Note 3 with HW3

Wharton school video on stock-based compensation:

https://www.youtube.com/watch?v=BXejPU1tapQ
Pay-for-Performance
Employee Stock Grants or Options
Enterprise Management Incentives in UK
Taxation and Accounting Issues
Delayed Tax Payment
Pension System

Questions related to employee stock options:

According to Warren Buffett, investor Chairman & CEO of Berkshire Hathaway, "[T.]here is no question in my mind that mediocre CEOs are getting incredibly overpaid. And the way it’s being done is through stock options."

Other criticisms include:

Dilution can be very costly to shareholder over the long run.

Stock options are difficult to value.

Stock options can result in egregious compensation of executive for mediocre business results.

An individual employee is dependent on the collective output of all employees and management for a bonus.[citation needed]

financial crisis and pay-for-performance

Other critics of (conventional) stock option plans in the US include supporters of "reduced-windfall" or indexed options for executive/management compensation. Reduced-windfall options would adjust option prices to exclude "windfalls" such as falling interest rates, market and sector-wide share price movements, and other factors unrelated to the managers’ own efforts.

Broad-based options remain the norm in high-technology companies and have become more widely used in other industries as well. Larger, publicly traded companies such as Starbucks, Southwest Airlines, and Cisco now give stock options to most or all of their employees. Many non-high tech, closely held companies are joining the ranks as well.

Modelling uncertainty

Uncertainly about the value of a random variable modelled by a cumulative distribution

$F(x)$ over some interval $[\alpha, \beta]$

$F(x) \in [0,1]$ means the probability the variable is below or equal to the value x.

The variable could be the buyer's willingness to pay, the income levels of the buyers, the number of participants in an auction, the date of expiration of a contract, whether an action will be taken by a rival buyer, etc.

Discrete distribution: If we don't know whether some action will be taken, it is a 0-1 discrete case: Either the action is taken or it is not. It is easy do describe the uncertainty in this case. We just let p be the probability the action will be taken. Let $x = 0$ means the action is not taken, and $x = 1$ means it is taken. When we describe it by a cumulative distribution, $F(.)$ in this case is a step function defined by

$$F(x) = 1 - p \text{ for } x < 1,$$
$$F(x) = 1 \text{ for } x \geq 1.$$

With a discrete variable which may take n possible values, arranged in increasing sequence

$$x_1, x_2, ,, ., x_n,$$

with assigned probabilities

$$p_1, p_2, \ldots, p_n,$$

the function $F(.)$ is given by

$$\begin{aligned} F(x) &= p_1, \text{ for } x < p_2, \\ &= p_1 + p_2 \text{ for } x \in [p_2, p_3), \\ &= p_1 + p_2 + p_3 \text{ for } x \in [p_3, p_4). \\ &\quad \ldots. \\ &= 1 - p_n \text{ for } x < p_n, \\ &= 1 \text{ for } x \geq p_n. \end{aligned}$$

More generally, when x is a continuous variable, the function $F(.)$ is an increasing function defined over the range of the variable $[\alpha, \beta]$, with $F(\alpha) = 0, F(\beta) = 1$. The derivative $f(x) = \frac{d}{dx}F(x)$ is called the density of the distribution $F(.)$. A uniform distribution over $[\alpha, \beta]$ has the form

$$F(x) = \frac{x - \alpha}{\beta - \alpha},$$

which is the linear function determined by the boundary condition $F(\alpha) = 0, F(\beta) = 1$. A uniform distribution $F(.)$ has a constant density $f(x) = \frac{1}{\beta - \alpha}$. When $\beta - \alpha$ has unit length, the density is 1. A normal distribution has an well-defined density. Any uncertainty about a variable can be represented by either $F(.)$ or $f(.)$. The expected value of a distribution is given by

$$E[F] = \int_{\alpha}^{\beta} x dF(x) = \int_{\alpha}^{\beta} x f(x) dx.$$

Using integration by parts, we can also write

$$E[F] = \beta - \int_{\alpha}^{\beta} F(x)dx.$$

When $F(.)$ is a uniform distribution, the expected value of $F(.)$ can be computed by the simple formula

$$E[F] = \frac{\alpha + \beta}{2}.$$

So you simply compute the average of the highest and lowest value, and you get the expected value of the random variable. To see this, just compute

$$\int_{\alpha}^{\beta} xdF(x) = \int_{\alpha}^{\beta} xf(x)dx = \frac{1}{\beta - \alpha}\int_{\alpha}^{\beta} xdx = \frac{1}{\beta - \alpha}\frac{1}{2}(\beta^2 - \alpha^2)$$
$$= \frac{\alpha + \beta}{2}.$$

