MATH 425b FINAL EXAM SOLUTIONS
SPRING 2016
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IN CLASS FINAL:

(1)(a) $d\omega = \frac{\partial f}{\partial x_3} dx_1 \wedge dx_2 \wedge dx_3$ so $d\omega = 0 \iff \frac{\partial f}{\partial x_3} \equiv 0 \iff f$ is constant in x_3 for each fixed $(x_1, x_2) \iff f$ is a function of x_1, x_2 only.

(b) $d\lambda = 0$ means λ is closed in $\mathbb{R}^3$, hence exact by 10.39: $\lambda = dh$ for some $\mathcal{C}''$ function h. Hence by Stokes Theorem 10.33,

$$\int_\beta \lambda = \int_\beta dh = \int_{\partial\beta} h$$

Here $\partial\beta$ is $[\mathbf{b}] - [\mathbf{a}]$ so $\int_\beta h = h(\mathbf{b}) - h(\mathbf{a})$. The same is true for γ so the integrals are equal.

(c) The Jacobian is $\frac{\partial(x_1,x_2)}{\partial(u_1,u_2)} = \det I = 1$, and (since f depends only on the first two coordinates) $f(\Phi(\mathbf{u})) = f(u_1, u_2, 0)$. Therefore

$$\int_\Phi \omega = \int_{[0,1]^2} f(\Phi(\mathbf{u})) \frac{\partial(x_1, x_2)}{\partial(u_1, u_2)}\ d\mathbf{u} = \int_0^1 \int_0^1 f(u_1, u_2, 0)\ du_1\ du_2,$$

which doesn't depend on g.

(d) Here the Jacobian is

$$\frac{\partial(x_1, x_3)}{\partial(u_1, u_2)} = \det \begin{bmatrix} 1 & 0 \\ D_1 g & D_2 g \end{bmatrix} = D_2 g(\mathbf{u}) = 2u_2,$$

so

$$\begin{aligned} \int_\Phi \xi &= \int_{[0,1]^2} u_2(u_1 + u_2^2) \frac{\partial(x_1, x_3)}{\partial(u_1, u_2)}\ d\mathbf{u} \\ &= \int_0^1 \int_0^1 u_2(u_1 + u_2^2) \cdot 2u_2\ du_1\ du_2 \\ &= \int_0^1 \int_0^1 (2u_1 u_2^2 + 2u_2^4)\ du_1\ du_2 \\ &= \int_0^1 (u_2^2 + 2u_2^4)\ du_2 \\ &= \frac{1}{3} + \frac{2}{5} = \frac{11}{15}. \end{aligned}$$

(2) Since f is differentiable at $\mathbf{x}$, we know $f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = f'(\mathbf{x})\mathbf{h} + o(|\mathbf{h}|)$. Also, since f

is continuous, there exist ϵ and M such that $|\mathbf{h}| < \epsilon \implies |f(\mathbf{x}+\mathbf{h})| \le M$. Therefore

$$\begin{aligned}
&\Big| f(\mathbf{x}+\mathbf{h})^2 - f(\mathbf{x})^2 - 2f(\mathbf{x})f'(\mathbf{x})\mathbf{h} \Big| \\
&\qquad = \Big| \big(f(\mathbf{x}+\mathbf{h}) + f(\mathbf{x})\big)\big(f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})\big) - 2f(\mathbf{x})f'(\mathbf{x})\mathbf{h} \Big| \\
&\qquad = \Big| \big(f(\mathbf{x}+\mathbf{h}) + f(\mathbf{x})\big)\big(f'(\mathbf{x})\mathbf{h} + o(|\mathbf{h}|)\big) - 2f(\mathbf{x})f'(\mathbf{x})\mathbf{h} \Big| \\
&\qquad \le \Big| \big(f(\mathbf{x}+\mathbf{h}) + f(\mathbf{x})\big) o(|\mathbf{h}|) \Big| + \Big| \big(f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})\big) f'(\mathbf{x})\mathbf{h} \Big| \\
&\qquad \le 2Mo(|\mathbf{h}|) + \Big| f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) \Big| \cdot \|f'(\mathbf{x})\| \cdot |\mathbf{h}|
\end{aligned}$$

