Mechanism Design

An auction is one of many ways that a seller can use to sell an object to potential buyers with unknown values. In an auction, the object is sold at a price determined by competition among the buyers according to rules set out by the seller—the auction format—but the seller could use other methods. The seller could post a fixed price and sell the object to the first arrival. Or the seller could negotiate with one of the buyers—say, one chosen at random. The seller could also hold an auction and then negotiate with the winner. The range of options is virtually unlimited. This chapter considers the underlying allocation problem by abstracting away from the details of any particular selling format and asking the question: What is the best way to allocate an object?

As before, a seller has one indivisible object to sell and there are N risk-neutral potential buyers (or bidders) from the set $\mathcal{N} = \{1, 2, \ldots, N\}$. Again, buyers have private values and these are independently distributed. Buyer i's value X_i is distributed over the interval $\mathcal{X}_i = [0, \omega_i]$ according to the distribution function F_i with associated density function f_i. Notice that, as in Section 4.3, we allow for asymmetries among the buyers: the distributions of values need not be the same for all buyers.

For the sake of simplicity, we suppose that the value of the object to the seller is 0. Let $\mathcal{X} = \times_{j=1}^{N} \mathcal{X}_j$ denote the product of the sets of buyers' values, and for all i, let $\mathcal{X}_{-i} = \times_{j \neq i} \mathcal{X}_j$. Define $f(\mathbf{x})$ to be the joint density of $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. Since the values are independently distributed, $f(\mathbf{x}) = f_1(x_1) \times f_2(x_2) \times \cdots \times f_N(x_N)$. Similarly, define $f_{-i}(\mathbf{x}_{-i})$ to be the joint density of $\mathbf{x}_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$.

5.1 MECHANISMS

A selling *mechanism* $(\mathcal{B}, \boldsymbol{\pi}, \boldsymbol{\mu})$ has the following components: a set of possible *messages* (or "bids") $\mathcal{B}_i$ for each buyer; an *allocation rule* $\boldsymbol{\pi} : \mathcal{B} \to \Delta$, where Δ is the set of probability distributions over the set of buyers $\mathcal{N}$; and a *payment rule* $\boldsymbol{\mu} : \mathcal{B} \to \mathbb{R}^N$. An allocation rule determines, as a function of all N messages, the probability $\pi_i(\mathbf{b})$ that i will get the object. A payment rule determines, again as a function of all N messages, for each buyer i, the expected payment $\mu_i(\mathbf{b})$ that i must make.

Notice that both first- and second-price auctions are mechanisms. The set of possible bids $\mathcal{B}_i$ in both can be safely assumed to be $\mathcal{X}_i$. Assuming that there is no reservation price, the allocation rule for both is $\pi_i(\mathbf{b}) = 1$ if $b_i > \max_{j \neq i} b_j$ and $\pi_j(\mathbf{b}) = 0$ for $j \neq i$. They differ only in the associated payment rules. For a first-price auction, $\mu_i^{\mathrm{I}}(\mathbf{b}) = b_i$ if $b_i > \max_{j \neq i} b_j$ and $\mu_j^{\mathrm{I}}(\mathbf{b}) = 0$ for $j \neq i$. For a second-price auction, $\mu_i^{\mathrm{II}}(\mathbf{b}) = \max_{j \neq i} b_j$ if $b_i > \max_{j \neq i} b_j$ and $\mu_j^{\mathrm{II}}(\mathbf{b}) = 0$ for $j \neq i$. If there are ties, each winning bidder has an equal likelihood of being awarded the object, so the π_j have to take account of this.

Every mechanism defines a game of incomplete information among the buyers. An N-tuple of strategies $\beta_i : [0, \omega_i] \to \mathcal{B}_i$ is an *equilibrium* of a mechanism if for all i and for all x_i, given the strategies $\boldsymbol{\beta}_{-i}$ of other buyers, $\beta_i(x_i)$ maximizes i's expected payoff.

5.1.1 The Revelation Principle

A mechanism could, in principle, be quite complicated since we have made no assumptions on the sets $\mathcal{B}_i$ of "bids" or "messages." A smaller and simpler class consists of those mechanisms for which the set of messages is the same as the set of values—that is, for all i, $\mathcal{B}_i = \mathcal{X}_i$. Such mechanisms are called *direct*, since, in effect, every buyer is asked to directly report a value. Formally, a *direct mechanism* $(\mathbf{Q}, \mathbf{M})$ consists of a pair of functions $\mathbf{Q} : \mathcal{X} \to \Delta$ and $\mathbf{M} : \mathcal{X} \to \mathbb{R}^N$, where $Q_i(\mathbf{x})$ is the probability that i will get the object and $M_i(\mathbf{x})$ is the expected payment by i. If it is an equilibrium for each buyer to reveal his or her true value, then the direct mechanism is said to have a truthful equilibrium. We will refer to the pair $(\mathbf{Q}(\mathbf{x}), \mathbf{M}(\mathbf{x}))$ as the *outcome* of the mechanism at $\mathbf{x}$.

The following result, referred to as the *revelation principle*, shows that the outcomes resulting from any equilibrium of any mechanism can be replicated by a truthful equilibrium of some direct mechanism. In this sense, there is no loss of generality in restricting attention to direct mechanisms.

Proposition 5.1. (Revelation Principle) *Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (1) it is an equilibrium for each buyer to report his or her value truthfully and (2) the outcomes are the same as in the given equilibrium of the original mechanism.*

Proof. Let $\mathbf{Q} : \mathcal{X} \to \boldsymbol{\Delta}$ and $\mathbf{M} : \mathcal{X} \to \mathbb{R}^N$ be defined as follows: $\mathbf{Q}(\mathbf{x}) = \boldsymbol{\pi}(\boldsymbol{\beta}(\boldsymbol{x}))$ and $\mathbf{M}(\mathbf{x}) = \boldsymbol{\mu}(\boldsymbol{\beta}(\boldsymbol{x}))$. In other words, as depicted in Figure 5.1, the direct mechanism $(\mathbf{Q}, \mathbf{M})$ is a composition of $(\boldsymbol{\pi}, \boldsymbol{\mu})$ and $\boldsymbol{\beta}$. Conclusions (1) and (2) can now be verified routinely. ■

The idea underlying the revelation principle is very simple. Fix a mechanism and an equilibrium $\boldsymbol{\beta}$ of the mechanism. Now instead of having the buyers submit messages $b_i = \beta_i(x_i)$ and then applying the rules of the mechanism in order to determine the outcome—who gets the object and who pays what—we could

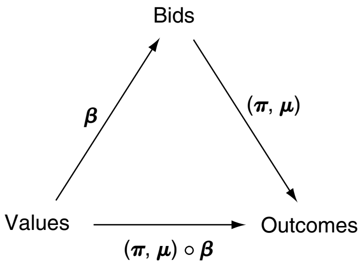

FIGURE 5.1 The revelation principle.

directly ask the buyers to "report" their values x_i and then make sure that the outcome is the same as if they had submitted bids $\beta_i(x_i)$. Put another way, a direct mechanism does the "equilibrium calculations" for the buyers automatically. Now suppose that some buyer finds it profitable to be untruthful and report a value of z_i when his true value is x_i. Then in the original mechanism the same buyer would have found it profitable to submit a bid of $\beta_i(z_i)$ instead of $\beta_i(x_i)$. But since the β_i constitute an equilibrium, this is impossible.

