Chapter | Twelve

# An Introduction to Multiple-Object Auctions

In this part of the book we turn to the study of situations in which multiple, related objects are to be sold. The objects may be physically identical, say multiple cases of the same wine or treasury bills of the same denomination, or they may be physically distinct but still be good substitutes, say different apartments in the same building or different paintings by the same artist, so that the marginal value of acquiring a second item, say, is lower than the value of the first. Alternatively, the objects may be complements—that is, the value derived from a particular object may be greater if another has already been obtained. For instance, a philatelist may value a collection of stamps more than the sum of the values of the individual stamps. Similarly, how much an airline values an airport landing slot may increase with the number of slots it has already acquired.

Not surprisingly, when multiple objects are to be sold, many options are open to the seller. First, the seller must decide whether to sell the objects separately in *multiple auctions* or jointly in a *single auction*. In the former case, the objects are sold one at a time in separate auctions—conducted sequentially, say—in a way that the bids in the auction for one of the objects do not directly influence the outcome of the auction for another. In the latter case, the objects are sold at one go in a single auction, but not necessarily all to the same bidder, and the bids on the various objects collectively influence the overall allocation.

Second, the seller must choose among a variety of auction formats, and there is a wide range of possibilities to choose from. For instance, if the seller decides to sell the objects one at a time in a sequence of single-object auctions, there is still the question of the particular auction form—first-price, second-price, or some other format—to adopt. If the seller decides to sell the objects at one go

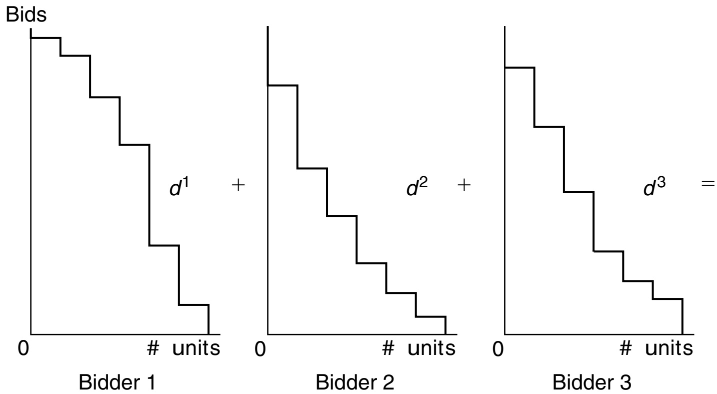


**FIGURE 12.1** Individual demand functions.

in a single auction, there are also many possibilities. We begin by outlining the workings of a few auction forms for the sale of multiple units of the same good at one go, returning to study multiple one at a time, sequential, or simultaneous auctions later.

## 12.1 SEALED-BID AUCTIONS FOR SELLING IDENTICAL UNITS

Three sealed-bid auction formats for the sale of $K$ identical objects are of particular interest. The first two are important on practical grounds—they are widely used in real-world auctions—and the last, although not widely used, is of special interest for theoretical reasons. All three are intended to be used in situations in which the marginal values are declining—that is, the value of an additional unit decreases with the number of units already obtained.

**D.** The *discriminatory* (or "pay-your-bid") auction.
**U.** The *uniform-price* auction.
**V.** The *Vickrey* auction.

In each of these auctions, a bidder is asked to submit $K$ bids $b_k^i$, satisfying $b_1^i \geq b_2^i \geq \cdots \geq b_K^i$, to indicate how much he is willing to pay for each additional unit. Thus, $b_1^i$ is the amount $i$ is willing to pay for one unit, $b_1^i + b_2^i$ is the amount he is willing to pay for two units, $b_1^i + b_2^i + b_3^i$ is the amount he is willing to pay for three units, and so on. We will refer to $\mathbf{b}^i = \left(b_1^i, b_2^i, \ldots, b_K^i\right)$ as a *bid vector.*[1]

A bid vector $\mathbf{b}^i$ can be usefully thought of as an "inverse demand function" and can be inverted to obtain $i$'s *demand function* $d^i : \mathbb{R}_+ \to \{1, 2, \ldots, K\}$:

$$d^i(p) \equiv \max\{k : p \leq b_k^i\} \tag{12.1}$$

[1]In this part of the book, superscripts identify bidders and subscripts identify units.