In the last line, since $f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) \to 0$ as $\mathbf{h} \to 0$, the second term is $o(|\mathbf{h}|)$ like the first one. Thus the first line is $o(|\mathbf{h}|)$, which shows $(f^2)'(\mathbf{x}) = 2f(\mathbf{x})f'(\mathbf{x})$.

(3)(a) $\int_{-\pi}^{\pi} g(x) \cdot \frac{1}{\sqrt{\pi}} \sin nx \; dx = 0$ since $g(x)\sin nx$ is odd.

(b) No. For any g as in part (a) with $g \neq 0$, for all n we have $c_n = \int_{-\pi}^{\pi} g(x) \cdot \frac{1}{\sqrt{\pi}} \sin nx \; dx = 0$ but $\int_{-\pi}^{\pi} g(x)^2 \; dx \neq 0$.

(4)(a) Since the traces are unequal, there exists some $\mathbf{x}_0$ in one trace only, say $\mathbf{x}_0 \in \text{trace}(\sigma^1), \mathbf{x}_0 \notin \text{trace}(\sigma^2)$. Trace$(\sigma^2)$ is closed so there exists a neighborhood U of $\mathbf{x}_0$ which is outside trace(σ^2). As in the hint, there exists a continuous function f which is positive near $\mathbf{x}_0$ and 0 outside U. Let $I = \{i_1, \ldots, i_k\}$ be such that the Jacobian $\frac{\partial(\sigma^1_{i_1}, \ldots, \sigma^1_{i_k})}{\partial(u_1, \ldots, u_k)} \equiv c \neq 0$, and let $\omega = f \; d\mathbf{x}_I$. Then

$$\int_{\sigma^1} \omega = \int_{Q^k} f(\sigma^1(\mathbf{u})) \frac{\partial(\sigma^1_{i_1}, \ldots, \sigma^1_{i_k})}{\partial(u_1, \ldots, u_k)} d\mathbf{u} = c \int_{Q^k} f(\sigma^1(\mathbf{u})) \; d\mathbf{u}.$$

Since $f \circ \sigma^1$ is continuous, nonnegative, and strictly positive near $(\sigma^1)^{-1}(\mathbf{x}_0)$, the last integral is strictly positive.

But $\int_{\sigma^2} \omega = 0$ since $f = 0$ on trace(σ^2). Thus $\int_{\sigma^1} \omega + \int_{\sigma^2} \omega \neq 0$.

(b) By part (a) we have trace$(\sigma^1) =$ trace(σ^2), so σ^1, σ^2 are two parametrizations of the same surface. This means that either $\int_{\sigma^1} \omega = \int_{\sigma^2} \omega$ for all ω (if orientation is the same) or $\int_{\sigma^1} \omega = -\int_{\sigma^2} \omega$ for all ω (if orientation is opposite.) Opposite orientation means $q_0, \ldots, q_k$ is an odd permutation of $p_0, \ldots, p_k$, our desired conclusion. If the orientation is the same, then $0 = \int_{\sigma^1} \omega + \int_{\sigma^2} \omega = 2\int_{\sigma^1} \omega$ for all ω. But this is false: we can take I, ω as in part (a), with f continuous and strictly positive, and as in part (a) we get $\int_{\sigma^1} \omega \neq 0$. Thus the orientation is opposite.