5.1.2 Incentive Compatibility

Given a direct mechanism $(\mathbf{Q}, \mathbf{M})$, define

$$q_i(z_i) = \int_{\mathcal{X}_{-i}} Q_i\left(z_i, \mathbf{x}_{-i}\right) f_{-i}\left(\mathbf{x}_{-i}\right) \, \mathbf{dx}_{-i} \tag{5.1}$$

to be the probability that i will get the object when he reports his value to be z_i and all other buyers report their values truthfully. Similarly, define

$$m_i(z_i) = \int_{\mathcal{X}_{-i}} M_i\left(z_i, \mathbf{x}_{-i}\right) f_{-i}\left(\mathbf{x}_{-i}\right) \, \mathbf{dx}_{-i} \tag{5.2}$$

to be the expected payment of i when his report is z_i and all other buyers tell the truth. It is important to note that because the values are independently distributed, both the probability of getting the object and the expected payment depend only on the *reported* value z_i and not on the *true* value, say x_i. The expected payoff of buyer i when his true value is x_i and he reports z_i, again assuming that all other buyers tell the truth, can then be written as

$$q_i(z_i)\, x_i - m_i(z_i) \tag{5.3}$$

The direct revelation mechanism $(\mathbf{Q}, \mathbf{M})$ is said to be *incentive compatible* (IC) if for all i, for all x_i and for all z_i,

$$U_i(x_i) \equiv q_i(x_i)\, x_i - m_i(x_i) \geq q_i(z_i)\, x_i - m_i(z_i) \tag{5.4}$$

We will refer to U_i as the *equilibrium payoff function*.

Incentive compatibility has some simple but powerful implications. First, for each reported value z_i, the expected payoff $q_i(z_i)x_i - m_i(z_i)$ is an affine function of the true value x_i. Incentive compatibility implies that

$$U_i(x_i) = \max_{z_i \in \mathcal{X}_i} \{q_i(z_i)x_i - m_i(z_i)\}$$

—that is, U_i is a maximum of a family of affine functions, therefore *U_i is a convex function.*

Second, we can write for all x_i and z_i,

$$\begin{aligned} q_i(x_i)z_i - m_i(x_i) &= q_i(x_i)x_i - m_i(x_i) + q_i(x_i)(z_i - x_i) \\ &= U_i(x_i) + q_i(x_i)(z_i - x_i) \end{aligned}$$

so incentive compatibility is equivalent to the requirement that for all x_i and z_i,

$$U_i(z_i) \geq U_i(x_i) + q_i(x_i)(z_i - x_i) \tag{5.5}$$

This implies that for all x_i, $q_i(x_i)$ is the slope of a line that supports the function U_i at the point x_i. (See Figure 5.2.) A convex function is absolutely continuous[1] and thus it is differentiable almost everywhere in the interior of its domain (in Figure 5.2, U_i is not differentiable at x_i). Thus, at every point that U_i is

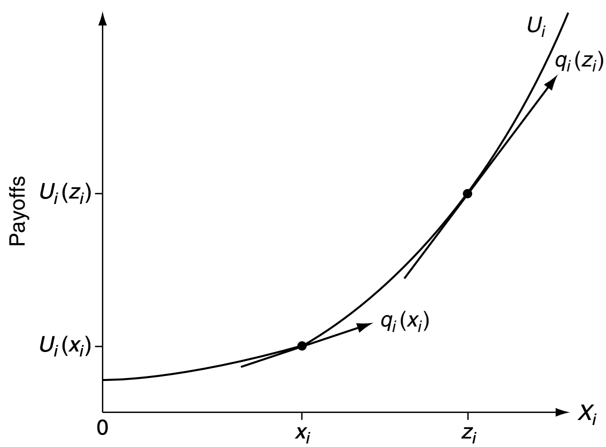

FIGURE 5.2 Implications of incentive compatibility.

[1]For a definition of an absolutely continuous function, see the notes at the end of this chapter (page 82).

differentiable,

$$U_i'(x_i) = q_i(x_i) \tag{5.6}$$

Since U_i is convex, this implies that q_i *is a nondecreasing function.*

Third, every absolutely continuous function is the definite integral of its derivative, so we have

$$U_i(x_i) = U_i(0) + \int_0^{x_i} q_i(t_i)\, dt_i, \tag{5.7}$$

which implies that up to an additive constant, the expected payoff to a buyer in an incentive compatible direct mechanism $(\mathbf{Q}, \mathbf{M})$ depends *only* on the allocation rule $\mathbf{Q}$. If $(\mathbf{Q}, \mathbf{M})$ and $\left(\mathbf{Q}, \overline{\mathbf{M}}\right)$ are two incentive compatible mechanisms with the same allocation rule $\mathbf{Q}$ but different payment rules, then the expected payoff functions associated with the two mechanisms, U_i and $\overline{U}_i$, respectively, differ by at most a constant; the two mechanisms are *payoff equivalent*. Put another way, the "shape" of the expected payoff function is completely determined by the allocation rule $\mathbf{Q}$ alone. The payment rule $\mathbf{M}$ only serves to determine the constants $U_i(0)$. See Figure 5.3, where, despite appearances to the contrary, $\overline{U}_i - U_i$ is a constant.

Finally, note that a mechanism is incentive compatible if and only if the associated q_i is nondecreasing. We have already argued that incentive compatibility implies that q_i is nondecreasing. To see the converse, note that (5.5) can be rewritten as

$$\int_{x_i}^{z_i} q_i(t_i)\, dt_i \geq q_i(x_i)(z_i - x_i)$$

by using (5.7) and this certainly holds if q_i is nondecreasing.

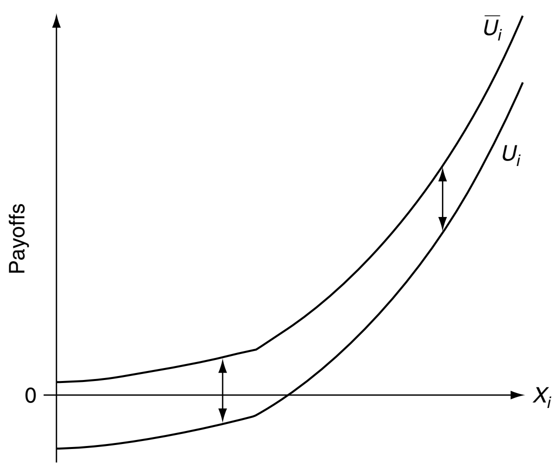

FIGURE 5.3 Payoff equivalence.

REVENUE EQUIVALENCE (REDUX)

The payoff equivalence derived here immediately implies a general form of the revenue equivalence principle.

Proposition 5.2. (Revenue Equivalence) *If the direct mechanism* $(\mathbf{Q}, \mathbf{M})$ *is incentive compatible, then for all i and* x_i *the expected payment is*

$$m_i(x_i) = m_i(0) + q_i(x_i)x_i - \int_0^{x_i} q_i(t_i)\ dt_i \tag{5.8}$$

Thus, the expected payments in any two incentive compatible mechanisms with the same allocation rule are equivalent up to a constant.