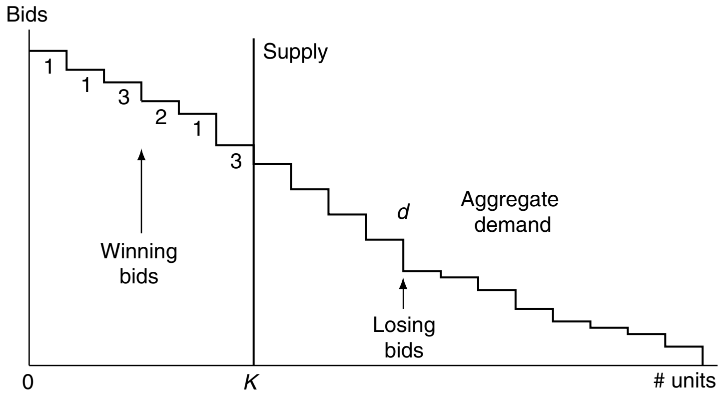


**FIGURE 12.2** Aggregate demand and supply.

In particular, if $b_k^i > b_{k+1}^i$, then at any price $p$ lying between $b_k^i$ and $b_{k+1}^i$, bidder $i$ is willing to buy exactly $k$ units. A bidder's demand is clearly nonincreasing in the price. Since the demand function is just the "inverse" of the bid vector, and vice versa, submitting the bid vector $\mathbf{b}^i$ is equivalent to submitting the demand function $d^i$. We will thus use these interchangeably.

In all three of the auction formats considered here, a total of $N \times K$ bids $\{b_k^i : i = 1, 2, \ldots N;\ k = 1, 2, \ldots, K\}$ are collected and the $K$ units are awarded to the $K$ highest of these bids—that is, if bidder $i$ has $k \leq K$ of the $K$ highest bids, then $i$ is awarded $k$ units.

As an example, consider a situation in which there are six units ($K = 6$) to be sold to three bidders and the submitted bid vectors are

$$\mathbf{b}^1 = (50, 47, 40, 32, 15, 5)$$
$$\mathbf{b}^2 = (42, 28, 20, 12, 7, 3)$$
$$\mathbf{b}^3 = (45, 35, 24, 14, 9, 6)$$

The three bid vectors—equivalently, the three demand functions—are depicted in Figure 12.1. In this case, the six highest bids are

$$\left(b_1^1, b_2^1, b_1^3, b_1^2, b_3^1, b_2^3\right) = (50, 47, 45, 42, 40, 35)$$

so bidder 1 is awarded three units, bidder 2 is awarded one unit, and bidder 3 is awarded two units.

The allocation rule implicit in all three auctions may be framed in conventional supply and demand terms. First, an *aggregate demand* function $d$ is obtained by "horizontally adding" the $N$ individual demand functions, as depicted. For example, the demand function depicted in Figure 12.2 is the aggregate of the three individual demand functions in Figure 12.1. As usual, the aggregate demand function determines how many units are demanded *in* *toto* at different prices, so that for any $p$, $d(p)=\sum_i d^i(p)$. Since the number of units to be sold is fixed, the supply function is just a vertical line. All bids to the left of the intersection of the aggregate demand and supply functions—the $K$ highest bids—are deemed "winning bids" and the number of units awarded to a bidder is equal to the number of winning bids submitted by him. All other bids are deemed "losing bids." In the figure each winning bid is labeled with the identity of the bidder who submitted the bid.

We will refer to an auction in which the $K$ highest bids are deemed winning and awarded objects as a *standard auction*. The three auctions introduced next are all standard but differ in terms of their pricing rules—how much each bidder is asked to pay for the units he is awarded.

### 12.1.1 Discriminatory Auctions

In a discriminatory auction, each bidder pays an amount equal to the sum of his bids that are deemed to be winning—that is, the sum of his bids that are among the $K$ highest of the $N \times K$ bids submitted in all. Formally, if exactly $k^i$ of the $i$th bidder's $K$ bids $b_k^i$ are among the $K$ highest of all bids received, then $i$ pays

$$\sum_{k=1}^{k^i} b_k^i$$

This amounts to perfect price discrimination relative to the submitted demand functions; hence the name of the auction.

The discriminatory pricing rule can also be framed in terms of the *residual supply* function facing each bidder. At any price $p$ the residual supply facing bidder $i$, denoted by $s^{-i}(p)$, is equal to the total supply $K$ less the sum of the amounts demanded by other bidders, provided that this is nonnegative. Formally,

$$s^{-i}(p) \equiv \max\left\{K - \sum_{j \neq i} d^j(p),\ 0\right\} \tag{12.2}$$

and this is clearly a nondecreasing function of the price. The discriminatory auction asks each bidder to pay an amount equal to the area under his own demand function up to the point where it intersects the residual supply curve.

