MATH 425b ASSIGNMENT 4
SPRING 2016
Prof. Alexander
Due Monday February 22.

Rudin Chapter 8 #13, 19, and (A)–(D) below.

The midterm will cover Ch. 7 and 8, excluding the following sections of Ch. 8: The Exponential and Logarithmic Functions, The Trigonometric Functions, The Gamma Function.

(A) For this problem we use the L^2 inner product on all of $\mathbb{R}$:

$$(*) \qquad \langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\, dx.$$

(a) Show that the functions $\varphi_1(x) = 2xe^{-x^2/2}, \varphi_3(x) = (8x^3 - 12x)e^{-x^2/2}$ on $\mathbb{R}$ are orthogonal.

(b) Show that there are constants $c_1, c_3 > 0$ such that if we define functions $\psi_i = c_i\varphi_i$ then $\{\psi_1, \psi_3\}$ form an orthonormal system. You do NOT need to find c_1, c_3, just show they exist. The actual values are $c_1 = \sqrt{2}\pi^{-1/4}, c_3 = (\sqrt{3}\pi^{1/4})^{-1}$.

(c) Let $g(x) = x$. Find the coefficients a_1, a_3 for which the function $a_1\psi_1(x) + a_3\psi_3(x)$ is closest to g. Here "closest" means in the sense of the L^2 distance

$$d_2(f, g) = \int_{-\infty}^{\infty} |f(x) - g(x)|^2\, dx.$$

As an added note, the polynomials $H_1(x) = 2x, H_3(x) = 8x^3 - 12x$ appearing in φ_1, φ_3 are part of a whole sequence $\{H_n(x), n \geq 0\}$ of *Hermite polynomials*, for which the corresponding functions $\varphi_n(x) = H_n(x)e^{-x^2/2}$ are orthogonal, and they arise in the context of certain differential equations in quantum mechanics.

(B)(a) Find the Fourier coefficients c_n of the function $f : [-\pi, \pi] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & \text{if } -\pi \leq x \leq 0, \\ 0 & \text{if } 0 \leq x \leq \pi. \end{cases}$$

(b) Show that

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Here the sum is over odd positive integers.

(C) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and has period 2π, and its Fourier series on $[-\pi, \pi]$ is $\sum_{n=1}^{\infty} c_n e^{inx}$. Show that the Fourier series of f' is $\sum_{n=1}^{\infty} inc_n e^{inx}$, that is, one can differentiate term by term.

(D) Suppose that the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n = A(z) + iB(z)$ converes for all $z \in D = \{z \in \mathbb{C} : |z| < 1\}$. Show that $A(z)$ does not have a local maximum in D. Here $A(z)$ and $B(z)$ are the real and imaginary parts of $f(z)$.

As an added note, the real part (like $A(z)$ here) of an analytic function on a region in $\mathbb{C}$ is what is called a *harmonic function*, meaning that if we write $z = x + iy$ it satisfies $\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = 0$. Many quantities of physical interest are harmonic functions, such as the distribution of heat at equilibrium in a flat metal plate, so the lack of local maxima has physical significance.

HINTS:

GENERAL HINT: There are two basic ways to get sums of infinite series as in #13 and the second half of #14. One is to use Parseval's Theorem, the other is to plug a specific value of x into a Fourier series, like the one in the first half of #14. For this second approach, you have to choose the right x to make it work, and you have to justify why the convergence is valid *at that particular* x.

(13) Be careful with c_0, it's calculated differently from the other c_n's.

(19) This one is hard, but see what you can do! After following Rudin's hint, use Stone-Weierstass.

Another added note: You can think of f as a function of angle (or distance), on the unit circle. Think of a particle that moves an angle α each unit of time, starting from x at time 0, so at time n it's at angle $x + n\alpha$. Then $\frac{1}{N}\sum_{n=1}^{N} f(x + n\alpha)$ is the average value of f "observed" by the particle up to time N, which we can call the limit the "time average" of f as seen by the particle. The right side $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt$ is the overall "space average" of f on the whole circle. So the problem says time average = space average. Such a result is called an *ergodic theorem*, and these are important in physics. Note the result fails for rational α/π, because the particle then only visits finitely many points, ever.

(A)(a) To show an integral of form $\int_{-\infty}^{\infty}(ax^j - bx^k)e^{-x^2/2}\, dx = 0$, one way is to apply integration by parts to $\int_{-\infty}^{\infty} ax^j e^{-x^2/2}\, dx$ to show it's the same as $\int_{-\infty}^{\infty} bx^k e^{-x^2/2}\, dx$, or vice versa. Also, remember that the integral over $\mathbb{R}$ of an odd function is 0—don't do unnecessary calculations!

(b) Virtually no calculations are needed for this.

(c) To calculate $\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\, dx$, think of the integrand as a product $x \cdot xe^{-x^2/2}$.

(B)(b) Using Parseval is not practical here.

(C) Write c_n as $\alpha_n + i\beta_n$. From the formula for c_n, what are the formulas for α_n and β_n? Use this, together with integration by parts, to calculate the Fourier coefficients of f', expressed in terms of α_n and β_n. Remember that $e^{inx} = \cos nx + i \sin nx$. Note that periodicity means $f(\pi) = f(-\pi)$.

(D) This is quite similar to the proof of Theorem 8.8. Suppose $A(z)$ has a local maximum at $z = z_0 \in D$, to get a contradiction. Show that you can express $f(z)$ in a neighborhood of z_0 as

$$f(z) = f(z_0) + \sum_{n \geq 1} a_n (z - z_0)^n = f(z_0) + a_k (z - z_0)^k \left[1 + \sum_{n > k} \frac{a_n}{a_k} (z - z_0)^{n-k} \right],$$

with a_k being the first nonzero coefficient in the series. Writing z as $z_0 + re^{i\theta}$, how can you choose r and θ so that $A(z) > A(z_0)$, that is, $f(z)$ is to the right of $f(z_0)$ in the complex plane? Think of the complex number $a_k(z - z_0)^k$ as a vector in the plane and decide which way you want it to point, to accomplish this. Then show that the sum over $n > k$ is small enough that it doesn't mess up what you're trying to do.