Proof. Since $U_i(x_i) = q_i(x_i)x_i - m_i(x_i)$ and $U_i(0) = -m_i(0)$, the equality in (5.7) can be rewritten as (5.8). ■

Proposition 5.2 generalizes the revenue equivalence principle from Chapter 3, Proposition 3.1, to situations where buyers may be asymmetric. To see this, note that if buyers are symmetric and there is an increasing symmetric equilibrium, then the object is allocated to the buyer with the highest value; thus, for all such auctions, the allocation rule is the same. The remaining hypothesis in Proposition 3.1 pins down the expected payments completely by supposing that $m_i(0) = 0$.

At first glance, Proposition 5.2 seems to contradict the finding in the previous chapter that with asymmetric buyers, revenue equivalence between the first- and second-price auctions did not hold. Notice, however, that Proposition 5.2 implies only that, given the possible asymmetries, if there are two auctions with the *same* allocation rule, then they are revenue equivalent. When buyers are asymmetric, the first- and second-price auctions allocate differently: A second-price auction allocates efficiently, whereas a first-price auction typically does not. This accounts for differences in the resulting payments and revenues in the two auctions when buyers are not symmetric.

The general revenue equivalence principle can be usefully restated as follows: In any incentive compatible mechanism the expected payment of a buyer depends, up to an additive constant, on the allocation rule alone.

5.1.3 Individual Rationality

A mechanism in which the payments are so high that a buyer is better off by not participating will not attract this potential buyer. Specifically, we will say that a direct mechanism $(\mathbf{Q}, \mathbf{M})$ is *individually rational* if for all i and x_i, the equilibrium expected payoff $U_i(x_i) \geq 0$. We are implicitly assuming here that by not participating, a buyer can guarantee himself or herself a payoff of zero.

If the mechanism is incentive compatible, then from (5.7) individual rationality is equivalent to the requirement that $U_i(0) \geq 0$, and since $U_i(0) = -m_i(0)$, this is equivalent to the requirement that $m_i(0) \leq 0$.

5.2 OPTIMAL MECHANISMS

In this section we view the seller as the designer of the mechanism and examine mechanisms that maximize the expected revenue—the sum of the expected payments of the buyers—among all mechanisms that are incentive compatible and individually rational. We reiterate that when carrying out this exercise, the revelation principle guarantees that there is no loss of generality in restricting attention to direct mechanisms.

We will refer to a mechanism that maximizes expected revenue, subject to the incentive compatibility and individual rationality constraints, as an *optimal mechanism*.

5.2.1 Setup

Suppose that the seller uses the direct mechanism $(\mathbf{Q}, \mathbf{M})$. The expected revenue of the seller is

$$E[R] = \sum_{i \in \mathcal{N}} E[m_i(X_i)],$$

where the *ex ante* expected payment of buyer i is

$$\begin{aligned} E[m_i(X_i)] &= \int_0^{\omega_i} m_i(x_i) f_i(x_i)\ dx_i \\ &= m_i(0) + \int_0^{\omega_i} q_i(x_i) x_i f_i(x_i)\ dx_i \\ &\quad - \int_0^{\omega_i} \int_0^{x_i} q_i(t_i) f_i(x_i)\ dt_i dx_i \end{aligned}$$

by substituting from (5.8). Interchanging the order of integration in the last term results in

$$\begin{aligned} \int_0^{\omega_i} \int_0^{x_i} q_i(t_i) f_i(x_i)\ dt_i dx_i &= \int_0^{\omega_i} \int_{t_i}^{\omega_i} q_i(t_i) f_i(x_i)\ dx_i dt_i \\ &= \int_0^{\omega_i} (1 - F_i(t_i))\, q_i(t_i)\ dt_i \end{aligned}$$

Thus, we can write

$$\begin{aligned} E[m_i(X_i)] &= m_i(0) + \int_0^{\omega_i} \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) q_i(x_i) f_i(x_i)\ dx_i \\ &= m_i(0) + \int_{\mathcal{X}} \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) Q_i(\mathbf{x}) f(\mathbf{x})\ \mathbf{dx} \end{aligned}$$

using the definition of $q_i(x_i)$ from (5.1).

The seller's objective therefore is to find a mechanism that maximizes

$$\sum_{i\in\mathcal{N}} m_i(0) + \sum_{i\in\mathcal{N}} \int_{\mathcal{X}} \left(x_i - \frac{1-F_i(x_i)}{f_i(x_i)} \right) Q_i(\mathbf{x}) f(\mathbf{x})\ \mathbf{dx} \tag{5.9}$$

subject to the constraint that the mechanism is

IC. incentive compatible, which is equivalent to the requirement that q_i be nondecreasing and that (5.7) be satisfied; and

IR. individually rational, which is equivalent to the requirement that $m_i(0) \leq 0$.

5.2.2 Solution

We now make a simplification that allows an explicit derivation of the optimal selling mechanism. Define

$$\psi_i(x_i) \equiv x_i - \frac{1-F_i(x_i)}{f_i(x_i)} \tag{5.10}$$

to be the *virtual valuation* of a buyer with value x_i. It is routine to verify that for all i,

$$E[\psi_i(X_i)] = 0 \tag{5.11}$$

The design problem is said to be *regular* if for all i, the virtual valuation $\psi_i(\cdot)$ is an increasing function of the true value x_i. Since

$$\psi_i(x_i) = x_i - \frac{1}{\lambda_i(x_i)},$$

where $\lambda_i \equiv f_i/(1-F_i)$ is the hazard rate function associated with F_i, a sufficient condition for regularity is that for all i, $\lambda_i(\cdot)$ is increasing. In what follows, we will assume that the design problem is regular.

Thus, the seller should choose $(\mathbf{Q}, \mathbf{M})$ to maximize

$$\sum_{i\in\mathcal{N}} m_i(0) + \int_{\mathcal{X}} \left(\sum_{i\in\mathcal{N}} \psi_i(x_i) Q_i(\mathbf{x}) \right) f(\mathbf{x})\ \mathbf{dx} \tag{5.12}$$

Temporarily neglect the IC and IR constraints, and consider the expression

$$\sum_{i\in\mathcal{N}} \psi_i(x_i) Q_i(\mathbf{x}) \tag{5.13}$$

from the second term in (5.12). The function $\mathbf{Q}$ is then like a weighting function, and clearly it is best to give weight only to those $\psi_i(x_i)$ that are maximal, provided they are positive. This would maximize the function in (5.13) at *every* point $\mathbf{x}$ and so also maximize its integral.

With this in mind, consider a mechanism $(\mathbf{Q},\mathbf{M})$ where

- the allocation rule $\mathbf{Q}$ is that the object goes to buyer i with positive probability if and only if $\psi_i(x_i) = \max_{j\in\mathcal{N}} \psi_j(x_j)$; thus,

$$Q_i(\mathbf{x}) > 0 \Leftrightarrow \psi_i(x_i) = \max_{j\in\mathcal{N}} \psi_j(x_j) \geq 0 \tag{5.14}$$

- the payment rule $\mathbf{M}$ is

$$M_i(\mathbf{x}) = Q_i(\mathbf{x})\,x_i - \int_0^{x_i} Q_i(z_i,\mathbf{x}_{-i})\,dz_i \tag{5.15}$$

We claim that (5.14) and (5.15) define an optimal mechanism.

First, notice that the resulting q_i is a nondecreasing function. Suppose $z_i < x_i$. Then by the regularity condition, $\psi_i(z_i) < \psi_i(x_i)$ and thus for all $\mathbf{x}_{-i}$, it is also the case that $Q_i(z_i,\mathbf{x}_{-i}) \leq Q_i(x_i,\mathbf{x}_{-i})$. Thus, q_i is a nondecreasing function.

