MATH 425a ASSIGNMENT 9 SOLUTIONS
FALL 2015 Prof. Alexander

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Rudin Chapter 5:

(1) For each fixed x, for all $y \neq x$,

$$\left|\frac{f(y)-f(x)}{y-x}\right| \leq |y-x| \to 0 \text{ as } y \to x,$$

so $f'(x) = 0$ for all x. By 5.11b, f is constant.

(6) By 5.3c (quotient rule), we have $g'(x) = \frac{xf'(x)-f(x)}{x^2}$ for all $x \neq 0$, so to show $g'(x) > 0$, it is enough to show $f'(x) > \frac{f(x)}{x}$. But in fact, by the Mean Value Theorem 5.10, for some $t \in (0,x)$,

$$\frac{f(x)}{x} = \frac{f(x)-f(0)}{x-0} = f'(t) \leq f'(x),$$

since $t < x$ and f' is monotone increasing.

(7) Suppose $f(x) = g(x) = 0$ and $g'(x) \neq 0$. Then for $t \neq x$,

$$\frac{f(t)}{g(t)} = \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} \to \frac{f'(x)}{g'(x)} \quad \text{as } t \to x.$$

(13)(a) For $x \neq 0$, f is a composition and product of continuous functions, so it is continuous. Thus we need only consider continuity at $x = 0$.

If $a < 0$ then for $x_n = ((2n+\frac{1}{2})\pi)^{-1/c}$ we have $x_n \to 0$, while $\sin(x_n^{-c}) = 1$ so $f(x_n) = ((2n+\frac{1}{2})\pi)^{-a/c} \to \infty$. Therefore f is not continuous at 0.

If $a = 0$ then for $x_n' = (2n\pi)^{-1/c}$ we have $x_n \to 0, x_n' \to 0$ but $f(x_n) = 1, f(x_n') = 0$. This shows f is not continuous at 0.

If $a > 0$, then $x^a \to 0$ as $x \to 0$, and $\sin(x^{-c})$ is bounded, so $f(x) \to 0$ as $x \to 0$, meaning f is continuous at 0.

(b) For $x \neq 0$ we have

$$\frac{f(x)-f(0)}{x-0} = x^{a-1}\sin(x^{-c}).$$

By the proof of (a), this does not have a limit as $x \to 0$ if $a-1 \leq 0$, so $f'(0)$ does not exist, while if $a-1 > 0$ it converges to 0 as $x \to 0$, meaning $f'(0) = 0$.

(A) Since f is differentiable at 0, it has the same derivative from the left as from the right, so we have

$$f'(0) = \lim_{x \searrow 0} \frac{f(-x)-f(0)}{-x} = \lim_{x \searrow 0} \frac{f(x)-f(0)}{-x} = -f'(0),$$

and therefore $f'(0)$ must be 0.

(B) Let $h = g - f$, so $h' \geq 0$ on $[a, b]$. Applying the Mean Value Theorem 5.10 on $[a, x]$ we get

$$h(x) = h(x) - h(a) = h'(\xi)(x - a) \geq 0,$$

so $g(x) \geq f(x)$.

(C) For $t \neq x$,

$$\begin{aligned}
\frac{g(t) - g(x)}{t - x} &= \frac{f(t)^3 - f(x)^3}{t - x} \\
&= \frac{f(t) - f(x)}{t - x}(f(t)^2 + f(t)f(x) + f(x)^2) \\
&\to f'(x) \cdot 3f(x)^2 \quad \text{as } t \to x.
\end{aligned}$$

This shows that $g'(x) = 3f(x)^2 f'(x)$.

(D) Suppose there are 3 values $x_1 < x_2 < x_3$ with $f(x_i) = c$. By the Mean Value Theorem 5.10, there exist ξ_1, ξ_2 with $x_1 < \xi_1 < x_2 < \xi_2 < x_3$ for which $f'(\xi_1) = f'(\xi_2) = 0$. Applying 5.10 again to the function f' we get that there exists $t \in (\xi_1, \xi_2)$ with $f''(t) = 0$, which is a contradiction. Thus there are at most two such x_i.

(E) We have $|x| = x$ for $x > 0$ and $|x| = -x$ for $x < 0$, so $|x|$ is a differentiable function of x on $(-\infty, 0) \cup (0, \infty)$. Therefore we can apply the chain rule where $f(x) \neq 0$:

$$(*) \quad \frac{d}{dx}|f(x)| = \begin{cases} f'(x) & \text{if } f(x) > 0; \\ -f'(x) & \text{if } f(x) < 0. \end{cases}$$

This suggests we should require $f'(x) = 0$ at points where $f(x) = 0$. In fact let us prove the following:

Theorem. Suppose f is differentiable in (a, b), and $f'(x) = 0$ at all points $x \in (a, b)$ where $f(x) = 0$. Then $|f|$ is differentiable in (a, b).

Proof. By $(*)$ we need only establish differentiability at points x where $f(x) = 0$. Fix such a point x. Then for $t \neq x$,

$$\frac{|f(t)| - |f(x)|}{t - x} = \frac{|f(t) - f(x)|}{t - x} = \begin{cases} \left|\frac{f(t)-f(x)}{t-x}\right| & \text{if } t > x, \\ -\left|\frac{f(t)-f(x)}{t-x}\right| & \text{if } t < x. \end{cases}$$

Since $f'(x) = 0$, both the quantities on the right approach 0 as $t \to x$, which shows that $\frac{d}{dx}|f(x)| = 0$ at x.

(F) Suppose $f : (a, b) \to \mathbb{R}$ is differentiable, with $-\infty \leq a < b \leq \infty$, and suppose $\lim_{x \to a} f(x) = \lim_{x \to b} f(x) = L$ for some finite L. If f is constant, that is, $f(x) = L$ for all x, then $f'(x) = 0$ for all x so we are done.

If f is not constant, then either its sup is $> L$ or its inf is $< L$. So suppose $\sup_{x\in(a,b)} f(x) = \alpha > L$. Fix $\epsilon > 0$ with $L < L + \epsilon < \alpha$. Since $\lim_{x\to a} f(x) = \lim_{x\to b} f(x) = L$, there exist $c < d$ in (a, b) such that $f(x) < L+\epsilon$ for all $x \leq c$ and for all $x \geq d$. This means $\sup_{x\in[c,d]} f(x)$ must be α, and this sup is not achieved at c or d. Since f is continuous and $[c, d]$ is compact, the sup must therefore be achieved at some $x \in (c, d)$. Since f is differentiable, this means $f'(x) = 0$.

The proof when the inf is $< L$ is similar.

(G) For $x > 0$ we have $f'(x) = 2x$ and $f''(x) = 2$. For $x < 0$ we have $f'(x) = -2x$ and $f''(x) = -2$. This means that at $x = 0$, f'' has limits from the left and right, but these limits are not equal. If $f''(0)$ existed this would mean f'' had a discontinuity of the first kind at $x = 0$, but this is impossible by the Corollary to Theorem 5.12, so $f''(0)$ does not exist. Since f is continuous at $x = 0$ and $\lim_{x\to 0} f'(x) = 0$, by a lemma from lecture (similar to Ch. 5 exercise 9), we have $f'(0) = 0$.