Figure 12.3 depicts the residual supply function facing bidder 1 in the example described above. The shaded area in the first panel of the figure is the total amount paid by bidder 1 in a discriminatory auction. In the example, bidder 1 wins three units, so the total amount he pays for these is $b_1^1 + b_2^1 + b_3^1 = 50 + 47 + 40 = 137$.

At the risk of stating the obvious, we caution the reader not to infer from Figure 12.3 that the discriminatory auction raises more revenue than do the others. The reason is that so far nothing has been said about equilibrium bidding behavior in the three formats—this is the subject of the next chapter—and the figure illustrates only what would happen if the *same* set of bids were submitted

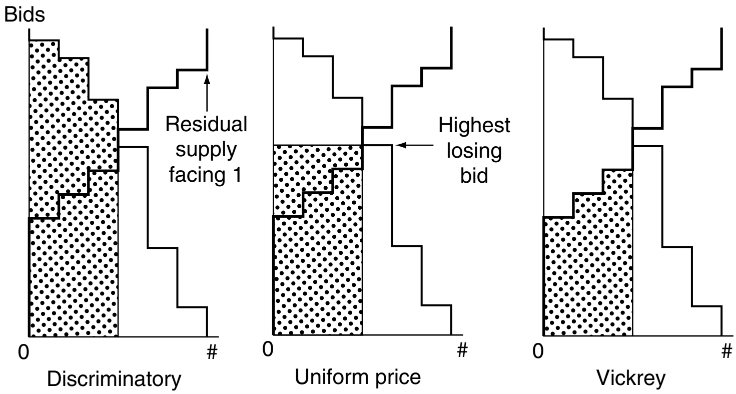


**FIGURE 12.3** Bidder 1's payments under different pricing rules.

in all three auctions. As we will see, bidding behavior in the three auctions differs substantially and reaching any conclusions regarding revenue is a delicate matter.

The discriminatory auction is the natural multiunit extension of the first-price sealed-bid auction. In particular, if there is only a single unit for sale ($K = 1$), then the discriminatory auction reduces to a first-price auction.

### 12.1.2 Uniform-Price Auctions

In a uniform-price auction all $K$ units are sold at a "market-clearing" price such that the total amount demanded is equal to the total amount supplied. In the discrete model studied here, there is some leeway in defining the price that clears the market—any price lying between the highest losing bid and the lowest winning bid equates demand and supply. We adopt the rule that the *market-clearing price* is the same as the highest losing bid.

Denote by $\mathbf{c}^{-i}$ the $K$-vector of *competing bids* facing bidder $i$. This is obtained by rearranging the $(N-1)K$ bids $b_k^j$ of bidders $j \neq i$ in decreasing order and selecting the first $K$ of these. Thus, $c_1^{-i}$ is the highest of the other bids, $c_2^{-i}$ is the second-highest, and so on. The number of units that bidder $i$ wins is just the number of competing bids he defeats. For instance, in order for $i$ to win exactly one unit it must be the case that $b_1^i > c_K^{-i}$ and $b_2^i < c_{K-1}^{-i}$; that is, he must defeat the lowest competing bid but not the second-lowest. Similarly, in order to win exactly two units, bidder $i$ must defeat the two lowest competing bids but not the third-lowest. More generally, bidder $i$ wins exactly $k^i > 0$ units if and only if

$$b_{k^i}^i > c_{K-k^i+1}^{-i} \quad \text{and} \quad b_{k^i+1}^i < c_{K-k^i}^{-i}$$