Second, from (5.15), it is clear that $M_i(0,\mathbf{x}_{-i}) = 0$, for all $\mathbf{x}_{-i}$, and thus $m_i(0) = 0$.

Thus, the proposed mechanism is *both* incentive compatible and individually rational. It is optimal, since it separately maximizes the two terms in (5.12) over all $\mathbf{Q}(\mathbf{x}) \in \Delta$. In particular, it gives positive weight only to nonnegative and maximal terms in (5.13). This implies that the maximized value of the expected revenue (5.12) is

$$E\left[\max\left\{\psi_1(X_1),\psi_2(X_2),\ldots,\psi_N(X_N),0\right\}\right] \tag{5.16}$$

In other words, it is the expectation of the highest virtual valuation, provided it is nonnegative.

A more intuitive formula may be obtained by writing

$$y_i(\mathbf{x}_{-i}) = \inf\left\{z_i : \psi_i(z_i) \geq 0 \text{ and } \forall j \neq i, \psi_i(z_i) \geq \psi_j(x_j)\right\}$$

as the smallest value for i that "wins" against $\mathbf{x}_{-i}$. Thus, we can rewrite (5.14) as

$$Q_i(z_i,\mathbf{x}_{-i}) = \begin{cases} 1 & \text{if } z_i > y_i(\mathbf{x}_{-i}) \\ 0 & \text{if } z_i < y_i(\mathbf{x}_{-i}), \end{cases}$$

which results in

$$\int_0^{x_i} Q_i(z_i,\mathbf{x}_{-i})\,dz_i = \begin{cases} x_i - y_i(\mathbf{x}_{-i}) & \text{if } x_i > y_i(\mathbf{x}_{-i}) \\ 0 & \text{if } x_i < y_i(\mathbf{x}_{-i}) \end{cases}$$

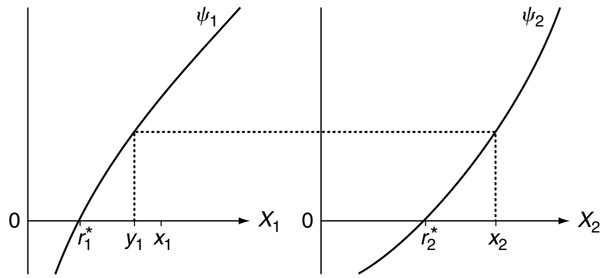

FIGURE 5.4 An optimal mechanism.

and so (5.15) becomes

$$M_i(\mathbf{x}) = \begin{cases} y_i(\mathbf{x}_{-i}) & \text{if } Q_i(\mathbf{x}) = 1 \\ 0 & \text{if } Q_i(\mathbf{x}) = 0 \end{cases}$$

Thus, only the "winning" buyer pays anything; he pays the smallest value that would result in his winning.

Figure 5.4 illustrates the workings of mechanism when there are two buyers. The virtual valuation curves, ψ_1 and ψ_2, are depicted, and since these are different, the two buyers are subject to different reserve prices of $r_1^* = \psi_1^{-1}(0)$ and $r_2^* = \psi_2^{-1}(0)$, respectively. With the given values x_1 and x_2, it is the case that $\psi_1(x_1) > \psi_2(x_2) > 0$ so the reserve prices do not come into play. The object goes to buyer 1 who is asked to pay an amount y_1.

Thus, we obtain the main result:

Proposition 5.3. *Suppose the design problem is regular. Then the following is an optimal mechanism:*

$$Q_i(\mathbf{x}) = \begin{cases} 1 & \text{if } \psi_i(x_i) > \max_{j \neq i} \psi_j(x_j) \text{ and } \psi_i(x_i) \geq 0 \\ 0 & \text{if } \psi_i(x_i) < \max_{j \neq i} \psi_j(x_j) \end{cases}$$

and

$$M_i(\mathbf{x}) = \begin{cases} y_i(\mathbf{x}_{-i}) & \text{if } Q_i(\mathbf{x}) = 1 \\ 0 & \text{if } Q_i(\mathbf{x}) = 0 \end{cases}$$

THE SYMMETRIC CASE

Suppose that we have a symmetric problem so the distributions of values are identical across buyers. In other words, for all $i, f_i = f$, and hence for all i, $\psi_i = \psi$. Now we have that,

$$y_i(\mathbf{x}_{-i}) = \max\left\{\psi^{-1}(0), \max_{j \neq i} x_j\right\}$$

Thus, the optimal mechanism is a second-price auction with a reserve price $r^* = \psi^{-1}(0)$.

Proposition 5.4. *Suppose the design problem is regular and symmetric. Then a second-price auction with a reserve price $r^* = \psi^{-1}(0)$ is an optimal mechanism.*

Note that $\psi^{-1}(0)$ is the same as the optimal reserve price derived in Chapter 2 (see (2.12) on page 23).

5.2.3 Discussion and Interpretation

The optimal mechanism derived in Proposition 5.3 is typically inefficient, and there are two separate sources of inefficiency. First, the optimal mechanism calls on the seller to retain the object if the highest virtual valuation is negative. Since buyers' values are always nonnegative and the value to the seller is 0, this means that with positive probability the object is not allocated to one of the buyers even though there would be social gains from doing so. Second, even when the object is allocated, it is allocated to the buyer with the highest *virtual* valuation and, in the asymmetric case, this need not be the buyer with the highest value.

Why is it optimal to allocate the object on the basis of virtual valuations and, more to the point, what are virtual valuations? Consider a particular buyer in isolation whose values are distributed according to the function F, say. Suppose the seller makes a take-it-or-leave-it offer to this buyer at a price of p. The probability that the buyer will accept the offer is just $1 - F(p)$, the probability that his or her value exceeds p. We can think of the probability of purchase as the "quantity" demanded by i and thus write the buyer's implied "demand curve" as $q(p) \equiv 1 - F(p)$. The inverse demand curve is then $p(q) \equiv F^{-1}(1-q)$, where q is the "quantity" purchased (or equivalently, the probability of purchase). The resulting "revenue function" facing the seller is

$$p(q) \times q = qF^{-1}(1-q)$$

and differentiating the revenue with respect to q

$$\frac{d}{dq}(p(q) \times q) = F^{-1}(1-q) - \frac{q}{F'(F^{-1}(1-q))}$$

Since $F^{-1}(1-q) = p$, we have that

$$\begin{aligned} MR(p) &\equiv p - \frac{1-F(p)}{f(p)} \\ &= \psi(p) \end{aligned}$$

the virtual valuation of i at $p(q) = p$. Thus, the virtual valuation of a buyer $\psi(p)$ can be interpreted as a *marginal revenue*, and recall that we have assumed that ψ is strictly increasing. Facing this buyer in isolation, the seller would set a

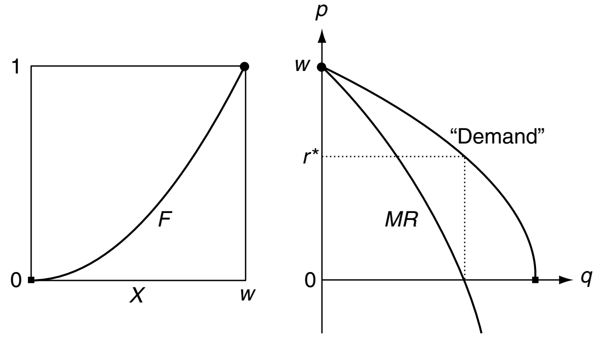

FIGURE 5.5 Virtual valuations as marginal revenues.

