects, $k+1$ price auctions, where all winning bidders pay the $k+1$st highest bid). The basic question is: as the number of bidders gets large, will the auction price converge to the true value of the object(s) for sale?

3.1 Wilson (1977, RES)

Wilson considers information aggregation in a setting with a special information structure. In his setting, bidders learn a lower bound on the object's value. The model has:

- Bidder's $1, ..., n$
- A single object with common value $V \sim U[0, 1]$
- Bidder's signals $S_1, ..., S_n$ are iid with $S_i \sim U[0, v]$ (so signal $s_i \Rightarrow V \geq s_i$).

Wilson shows that as $n \to \infty$, the expected price converges to v. This is easy to see: with lots of bidders, someone will have a signal close to v, and you have to bid very close to your signal to have any chance of winning.

3.2 Milgrom (1979, EMA)

Milgrom goes on to identify a necessary and sufficient condition for information aggregation with many bidders and a fixed number of objects. Milgrom's basic requirement is, in the limit as $n \to \infty$, that for all $v' \in \Omega$ and all $v < v'$, and $M > 0$, there exists some $s' \in S$ such that

$$\frac{P(s'|v')}{P(s'|v)} > M.$$

That is, for any possible value v', there must be arbitrarily strong signals that effectively rule out any value $v < v'$.

This condition builds on Wilson, and the intuition is again quite easy. As $n \to \infty$, there is a very strong winner's curse. In a second price auction, the high bid is:

$$\mathbb{E}_V \left[V \mid s^{1:n}, \max_{j \neq i} S_j = s^{1:n} \right].$$

While it is clear that $s^{1:n}$ is likely to be very high as $n \to \infty$, the only way anyone would ever bid $\overline{v}$ would be to have a signal so strong that conditioning on millions of other signals being lower than your own would not push down your estimate too much.

3.3 Pesendorfer and Swinkels (1997, EMA)

Pesendorfer and Swinkels consider a model that is quite similar to Milgrom.

- n bidders, k objects
- Each bidder has value $V \sim F(\cdot)$ on $[0, 1]$
- Signals $S_i \sim G(\cdot|v)$ on $[0, 1]$, where $g(\cdot|\cdot)$ has the MLRP.
- Signals $S_1, ..., S_n$ are independent conditional on $V = v$.

Remark 3 *Clearly with a large number of independent signals, there is enough information to consistently estimate v. The question then is whether the bids will accurately aggregate this information — i.e. will the price be a consistent estimator?*

Pesendorfer and Swinkel's big idea is the following. While the winner's curse will make a bidder with $s = 1$ shade her bid as $n \to \infty$ with a fixed number of objects k, as $k \to \infty$, there is also a *loser's curse*: if lowering your bid ε matters there must be k people with *higher valuations*. This operates against the winner's curse, counterbalancing it.

Proposition 3 *The unique equilibrium of* $k+1^{st}$ *price auction is symmetric:*

$$b(s_i) = \mathbb{E}_V \left[V \mid s_i, kth \text{ highest of other signals is also } s_i \right].$$

Definition 1 *A sequence of auctions* (n_m, k_m) *has information aggregation if the price converges in probability to* v *as* $m \to \infty$*.*

PS first consider *necessary* conditions for information aggregation. Clearly, a requirement for information aggregation is that:

$$\lim_{m\to\infty} b_m(0) = 0 \qquad \text{and} \qquad \lim_{m\to\infty} b_m(1) = 1$$

Without this, there will be no convergence when $v = 0$ or when $v = 1$. Now,

$$b_m(1) = \mathbb{E}_V \left[V \mid s_i = 1, k\text{th highest of other signals is } 1 \right]$$

If $k_m \nrightarrow \infty$, then this expectation will shrink below 1. So we must have $k_m \to \infty$. And similarly, looking at $b_m(0)$, we must have $n_m - k_m \to \infty$.

PS go on to show that these conditions are not just necessary, but sufficient for information aggregation.

Proposition 4 *A sequence of auctions has information aggregation if and only if* $n_m - k_m \to \infty$ *and* $k_m \to \infty$*.*

The basic idea is that conditioning on $k_m - 1$ bidders having higher signals than s_i gives a very strong signal about true value. So the bidder who actually *has* the kth highest sighnal will tend to be right on when he bids $\mathbb{E}_V [V \mid s_i, k\text{th highest of other signals is } s_i]$. Thus, we will end up with information aggregation.

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Repeated Games I: Perfect Monitoring

Jonathan Levin

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A crucial feature of many strategic situations is that players interact repeatedly over time, not just once. For instance, American Airlines and United compete for business every day, bosses try to motivate workers on an ongoing basis, suppliers and buyers make deals repeatedly, nations engage in ongoing trade, and so on. The repeated game model is perhaps the simplest model that captures this notion of ongoing interaction. Of course, in all these examples, there is a strong argument to be made that the game itself changes over time. The basic repeated game model abstracts from this issue, and focuses just on the effect of repetition.

1 Some Examples

1.1 Example 1: Cooperation in the Prisoners' Dilemma

Probably the best-known repeated game argument is that ongoing interaction can explain why people might behave cooperatively when it is against their self-interest in the short run. The classic example is the repeated prisoners' dilemma.

	C	D
C	$1,1$	$-1,2$
D	$2,-1$	$0,0$

The unique Nash equilibrium if the game is played once is (D,D).

Suppose that players 1 and 2 play the game repeatedly at time $t = 0,1,2...$ and that i's payoff for the entire repeated game is:

$$u_i\left(\{a^1,a^2,....\}\right) = (1-\delta)\sum_{t=0}^{\infty}\delta^t g_i(a_i^t,a_{-i}^t)$$

where $\delta \in [0,1)$. The fact that $\delta < 1$ means that players discount the future — a dollar tomorrow is less than a dollar today. The overall payoffs are multiplied by $(1-\delta)$ to get a per-period average payoff for the game (note that this makes the repeated game payoff comparable to the stage game payoffs).

Proposition 1 *If $\delta \geq \frac{1}{2}$, the repeated prisoners' dilemma game has a subgame perfect equilibrium in which (C, C) is played in every period.*

Proof. Suppose the players use "grim trigger" strategies:

I. Play C in every period unless someone plays D, in which go to II.

II. Play D forever.

To check that these strategies form a subgame perfect equilibrium if $\delta \geq \frac{1}{2}$, we need to verify that there is no single period where i can make a profitable deviation (this is the "one-stage deviation principle" — for details, see Fudenberg and Tirole, p. 108–110).

Suppose up to time t, D has never been played. Then i's payoffs looking forward are:

$$\begin{aligned} \text{Play } C &\Rightarrow (1-\delta)\left[1+\delta+\delta^2+...\right]=1 \\ \text{Play } D &\Rightarrow (1-\delta)\left[2+\delta\cdot 0+\delta^2\cdot 0+...\right]=(1-\delta)2 \end{aligned}$$

so if $\delta \geq \frac{1}{2}$, C is optimal.

Suppose that at time t, D has already been played. Then j will play D and no matter what will continue to play D, so i's payoffs are:

$$\begin{aligned} \text{Play } C &\Rightarrow (1-\delta)\left[-1+\delta\cdot 0+\delta^2\cdot 0+...\right]=(1-\delta)(-1) \\ \text{Play } D &\Rightarrow (1-\delta)\left[0+\delta\cdot 0+\delta^2\cdot 0+...\right]=0 \end{aligned}$$

So D is definitely optimal. *Q.E.D.*

1.2 Example 2: Use of Non-Nash Reversion

A second example shows that repeated play can lead to *worse* outcomes than in the one-shot game.

	A	B	C
A	$2,2$	$2,1$	$0,0$
B	$1,2$	$1,1$	$-1,0$
C	$0,0$	$0,-1$	$-1,-1$

In this game, A is strictly dominant, and the unique Nash Equilibrium is (A, A).

Proposition 2 *If $\delta \geq \frac{1}{2}$, this game has a subgame perfect equilibrium in which (B, B) is played in each period.*

Proof. Here, we construct slightly more complicated strategies than grim trigger.

I. Play B in every period unless someone deviates, in which case go to II.

II. Play C. If no one deviates, go to I. If someone deviates, stay in II.

These strategies have what Abreu (1988) calls a "stick" (threatening to play C if someone deviates from B) *and* a "carrot" (promising to go back to B if everyone carries out the C punishment). Let's check that this is a SPE.

Suppose no one deviated at $t-1$, so players should play B at time t (i.e. they're in phase I):

$$\begin{aligned} \text{Play } B \quad &\Rightarrow \quad (1-\delta)\left[1+\delta+\delta^2+\delta^3+...\right] = 1 \\ \text{Best Dev. } (A) \quad &\Rightarrow \quad (1-\delta)\left[2+\delta(-1)+\delta^2+\delta^3+...\right] = 1+(1-\delta)(1-2\delta) \end{aligned}$$

so it's optimal to play B if $\delta \geq \frac{1}{2}$.

Suppose someone deviated at $t-1$, so players should play C at time t (i.e. they're in phase II)

$$\begin{aligned} \text{Play } C \quad &\Rightarrow \quad (1-\delta)\left[-1+\delta+\delta^2+\delta^3+...\right] = 1-(1-\delta)(2) \\ \text{Best Dev. } (A) \quad &\Rightarrow \quad (1-\delta)\left[0+\delta(-1)+\delta^2+\delta^3+...\right] = 1-(1-\delta)(1+2\delta) \end{aligned}$$

so it's optimal to play C if $\delta \geq \frac{1}{2}$. *Q.E.D.*

2 A General Model

- Let G be a normal form game with action spaces $A_1, ..., A_I$, payoff functions $g_i : A \to \mathbb{R}$, where $A = A_1 \times ... \times A_I$.
- Let $G^\infty(\delta)$ be the infinitely repeated version of G played at $t = 0, 1, 2, ...$ where players discount at δ and observe all previous actions.
- A *history* is $H^t = \{(a_1^0, ..., a_I^0), ..., (a_1^{t-1}, ..., a_I^{t-1})\}$.
- A *strategy* is $s_{it} : H^t \to A_i$.

- Average payoffs for i are:

$$u_i(s_i, s_{-i}) = (1-\delta)\sum_{t=0}^{\infty} \delta^t g_i(a_i, a_{-i}).$$

We now investigate what average payoffs could result from different equilibria when δ is near 1. That is, what can happen in equilibrium when players are very patient?

Fact 1 (Feasibility) If $(v_1, ..., v_I)$ are average payoffs in a Nash equilibrium, then

$$(v_1, ..., v_I) \in Conv\left\{(x_1, ..., x_I) : \exists (a_1, ..., a_I) \text{ with } g_i(a) = x_i \text{ for all } i\right\}$$

Definition 1 *Player i's min-max payoff is*

$$\underline{v}_i = \min_{\sigma_{-i}} \max_{\sigma_i} g_i(\sigma_i, \sigma_{-i})$$

Fact 2 (Individual Rationality) In any Nash equilibrium, player i must receive at least $\underline{v}_i$.

Proof. Suppose (σ_i, σ_{-i}) is a Nash equilibrium. Then let σ_i' be the strategy of playing a static best-response to σ_{-i} in each period. Then (σ_i', σ_{-i}) will give i a payoff of at least $\underline{v}_i$, and playing σ_i must give at least this much. *Q.E.D.*

3 The Folk Theorem

The first result is the (Nash) folk theorem which states that any feasible and strictly individually rational payoff vector can be achieved as a *Nash equilibrium* of the repeated game, provided players are sufficienly patient.

Theorem 1 *(Nash Folk Theorem) If* $(v_1, ..., v_I)$ *is feasible and strictly individually rational, then there exists* $\underline{\delta} < 1$ *such that for all* $\delta > \underline{\delta}$*, there is a Nash Equilibrium of* $G^\infty(\delta)$ *with average payoffs* $(v_1, ..., v_I)$.

Proof. Assume there exists a profile $a = (a_1, ..., a_I)$ such that $g_i(a) = v_i$ for all i (I'll comment on this assumption later.) Let m_{-i}^i denote the strategy profile of players other than i that holds i to at most $\underline{v}_i$ and write m_i^i for i's best-response to m_{-i}^i.

Now consider the following strategies:

I. Play $(a_1, ..., a_I)$ as long as no one deviates.

II. If some player j deviates, play m_i^j thereafter.

If i plays this strategy, he gets v_i. If he deviates in some period t, then if $\overline{v}_i = \sup_a g_i(a)$, the *most* that i could get is:

$$(1-\delta)\left[v_i + \delta v_i + ... + \delta^{t-1} v_i + \delta^t \overline{v}_i + \delta^{t+1} \underline{v}_i + \delta^{t+2} \underline{v}_i + ...\right]$$

Following the suggested strategy will be optimal if:

$$\frac{\delta}{1-\delta}\left(v_i - \underline{v}_i\right) \geq \left(\overline{v}_i - v_i\right)$$

As $\delta \to 1$, the ratio $\frac{\delta}{1-\delta} \to \infty$, so simply pick $\underline{\delta} = \max_i \left(\overline{v}_i - v_i\right)/(\overline{v}_i - \underline{v}_i)$. *Q.E.D.*

This Nash folk theorem says that essentially anything goes as a Nash equilibrium when players are sufficiently patient. Of course, we should be a little bit cautious about using Nash equilibrium as our solution concept since it might specify punishment behavior that is implausible. For example, consider the game

	L	R
U	$6,6$	$0,-100$
D	$7,1$	$0,-100$

.

The Folk Theorem says that $(6,6)$ is possible as a Nash equilibrium payoff, but the strategies suggested in the proof require the column player to play R in every period following a deviation. While this will hurt Row, it will hurt Column a lot — it seems unreasonable to expect her to carry ought the threat.

What we'd like to do is get $(6,6)$, or more generally, the whole set of feasible and individually rational payoff vectors as subgame perfect equilibrium payoffs. The Fudenberg and Maskin (1986) folk theorem says that this possible.

Theorem 2 *(Folk Theorem) Let V^* be the set of feasible and strictly individually rational payoffs. Assume that* $\dim V^* = I$*. Then for any $(v_1, ..., v_I) \in V^*$, there exists a $\underline{\delta} < 1$, such that for any $\delta > \underline{\delta}$, there is a subgame perfect equilibrium of $G^\infty(\delta)$ with average payoffs $(v_1, ..., v_I)$.*

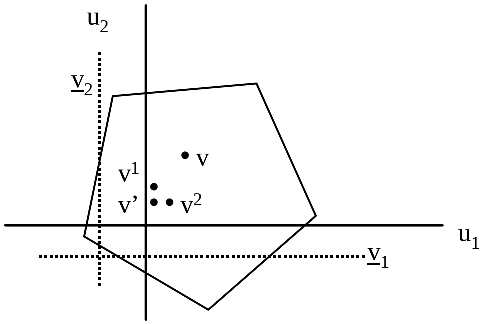
Proof. Fixing a payoff vector $(v_1, ..., v_I) \in V^*$, we construct a SPE that achieves it. For convenience, let's assume that there is some profile

$(a_1, ..., a_I)$ such that $g_i(a) = v_i$ for all i. The key to the proof is find payoffs that allow us to "reward" all agents $j \neq i$ in the event that i deviates and has to be min-maxed for some length of time.

- Choose $v' \in Int(V^*)$ such that $v_i' < v_i$ for all i.
- Choose T such that $\max_a g_i(a) + T\underline{v}_i < \min_a g_i(a) + Tv_i'$
- Choose $\varepsilon > 0$ such that for each i,

$$v^i(\varepsilon) = \left(v_1' + \varepsilon, ..., v_{i-1}' + \varepsilon, v_i', v_{i+1}' + \varepsilon, ..., v_I' + \varepsilon\right).$$

- Let a^i be the profile with $g(a^i) = v^i(\varepsilon)$
- Let m^i be the profile that min-maxes i, so $g_i(m^i) = \underline{v}^i$.



Consider the following strategies for $i = 1, 2, ..., I$.

I. Play a_i so long as no player deviates from $(a_1, ..., a_I)$. If j alone deviates, go to II_j. (If two or more players simultaneously deviate, play stays in I.)

II$_j$. Play m_i^j for T periods, then go to III_j if no one deviates. If k deviates, re-start II_k.

III$_j$. Play a_i^j so long as no one deviates. If k deviates, go to II_k.

Note that strategies involve both punishments (the stick) and rewards (the carrot). Let's check that they are indeed a subgame perfect equilibrium using the one-shot deviation principle. We need to check for each of the different subgames.

Subgame I. Consider i's payoff to playing the strategy and deviating:

$$
\begin{aligned}
i \text{ follows strategy} &: \quad (1-\delta)\left[v_i + \delta v_i + \ldots\right] = v_i \\
i \text{ deviates} &: \quad (1-\delta)\left[\overline{v}_i + \delta \underline{v}_i + \ldots + \delta^T \underline{v}_i + \delta^{T+1} v_i' + \ldots\right]
\end{aligned}
$$

Subgame II$_i$. (suppose there are $T' \leq T$ periods left)

$$
\begin{aligned}
i \text{ follows strategy} &: \quad (1-\delta^{T'})\underline{v}_i + \delta^{T'} v_i' \\
i \text{ deviates} &: \quad (1-\delta)\overline{v}_i + \delta(1-\delta^T)\underline{v}_i + \delta^{T+1} v_i'
\end{aligned}
$$

Subgame II$_j$. (suppose there are $T' \leq T$ periods left)

$$
\begin{aligned}
i \text{ follows strategy} &: \quad (1-\delta^{T'}) g_i(m^j) + \delta^{T'} (v_i' + \varepsilon) \\
i \text{ deviates} &: \quad (1-\delta)\overline{v}_i + \delta(1-\delta^T)\underline{v}_i + \delta^{T+1} v_i'
\end{aligned}
$$

Subgame III$_i$,III$_j$. Consider i's payoff to playing the strateg and deviating:

$$
\begin{aligned}
i \text{ follows strategy} &: \quad v_i' \\
i \text{ deviates} &: \quad (1-\delta)\overline{v}_i + \delta(1-\delta^T)\underline{v}_i + \delta^{T+1} v_i'
\end{aligned}
$$

The payoffs here are the least i could get if he follows the strategy and the most he could get if he deviates. A small amount of algebra shows that for $\delta \approx 1$, it is best not to deviate. *Q.E.D.*

Remark 1 *The equilibrium constructed in the above proof involves both the "stick" (Phase II) and the "carrot" (Phase III). Often, however, only the stick is necessary. The carrot phase is needed only if the parties punishing in Phase II get less than their min-max payoffs.*

Note that the (perfect) Folk Theorem requires an extra (though relatively mild) assumption, namely that $\dim V^* = I$. The assumption ensures that each player i can, in the event of a deviation, be singled out for punishment. It rules out special games like the one in the following example.

Example 1 Consider the following game with three players: 1 chooses Row, 2 chooses column, and 3 chooses matrix.

$$
\begin{array}{c|c|c|}
\multicolumn{3}{c}{\underline{A}} \\
\multicolumn{1}{c}{} & \multicolumn{1}{c}{A} & \multicolumn{1}{c}{B} \\
\cline{2-3}
A & 1,1,1 & 0,0,0 \\
\cline{2-3}
B & 0,0,0 & 0,0,0 \\
\cline{2-3}
\end{array}
\qquad\qquad
\begin{array}{c|c|c|}
\multicolumn{3}{c}{\underline{B}} \\
\multicolumn{1}{c}{} & \multicolumn{1}{c}{A} & \multicolumn{1}{c}{B} \\
\cline{2-3}
A & 0,0,0 & 0,0,0 \\
\cline{2-3}
B & 0,0,0 & 1,1,1 \\
\cline{2-3}
\end{array}
$$

In this game the min-max level is zero. To min-max i, j and k just need to mis-coordinate. The set of feasible and individually rational payoffs is:

$$V^* = \{(v,v,v) : v \in [0,1]\}$$

Claim. For any $\delta \in (0,1)$, there is no SPE of $G^\infty(\delta)$ with average payoff less than $\frac{1}{4}$.

