Chapter | Sixteen

Nonidentical Objects

In considering multiple object auctions we have thus far restricted attention to situations in which the objects are identical. Furthermore, we supposed that the marginal value of an additional unit declined with the number of units in hand. We now turn to the case where the objects, while related, are not identical.

16.1 THE MODEL

Let $K = \{a, b, c, \ldots\}$ be a finite set of distinct objects for sale and, as usual, let $\mathcal{N}$ be the set of buyers. Each buyer $i \in \mathcal{N}$ is assumed to assign a value $x^i(S)$ to each subset $S \subseteq K$. Subsets of K are called *packages* or *combinations*. The set of possible packages is 2^K. We can then think of a buyer's *value* as a vector

$$\mathbf{x}^i = \left(x^i(S)\right)_{S \subseteq K}$$

We will suppose that $x^i(\emptyset) = 0$ and if $S \subset T$, then $x^i(S) \leq x^i(T)$. The set of possible value vectors for i, $\mathcal{X}^i$, is assumed to be a closed and convex subset of the nonnegative orthant and $\mathbf{0} \in \mathcal{X}^i$. Notice that this is still a setting with private values.

Observe that the interpretation of the value vector is now somewhat different from that in the case of identical objects. With identical objects, the different components of $\mathbf{x}^i$ represented the *marginal* value of an additional unit. In the current specification, the component $x^i(S)$ represents the total value derived from a package S of the objects.

An *allocation* $\left\langle S^1, S^2, \ldots, S^N \right\rangle$ is an ordered collection of N packages, which forms a partition—that is, $\cup_{i \in \mathcal{N}} S^i = K$ and for all $i \neq j$, $S^i \cap S^j = \emptyset$. The interpretation is, of course, that buyer i is allocated package S^i, so the requirement that an allocation be a partition amounts to saying that (1) every object must be allocated to some buyer, and (2) no object can be allocated to more than one buyer. Let $\mathbb{K}$ denote the set of all allocations.

We adopt a mechanism design perspective and define an *allocation rule* $\mathbb{S} : \mathcal{X} \to \mathbb{K}$ as a function that assigns an allocation

$$\mathbb{S}(\mathbf{x}) = \left\langle S^1(\mathbf{x}), S^2(\mathbf{x}), \ldots, S^N(\mathbf{x}) \right\rangle$$

to each N-tuple of value vectors $\mathbf{x} = \left(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right)$. As usual, a *direct mechanism* $(\mathbb{S}, \mathbf{M})$ consists of an allocation rule together with a payment rule.[1]

Given an allocation rule $\mathbb{S}$, let $q^i\left(S, \mathbf{z}^i\right)$ denote the probability that buyer i will be allocated the set of objects S when he reports his value vector as $\mathbf{z}^i$. Let

$$\mathbf{q}^i\left(\mathbf{z}^i\right) = \left(q^i\left(S, \mathbf{z}^i\right)\right)_{S \subseteq K}$$

be the 2^K-vector of allocation probabilities. Define $m^i\left(\mathbf{z}^i\right)$ to be buyer i's expected payment when reporting $\mathbf{z}^i$. His expected payoff from reporting $\mathbf{z}^i$ when his true value vector is $\mathbf{x}^i$ can then be written in the usual fashion as

$$\mathbf{q}^i\left(\mathbf{z}^i\right) \cdot \mathbf{x}^i - m^i\left(\mathbf{z}^i\right)$$

As in Chapter 5, a direct mechanism is said to be incentive compatible if truth-telling is an equilibrium—that is,

$$U^i\left(\mathbf{x}^i\right) \equiv \mathbf{q}^i\left(\mathbf{x}^i\right) \cdot \mathbf{x}^i - m^i\left(\mathbf{x}^i\right) \geq \mathbf{q}^i\left(\mathbf{z}^i\right) \cdot \mathbf{x}^i - m^i\left(\mathbf{z}^i\right)$$

The following "revenue equivalence" result then follows immediately.