"monopoly price" of r^* by setting $MR(p) = MC$, the marginal cost. Since the latter is assumed to be zero, $MR(r^*) = \psi(r^*) = 0$, or $r^* = \psi^{-1}(0)$.

Figure 5.5 depicts how the distribution function of a particular buyer can be rotated to obtain his or her "demand curve" by identifying corresponding points on the two curves. The associated "marginal revenue" curve and the "monopoly price," r^*, are also shown.

When facing many buyers, the optimal mechanism calls for the seller to set *discriminatory reserve prices* of $r_i^* = \psi_i^{-1}(0)$ for the buyers. If no buyer's value x_i exceeds his reserve price r_i^*, the seller keeps the object. Otherwise, it is allocated to the buyer with the *highest marginal revenue* and this "winning" buyer is asked to pay $p_i = y_i(\mathbf{x}_{-i})$, the smallest value such that he or she would still win.

In general, the optimal mechanism discriminates in favor of "disadvantaged" buyers in the following sense. Suppose that there are two buyers whose values are drawn from F_1 and F_2, respectively, with the same support $[0, \omega]$. Suppose further that for all x, the associated hazard rates satisfy $\lambda_1(x) \leq \lambda_2(x)$. Then buyer 2 is relatively disadvantaged, since his values are likely to be lower; in particular, F_1 stochastically dominates F_2. But given that both have the same realized value x, the virtual valuation of buyer 2 will be higher because

$$\psi_1(x) = x - \frac{1}{\lambda_1(x)} \leq x - \frac{1}{\lambda_2(x)} = \psi_2(x)$$

Since the optimal mechanism awards the object on the basis of virtual valuations, buyer 2 will "win" more often than dictated by a comparison of actual values alone.

Finally, note that in the optimal mechanism, buyers whose values x are positive will walk away with some positive surplus. This is because the "winning buyer" pays an amount $y_i(\mathbf{x}_{-i})$ that is typically less than his value x_i. The expectation of the resulting surplus, $E[X_i - y_i(\mathbf{X}_{-i})]$, is sometimes referred to as the *informational rent* accruing to buyer i by virtue of the fact that he is the only one in the system who knows the value of X_i. The seller is therefore unable to perfectly price discriminate and extract all the surplus; buyers must be given some informational rents in order to get them to reveal their private information.

AUCTIONS VERSUS NEGOTIATIONS

The characterization of an optimal mechanism as just obtained offers some insight into the value of competition. Specifically, suppose that a single seller confronts a single buyer in some sort of *negotiation* for the sale of an object. The buyer's value X_1 is drawn from a distribution F. The preceding analysis shows that the optimal mechanism involves a take-it-or-leave-it offer of $r_1^* = \psi^{-1}(0)$ on the part of the seller. All buyers with valuations X_1 above r^* accept the offer, and all those with valuations X_1 below r^* reject it. From (5.16), the expected revenue from the optimal mechanism can be written as

$$E[\max\{\psi(X_1), 0\}] \tag{5.17}$$

and this is an upper bound to the revenue from any negotiation or, for that matter, any other selling mechanism.

As an alternative to negotiating with a single buyer, suppose that the seller is able to attract the interest of another buyer whose value X_2 is also drawn from the same distribution F. Consider what happens if the seller were to sell an object by means of some standard *auction*—say, of the second-price variety—without setting a reserve price. Following the reasoning underlying (5.16), it is easy to see that in the symmetric case the expected revenue from a second-price auction is

$$E[\max\{\psi(X_1), \psi(X_2)\}] \tag{5.18}$$

We now argue that (5.18) exceeds (5.17). First, consider the event that $X_1 \geq r^*$—that is, when buyer 1 would accept the seller's offer. Because of regularity, this is equivalent to the event that $\psi(X_1) \geq 0$, so we have

$$\begin{aligned}
&E[\max\{\psi(X_1), \psi(X_2)\} \mid \psi(X_1) \geq 0] \\
&\quad > E[\psi(X_1) \mid \psi(X_1) \geq 0] \\
&\quad = E[\max\{\psi(X_1), 0\} \mid \psi(X_1) \geq 0]
\end{aligned}$$

Next, consider the event that $X_1 < r^*$, so buyer 1 would reject the seller's offer. Now since "max" is a convex function,

$$\begin{aligned}
&E[\max\{\psi(X_1), \psi(X_2)\} \mid \psi(X_1) < 0] \\
&\quad > \max\{E[\psi(X_1) \mid \psi(X_1) < 0],\ E[\psi(X_2) \mid \psi(X_1) < 0]\}
\end{aligned}$$

Since the X_1 and X_2 are independent, the right-hand side of the preceding inequality is the same as

$$\begin{aligned}&\max\{E[\psi(X_1) \mid \psi(X_1) < 0], E[\psi(X_2)]\}\\&= \max\{E[\psi(X_1) \mid \psi(X_1) < 0], 0\}\\&= 0\\&= E[\max\{\psi(X_1), 0\} \mid \psi(X_1) < 0],\end{aligned}$$

where we have used the fact that $E[\psi(X_2)] = 0$ (see (5.11)). Since in both the event $X_1 \geq r^*$ and the event $X_1 < r^*$ the conditional expectation of $\max\{\psi(X_1), \psi(X_2)\}$ exceeds the conditional expectation of $\max\{\psi(X_1), 0\}$, we deduce that (5.18) exceeds (5.17).

We have thus argued that no matter how the negotiations with a single buyer are conducted, it is better to invite a second buyer and hold an auction without a reserve price. The significance of this result stems from the fact that an auction without a reserve price is a "detail-free" mechanism. The seller need not know the exact distribution of values, only that both buyers' values come from the same distribution—while the optimal take-it-or-leave-it offer is not—finding r^* requires knowing F. Instead of worrying about what the optimal take-it-or-leave-it offer is, the seller is better off inviting a second interested buyer and letting competition do the work.

5.2.4 Auctions versus Mechanisms

In general, a mechanism may be tailored to a specific situation. For instance, the optimal mechanism identified in Section 5.2 depends on the specific distributions of buyers' values—that is, the F_i's. Both the allocation and the payment rules of the optimal auction depend on a comparison of virtual valuations, which, in turn, depend on buyers' value distributions. Moreover, as pointed out earlier, the optimal mechanism does not treat different buyers in the same way—buyers with different virtual valuations are treated differently. Thus, the optimal mechanism is neither universal (its rules are specific, via the value distributions, to the item for sale), nor is it anonymous (buyers' identities matter). The optimal mechanism does not satisfy the two important properties we set out in the Introduction as being characteristic of auctions.

From a practical standpoint, the restriction to mechanisms that satisfy these two properties is an important consideration. Any mechanism that depends on the fine details of buyers' distributions would be difficult to implement in practice. The admonition that we should primarily be concerned with mechanisms that do not depend on such fine detail has been called the "Wilson doctrine," named after the leading proponent of this point of view. In this book, we have defined auctions to be mechanisms that adhere to this doctrine.[2]

[2]Notice that even determining the optimal reserve price depends on a detailed knowledge of the distributions of values.