Proof. Fix δ, and let $x = \inf\{v : (v,v,v) \text{ is a SPE payoff}\}$. The first step of the proof is to show that it (v,v,v) is an SPE payoff then:

$$v \geq (1-\delta)\frac{1}{4} + \delta x.$$

To see this, let $(\sigma_1, \sigma_2, \sigma_3)$ denote the first period mixtures used in a SPE with payoff v. Then there must exist either two players with $\sigma_i(A) \geq \frac{1}{2}$ or two players with $\sigma_i(B) \geq \frac{1}{2}$. Assume the former, and suppose $\sigma_1(A), \sigma_2(A) \geq \frac{1}{2}$.

Suppose 3 plays A in the first period and then follows his equilibrium strategy. His payoff from this will be *at least* $(1-\delta)\frac{1}{4} + \delta x$ — since $\sigma_1(A), \sigma_2(A) \geq \frac{1}{2}$, he gets at least $\frac{1}{4}$ in the first period, and over all future periods he must average at least x, given that continuation play will be an SPE. Since this deviation is unprofitable, the initial claim holds.

But now we're essentially done, since

$$x = \inf_{v \text{ is } SPE} v \geq (1-\delta)\frac{1}{4} + \delta x \qquad \Longrightarrow \qquad x \geq \frac{1}{4}.$$

The problem is that no individual can be punished for deviating without punishing everyone, so there is no way to "reward" the punishers. *Q.E.D.*

4 Comments

There are many variations and strengthenings of the perfect monitoring folk theorem.

1. The assumption that $\dim V^* = I$ (full-dimensionality) can be relaxed. Abreu, Dutta and Smith (1994) show that what is really needed is that no two players have payoffs that are affine transformations of each other — in this case it is always possible to single out individuals for punishment.

2. The assumption of strict individual rationality can also be relaxed.

3. There are some subtle issues involving randomization. The proof above assumes that there are profiles a, a^i, m^i to achieve the given payoff vectors in each period, and that deviations from these profiles are observable. There are several ways to justify this. The simplest is to assume players can carry out a public coin toss (public randomization) before each period, and that mixed strategies are observable. Randomization can also be replaced by a deterministic variation in play over time (which is more tricky).

4. Benoit and Krishna (1986) prove a folk theorem for finitely repeated games. Clearly this can't be done in the prisoners' dilemma where backward induction says that (D, D) will be played in each period. The stage game must have multiple nash equilibria to allow for rewards and punishments towards the end of the game.

5. There are also folk theorems for games where some players are long-run (infinite-horizon) and others are short-run (myopic), for games with overlapping generations of players, and for games where players face a new opponent randomly drawn from the population in each period (Kandori, 1992; Ellison, 1994).

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Relational Incentive Contracts

Jonathan Levin

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These notes consider Levin's (2003) paper on relational incentive contracts, which studies how self-enforcing contracts can provide incentives in agency settings. The principal applied motivation is that contractual relationships often function quite well without every detail of the relationship being codified in a contingent court-enforceable contract. There are many reasons for this: information may be available to the parties that is unverifiable in court; writing or enforcing contracts can be time-consuming and expensive; some contracts (such a vote-buying) may be illegal; the court system may be non-existent or poorly functioning. The paper tries to identify the extent to which a long-term relationship sustained by goodwill or reputation can substitute for the kind of perfect enforcement mechanisms typically assumed in incentive theory.

The analysis uses the repeated game methods of Abreu, Pearce and Stacchetti (1990) and Fudenberg, Levine and Maskin (1994). Relational contracts are a special class of repeated games, however, because the ability to make monetary transfers simplifies the structure of optimal repeated game equilibria. Because parties can "settle up" period by period, rather than moving to a different continuation equilibrium, the analysis becomes much more straightforward.

1 The Model

There are a principal and an agent, both risk-neutral, who share a common discount factor $\delta < 1$. At each time t, they interact as follows. The principal first offers the agent a salary w_t. The parties then simultaneously choose whether to transact or separate. If they separate, they receive period payoffs $\overline{\pi}$ and $\overline{u}$. If both choose to trade, the agent chooses an effort $e_t \geq 0$ at cost $c(e_t, \theta_t)$, where $c_e, c_{ee} > 0$. Effort is not directly observed, but generated an output y_t, drawn from a distribution with density $f(\bullet|e)$. The principal receives the output, pays the salary w_t and may also make a discretionary payment b_t (we also allow b_t to be negative in which case the agent makes the discretionary payment). Let $W_t = w_t + b_t$ denote the total payment.

The realized payments at time t are then $y_t - W_t$ for the principal, $W_t - c(e_t)$ for the agent. The joint surplus is $y_t - c(e_t)$. The first best effort level is e^{FB}, the solution to $\max_e s(e) = E[y|e] - c(e)$. Average payoffs in the repeated game are given by:

$$\begin{aligned} \pi &= (1-\delta)E\left\{\sum_{t=0}^{\infty}\delta^t\left(y_t - W_t\right)\right\} \\ u &= (1-\delta)E\left\{\sum_{t=0}^{\infty}\delta^t\left(W_t - c\left(e_t\right)\right)\right\} \end{aligned}$$

It is useful to define $s = \pi + u$ to be the average surplus in the repeated game, and $\overline{s} = \overline{\pi} + \overline{u}$ to be the surplus realized if the agents do not transact. We'll assume for simplicity that $s(e^{FB}) > \overline{s} > s(0)$.

Observe that the unique nash (and subgame perfect) equilibrium in a one-shot game (or if $\delta = 0$) is for the parties not to trade. Why? If they do, the principal certainly will not make a discretionary payment, so $b = 0$. So the agent has no incentive to exert effort and will choose $e = 0$. But then a greater surplus would be realized by not trading, so no salary can make trade desirable for both parties.

More is possible in the repeated game. To define repeated game strategies, let $h^t = (y_0, w_0, b_0, ..., y_{t-1}, w_{t-1}, b_{t-1})$ denote a time t history. A strategy for the principal specifies a salary $w_t(h^t)$, a decision about whether or not to trade, and a contingent bonus $b_t(h^t, y_t)$, A strategy for the agent specifies whether or not to trade and an effort level $e_t(h^t, w_t)$. A *relational contract* specifies for any history h^t, an effort e_t, a salary w_t and a contingent bonus $b_t(y)$. A relational contract is *self-enforcing* it corresponds to some perfect public equilibrium of the repeated game.

A useful observation is that if there is some self-enforcing contract (or PPE) that achieves a joint surplus s, then there are self-enforcing contracts that achieve any individually rational split of this surplus.

Proposition 1 *Suppose there is some self-enforcing contract with expected surplus s. Then any payoff vector u, π with $u + \pi = s$, $u \geq \overline{u}$ and $\pi \geq \overline{\pi}$ can be acheived with a self-enforcing contract.*

Proof. Suppose the PPE that has expected surplus s gives expected payoffs $\hat{u}, \hat{\pi}$ and involves a salary $\hat{w}$ in the first period. Let $u \geq \overline{u}, \pi =$

$s - u \geq \overline{\pi}$ be given. Construct a new contract as follows. In the first period, the principal offers a salary $w = \hat{w} + (u - \hat{u})/(1 - \delta)$, following which play exactly follows the PPE that gives payoffs $\hat{u}, \hat{\pi}$. If the principal deviates, the players do not transact at date 0 or any following date. *Q.E.D.*

2 Stationary Contracts

A relational contract is *optimal* if no other self-enforcing contract acheives a higher surplus. A key simplifying result is that in searching for optimal contracts, it suffices to consider contracts that are stationary: i.e. that involve the same salary, effort and contingent bonus plan in every period on the equilibrium path.

Definition 1 *A contract is stationary if in every period on the equilibrium path $e_t = e$, $w_t = w$, and $b_t = b(y_t)$ for some $(e, w, b(y))$.*

Notice that stationary contracts assign the same continuation payoffs (and same continuation play) after every history on the eqm path. This is similar to optimal equilibria in games with perfect monitoring, but in sharp contrast to, say, equilibria in the Green and Porter model.

Proposition 2 *If an optimal contract exists, there is a stationary contract that is optimal.*

Proof. Let s^* denote the surplus achieved by an optimal contract. Suppose there is some optimal contract that achieves payoffs u, π, with $u + \pi = s^*$, involves a salary w_0, effort e_0 and bonus payments $b_0(y_0)$ at $t = 0$ and specifies continuation payoffs $u_1(h^1), \pi_1(h^1)$ starting at $t = 1$. Note that for histories off the equilibrium payoff, we can without loss generality specify that the players cease to transact forever, as this is the worst possible punishment.

The first claim is that any optimal contract must be sequentially optimal. That is $s(e_0) = s^*$ and moreover $u_1(h^1) + \pi_1(h^1) = s^*$ for any h^1 on the equilibrium path. To see why the latter must be so, notice that increasing $\pi_1(h^1)$ improves the principal's incentives to deliver on discretionary payments without changing the agent's incentives at all. Therefore if $u_1(h^1) + \pi_1(h^1) < s^*$ for some h^1 on the equilibrium path, it would be possible to increase $\pi_1(h^1)$ and have a new self-enforcing contract with higher initial surplus. Therefore starting at time $t = 1$, any optimal contract much achieve surplus s^* for any history h^1 on the equilibrium path. Clearly a higher surplus is not possible starting at $t = 1$ as s^* is the highest possible equilibrium surplus. But then to achieve s^* on average from date $t = 0$, it must be the case that $s(e_0) = s^*$.

Having established sequential optimality, we now use the (possibly non-stationary) optimal contract to construct a stationary contract that achieves the same surplus. Let u, π be individually rational payoff vectors with $u \geq \overline{u}$, $\pi \geq \overline{\pi}$ and $u + \pi = s^*$. Let $e = e_0$, so that $s(e) = s^*$. Define payments $w, b(y)$ to satisfy:

$$\begin{aligned} u &= w + \mathbb{E}_y[b(y)|e] - c(e) \\ b(y) + \frac{\delta}{1-\delta}u &= b_0(y_0) + \frac{\delta}{1-\delta}u_1(w_0, e_0, y). \end{aligned}$$

Consider the agent's expected future payoff at the point in time he chooses his action. By construction it is the same under the stationary contract as it is in the first period of the optimal contract. As $e_0 = e$ was optimal in the first period of the optimal contract, the same is true in the stationary contract.

Next consider the parties' expected future payoff at the point in time they choose whether or not to make the discretionary payment. The agent's payoff is:

$$b(y) + \frac{\delta}{1-\delta}u = b_0(y_0) + \frac{\delta}{1-\delta}u_1(w_0, e_0, y),$$

i.e. identical to in the first period of the optimal contract. The principal's payoff is:

$$-b(y) + \frac{\delta}{1-\delta}\pi = -b_0(y_0) + \frac{\delta}{1-\delta}\pi_1(w_0, e_0, y),$$

i.e. identical to in the first period of the optimal contract (note we've used the fact that $\pi_1 + u_1 = \pi + u = s^*$. So both parties are willing to make the discretionary payment rather than walk away, and we have identified a stationary contract that is self-enforcing and generates the optimal surplus s^* (indeed with an arbitrary individually rational split). *Q.E.D.*

An implication of this result is that to characterize optimal contracts, one can consider only stationary contracts. The basic logic of the result is very simple. In the model, the parties have two instruments to provide incentives: contingent transfers made today and continuation payoffs. These instruments are perfect substitutes. If we start with an optimal contract where the principal provides incentives using variation in continuation payoffs, we can always replace this variation with variation in transfers payments today yielding a stationary contract that provides the same incentives.

Remark 1 *The optimal stationary contract constructed in the proof of Proposition 2 is not renegotiation proof because observable deviations (e.g. refusal to make specified payments) are punished with termination of the relationship. Levin (2003) argues that one can construct optimal contracts that are strongly renegotiation proof, however. The reason is that among the set of optimal (stationary) contracts are contracts that hold each of the two parties to their outside options,* $\overline{u}$ *and* $\overline{\pi}$ *respectively.*

3 Optimal Contracts

The next step is to characterize optimal stationary contracts. A stationary consists of an effort level e, a salary w and a contingent payment plan $b(y)$. The next result explains exactly what stationary contracts are self-enforcing.

Proposition 3 *There exists a stationary contract that implements effort* e *if and only if there is some payment schedule* $W(y)$ *such that*

$$e \in \arg\max_{\hat{e}} \mathbb{E}_y[W(y)|\hat{e}] - c(\hat{e}) \tag{IC}$$

and

$$\frac{\delta}{1-\delta}(s(e) - \overline{s}) \geq \max_y W(y) - \min_y W(y) \tag{DE}$$

Proof. Note that to construct a self-enforcing contract it is natural to punish any departure from the contract with the worst possible continuation payoff, namely the separation payoffs $\overline{u}, \overline{\pi}$. Given this, a stationary contract $\{e, w, b(y)\}$ will be self-enforcing if and only if it (1) gives the principal a period expected utility $\pi \geq \overline{\pi}$ and the agent a period expected utility $u \geq \overline{u}$, (2) satisfies the incentive compatibility constraint

$$e \in \arg\max_{\hat{e}} \mathbb{E}_y[w + b(y)|\hat{e}] - c(\hat{e}),$$

and (3) satisfies two constraints on the discretionary transfer payment $b(y)$: for all y,

$$\begin{aligned} b(y) &\leq \frac{\delta}{1-\delta}(\pi - \overline{\pi}) \\ -b(y) &\leq \frac{\delta}{1-\delta}(u - \overline{u}) \end{aligned}$$

To prove the result, we first show that (IC) and (DE) are necessary for there to be a self-enforcing contract that implements e. Suppose $\{e, w, b(y)\}$

is self-enforcing. Define $W(y) = w + b(y)$. Then $e, W(y)$ satisfies (IC) and (DE).

Conversely, suppose $e, W(y)$ satisfies (IC), (DE). Let u, π be given with $u + \pi = s(e)$, $u \geq \overline{u}$ and $\pi \geq \overline{\pi}$. Construct stationary payments $w, b(y)$ that satisfy::

$$
\begin{aligned}
u &= \mathbb{E}_y[w + b(y)|e] - c(e) \\
b(y) &= W(y) - \min_{\tilde{y}} W(\tilde{y}).
\end{aligned}
$$

By construction, the stationary contract $\{e, w, b(y)\}$ (1) generates individually rational payoffs $u \geq \overline{u}$, $\pi \geq \overline{\pi}$ with $u + \pi = s(e)$, (2) as a consequence of (IC), satisfies the incentive compatibility constraint above, and (3) as a consequence of (DE), satisfies the restrictions on discretionary transfers. *Q.E.D.*

The result says that stationary contracts must satisfy two natural constraints: a standard incentive compatibility constraint for the agent's effort choice and a dynamic enforcement constraint. The latter requires that discretionary payments are not too small (to prevent the agent from walking away), nor too large (to prevent the principal from walking away). This limited variation in payments is what distinguishes optimal self-enforced contract from optimal contracts that are court-enforced.

Given the above result, it's pretty straightforward to characterize optimal contracts. To do so, it's useful to impose two asumptions: namely that the distribution of output as a function of effort, $F(y|e)$, satisfies the monotone likelihood ratio property (MLRP) and is concave in effort (CDFC). These assumptions are strong, but fairly standard in incentive theory. They imply that the incentive constraint above can be replaced by a first-order condition for the agent's optimal effort choice.

The optimal contract $\{e, W(y) = w + b(y)\}$ is then the solution to the following problem:

$$
\begin{aligned}
&\max_{e, W(y)} s = \mathbb{E}[y|e] - c(e) \\
\text{s.t.} \quad &\int_Y W(y) \frac{f_e}{f}(y|e) dF(y|e) - c'(e) = 0 \\
&\frac{\delta}{1-\delta}(s - \overline{s}) \geq \max_y W(y) - \min_y W(y)
\end{aligned}
$$

The optimal contract take a very simple form: a fixed salary plus a bonus if output exceeds some threshold. The size of the salary can be varied to achieve different divisions of the joint surplus.

Proposition 4 *Under MLRP and CDFC, the optimal contracts are "one-step", i.e. there is some* $\widehat{y}$ *with* $W(y) = \underline{W}$ *if* $y \leq \widehat{y}$ *and* $W(y) = \overline{W}$ *if* $y \geq \widehat{y}$.

Proof. The marginal benefit to raising $W(y)$ for some y is $\min W < W(y) < \max W$ is

$$\mu \cdot \frac{f_e}{f}(y|e),$$

where $\mu > 0$ is the Lagrange multiplier on the incentive compatibility constraint. The MLRP assumption means that $(f_e/f)(y|e)$ is increasing in y for a fixed e, so there will be some $\hat{y}$ s.t. the marginal benefit is positive for all $y > \hat{y}$ and negative for all $y < \hat{y}$. The result follows immediately. *Q.E.D.*

4 Comments

1. The stationarity result is more general than is outlined here. Essentially it follows from two observations. The first is that the combination of risk-neutrality (quasi-linear utility) and monetary transfers allow the parties to replace variation in continuation payoffs with variation in present transfers, i.e. to "settle up" immediately. The second is that in a model where the principal's actions are observable, optimal contracts will be sequentially optinal, so transfers can be balanced. More generally, for instance in some moral hazard in teams problems, optimal contracts might involve money-burning (a deliberate destruction of surplus).

2. As a result, the only difference between standard (static) incentive theory and relational incentive theory is the presence of the dynamic enforcement constraint. As a consequence, many applications are possible: hidden action as above, hidden information, multiple agents (Levin, 2002), the use of both verifiable and observable but non-verifiable information (in Baker, Gibbons and Murphy 1993), team production (Rayo '01). It is also possible to incorporate explicit (payoff-relevant) state variables.

3. Self-enforcement has interesting implications for the use of hidden information screening contracts. To study hidden information we assume that the agent privately observes some iid cost shock θ_t drawn from a distribtion $P(\cdot)$ and chooses output y_t at cost $c(y_t, \theta_t)$. Levin (2003) shows that in this setting optimal contracts either achieve the first-best or they will involve production distortions for *all* cost types. Moreover, second-best contracts always involve pooling. For low discount factors, optimal contracts require all types to pool on a single level of effort. For medium discount factors, optimal contracts separate high cost types but pool low-cost types. For high discount factors, the first best separating contract is possible.

4. The last section of Levin (2003) considers a variant of the hidden action model where output is privately observed by the principal, rather than commonly observed. The principal can then issue a report about the agent's performance (a subjective evaluation). This model is much more complicated because it involves *private monitoring.* Two issue arise. The first is that an optimal contract must provide incentives for the agent to exert effort and for the principal to monitory honestly. As a result, equilibrium contracts cannot be sequentially optimal; joint surplus must vary over time. The second question that arises is how the principal should release information over time. Discounting means that the principal cannot wait forever to make payments. But concealing information makes it easier to provide incentives for the agent (this is in insight of Abreu, Milgrom and Pearce, 1991). Levin (2003) restricts attention to "full-review" contracts (i.e. PPE) and shows that one-step termination contracts are optimal. MacLeod (2003) and Fuchs (2005) provide further analyses.

5. An important early paper on relational contracts by MacLeod and Malcomson (1989) provides a full characterization of self-enforcing contracts under the assumption of perfect information. Not surprisingly, stationary contracts are optimal, the key enforcement condition being that $s(e) \geq \frac{\delta}{1-\delta} c(e)$. Their paper also goes a step further by nesting the agency model in a market equilibrium setting where principals and agents match to start relationships. This is a great paper that didn't get nearly the attention it deserved when it was published.

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Repeated Games II: Imperfect Public Monitoring

Jonathan Levin

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We now take up the problem of repeated games where players' actions may not be directly observable. This is a rich class of problems, with many economic applications. Abreu, Pearce and Stacchetti (1990) and Fudenberg, Levine and Maskin (1994) have developed a beautiful and powerful set of techniques for these games.

1 A Few Examples

1. Cournot competition with noisy demand (Green and Porter, 1984). Firms set outputs $q_{1t}, ...q_{It}$, chosen privately. Demand conditions then determine $p_t = P(q_{1t}, ..., q_{It}, \varepsilon)$, which is observed publicly.

2. "Reputation" for quality. A single firm sets price p_t and chooses an effort e_t at some cost $c(e_t)$. Product quality is "High" with probability $p(e_t) \in (0, 1)$, where p is increasing in e_t. Consumers are willing to pay more for high quality.

3. Noisy prisoners' dilemma. Players choose from $\{C, D\}$. But instead of these actions being observed, some noisy signal of these actions is observed instead (see below).