Observe that the residual supply function $s^{-i}$ facing bidder $i$, defined in (12.2), can also be obtained from the vector of competing bids $\mathbf{c}^{-i}$, since

$$s^{-i}(p) = K - \max\left\{k : c_k^{-i} \geq p\right\} \tag{12.3}$$

The highest losing bid—the market-clearing price—is then just

$$p = \max\left\{b^i_{k^i+1}, c^{-i}_{K-k^i+1}\right\}$$

and in a uniform-price auction, if bidder $i$ wins $k^i$ units, then he pays $k^i$ times $p$. The market-clearing price can also be written as

$$p = \max_i \left\{b^i_{k^i+1}\right\}$$

In the example considered above, the vector of competing bids facing bidder 1 is

$$\mathbf{c}^{-1} = (45, 42, 35, 28, 24, 20)$$

whereas his own bid vector is

$$\mathbf{b}^1 = (50, 47, 40, 32, 15, 5)$$

and since $b^1_3 > c^{-1}_4$ but $b^1_4 < c^{-1}_3$, bidder 1 wins three units. The market-clearing price in this case is $\max\left\{b^1_4, c^{-1}_4\right\} = b^1_4 = 32$, so bidder 1 pays a total of 96. The shaded area in the second panel of Figure 12.3 depicts the total amount bidder 1 pays in a uniform-price auction.

The uniform-price auction reduces to a second-price sealed-bid auction when there is only a single unit for sale ($K = 1$).[2] It thus seems that it is a natural extension of the second-price auction to the multiunit case. As we will see, however, it does not share many important properties with the second-price auction, so the analogy is imperfect.

### 12.1.3 Vickrey Auctions

In a Vickrey auction, a bidder who wins $k^i$ units pays the $k^i$ highest losing bids of the *other* bidders—that is, the $k^i$ highest losing bids not including his own. As before, denote by $\mathbf{c}^{-i}$ the $K$-vector of competing bids facing bidder $i$, so that $c^{-i}_1$ is the highest of the other bids, $c^{-i}_2$ is the second-highest, and so on.

To win one unit, bidder $i$'s highest bid must defeat the lowest competing bid—that is, $b^i_1 > c^{-i}_K$. To win a second unit, $i$'s second-highest bid must defeat the second-lowest competing bid—that is, $b^i_2 > c^{-i}_{K-1}$. To win the $k$th unit, $i$'s $k$th-highest bid must defeat the $k$th-lowest competing bid. The Vickrey pricing rule is the following. Bidder $i$ is asked to pay $c^{-i}_K$ for the first unit he wins, $c^{-i}_{K-1}$ for the second unit, $c^{-i}_{K-2}$ for the third unit, and so on. Thus, if bidder $i$ wins $k^i$

[2]Recall that the market-clearing price was defined as the highest losing bid.

units, then the amount he pays is

$$\sum_{k=1}^{k^i} c_{K-k^i+k}^{-i}$$

Continuing with the example, bidder 1's payment is

$$c_6^{-1} + c_5^{-1} + c_4^{-1} = b_3^2 + b_3^3 + b_2^2$$

The Vickrey pricing rule is illustrated in the last panel of Figure 12.3. The shaded area is the total amount, $b_3^2 + b_3^3 + b_2^2 = 20 + 24 + 28 = 72$, paid by bidder 1; as depicted, this is the area lying under the residual supply function facing bidder 1.

The basic principle underlying the Vickrey auction is the same as the one underlying the Vickrey-Clarke-Groves mechanism discussed in Chapter 5: Each bidder is asked to pay an amount equal to the externality he exerts on other competing bidders. In the example, had bidder 1 been absent, the three units allocated to him would have gone to the other bidders: two to bidder 2 and one to bidder 3. According to the demand function submitted by him, bidder 2 is willing to pay $b_2^2$ and $b_3^2$, respectively, for two additional units. Similarly, bidder 3 is willing to pay $b_3^3$ for one additional unit. Bidder 1 is asked to pay the sum of these amounts. The amounts that bidders 2 and 3 are asked to pay are determined in similar fashion.

Like the uniform-price auction, the Vickrey auction also reduces to a second-price sealed-bid auction when there is only a single unit for sale ($K = 1$). Unlike the uniform-price auction, however, it shares many important properties with the second-price auction and is, as we will argue, the appropriate extension of the second-price auction to the case of multiple units.

Other possible pricing rules exist; the range of available options is virtually unlimited. For example, in one variant of the uniform pricing rule, all units are sold at a price equal to the average of all the winning bids.

## 12.2 SOME OPEN AUCTIONS

Each of the three sealed-bid auction formats introduced here has a corresponding open format.

### 12.2.1 Dutch Auctions

In the *multiunit Dutch* (or open descending price) auction, as in its single unit counterpart, the auctioneer begins by calling out a price high enough so that no bidder is willing to buy any units at that price. The price is then gradually lowered until a bidder indicates that he is willing to buy a unit at the current price. This bidder is then sold an object at that price and the auction continues—the price is lowered further until another unit is sold, and so on. This continues until all $K$ units have been sold.