5.3 EFFICIENT MECHANISMS

In the context of a sale of an object to many potential buyers, we have already argued that a second-price auction (without a reserve price) will always allocate the object efficiently. This section concerns a generalization of the second-price auction that is applicable to other contexts. As an example, in the next section we consider the possibility of efficient trade between a single seller with privately known costs of production and a single buyer with privately known values. (In the auction context, the value of the object to the seller is assumed to be commonly known.)

We first generalize our setup very slightly to allow the values of agents to lie in some interval $\mathcal{X}_i = [\alpha_i, \omega_i] \subset \mathbb{R}$, thereby allowing, when $\alpha_i < 0$, for the possibility of negative values (or positive costs).

An allocation rule $\mathbf{Q}^* : \mathcal{X} \to \mathbf{\Delta}$ is said to be *efficient* if it maximizes "social welfare"—that is, for all $\mathbf{x} \in \mathcal{X}$,

$$\mathbf{Q}^*(\mathbf{x}) \in \arg\max_{\mathbf{Q} \in \mathbf{\Delta}} \sum_{j \in \mathcal{N}} Q_j x_j \tag{5.19}$$

When there are no ties, an efficient rule allocates the object to the person who values it the most.[3] Any mechanism with an efficient allocation rule is said to be efficient. Given an efficient allocation rule $\mathbf{Q}^*$, define the maximized value of social welfare by

$$W(\mathbf{x}) \equiv \sum_{j \in \mathcal{N}} Q_j^*(\mathbf{x})\, x_j \tag{5.20}$$

when the values are $\mathbf{x}$. Similarly, define

$$W_{-i}(\mathbf{x}) \equiv \sum_{j \neq i} Q_j^*(\mathbf{x})\, x_j \tag{5.21}$$

as the welfare of agents other than i.

5.3.1 The VCG Mechanism

The Vickrey-Clarke-Groves, or *VCG mechanism* $(\mathbf{Q}^*, \mathbf{M}^{\mathrm{V}})$, is an efficient mechanism with the payment rule $\mathbf{M}^{\mathrm{V}} : \mathcal{X} \to \mathbb{R}^N$ given by

$$M_i^{\mathrm{V}}(\mathbf{x}) = W\big(\alpha_i, \mathbf{x}_{-i}\big) - W_{-i}(\mathbf{x}) \tag{5.22}$$

$M_i^{\mathrm{V}}(\mathbf{x})$ is thus the difference between social welfare at i's lowest possible value α_i and the welfare of *other* agents at i's reported value x_i; assuming in both cases that the efficient allocation rule $\mathbf{Q}^*$ is employed.

[3] There may be more than one efficient rule depending on how ties are resolved.

In the context of auctions, $\alpha_i = 0$ and it is routine to see that the VCG mechanism is the same as a second-price auction. In the auction context, $M_i^{\mathrm{V}}(\mathbf{x}) = W_{-i}\left(0, \mathbf{x}_{-i}\right) - W_{-i}(\mathbf{x})$, and this is positive if and only if $x_i \geq \max_{j \neq i} x_j$. In that case, $M_i^{\mathrm{V}}(\mathbf{x})$ is equal to $\max_{j \neq i} x_j$, the second-highest value.

The VCG mechanism is incentive compatible. Indeed, it is easy to see that, as in the second-price auction, truth-telling is a weakly dominant strategy in the VCG mechanism. If the other buyers report values $\mathbf{x}_{-i}$, then by reporting a value of z_i, agent i's payoff is

$$Q_i^*\left(z_i, \mathbf{x}_{-i}\right) x_i - M_i^{\mathrm{V}}(z_i, \mathbf{x}_{-i}) = \sum_{j \in \mathcal{N}} Q_j^*\left(z_i, \mathbf{x}_{-i}\right) x_j - W\left(\alpha_i, \mathbf{x}_{-i}\right)$$

by using (5.20) and (5.21). The definition of $\mathbf{Q}^*$ in (5.19) implies that for all $\mathbf{x}_{-i}$, the first term is maximized by choosing $z_i = x_i$; and since the second term does not depend on z_i, it is optimal to report $z_i = x_i$. Thus, i's equilibrium payoff when the values are $\mathbf{x}$ is

$$Q_i^*(\mathbf{x})\, x_i - M_i^{\mathrm{V}}(\mathbf{x}) = W(\mathbf{x}) - W\left(\alpha_i, \mathbf{x}_{-i}\right),$$

which is just the difference in social welfare induced by i when he reports his true value x_i as opposed to his lowest possible value α_i.

Since the VCG mechanism is incentive compatible, it has the properties derived in Section 5.1. In particular, the equilibrium expected payoff function U_i^{V} associated with the VCG mechanism,

$$U_i^{\mathrm{V}}(x_i) = E\left[W\left(x_i, \mathbf{X}_{-i}\right) - W\left(\alpha_i, \mathbf{X}_{-i}\right)\right]$$

is convex and increasing. Clearly, $U_i^{\mathrm{V}}(\alpha_i) = 0$, and the monotonicity of U_i^{V} now implies that the VCG mechanism is also individually rational.

If $(\mathbf{Q}^*, \mathbf{M})$ is some other efficient mechanism that is also incentive compatible, then by the revenue equivalence principle we know that for all i, the expected payoff functions for this mechanism, say U_i, differ from U_i^{V} by at most an additive constant, say c_i. If $(\mathbf{Q}^*, \mathbf{M})$ is also individually rational, then this constant must be nonnegative—that is, $c_i = U_i(x_i) - U_i^{\mathrm{V}}(x_i) \geq 0$. This is because otherwise we would have $U_i(\alpha_i) < U_i^{\mathrm{V}}(\alpha_i) = 0$, contradicting that $(\mathbf{Q}^*, \mathbf{M})$ was individually rational. Since the expected payoffs in $(\mathbf{Q}^*, \mathbf{M})$ are greater than in the VCG mechanism, and the two have the same allocation rule, the expected payments must be lower.

Proposition 5.5. *Among all mechanisms for allocating a single object that are efficient, incentive compatible, and individually rational, the VCG mechanism maximizes the expected payment of each agent.*

In many economic problems, it is desirable to consider mechanisms that do not require an injection of funds from the mechanism designer—that is, the mechanism designer's budget is exactly balanced *ex post*. While the VCG mechanism typically does not have this property, it is still an important guide in determining when this is feasible.

5.3.2 Budget Balance

In our notation, a mechanism is said to balance the budget if for every realization of values, the net payments from agents sum to zero—that is, for all $\mathbf{x}$,

$$\sum_{i \in \mathcal{N}} M_i(\mathbf{x}) = 0$$

The Arrow-d'Aspremont-Gérard-Varet or *AGV mechanism* (also called the "expected externality" mechanism) $\left(\mathbf{Q}^*, \mathbf{M}^A\right)$ is defined by

$$\begin{aligned} M_i^{\mathrm{A}}(\mathbf{x}) &= \frac{1}{N-1} \sum_{j \neq i} E_{\mathbf{X}_{-j}} \left[W_{-j}\left(x_j, \mathbf{X}_{-j}\right) \right] \\ &\quad - E_{\mathbf{X}_{-i}} \left[W_{-i}\left(x_i, \mathbf{X}_{-i}\right) \right] \end{aligned} \tag{5.23}$$

so that for all $\mathbf{x}$,

$$\sum_{i \in \mathcal{N}} M_i^{\mathrm{A}}(\mathbf{x}) = 0$$

To see that the AGV mechanism is incentive compatible, suppose that all other agents are reporting their values $\mathbf{x}_{-i}$ truthfully. The expected payoff to i from reporting z_i when his true value is x_i is

$$\begin{aligned} &E_{\mathbf{X}_{-i}} \left[Q_i^*\left(z_i, \mathbf{X}_{-i}\right) x_i + W_{-i}\left(z_i, \mathbf{X}_{-i}\right) \right] \\ &- E_{\mathbf{X}_{-i}} \left[\frac{1}{N-1} \sum_{j \neq i} E_{\mathbf{X}_{-j}} \left[W_{-j}\left(X_j, \mathbf{X}_{-j}\right) \right] \right] \end{aligned}$$

and since the second term is independent of z_i, this is maximized by setting $z_i = x_i$.