4. Team Production. Players choose efforts $e_1 \in \{e_L, e_H\}$, as part of a joint project that succeeds with probability $p(e_1 + e_2)$. Only the joint outcome is observed publicly.

5. Self-enforced agency contracts (Levin, 2003). Each period, the agent privately observes a cost parameter θ_t, and produces output y_t at cost $c(\theta_t, y_t)$. The output, but not the cost, is observed. Alternatively, the agent chooses an effort e_t, and output is stochastic $y_t \sim f(\cdot|e)$.

6. Consumption smoothing and insurance (Green, 1987). There are a continuum of consumers, who each period get income shocks z_{it}. They then report their incomes and make transfers among themselves. Transfers must be balanced.

2 The General Model

- Let $A_1, ..., A_I$ be finite action sets.
- Let Y be a finite set of public outcomes.
- Let $\pi(y|a) = \Pr(y|a)$.
- Let $r_i(a_i, y)$ be i's payoff if she plays a_i and the public outcome is y.
- Player i's expected payoff is:

$$g_i(a) = \sum_{y \in Y} \pi(y|a) r_i(a_i, y).$$

- A mixed strategy is $\alpha_i \in \Delta(A_i)$. Payoffs are defined in the obvious way.

Example 1 In the Cournot game, a_i is quantity, y is price (here, one might want to have A, Y continuous rather than finite).

Example 2 In the PD game, a_i is intended action, y is actual actions.

Example 3 In the agency problem, a_i is a *vector* that specifies the agent's output y for each cost realization.

- The public information at the start of period $t : h^t = (y^0, ..., y^{t-1})$.
- Player i's private information is $h_i^t = (a_i^0, ..., a_i^{t-1})$.
- A strategy for i is a sequence of maps σ_i^t taking $(h^t, h_i^t) \to \Delta(A_i)$

Definition 1 *A* ***public strategy*** *for player i is a sequence of maps σ_{it} : $h^t \to \Delta(A_i)$.*

We focus on public strategies because they are simple and lead to a nice structure for the game. More on this later, however.

- Player i's average discounted payoff for the game if he gets a sequence of payoffs $\{g_i^t\}$ is:

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t g_i^t$$

Definition 2 *A profile* $(\sigma_1, .., \sigma_I)$ *is a* ***perfect public equilibrium*** *if*

(i) σ_i *is a public strategy for all* i*.*

(ii) *For each date* t *and history* h^t*, the strategies are a Nash equilibrium from that point on.*

A crucial point about PPE is that after any history h^t, a PPE induces a PPE in the remaining game. However, note that like the set of subgame perfect equilibrium payoffs in a repeated game model with perfect monitoring, the set of PPE payoffs is *stationary* — i.e. it's the same starting from any period t. The first point is perhaps a little subtle. With imperfect monitoring, there are often no proper subgames (i.e. a player may be uncertain as to which of many information nodes he is at), so SPE might have no bite. However, since opponents don't base their strategies on private information, all possible nodes have the same distribution over opponent play, so there's no need to distinguish. The perfection condition *(ii)* is well-defined because the public history is commonly known.

Note that if the game *happens* to have perfect monitoring, so $Y = A$, and $\pi(y|a)$ puts probability 1 on a, then PPE *coincides* with SPE. More generally, a PPE is a perfect bayesian equilibrium of the repeated game, but not all perfect bayesian equilibria are PPEs. What do I mean by this? In PPE, everyone uses a public strategy. Given that opponent's are using public strategies, it doesn't help i to use a non-public strategy, since any private information he might have is not payoff-relevant (since preferences don't depend on the private information). However, if $j \neq i$ are using their private information, i might want to use his. This non-public strategy case is not nearly as well-understood.

Example 4 Green and Porter (1984) suggest the following type of "trigger strategies" for the noisy Cournot model:

1. Play $q_1, ..., q_I$. If $p_t < \underline{p}$, go to phase 2.
2. Play $q_1^c, ..., q_I^c$ (cournot) for T periods. Then return to phase I.

GP verify that if the players are sufficiently patient, there is an equilibrium of this form where $q_1, ..., q_I$ are less than the static Cournot levels (e.g. equal to q^M/I where q^M is the monopoly quantity) and $\underline{p}$ and T are chosen appropriately. This equilibrium is a PPE. Strategies are "public" and play is Nash from every time forward.

Note that a lower trigger price means less chance of punishment and more incentive to deviate. A longer punishment periods mean less incentive to deviate, but less efficiency. GP's main result is that firms can't achieve the first-best monopoly profits as there will be "price wars" in equilibrium.

3 Self-Generation

We now develop a set of powerful techniques for characterizing perfect public equilibria. In contrast to the Green-Porter analysis, we will think not in terms of *strategies*, but in terms of *payoffs*. The idea is that to "enforce" certain actions at time t, we will attach continuation payoffs from time $t+1$ on to each time t outcome. You can think of this as analogous to a principal–agent problem, where to motivate the agent, the principal promises certain rewards or punishments. The subtlety here is that the promised rewards and punishments must themselves correspond to payoffs in a PPE of the continuation game (rather than being monetary payoffs specified in a court-enforced contract).

Definition 3 *The pair (α, v) is* ***enforceable*** *with respect to δ and $W \subset \mathbb{R}^I$ if there exists a function $w : Y \to W$ such that for all i,*

(i) $v_i = (1-\delta)g_i(\alpha) + \delta \sum_y \pi(y|a)w_i(y)$

(ii) $\alpha_i \in \arg\max_{\alpha_i' \in \Delta(A_i)} (1-\delta)g_i(\alpha_i', \alpha_{-i}) + \delta \sum_y \pi(y|\alpha_i', \alpha_{-i})w_i(y)$

Condition *(i)* says that the target payoff v can be *decomposed* into today's payoff $g_i(\alpha)$ and the expected continuation payoff (macroeconomists call this the "promised utility"). Condition *(ii)* is essentially an incentive compatibility constraint. These conditions ought to remind you of Bellman's equation for dynamic programming.

Definition 4 *Let $B(\delta, W)$ be the set of payoffs v such that for some α, (α, v) is enforced with respect to δ and W. Then $B(\delta, W)$ is the payoff set* ***generated*** *by δ, W.*

Definition 5 *$E(\delta)$ is the set of PPE payoffs.*

Proposition 1 *$E(\delta) = B(\delta, E(\delta))$.*

Proof. ($\supseteq$) Fix $v \in B(\delta, E(\delta))$. Pick α, $w : Y \to E(\delta)$ such that w enforces (α, v). Now consider the following strategies. In period 0, play α. Then starting in period 1, play the perfect public equilibrium that gives payoffs $w(y_0)$. This is a PPE, so $v \in E(\delta)$.

($\subseteq$) Fix $v \in E(\delta)$. There is some PPE that gives payoffs v. Suppose in this PPE, play in period 0 is α, and continuation payoffs are $w(y_0) \in E(\delta)$, since continuation corresponds to PPE play. The fact that no one wants to deviate means that (α, v) is enforced by $w : Y \to E(\delta)$, so $v \in B(\delta, E(\delta))$. *Q.E.D.*

Abreu, Pearce and Stacchetti (1986, 1990) call this factorization. The idea is that for any PPE, the payoffs can be decomposed or factored into today's payoffs and continuation payoffs. In a PPE, all the continuation payoffs have to themselves correspond to PPE profiles. So those can be decomposed, and so on. So they have a recursive structure.

Definition 6 *W is **self-generating** if $W \subset B(\delta, W)$.*

The interpretation is that it is possible to sustain average payoffs in W by promising different continuation payoffs in W. Note that $E(\delta)$ is self-generating. The set of static Nash equilibrium payoffs is also self-generating.

Proposition 2 *If W is self-generating, then $W \subset E(\delta)$.*

Proof. Fix $v \in W$. Then $v \in B(\delta, W)$, so there is some $w : Y \to W$ and some α such that (α, v) is enforced by w. We construct an equilibrium that gives v. In period 0, play α, and for an outcome y_0, set $v_1 = w(y_0)$. Then $v_1 \in W \subset B(\delta, W)$, so again there is some α_1 and some $w_1 : Y \to W$ such that (α_1, v_1) is enforced by w_1. Continue with this argument ad infinitum, to obtain recommended strategies after each public history such that there are no profitable deviations, and which by construction give payoff v from time 0. *Q.E.D.*

Corollary 1 *$E(\delta)$ is the largest self-generating set.*

We'll now go on to discuss some applications of these ideas.

4 Examples of Self-Generation

Let's try out self-generation in two variants of the prisoners' dilemma.

4.1 Prisoners' Dilemma

Consider the prisoner's dilemma with *perfect monitoring.*

	C	D
C	$1,1$	$-1,2$
D	$2,-1$	$0,0$

With perfect monitoring, $Y = \{(C,C), (C,D), (D,C), (D,D)\}$.

Claim If $\delta \geq 1/2$, the set $W = \{(0,0), (1,1)\}$ is self-generating.

Proof. We want to show that $(0,0) \in B(\delta, W)$, and $(1,1) \in B(\delta, W)$ for $\delta \geq 1/2$. Consider $(0,0)$ first. It is easy to see that the strategy profile (D,D), and payoff profile $(0,0)$ are enforced by *any* δ and the function $w(y) = (0,0)$ since

$$0 = (1-\delta)g_i(D,D) + \delta w_i(D,D)$$

and for all $a_i \in \{C, D\}$,

$$0 \geq (1-\delta)g_i(a_i, D) + \delta w_i(a_i, D).$$

Now consider $(1,1)$. We show that the strategy profile (C,C) and payoff profile $(1,1)$ are enforced by $\delta \geq 1/2$ and W. Let $w(C,C) = (1,1)$ and $w(y) = (0,0)$ for all $y \neq (C,C)$. Then

$$1 = (1-\delta)g_i(C,C) + \delta w_i(C,C)$$

and for all $a_i \in \{C, D\}$, if $\delta \geq 1/2$

$$1 \geq (1-\delta)g_i(a_i, C) + \delta w_i(a_i, C).$$

So $W \subset B(\delta, W)$ for $\delta \geq 1/2$, meaning that W is self-generating. *Q.E.D.*

Exercise 1 *Try showing that if $\delta = 1/2$, then $(3/2, 0)$ is also in $B(\delta, W)$.*

4.2 Noisy Prisoners' Dilemma

There are two players $i = 1, 2$ and $A_i = \{C, D\}$. The observed outcomes are $Y = \{G, B\}$ (good and bad) where

$$\Pr(G \mid a) = \begin{cases} p & \text{if } a = (C,C) \\ q & \text{if } a = (C,D),(D,C) \\ r & \text{if } a = (D,D) \end{cases},$$

with $p > q > r$. We assume that $p - q > q - r$. Payoffs are given by

$$r_i(a_i, y) = \begin{cases} 1 + \frac{2-2p}{p-q} & \text{if } (C,G) \\ 1 - \frac{2p}{p-q} & \text{if } (C,B) \\ \frac{2-2r}{q-r} & \text{if } (D,G) \\ \frac{-2r}{q-r} & \text{if } (D,B) \end{cases},$$

which means that *expected payoffs* $g_i(a)$ are given by the standard prisoners' dilemma matrix above.

Claim If $\frac{1}{(p-r)+(q-r)} \geq \delta \geq \frac{1}{(p-q)+(p-r)}$, the set $W = \{\frac{\delta r}{1-\delta(p-r)}, \frac{1-\delta+\delta r}{1-\delta(p-r)}\}$ is self-generating.

Proof. Recall that to enforce a payoff v, we need a profile a and a map $w : Y \to W$ from outcomes to continuation payoffs such that:

$$v = (1-\delta)g_i(a) + \delta \mathbb{E}\left[w_i(y) \mid a_i', a_{-i}\right]$$

and for all $a_i' \neq a_i$

$$v \geq (1-\delta)g_i(a_i', a_{-i}) + \delta \mathbb{E}\left[w_i(y) \mid a_i', a_{-i}\right].$$

Now, let $v = \frac{\delta r}{1-\delta(p-r)}$, and $v' = \frac{1-\delta+\delta r}{1-\delta(p-r)}$ (the bad and good continuation payoffs). To enforce v, use the profile (D, D) and

$$w(y) = \begin{cases} v' & \text{if } y = G \\ v & \text{if } y = B \end{cases}$$

We need to check that:

$$\begin{aligned} v &= (1-\delta)(0) + \delta v + \delta r(v' - v), \\ v &\geq (1-\delta)(-1) + \delta v + \delta q(v' - v) \end{aligned}$$

The first constraint is just algebra. The second constraint holds so long as:

$$\delta(q-r)(v'-v) \geq 1-\delta,$$

that is, if $\delta \geq 1/(p+q-2r)$.

To enforce v', use the profile (C, C) and the *same* $w : Y \to W$. We need to check

$$\begin{aligned} v' &= (1-\delta)(1) + \delta v + \delta p(v'-v) \\ v' &\geq (1-\delta)(2) + \delta v + \delta q(v'-v) \end{aligned}$$

The first condition is again just algebra, while the second holds so long as:

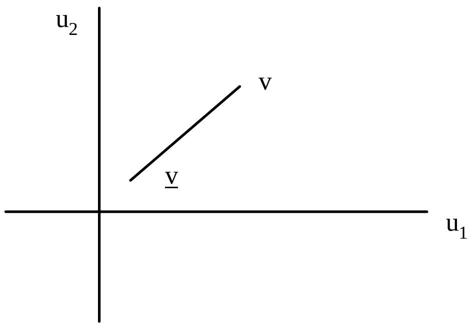
$$\delta(p-q)(v'-v) \geq 1-\delta$$

which requires that $\delta \geq 1/(2p-q-r)$. *Q.E.D.*

4.3 Strongly Symmetric PPE

The examples above are illustrative but they don't characterize the set of PPE. In general models, this tends to be quite hard. A useful simplification in symmetric games (introduced by Abreu, Pearce and Stacchetti, 1986) is to focus on PPE that are *strongly symmetric* in the sense that each player uses the same strategy after *every* history. In the two examples above, the strategies were strongly symmetric.

More generally, if we allow for public randomization, the set of strongly symmetric PPE payoffs will be an interval $[\underline{v}, \overline{v}]$, where $\underline{v}$ is the lowest and $\overline{v}$ the highest strongly symmetric PPE payoffs. Therefore, solving for the equilibrium set boils down to finding the best and worst equilibrium payoffs.



Set of Symmetric PPE

To actually characterize the highest and lowest payoffs, we need to solve a fixed point problem. We can think about fixing future symmetric payoffs to lie on some interval and finding maximal and minimal present payoffs subject to incentive compatibility, promise-keeping and the constraint that continuation payoffs are chosen from the allowable interval. If we start with a very large interval, we will find a somewhat smaller interval; continuing the process we will eventually converge to the equilibrium payoff set. Alternatively, we can jump right to the fixed point by solving the following problem:

$$
\begin{aligned}
\overline{v} &= \max_{\overline{a},\underline{a},\overline{v},\underline{v},w:Y\to\mathbb{R}} (1-\delta)g(\overline{a}) + \delta \sum_y w(y)\pi(y|\overline{a}) \\
\text{s.t. } \overline{v} &= (1-\delta)g(\overline{a}) + \delta \sum_y w(y)\pi(y|\overline{a}) \\
\underline{v} &= (1-\delta)g(\underline{a}) + \delta \sum_y w(y)\pi(y|\underline{a}) \\
\overline{v} &\geq (1-\delta)g(a,\overline{a}) + \delta \sum_y w(y)\pi(y|a,\overline{a}) \text{ for all } a \in A \\
\underline{v} &\geq (1-\delta)g(a,\underline{a}) + \delta \sum_y w(y)\pi(y|a,\underline{a}) \text{ for all } a \in A \\
\overline{v} &\geq w(y) \geq \underline{v} \text{ for all } y \in Y
\end{aligned}
$$

Note that here we find maximixal and minimal payoffs in one step using the fact that a lower minimum will automatically allow a higher minimum and vice versa.

While this is simpler than solving for PPE in general, it's still pretty complicated. Abreu, Pearce and Stacchetti (1986) characterize strongly symmetric equilibria in the Green-Porter oligopoly game where players chooses quantities and price is a noisy function of the aggregate quantity. Athey, Bagwell and Sanchirico (2004) study strongly symmetric equilibria in a repeated Bertrand pricing game where firms have private cost information.

5 The Folk Theorem

So far, we've argued that strongly symmetric strategies lead to inefficient outcomes (because of equilibrium "price wars"). Nevertheless, Fudenberg, Levine and Maskin (1994) show that this inefficiency arises because GP '84 and APS '86 limit the space of strategies, and go on to prove a version of the Folk Theorem.

Fudenberg, Levine and Maskin's result requires two "observability" or "identification" conditions.

(I1) For all i, and a_{-i}, the $|A_i|$ vectors $\pi(\cdot|a_i, a_{-i})$ are linearly independent.

(I2) For all i, j, there is some profile α such that the $|A_i| + |A_j|$ vectors $\pi(\cdot|a_i, \alpha_{-i})$ and $\pi(\cdot|a_j|\alpha_{-j})$ admit only one linear dependency.

The first condition requires that i's actions can be statistically identified — that is, they do not induce the same probability distribution on outcomes. Note that for (I1) to hold, it *must* be the case that $|Y| \geq |A_i|$ — which is arguably quite a strong requirement. The second condition says that if everyone is playing α, then not only can i's actions be distinguished, and j's actions being distinguished, but i's actions can be distinguished from j's actions. If you're a statistics/econometrics type, you can think of this just like statistical identification. Here i's action is the parameter. To identify it, you need the probability distribution over observables to change when it changes. The second condition is what you need to identify both a_i and a_j at the same time — we only require this for *some* profile played by the others, $k \neq i, j$.

Let V^* be the set of feasible and strictly individually rational payoff vectors.

Proposition 3 *Suppose* $\dim V = I$*, and* $(I1), (I2)$ *hold. Then for any closed set* $W \subset int(V^*)$*, there exists some* $\underline{\delta} < 1$ *such that for all* $\delta \geq \underline{\delta}$*,* $W \subset E(\delta)$*.*

Proof. I'll sketch the idea in class; you'll have to read the paper for details! *Q.E.D.*

The folk theorem applies to payoff vectors in the interior of V^*. Generally you can't get *exact* efficiency with imperfect monitoring. The argument is simple and illustrative. Suppose that $\pi(\cdot|a)$ has a support that is independent of a (as in APS). And suppose that v is extremal but not a static Nash equilibrium payoff. Because v is extremal, the only sequence of payoffs that gives average value v must have payoffs v in *every* period. So if a PPE gives v, the first period strategies must specify a profile a with $g(a) = v$, and for any outcome y, the continuation payoffs must be $w(y) = v$. But then continuation payoffs are independent of today's outcome, so unless a happens to be a static Nash equilibrium (which it isn't by assumption), someone will want to deviate.

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Repeated Games III: Imperfect Private Monitoring

Jonathan Levin

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In the last couple of classes, we looked at repeated games with imperfect public monitoring, and studied perfect public equilibria of these games. These equilibria had a very nice recursive structure. But what happens if the outcome in each period is *not* commonly observed? In this case, we have a repeated game with imperfect *private* monitoring. Relatively little is known about the structure of equilibria in these games. These notes survey a few recent ideas on the topic.

1 Private Monitoring: An Example

To get a sense of the problems that arise when there is private monitoring, let's start by working through some variations on a simple two-period model (based on Mailath and Morris, 1998; Bhaskar and van Damme, 2002).

There are two players. In the first period, they play a prisoners' dilemma.

	C	D
C	$1,1$	$-1,2$
D	$2,-1$	$0,0$

In the second period, they play a coordination game

	G	B
G	k,k	$0,0$
B	$0,0$	$1,1$

It is assumed that $k > 2$. All the payoffs accrue after the second period, and there is no discounting. While this isn't a repeated game, it is like one in the sense that the players can try to enforce cooperation in the first period by coordinating on either a good or bad outcome later on.

1.1 Perfect Monitoring

With perfect monitoring, the following strategies will support first period cooperation:

$$\begin{aligned} \text{Period 1} &: \quad \text{Play } C \\ \text{Period 2} &: \quad \text{Play } G \text{ if first period outcome is } (C, C), \text{ else play } B. \end{aligned}$$

Clearly these strategies lead to an equilibrium in the second period, regardless of the first period outcome. In the first period, consider i's incentives:

$$\begin{aligned} \text{Play } C &\Rightarrow \quad 1 + k \\ \text{Play } D &\Rightarrow \quad 2 + 1 \end{aligned}$$

Since $k > 2$, it is optimal to follow the specified strategy strategy.

