MATH 425b ASSIGNMENT 1 SOLUTIONS
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Prof. Alexander

Chapter 7

(3) Here is one example: let $f_n(x) = f(x) = g(x) = x, g_n(x) = x - \frac{1}{n}$, for $n \geq 1$ and $x \geq 0$. Then $f_n \to f$ uniformly and $g_n \to g$ uniformly, but $\sup_{x\geq 0} |f_n(x)g_n(x) - f(x)g(x)| = \sup_{x\geq 0} |x/n| = \infty$. Thus $f_n g_n \not\to fg$ uniformly.

(6) We decompose into two series:

$$\sum_n (-1)^n \frac{x^2+n}{n^2} = \sum_n (-1)^n \frac{x^2}{n^2} + \sum_n (-1)^n \frac{1}{n}, \tag{1}$$

which is legitimate provided the two series on the right both converge. For each term in the first series on the right side of (1), the maximum over x necessarily occurs at an endpoint:

$$\left|(-1)^n \frac{x^2}{n^2}\right| \leq \frac{\max(|a|^2, |b|^2)}{n^2} \quad \text{for all } x \in [a,b],$$

and

$$\sum_n \frac{\max(|a|^2, |b|^2)}{n^2}$$

converges, so by the Weierstrass M-test, the first series on the right side of (1) converges uniformly. The second series converges by the Alternating Series Test, and the series doesn't depend on x so the convergence is necessarily uniform in x. Therefore by problem 2, the sum of the two series (the left side of (1)) also converges uniformly in $[a,b]$, for all $a < b$.

For a fixed x,

$$\left|(-1)^n \frac{x^2+n}{n^2}\right| = \frac{x^2+n}{n^2} \geq \frac{n}{n^2} = \frac{1}{n},$$

and $\sum_n 1/n$ diverges, so by the comparison test,

$$\sum_n \left|(-1)^n \frac{x^2+n}{n^2}\right|$$

diverges. This means the series on the left side of (1) does not converge absolutely for any x.

(8) Since $|c_n I(x - x_n)| \leq |c_n|$ and $\sum |c_n| < \infty$, the series defining f converges uniformly by the Weierstrass M-test (Theorem 7.10.) Letting $f_n(t) = \sum_{k=1}^n c_n I(x - x_n)$, this means that $f_n \to f$ uniformly. Let x be a point that is not one of the x_n's; then each f_n is continuous at x, that is, $\lim_{t\to x} f_n(t) = f_n(x)$. Since $f_n(x) \to f(x)$ as $n \to \infty$, Theorem 7.11 (applied with $A_n = f_n(x)$) says

$$\lim_{t\to x} f(t) = \lim_{n\to\infty} f_n(x) = f(x),$$

that is, f is continuous at x.

(I)(a) For fixed $x > 0$, since e^{-nx^2} decreases exponentially in n while nx increases only linearly, the product, which is $f_n(x)$, converges to 0. Since $f_n(0) = 0$ for all n, this shows that $f_n \to f \equiv 0$ pointwise on $[0, 1]$.

(b) Using $u = nx^2, du = 2nx\ dx$,

$$\int_0^1 f_n(x)\ dx = \int_0^1 nxe^{-nx^2}\ dx = \frac{1}{2}\int_0^n e^{-u}\ du \to \frac{1}{2}\int_0^\infty e^{-u}\ du = \frac{1}{2} \neq 0 = \int_0^1 f(x)\ dx.$$

(II) f_n is continuous, but f is not. If the convergence were uniform, the limit would have to be continuous, so the convergence in this case cannot be uniform.

(III)(a) This really only requires that ψ be one-to-one, not necessarily a bijection.

First we have $\rho(x, x) = d(\psi(x), \psi(x)) = 0$ for all x, and for $x \neq y$ we have $\psi(x) \neq \psi(y)$, so $\rho(x, y) = d(\psi(x), \psi(y)) \neq 0$. Next we have $\rho(x, y) = d(\psi(x), \psi(y)) = d(\psi(y), \psi(x)) = \rho(y, x)$. Finally, for the triangle inequality, given x, y, z, since d is a metric we have

$$\rho(x, y) = d(\psi(x), \psi(y)) \leq d(\psi(x), \psi(z)) + d(\psi(z), \psi(y)) = \rho(x, z) + \rho(z, y).$$

Thus ρ is a metric.

(b) The function $\psi(x) = x/(1 + |x|)$ is continuous on $\mathbb{R}$. You can check (separately for positive and negative x) that $\psi'(x) > 0$ for all x, so ψ is strictly increasing, and $\psi(x) \to -1$ as $x \to -\infty$, $\psi(x) \to 1$ as $x \to +\infty$. Therefore ψ is a bijection from $\mathbb{R}$ to $(-1, 1)$. Applying part (a), with d being Euclidean distance, shows that ν is a metric.

Since ψ is continuous, if $|x_n - x| \to 0$ then $\psi(x_n) \to \psi(x)$ so $\nu(x_n, x) = |\psi(x_n) - \psi(x)| \to 0$. In the other direction, you can calculate that the inverse function is

$$\psi^{-1}(y) = \begin{cases} \frac{y}{1-y} & \text{if } y \in [0, 1), \\ \frac{y}{1+y} & \text{if } y \in (-1, 0), \end{cases}$$

which is also continuous. Therefore if $\nu(x_n, x) = |\psi(x_n) - \psi(x)| \to 0$ then $|x_n - x| = |\psi^{-1}(\psi(x_n)) - \psi^{-1}(\psi(x))| \to 0$.