It is easy to see that the AGV mechanism may not satisfy the individual rationality constraint. The question of whether there are efficient, incentive compatible, individually rational mechanisms that, at the same time, balance the budget can also be answered by means of the VCG mechanism.

Proposition 5.6. *There exists an efficient, incentive compatible, and individually rational mechanism that balances the budget if and only if the VCG mechanism results in an expected surplus.*

Proof. The fact that it is necessary for the VCG mechanism to run an expected surplus follows from Proposition 5.5: If the VCG mechanism runs a deficit, then all efficient, incentive compatible, and individually rational mechanisms must run a deficit.

We now show that the condition is sufficient by explicitly constructing an efficient, incentive compatible mechanism that balances the budget and is individually rational.

First, consider the AGV mechanism $\mathbf{M}^{\mathrm{A}}$ defined in (5.23). From the revenue equivalence principle we know that there exist constants c_i^{A} such that

$$U_i^{\mathrm{A}}(x_i) = E\left[W\left(x_i, \mathbf{X}_{-i}\right)\right] - c_i^{\mathrm{A}}$$

Next, consider the VCG mechanism defined in (5.22). Again, from the revenue equivalence principle, there also exist constants c_i^{V} such that

$$U_i^{\mathrm{V}}(x_i) = E\left[W\left(x_i, \mathbf{X}_{-i}\right)\right] - c_i^{\mathrm{V}}$$

Suppose that the VCG mechanism runs an expected surplus—that is,

$$E\left[\sum_{i \in \mathcal{N}} M_i^{\mathrm{V}}(\mathbf{X})\right] \geq 0$$

Then

$$E\left[\sum_{i \in \mathcal{N}} M_i^{\mathrm{V}}(\mathbf{X})\right] \geq E\left[\sum_{i \in \mathcal{N}} M_i^{\mathrm{A}}(\mathbf{X})\right]$$

since the right-hand side is exactly 0. Equivalently,

$$\sum_{i \in \mathcal{N}} c_i^{\mathrm{V}} \geq \sum_{i \in \mathcal{N}} c_i^{\mathrm{A}} \tag{5.24}$$

For all $i > 1$, define $d_i = c_i^{\mathrm{A}} - c_i^{\mathrm{V}}$ and let $d_1 = -\sum_{i=2}^{N} d_i$. Consider the mechanism $\overline{\mathbf{M}}$ defined by

$$\overline{M}_i(\mathbf{x}) = M_i^{\mathrm{A}}(\mathbf{x}) - d_i$$

Clearly, $\overline{\mathbf{M}}$ balances the budget. It is also incentive compatible, since the payoff to each agent in the mechanism $\overline{\mathbf{M}}$ differs from the payoff from an incentive compatible mechanism, $\mathbf{M}^{\mathrm{A}}$, by an additive constant. Thus, it is only necessary to verify that $\overline{\mathbf{M}}$ is individually rational.

For all $i \neq 1$,

$$\begin{aligned}
\overline{U}_i(x_i) &= U_i^{\mathrm{A}}(x_i) + d_i \\
&= U_i^{\mathrm{A}}(x_i) + c_i^{\mathrm{A}} - c_i^{\mathrm{V}} \\
&= U_i^{\mathrm{V}}(x_i) \\
&\geq 0
\end{aligned}$$

By construction $\sum_{i=1}^{N} d_i = 0$ and observe from (5.24) that

$$d_1 = -\sum_{i>1} d_i = \sum_{i>1}\left(c_i^{\mathrm{V}} - c_i^{\mathrm{A}}\right) \geq \left(c_1^{\mathrm{A}} - c_1^{\mathrm{V}}\right)$$

Thus,

$$\begin{aligned}\overline{U}_1(x_1) &= U_1^{\mathrm{A}}(x_1) + d_1 \\ &\geq U_1^{\mathrm{A}}(x_1) + c_1^{\mathrm{A}} - c_1^{\mathrm{V}} \\ &= U_1^{\mathrm{V}}(x_1) \\ &\geq 0\end{aligned}$$

so that $\overline{\mathbf{M}}$ is also individually rational. ■

While Proposition 5.5 results from a simple application of the revenue equivalence principle, it is nevertheless a very useful tool and can be used to consider a variety of problems—outside the realm of auction theory—concerning efficient allocations. As an example, we now turn to one such application.

5.3.3 An Application to Bilateral Trade

Suppose that there is a seller with a privately known cost $C \in [\underline{c}, \overline{c}]$ of producing a single indivisible good. Suppose also that there is a buyer with a privately known value $V \in [\underline{v}, \overline{v}]$ of consuming the good. The cost C and value V are independently distributed, and the prior distributions are commonly known and have full support on the respective intervals. Thus, there is incomplete information on both sides of the market. Finally, suppose that $\underline{v} < \overline{c}$ and $\overline{v} \geq \underline{c}$, so the supports overlap and sometimes it is efficient not to trade. Is there some way to guarantee that trade will take place whenever it should? To answer this question, it is natural to adopt a mechanism design perspective.

A mechanism decides whether or not the good is traded. It also decides the amount P the buyer pays for the good and the amount R the seller receives. If the good is traded, the net gain to the buyer is $V - P$, and the net gain to the seller is $R - C$. At the moment, we do not restrict P or R to be positive or negative, nor do we assume that the budget is balanced—that is, $P = R$. A mechanism is efficient if whenever $V > C$, the object is produced and allocated to the buyer.

Proposition 5.7. *In the bilateral trade problem, there is no mechanism that is efficient, incentive compatible, individually rational, and at the same time balances the budget.*

Proof. First, consider the VCG mechanism, whose operation in this context is as follows:

The buyer announces a valuation V and the seller announces a cost C.

1. If $V \leq C$, the object is not exchanged and no payments are made.
2. If $V > C$, the object is exchanged. The buyer pays $\max\{C, \underline{v}\}$ and the seller receives $\min\{V, \overline{c}\}$.

It is routine to verify that it is a weakly dominant strategy for the buyer to announce $V = v$ and the seller to announce $C = c$. This mechanism is efficient, since, in equilibrium, the object is transferred whenever $v > c$.

A buyer with value $\underline{v}$ has an expected payoff of 0, and any buyer with value $v > \underline{v}$ has a positive expected payoff. Similarly, a seller with cost $\overline{c}$ has an expected payoff of 0, and any seller with cost $c < \overline{c}$ has a positive expected payoff. Thus, the mechanism is individually rational.