1.2 Private (Independent) Monitoring

Now suppose that first period actions (a_1, a_2) are not observed. Rather, each player i observes a signal $y_i \in \{c, d\}$ about her opponent's action. Suppose that

$$\Pr(y_i = c \mid a_j) = \begin{cases} 1 - \varepsilon & \text{if } a_j = C \\ \varepsilon & \text{if } a_j = D \end{cases}$$

If ε is small, monitoring is almost perfect. However, the striking fact is that *no pure-strategy equilibrium supports (C,C) in the first period.* Why? Observe that in the second period, i will want to play G if and only if she assigns probability $1/(k + 1)$ or greater to j playing G. Consider strategies that call for each player to play C in the first period, and G in the second period if and only if $y_i = c$. If i plays C in the first period, she assigns probability $1 - \varepsilon$ to j observing c, and hence to j playing G. But then *regardless* of the signal she observes, she will want to play G, and so she won't follow the strategy.

Essentially, the problem is that i and j's second period information is independent. So long as i cooperates in the first period, she will assign high probability to j observing a good signal regardless of her own signal. Consequently, she prefers to just keep cooperating. In short, the private monitoring means there is no way to coordinate on the punishment equilibrium in period two following a bad outcome, and on the good equilibrium otherwise. Thus, there's no way to enforce cooperation in the first period.

1.3 Private (Correlated) Monitoring

The coordination problem is lessened if the private signals are correlated rather than independent.To see this, let's consider the extreme case of perfect correlation. Suppose that $y_i = y_j = y \in \{c, d\}$, where

$$\Pr(y = c \mid (a_1, a_2)) = \begin{cases} 1-\varepsilon & a_i = a_j = C \\ \varepsilon & \text{otherwise} \end{cases}$$

What we have now is really a game with imperfect public monitoring.

Consider strategies that call for each player to play C in the first period, and to play G in the second period if and only if they observe c. This clearly this gives an equilibrium in the second period, and checking first period incentives:

$$\begin{aligned} \text{Play } C \quad &\Rightarrow \quad 1 + (1-\varepsilon)(k-1) + 1 \\ \text{Play } D \quad &\Rightarrow \quad 2 + \varepsilon(k-1) + 1 \end{aligned}$$

It is optimal to follow the specified strategy if $k \geq 1 + 1/(1-2\varepsilon)$, which is ensured for ε small.

More generally, if y_i and y_j are highly, but not perfectly, correlated, it will be possible to support cooperation in the first period by coordinating on different second period play depending on the signals.

1.4 Mixed Strategies

Interestingly, even if the private signals are independent, the players may be able to correlate their beliefs by playing mixed strategies in the first period. Returning to the independent signals set-up from above, consider the following strategies.

$$\begin{aligned} \text{Period 1} \quad &: \quad \text{Play } C, D \text{ with probabilities } \alpha, 1-\alpha \\ \text{Period 2} \quad &: \quad \text{Play } G \text{ if and only if } a_i = C \text{ and } y_i = c. \end{aligned}$$

In the second period, i will want to play G if and only if she assigns probability $1/(k+1)$ or greater to j playing G. She assigns this probability by conditioning on what she knows, her action a_i and her signal y_i.

$$\Pr(j \text{ will play } G \mid a_i, y_i) = \Pr(y_j = c \mid a_i) \times \Pr(a_j = C \mid y_i)$$

Applying Bayes' rule to derive the corresponding probabilities:

(a_i, y_i)	$\Pr(j \text{ will play } G)$
(C, c)	$(1-\varepsilon)\frac{1-\varepsilon}{\alpha(1-\varepsilon)+(1-\alpha)\varepsilon}\alpha$
(C, d)	$(1-\varepsilon)\frac{\varepsilon}{\alpha\varepsilon+(1-\alpha)(1-\varepsilon)}\alpha$
(D, c)	$\varepsilon\frac{1-\varepsilon}{\alpha(1-\varepsilon)+(1-\alpha)\varepsilon}\alpha$
(D, d)	$\varepsilon\frac{\varepsilon}{\alpha\varepsilon+(1-\alpha)(1-\varepsilon)}\alpha$

Aside for the (C, c) case, the probability that j will play G is of order ε. So, for sufficiently small values of ε, i will be willing to following the prescribed strategy in the second period.

Now consider i's incentives in the first period. Her expected payoffs are:

$$\begin{aligned} \text{Play } C &\Rightarrow \alpha + (1-\alpha)(-1) + \alpha(1-\varepsilon)^2(k-1) + 1 \\ \text{Play } D &\Rightarrow 2\alpha + 1 \end{aligned}$$

We can make her just indifferent between C and D by setting:

$$\alpha = \frac{1}{(k-1)(1-\varepsilon)^2}.$$

Note that a key to the randomization equilibrium is that player i conditions his second period behavior on the result of his first period randomization. Because of this, player j's first period signal is informative about i's second period behavior. This means that player j will want to condition his second period action on his first period signal, which means in turn that some incentive can be provided for i to cooperate in the first period.

2 Private Monitoring: Different Approaches

The example above illustrates some of the issues that arise with private monitoring. Let's now consider some more general approaches to the repeated game problem, and what has been shown. Most of the existing papers focus on infinite repetitions of the prisoner's dilemma (with $l \geq g > 0$):

	C	D
C	$1, 1$	$-l, 1+g$
D	$1+g, -l$	$0, 0$

.

2.1 Failure of Grim Trigger Strategies

An early paper by Compte (2002, originally 1996) shows a strong negative result. Compte argues that if private signals are independent conditional on the action profile that was played (i.e. conditionally independent), then grim trigger strategies won't work to support cooperation. The problem is that when i observes a signal that is supposed to trigger entry to the punishment phase, she will be very reluctant to initiate punishment unless she assigns high probability to j having already entered the punishment phase. But if she assigns high probability to j having already entered the punishment phase, she would not have been willing to play C to begin with. Indeed, similar to the example above, it can be shown that with conditionally independent signals, the only pure strategy equilibrium is to play D in every period. The approaches below will try to get around this in different ways.

2.2 "Almost Public" Monitoring

Mailath and Morris (2002) explore what might happen if there is private monitoring, but the signals are (highly) correlated. Their idea is to start by considering a public monitoring technology and some *strict* perfect public equilibrium under that technology (a PPE is strict if after every history, each player strictly prefers her specified action to any other). Mailath and Morris then ask whether such a perfect public equilibrium is *robust* to the introduction of a small amount of private monitoring. Let's consider their argument.

Recall that a public monitoring technology has a set of possible signals Y and a probability distribution over those signals conditional on the actions played, $\rho(y|a)$. Assume this distribution has *full support.* With private monitoring, each player i observes a signal $y_i \in Y$, and there is a probability distribution over signals $\pi(\mathbf{y}|a)$. The private monitoring technology is said to be *ε-close* to ρ if for all a and $y \in Y$, $|\pi(y, .., y|a) - \rho(y|a)| < \varepsilon$. This is MM's notion of "almost public" monitoring.

In the prisoner's dilemma, an example of a public monitoring technology has $Y = \{\underline{y}, \overline{y}\}$, and

$$\rho\left(\overline{y} \mid a_1, a_2\right) = \begin{cases} p & \text{if } (C, C) \\ r & \text{otherwise} \end{cases}.$$

Mailath and Morris consider strategies that can be represented as finite automata. In the case of the prisoners' dilemma, let's consider strategies defined by a two-state automaton with state space $W = \{w_C, w_D\}$, where i

is supposed to play C in state w_C and D in state w_D. A symmetric strategy profile is defined by an initial state, and a transition function $\sigma(yw)$. For example, grim trigger has initial state w_C, and transition $\sigma(yw) = w_C$ if and only if $w = w_C$ and $y = \overline{y}$. Mailath and Morris also consider the following example, where the initial state is w_C and

$$\sigma(yw) = \begin{cases} w_C & \text{if } y = \overline{y} \\ w_D & \text{if } y = \underline{y} \end{cases}.$$

The strategies that result are a strict equilibrium of the public monitoring game if $\delta \geq \frac{1}{3(p-r)}$. Checking this requires showing that if i, j are in state w_C, then i strictly wants to play C, while if i, j are in state w_D, i strictly prefers D.

When we introduce private monitoring, it is natural to think of each player having her own *private* state $w_i \in \{w_C, w_D\}$, where i is supposed to play C if $w_i = w_C$ and play D if $w_i = w_D$. We can consider the same strategies — same initial state and transitions functions — only now the transitions operate on the private states, so $\sigma(y_i w_i)$. To check the relevant incentives, we need to know the probability that i will assign to her opponent being in different states after each history. (Of course, with public monitoring, i and j are always in the same state). If, regardless of what has happened, i believes that her opponents are in the same state as she is with very high probability (say $1-\zeta$), she will not want to deviate given that the original equilibrium was strict (since the private monitoring will just add a negligible term of order ζ to each side of the incentive constraint). MM try to identify conditions on strategies that will ensure that players end up almost certain of being in the same state at every point in time.

Interestingly, it turns out that one condition that ensures the necessary correlation across private states is that strategies depend only on a finite history of signals. The transition given above depends only on the last signal, so it satisfies this condition, and hence the strategies remain an equilibrium with almost-public monitoring. Grim trigger, however, depends on the whole infinite history of play. MM point out that with perfect monitoring, one can prove the folk theorem with finite history strategies. Thus, they obtain a folk theorem for games with almost-public, almost-perfect monitoring.

2.3 Initial Randomization

While Mailath and Morris try to identify which public monitoring strategies can be transported directly into games with a little bit of private monitoring, Sekiguchi (1997) explicitly constructs strategies for the repeated prisoner's dilemma with private monitoring, and shows that they allow cooperation provided that the monitoring is almost-perfect. Sekiguchi requires both players to randomize at the beginning of the game between playing grim trigger, and playing defect in every period. It turns out this is an equilibrium for intermediate discount factors provided monitoring is sufficiently accurate.

Sekiguchi has to tacle two issues to verify the equilibrium. First, if i has been cooperating, and has always received good signals, will she still want to cooperate? It turns out she will if δ is high enough and monitoring is sufficiently accurate. Second, will i want to defect as soon as a bad signal is received? If she receives the bad signal early on (e.g. the first period), she can conclude that her opponent started with the always defect strategy, so she is happy to defect. On the other hand, if she receives the defect signal only after some time, the most likely events are that either *(i)* the signal is erroneous, or *(ii)* j received a bad signal in the previous period and has just entered the punishment phase. These events have equal probability, but Sekiguchi manages to show that if δ is not too high, i will be willing to experiment by defecting. Thus, he obtains cooperation with almost-perfect (but not necessarily almost-public) monitoring.

2.4 Belief-Free Equilibria

Piccione (2002) and Ely and Valimaki (2002) consider a very different approach that is based on ongoing randomization. These authors observe that the above approaches require keeping track of players beliefs about what the others are doing over time. This can be very complicated. Their approach is to construct equilibria for games with almost-perfect (but not necessarily correlated) signals in which the players are *indifferent* between C, D at *every point in time.*

As an example, consider the repeated prisoner's dilemma:

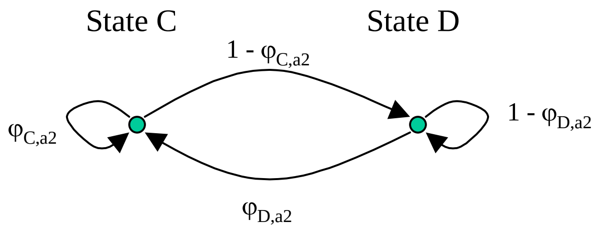
	C	D
C	$1,1$	$-1,2$
D	$2,-1$	$0,0$

where, after each period, player i observes a signal $Y_i \in \{c, d\}$ and:

$$\Pr(Y_i = c) = \begin{cases} 1-\varepsilon & \text{if } a_i = C \\ \varepsilon & \text{if } a_i = D \end{cases}$$

The P-EV approach is to construct sequential equilibria in which each player are *indifferent* between C, D at *every point in time* – i.e. after every possible history.

To see how this might be done, consider the perfect monitoring version of the repeated prisoner's dilemma ($\varepsilon = 0$). Following EV, consider strategies as follows. Player i plays C in the first period, and in any subsequent period t, plays a mixed strategy that depends only on the outcome at $t-1$. Let $\varphi^i_{a_i,a_j}$ denote the probability she plays C at t conditional on outcome (a_i, y_i) at $t-1$. We can think of this strategy as a two-state machine.



EL Machine Strategy for Player 1

EV claim that for any pair $(v_1, v_2) \in V = (0,1]^2$, there are strategies that achieve the pair. To see this, fix values $V^i_C, V^i_D \in (0,1]$, with $V^i_C > V^i_D$. We now show that we can find probabilities φ^j, such that (i) in any period where j plays C, i is indifferent between C, D and obtains continuation payoff V^i_C, and *(ii)* in any period where j plays D, i is indifferent between C, D and obtains continuation payoff V^i_D. This requires finding four probabilities $\varphi^j \in [0,1]$ to satisfy the four equations:

$$\begin{aligned}
V^i_C &= (1-\delta) + \delta\left[\varphi^j_{cc} V^i_C + (1-\varphi^j_{cc}) V^i_D\right] \\
V^i_C &= (1-\delta)2 + \delta\left[\varphi^j_{cd} V^i_C + (1-\varphi^j_{cd}) V^i_D\right] \\
V^i_D &= (1-\delta)(-1) + \delta\left[\varphi^j_{dc} V^i_C + (1-\varphi^j_{dc}) V^i_D\right] \\
V^i_D &= (1-\delta)0 + \delta\left[\varphi^j_{dd} V^i_C + (1-\varphi^j_{dd}) V^i_D\right]
\end{aligned}$$

These can be solved if δ is near 1. In particular, note that:

$$\begin{aligned}
\delta(\varphi_{dc} - \varphi_{dd})\left[V^i_C - V^i_D\right] &= (1-\delta) \\
\delta(\varphi_{cc} - \varphi_{cd})\left[V^i_C - V^i_D\right] &= (1-\delta).
\end{aligned}$$

As an example, let $V_C^i = 1$ and $V_D^i = 1/2$, and $\delta = 0.9$. Then we need:

$$\varphi_{cc} - \varphi_{cd} = \varphi_{dc} - \varphi_{dd} = \frac{2}{9}$$

and also from the first and last value function equations:

$$\begin{aligned} 1 &= 0.1 + 0.9 \cdot 0.5 \cdot (1 + \varphi_{cc}) \\ 0.5 &= 0.9 \cdot 0.5 \cdot (1 + \varphi_{dd}) \end{aligned}$$

which gives:

$$\begin{aligned} \varphi_{cc} &= 1 \\ \varphi_{cd} &= 7/9 \\ \varphi_{dc} &= 3/9 \\ \varphi_{dd} &= 1/9 \end{aligned}$$

Similar calculations can be done for any $1 \geq V_C > V_D \geq 0$ (provided we can pick δ near enough to 1). So any payoff in the unit square can be obtained in an equilibrium where players are indifferent between actions after every history.

Essentially the same construction continues to work if monitoring is imperfect ($\varepsilon > 0$). To prove a folk theorem for the Prisoners' Dilemma, P–EV must show that payoffs that are in the feasible strictly individually rational set, but not in the unit square can also be obtained. To do this they add a preliminary phase of finite length and support different payoffs in this preliminary phases with different continuation payoffs from the unit square. This gives the folk theorem for the case where monitoring is nearly perfect.

Matsushima (2002) uses the P-EV belief-free approach to show that if signals are conditionally independent, a Folk Theorem holds in the prisoner's dilemma even if monitoring is not nearly perfect. More recently, Ely, Horner and Olszewski (2004) characterize the entire set of payoffs that can be obtained with a belief-free equilibrium. They show that these equilibria cannot in general generate all feasible and individually rational payoffs even when players are very patient. So proving a general Folk Theorem for games with private monitoring will require something more.

3 Comments

1. Ely, Horner and Olszewski show that belief-free equilibria can be characterized in a manner similar to APS self-generation, which lends them

some appeal. A natural question, however, is whether these equilibria are fragile, in the sense that players may have to alter their mixing behavior depending on the history of play even if their continuation payoffs do not change. Bhaskar, Mailath and Morris (2004) alleviate this concern somewhat by showing that in the prisoner's dilemma case, it is possible to purify the EV equilibria.

2. Compte (1998) and Kandori and Matsushima (1998) suggest a very different approach to private monitoring. They allow the players to communicate through cheap talk following each period. They study perfect public equilibria where the players condition on the public announcements. Both papers prove elegant extensions of the FLM folk theorem to private monitoring games with communication.

3. Horner and Olszewski (2006) prove the most general Folk Theorem to date, for general games with private almost-perfect monitoring. Their approach is to break the game into T-period blocks and construct an equilibrium where player's beliefs matter only within each block but not across blocks. The equilibrium construction in their paper is not a trivial accomplishment.

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Reputation in Repeated Interaction

Jonathan Levin

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We now investigate the idea that if there is some uncertainty about strategic motives, a player who plays a game repeatedly may be able to capitalize on it by building a reputation for a certain sort of behavior. As we will see, these effects can be quite powerful in some situations — even a small amount of uncertainty about a player's intentions can allow the player to completely dictate how the game will be played.

1 Examples

Reputation for Quality A firm that produces an experience good faces a series of customers. Each decides whether or not to purchase; upon purchase, the firm decides on high or low quality $q \in \{0, 1\}$. If a consumer doesn't purchase, both firm and customer get 0. If the customer buys, he gets $vq-p$, while the firm gets $p - cq$. We assume that $v > c$, so that high quality is efficient.

Our intuition says that the firm should be able to establish a reputation for high quality by actually producing high quality a few times. However, in the one-shot game or in a finite repetition, backward induction tells us that the firm will always produce low quality, and hence that the customer won't buy the good. And in the infinite horizon game, the folk theorem says that essentially anything can happen.

Chain Store Game For this model, imagine a drug company being sued by patients who claim to have been harmed by a drug. There is a series of potential litigants. If sued, the drug company can either fight of settle. If a litigant doesn't sue, she gets zero, while the company gets 2. If she sues and the company concedes, both get 1. If she sues and the company fights, both get -1.

Again, it seems reasonable that the drug company will fight to establish a reputation for being hard to sue. But in a finite version of this game, it will settle every suit from the start. And in an infinitely repeated version, anything can happen.

Monetary Policy Reputational models are often used to capture central bank behavior. The canonical problem is that a central bank wants agents to believe it will have a tight money supply (in order to keep inflation down), but once agents have set prices, the bank is tempted to raise money supply to boost output. Thus, central bankers need to establish a reputation for being tough on inflation.

Credible Advice. Consultants, political advisors, and others would often like to build a reputation for giving unbiased advice. However, they may have certain biases, or may just be worried about being labeled as biased or extremist. The question is whether such advisors can successfully acquire a reputation for giving unbiased credible advice.

2 Reputation as Commitment

We start with a fairly general model with one long-run player facing a sequence of small opponents. The development here follows Fudenberg and Levine (1989). I focus on finitely-repeated games, but the same argument applies to infinitely repeated games if the long-run player has a sufficiently high discount factor.

Let G be a two-player simultaneous move game. Suppose that players $1, 2$ have action sets A_1, A_2, and sssume that $BR_2(a_1)$ is unique for all $a_1 \in A_1$. We consider a T-period repetition of this game with incomplete information. Prior to play, nature chooses a type $\theta \in \Theta$ for player 1, and then player 1 meets a sequence of player 2, in periods $t = 1, 2, ..., T$. Suppose that Θ consists of:

- A "rational" player 1, who maximizes

$$\frac{1}{T}\sum_{t=1}^{T} u_1(a_1, a_2)$$

- For each $a_1 \in A_1$, a type θ^{a_1}, who plays a_1 in every period.

Assume that Nature chooses $\theta \in \Theta$ with probability $\mu(\theta) > 0$ for all θ.