To say $F_1(.)$ is higher than $F_2(.)$ (for example one stock has higher value than another), we can use the concept of first-order stochastic dominance. We say that $F_1(.)$ over $[\alpha_1, \beta]$ dominates $F_2(.)$ over $[\alpha_2, \beta]$ in the first-order sense if $\alpha_1 \geq \alpha_2$, and

$$F_1(x) \leq F_2(x) \text{ for all } x \in [\alpha_2, \beta].$$

In this case, $F_1(.)$ has higher expected value than $F_2(.)$. Intuitively, the inequality means that for $F_1(.)$, the probability is more concentrated in the higher value range than $F_2(.)$. For instance $F_1(x) = x^2$ dominates $F(x)$ over $[0,1]$ in the first-order sense because $x^2 \leq x$. In general, over the interval $[0,1]$, higher power x^n means higher rank in the first-order stochastic dominance. Convexity of $F(.)$ means that $F(.)$ has higher density in higher range. Concavity means that it has lower density in higher range.

A uniform distribution over $[2,4]$ can be described by

$$F(x) = \frac{1}{2}(x-2) = -1 + \frac{1}{2}x.$$

If y is related to x by $y = h(x)$, then a distribution $F(.)$ over $[\alpha, \beta]$ of x implies a distribution of y, and in this case, y is distributed over $[h(\alpha), h(\beta)]$, given by

$$F_h(x) = F(h^{-1}(x)).$$

For instance, if x is uniformly distributed over $[0,1]$, and $y = h(x) = 2x$, then y is distributed over $[0,2]$ with

$$F_h(x) = F(\tfrac{x}{2}) = \frac{x}{2},$$

which is the uniform distribution over $[0,2]$. if $h(.)$ is an increasing linear function, and $F(.)$ is a uniform distribution, then $F_h(.)$ is also a uniform distribution. When x is the productivity parameter of an agent, the function $h(x)$ can be the outside option of the agent, or the revenue generated by the agent or net profit of the principal.

We say that $F_1(.)$ dominates over $F_2(.)$ in the sense of second-order stochastic dominance if $E(F_1) = E(F_2)$ and $F_2(.)$ is a mean preserving spread of $F_1(.)$. This means that $F_1(.)$ is a c.d.f. of a random variable X_1, ε is an independent variable of X_1 with zero mean (We call ε a white noise), and $F_2(.)$ is a c.d.f. of $X_1 + \varepsilon$. For normal random variables, it just means that $F_2(.)$ has higher variance than $F_1(.)$, and variance is defined as

$$var(F(.)) \;=\int_{\alpha}^{\beta}(x-E(F))^2 dF(x).$$

For uniform distributions, $F_1(.)$ over $[\alpha_1, \beta_1]$, and $F_2(.)$ over $[\alpha_2, \beta_2]$, $F_1(.)$ dominates $F_2(.)$ in the sense of second-order stochastic dominance if and only if $\frac{\alpha_1+\beta_1}{2} = \frac{\alpha_2+\beta_2}{2}$ and $\beta_1-\alpha_1 < \beta_2-\alpha_2$. For example, when $F_1(v) = \frac{v-4}{2}$ over $[4.6]$, and $F_2(v) = \frac{v-2}{6}$ over $[2,8]$, we $F_1(.)$ dominates $F_2(.)$ in the sense of second-order stochastic dominance. In finance literature, we say that $F_2(.)$ has higher risk than $F_1(.)$.

First-order stochastic dominance: higher return

Second-order stochastic dominance: higher risk

The following example illustrates the simple theory of updating of beliefs based on new information, and the importance of updating of beliefs in decision making.

