MATH 425b ASSIGNMENT 10 SOLUTIONS
SPRING 2016
Prof. Alexander

Chapter 10:

(16) For $k = 2$: Let $\sigma = [p_0, p_1, p_2]$. Then $\partial\sigma = [p_1, p_2] - [p_0, p_2] + [p_0, p_1]$ so

$$\partial^2\sigma = [p_2] - [p_1] - ([p_2] - [p_0]) + [p_1] - [p_0] = 0$$

since all terms cancel.

For $k = 3$: let $\sigma = [p_0, p_1, p_2, p_3]$. Then

$$\partial\sigma = [p_1, p_2, p_3] - [p_0, p_2, p_3] + [p_0, p_1, p_3] - [p_0, p_1, p_2]$$

so

$$\begin{aligned}\partial^2\sigma &= \Big([p_2, p_3] - [p_1, p_3] + [p_1, p_2]\Big) - \Big([p_2, p_3] - [p_0, p_3] + [p_0, p_2]\Big) \\ &\quad + \Big([p_1, p_3] - [p_0, p_3] + [p_0, p_1]\Big) - \Big([p_1, p_2] - [p_0, p_2] + [p_0, p_1]\Big) \\ &= 0\end{aligned}$$

since again all terms cancel.

For a k-chain $\Psi = \Phi_1 + .. + \Phi_r$ in $\mathbb{R}^n$ (with $k = 2, 3$), we have $\partial^2\Psi = \sum_{i=1}^r \partial^2\Phi_i$ so it's enough to show $\partial^2\Phi = 0$ for all surfaces $\Phi = T \circ \sigma$ (where σ is affine and T is $\mathcal{C}''$, as in 10.30.) By definition, $\partial^2\Phi = T(\partial^2\sigma)$, so for every $\mathcal{C}''$ $(k-2)$-form ω,

$$\int_{\partial^2\Phi} \omega = \int_{T(\partial^2\sigma)} \omega = \int_{\partial^2\sigma} \omega_T = 0.$$

This shows that $\partial^2\Phi = 0$.

(20) Suppose Φ is a k-surface of class $\mathcal{C}''$ in an open $V \subset \mathbb{R}^n$, ω is a $(k-1)$-form of class $\mathcal{C}'$ in V, and f is a $\mathcal{C}'$ function on V. Then $f\omega$ is a $(k-1)$-form of class $\mathcal{C}'$ on V so by Stokes Theorem and 10.20a,

$$\int_{\partial\Phi} f\omega = \int_{\Phi} d(f\omega) = \int_{\Phi} df \wedge \omega + \int_{\Phi} f\ d\omega.$$

(24) Let $\omega = \sum_i a_i(\mathbf{x})\ dx_i$ be a 1-form of class $\mathcal{C}''$ in E with $d\omega = 0$, and let $\mathbf{p} \in E$. Define $f(\mathbf{x}) = \int_{[\mathbf{p},\mathbf{x}]} \omega$, for $x \in E$. By Stokes Theorem, for $\mathbf{x} \neq \mathbf{y}$ and $\gamma(t) = (1-t)\mathbf{x} + t\mathbf{y}$,

$$0 = \int_{[\mathbf{p},\mathbf{x},\mathbf{y}]} d\omega = \int_{\partial[\mathbf{p},\mathbf{x},\mathbf{y}]} \omega = \int_{[\mathbf{p},\mathbf{x}]} \omega - \int_{[\mathbf{p},\mathbf{y}]} \omega + \int_{[\mathbf{x},\mathbf{y}]} \omega$$

$$= f(\mathbf{x}) - f(\mathbf{y}) + \int_0^1 \sum_{i=1}^n a_i(\gamma(t))\gamma_i'(t)\ dt = f(\mathbf{x}) - f(\mathbf{y}) + \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i(\gamma(t))\ dt.$$

Taking $\mathbf{y} = \mathbf{x} + he_i$ we get

$$\frac{f(\mathbf{y}) - f(\mathbf{x})}{h} = \int_0^1 a_i(\mathbf{x} + the_i)\ dt \to a_i(\mathbf{x}) \quad \text{as } h \to 0,$$

since $the_i \to 0$ uniformly in t, as $h \to 0$. Thus $(D_i f)(\mathbf{x}) = a_i(\mathbf{x})$, so $\omega = df$.