Proposition 1 *For all $\varepsilon > 0$, there exists $\underline{T}$ such that for all $T \geq \underline{T}$, all perfect bayesian equilibria of the T-period game with incomplete information give the rational player a payoff of at least:*

$$\max_{a_1 \in A_1} u_1 \left(a_1, BR_2(a_1)\right) - \varepsilon.$$

Proof. Define

$$a_1^* = \arg \max_{a_1 \in A_1} u_1(a_1, BR_2(a_1)).$$

Pick $\alpha > 0$ such that if $\sigma(a_1^*) \geq 1 - \alpha$, then $BR_2(\sigma) = BR_2(a_1^*)$. Also, note that following any history h_t, the following identity applies:

$$\mu_2 \left(\theta^{\text{Rat}} | h_t\right) + \mu_2 \left(\theta^{a_1^*} | h_t\right) = 1$$

Now consider some PBE where player 2 doesn't play $BR_2(a_1^*)$ following a history h_t. It must be that $\sigma \left(a_1^* | h_t, \theta = \theta^{\text{rat}}\right) < 1 - \alpha$. Consider the updating that player 2 would do if she saw h_t followed by a_1^*. Using Bayes rule,

$$\frac{\mu_2 \left(\theta^{\text{rat}} | h_{t+1}\right)}{\mu_2 \left(\theta^{a_1^*} | h_{t+1}\right)} < (1 - \alpha) \frac{\mu_2 \left(\theta^{\text{rat}} | h_t\right)}{\mu_2 \left(\theta^{a_1^*} | h_t\right)}$$

Let k be such that

$$\mu \left(\theta^{\text{rat}}\right) (1 - \alpha)^k < \alpha \mu \left(\theta^{a_1^*}\right).$$

If player 1 follows the strategy of playing a_1^* in every period, there can be at most k periods in which player 2 does not play $BR_2(a_1^*)$ because after that point player 2 will assign probability at least $1 - \alpha$ to type $\theta^{a_1^*}$, and then will play $BR_2(a_1^*)$ as a best-response.

Now, pick $\underline{T}$ such that:

$$(\underline{T} - k) \, u_1 \left(a_1^*, BR_2(a_1^*)\right) + k \min_{a_2} u_1 \left(a_1^*, a_2\right) > \underline{T} \left[u_1 \left(a_1^*, BR_2(a_1^*)\right) - \varepsilon\right]$$

Thus, if $T > \underline{T}$, player 1 can always ensure a payoff $u_1 \left(a_1^*, BR_2(a_1^*)\right) - \varepsilon$ by deviating, and hence must get at least this in equilibrium. *Q.E.D.*

Note that when the game is played, player 1 does not actually "build" a reputation. That is, as the game goes on, opponents do not put more weight on player 1 being committed. Rather, they know that even if he is rational, player 1 will act as if he is committed. One point to note here is that the when $\Pr(\theta^{a_1^*})$ is small, then we may need to pick $\underline{T}$ very large.

You might think that it is important here for the short-run players to be able to observe perfectly what the long-run player is doing. In fact, that need not be the case. Fudenberg and Levine (1992) show that a version of their "bounds" argument applies even if the long-run player's behavior is imperfectly observed. Basically the idea is that short-run players will be actively learning about the long-run player's behavior over time, with some outcomes making it more likely that the long-run player is a Stackelberg type and some making it less likely. A "normal" type will have an incentive to play non-Stackelberg actions, but Fudenberg and Levine show that as the horizon becomes sufficiently long, the normal types will still mimic the Stackelberg type to get a payoff nearly equal to the Stackelberg payoff. Cripps, Mailath and Samuelson (2003) study the long-run, or asymptotic, pattern of behavior in this model.

3 Many Long-Run Players

Fudenberg and Levine's result shows that if there is a single long-run player and a sequence of short run players, then the possibility of building a reputation completely determines how the game will be played (or at least the payoffs). Fudenberg and Maskin (1986) show that if there are two long-run players and both are potentially "crazy" then anything can happen. Their result extends Kreps et al.'s (1982) famous paper showing that cooperation was possible in a repeated prisoner's dilemma where there was some chance one player was committed to tit-for-tat.

Proposition 2 *Let G be a two-player game. Suppose that $(v_1, v_2) \in V^*$. Then for any $\varepsilon > 0$, there exists $\underline{T}$ and a form of behavior for crazy types, θ_1^c, θ_2^c such that in the T-period game with* $\Pr\left(\theta_1 = \theta_1^{rational}\right) = \Pr\left(\theta_2 = \theta_2^{rational}\right) = 1 - \varepsilon$, *and* $\Pr\left[\theta_1 = \theta_1^c\right] = \Pr\left[\theta_2 = \theta_2^c\right] = \varepsilon$ *there exists a PBE in which the rational players' average payoffs are within ε of (v_1, v_2).*

Proof. (with some loss of generality) Suppose that (v_1, v_2) pareto dominate some Nash equilibrium s^* of G with payoffs (e_1, e_2). And assume that there is some (a_1, a_2) such that $u_i(a_1, a_2) = v_i$. Suppose the crazy types play a_i until someone plays something other than (a_1, a_2), and then plays s_i^* ever after. Consider the strategy for rational types:

I. In periods $1, 2, ..., T - \hat{T}$, play a_1 so long as no one deviates.

II. If someone deviated from phase I, play s^* for the rest of the game.

III. If no one deviated from phase I, then from $T - \hat{T} + 1$ on, play some PBE σ^* of the $\hat{T}$-period game, which has the property that s^* is played whenever players 1 and 2 are known to be rational.

We want to show this is a PBE. Clearly, there are no profitable deviations in phase III for rational types since this is a PBE, and similarly in phase II since this is a NE in every period. So consider phase I.

Before $T - \hat{T}$, if rational player i deviates, he gains at most $\overline{v} - v_i$ for one period, but loses $v_i - e_i$ in periods $t+1, ..., T - \hat{T}$, and loses something more in the last $\hat{T}$ periods. How much does he lose in the last $\hat{T}$ periods? In these last $\hat{T}$ periods, if the rational player has not deviated before $T - \hat{T}$, he could follow the strategy: play a_i until his opponent plays something other than a_{-i}, then play something random to reveal his rationality, then play s_i^* until the end. This gives a payoff of at least:

$$(1-\varepsilon)\left(2\underline{v} + (\hat{T} - 2)e_i\right) + \varepsilon \hat{T} v_i$$

On the other hand, if rational i deviates in the first $T - \hat{T}$ period, he will get $\hat{T} e_i$ in the last $\hat{T}$ periods. The loss in the last $\hat{T}$ periods from deviating in Phase I is thus at least:

$$\varepsilon \hat{T}(v_i - e_i) - (1-\varepsilon)2(e_i - \underline{v})$$

Taking $\hat{T}$ large enough, we can ensure that this is greater than or equal to $\overline{v} - v_i$. Then rational i will not deviate in phase I. Finally, since $\hat{T}$ is now fixed, we can simply take T large enough to make the payoffs from this equilibrium be within ε of (v_1, v_2). *Q.E.D.*

1. The result says that even in long finite games, essentially "anything can happen" if there is a small chance players aren't rational. It suggests that (i) the backward induction solution to these games may not be "robust" to the introduction of particular types of irrationality, and also that long finite games may behave much like infinitely repeated games.

2. Of course, the full strength of the result depends on being able to find just the right kind of crazy type. The result depends crucially on this. FM suggest that some kinds of craziness may be more reasonable than others.

3.1 Extensions

The two results we have shown have an interesting contrast in that the equilibrium payoffs are sharply constrained if there is just a single long-run player who may be committed, but not constrained at all if there are two long-run players who may be committed. A natural question is whether there are situations where payoffs can be pinned down by reputation even with two long-run players.

- Schmidt (1993) observes that with two long-run players, it is hard to use reputation to pin down payoffs. The reason is that once both players have revealed to be rational, the folk theorem kicks in and anything can happen. The problem spills over to situations where one player has revealed rationality. Schmidt identifies a class of games with conflicting interest payoffs where a Fudenberg-Levine type result applies when one player is much more patient than the other.

- Abreu and Pearce (2003) consider infinitely repeated two-player games, where at the outset players can announce a repeated game strategy and with $\varepsilon \to 0$ probability remain committed to it. They ask whether in such a game, there exists a payoff profile (v_1, v_2) such that if, in all subgames where rationality has been revealed, play yields (v_1, v_2), then (v_1, v_2) will be the payoff that results from the start of play. They show that there is exactly one such payoff vector, which remarkably, coincides with the solution to the Nash demand game with endogenous threats.

4 Bad Reputation

Our analysis suggests that a long-run player interacting with a sequence of a short-run players will typically benefit from reputation effects since he has the option of acting committed to a certain strategy. But this need not be the case if there is imperfect monitoring! We now consider an example of Ely and Valimaki (2003) in which reputation has perverse consequences.

The main character is a mechanic who interacts with a motorist. The motorist's car needs either a tune-up or an engine replacement with equal probability. Denote these possibilities as $\theta \in \{\theta_t, \theta_e\}$. The motorist can't tell which is needed, but if he hires the mechanic, the mechanic will be able to tell. The mechanic will then choose a repair $a \in \{t, e\}$. The motorist's payoff depends on the treatment and the state:

	θ_t	θ_e
t	u	$-w$
e	$-w$	u

We assume that $w > u > 0$ and that the motorist has some outside option that gives a payoff zero.

Let (β_t, β_e) denote the probability hte mechanic will perform the right repair in each of two states, i.e. β_a is the probability of repair a in state θ_a. The motorist's payoff from hiring is then:

$$-w + \frac{1}{2}(\beta_t + \beta_e)(u + w)$$

The motorist can get ensure zero by not hiring, so a *necessary* condition for hiring is that $\beta_t, \beta_e \geq \beta^* = \frac{w-u}{w+u}$.

In the benchmark case, we assume the mechanic is "good" and has the same preferences as the motorist. In the unique sequential equilibrium of the one-shot interaction, the motorist will hire the mechanic and he'll do the right repair. This remains essentially correct even if there is a small probability μ that the mechanic is a "bad" type who always replaces the engine. Given that the good mechanic does the right thing, the motorist's expected payoff is:

$$-w + (1 - \mu)(u + w) + \mu\frac{1}{2}(u + w) = u - \mu\frac{1}{2}(u + w).$$

Thus if $\mu \leq p^* = \frac{2u}{u+w}$, the motorist will hire the mechanic in a one-shot interaction and the good mechanic will do the right thing.

We now investigate the idea that even if motorists assign only a small probability of the mechanic being bad, this can distort the reputational incentives of a good mechanic in such a way that the motorist may not want to hire. To model this, we imagine an infinite sequnce of motorists who decide in turn whether to hire the mechanic after observing earlier repairs (but not what was actually needed).

In this game, a bad mechanic always chooses $a = e$. The good mechanic maximizes his average payoff using discount factor $\delta \in (0, 1)$. His strategy specifies for each date k and history h_k the probabilities $\beta_t^k(h^k)$ and $\beta_e^k(h^k)$ of doing the right repair.

Since each motorist is a short run player, motorist k will want to hire given history h^k if she expects the right repair with sufficient probability. Her decision will be based on the probability $\mu^k(h^k)$ she assigns to the mechanic being bad and the expected behavior β_t^k, β_e^k of the good mechanic. Of course if $\mu^k(h^k) > p^*$, she will certainly choose not to hire, and if $\mu^k(h^k) \leq p^*$ a necessary condition for her to hire is that $\beta_t^k, \beta_e^k \geq \beta^*$.

EV's surprising result is that if the mechanic is sufficiently patient, then he will be unable to realize positive profits over the course of the game even if μ is quite small. Specifically, let $\overline{V}(\mu, \delta)$ denote the supremum of the mechanic's Nash equilibrium average payoffs in the game with discount factor δ and prior μ.

Proposition 3 *For any* $\mu > 0$*,* $\lim_{\delta \to 1} \overline{V}(\mu, \delta) = 0$*.*

Proof. To begin, note that if $\mu > p^*$, there is a unique Nash equilibrium in which the mechanic is never hired. The first motorist does not hire, so beliefs are not updated. The second motorist then doesn't hire and so on.

Suppose that $\mu \leq p^*$ amd consider a Nash equilibrium in which the mechanic is hired in the first period. The updated probability of being bad depends on the first period behavior. In particular,

$$\mu^1(t) = 0$$

and

$$\mu^1(e) = \frac{\mu}{\mu + (1-\mu)\left(\frac{1}{2}\beta_e + \frac{1}{2}(1-\beta_t)\right)}.$$

Recall that the mechanic will only be hired if $\beta_t \geq \beta^* > 0$. So $\mu^1(e) > \mu > \mu^1(t) = 0$. Note that $\mu^1(e)$ is lower (i.e. the motorist has a better opinion of the mechanic following an engine replacement) when either β_t is low or β_e is high. Letting $\beta_t = \beta^*$ and $\beta_e = 1$, define

$$\Upsilon(\mu) = \frac{\mu}{\mu + (1-\mu)(1 - \frac{1}{2}\beta^*)}$$

to be the smallest possible posterior probability of a bad mechanic given an engine replacement and prior μ. We know that $\Upsilon(\mu) > \mu$ for all $\mu \in (0, p^*)$. Also Υ is continuous and strictly increasing in μ.

Now, let $p_1 = p^*$ and define p_m such that $\Upsilon(p_m) = p_{m-1}$. Under this definition, if the prior $\mu \geq p_{m+1}$, and an engine replacement is observed, then the posterior will be *at least* p_m. The sequence $p_1, p_2, ...$ is strictly decreasing and $\lim_{m \to \infty} p_m = 0$. We will now use an induction argument on m to bound the mechanic's Nash equilibrium payoffs as $\delta \to 1$.

We have already seen that if the prior μ exceeds p^*, the mechanic gets zero payoff. For the induction step assume that if the prior μ exceeds p_m, the mechanic's payoff is bounded above by some $\overline{V}_m(\delta)$ with $\lim_{\delta\to 1}\overline{V}_m(\delta) = 0$. To complete the induction argument, assume that the prior μ exceeds p_{m+1} and consider a Nash equilibrium where the mechanic is hired in the first period (this is wlog since if he isn't hired until the second period, the game starting in the second period has the same prior and a higher payoff).

Since in the first period the mechanic must be choosing the correct action with probability at least β^* in each state (otherwise the motorist wouldn't hire him), his payoff is bounded above by his payoff from choosing the correct action with probability 1 in each state (either this is his optimal strategy or he mixes in which case it's one of his best responses). Letting $z(a|\tau)$ denote the continuation payoff if the state is τ and the good mechanic chooses action a:

$$\overline{V}(\mu,\delta) \leq (1-\delta)u + \delta\frac{z(e|e)+z(t|t)}{2} \tag{1}$$

We have assumed that $\mu > p_{m+1}$, so if the mechanic chooses e, then $\mu^1(e) > p_m$ and hence by the induction assumption:

$$z(e|e) \leq \overline{V}_m(\delta).$$

Now, consider the incentive compatibility constraint for the mechanic. He must be willing to choose e when the state is actually e rather than deviating to t. So

$$(1-\delta)u + \delta z(e|e) \geq -(1-\delta)w + \delta z(t|e)$$

or, re-arranging:

$$z(t|e) \leq \frac{1-\delta}{\delta}(u+w) + z(e,e) \tag{2}$$

Combining the inequalities:

$$\overline{V}(\mu,\delta) \leq (1-\delta)\frac{3u+w}{2} + \delta\overline{V}_m(\delta) = \overline{V}_{m+1}(\delta).$$

Since $\lim_{\delta\to 1}\overline{V}_m(\delta) = 0$, then evidently $\lim \overline{V}_{m+1}(\delta) = 0$. By induction, this holds for all m, so we have shown that $\lim_{\delta\to 1}\overline{V}(\mu,\delta) = 0$ for any μ that is greater than some p_m. Since $\inf_m p_m = 0$, the proof is complete. *Q.E.D.*

Intuitively, the problem is that once there is a sufficiently high belief that the mechanic is bad, motorists will stop hiring and the game will effectively end. This means that if beliefs are such that the mechanic is only one engine replacement away from this region, and he cares about future payoff enough, he will be exceedingly averse to doing a replacement today even if one is needed — *unless the continuation payoff from a tune-up is also exceedingly low.* The dilemma is that if the mechanic is not willing to do a replacement, the motorist will anticipate this and refuse to hire him because she only wants to hire if he's willing to do engine replacements when they're needed. Thus the only way the mechanic will be hired when she's one e away from being fired forever is if the continuations from a tune-up are also close to zero. If δ is high, this means the overall expected payoff in this region must be very low. But now, this expands the region of beliefs that the mechanic wants to avoid and creates a new region where he's just one engine replacement away from a bad belief region. Sadly for the mechanic, this unraveling continues until we've shown that his payoffs must be low for any prior belief!

4.1 Comments

- The result doesn't imply the motorist will never get hired. For instance, if $\mu < p^*$, there is a Nash equilibrium in which the first motorist hires the mechanic, but no future motorist ever hires him regardless of what he does with the car. Since future motorists ignore his behavior, the mechanic will do the right thing in the first period. The problem with this equilibrium is that if the second motorist see t, she'll believe with probability 1 that the mechanic is good. So it seems quite unreasonable for her not to hire.

- A key problem here is that each motorist is a short-run player. She does not internalize the benefits she creates for later motorists if she hires and gives the mechanic a chance to signal his goodness by performing a tune-up. EV show that if there is a single long-run motorist, there is an equilibrium that essentially gets the first-best outcome, even if the bad mechanic may be a strategic player who can try to imitate a good mechanic rather than automatically choosing e each period.

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Bargaining and Reputation

Jonathan Levin

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These notes consider Abreu and Gul's (2000) reputational model of bargaining. In the 1980s, following Rubinstein's paper, there was a big push to study sequential offer models of bargaining with incomplete information. The idea of these models is that (as in Rubinstein) engage in a series of monopoly offers. A central observation of this work was that the specific structure of the game (who makes the offers, and how much time elapses between offers), and the patience of the players was critically important. Abreu and Gul's model stresses the "strategic posture" of players who with slight probability may be committed to holding out for a certain share of the pie. The resulting model looks more like a war of attrition.

1 The Model

We will look at a simplified version of Abreu and Gul's model. There are two agents, who bargain over a pie of size 1. At time 0, player 1 makes an initial demand $a_1 \in (0,1)$. Player two then makes a demand $a_2 \in (0,1)$. If $a_1 + a_2 \leq 1$, the game ends immediately, if $a_1 + a_2 \geq 1$, we proceed to a concession game. The concession game takes place in continuous time $t \in [0,\infty)$. At each point in time, both players choose whether to concede or to hold out. If i concedes, he receives $1 - a_j$, while if his opponent j concedes, he receives a_i.

Each player may be either rational or, with probability z_i, irrational. If i is irrational, he insists from the start on a particular demand α_i. We suppose that $\alpha_i + \alpha_j > 1$, so if i and j are both irrational, they hold out forever and never agree. If player i is rational, he has discount rate r_i, so if agreement is reached at time t, and he gets a share a, his payoff is $e^{-r_i t}a$.

1.1 The Concession Game

Let us first consider the concession game that arises if, at time zero, both players always make their irrational demands α_1, α_2. We describe i's behavior in the concession game by a probability distribution over stopping times, $F_i(t) = \Pr[i \text{ will concede prior to } t]$, where we allow $F_i(0) > 1$, so i may concede immediately with positive probability.

Suppose player j's behavior is given by $F_j(t)$. We look for an equilibrium where player i mixes between conceding and not conceding. For a rational i to be indifferent, it must be that:

$$r_i (1 - \alpha_j) = (\alpha_i - (1 - \alpha_j))_i \frac{f_j(t)}{1 - F_j(t)} = \alpha_i \lambda_j(t).$$

A similar equation holds for j. Thus, in such an equilibrium,

$$\lambda_i(t) = \lambda_i = \frac{r_j(1 - \alpha_i)}{\alpha_i + \alpha_j - 1}.$$

Integrating up the hazard rate gives

$$F_i(t) = 1 - (1 - F_i(0))\, e^{-\lambda_i t}.$$

If i does not concede with positive probability at time 0,then $F_i(0) = 0$ and

$$F_i(t) = 1 - e^{-\lambda_i t}.$$

Now, observe that if i is irrational, he will never concede. It follows that $F_i(t) \leq 1 - z_i$ for all t. Define T_i to be the value of t that solves $1 - e^{-\lambda_i t} = 1 - z_i$,

$$T_i = -\frac{1}{\lambda_i} \log z_i.$$

Note that $T_i > T_j$ if and only if $\lambda_i < \lambda_j$. Let $T = \min\{T_1, T_2\}$.