The Three Door puzzle

There is a prize hidden behind one of the three doors. You choose one of the doors, and then one door that does not contain the prize is removed. You are then asked to decide whether you want to switch your choice. Most people don't, but here is the computation of the likelihood of winning the prize when you make different choices:

If you don't switch, the removal of one of the doors does not help you. Your probability of winning is 1/3 before the removal or after the removal.

If you switch, the removal of one door that contains no prize means that you will get that prize if your initial choice wasn't correct. Since your initial choice wasn't correct with probability 2/3, you will get the price with probability 2/3.

Here we compute conditional probabilities on different events:

Event one: your initial choice is correct. This occurs with probability 1/3.

If you don't switch, you have a probability 1 of success conditional on the event one. If you don't switch, you have a probability of 0 of winning in this event.

Event two: your initial choice is wrong. This occurs when probability 2/3.

If you don't switch, you have a probability 0 of winning conditional on this event. If you don't switch, you have a probability 1 of winning in this event.

Four-door paradox:

There is a prize hidden behind one of the four doors. You choose one of the doors, and then one door that does not contain the prize is removed. You are then asked to decide whether you want to switch your choice. Here is the computation of the likelihood of winning the prize when you make different choices:

If you don't switch, the removal of one of the doors does not help you. Your probability of winning is 1/4 before the removal or after the removal.

If you switch, the removal of one door that contains no prize means that you will get that prize with probability 1/2, if your initial choice wasn't correct. Since your initial choice wasn't correct with probability 3/4, you will get the prize with probability $\frac{3}{4} * \frac{1}{2}$.

Here we compute conditional probabilities on different events:

Event one: your initial choice is correct. This occurs with probability 1/4.

If you don't switch, you have a probability 1 of success conditional on the event one. If you do switch, you have a probability of 0 of winning in this event.

Event two: your initial choice is wrong. This occurs when probability 3/4.

If you don't switch, you have a probability 0 of winning conditional on this event. If you do switch, you have a probability 1/2 of winning in this event.

Conditional probability: Let $prob(E_i)$ be the probability of event, and $prob(W \cap E_i)$ be the probability of winning and event E_i holds (join occurrence). Then the conditional (on the event E_i) probability of winning is

$$\frac{prob(W \cap E_i)}{prob(E_i)}$$

Let $\cup E_i$ have probability one, and $prob(E_i \cap E_j) = 0$ for all $i \neq j$. We call these E_i a partition. The over all probability of winning is the probability of event times conditional probability of winning after the event, summed over all possible events in a partition.

$$\sum_i probality(E_i) * (winning\ probability\ conditional\ on\ E_i)$$

Nash equilibrium concept in a tort law example

Normal form game (or strategic form):

A game of n players, in which we have a list of strategies for each player $a_i \in A_i$, and a payoff for each player from any combination of strategies $u_i(a_1, a_2, \ldots, a_n)$

matrix form for two players:

		player 2	
		wage1	wage2
player 1	accept	(4 ,5)	(7, 2)
	refuse	(0, 3)	(0 ,3)

A Nash equilibrium (a_1^*, a_2^*) is one strategy pair that satisfies the following conditions

$$u_1(a_1^*, a_2^*) \geq u_1(a_1, a_2^*) \text{ for all } a_1$$

$$u_2(a_1^*, a_2^*) \geq u_2(a_1^*, a_2) \text{ for all } a_2$$

No one player has an incentive to deviate (as long as the other player does not respond to his deviation). In a dynamic context, this concept can be applied, but may need refinements. A strategy is a sequence of planned actions in the dynamic context. When there is uncertainty about payoffs, this concept can be extended to Bayesian Nash equilibrium in which each action is a strategy contingent on different events.

Checking for a pure strategy Nash equilibrium is simple. In the above game (accept, wage 1) is a pure strategy Nash equilibrium. It is also a dominant strategy equilibrium. A dominant strategy Nash equilibrium satisfies the following conditions

$$u_1(a_1^*, a_2) \geq u_1(a_1, a_2) \text{ for all } a_1, a_2$$

$$u_2(a_1, a_2^*) \geq u_2(a_1, a_2) \text{ for all } a_2$$

so that the optimal property of a_i^* is independent of the action of the other player. In the game above, the action "accept" is a dominant strategy, because it is the best action, independent of what player two is doing.