Proposition 16.1. *The expected payoff (and payment) functions of a buyer in any two incentive compatible mechanisms with the same allocation rule differ by at most an additive constant.*

Proof. The proof is *identical* to that of Proposition 14.1. The only difference is that all the vectors are of size $2^{|K|}$ instead of size K. In particular, (14.5) holds—that is, for all $\mathbf{x}^i \in \mathcal{X}^i$,

$$U^i\left(\mathbf{x}^i\right) = U^i(\mathbf{0}) + \int_0^1 \mathbf{q}^i\left(t\mathbf{x}^i\right) \cdot \mathbf{x}^i \, dt \tag{16.1}$$

and the conclusion of the proposition follows immediately. ■

A mechanism $(\mathbb{S}, \mathbf{M})$ is said to be *individually rational* if for all i and $\mathbf{x}^i$, the expected payoff of buyer i is nonnegative—that is, if $U^i\left(\mathbf{x}^i\right) \geq 0$.

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[1] Notice that we are restricting attention to deterministic allocation rules. Given the concerns of this chapter, allowing for randomized rules would not affect what follows.

16.2 EFFICIENT ALLOCATIONS

Our first concern is with the possibility of achieving efficient allocations via an incentive compatible mechanism. The Vickrey-Clarke-Groves (VCG) mechanism, a natural extension of the Vickrey multiunit auction and already familiar from Chapter 5, provides a ready means of achieving efficiency. It is worthwhile to review its operation in the current context.

An allocation rule $\mathbb{S}$ is *efficient* if for every $\mathbf{x} \in \mathcal{X}$, the allocation $\mathbb{S}(\mathbf{x})$ maximizes *social welfare*—that is, the sum of buyers' values—over all allocations, so

$$\mathbb{S}(\mathbf{x}) \in \arg \max_{\langle T^1, T^2, \ldots, T^N \rangle} \sum_{i \in \mathcal{N}} x^i\left(T^i\right)$$

Define

$$W(\mathbf{x}) = \sum_{i \in \mathcal{N}} x^i\left(S^i(\mathbf{x})\right) \tag{16.2}$$

to be the social welfare from an efficient allocation $\mathbb{S}(\mathbf{x})$ when the values are $\mathbf{x}$, and define

$$W^{-i}(\mathbf{x}) = \sum_{j \neq i} x^j\left(S^j(\mathbf{x})\right) \tag{16.3}$$

to be the social welfare of individuals other than i from an efficient allocation $\mathbb{S}(\mathbf{x})$ when the values are $\mathbf{x}$.

The *VCG mechanism* is an efficient mechanism defined by the payment rule:

$$\begin{aligned} \overline{M}^i(\mathbf{x}) &= \sum_{j \neq i} x^j\left(S^j\left(\mathbf{0}, \mathbf{x}^{-i}\right)\right) - \sum_{j \neq i} x^j\left(S^j(\mathbf{x})\right) \\ &= W^{-i}\left(\mathbf{0}, \mathbf{x}^{-i}\right) - W^{-i}(\mathbf{x}), \end{aligned} \tag{16.4}$$

where, for all j, $S^j(\mathbf{0}, \mathbf{x}^{-i})$ is the jth component of an efficient allocation $\mathbb{S}(\mathbf{0}, \mathbf{x}^{-i})$ that would result if i were to report $\mathbf{x}^i = \mathbf{0}$ (or equivalently, in many settings, if i were not present). Recall from Chapter 5 that the amount $M^i(\mathbf{x})$ buyer i pays in the VCG mechanism represents the *externality* that i exerts on the other $N-1$ agents by his presence in society. It is the difference between the welfare of the others "without him" and the welfare of the others "with him."

Fix some $\mathbf{x}^{-i}$, the values of agents other than i. In the VCG mechanism, the *ex post* payoff to i with value $\mathbf{x}^i$ when he reports that it is $\mathbf{z}^i$ is

$$x^i\left(S^i\left(\mathbf{z}^i, \mathbf{x}^{-i}\right)\right) - \overline{M}^i\left(\mathbf{z}^i, \mathbf{x}^{-i}\right) = W\left(\mathbf{z}^i, \mathbf{x}^{-i}\right) - W\left(\mathbf{0}, \mathbf{x}^{-i}\right) \tag{16.5}$$

The second term is independent of the reported value $\mathbf{z}^i$, and by definition, the first term just represents the social welfare, so it is maximized by choosing $\mathbf{z}^i = \mathbf{x}^i$.

