MATH 425a SAMPLE MIDTERM EXAM 2
Fall 2015
Prof. Alexander

(1)(a) State what it means for a function $f : X \to Y$ to be uniformly continuous.

(b) State what it means for a sequence $\{x_n\}$ to be Cauchy.

(c) Prove the following part of Theorem 4.8: Suppose $f : X \to Y$, and $f^{-1}(V)$ is open for every open set V in Y. Show that f is continuous.

(2)(a) Establish convergence or divergence: $\sum \frac{1}{n(\log n)^{1/2}}$

(b) Establish convergence or divergence: $\sum \frac{\cos(n^2)}{n^2}$

(c) Determine whether the power series $\sum(-1)^n \frac{n^2}{4^{n+1}} z^n$ converges at the point $z = 2 + 3i$. HINT: It's easier not to do this "directly"–instead, use properties of power series related to where they converge.

(3)(a) Let K be a compact set, and $g : K \to Y$ a continuous bijection. Let $\{x_n\}$ be a sequence in K with $g(x_n) \to q$ as $n \to \infty$. Show that $\{x_n\}$ converges.

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x_0) < y_0$. Show that there exist $b > 0$ and $\delta > 0$ such that $|x - x_0| < \delta \implies f(x) < y_0 - b$.

NOTE: (b) is not related to (a).

(4) Let $\{x_j\}$ be a bounded sequence in $\mathbb{R}$, and $M_n = \sup\{x_n, x_{n+1}, ...\}$.

(a) Show that $L = \lim_n M_n$ exists in $\mathbb{R}$. HINT: Don't do a "Let $\epsilon > 0$..." proof, instead compare M_n and M_{n+1}.

(b) Suppose y is a subsequential limit of $\{x_j\}$. Show that $y \le M_n$ for all n.

(c) Show that $\limsup x_j \le L$. HINT: Use (b).

(d)(Harder!) Show that $\{x_j\}$ has a subsequence converging to L, and therefore $\limsup x_j = L$.

You will be asked to prove one or more of the following theorems on the exam:

3.25a, 3.28, 3.33, 4.8, 4.14, 4.17.

As in Midterm 1, your proof does not have to be the same as what's in the text, just a correct proof (but don't cite a theorem in your proof that appears in the text AFTER the theorem you're proving.) The following will be included with the exam:

REMINDER LIST–TESTS FOR CONVERGENCE

(1) Cauchy criterion ($\{s_n\}$ must be a Cauchy sequence, where s_n is the nth partial sum)
(2) Comparison test

(3) Cauchy condensation test
(4) Root test
(5) Ratio test
(6) Alternating series test
(7) "Two bounds" method

It is up to you to know what these tests are, and what kind of series they apply to (nonnegative terms only, monotone terms, etc.)