Concept of mixed strategy Nash equilibrium. Checking the mixed strategy equilibrium in the 2-by-2 case is quite easy. If a player uses two strategies with positive probabilities, the two strategies should yield the same payoff to the player. This will be enough to allow us to check whether a mixed strategy satisfies the equilibrium properties. For example in the matching pennies game, player one wins one dollar from player two, if the two players show the same side of the coin, otherwise, player two pays one dollar to player two:

		player 2	
		head	tail
player 1	head	(1 ,-1)	(-1, 1)
	tail	(-1, 1)	(1 ,-1)

This game has no pure strategy Nash equilibrium, but it has a mixed strategy Nash equilibrium in which both players randomize between showing head and tail with probability of 0.5 for each. To show that this is a Nash equilibrium, It is enough to show that the two strategies yield the same payoff to each player.

Tort law example:

Regime of no liability: Player one is the pedestrian, and player two is the driver. Each player has two actions: no care of due care. If any player exerts no care, accident occurs for sure. If both exercise due care, then accident occurs with probability 0.1. The loss to the pedestrian is $100 when accident occurs. The motorist suffers no loss from accident, but we can assign legal responsibility of compensation to the driver. No liability means that there is no compensation by the motorist to the pedestrian. The cost of due care is $10 to each player. The normal form game is given by

		motorist	
		no care	due care
pedestrian	no care	(-100 ,0)	(-100 , -10)
	due care	(-110 ,0)	(-20 ,- 10)

Here (no care, no care) is a Nash equilibrium. This outcome is not efficient (the one with the highest sum of payoffs). Intuitively, the driver has no incentive to drive carefully ("no care" is a dominant strategy), and knowing that the pedestrian has no incentive to exercise due care.

Regime of pure strict liability: the motorist is liable for any loss from the accident, no matter who is at fault

		motorist	
		no care	due care
pedestrian	no care	(0 ,-100)	(0 , -110)
	due care	(-10 ,-100)	(-10 ,- 20)

Here the pedestrian has no incentive to exercise due care ("no care" is a dominant strategy), and knowing that the driver has no incentive to exercise due care either. (no care, no care) is a Nash equilibrium, and is not efficient.

Regime of negligence with contributory negligence: the pedestrian is entitled to compensation only if the motorist is negligent, and the pedestrian is not:

		motorist	
		no care	due care
pedestrian	no care	(-100 ,0)	(-100 , -10)
	due care	(-10 ,-100)	(-20 ,- 10)

(due care, due care) now is a Nash equilibrium). It is an efficient outcome.

Efficient outcome: the outcome in which the total payoff is the highest possible

Nash Equilibrium: the outcome in which no one has the incentive to change the strategy, given the others' strategies:

Nash equilibrium may or may not be efficient
Checking whether we have an efficient outcome

Risk aversion:
Strict liability
Initial wealth for motorist w, cost of due care c, probability of accident p_0 for no care, and $p_1 < p_0$ for due care, damage amount if accident occurs D
utility function $u(I) = \sqrt{I}$ for the pedestrian

normal form game

		motorist	
		no care	due care
pedestrian	no care	$(0,\ (1-p_0)\sqrt{w} + p_0\sqrt{w-D})$	$(0, (1-p_1)\sqrt{w-c} + p_1\sqrt{w-D-c})$
	due care	$(-c,\ (1-p_0)\sqrt{w} + p_0\sqrt{w-D})$	$(-c,\ (1-p_1)\sqrt{w-c} + p_1\sqrt{w-D-c}))$

When do you get equilibrium with due care?
When do you get equilibrium with no care?
What is the efficient outcome?
When do you get efficient equilibrium?