Thus, "truth-telling" is a weakly dominant strategy in the VCG mechanism. *A fortiori*, the VCG mechanism is incentive compatible. Using (16.5) agent i's payoff in equilibrium (when $\mathbf{z}^i = \mathbf{x}^i$) is just $W(\mathbf{x}) - W(\mathbf{0}, \mathbf{x}^{-i})$—that is, the difference in social welfare when i reports $\mathbf{x}^i$ versus when he reports $\mathbf{0}$. Moreover, (16.5) implies that the payoff to a buyer with value vector $\mathbf{0}$, is 0 and for any $\mathbf{x}^i$ the expected payoff in the VCG mechanism is nonnegative. Thus, the VCG mechanism is also individually rational.

Now suppose that the value vectors $\mathbf{x}^i$ are independently distributed. The following result is a generalization of Proposition 5.5 on page 76 to the case of multiple objects. Its proof is identical to that of Proposition 5.5.

Proposition 16.2. *Among all mechanisms for allocating multiple objects that are efficient, incentive compatible and individually rational, the VCG mechanism maximizes the expected payment of each agent.*

The VCG mechanism not only achieves efficiency, but from the perspective of the seller, it does so in the most advantageous way. It raises the highest revenue among all efficient mechanisms. In the remainder of this chapter we turn to some practical matters.

16.3 SUBSTITUTES AND COMPLEMENTS

We have said little about the nature of the objects being sold other than to say that they are not necessarily identical. Indeed, the results of the preceding section do not rely on any specific relationship among the set of objects.

In previous chapters we considered multiple object auctions in which the objects were different units of the same good. Different units of the same good are, of course, perfect substitutes and we assumed that the marginal value of a unit to a buyer declined with the number of units already in hand. Generalizing this property to the case of nonidentical objects, we will say that the objects being sold are *substitutes* if the marginal value of obtaining a particular object a is smaller if the set of objects already in hand is "larger." Formally, buyer i considers the objects in K to be substitutes if for all $a \in K$ and packages S and T not containing a, such that $S \subset T$,

$$x^i(S \cup \{a\}) - x^i(S) \geq x^i(T \cup \{a\}) - x^i(T) \tag{16.6}$$

It can be shown that (16.6) is equivalent to requiring that for all packages S and T,

$$x^i(S) + x^i(T) \geq x^i(S \cup T) + x^i(S \cap T) \tag{16.7}$$

Functions satisfying (16.7) are called *submodular*. In particular, if $S \cap T = \emptyset$, then, since $x^i(\emptyset) = 0$, the inequality in (16.7) reduces to

$$x^i(S) + x^i(T) \geq x^i(S \cup T)$$

so the substitute property implies that $x^i(\cdot)$ is a *subadditive* function over the set of packages. When we say that the objects in K are substitutes, we mean that all buyers consider them as such.

In analogous fashion, we will say that the objects are *complements* if the marginal value of obtaining a particular object a is larger if the set of objects already in hand is "larger." Formally, buyer i considers the objects in K to be complements if for all $a \in K$ and packages S and T not containing a, such that $S \subset T$,

$$x^i(S \cup \{a\}) - x^i(S) \leq x^i(T \cup \{a\}) - x^i(T) \tag{16.8}$$

This is equivalent to requiring that for all packages S and T,

$$x^i(S) + x^i(T) \leq x^i(S \cup T) + x^i(S \cap T) \tag{16.9}$$

Functions satisfying (16.9) are called *supermodular*.[2] Again, if $S \cap T = \emptyset$, then we have

$$x^i(S) + x^i(T) \leq x^i(S \cup T)$$

implying that $x^i(\cdot)$ is a *superadditive* function. Again, when we say that the objects in K are complements we mean that all buyers consider them as such.

If both (16.6) and (16.8) hold, then the values are *additive*—that is, the value of any package S is simply the sum of the values of the individual objects in that package. In this case, it is useful to think of the different objects as being completely unrelated since the value derived from a particular object a does not depend on whether another object b is obtained or not.

