MATH 425b ASSIGNMENT 6 SOLUTIONS
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Chapter 9:

(7) Suppose $E \subset \mathbb{R}$ is open, $f : E \to \mathbb{R}$, and there exists M such that $|(D_j f)(x)| \leq M$ for all $x \in E$ and all $j \leq M$. Let $x \in E$ and let B be a neighborhood of x with $B \subset E$. Suppose $x + h \in B$ for some h, and let $v_k = x + \sum_{i=1}^{k} h_i e_i$, so $v_0 = x, v_n = y$, and all $v_k \in B$. Fix some $k \leq n$. Since $D_k f$ exists in B, the function $g(t) = f(v_{k-1} + th_k e_k)$ is differentiable in $[0, 1]$ with $g'(t) = (D_k f)(v_{k-1} + th_k e_k)\ h_k$. By the mean value theorem, there exists $t \in (0, 1)$ such that

$$f(v_k) - f(v_{k-1}) = g(1) - g(0) = g'(t)(1 - 0),$$

so $|f(v_k) - f(v_{k-1})| \leq |g'(t)| \leq M|h_k|$. Summing, we get

$$|f(x + h) - f(x)| \leq \left| \sum_{k=1}^{n} (f(v_k) - f(v_{k-1})) \right| \leq M \sum_{k=1}^{n} |h_k|,$$

which approaches 0 as $h \to 0$, so f is continuous.

(8) If f has a local maximum, the for every i, the function $g(t) = f(x + te_i)$ has a local maximum at $t = 0$. By Theorem 5.8, $g'(0) = 0$. But $g'(0) = (D_i f)(x)$ so $(D_i f(x) = 0$ for all i, so all entries in the $1 \times n$ matrix of $f'(x)$ are 0, that is, $f'(x) = 0$.

(I) $T'(0) = \cos 0 = 1$, so given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned}
|x| < \delta \implies & \left| \frac{\sin x - \sin 0}{x - 0} - 1 \right| < \epsilon \\
\implies & \left| \frac{\sin x - \sin 0}{x - 0} \right| > 1 - \epsilon \\
\implies & |\sin x - \sin 0| > (1 - \epsilon)|x - 0| \\
\implies & |T(x) - T(0)| > (1 - \epsilon)|x - 0|.
\end{aligned}$$

Thus there is no $c < 1$ such that $T(y) - T(x) < c|y - x|$ for all x, y, which means T is not a contraction.

(II)(a) For example, let $f(x) = x + c \sin \pi x$ for some $c > 0$. Then $f(x) = x \iff \sin \pi x = 0 \iff x \in \mathbb{Z}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with $\mathbb{Z} = \{\text{fixed points of } T\}$. Let $g(x) = f(x) - x$. Then $g(0) = g(1) = 0$ and (since there are no fixed points of f in $(0, 1)$) g is not constant on $[0, 1]$, so there exists $z \in (0, 1)$ where $g(z) \neq 0$.

If $g(z) > 0$ then by the Mean Value Theorem there exists $\xi \in (0, z)$ where

$$g'(\xi) = \frac{g(z) - g(0)}{z - 0} = \frac{g(z)}{z} > 0.$$

But then $f'(\xi) = g'(\xi) + 1 > 1$.

Alternatively, if $g(z) < 0$ then by the Mean Value Theorem there exists $\eta \in (z, 1)$ where

$$g'(\eta) = \frac{g(1) - g(z)}{1 - z} = -\frac{g(z)}{1 - z} > 0.$$

But then $f'(\eta) = g'(\eta) + 1 > 1$.

Thus is both cases there is a point where $f' > 1$.

(III) Let $A = f'(x)$. Then

$$\begin{aligned}
f(x + h_n + k_n) - f(x) &= A(h_n + k_n) + o(|h_n + k_n|) \\
&= Ah_n + Ak_n + o(|h_n| + |k_n|) \\
&= [f(x + h_n) - f(x) + o(|h_n|)] + [f(x + k_n) - f(x) + o(|k_n|) \\
&= [f(x + h_n) - f(x)] + [f(x + k_n) - f(x)] + o(|h_n| + |k_n|).
\end{aligned}$$

(IV) The full formula for f is

$$f(x_1, x_2, x_3, x_4) = (x_1^2 + x_2x_3,\ x_1x_2 + x_2x_4,\ x_1x_3 + x_3x_4,\ x_2x_3 + x_4^2)$$

which has continuous partials, so we can get the derivative matrix by calculating these partials—the matrix of $f'(x)$ is

$$\begin{bmatrix} 2x_1 & x_3 & x_2 & 0 \\ x_2 & x_1 + x_4 & 0 & x_2 \\ x_3 & 0 & x_1 + x_4 & x_3 \\ 0 & x_3 & x_2 & 2x_4 \end{bmatrix}.$$

At the identity (that is, at $x = (1, 0, 0, 1)$) this becomes

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Now f is $\mathcal{C}'$ and this derivative is invertible, so the Inverse Function Theorem applies. At the identity we have $x = (1, 0, 0, 1) = f(x)$ so the Inverse Function Theorem guarantees the desired neighborhoods exist.

(V)(a) f' is continuous, since f is $\mathcal{C}'$. Therefore, as a composition of f' and $||\cdot||$, $||f'(x)||$ is a continuous function of x. Since this function is finite at a, there is a neighborhood W of a where $||f'(x)|| \leq ||f'(a)|| + 1$, which we call M.

(b) By the Inverse Function Theorem, there exists neighborhoods U of a and V of $f(a)$ such that f is a bijection of U and V. By (a) we can assume $||f'|| \leq M$ on U. The inverse g of f is $\mathcal{C}'$ on V. Let $b = f(a)$ and $b + k = f(a + h)$. Then

$$|g(b + k) - g(b) - g'(b)k| = o(|k|),$$

that is,

$$|(a + h) - a - g'(b)(f(a + h) - f(a))| = o(|k|).$$

For $T = g'(b)$ this says

$$|h - T(f(a + h) - f(a))| = o(|k|).$$

By shrinking U we may assume it is a ball, meaning U is convex. Then by Theorem 9.19, we then have

$$|k| = |f(a + h) - f(a)| \leq M|h|,$$

so anything $o(|k|)$ is also $o(|h|)$.

(VI) Notice that $f(x, y) = 0$ whenever either x or y is 0. Therefore

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{x\to 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x\to 0} \frac{0}{x} = 0,$$

and similarly $\frac{\partial f}{\partial y}(0, 0) = 0$.

Fix c and consider $f(x, cx)$. Plugging into the formula for f shows that

$$f(x, cx) = \frac{4c(1 - c^2)}{(1 + c^2)^2},$$

which does not depend on x, but only on c, so let us call it $g(c)$. $g(c)$ is different for different c, for example, at $c_1 = 0$ we have $g(c_1) = 0$, but at $c_2 = 1/2$ we have $g(c_2) = 24/25$. Therefore

$$\lim_{x\to 0} f(x, c_1x) = g(c_1) = 0, \quad \lim_{x\to 0} f(x, c_2x) = g(c_2) = \frac{24}{25}.$$

These are not the same, which means the overall limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, so f is not continuous at $(0, 0)$.

You can understand this function better if you write it in polar coordinates: $f(r, \theta) = \sin 4\theta$ whenever $r > 0$, which depends only on the direction θ, so it is constant along rays outward from the origin. But the constant values are different on different rays. Along the axes, the constant value is 0, so the partial derivatives exist at the origin and are 0. These partial derivatives only involve values along the axes