MATH 425b ASSIGNMENT 3 SOLUTIONS
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Chapter 7

(20) Let $\epsilon > 0$. There exists a polynomial with $||P - f|| < \epsilon$ (sup norm), say $P(x) = \sum_{n=0}^{N} c_n x^n$. f is bounded since it is continuous on the compact set $[0, 1]$, so there exists M such that $|f(x)| \leq M$ for all x. Therefore

$$\int_0^1 f(x)P(x)\ dx = \sum_{n=0}^{N} c_n \int_0^1 f(x)x^n\ dx = 0$$

and

$$\begin{aligned} 0 \leq \int_0^1 f(x)^2\ dx &= \left| \int_0^1 f(x)^2\ dx - \int_0^1 f(x)P(x)\ dx \right| \\ &= \left| \int_0^1 f(x)(f(x) - P(x))\ dx \right| \\ &\leq \int_0^1 |f(x)||(f(x) - P(x)|\ dx \\ &\leq \int_0^1 M\epsilon\ dx \\ &= M\epsilon. \end{aligned}$$

Since ϵ is arbitrary, this shows $\int_0^1 f(x)^2\ dx = 0$. By Exercise 2 of chapter 6, this means $f(x)^2 = 0$ for all x, so $f(x) = 0$ for all x.

(21) The constant function $f(e^{i\theta}) \equiv 1$ for all θ is in $\mathcal{A}$, and vanishes nowhere, so $\mathcal{A}$ vanishes at no point of K. The identity function $f(e^{i\theta}) = e^{i\theta}$ is in $\mathcal{A}$, and is one-to-one, so $\mathcal{A}$ separates points.

To prove Rudin's hint, for any function $f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}$ in $\mathcal{A}$ we have

$$\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\ d\theta = \sum_{n=0}^{N} c_n \int_0^{2\pi} e^{i(n+1)\theta}\ d\theta = 0. \tag{1}$$

For $f \in \overline{\mathcal{A}}$ there exists a sequence $\{f_n\} \subset \mathcal{A}$ with $f_n \to f$ uniformly. Hence applying (1) to

f_n,

$$\begin{aligned}\left|\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\ d\theta\right| &= \left|\int_0^{2\pi} (f(e^{i\theta}) - f_n(e^{i\theta}))e^{i\theta}\ d\theta\right| \\ &\le \int_0^{2\pi} |f(e^{i\theta}) - f_n(e^{i\theta})|\ |e^{i\theta}|\ d\theta \\ &\le 2\pi ||f - f_n||_\infty \quad \text{(sup norm)} \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{2}$$

so we must have $\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\ d\theta = 0$, for all $f \in \overline{\mathcal{A}}$. But for the particular choice $f(e^{i\theta}) = e^{-i\theta}$ we have $\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\ d\theta = \int_0^{2\pi} 1\ d\theta = 2\pi$, so $f \notin \overline{\mathcal{A}}$, though f is continuous on K.

Chapter 8

(4)(a) Let $f(x) = b^x = e^{(\log b)x}$, so $f'(x) = (\log b)e^{(\log b)x}$. Then

$$\lim_{x\to 0} \frac{b^x - 1}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = \log b.$$

(b) Use L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1.$$

(c) Use (b):

$$\lim_{x\to 0} (1+x)^{1/x} = \lim_{x\to 0} e^{\frac{\log(1+x)}{x}} = e^1 = e.$$

(d) By (c), $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{n/x} = e$. Since y^x is a continuous function of y, this shows

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty} \left(\left(1 + \frac{x}{n}\right)^{n/x}\right)^x = e^x.$$

(6)(a) Taking $x = y = 0$ shows $f(0)^2 = f(0)$ so $f(0) = 0$ or 1 for all x. But $f(x) = f(x+0) = f(x)f(0)$ so if $f(0) = 0$ then $f(x)$ would be 0 for all x. Therefore $f(0) = 1$.

Let $g(x) = \log f(x)$, so $g(0) = 0$ and $g(x+y) = g(x) + g(y)$. Then

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{g(h)}{h} = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = g'(0) \quad \text{for all } x.$$

Letting $c = g'(0)$, this shows that $g(x) = cx + c'$ for some c'. Since $g(0) = 0$ we must have $c' = 0$, so $g(x) = cx$, which means $f(x) = e^{cx}$.

(b) Rudin is a bit unclear—we still assume f is not 0, we just replace differentiability with continuity.

Since $g(x+y) = g(x) + g(y)$, taking $x = y$ shows $g(2x) = 2g(x)$, and then by easy induction on m,

$$g(mx) = g((m-1)x + x) = g((m-1)x) + g(x) = (m-1)g(x) + g(x) = mg(x),$$

for all m and x. Hence also

$$g(x) = g\left(n \cdot \frac{x}{n}\right) = ng\left(\frac{x}{n}\right)$$

for all n, x, so $g(\frac{x}{n}) = \frac{1}{n}g(x)$. Therefore for all m, n,

$$g\left(\frac{m}{n}\right) = g\left(m \cdot \frac{1}{n}\right) = mg\left(\frac{1}{n}\right) = mg\left(\frac{1}{n} \cdot 1\right) = \frac{m}{n}g(1).$$

Letting $a = g(1)$ we thus have $g(x) = ax$ for all rational x. Since g is continuous, for irrational x we can take a sequence of rationals $x_k \to x$ and

$$g(x) = \lim_k g(x_k) = \lim_k ax_k = ax.$$

Thus $f(x) = e^{ax}$.