Whenever $V > C$, so there is trade, the fact that $\underline{v} < \overline{c}$ implies that the amount the seller receives $R = \min\{V, \overline{c}\}$ is *greater* than the amount the buyer pays $P = \max\{C, \underline{v}\}$. In this context, the VCG mechanism always runs a deficit. Indeed, for any realization of V and C such that $V > C$, the deficit $R - P = V - C$, which is exactly equal to the *ex post* gains that result from trade.

Now suppose that we have some other mechanism that is incentive compatible and efficient. By the revenue equivalence principle, there is a constant K such that the expected payment for any buyer with value v under this mechanism differs from his expected payment under the VCG mechanism by exactly K. Similarly, there is a constant L such that the expected receipts of any seller with cost c under this mechanism differ from her expected receipts under the VCG mechanism by exactly L.

Suppose the other mechanism is individually rational. Since in the VCG mechanism a buyer with value $\underline{v}$ gets an expected payoff of 0, we must have that $K \leq 0$. Similarly, since a seller with costs $\overline{c}$ gets an expected payoff of 0, we have $L \geq 0$. (This is just the same as the argument in Proposition 5.5.)

The expected deficit under the other mechanism is just the expected deficit under the VCG mechanism plus $L - K \geq 0$. But since the VCG mechanism runs a deficit, we have argued that every other mechanism also runs a deficit. Thus, there does not exist an efficient mechanism that is incentive compatible, individually rational, and balances the budget. ■

PROBLEMS

5.1. (Surplus extraction) Show that if buyers' values are independently distributed, then the seller cannot design an incentive compatible and individually rational mechanism that extracts the whole surplus from buyers. (In doing this problem, use only the results of Section 5.1 and not those from Section 5.2.)

5.2. (Optimal auction) There is a single object for sale and there are two potential buyers. The value assigned by buyer 1 to the object X_1 is uniformly drawn from the interval $[0, 1+k]$ whereas the value assigned by buyer 2 to the object X_2 is uniformly drawn from the interval $[0, 1-k]$, where k is a parameter satisfying $0 \leq k < 1$. The two values X_1 and X_2 are independently distributed.

a. Suppose the seller decides to sell the object using a second-price auction with a reserve price r. What is the optimal value of r and what is the expected revenue of the seller?

b. What is the optimal auction associated with this problem?

5.3. (Dissolving a partnership) Two agents jointly own a firm and each has an equal share. The value of the *whole* firm to each is a random variable X_i which is independently and uniformly distributed on $[0,1]$. Thus in the current situation, agent 1 derives a value $\frac{1}{2}X_1$ from the firm and agent 2 derives a value $\frac{1}{2}X_2$ from the firm. Suppose that the two agents wish to dissolve their partnership and since the firm cannot be subdivided, ownership of the whole firm would have go to one of the two agents.

a. Consider the following procedure for reallocating the firm. Both agents bid amounts b_1 and b_2 and if $b_i > b_j$, then i gets ownership of the whole firm and pays the other agent j the amount b_i. Find a symmetric equilibrium bidding strategy in this auction.

b. Is the procedure outlined above efficient? Is it individually rational?

c. Calculate each agent's payments in the VCG mechanism associated with this problem.

5.4. (Negative externality) The holder of a patent on a cost reducing process is considering the possibility of licensing it to one of two firms. The two firms are competitors in the same industry and so if firm 1 obtains the license, its profits will *increase* by X_1 while those of firm 2 will *decrease* by αX_2, where α is a known parameter satisfying $0 < \alpha < 1$. This is because if firm 1 gets the license, firm 2 will have a cost disadvantage relative to firm 1. Similarly, if firm 2 obtains the license, its profits will increase by X_2 while those of firm 1 will decrease by αX_1. The variables X_1 and X_2 are uniformly and independently distributed on $[0,1]$. Firm 1 knows the realized value x_1 of X_1 and only that X_2 is uniformly distributed, and similarly for firm 2.

a. Suppose that the license will be awarded on the basis of a first-price auction. What are the equilibrium bidding strategies? What is the expected revenue of the seller, that is, the holder of the patent?

b. Find the payments in the VCG mechanism associated with this problem. Are the expected payments the same as in a first-price auction?

c. Suppose that the patent holder is a government laboratory and it wants to ensure that the license is allocated efficiently and that the net payments of the buyers add up to zero—that is, the "budget" is balanced. What is the associated "expected externality" mechanism for this problem? Is it individually rational? Does there exist an efficient, incentive compatible, and individually rational mechanism that also balances the budget?

5.5. (A nonstandard selling method) There is a single object for sale, and there are two interested buyers. The values assigned by the buyers to the object are independently and uniformly distributed on $[0,1]$. As always, each buyer knows the value he or she assigns to the object but the seller knows only that each is independently and uniformly distributed. The seller assigns a value of 0 to the object. Suppose that the seller adopts the following selling strategy. She approaches one of the buyers (chosen at random) and makes a "take-it-or-leave-it" offer at a fixed price p_1. If the first buyer accepts the offer, the object is sold to him at the offered price. If the first buyer declines the offer, the seller then approaches the other buyer with a "take-it-or-leave-it" offer at a fixed price p_2. If the second buyer accepts the offer, the object is sold to him at the offered price. If neither buyer accepts then the seller keeps the object.

a. What are the optimal values of p_1 and p_2?

b. What is the expected revenue to the seller if he adopts this selling scheme?

c. How does it compare to a standard auction? In particular, does the revenue equivalence principle apply?

CHAPTER NOTES

The formulation of the mechanism design problem is due to Myerson (1981) in a now classic paper. The revelation principle, general revenue equivalence, and the derivation of the optimal selling mechanism were all developed there. The interpretation of virtual valuations as "marginal revenues" and the analogy of the optimal mechanism design problem with the problem facing a discriminating monopolist was provided by Bulow and Roberts (1989).

The comparison of a two-bidder auction without a reserve price and optimal negotiations with a single buyer is based on the work of Bulow and Klemperer (1996). In this paper the authors show a more general result: From the seller's perspective, an $N+1$ bidder English auction (without a reserve price) is superior to the mechanism which consists of an N bidder English auction followed by the seller, armed with information that is revealed during the auction, optimally negotiating with the winner.

The origins of the VCG mechanism can be traced to Vickrey (1961) who proposed the second-price auction and an extension to the case of multiple identical goods (see Chapter 12). Clarke (1971) proposed a similar mechanism in the context of public goods. These ideas were further generalized by Groves (1973). The balanced budget AGV mechanism was independently suggested by Arrow (1979) and d'Aspremont and Gérard-Varet (1979a,b). Propositions 5.5 and 5.6 are due to Krishna and Perry (1998).

The result on the impossibility of efficient exchange in the bilateral trade problem, Proposition 5.7, is due to Myerson and Satterthwaite (1983). The treatment here follows that in Krishna and Perry (1998).

A real-valued function U defined on a bounded interval $[0, \omega]$ is said to be *absolutely continuous* if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} \left| U(x_i') - U(x_i) \right| < \varepsilon$$

for every finite collection $\left\{ (x_i, x_i') \right\}$ of nonoverlapping intervals satisfying

$$\sum_{i=1}^{n} \left| x_i' - x_i \right| < \delta$$

Every convex function is absolutely continuous. An absolutely continuous function is differentiable almost everywhere and is the integral of its derivative. These are fairly standard results in real analysis and can be found, for instance, in Royden (1968).