MATH 425b MIDTERM EXAM 1 SOLUTIONS
Spring 2016
Prof. Alexander

(1) Let $t_N = \sum_{n=1}^N a_n \phi_n$. Then from the proof of 8.11,

$$(*) \qquad \int_a^b |f - t_N|^2 \, dx = \int_a^b |f|^2 \, dx - \sum_{n=1}^N |c_n|^2 + \sum_{n=1}^N |c_n - a_n|^2.$$

By 8.12, we have $\int_a^b |f|^2 \, dx \geq \sum_{n=1}^N |c_n|^2$ so by $(*)$,

$$(**) \qquad \int_a^b |f - t_N|^2 \, dx \geq \sum_{n=1}^N |c_n - a_n|^2.$$

By assumption the sum on the right in $(**)$ includes a strictly positive term, so it does not approach 0 as $N \to \infty$. Therefore also the left side $\int_a^b |f - t_N|^2 \, dx \not\to 0$ as $N \to \infty$, that is, $t_N \not\to f$ in L^2 as $N \to \infty$.

(2)(a) Since $a_n \to 0$, for all large n we have $|a_n| < 1$, so $|a_n|^{1/n} < 1$. Therefore $\limsup_n |a_n|^{1/n} \leq 1$, so $R \geq 1$.

(b) If $\sum_n a_n$ converges, then by Theorem 8.2, $f(x) \to \sum_n a_n$ as $x \nearrow 1$. But from the formula $f(x) = (1-x)^{-2/3}$, we have $f(x) \to \infty$ as $x \nearrow 1$, a contradiction. Therefore $\sum_n a_n$ diverges.

(3) Note the assumption $f(0) = 0$ for all $f \in \mathcal{F}$ was added during the exam.

By equicontinuity, there exists $\delta > 0$ such that

$$(\#) \qquad |y - x| \leq \delta \implies |f(y) - f(x)| \leq 1 \quad \text{for all } f \in \mathcal{F}.$$

Hence $|f(n\delta) - f((n-1)\delta)| \leq 1$ for all n, so by the triangle inequality and $(\#)$,

$$\begin{aligned} |f(n\delta)| &\leq |f(0)| + \sum_{j=1}^n |f(j\delta) - f((j-1)\delta)| \\ &\leq 0 + \sum_{j=1}^n 1 \\ &= n = \frac{1}{\delta} \cdot n\delta. \end{aligned}$$

For general x we have $n\delta \leq x < (n+1)\delta$ for some n, so $|x - n\delta| < \delta$, so by $(\#)$, for all

$f \in \mathcal{F}$,

$$\begin{aligned} |f(x)| &\le |f(n\delta)| + |f(x) - f(n\delta)| \\ &\le |f(n\delta)| + 1 \\ &\le \frac{1}{\delta} \cdot n\delta + 1 \\ &\le \frac{1}{\delta} x + 1. \end{aligned}$$

This proves the result with $A = 1, B = 1/\delta$.

(4)(a) Choose any function f in $\mathcal{A}$ which is one-to-one and never 0 on $[0, 1)$, for example $f(x) = 1 - x$. Since $f(x) \neq f(y)$ for all $x \neq y$, $\mathcal{A}$ separates points. Since $f(x) \neq 0$ for all $x \in [0, 1)$, $\mathcal{A}$ vanishes at no point of $[0, 1)$.

(b) Take any $g \notin \mathcal{A}$, for example $g(x) \equiv 1$. Given $f \in \mathcal{A}$, since $\lim_{x\to 1} f(x) = 0$ there exists $x \in [0, 1)$ with $f(x) < 1/2$, and therefore

$$\|f - g\|_\infty \ge |f(x) - g(x)| > |\tfrac{1}{2} - 1| = \frac{1}{2}.$$

Thus there is no $f \in \mathcal{A}$ with $\|f - g\|_\infty < \frac{1}{2}$, so $\mathcal{A}$ is not dense in $C_B[0, 1)$.

This doesn't violate Stone-Weierstrass because the domain $[0, 1)$ is not compact.