16.4 BUNDLING

In some circumstances it may be in the interests of the seller to *bundle* some or all the goods together—that is, to sell some objects only as part of a specific larger package. Specifically, the seller may partition the objects in K into bundles $B_1, B_2, \ldots, B_L$. The objects in any B_l must be sold to the same buyer. In effect, bundling forces the buyers to treat each B_l as a single object.

Once the seller has decided to bundle the objects, the only possible subsets that the buyers can obtain are those of the form $\cup B_l$ and consequently only the values for these subsets are relevant. The VCG mechanism can then be used to allocate the bundles and given any partition of K into bundles it will allocate the bundles efficiently. Of course, full efficiency will not be achieved in general because social welfare may increase if some of the goods in a particular bundle

[2]In some contexts—for instance, the theory of cooperative games—supermodular functions are also called *convex*.

are allocated to separate buyers. The VCG mechanism would, nevertheless, be constrained efficient—that is, welfare maximizing subject to the bundling constraint.

Is bundling advantageous for the seller? A striking example illustrating its benefits occurs when there are only two buyers.

Proposition 16.3. *Suppose there are two buyers ($N=2$). In the VCG mechanism, the revenue to the seller from selling all the objects as a single bundle exceeds the revenue derived from bundling them in any other way.*

Proof. Let $B_1, B_2, \ldots, B_L$ be a partition of K into L bundles. Suppose that in the VCG mechanism with the bundles it is constrained efficient to allocate set S^1 to buyer 1 and $S^2 = K \setminus S^1$ to buyer 2 where each S^i is a union of some bundles. Then, in particular we have

$$x^1(S^1) + x^2(S^2) \geq \max\left\{x^1(K), x^2(K)\right\} \tag{16.10}$$

In the VCG mechanism buyer 1 would pay the externality he exerts on buyer 2—that is, $x^2(K) - x^2(S^2)$. Similarly, buyer 2 would pay $x^1(K) - x^1(S^1)$. The revenue accruing to the seller is

$$x^1(K) + x^2(K) - x^1(S^1) - x^2(S^2)$$

Now if the seller were to require that all the objects be sold as a single bundle—in this case, the VCG mechanism is equivalent to a second-price auction—then the revenue would be $\min\left\{x^1(K), x^2(K)\right\}$. But

$$\min\left\{x^1(K), x^2(K)\right\} \geq x^1(K) + x^2(K) - x^1(S^1) - x^2(S^2)$$

because of (16.10). Thus, the revenue from bundling all the objects as one is greater than the revenue from any other form of bundling. In particular, it is greater than the revenue from not bundling at all and selling the objects individually—that is, in the case of $L = K$.

As long as it is not efficient to allocate all the objects to one of the buyers—that is, the inequality in (16.10) is strict—selling the objects as a single bundle is strictly better for the seller. ■

We have shown that with only two buyers, it is always better to sell the objects as a single bundle. Remarkably, this holds regardless of whether the objects are substitutes or complements.

One may reasonably conjecture that when there are more than three buyers it may not be advantageous to bundle if the objects are substitutes. An example shows that bundling may not be advantageous even when the objects are complements.

Example 16.1. *With three or more buyers, bundling may not be optimal.*

Suppose that there are two objects, a and b, and three buyers with values given in the following table.

	a	b	ab
$\mathbf{x}^1$	8	4	14
$\mathbf{x}^2$	4	7	12
$\mathbf{x}^3$	7	1	10

Notice that a and b are complements since for each buyer $x^i(ab) > x^i(a) + x^i(b)$.[3]

Without bundling, the efficient allocation is to give a to buyer 1 and b to buyer 2. The welfare from this allocation is 15 and this exceeds the welfare from any other allocation. If buyer 1 were not present, or equivalently if he reported $\mathbf{x}^i = 0$, then it would be efficient to give a to buyer 3 and b to buyer 2 for a total welfare $W^{-1}(\mathbf{0}, \mathbf{x}^{-1}) = 14$, so in the VCG mechanism buyer 1 would pay $W^{-1}(\mathbf{0}, \mathbf{x}^{-1}) - W^{-1}(\mathbf{x}^1, \mathbf{x}^{-1}) = 7$. If buyer 2 were not present, then it would be efficient to give both objects to buyer 1, so buyer 2 would pay $W^{-2}(\mathbf{0}, \mathbf{x}^{-2}) - W^{-2}(\mathbf{x}^2, \mathbf{x}^{-2}) = 14 - 8 = 6$. The total revenue in the VCG mechanism would be 13.