The multiunit Dutch auction is *outcome equivalent* to the discriminatory auction in the sense that if each bidder behaves according to a bid vector $\mathbf{b}^i$, indicating his interest in purchasing one unit when the price reaches $b_1^i$, another when the price reaches $b_2^i$, and so on, then the outcome is the same as when each bidder submits the bid vector $\mathbf{b}^i$ in a discriminatory auction. Recall from Chapter 1 that when a single object was for sale, the Dutch descending price auction was also equivalent to a first-price sealed-bid auction in a stronger sense—it was strategically equivalent and this equivalence held regardless of the informational environment. The multiunit extension of the Dutch auction, however, is not strategically equivalent to the multiunit extension of the first-price auction, the discriminatory auction. The reason is that if bidders' values are interdependent—the information available to one bidder may affect the valuation of objects by other bidders—then in the multiunit Dutch auction, once a bidder indicates a willingness to buy an object, this fact can be used by other bidders to update their own valuations. In a sealed-bid discriminatory auction, no such information is available. The two auctions are, however, weakly equivalent in the sense of Chapter 1—with private values the information acquired from the fact that one bidder is willing to buy at some price, while available, is irrelevant.

### 12.2.2 English Auctions

In the *multiunit English* (or open ascending-price) auction, the auctioneer begins by calling out a low price and then gradually raises it. Each bidder indicates—by using hand signals, by holding up numbered cards, or electronically—how many units he is willing to buy at that price—in other words, his demand at that price. As the price rises, bidders naturally reduce the number of units they are willing to buy. The auction ends when the total number of units demanded is exactly $K$ and all units are sold at the price where the total demand changes from $K+1$ to $K$.

The multiunit English auction bears the same relation to the uniform-price auction as the ordinary English auction does to the second-price sealed-bid auction—the two are outcome equivalent. The equivalence between the two multiunit auctions is weak for the same reason that the equivalence between the single unit auctions was weak—potentially useful information is available in the open auctions that is not available in the sealed-bid formats. Once again, with private values, this information is irrelevant.

### 12.2.3 Ausubel Auctions

The *Ausubel* auction is an alternative ascending-price format that is outcome equivalent to the Vickrey auction. As in the English auction, the auctioneer begins by calling out a low price and then raises it. Each bidder indicates his demand $d^i(p)$ at the current price $p$ and the quantity demanded is reduced as the price rises. The goods are sold according to the following procedure.

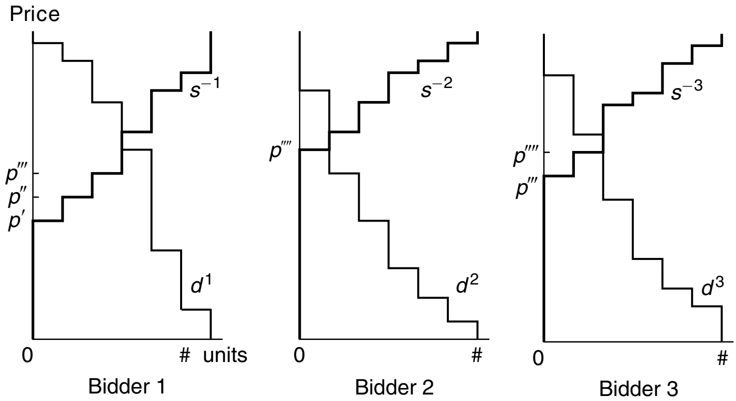


**FIGURE 12.4** Prices in the Vickrey and Ausubel auctions.

At every price, the residual supply facing every bidder $i$, $s^{-i}(p)$ is computed according to (12.2). Of course, when the price is very low, then for all $i$, $s^{-i}(p) = 0$, whereas $d^i(p) > 0$. The price is raised until it reaches a level $p'$ such that for at least one bidder, $s^{-i}(p') > 0$. Now any bidder $i$ for whom $s^{-i}(p') > 0$ is sold $s^{-i}(p')$ units at a price of $p'$. The price is raised again until it reaches a level $p''$ such that for at least one bidder the residual supply is greater—that is, $s^{-i}(p'') > s^{-i}(p')$. Now each such bidder is sold $s^{-i}(p'') - s^{-i}(p')$ units at a price of $p''$. The price is raised again until the residual supply facing some bidder increases, and so on. Thus, objects are sold whenever there is a jump in any bidder's residual supply because one or more bidders reduce the amount they demand.[3] The pricing rule implicit here is the *same* as that in a Vickrey auction—each bidder pays the area under the residual supply function he faces up until the point that the residual supply intersects his demand function.

