MATH 425b ASSIGNMENT 7
SPRING 2016
Prof. Alexander
Due Wednesday April 6.

To avoid the assignment being too long—you may consider (B)(b) and (C) as "practice problems" meaning they won't be graded, but they may help you prepare for the exam.

Rudin Chapter 9 #29 and:

(A) This problem tests your understanding of the definition of derivative. A system has 3 inputs x_1, x_2, x_3, and the corresponding output $f(\mathbf{x})$ has 3 components; more precisely $f : \mathbb{R}^3 \to \mathbb{R}^3$ is $\mathcal{C}'$. The function f is not known to the researcher, but he can input values $\mathbf{x}$ and observe the output values $f(\mathbf{x})$. So far he has the following observations:

$$\begin{aligned} f(2,3,4) &= (7,6,5) \\ f(2.01,3,4) &= (6.99,6.03,5.04) \\ f(2.01,3.01,4) &= (7.01,6.06,5.05) \\ f(2.01,3.01,4.01) &= (7.01,6.02,5) \end{aligned}$$

Use this information to find the approximate value of the total derivative $f'(2,3,4)$ (expressed as a matrix) and to estimate $f(2,3.01,4.01)$.

(B) Let U be a connected open subset of $\mathbb{R}^2$ and let $f : U \to \mathbb{R}$ be a $\mathcal{C}'$ function with $\frac{\partial f}{\partial y} = 0$ everywhere in U.

(a) Recall that U is called *convex* if for any two points in U, the line segment connected these two points is contained in U. If U is convex, show that f does not depend on y, that is, $f(x,y) = g(x)$ for all x, y, for some $g(x)$.

(b) In contrast to (a), find an example of a non-convex U and an f as above, in which f does depend on y.

(C) Let $E \subset \mathbb{R}^2$ be open and let $f : E \to \mathbb{R}$. Suppose that $D_1 f$ exists everywhere in E and is continuous at some point $\mathbf{a} \in E$, while $D_2 f$ exists only at $\mathbf{a}$. Show that f is differentiable at $\mathbf{a}$.

(D)(a) For $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$, show that $\big| \, ||A|| - ||B|| \, \big| \le ||A - B||$.

(b) Show that $\psi(A) = ||A||$ defines a continuous function on $L(\mathbb{R}^n, \mathbb{R}^m)$.

(c) Suppose $E \subset \mathbb{R}^n$ is open, $f : E \to \mathbb{R}^n$ is $\mathcal{C}'$, and $f'(\mathbf{a})$ is invertible. Show that there exist $c > 0$ and a neighborhood U of $\mathbf{a}$ such that $\mathbf{x}, \mathbf{y} \in U$ implies $|f(\mathbf{x}) - f(\mathbf{y})| \ge c|\mathbf{x} - \mathbf{y}|$.

(E) Show that there exists a function $f(x,y)$ differentiable at $(-1,1)$ with $f(-1,1) = 0$ such that $x^3 + y^3 + f(x,y)^3 = 3xyf(x,y)$ at all points in some neighborhood B of $(-1,1)$.

(F) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, $f(a,b) = 0$, $(D_1 f)(a) \neq 0$. According to the implicit function theorem, in a neighborhood of b the curve $f(x,y) = 0$ is given by $x = g(y)$. The tangent line to the curve at (a,b) therefore goes in the direction $(\frac{dx}{dy}, 1) = (g'(b), 1)$. (You need not prove this.) Show that the gradient of f at (a, b) is perpendicular to the tangent direction $(g'(b), 1)$.

(G) The expression in the Implicit Function Theorem for the derivative of g ((58) page 225) can be viewed as a relation between increments: if we move $\mathbf{y}$ from $\mathbf{b}$ to $\mathbf{b} + \Delta\mathbf{y}$, then the corresponding $\mathbf{x} = g(\mathbf{y})$ moves from $\mathbf{a}$ to $\mathbf{a} + \Delta\mathbf{x}$, and $g'(\mathbf{b})$ gives the approximate relation between $\Delta\mathbf{x}$ and $\Delta\mathbf{y}$. Use this idea to solve the following. Let $\mathbf{y} = (y_1, y_2, y_3)$, $\mathbf{x} = (x_1, x_2)$, and

$$f_1(\mathbf{x}, \mathbf{y}) = y_1^2 + y_2^2 + y_3^2 - x_1^2 + x_2^2 - 1, \quad f_2(\mathbf{x}, \mathbf{y}) = y_1^2 - y_2^2 + y_3^2 + x_1^2 + 2x_2^2 - 21.$$

The point $(\mathbf{x}, \mathbf{y}) = (3, 2, 1, 1, 2)$ is then on the surface given by $(f_1, f_2) = (0, 0)$. Suppose we move $\mathbf{y}$ away from $(1, 1, 2)$ in the direction $(0, 1, 1)$. What direction must $\mathbf{x}$ move away from $(3, 2)$, to keep $(\mathbf{x}, \mathbf{y})$ on the surface $(f_1, f_2) = (0, 0)$? (Here when we refer to moving in a direction, we mean the instantaneous direction we start out in—if we move $\mathbf{x}$ and $\mathbf{y}$ in straight lines for a positive amount of time, then we will leave the surface because the surface is curved.)

HINTS:

(29) Any permutation can be constructed by a number of switches of adjacent elements, for example $21341 \to 23141$ switches the adjacent 1 and 3. So it's enough to prove that any kth order derivative is unchanged if you switch two adjacent indices.

(A) The information given lets you approximate certain partial derivatives, or directional derivatives which are related (how?) to partial derivatives. What information do you need from this, to find the matrix of the total derivative?

(B)(b) This is a little tricky—you have to think geometrically about the graph of such an f. Use a set U which is actually "U" shaped (not necessarily opening upward, though.) To build your f, you may want to make use of functions on $\mathbb{R}$ which are $\mathcal{C}'$ and not constant, but are constant on an interval, for example,

$$h(t) = \begin{cases} 0, & t \le 0, \\ t^2, & t > 0. \end{cases}$$

(C) Express $f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a})$ as a sum of two increments. Also, if $f'(\mathbf{a})$ does exist its matrix must have entries $D_1 f(\mathbf{a}), D_2 f(\mathbf{a})$ so you just have to show that the transformation with this matrix satisfies the definition of derivative.

(D)(a) Use the triangle inequality. What happens if you take away the outer $|\cdot|$ on the left?

(b) Use (a).

(c) Use (b) to show that $||f'(\mathbf{x})^{-1}||$ is bounded on U, if you choose the right U. Also, use the Inverse Function Theorem and Theorem 9.19. You may assume the fact that for matrices A, the inverse A^{-1} is a continuous function of A, wherever the inverse exists.

(F) Use the last part of the Implicit Function Theorem. Consider properties of $f(g(y), y)$ and its derivative in y.