If a and b were sold as a single bundle, then it would go to buyer 1 and he would pay 12.

Thus, in this example it is better for the seller to sell the objects separately even though all buyers consider them to be complements. ▲

The distinguishing feature of the two-buyer case is that if one of the buyers is not present, the K objects must necessarily be awarded to the other buyer. As is apparent in the preceding example, this is not true once there are three or more buyers. A second feature of the two-buyer case is that bundling is advantageous for every realization of buyers' values, so this advantage holds no matter what the distribution of values. This is also special to the two-buyer situation. In general, whether or not bundling is advantageous depends on how buyers' values are distributed.

16.5 SOME COMPUTATIONAL ISSUES

The VCG mechanism is an effective means of achieving efficiency, but it imposes a substantial computational burden on the buyers and the seller. First, each buyer is asked to submit a value for each subset S of K and thus the vector $\mathbf{x}^i$ that is submitted is of size $2^{|K|}$. Second, the seller is asked to determine (1) an efficient allocation based on the submitted value vectors and (2) each buyer's payment.

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[3]We write $x^i(ab)$ to denote $x^i(\{a,b\})$.

To find an efficient allocation the seller needs to solve the following integer programming problem: for all i and S choose variables $\theta^i(S) \in \{0,1\}$ to maximize

$$\sum_{i \in \mathcal{N}} \sum_{S \subseteq K} x^i(S)\,\theta^i(S) \tag{16.11}$$

subject to the constraints

$$\forall i, \ \sum_{S \subseteq K} \theta^i(S) \ \leq \ 1 \tag{16.12}$$

$$\forall a, \ \sum_{S \ni a} \sum_{i \in \mathcal{N}} \theta^i(S) \ \leq \ 1, \tag{16.13}$$

where $S \ni a$ denotes that the sum is taken over all subsets $S \subseteq K$ that contain the object a. If the variable $\theta^i(S) = 1$, then package S is allocated to buyer i. The objective function of the integer program is just the total value—the social welfare—derived from the allocation and its maximized value is just $W(\mathbf{x})$. The N constraints (16.12) ensure that each buyer is allocated at most one package. The $|K|$ constraints (16.13) ensure that each object is allocated to at most one buyer. The total number of variables $\theta^i(S)$ is $N \times 2^{|K|}$. We will refer to this as the *allocation problem*.

Determining each buyer's payments in the VCG mechanism requires, in addition, that a similar allocation problem be solved once the buyer in question has been removed from the picture. If each buyer receives at least one object, then this means that N additional allocation problems, similar to the preceding one, need to be solved.

Since K is a finite set one could, in principle, list all possible allocations to find one that is optimal. As a practical method, however, this would be too cumbersome since even when relatively few objects are sold the number of possible allocations grows exponentially with the number of objects. While other, more efficient, algorithms can be used in specific circumstances, it is not known whether there is a general polynomial time algorithm—one in which the number of steps required to reach a solution grows as a polynomial function of the size of the problem. The allocation problem belongs to a class of problems computer scientists call *NP-hard*.

Of course, these theoretical measures of the underlying computational complexity are based on a general "worst-case" analysis. While it may not be possible to rule out that in some problems the number of steps grows exponentially, it may well be that in "typical" problems the number of steps is not prohibitively large. Conversely, even knowing that there is a polynomial algorithm does not ensure that it is practical; it may well be that the number of steps grows as a polynomial of high degree.

In many situations, there may be some natural structure that limits the set of packages that bidders consider valuable, thereby simplifying the allocation problem. Let us examine a few examples.

As a first step, notice that the allocation problem is simple in two extreme cases. First, if all the objects are identical so that as in Chapter 12 each buyer cares only about the number of objects he receives, then efficient allocations are relatively simple to determine. Second, if the objects are completely unrelated so that the value of a package is just the sum of the values of the objects in that package—the case of additive values—once again the allocation problem is relatively simple: each object can be efficiently allocated independently of how other objects are allocated.