The workings of the auction are illustrated in Figure 12.4. In addition to the three demand functions from the previous section, the residual supply functions facing the bidders are depicted. When the price is zero, all $s^{-i}(0) = 0$ and remain zero at low prices. When the price reaches 20, there is a jump in the residual supply facing bidder 1, since $s^{-1}(20) = 1$, whereas $s^{-2}(20)$ and $s^{-3}(20)$ are still zero. Bidder 1 is thus sold one unit at a price of $p' = 20$. The price is raised further and no change occurs until it reaches 24, when there is another jump in the residual supply facing bidder 1, since $s^{-1}(24) = 2$. For the other bidders, the residual supply is still 0. Bidder 1 is thus sold another unit at a price of $p'' = 24$. The next change occurs at $p''' = 28$ and now the residual supply facing bidder 1 jumps to $s^{-1}(28) = 3$ and that facing bidder 3 jumps to $s^{-3}(28) = 1$.

---

[3]At any price, the residual supply jumps by at most one unit unless more than one bidder reduces his demand or some bidder reduces his demand by more than one unit. Both require that there be some ties in the bid vectors.

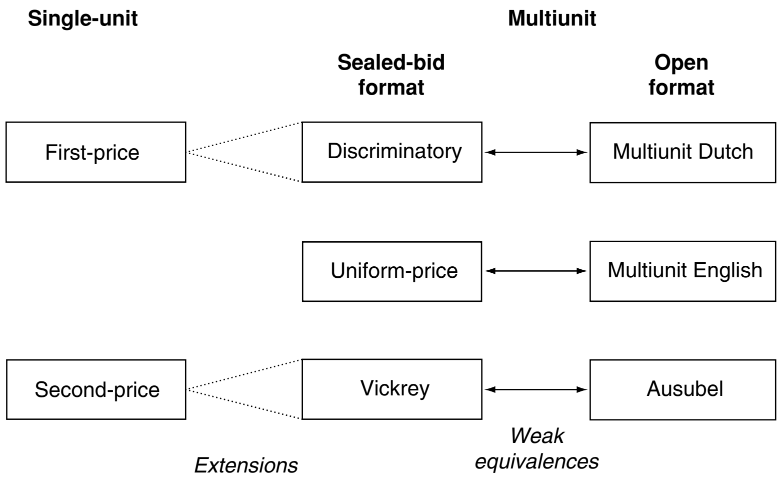


**FIGURE 12.5** Extensions and equivalences of auction formats.

Both bidders 1 and 3 are sold one unit each at a price of $p''' = 28$. Finally, when the price reaches 32, there are jumps in the residual supply functions facing both bidders 2 and 3: $s^{-2}(32) = 1$ and $s^{-3}(32) = 2$. Each is sold one unit each at a price of $p'''' = 32$. All six units have now been sold, so the auction is over.

Another way to formulate the workings of the Ausubel auction is to think of each unit of demand as a "claim" on a unit. When the price is low, the number of units claimed exceeds the total supply, so no units are awarded. As the price rises, the number of units claimed decreases until the number of units claimed by other bidders is less than the supply. A bidder is then said to have "clinched" a unit since, regardless of what happens in the remainder of the auction, there is at least one unit that is not claimed by the other bidders.[4]

Figure 12.5 summarizes the relationships between the three sealed-bid formats for multiunit auctions and their open counterparts. As noted earlier, all the equivalences are *weak*—in general, they hold only if values are private, so the information conveyed in open formats is not useful. The figure also shows the single-unit antecedents of the multiunit auctions. Despite appearances, the uniform-price auction is not the appropriate extension of the single-unit second-price auction; it does not inherit the strategic and economic properties of the second-price auction. The Vickrey auction does and is, in fact, the appropriate extension.

[4]Toward the end of the baseball season, a team is said to have "clinched" its division if its won-lost record is such that regardless of what happens in the remainder of the season, it will finish first.

OUTLINE OF PART II

In this chapter we introduced three basic auction formats for the sale of multiple, identical units. In Chapter 13 we study equilibrium bidding behavior in the three auctions in a setting with independent private values. Bidding strategies cannot be explicitly derived, except in some special cases, so instead we highlight some basic strategic considerations in each of the three auctions. The analysis centers on the question of the efficiency of the different formats. Chapter 14 is then concerned with the question of revenue. It derives the revenue equivalence principle in the multiple object context and illustrates its use.