Proposition 1 *There is a unique sequential equilibrium to the concession game, described as follows:*

- *If $\lambda_i \geq \lambda_j$, then i never concedes immediately, and concedes between $(0, T]$ at constant rate λ_i.*
- *If $\lambda_i < \lambda_j$, then i concedes immediately with probability $1 - z_i z_j^{-\lambda_i/\lambda_j}$, and concedes between $(0, T]$ at constant rate λ_i.*

- *After time T, both players are known to be irrational and never concede.*

Proof. (Sketch) The proof proceeds by observing that any sequential equilibrium pair (F_1, F_2) must have the following properties.

(i) A rational player will not hesitate to concede once he knows his opponent is irrational. Thus either $F_i(t) < 1 - z_i$ for all t, and $i = 1, 2$ or there is some $T < \infty$ such that $F_i(t) < 1 - z_i$ for all $t < T$ and $F_i(T) = 1 - z_i$ for $i = 1, 2$.

(ii) If F_i jumps at t, then F_j is constant at t. The reason is that j would always want to incur the rdt loss from waiting in order to enjoy the discrete chance of i conceding.

(iii) If F_i is constant between (t', t''), then so is F_j. If i will not concede between (t', t''), then if j plans to concede in this interval, he does better to concede immediately at t' rather than wait to some time $t > t'$.

(iv) There is no interval (t', t'') with $t'' < T$ on which F_i and F_j are constant. If so, i would do better to concede at $t'' - \varepsilon$ than to concede at t'', leading to contradiction.

From *(i)–(iv)* it follows that F_i, F_j will be continuous and strictly increasing on $[0, T]$ or $[0, \infty)$. But if both are conceding with positive probability, then they must be conceding at hazard rates λ_i, λ_j as defined above, so $F_i(t) = 1 - (1 - F_i(0))e^{-\lambda_i t}$ as defined above. This rules out the latter case (where concession goes on indefinitely). Finally, the fact that both must stop conceding at the same time, so $F_i(t)$ must reach $1 - z_i$ at the same time T as $F_j(t)$ reaches $1 - z_j$, means that at least one of $F_i(0)$,$F_j(0)$ is strictly positive. By *(ii)*, they can't both be positive. So then $T = \min\{T_1, T_2\}$. But then, if $T_i \leq T_j$,

$$F_i(0) = 1 - e^{\lambda_i T_i} z_i = 0$$

while if $T_i > T_j$,

$$F_i(0) = 1 - e^{\lambda_i T_j} z_i = 1 - z_j^{-\lambda_i/\lambda_j} z_i.$$

So the unique equilibrium must be as described. *Q.E.D.*

Remark 1 *Note that $F_i(t)$ gives the probability i will concede prior to t. The probabilility that i will concede prior to t given that he is rational is higher, equal to $F_i(t)/(1 - z_i)$.*

1.2 Properties of Equilibrium

Equililibrium payoffs for a rational player i are

$$u_i = F_j(0)\alpha_i + (1 - F_j(0))(1 - \alpha_j)$$

Equilibrium has several interesting properties.

1. Bargaining is inefficient because there is delay in reaching agreement. For instance, if the model is symmetric, so $\alpha_i = \alpha_j$ and $r_i = r_j$, then $\lambda_i = \lambda_j$ and so $F_i(0) = F_j(0) = 0$. However, despite delay, there is only a small chance (z^2) of perpetual disagreement.

2. It is fairly natural to think of $T_i = -\frac{1}{\lambda_i}\log z$ as a measure of how "weak" player i is, since if $T_i > T_j$, then i will have to concede with positive probability right at the start, and the greater is T_i, the lower is i's payoff. Substituting for λ_i, we have:

$$T_i = -\frac{1}{\lambda_i}\log z_i = -\frac{\alpha_i + \alpha_j - 1}{r_j(1 - \alpha_i)}\log z_i$$

   So, for instance, i is weaker in the game when r_i is greater or z_i is smaller. In addition, i is weaker when either α_i or α_j is larger.

3. An interesting point, observed by Kambe (1999) is that if $z_i = z_j = z \to 0$,

$$T_i = -\frac{1}{\lambda_j}\log z \to \infty,$$

   but, if $\lambda_i > \lambda_j$, then $F_i(0) = 1 - z^{1-\lambda_i/\lambda_j} \to 1$, and so

$$u_i \to 1 - \alpha_j \text{ and } u_j \to \alpha_j.$$

   The "weak" player must concede immediately. We return to this below.

1.3 The Demand Game

Given the equililibrium for the concession game derived above, it is possible to characterize equilibrium in the demand game at time 0. If there is only one irrational type for each player, then it is fairly straightforward to see that both players will choose with probability one to mimic their irrational types at the start. In particular, if one player reveals rationality, but the other does not, then the player who has revealed rationality will concede immediately in the concession game.

With multiple irrational types for each player, then players will mix over their different irrational types. The crucial property of equilibrium is that i must obtain the same payoff from each irrational type. Since i's payoff depends on the posterior probability that he is irrational given his initial demand, equilibrium mixtures must reflect this. In particular, if i puts weight on some irratioinal demand α_i, he will put weight on all irrational demands $\alpha_i' > \alpha_i$.

Kambe (1999) and Abreu and Gul study the limit of this bargaining game as $z \to 0$. The key point (noted above) is that as $z \to 0$, if $\alpha_1 + \alpha_2 > 0$, then player i obtains his demand exactly if and only if:

$$\frac{r_i}{(1-\alpha_i)} < \frac{r_j}{(1-\alpha_j)}$$

and otherwise must concede immediately. By demanding

$$v_i = \frac{r_i}{r_i + r_j},$$

player i can ensure himself at least v_i, and j can similarly ensure v_j. Thus, in the limit as $z \to 0$, the equilibrium shares are $(v_i, v_j = 1 - v_i)$. Interestingly, this corresponds to the Nash bargaining solution.

2 Remarks

1. Note that two-sided reputation building is quite different than one-sided. If only one player can build a reputation, then if he is patient (or if moves are frequent), this asymmetric information tends to "take over" the game, as in Fudenberg-Levine. With two-sided reputation-building, neither player wants to reveal rationality, since this basically means admitting defeat. This generates the war of attrition type situation.

2. Abreu and Gul follow Kreps and Wilson (1982) and the literature on the war of attrition in studying the concession game in continuous time. But they also show that it is the limit of a sequence of models where players make offers in discrete time.

3. Abreu and Pearce (2006) use the Abreu-Gul and Kambe logic to study bargaining problems where players strategically interact during the

course of bargaining. For the model they consider, there is a Folk Theorem in the absence of reputational perturbations. They assume, however, that players first announce strategies for the bargaining game and with small probability become committed to these "bargaining postures". They show a remarkable generalization of Kambe's result. In the limit as the probability of commitment disappears, payoffs are given by the Nash bargaining solution with endogenous threat points, defined by Nash in his second paper on bargaining.

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Learning in Games

Jonathan Levin

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Most economic theory relies on an equilibrium analysis, making use of either Nash equilibrium or one of its refinements. As we discussed earlier, one defense of this is to argue that Nash equilibrium might arise as a result of learning and adaptation. Some laboratory evidence — such as Nagel's beauty contest experiment — seems consistent with this. In these notes, we investigate theoretical models of learning in games.

A variety of learning models have been proposed, with different motivations. Some models are explicit attempts to define dynamic processes that lead to Nash equilibrium play. Other learning models, such as stimulus-response or re-inforcement models, were introduced to capture laboratory behavior. These models differ widely in terms of what prompts players to make decisions and how sophisticated players are assumed to be. In the simplest models, players are just machines who use strategies that have worked in the past. They may not even realize they're in a game. In other models, players explicitly maximize payoffs given beliefs; these beliefs may involve varying levels of sophistication. Thus we will look at several approaches.

1 Fictitious Play

One of the earliest learning rules to be studied is fictitious play. It is a "belief-based" learning rule, meaning that players form beliefs about opponent play and behave rationally with respect to these beliefs.

Fictitious play works as follows. Two players, $i = 1, 2$, play the game G at times $t = 0, 1, 2, ...$ Define $\eta_i^t : S_{-i} \to \mathbb{N}$ to be the number of times i has observed s_{-i} in the past, and let $\eta_i^0(s_{-i})$ represent a starting point (or fictitious past). For example, if $\eta_1^0(U) = 3$ and $\eta_1^0(D) = 5$, and player two plays U, U, D in the first three periods, then $\eta_1^3(U) = 5$ and $\eta_1^3(D) = 6$.

Each player *assumes* that his opponent is using a stationary mixed strategy. So beliefs in the model are given by a distribution ν_i^t on $\Delta(S_j)$. The standard assumption is that ν_i^0 has a Dirichlet distribution, so

$$\nu_i^0(\sigma_{-i}) = k \prod_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})^{\eta_i^0(s_{-i})}$$

Expected play can then be defined as:

$$\mu_i^t(s_{-i}) = \mathbb{E}_{\nu_i^t} \sigma_{-i}(s_{-i})$$

The Dirichlet distribution has particularly nice updating properties, so that Bayesian udating implies that

$$\mu_i^t(s_{-i}) = \frac{\eta_i^t(s_{-i})}{\sum_{s_{-i} \in S_{-i}} \eta_i^t(s_{-i})}. \tag{1}$$

In other words, this says that i forecasts j's strategy at time t to be the empirical frequency distribution of past play.[1]

Note that even though updating is done correctly, forecasting is not fully rational. The reason is that i assumes (incorrectly) that j is playing a stationary mixed strategy. One way to think about this is that i's *prior* belief about j's strategy is wrong, even though he updates correctly from this prior.

Given i's forecast rule, he chooses his action at time t to maximize his payoffs, so

$$s_i^t \in \arg\max_{s_i \in S_{-i}} g_i(s_i, \mu_i^t)$$

This choice is myopic. However, note that myopia is consistent with the assumption that opponents are using stationary mixed strategies. Under this assumption, there is no reason to do anything else.

Example Consider fictitious play of the following game:

	L	R
U	$3,3$	$0,0$
D	$4,0$	$1,1$

Period 0: Suppose $\eta_1^0 = (3,0)$ and $\eta_2^0 = (1, 2.5)$. Then $\mu_1^0 = L$ with probability 1, and $\mu_2^0 = \frac{1}{3.5}U + \frac{2.5}{3.5}D$, so play follows $s_1^0 = D$ and $s_2^0 = L$.

[1] The whole updating story can also be dropped in favor of the direct assumption that players just forecast today's play using the naive forecast rule (1).

Period 1: $\eta_1^1 = (4, 0)$ and $\eta_2^1 = (1, 3.5)$, so $\mu_1^1 = L$ and $\mu_2^1 = \frac{1}{4.5}U + \frac{3.5}{4.5}D$. Play follows $s_1^1 = D$ and $s_2^1 = R$.

Period 2: $\eta_1^2 = (4, 1)$ and $\eta_2^2 = (1, 4.5)$, so $\mu_1^2 = \frac{4}{5}L + \frac{1}{5}R$ and $\mu_2^2 = \frac{1}{5.5}U + \frac{4.5}{5.5}D$. Play follows $s_1^1 = D$ and $s_2^1 = R$.

Basically, D is a dominant strategy for player 1, so he *always* plays D, and eventually $\mu_2^t \to D$ with probability 1. At this point, player will end up playing R.

Remark 1 *One striking feature of fictitious play is that players don't have to know anything at all about their opponent's payoffs. All they form beliefs about is how their opponents will play.*

An important question about fictitious play is what happens to the sequence of play $s^0, s^1, s^2, ...$ Does it converge? And to what?

Definition 1 *The sequence $\{s^t\}$ converges to s if there exists T such that $s^t = s$ for all $t \geq T$.*

Definition 2 *The sequence $\{s^t\}$ converges to σ in the time-average sense if for all i, s_i :*

$$\lim_{T\to\infty} \frac{1}{T+1} \left[\# \textit{ times } s_i^t = s_i \textit{ in } \{0, 1, ..., T\}\right] = \sigma_i(s_i)$$

Note that the former notion of convergence only applies to pure strategies. The latter is somewhat more standard, though it doesn't mean that players will actually every play a Nash equilibrium in any given period.

Proposition 1 *Suppose a fictitious play sequence $\{s^t\}$ converges to σ in the time-average sense. Then σ is a Nash equilibrium of G.*

Proof. Suppose $s^t \to \sigma$ in the time-average sense and σ is not a NE. Then there is some i, s_i, s_i' such that $\sigma_i(s_i) > 0$ and $g_i(s_i', \sigma_{-i}) > g_i(s_i, \sigma_{-i})$. Pick $\varepsilon > 0$ such that $\varepsilon < \frac{1}{2}\left[g_i(s_i', \sigma_{-i}) - g_i(s_i, \sigma_{-i})\right]$ and choose T such that whenever $t \geq T$,

$$\left|\mu_i^t(s_{-i}) - \sigma_{-i}(s_{-i})\right| < \frac{\varepsilon}{2N}$$

where N is the number of pure strategies. We can find such a T since $\mu_i^t \to \sigma_{-i}$. But then for any $t \geq T$,

$$\begin{aligned}
g_i(s_i, \mu_i^t) &= \sum g_i(s_i, s_{-i})\mu_i^t(s_{-i}) \\
&\leq \sum g_i(s_i, s_{-i})\sigma_{-i}(s_{-i}) + \varepsilon \\
&< \sum g_i(s_i', s_{-i})\sigma_{-i}(s_{-i}) - \varepsilon \\
&\leq \sum g_i(s_i', s_{-i})\mu_i^t(s_{-i}) = g_i(s_i', \mu_i^t)
\end{aligned}$$

So after t, s_i is never played, which implies that as $T \to \infty$, $\mu_j^t(s_i) \to 0$ for all $j \neq i$. But then it can't be that $\sigma_i(s_i) > 0$, so we have a contradiction. *Q.E.D.*

Remark 2 *The proposition is intuitive if one thinks about it in the following way. Recall that Nash equilibrium requires that (i) playes optimize given their beliefs about opponents play, and (ii) beliefs are correct. Under fictitious play, if play converges, then beliefs do as well, and in the limit they must be correct.*

It is important to realize that convergence in the time-average sense is not necessarily a natural convergence notion, as the following example demonstrates.

Example (Matching Pennies)

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Consider the following sequence of play:

	η_1^t	η_2^t	Play
0	$(0,0)$	$(0,2)$	(H,H)
1	$(1,0)$	$(1,2)$	(H,H)
2	$(2,0)$	$(2,2)$	(H,T)
3	$(2,1)$	$(3,2)$	(H,T)
4	$(2,2)$	$(4,2)$	(T,T)
5	$(2,3)$	$(4,3)$	(T,T)
6	...	...	(T,H)

Play continues as $(T,H),(H,H),(H,H)$ — a deterministic cycle. The time average converges to $(\frac{1}{2}H+\frac{1}{2}T,\frac{1}{2}H+\frac{1}{2}T)$, but players never actually use mixed strategies, so players never end up playing Nash.

Here's another example, where fictitious play leads to really perverse behavior!

Example (Mis-coordination)

	A	B
A	$1,1$	$0,0$
B	$0,0$	$1,1$

Consider the following sequence of play:

	η_1^t	η_2^t	Play
0	$(1/2,0)$	$(0,1/2)$	(A,B)
1	$(1/2,1)$	$(1,1/2)$	(B,A)
2	$(3/2,1)$	$(1,3/2)$	(A,B)
3	...	...	(B,A)

Play continues as $(A,B),(B,A),...$ — again a deterministic cycle. The time average converges to $(\frac{1}{2}A+\frac{1}{2}B,\frac{1}{2}A+\frac{1}{2}B)$, which is a mixed strategy equilibrium of the game. But players never successfully coordinate!!

A few more results for convergence of fictitious play.

Proposition 2 *Let $\{s^t\}$ be a fictitious play path, and suppose that for some t, $s_t = s^*$, where s^* is a strict Nash equilibrium of G. Then $s_\tau = s^*$ for all $\tau > t$.*

Proof. We'll just show that $s_{t+1} = s^*$, the rest follows from induction. Note that:

$$\mu_i^{t+1} = (1-\alpha)\mu_i^t + \alpha s_{-it}$$

where $\alpha = 1/\left(\sum_{s_{-i}} \eta_t^t(s_{-i}) + 1\right)$, so:

$$g_i(a_i;\mu_i^{t+1}) = (1-\alpha)g_i(a_i,\mu_i^t) + \alpha g_i(a_i,s_{-i}^*)$$

but s_i^* maximizes both terms, to s_i^* will be played at $t+1$. *Q.E.D.*

Proposition 3 *(Robinson, 1951; Miyasawa, 1951) If G is a zero-sum game, of if G is a 2×2 game, then fictitious play always converges in the time-average sense.*

An example of this is matching pennies, as studied above.

Proposition 4 *(Shapley, 1964) In a modified version of Rock-Scissors-Paper, fictitious play does not converge.*

Example Shapley's Rock-Scissors-Papers game has payoffs:

	R	S	P
R	$0,0$	$1,0$	$0,1$
S	$0,1$	$0,0$	$1,0$
P	$1,0$	$0,1$	$0,0$

Suppose that $\eta_1^0 = (1,0,0)$ and that $\eta_2^0 = (0,1,0)$. Then in Period 0: play is (P,R). In Period 1, player 1 expects R, and 2 expects S, so play is (P,R). Play then continues to follow (P,R) until player 2 switches to S. Suppose this takes k periods. Then play follows (P,S), until player 1 switches to R. This will take βk periods, with $\beta > 1$. Play then follows (R,S), until player 2 switches to P. This takes $\beta^2 k$ periods. And so on, with the key being that each switch takes longer than the last.

To prove Shapley's result, we'll need one Lemma. Define the time-average of payoffs through time t as:

$$U_i^t = \frac{1}{t+1} \sum_{\tau=0}^{t} g_i(s_i^\tau, s_{-i}^\tau).$$

Define the expected payoffs at time t as:

$$\tilde{U}_i^t = g_i(s_i^t, \mu_i^t) = \max_{s_i} g_i(s_i, \mu_i^t)$$

Lemma 1 *For any $\varepsilon > 0$, there exists T s.t. for any $t \geq T$, $\tilde{U}_i^t \geq U_i^t - \varepsilon$.*

Proof. Note that:

$$\begin{aligned}
\tilde{U}_i^t = g_i(s_i^t, \mu_i^t) &\geq g_i(s_i^{t-1}, \mu_i^t) \\
&= \frac{1}{t+1} g_i(s_i^{t-1}, s_i^{t-1}) + \frac{t}{t+1} g_i(s_i^{t-1}, \mu_i^{t-1}) \\
&= \frac{1}{t+1} g_i(s_i^{t-1}, s_i^{t-1}) + \frac{t}{t+1} \tilde{U}_i^{t-1}
\end{aligned}$$

Expanding $\tilde{U}_i^{t-1}$ in the same way:

$$\tilde{U}_i^t \geq \frac{1}{t+1} g_i(s_i^{t-1}, s_i^{t-1}) + \frac{1}{t+1} g_i(s_i^{t-2}, s_i^{t-2}) + \frac{t-1}{t+1} \tilde{U}_i^{t-2}$$

and iterating:

$$\tilde{U}_i^t \geq \frac{1}{t+1} \sum_{\tau=0}^{t-1} g_i(s_i^\tau, s_{-i}^\tau) + \frac{1}{t+1} g_i(s_i^0, \mu_i^0)$$

Define T such that

$$\varepsilon > \frac{1}{T+1} \max_s g_i(s)$$

and we're done. *Q.E.D.*

The Lemma says that if the game goes on long enough, expected payoffs must ultimately be almost as big as time-average payoffs. Of course, they can in principle be a lot bigger. In the mis-coordination example above, expect payoffs converge to 1/2. But actual payoffs are always zero.

We can use the Lemma to prove Shapley's result.