A more interesting special case is the following. Suppose that the K objects can be linearly ordered as

$$a_1, a_2, a_3, \ldots, a_k, a_{k+1}, \ldots, a_{|K|}$$

and it is the case that these are complements but that for all buyers the values are strictly superadditive only among "adjacent" objects. Thus, for example, while the possibility that $x^i(a_1a_2a_3) > x^i(a_1a_2) + x^i(a_3)$ is allowed, it is required that $x^i(a_1a_2a_4) = x^i(a_1a_2) + x^i(a_4)$. In this case the allocation problems are computationally manageable. The reason is that the integer program specified above can be safely solved as a linear program. In other words, we can neglect the requirement that each $\theta^i(S)$ is either 0 or 1. The solution to the resulting linear program will automatically be such that the $\theta^i(S)$ will satisfy this requirement. Linear programming problems, unlike their integer programming counterparts, can, of course, be efficiently solved.

16.6 BUDGET CONSTRAINTS

We saw in Chapter 4 that when bidders are subject to budget constraints, the first- and second-price auctions need not yield the same expected revenue. Here we briefly look at the effects of budget constraints on bidding behavior in multiple-object auctions. Unlike in Chapter 4, however, we suppose that there is complete information regarding values and budgets. Thus, what follows is only meant to indicate some novel issues that arise in the multiple-object context.

Suppose that there are two objects, a and b, and these are sold sequentially by means of two English auctions. Specifically, a is brought up for sale in the first auction, whose results are announced in public. After that, b is sold in a second auction. There are two buyers with values given in the following table.

	a	b	ab
$\mathbf{x}^1$	14	3	16
$\mathbf{x}^2$	8	4	12

Suppose further that these values are commonly known—there is *no* incomplete information regarding values.

In the first auction, since $x^1(a) > x^2(a)$, bidder 1 wins object a at a price of 8. In the second auction, bidder 2 attaches a greater value to object b than does bidder 1, so bidder 2 wins b at a price of 3. The total revenue accruing to the seller is 11.

Now suppose that bidder 1 has a budget of $w^1 = 13$ and bidder 2 has a budget of $w^2 = 11$. These budgets are inflexible—neither bidder can pay more than his budget—and suppose that these are also commonly known. Once again suppose that a and b are sold, in that order, by means of two separate English auctions. What constitutes equilibrium bidding behavior once budgets come into play?

First, consider the second auction, in which b is offered for sale. Bidding behavior in this auction is clear. Let $\widehat{w}^i$ be the amount of money bidder i has left over after the first auction is over and call this i's *residual budget*. The residual budget is just the original budget w^i less the amount, if any, that i spent in the first auction. It is clear that in the second auction, bidder i should stay in until the price reaches $p^i(b) \equiv \min\{\widehat{w}^i, x^i(b)\}$.

Now consider the first auction—that is, the one in which a is sold. The price cannot exceed 11 since that is bidder 2's budget. Moreover, as long as the current price $p \leq 11$, it is not in bidder 1's interest to drop out. This is because the largest profit he can make in the second auction is 3, and if $p \leq 11$, the profit from buying a at p exceeds this amount. Thus, bidder 1 is not the first to drop out. How high is bidder 2 willing to go? Bidder 2 values object a at $x^2(a) = 8$, but if he drops out at 8, this will leave bidder 1 with a residual budget of $\widehat{w}^1 = 13 - 8 = 5$, so bidder 2 will end up paying $\min\{\widehat{w}^1, x^1(b)\} = 3$ for b. But bidder 2 can reduce the price he pays in the second auction by staying in after the price in the first auction reaches 10. The longer he stays in, the more he depletes bidder 1's budget, thereby reducing the amount bidder 1 is able to bid in the second auction. Thus, it is in bidder 2's interest to stay in as long as possible—that is, until the price reaches 11. Moreover, bidder 2 is confident that bidder 1 will not drop out before then. By running up the price in this manner, bidder 2 is able to reduce bidder 1's residual budget to $13 - 11 = 2$, and this is the price bidder 2 would pay for b in the second auction. Thus, in equilibrium, bidder 1 would win a at a price of 11 and bidder 2 would then win b for 2. The total revenue of the seller is now 13.