(c) Notice that $\rho(f, g) = \sum_{k=1}^\infty 2^{-k}\nu(f(a_k), g(a_k))$. Therefore $\rho(f, f) = \sum_{k=1}^\infty 2^{-k}\nu(f(a_k), f(a_k)) = \sum_{k=1}^\infty 2^{-k} \cdot 0 = 0$. For $f \neq g$, there must be some j with $f(a_j) \neq g(a_j)$, so

$$\rho(f, g) = \sum_{k=1}^{\infty} 2^{-k}\nu(f(a_k), g(a_k)) \geq 2^{-j}\nu(f(a_j), g(a_j)) > 0.$$

Since ν is a metric, we have $\nu(f(a_k), g(a_k)) = \nu(g(a_k), f(a_k))$ for all k, so $\rho(f, g) = \rho(g, f)$. For the triangle inequality, given functions f, g, h, since ν is a metric we have for all k:

$$\nu(f(a_k), g(a_k)) \leq \nu(f(a_k), h(a_k)) + \nu(h(a_k), g(a_k)).$$

Multiplying both sides by 2^{-k} and summing over k shows that $\rho(f,g) \le \rho(f,h) + \rho(h,g)$. Thus ρ is a metric.

Now suppose $f_n \to f$ pointwise on A, and let $\epsilon > 0$. Observe first that since $\psi(x) \in (-1,1)$ for all x, we have

$$(*) \qquad \nu(x,y) = |\psi(x) - \psi(y)| < 2 \quad \text{for all } x, y.$$

Second, summing the geometric series shows that for all K, $\sum_{k>K} 2^{-k} = 2^{-K}$. We choose K so that $2^{-K} < \epsilon/4$. For each fixed $k \le K$ we have $f_n(a_k) \to f(a_k)$ as $n \to \infty$ by pointwise convergence, so by part (b), $\nu(f_n(a_k), f(a_k)) \to 0$ as $n \to \infty$. Therefore the finite sum

$$(**) \qquad \sum_{k=1}^{K} 2^{-k}\nu(f_n(a_k), f(a_k)) \to 0 \quad \text{as } n \to \infty,$$

so for large n, using $(*)$ and $(**)$,

$$\begin{aligned}
\rho(f_n, f) &= \sum_{k=1}^{K} 2^{-k}\nu(f_n(a_k), f(a_k)) + \sum_{k=K+1}^{\infty} 2^{-k}\nu(f_n(a_k), f(a_k)) \\
&< \frac{\epsilon}{2} + \sum_{k=K+1}^{\infty} 2^{-k} \cdot 2 \\
&= \frac{\epsilon}{2} + 2^{-K} \cdot 2 \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, this shows $\rho(f_n, f) \to 0$.

Conversely suppose $\rho(f_n, f) \to 0$. For each k we have $\rho(f_n, f) \ge 2^{-k}\nu(f_n(a_k), f(a_k))$ so $\nu(f_n(a_k), f(a_k)) \to 0$ as $n \to \infty$. By part (b) this means $f_n(a_k) \to f(a_k)$ as $n \to \infty$ for all k, that is, $f_n \to f$ pointwise on A.

(IV) Let g_n, g be the functions in problem (I); note that $g_0 \equiv 0$. For $n \ge 1$ let $f_n = g_n - g_{n-1}$. We then have the telescoping sum $\sum_{n=1}^{N} f_n = \sum_{n=1}^{N}(g_n - g_{n-1}) = g_N - g_0 = g_N$, and hence the series converges pointwise by (I)(a), meaning that for each x, the limit exists in the following:

$$\sum_{n=1}^{\infty} f_n(x) = \lim_{N\to\infty} \sum_{n=1}^{N} f_n(x) = \lim_{N\to\infty} g_N(x) = g(x).$$

But, by problem (I)(b),

$$\sum_{n=1}^{N} \int_0^1 f_n \, dx = \int_0^1 \left(\sum_{n=1}^{N} f_n \right) \, dx = \int_0^1 g_N \, dx \not\to \int_0^1 g \, dx,$$

which is the same as $\sum_{n=1}^{\infty} \int_0^1 f_n \, dx \ne \int_0^1 g \, dx$.

(V)(a) Given $\epsilon > 0$ let $\delta = \epsilon^{1/\alpha}$. Then

$$f \in E, |y - x| < \delta \implies |f(y) - f(x)| \le |y - x|^\alpha < \delta^\alpha = \epsilon,$$

i.e. this δ "works" uniformly over E. This shows E is equicontinuous.

(b) Suppose f is a limit point of E. Then there is a sequence $f_n \in E$ with $f_n \to f$ uniformly. (The uniformity is because we are dealing in the uniform metric, i.e. sup norm distance.) Then $f(0) = \lim_n f_n(0) = 0$ and for all $x, y \in [0, 1]$,

$$|f(x) - f(y)| = \lim_n |f_n(x) - f_n(y)| \le |y - x|^\alpha,$$

so $f \in E$.

(VI) Since $[a, b]$ is compact, each f_i is uniformly continuous. Hence given $\epsilon > 0$, for each $i \le n$ there is a $\delta_i > 0$ such that $|y - x| < \delta_i$ implies $|f_i(y) - f_i(x)| < \epsilon/n$. Let $\delta = \min(\delta_1, .., \delta_n)$. Then for $f = \sum_{i=1}^n c_i f_i \in \mathcal{F}$ and $x, y \in [a, b]$ with $|y - x| < \delta$, we have

$$|f(y) - f(x)| \le \sum_{i=1}^n |c_i||f_i(y) - f_i(x)| < \sum_{i=1}^n 1 \cdot \frac{\epsilon}{n} = \epsilon.$$

This shows $\mathcal{F}$ is equicontinuous.