Proof of Shapley Non-convergence. In the Rock-Scissors-Paper game, there is a unique Nash equilibrium with expected payoffs of 1/3 for both players. Therefore, if fictitious play converged, then ultimately $\tilde{U}_i^t \to 1/3$, meaning that $\tilde{U}_1^t + \tilde{U}_2^t \to 1/3 + 1/3 = 2/3$. But under fictitious play, the empirical payoffs *always* sum to 1, so $U_1^t + U_2^t = 1$ for all t. This contradicts the Lemma, meaning fictitious play can't converge. *Q.E.D.*

Shapley's example highlights a few key points about fictitious play. One is that because it jumps around, it is not particularly well-suited for learning mixed equilibria. Another is that because players only are thinking about their opponent's actions, they're not paying attention to whether they've actually been doing well.

More recent work on belief-based learning has tried to get around these difficulties in various ways. A good reference for this material is the book by Fudenberg and Levine (1999). Interesting papers include Fudenberg and Kreps (1993, 1995) on smoothed fictitious play, Kalai and Lehrer (1993), and also Nachbar (2003) on Bayesian learning, and Foster and Vohra (1998) on calibrated learning rules. I'll have something to say about this literature in class.

2 Reinforcement Learning

A different style of learning model derives from psychology and builds on the idea that people will tend to use strategies that have worked well in the past. These adaptive learning models do not incorporate beliefs about opponent's strategies or require players to have a 'model' of the game. Instead player respond to positive or negative stimula. References include Bush and Mosteller (1955) and Borgers and Sarin (1996).

The following model is based on Erev and Roth (1998). Let $q_{ik}(t)$ denote player i's propensity to play his kth pure strategy at time t. Initially, player i has the uniform propensities across strategies:

$$q_{i1}(1) = q_{i2}(1) = ... = q_{iK}(1)$$

After each period, propensities are updated using a *reinforcement* function. Suppose that at time t, player i played strategy k_t and obtained a payoff x. Then,

$$q_{ik}(t+1) = \begin{cases} q_{ik}(t) + R(x) & \text{if } k = k_t \\ q_{ik}(t) & \text{otherwise} \end{cases},$$

for some increasing function $R(\cdot)$. The idea is that if k_t was successful, the player is more likely to use it again. If it was unsuccessful, he will be less likely to use it.

Propensities are mapped into choices using a choice rule. For instance, letting $p_{ik}(t)$ denote the probability that i will choose k at time t, a simple rule would be:

$$p_{ik}(t) = \frac{q_{ik}(t)}{\sum_j q_{ij}(t)}.$$

While this sort of model is very simple, it can sometimes do very well explaining experimental results. Not surprisingly, these models tend to fit the data better with more free parameters.

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Experimental Evidence

Jonathan Levin

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Dating back to the sixties, researchers have used laboratory experiments to study how people actually might play particular games. Early work focused on simple games such as the prisoners' dilemma and market trading environments. More recent work has broadened the scope of inquiry. These notes briefly survey a few of the many interesting experimental findings.

Experimental work is useful for exploring both human behavior and the performance of institutions. From a behavioral viewpoint, two of the interesting questions for game theory are:

- How do people react when faced with a strategic situation? Do they play dominated strategies? Perform backward induction or reason strategically? Attempt to maximize their payoff?
- How do people learn over time to play games? Does behavior converge toward Nash equilibrium or one of its refinements?

One thing we will see immediately is that people don't just walk into the laboratory and play Nash equilibrium. Sometimes they play dominated strategies. Over time, laboratory play often converges toward equilibrium, but not always. In a sense, this is not surprising. Experimental subjects don't know game theory; they're just given a sheet of paper with numbers. The interest in the experiments comes from finding behavior that is consistent across different games, or that leads to some insight into how people approach problem solving, or how they learn to behave over time.

Beyond these behavioral questions, there is another motivation for experiments, which is to test the performance of different institutions. A basic question here is:

- Do different institutions — auctions, market mechanisms, matching algorithms, bargaining protocols — achieve good outcomes even when players are inexperienced? How do changes in the rules governing these institutions affect outcomes?

1 Bargaining: The Ultimatum Game

One of the most debated results in experimental economics comes from studies of the so-called "ultimatum" game. In this game, one player (the "proposer") goes first and offers a split of a pie of given size. The second player chooses whether to accept or reject. If the second player rejects, both get noting. If he accepts, they split the pie as was proposed. The ultimatum game has a unique subgame perfect equilibrium where the proposer gets (essentially) the whole pie. It also has many other Nash equilibria, where the proposer offers a more generous split, fearing an aggressive offer will be rejected.

The experimental results are in stark contrast to the backward induction solution. On average, proposers offer about 40% of the pie to the second player. Moreover, offers are rejected about 15-20% of the time. Perhaps not surprisingly, lower offers are more likely to be rejected. Table 1 shows data from an experiment by Roth et. al. (1991). The modal proposal is about 500 (out of 1000). Offers are a bit dispersed at the outset, but tend to cluster at the mode over time.

Roth et. al. also ran the same experiment in three other countries — Yugoslavia, Japan and Israel (see table 2). The modal proposal in Yugoslavia looks similar to the U.S., while in Japan and Israel it was closer to 400. Interestingly, countries with lower offers did not have higher rates of disagreement. Instead, somewhat lower offers were accepted in countries where low offers were made.

Discussion of these results often raises two issues:

- First, why do responders reject offers?
- Second, does it seem that proposers are behaving "rationally" given responder behavior?

There are many interpretations for why offers are rejected. One leading explanation is that players punish greed – their behavior exhibits some sort of concern for fair outcomes or reciprocity motive. A striking finding is that these empirical patterns do not seem to be due to low stakes. Hoffman et al. (1998) ran ultimatum experiments in the U.S. with \$10 and \$100 stakes. While they found somewhat lower offers and more willingness to accept in the \$100 treatment, their results are quite far from the backward induction prediction.

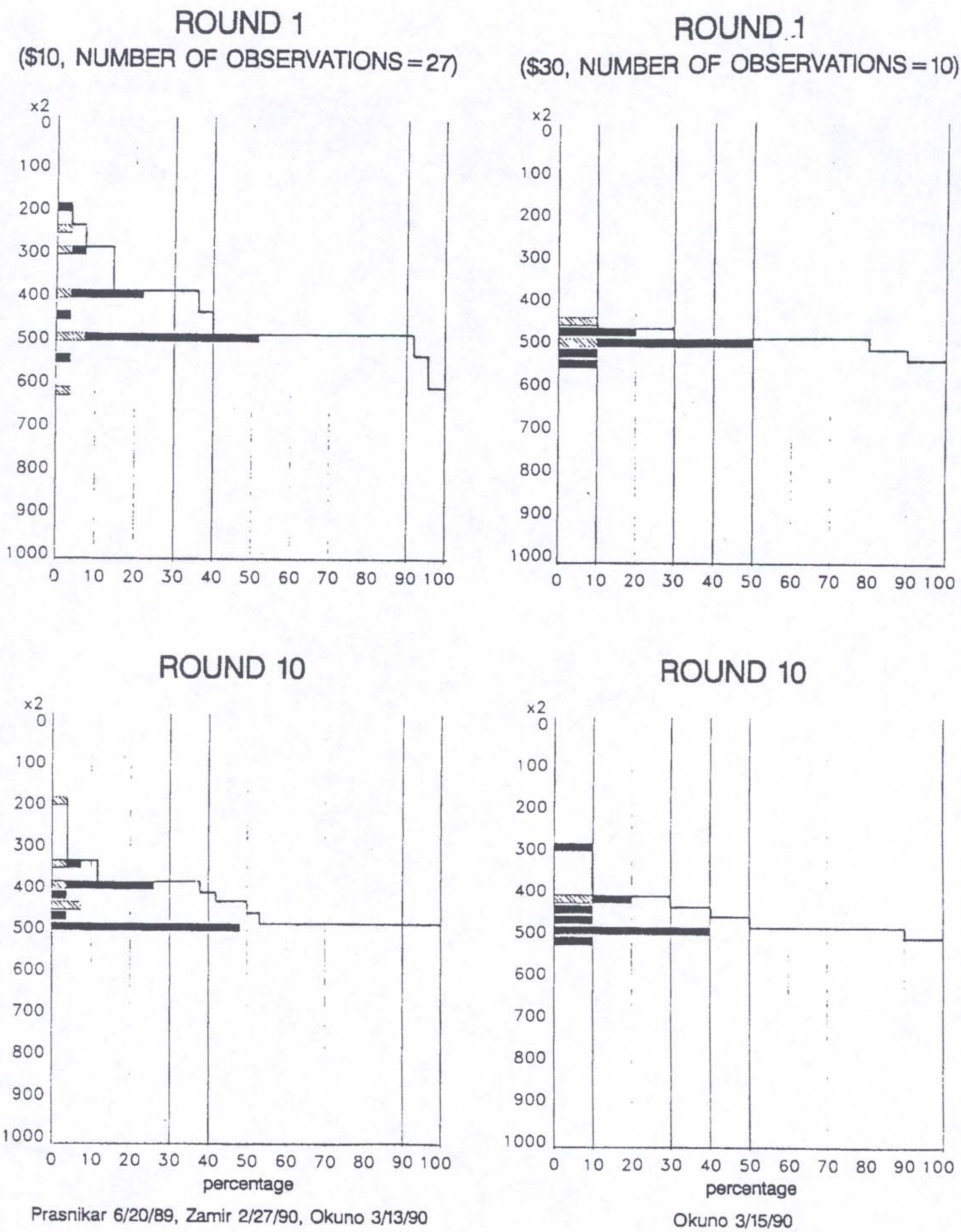


FIGURE 3. DISTRIBUTION OF BARGAINING OFFERS IN THE UNITED STATES (SOLID BARS = ACCEPTED OFFERS; STRIPED BARS = REJECTED OFFERS)

Figure 1: Roth et. al. (1991, *AER)*

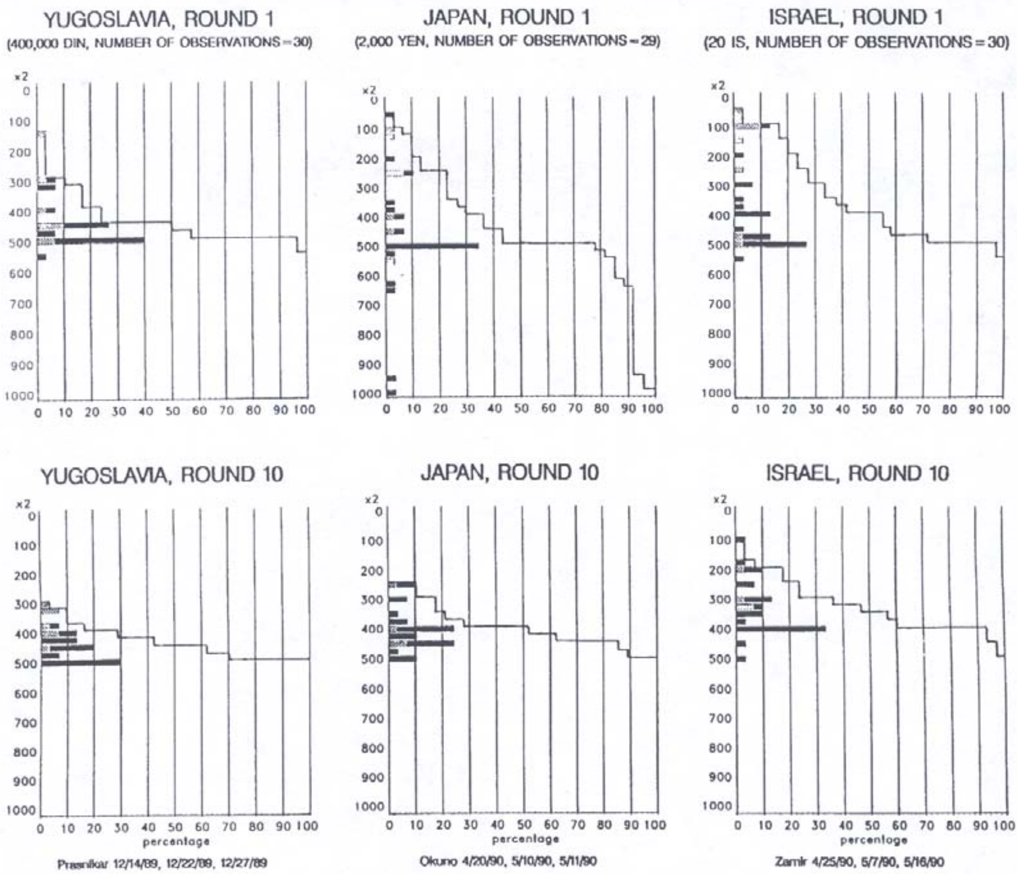
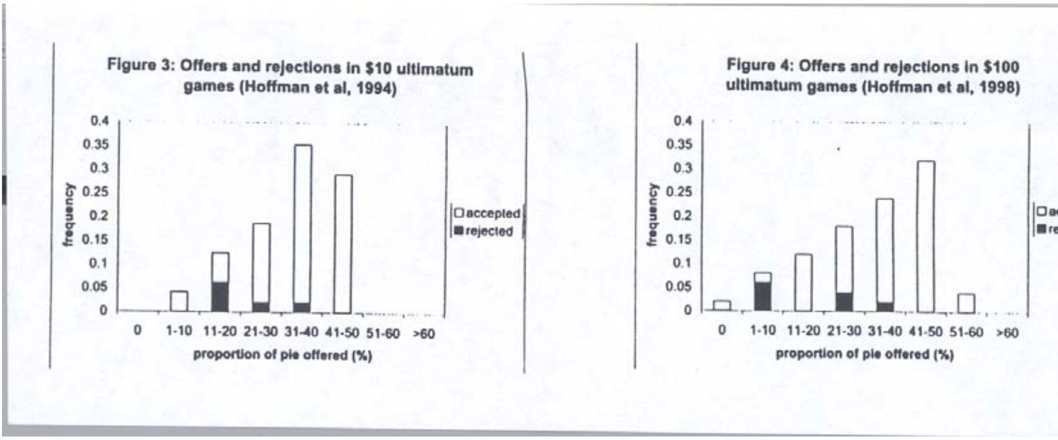


FIGURE 4. DISTRIBUTIONS OF BARGAINING OFFERS IN YUGOSLAVIA, JAPAN, AND ISRAEL (SOLID BARS = ACCEPTED OFFERS; STRIPED BARS = REJECTED OFFERS)

Figure 2: Roth et. al. (1991, *AER*)



Ultimatum Game with High Stakes

As Colin Camerer (2000) points out, a weakness of fairness-based explanations is that they typically don't address the question of where such preferences might come from. One possibility is social conditioning. Camerer discusses an experiment done by anthropologists in primitive cultures in Africa, the Amazon, Papua New Guinea, Indonesia and Mongolia. In most cases, offers were very low (about $1.50 out of $10) and offers were accepted practically every time. This suggests that cultural forces may need to be incorporated to generate a satisfactory theory for the data.

2 Market Games

Arguably, the results in the ultimatum game contradict the standard theory. In contrast, experiments on various forms of market games have tended to be strongly supportive of theoretical predictions. There is a long history of market experiments (this is what Vernon Smith won the Nobel Prize for in 2002). The remarkable finding is that even with relatively few players, and a relatively short number of rounds, market outcomes tend to look like competitive equilibria.

A nice example are Roth et al's (1991) experiments on university students in the United States, Yugoslavia, Israel and Japan. Their game works as follows. In their game, multiple buyers (nine in most sessions) submit an offer to a single seller to buy an indivisible object worth the same amount to each buyer. The seller can either reject the bids, in which case everyone gets zero, or accept the highest bid, in which case the seller gets the bid, and the high bidder gets the difference between the valuation and the bid.

The Roth et. al. game has a unique perfect equilibrium where players bid their value, and the seller accepts. The experimental evidence is strongly in line with this. In Roth's experiments, the seller *always* accepted the high bid and within a few rounds, the high bid converged to the buyers' value. Indeed (see Table 3), the bids ended up highly concentrated in this region. Moreover, the cross-country differences are minimal.

3 Iterated Dominance: The Beauty Contest

The ultimatum game seems to shed some light on preferences, while market games speak to the power of competition. Another interesting class of experiments are aimed at understanding the extent to which subjects use iterated dominance reasoning (i.e. think through what people think about what people think ... people will do).

An elegant example is Nagel's (1995) work on p-beauty contest games. In Nagel's experiment, players were asked to choose a number between 0 and 100, with a prize going to the player whose guess is closest to p times the average guess. When $p < 1$, this game can be solved using iterated strict dominance (first elimate strategies greater than $100p$, then greater than $100p^2$, and so on). Of course, the unique equilibrium has everyone guess zero.

In Nagel's experiment, when $p = \frac{1}{2}$, in the first round of play many players guessed between 10 and 30. When $p = \frac{2}{3}$, players tended to guess in the 20–35 range, although in both cases there were higher guesses as well (Table 4). Thus at the outset, people behave as if they are doing perhaps two rounds of iterated reasoning.

As players playes a few more times, play tended to converge toward the Nash equilibrium (Table 5). Interestingly, play converges faster than best-response dynamics would converge. That is, subjects guess lower than a direct best-response to the previous period's average — perhaps moving two steps at a time. Table 5 also shows that with enough time, players eventually play 0.