(A) ((a) $\implies$ (b)) Suppose $\sum_{n=0}^{\infty} a_n$ converges. Then the radius of convergence is at least 1, so f is defined at least on $[0,1]$. For $x \in [0,1]$ we have

$$|f(x) - \sum_{n=1}^{N} a_n x^n| \leq \sum_{n=N+1}^{\infty} |a_n||x|^n \leq \sum_{n=N+1}^{\infty} |a_n|.$$

The last sum does not depend on x, and approaches 0 as $N \to \infty$. Thus the series converges uniformly to $f(x)$ on $[0,1]$.

((b) $\implies$ (c)) Suppose $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[0,1]$. Then the limit $f(x)$ is a continuous function, so f is bounded on $[0,1]$, hence also on $[0,1)$.

((c) $\implies$ (a)) Suppose $\sum_{n=0}^{\infty} a_n = \infty$. Given $M > 0$ there exists N such that $\sum_{n=0}^{N} a_n > M$. Then for x sufficiently close to 1 we have $f(x) \geq \sum_{n=0}^{N} a_n x^n > M$. This shows that f is unbounded on $[0,1]$.

(B) Let $f(x) = \sum_{n=1}^{\infty} x^n/n$. Since $(1/n)^{1/n} \to 1$, the radius of convergence is 1 for this series, and plugging in $x = 0$ shows $f(0) = 0$. By Theorem 8.1 we can differentiate term-by-term for $|x| < 1$: $f'(x) = \sum_{n=1}^{\infty} x^{n-1} = \sum_{m=0}^{\infty} x^m = (1-x)^{-1}$, since $f'(x)$ is a geometric series. Integrating gives $f(x) = f(x) - f(0) = \int_0^x f'(t)\,dt = \int_0^x (1-t)^{-1}\,dt = -\log(1-x)$ for all $|x| < 1$.

(C)(a) Since f is never 0, $\mathcal{A}_1$ vanishes at no point of $[0,1]$. If $(x_1,y_1) \neq (x_2,y_2)$ then either $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$ then $g(x_1,y_1) \neq g(x_2,y_2)$. If $y_1 \neq Y_2$ then $f(x_1,y_1) \neq g(x_2,y_2)$. This shows that $\mathcal{A}_1$ separates points. By the Stone-Weierstrass Theorem, the uniform closure of $\mathcal{A}_1$ is all of $C([0,1]^2)$, so in particular it includes h.

(b) Every polynomial of form $c + (x-\frac{1}{2})^2 R(x)$, with R a polynomial and c a constant, is in $\mathcal{A}_2$. In particular the strictly increasing function $(x-\frac{1}{2})^3 \in \mathcal{A}_2$, which shows that $\mathcal{A}_2$ separates points. Taking $c > 0$ and $R \equiv 1$ we see that $\mathcal{A}_2$ vanishes at no point. By the Stone-Weierstrass Theorem, $\mathcal{A}_2$ is dense in $C[0,1]$.

(D)(a) Fix x and let $a_n = \binom{\alpha}{n} x^n$. Then

$$\frac{|a_{n+1}|}{|a_n|} = \left| \frac{\binom{\alpha}{n+1} x^{n+1}}{\binom{\alpha}{n} x^n} \right| = \frac{|\alpha - n|}{n+1} |x| \to |x| \quad \text{as } n \to \infty.$$

Hence by the ratio test, the series $S(x)$ converges if $|x| < 1$, and diverges if $|x| > 1$. This shows the radius of convergence is 1.

(b) By Theorem 8.1 we can differentiate term-by-term for $|x| < 1$:

$$(*) \qquad S'(x) = \sum_{n=0}^{\infty} n \binom{\alpha}{n} x^{n-1} = \sum_{n=1}^{\infty} n \binom{\alpha}{n} x^{n-1}.$$

From the formulas,

$$n \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} = (\alpha - n + 1) \binom{\alpha}{n-1},$$

so

$$S'(x) = \alpha \sum_{n=1}^{\infty} \binom{\alpha}{n-1} x^{n-1} - \sum_{n=1}^{\infty} (n-1) \binom{\alpha}{n-1} x^{n-1}.$$

Changing the index to $n = m - 1$ and using $(*)$ gives

$$S'(x) = \alpha \sum_{m=0}^{\infty} \binom{\alpha}{m} x^m - \sum_{m=0}^{\infty} m \binom{\alpha}{m} x^m = \alpha S(x) - x S'(x).$$

(c) Rearranging the conclusion of part (b) we get for $|x| < 1$:

$$(**) \quad S'(x) = \frac{\alpha}{1+x} S(x) \quad \text{so} \quad \frac{d}{dx} \log |S(x)| = \frac{S'(x)}{S(x)} = \frac{\alpha}{1+x} \text{ wherever } S(x) \neq 0.$$

We claim that in fact $S(x) > 0$ in the whole interval $(-1,1)$. If not, then since $S(1) = 1 > 0$, there must be a point $x_0 \in (-1,1)$ where $S(x_0) = 0$, so we must have $\log |S(x)| \to \infty$ as

$x \to x_0$. But our formula $(**)$ shows that the derivative of $\log |S(x)|$ remains bounded as $x \to x_0$, a contradiction. Thus there is no $x_0 \in (-1,1)$ where $S(x_0) = 0$, and therefore $S(x) > 0$ in the whole interval $(-1,1)$. Therefore by $(**)$,

$$\frac{d}{dx} \log S(x) = \frac{\alpha}{1+x} \quad \text{for all } x \in (-1,1).$$

Integrating gives $\log S(x) = \alpha \log(1+x) + C$, and then $S(0) = 1$ shows that $C = 0$, so $S(x) = (1+x)^\alpha$.