(I) Let $\omega = M\ dx + N\ dy$, so $d\omega = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \wedge\ dy$. Therefore the following are equivalent:

(i) the differential equation is exact;

(ii) $d\omega = 0$;

(iii) ω is closed;

(iv) ω is exact;

(v) $\omega = dF$ for some F;

(vi) $\omega = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$;

(vii) $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$.

(II) Define $\Phi : [0,1]^2 \to \mathbb{R}^3$ by $\Phi(x,y) = (x, y, x^4 + y^2)$. Let

$$J_1(x,y) = \text{ Jacobian of } (y, x^4 + y^2) = \det \begin{bmatrix} 0 & 1 \\ 4x^3 & 2y \end{bmatrix} = -4x^3,$$

$$J_2(x,y) = \text{ Jacobian of } (x^4 + y^2, x) = \det \begin{bmatrix} 4x^3 & 2y \\ 1 & 0 \end{bmatrix} = -2y,$$

$$J_3(x,y) = \text{ Jacobian of } (x,y) = \det I = 1.$$

Then

$$\begin{aligned}
\int_\Phi \omega &= \int_0^1 \int_0^1 \left[xJ_1(x,y) + yJ_2(x,y) + (x^4+y^2)J_3(x,y) \right] \, dx \, dy \\
&= \int_0^1 \int_0^1 \left[-4x^4 - 2y^2 + x^4 + y^2 \right] \, dx \, dy \\
&= \int_0^1 \left[\left(-3\frac{x^5}{5} - y^2 x \right) \Big|_0^1 \right] \, dy \\
&= \int_0^1 \left(-\frac{3}{5} - y^2 \right) \, dy \\
&= \left(-\frac{3}{5}y - \frac{1}{3}y^3 \right) \Big|_0^1 \\
&= -\frac{3}{5} - \frac{1}{3} \\
&= -\frac{14}{15}.
\end{aligned}$$

(III) Use Stokes Theorem: $\int_S \omega = \int_E d\omega$. We have $d\omega = (yz + 2y + 1) \, dx \wedge dy \wedge dz$ so

$$\int_E d\omega = \int_{-2}^2 \int_{x^2}^4 \int_0^1 (yz + 2y + 1) \, dz \, dy \, dx.$$

Now

$$\int_0^1 (yz + 2y + 1) \, dz = (y\frac{z^2}{2} + 2yz + z)\Big|_0^1 = \frac{5}{2}y + 1,$$

while

$$\int_{x^2}^4 (\frac{5}{2}y + 1) \, dy = (\frac{5}{4}y^2 + y)\Big|_{x^2}^4 = 24 - \frac{5}{4}x^4 - x^2,$$

so

$$\int_E d\omega = \int_{-2}^2 (24 - \frac{5}{4}x^4 - x^2) \, dx = (24x - \frac{1}{4}x^5 - \frac{1}{3}x^3)\Big|_{-2}^2 = \frac{224}{3}.$$

(IV)(a)

$$d\eta = \frac{\partial}{\partial x}\frac{x}{x^2+y^2} \, dx \wedge dy - \frac{\partial}{\partial y}\frac{y}{x^2+y^2} \, dy \wedge dx = \frac{y^2 - x^2}{x^2+y^2} \, dx \wedge dy + \frac{x^2-y^2}{x^2+y^2} \, dx \wedge dy = 0.$$

(b) Since $\arctan u$ has derivative $1/(1+u^2)$, wherever $x \neq 0$ we have by the chain rule

$$d \arctan \frac{y}{x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(\frac{-y}{x^2} \right) dx + \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(\frac{1}{x} \right) dy = \frac{x \, dy - y \, dx}{x^2 + y^2} = \eta.$$

Similarly, wherever $y \neq 0$ we have $-d \arctan \frac{x}{y} = \eta$.

(c)

$$\begin{aligned}
\int_\gamma \eta &= \int_0^{2\pi} \left[\frac{-\gamma_2(t)}{\gamma_1(t)^2 + \gamma_2(t)^2} \gamma_1'(t) + \frac{\gamma_1(t)}{\gamma_1(t)^2 + \gamma_2(t)^2} \gamma_2'(t) \right] \, dt \\
&= \int_0^{2\pi} [\sin^2 t + \cos^2 t] \, dt \\
&= \int_0^{2\pi} 1 \, dt = 2\pi.
\end{aligned}$$

(d) If $\eta = df$ in $\mathbb{R}^2/\{0\}$, then since γ is a closed loop we have $\int_\gamma \eta = \int_\gamma df = 0$, contradicting part (c). Thus η is not exact.