MATH 425a ASSIGNMENT 1 SOLUTIONS
FALL 2011 Prof. Alexander

These solutions are for the individual use of Math 425a students and are not to be distributed outside that group.

Rudin Chapter 1:

(1) If $r + x$ is rational, so is $x = (r + x) - r$. This is equivalent to the statement that if x is irrational, so is $r + x$.

Similarly, if rx is rational, then so is $x = (rx)/r$. This is equivalent to the statement that if x is irrational, so is rx.

(2) If $\sqrt{12} = p/q$ in lowest terms, then (*) $p^2 = 2^2 \cdot 3 \cdot q^2$, so 3 divides p^2, so 3 divides p. But then 3^2 divides p^2, so from (*), 3 must divide q^2, so 3 divides q. But this means p/q is not in lowest terms, a contradiction. Thus $\sqrt{12}$ must be irrational.

(4) Let x be any element of E. Since α is a lower bound and β is an upper bound, we must have $\alpha \leq x \leq \beta$, so $\alpha \leq \beta$.

(5) Let $\alpha = \inf A$. We want to show that $\sup(-A) = -\alpha$.

First we show $-\alpha$ is an upper bound. If $x \in -A$, then $-x \in A$ so $\alpha \leq -x$, so $-\alpha \geq x$. This shows $-\alpha$ is indeed an upper bound for $-A$.

To show $-\alpha$ is the *least* upper bound, we show every $\gamma < -\alpha$ is *not* an upper bound. Suppose $\gamma < -\alpha$. Then $-\gamma > \alpha = \inf A$, so $-\gamma$ is not a lower bound for A, meaning there exists $y \in A$ with $y < -\gamma$. Then $-y > \gamma$ and $-y \in -A$, so γ is not an upper bound for $-A$. Thus $-\alpha$ is the least upper bound for $-A$.

Handout:

(I)(a) True (take any $y < 0$.)

(b) True (take any $x > 0$.)

(c) False (the statement fails for $y < -x^2$.)

(d) False (the statement fails for $a = b = 0, c = 1$, for example.)

(II)(a) For every number, there is a number smaller than the square of the first number.

(b) For some number, adding any square to it gives a positive result.

(c) There exists a number such that if add any second number to its square, the result is positive.

(d) Every polynomial of degree at most two has a real root.

(III) It is equivalent to (a). The negation of the original statement "$\forall x\ \exists y : p(x, y)$ is true" is obtained by reversing the quantifiers, so it is given by the statement in parentheses in (a). Thus the negation of the statement in parentheses in (a) is the negation of the negation of the original statement, which is the same as the original statement.

(IV)(a) $\{2, 4, 8\}$
(b) $\{2, 4, 6, 8, 10, 12, 14, 16, 32\}$
(c) $\{x \in \mathbb{R} : 0 < x < 9\}$
(d) $\{2, 4, 6, 8, 10, 12, 14\}$

(V) Look at the proof of Chapter 1 #2 above. The thing that makes it work is that in the number $n = 12 = 2^2 \cdot 3$, the 3 is not a square, so when 3 divides p, it forces 3 to divide q as well. To have a non-square factor and do basically the same proof, we need the powers of primes that appear when we factor n to not all be even. But this just means n itself is not a square.

Here is a more formal proof. We can equivalently prove the contrapositive: if $\sqrt{n}$ is rational, then n must be a square, that is, $\sqrt{n}$ must be an integer. So suppose $\sqrt{n} = p/q$ is rational and expressed in lowest terms. This means $p^2 = q^2 n$. Since they are squares, every prime factor in p^2 or q^2 appears with an even exponent. If some prime k appears with an odd exponent in n, then since k has an even exponent (possibly 0) in q^2, it has an odd exponent in $q^2 n$, that is, in p^2. But no primes have odd exponents in p^2, since it's a square, so there can be no such k. Thus every prime k appears with an even exponent (possibly 0) in n, which makes n a square.