MATH 425a ASSIGNMENT 8
FALL 2015 Prof. Alexander
Due Monday November 9.

The due date is after the midterm, but this material IS covered on the midterm.

Rudin Chapter 4 #18, plus the problems (A)–(G) below:

(A) Suppose $f : [a, b] \to \mathbb{R}, c \in (a, b), f$ is uniformly continuous on $[a, c]$ and on $[c, b]$. Show directly from the definitions (not using compactness) that f is uniformly continuous on the full interval $[a, b]$.

(B)(a) If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are uniformly continuous and bounded, show that fg is uniformly continuous.

(b) Give an example to show that (a) can be false if we don't assume boundedness.

(C)(a) Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous, and for some $L \in \mathbb{R}$ we have $f(x) \to L$ as $x \to \infty$. Show that f is uniformly continuous on $\mathbb{R}$.

(b) Show that $f(x) = 1/(1 + x^2)$ is uniformly continuous on $[0, \infty)$.

(D) Suppose K is compact and $f : K \to \mathbb{R}$ is continuous. Show that either $f(x) = 0$ for some x, or f is bounded away from 0 (that is, there exists $\epsilon > 0$ such that $|f(x)| \geq \epsilon$ for all x.)

(E)(a) Give an example of a continuous function on $\mathbb{R}$ for which the inverse image of some compact set is not compact.

(b) Give an example of a continuous function on $\mathbb{R}$ for which the image of some closed set is not closed.

(F) Prove (by logic) or disprove (by example): Let $f : X \to Y$ be a continuous bijection. Then the inverse image of a convergent sequence in Y is a convergent sequence in X.

(G)(a) Let $\mathbb{Z}$ be the integers and let Y be any metric space. Show that all functions $f : \mathbb{Z} \to Y$ are continuous.

(b) Suppose the metric space X has a limit point. Show that there exists a function $f : X \to \mathbb{R}$ which is not continuous.

HINTS:

(18) Show that $\lim_{x\to 0} f(x) = 0$ for all $p \in \mathbb{R}$.

(A) Be careful—you can have points $x \in [a, c], y \in [c, b]$ with $|x - y| < \delta$. This means it's not enough to just find a δ that "works" for both $[a, c]$ and $[c, b]$.

(B)(b) $f = g$ can work.

(C)(a) For a given ϵ, find a value M, a δ_1 that "works" on $[M, \infty)$, and a δ_2 that "works" on $[0, M + 1]$. Note that overlapping these two intervals helps avoid the issue pointed out in hint (A).

(F) If p, q are close together in Y, does that force $f^{-1}(p), f^{-1}(q)$ to be close together in X?