Nagel's results suggest that while people don't leap immediately to the iterated dominance solution, they do "think through" the game to some extent. Further evidence on "strategic sophistication" is provided by Costa-Gomes, Crawford and Broseta (2002). They asked subjects to play games that were dominance-solvable or that had a unique pure-strategy equilibrium. Their basic conclusion is that many subjects look as if they are best-responding to what they perceive as a random choice on the part of

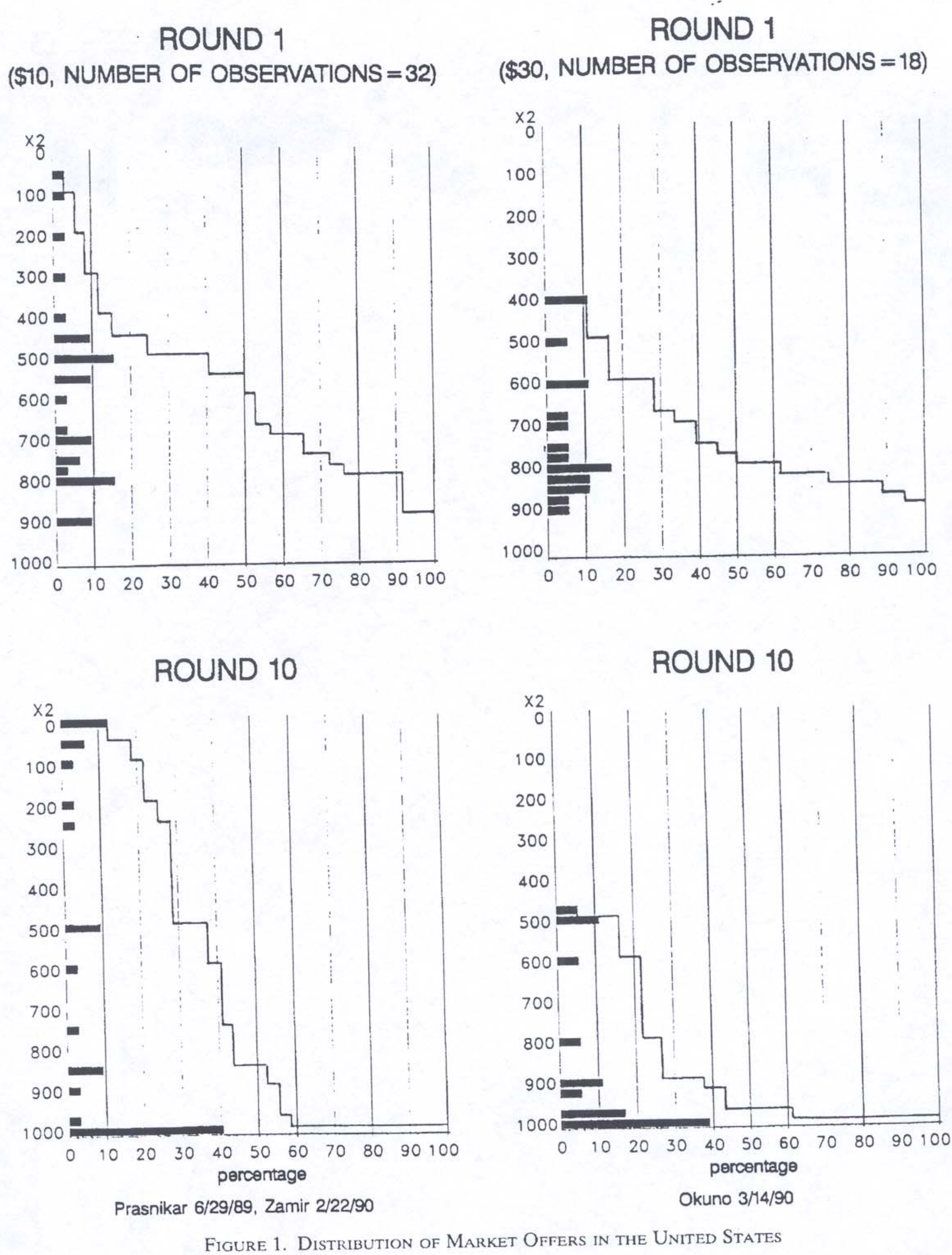


FIGURE 1. DISTRIBUTION OF MARKET OFFERS IN THE UNITED STATES

Figure 3: Roth et. al (1991, *AER)*

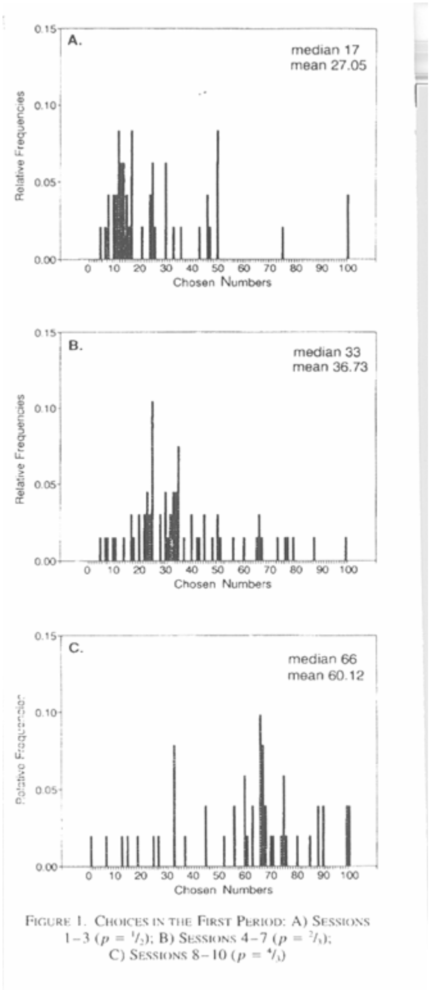


Figure 4: Nagel (1995, *AER*)

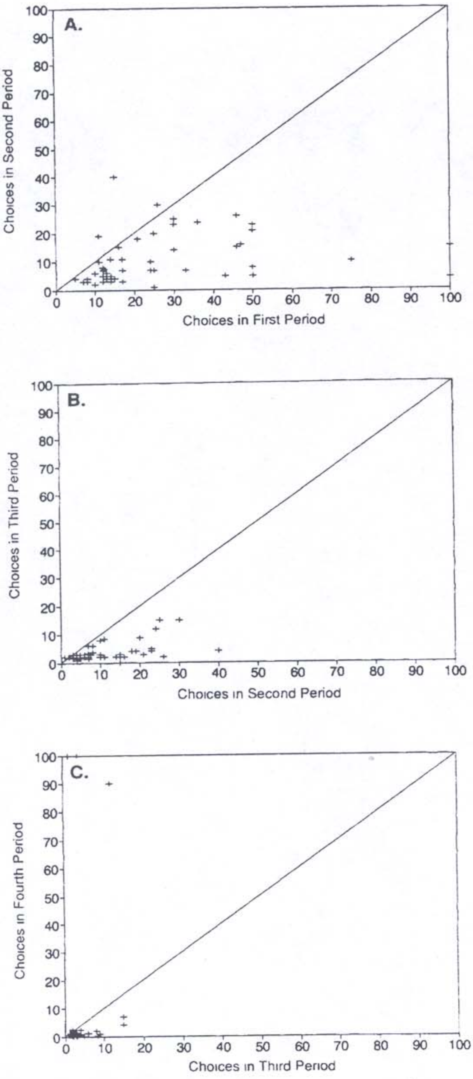


FIGURE 3. OBSERVATIONS OVER TIME FOR SESSIONS 1–3 ($p = {}^1/_2$): A) TRANSITION FROM FIRST TO SECOND PERIOD; B) TRANSITION FROM SECOND TO THIRD PERIOD; C) TRANSITION FROM THIRD TO FOURTH PERIOD

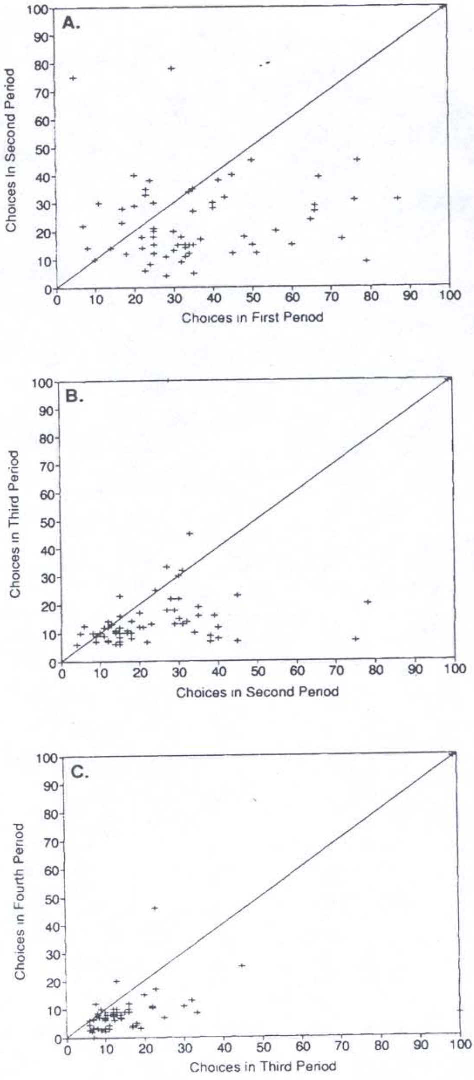


FIGURE 4. OBSERVATIONS OVER TIME FOR SESSIONS 4–7 ($p = {}^2/_3$): A) TRANSITION FROM FIRST TO SECOND PERIOD; B) TRANSITION FROM SECOND TO THIRD PERIOD; C) TRANSITION FROM THIRD TO FOURTH PERIOD

Figure 5: Nagel (1995, *AER*)

opponents, or perhaps a random choice from among undominated strategies.

4 Adaptive Behavior in Coordination Games

A striking finding in the beauty contest is that behavior seems to converge in a natural adaptive way toward the unique equilibrium. To the extent that behavior over time can be described as a process of learning and adaptation, a natural question is what will happen in games where there is no "obvious" outcome toward which behavior should converge. One clever experiment along these lines was done by van Huyck, Battalio and Cook (1998).

In their game, players in groups of seven chose numbers between 1 and 14. Payoffs depended on own choice and the median and were symmetric. Best responses are shown below. The game has two equilibrium, 3 and 12, where the higher equilibrium is Pareto-preferred.

	Median Choice													
	1	2	**3**	4	5	6	7	8	9	10	11	**12**	13	14
Best Response	2	3	**3**	3	4	5	6	9	10	11	12	**12**	12	13

This game is called the "continental divide" game because on either side of 7, the best-responses "flow downhill" toward an equilibrium.

The experiment called for players to play for fifteen rounds. Median choices for several runs are shown in Table 6.

Behavior seems to adjust roughly in the direction of best-responses, though perhaps at a slower or faster rate than the simplest best-response dynamics. Nevertheless, an adaptive learning story looks like a reasonable one to explain what is going one. A particularly interesting aspect of these results is that the long-run play depends a lot on the initial behavior. If the median starts out below seven, play tends to converge toward the low equilibrium. If the median starts out above seven, play tends toward the higher equilibrium. This is true despite the fact that the high equilibrium is Pareto-preferred.

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Figure 1: Median choices in 'continental divide' game, VHCB (in press)

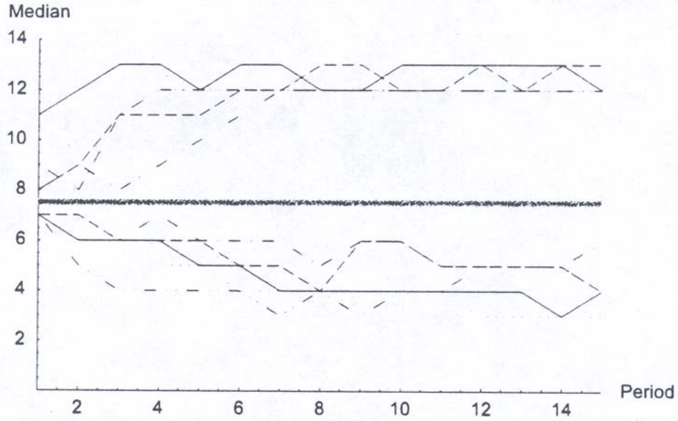

Figure 6: Adaptive Behavior in a Coordination Game

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Fairness and Reciprocity

Jonathan Levin

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Beyond the fact that laboratory play does not correspond to notions of equilibrium, many laboratory results seem strikingly at odds with the hypothesis that players act only to maximize their material payoffs.

- In the ultimatum game, responders turn down offers with positive probability and are particularly likely to turn down low offers.
- In public goods experiments where there are strong incentives to free-ride and contribute nothing, people tend to contribute non-zero amounts. This is especially true if it is possible to punish non-contributors.
- In efficiency wage experiments, agents exert effort even if they have a financial incentive not to do so, and exert more effort when paid a high wage.

How can this behavior be explained? One possibility is that players are simply confused about what is going on in the lab. Other explanations center on the idea that players are conditioned by real-world environments where there is repeated interaction and/or sanctions for behaving selfishly. Finally, a number of leading theories suggest that people care intrinsically about "fairness" — that is, either about the distribution of payoffs or about *how* the game is played. We now turn to discussing this idea.

1 Theories of Fairness: Payoff-Driven

Fehr and Schmidt (1999) and Bolton and Ockenfels (2000) propose models in which players care about their own payoffs and also the payoffs of others. In the Fehr-Schmidt model, given a vector of material payoffs $x = (x_1, ..., x_n)$, player i's utility is:

$$u_i(x) = x_i - \alpha_i \frac{1}{n-1} \sum_{j \neq i} \max\{x_j - x_i, 0\} - \beta_i \frac{1}{n-1} \sum_{j \neq i} \max\{x_i - x_j, 0\},$$

where $0 \leq \beta_i \leq 1$ and $\alpha_i \geq \beta_i$. For just two players, this reduces to:

$$u_i(x) = x_i - \alpha_i \max\{x_j - x_i, 0\} - \beta_i \max\{x_i - x_j, 0\},$$

The basic properties of the model are:

1. Players don't like inequality ($\alpha_i, \beta_i \geq 0$). Holding a player's own material payoff fixed, he likes everyone to get the same amount.

2. Players dislike being behind more than being ahead ($\alpha_i \geq \beta_i$).

3. Players aren't willing to throw money away to reduce inequality ($\beta_i \leq 1$), but they might give away a dollar away to reduce inequality (e.g. if $n = 2$ and $\beta_i > 1/2$).

Given these preferences, we can still use Nash or subgame perfect equilibrium as our solution concept.

Example 1: The Ultimatum Game. In the ultimatum game, player 1 proposes how to split a dollar. Player 2 either accepts the split or rejects it. In the latter case, both get zero.

Claim In the unique subgame equilibrium, the proposer offers $1/2$ if $\beta_1 > 1/2$ and offers $(1+\alpha_2)/(1+2\alpha_2)$ if $\beta_1 < 1/2$. The responder accepts.

If player 1 offers a split $(s, 1-s)$, player 2's best response is:

$$\begin{aligned}
\text{If } s &< 1/2\text{: Accept if } 1 - s - \beta_2(1-2s) \geq 0 \\
\text{If } s &= 1/2\text{: Accept} \\
\text{If } s &> 1/2\text{: Accept if } 1 - s - \alpha_2(2s-1) \geq 0
\end{aligned}$$

Player 1 does better to offer $s = 1/2$ rather than any $s < 1/2$. For offers $s > 1/2$, player 2 will accept if:

$$s \leq s^* = \frac{1+\alpha_2}{1+2\alpha_2}.$$

Since payoffs are linear, player 1 should either offer s^* or $1/2$. The former gives a higher direct payoff, but generates inequality. The equal split is better if and only if:

$$\frac{1}{2} \geq s^* - \beta_1(2s^* - 1) \qquad \Leftrightarrow \qquad \beta_1 > 1/2.$$

Thus if $\beta_1 < 1/2$, player 1 offers $s^*, 1-s^*$ and if $\beta_1 > 1/2$, player 1 offers $1/2, 1/2$. Either way, player 2 accepts.

Example 2: The Market Game. In the Roth et. al. market game, players $1, ..., n-1$ make proposals $s_i, 1-s_i$. Player n, the responder, then can accept or reject the lowest offer s^L. If he accepts, he gets $1-s^L$ and the winning proposer gets s^L. If several proposers make the low offer, we randomly select one.

Claim In the unique subgame perfect equilibrium, at least two proposers offer $s = 0$ and the responder accepts.

First note that the responder would certainly accept a low offer $s^L \leq 1/2$, since then:

$$\begin{aligned} u(1-s^L, s^L, 0, ..., 0) &= (1-s^L) - \beta\frac{n-2}{n-1}(1-s^L) - \beta\frac{1}{n-1}(1-2s^L) \\ &= (1-s^L)(1-\beta) + \beta\frac{1}{n-1}s^L > 0. \end{aligned}$$

Now, if the lowest offer was greater than 1/2, a losing proposer would get at most zero (if the offer were to be rejected), but could get positive utility by offering 1/2. So in equilibrium, we must have $s^L \leq 1/2$ and the offer accepted. Moreover, in equilibrium at least two proposers must offer s^L — otherwise the winning proposer would deviate to a slightly higher offer.

If the lowest offer is $s^L > 0$, a losing proposer will have utility:

$$u(0, s^L, 1-s^L, 0, ..., 0) = -\alpha\frac{1}{n-1}.$$

By offering just below s^L, he could have:

$$u(s^L, 1-s^L, 0, ..., 0) = s^L - \alpha\frac{1}{n-1}(1-2s^L) - \beta\frac{n-2}{n-1}s^L > -\alpha\frac{1}{n-1}.$$

Since proposers prefer to win at just below s^L rather than lose at s^L, competition implies that $s^L = 0$ in equilibrium.

Fehr and Schmidt show the theory can also explain lab findings in some other games such as public good games (see the assignment). When it comes to applying the theory outside the lab, however, there are some tough conceptual issues that await further research:

1. How should one define the relevant reference group? Should it include just a person's closest peers or a broader sampling?

2. How should one define material payoffs? Are "payoffs" total wealth or just changes in wealth? Should endowments matter in a definition of fair outcomes?

2 Theories of Fairness: Intention-Driven

Rabin (1993) proposes an alternative model of fairness motivations in which players care about both material payoffs and about the *intentions* of other players. People want to reward those who are nice to them, and hurt those who are mean to them. Rabin's players end up playing what Geanakoplos, Pearce and Stacchetti (1989) call a "psychological game" — one where payoffs depend on actions and on beliefs about actions.

Rabin develops his theory for two player games, with action sets A_1, A_2. Starting with the material payoffs $\pi_i : A_1 \times A_2 \to \mathbb{R}$, Rabin defines "fairness" functions. Suppose player i believes player j is choosing b_j. Then by choosing a_i, player i is choosing a payoff pair from the set:

$$\Pi(b_j) = \{\pi_i(a_i, b_j), \pi_j(a_i, b_j) :, \pi_j) : a_i \in A_i\}.$$

Let $\pi_j^h(b_j), \pi_j^l(b_j)$ be the be player j's highest and lowest payoff among points that are Pareto-efficient in $\Pi(b_j)$. Let:

$$\pi^e(b_j) = \frac{1}{2}\left(\pi_j^h(b_j) + \pi_j^l(b_j)\right)$$

be the equitable payoff, and let $\pi_j^{\min}(b_j)$ be j's minimum payoff in $\Pi(b_j)$. Player i's kindness to player j in choosing a_i is then:

$$f_i(a, b_j) = \frac{\pi_j(a, b_j) - \pi_j^e(a, b_j)}{\pi_j^h(a, b_j) - \pi_j^{\min}(a, b_j)}$$

Now, suppose player i believes that player j believes that i will play c_i. Let $\tilde{f}_j(c_i, b_i) = f_j(c_i, b_j)$ be player i's *belief* about how kind player j is being to him, given that i believes j will play b_i. Then player i will choose his action a_i to maximize:

$$U_i(a_i, b_j, c_i) = \pi_i(a_i, b_j) + \tilde{f}_j(c_i, b_i)\left[1 + f_i(a_i, b_j)\right].$$

Definition 1 *A pair of strategies* (a_1, a_2) *is a* ***fairness equilibrium*** *if, for* $i = 1, 2$ *and* $j \neq i$,

$$a_i \in \arg\max_{a \in S_i} U_i(a, b_j, c_i)$$

with $c_i = b_i = a_i$.

Now consider some examples of fairness equilibria.

Example 1 Battle of the Sexes. Payoffs are:

	F	B
F	$2x, x$	$0, 0$
B	$0, 0$	$x, 2x$

Both (F, F) and (B, B) are Nash equilibria, and also fairness equilibria. However, if x is small, (F, B) and (B, F) are also fairness equilibria. The reason is that each player feels the other is trying to hurt him by miscoordinating, so he wants to respond similarly.

Example 2 Prisoner's Dilemma. Payoffs are:

	C	D
C	$4x, 4x$	$0, 6x$
D	$6x, 0$	$0, 0$

In the prisoner's dilemma, (C, C) will be a fairness equilibrium if x is small enough (so the material stakes are small), while (D, D) is always a fairness equilibrium.

Example 3 Hawk-Dove. Payoffs are:

	H	D
H	$-2x, -2x$	$0, 2x$
D	$2x, 0$	x, x

Here, if x is small, the Nash equilibria (H, D) and (D, H) are not fairness equilibrium. The player playing D feels that the other is trying to hurt her, and wants to be mean back, which means playing H.

Rabin also proves a few more general results. In particular, say that (a_1, a_2) is a *mutual-max* outcome if for $i = 1, 2$, $a_i \in \arg\max_{a \in S_i} \pi_j(a, a_j)$. Similarly, one can define a *mutual-min* equilibrium. Rabin shows that if (a_1, a_2) is a Nash equilibrium and also a mutual-max or mutual-min profile, then it must be a fairness equilibrium. Moreover, if stakes are small, then a mutual-max or mutual-min profile will be a fairness equilibrium even if it is not Nash.

3 Open Questions

1. Some recent experimental papers by Fehr, Rabin and others suggest very strongly that a convincing model of fairness should include some role for intentions. But Rabin's paper seems somewhat hard to make operational outside of a limited class of games. The Fehr-Schmidt model has the attractive feature of being very simple, but it misses the idea that intentions matter. This leaves the problem open for future work.

2. Recent work on this problem has focused on the laboratory, but earlier research by George Akerlof, Robert Frank and others has considered the implications of fairness or concerns about relative status in broader economic contexts. This seems like another area where there is significant room for research.

3. Perceptions of whether a situation is fair can be very sensitive to framing. As an example, Gneezy (2003) discusses the results of several experiments in which it is found that using small monetary incentives is counter-productive relative to using large incentives or no incentives at all. An explanation he offers is that people are either insulted by small compensation, or interpret it as meaning that the task is odious without feeling that it makes the task worthwhile. But is seems quite easy to imagine situations where offering a small "token of appreciation" for a service could elicit a large response. This suggests that when it comes to incentive systems, the way in which a system is framed can be of crucial importance. Again, this is something economists have made minimal progress in understanding.

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