Chapter 15 concerns the sequential sale of multiple, identical units in a series of first- or second-price auctions, again in a private values setting. Chapter 16 concerns the sale of non-identical objects with particular emphasis on situations where the objects are complements. Chapter 18 then addresses some issues arising from interdependent values in a multiple object context.

## CHAPTER NOTES

Government securities are sold all over the world by means of discriminatory auctions. The U.S. Treasury has used discriminatory auctions since 1929 to sell short-term securities—called treasury *bills*—with maturities of 13, 26, and 52 weeks. Prior to the early 1970s, medium-term securities—called treasury *notes*—with maturities of 2, 3, 5, and 10 years and long-term securities—called treasury *bonds*—with a maturities of more than 10 years, were sold at fixed prices set by the Treasury. Since the early 1970s, these have also been sold by means of auction, typically using discriminatory pricing rules. In 1992, however, the treasury began to sell 2- and 5-year notes under uniform-price rules. This was intended as an experiment to see if, as argued by many economists, uniform-price auctions would reduce borrowing costs for the government and to ascertain the pros and cons of a switch to the uniform-price rules for selling all government securities, bills, notes, and bonds alike. A comparison of the data resulting from the two formats, however, does not show a clear advantage of one auction method over the other. A report by Archibald and Malvey (1998) summarizes the empirical findings.

In the United Kingdom electricity generators bid to sell their output on a daily basis. Since 1990 these electricity auctions were conducted using uniform-price rules, but in early 2000 the format was changed—in a direction opposite to that contemplated by the U.S. Treasury—to a discriminatory format. Note that electricity auctions are procurement auctions—the bidders are sellers rather than buyers—but the framework of this chapter is easily adapted to this setting.

The Vickrey multiunit auction was proposed by Vickrey (1961) as an antidote to some of the problems associated with the discriminatory and uniform-price auctions, in particular, the inefficiency of the latter two formats. The next chapter undertakes a detailed examination of these three auctions with a view to comparing their performance in terms of efficiency. The Ausubel ascending price auction was proposed by Ausubel (2004).

We have modeled multiunit auctions as the sale of multiple discrete units of the same good. Alternatively, one may think of a fixed supply of some good, normalized to, say, one unit, but suppose that it is perfectly divisible. A bidder's demand function can then be assumed to be continuous—at any price it specifies the share of the overall supply that the bidder is willing to purchase. All of pricing rules defined in this chapter have straightforward extensions to such a model. The continuous specification does not affect any essential properties of the various auctions but, for some purposes, proves to be analytically convenient. Auctions in such a setting have been called *share auctions* by Wilson (1979), who compares the expected revenues from the uniform price and discriminatory auctions to the expected revenue from the sale of the overall supply as one unit.

The auctions introduced in this chapter are all standard—each awards the $K$ objects to the $K$ highest bids. An example of a *nonstandard* auction format is the *voucher auction* scheme used to privatize industrial enterprises in the former Soviet Union, especially Russia. Under this scheme, all citizens were eligible to receive vouchers with a given face value—10,000 rubles in Russia—and typically about 25 percent of the shares of an enterprise were sold to the public. Vouchers were freely tradable and any person interested in purchasing shares of a particular enterprise could bid an amount $b_i$ in terms of these vouchers. The auction rules were of the "everybody wins" variety. If the amounts bid were $(b_1, b_2, \ldots, b_N)$, then bidder $i$ received a share

$$\theta_i = \frac{b_i}{\sum_j b_j}$$

of the portion of the enterprise for sale. Thus, every bidder received a positive share equal to the ratio of his bid to the total amount bid for that enterprise (see Boyko, Shleifer, and Vishny, 1997). Thus, there were no "losing" bidders in the sense defined in this chapter. Indeed, voucher auctions are equivalent to a lottery in the following sense. Think of $b_i$ as the number of lottery tickets, each sold at a unit of currency, purchased by bidder $i$. If the "winner" of the lottery is chosen by picking a ticket at random, then $\theta_i$ represents the probability that $i$ will have the winning ticket. While voucher auctions are clearly inefficient, equilibrium behavior in such auctions—and their equivalent lotteries—has not been fully explored to date.