Regime of negligence with contributory negligence:

Suppose accident occurs with probability p_0 if at least one is negligent, and with probability p_1 if every one is careful.

		motorist	
		no care	due care
pedestrian	no care	$(-p_0D,\ \sqrt{w})$	$(-p_0D, \sqrt{w-c})$
	due care	$(-c,\ (1-p_0)\sqrt{w} + p_0\sqrt{w-D})$	$(-c-p_1D,\ \sqrt{w-c})$

In the following game, there is no contract between General Foods and the diary. Each decides on its own whether to test high or low. There are two possible tests: high or low. The high test costs more, but is more accurate, the low test is cheaper but less accurate. First we assume that the diary chooses which test to use, and General Foods knows whether the test has been conducted and its results.

The diary can sell its milk to General Foods, or processes the milk itself to make other products. If it does so, it makes profit \$3, and General Foods makes no profit because of no transaction with the diary. The payoff vector is (3,0).

The diary can sell the dry milk to General Foods, but it chooses whether to test high or low. If it tests high, and General Foods tests high, the payoff is (1,1). If it tests high, but General Foods tests low, the payoff is (1,3), reflecting the cost saving of the low test. If it tests low, and General Foods tests high, the payoff is (5,1), reflecting cost saving and higher revenue potential of selling it to General Foods. It it tests low, and General Foods also tests low, the ensuing tort liability reduces their profits to (0,0).

		hh	hl	lh	ll
diary	sh	(1,1)	(1,3)	(1,1)	(1,3)
	sl	(5,1)	(0,0)	(5,1)	(0,0)
	no sale	(3,0)	(3,0)	(3,0)	(3,0)

There are two Nash equilibria: (sh,hl) with payoff (1,3), and (sl,lh) with payoff (5,1).

From the perspective of the diary, the best outcome is (low, high): it tests low, and sells to General Foods which tests high.

However, from the perspective of General Foods, the best outcome is for the diary to test high, and sells it, and General Foods tests low.

Without asymmetric information, in this game, the Nash equilibrium outcome is (low, high), which is the best outcome of the diary. Backward induction can be used to find this equilibrium. Tree representation is often used in dynamic games.

HW3 due Feb 1

1. Player one is the pedestrian, and player two is the driver. Each player has two actions:

no care of due care. If any player exerts no care, accident occurs with probability 0.3. If both exercise due care, then accident occurs with probability 0.1. The loss to the pedestrian is $100 when accident occurs. The motorist suffers no loss from accident, but we can assign legal responsibility of compensation to the driver. The cost of due care is $10 to each player.

(A) Construct the normal form game under each of the following regimes: (1) no liability (2) strict liability (3) negligence with contributory negligence.

(B) Find pure strategy Nash equilibrium for each game above and determine whether they are efficient or not.

2. In the matching pennies game, assume that the when player one wins (match case), he gets $100 from player two, but when player two (no match), the payment is only $50. In this case, find the mixed strategy equilibrium of the game.

3. Suppose we have a unilateral model in the pedestrian-motorist game (the problem one above is not). In product liability, often it is a unilateral model. This means that the accident is caused only by the motorist level of care, and the pedestrian care does not affect the outcome. Specifically, when the motorist exercises no care, there is accident for sure which causes injury $100 to the pedestrian. If the motorist exercises due care, then the accident occurs only with probability 0.1 with the same size of injury to the pedestrian. The cost of due care to each party is still $10. We also assume that the motorist initial wealth is $120. Assume we have a strict liability system for the motorist.

(A) Construct the norrmal game form.

(B) What is the Nash equilibrium? What is the efficient outcome?

(C) Is the strict liability system efficient? Give an intuitive explanation.

4. In problem 3, now assume that the motorist is risk averse with the utility function $\sqrt{c}$. We maintain the other assumptions. Construct the normal form game and find the Nash equilibrium.

5. In problem 4, assume that the probability of accident is 0.85 when he exercises due care, other things are unchanged. The story is that the motorist is getting senile, and due care can only reduce the probability of accident by a small amount.

(A) If the motorist is risk neutral, construct the normal form game. Find the Nash equilibrium in this case.

(B) If the motorist is risk averse (with the square root utility function), construct the normal form game, and find the Nash equilibrium in this case.