MATH 425b SAMPLE FINAL EXAM SOLUTIONS
SPRING 2016
Prof. Alexander

(1)(a) We have

$$\begin{aligned}\omega \wedge \omega &= f(x)g(x)dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + f(x)g(x)dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_2 \\ &= 2f(x)g(x)dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.\end{aligned}$$

This is $\neq 0$ if and only if there exists x where $f(x)g(x) \neq 0$, i.e. where both $f(x)$ and $g(x)$ are nonzero, since then there is a neighborhood of x where $f(x)g(x) > 0$ or where $f(x)g(x) < 0$, so for a 2-surface in that neighborhood, the integral is nonzero.

(b) If $I \cap J \neq \phi$ then clearly all 4 terms in $\omega \wedge \omega$ are 0, so suppose $I \cap J = \phi$. We have

$$\omega \wedge \omega = f(x)g(x)(dx_I \wedge dx_J + dx_J \wedge dx_I) = f(x)g(x)((-1)^\alpha + (-1)^\beta)dx_{[I,J]},$$

where α is the number of pairs (i,j) with $i \in I$, $j \in J$ and $i > j$, and β is the number of pairs (i,j) with $i \in I$, $j \in J$ and $j > i$. The total number of pairs of both types is thus $\alpha + \beta = k^2$ which is odd. Therefore one of α, β is odd and the other is even, so $(-1)^\alpha + (-1)^\beta = 0$, so $\omega \wedge \omega = 0$.

(c) ω exact means $\omega = d\xi$ for some $(k-1)$-form ξ. Then

$$d(\xi \wedge d\beta) = d\xi \wedge d\beta + (-1)^{k-1}\xi \wedge d^2\beta = \omega \wedge d\beta,$$

which shows $\omega \wedge d\beta$ is exact.

(2)(a) By Parseval, $\|f^{(k)} - f\|_2^2 = \sum_{n \in \mathbb{Z}} |c_n^{(k)} - c_n|^2$.

(b) We have $|c_n| = \lim_k |c_n^{(k)}| \leq b_n$ so $|c_n^{(k)} - c_n| \leq |c_n^{(k)}| + |c_n| \leq 2b_n$, and therefore for fixed N,

$$\sum_{n \notin [-N,N]} |c_n^{(k)} - c_n|^2 \leq 4 \sum_{n \notin [-N,N]} b_n^2.$$

Given $\epsilon > 0$ we can choose N so $4\sum_{n \notin [-N,N]} b_n^2 < \epsilon/2$. Then we can choose K so that

$$k \geq K \implies \sum_{n \in [-N,N]} |c_n^{(k)} - c_n|^2 < \frac{\epsilon}{2}.$$

Then for $k \geq K$,

$$\sum_{n \in \mathbb{Z}} |c_n^{(k)} - c_n|^2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which shows, since ϵ is arbitrary, that

$$\|f^{(k)} - f\|_2^2 = \sum_{n\in\mathbb{Z}} |c_n^{(k)} - c_n|^2 \to 0, \quad \text{as } k \to \infty.$$

(3) Solution not included since this is essentially the same as a Take-Home Final problem.

(4)(a) We calculate

$$d\omega = 2z\ dx \wedge dy \wedge dz,$$

all other terms being 0. By Stokes Theorem,

$$\int_{\partial A} \omega = \int_A d\omega.$$

We can parametrize A by the identity so

$$\int_A d\omega = \int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} z\ dx\ dy\ dz = \int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} 2z\ dz\ dx\ dy.$$

Since z is an odd function, the innermost integral is 0.

(b) It's sufficient for the above argument that the innermost integral be from $-a$ to a, so $z_0 = 0$ is sufficient.