An interesting aspect of equilibrium behavior in this situation is that it is rational for bidder 2 to stay in the first auction past the point that the price exceeds his value, so he is no longer interested in winning a. The motive for doing this is to weaken bidder 1 for the second auction. Such *overbidding* can, of course, be rational only in multiple-object auctions. This overbidding results in a higher revenue to the seller than if there were no budget constraints—the seller's revenue is 13 rather than 11. An increase in revenue resulting from the presence of budget constraints can also only arise in a multiple-object setting.

PROBLEMS

16.1. (Low revenue) Consider the problem of allocating a set of two objects in $K = \{a, b\}$ to three buyers with values as follows:

	a	b	ab
$\mathbf{x}^1$	0	0	$10+\varepsilon$
$\mathbf{x}^2$	10	10	10
$\mathbf{x}^2$	10	10	10

where $0 \leq \varepsilon < 1$.

a. Find an efficient allocation and the corresponding payments in the VCG mechanism.

b. What is the total revenue accruing to the seller?

16.2. (Complements) Consider the problem of allocating a set of four objects in $K = \{a_1, a_2, b_1, b_2\}$ to five buyers. Buyer 1 has use only for objects a_1 and b_1; buyers 2 and 3 have use only for objects a_2 and b_2; buyer 4 has use only for b_1 and b_2; and buyer 5 has use only for objects a_1 and a_2. Specifically, the values attached by the buyers to these bundles are

$$
\begin{aligned}
x^1(a_1 b_1) &= 10 \\
x^2(a_2 b_2) &= 20 \\
x^3(a_2 b_2) &= 25 \\
x^4(b_1 b_2) &= 10 \\
x^5(a_1 a_2) &= 10
\end{aligned}
$$

All other combinations (or packages) are valued at zero.

a. Find an efficient allocation and the corresponding payments in the VCG mechanism.

16.3. Show that the conditions (16.6) and (16.7) are equivalent.

CHAPTER NOTES

Most of the material in this chapter is well known. Proposition 16.2 is just an extension of Proposition 5.5 from Chapter 5 and is due to Krishna and Perry (1998).

Circumstances in which bundling is advantageous for a monopolist have been the subject of extensive study in the literature on industrial organization. Proposition 16.3, showing that bundling is always advantageous when there are only two bidders, generalizes previous results on multiunit auctions obtained by Palfrey (1983), concerning additive values, and Krishna and Tranæs (2002), concerning more general valuations. Whether or not it is profitable to bundle has also been studied in the context of procurement. In that context, the issue is whether or not the buyer should award the contract to a single supplier—as a bundle—or conduct a "split-award auction," in which more than one supplier is used. Anton and Yao (1992) have studied circumstances in which it is optimal for the buyer to split the award between two suppliers. In their model the buyer does not commit beforehand to a particular course of action; rather the decision of whether or not to split the award is made once the bids are received. Chakraborty (1999) studies a two-object model with additive values that are independently and identically distributed. He finds sufficient conditions so it is profitable to bundle if and only if the number of buyers is less than some threshold N^*. To some extent, his results support the general intuition that bundling is profitable when there is a small number of buyers.

Armstrong (2000) and Avery and Hendershott (2000) show the advantages of bundling in a setting in which two objects are for sale and there are two-point values for each object. These papers show that the optimal auction in this setting displays bundling-like features—the probability that a particular bidder will be allocated an object is larger if he has already been allocated the other one than if he has not. Unlike in the situation considered in this chapter, however, the seller need not commit to sell the objects as a bundle.

Issues surrounding the computational complexity of finding efficient allocations have been surveyed very nicely by de Vries and Vohra (2003). The computationally manageable example of linearly ordered objects with complementarities only among adjacent items is due to Rothkopf, Pekec, and Harstad (1998). A collection of readings on combinatorial auctions may be found in the book edited by Cramton, Shoham, and Steinberg (2006).

The question of how budget constraints affect equilibrium behavior in sequential multiple-object auctions was first studied by Pitchik and Schotter (1988). The example showing that budget constraints may actually increase the seller's revenue is taken from a paper by Benoît and Krishna (2001). Both of these papers consider the issue in the context of models with complete information.