TAKE HOME FINAL:

(1)(a) Suppose $f \in \text{Lip}_{c_1}(K), g \in \text{Lip}_{c_2}(K)$ for some c_1, c_2. Note $\|f\|_\infty$ and $\|g\|_\infty$ are finite since f, g are continuous and K is compact. Therefore for all $x, y \in K$ and $a \in \mathbb{R}$:

$$|af(x) - af(y)| \le |a| c_1 d(x,y),$$

so $af \in \mathrm{Lip}(K)$;

$$|(f+g)(x)-(f+g)(y)| \le |f(x)-f(y)| + |g(x)-g(y)| \le (c_1+c_2)d(x,y)$$

so $f+g \in \mathrm{Lip}(K)$;

$$\begin{aligned}|(fg)(x)-(fg)(y)| &\le |f(x)g(x)-f(x)g(y)| + |f(x)g(y)-f(y)g(y)|\\ &\le \|f\|_\infty |g(x)-g(y)| + \|g\|_\infty |f(x)-f(y)|\\ &\le (c_2\|f\|_\infty + c_1\|g\|_\infty)d(x,y),\end{aligned}$$

so $fg \in \mathrm{Lip}(K)$.

(b) By the Stone-Weierstrass Theorem, it is sufficient to show that $\mathrm{Lip}(K)$ separates points and vanishes at no point. Since $\mathrm{Lip}(K)$ contains all constant functions, it vanishes at no point. To show it separates points, we first show that for fixed $x_0 \in K$, the function $f(x) = d(x, x_0)$ is in $\mathrm{Lip}(K)$. In fact, by the triangle inequality, for all x, y,

$$f(x) \le d(x,y) + f(y), \quad f(y) \le d(x,y) + f(x), \quad \text{so } |f(x)-f(y)| \le d(x,y),$$

so indeed $f \in \mathrm{Lip}(K)$. Since $f(x) \ne 0 = f(x_0)$ for all $x \ne x_0$, and x_0 is arbitrary, this shows $\mathrm{Lip}(K)$ separates points.

(c) We need to show $F_{c,M}$ is closed, pointwise bounded, and equicontinuous. Since $F_{c,M}$ is uniformly bounded by M, it is pointwise bounded. Given $\epsilon > 0$, for all $f \in F_{c,M}$ we have

$$d(x,y) < \frac{\epsilon}{c} \implies |f(x)-f(y)| \le cd(x,y) < \epsilon,$$

so $F_{c,M}$ is equicontinuous. If $f_n \in F_{c,M}$ for all n, and $f_n \to f$ uniformly, then for all x, y,

$$|f(x)-f(y)| = \lim_n |f_n(x)-f_n(y)| \le cd(x,y),$$

and $|f(x)| = \lim_n |f_n(x)| \le M$ for all x, so $f \in F_{c,M}$. Thus $F_{c,M}$ is closed. Together these properties show $F_{c,M}$ is compact.

(d) $F_{c,M}$ is compact so it has a countable dense subset $D_{c,M}$, by the theorem we proved after 7.23. Let $D = \cup_{c=1}^\infty \cup_{M=1}^\infty D_{c,M} \subset \mathrm{Lip}(K)$. Then $f \in \mathrm{Lip}(K) \implies f \in F_{c,M}$ for some integers $c, M \implies f \in \overline{D_{c,M}} \subset \overline{D}$, so $\mathrm{Lip}(K) \subset \overline{D}$, so $C(K) = \overline{\mathrm{Lip}(K)} \subset \overline{D}$, meaning D is dense in $C(K)$. Also D is countable since each $D_{c,M}$ is countable. Here the reason we can take c, M to be integers is that if $f \in F_{c',M'}$ for some real c', M', then we have $f \in F_{c,M}$ for any integers $c \ge c'$ and $M \ge M'$. This means the union is the same over real c, M as over integer c, M.

(2)(a) Let S_j be the submatrix with the 1st column and the jth row both removed, so that $\mathcal{H}(\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n-1)}) = \sum_{j=1}^n (\det S_j) e_j$. If, for some i, we replace each e_j with $v_j^{(i)}$ in the matrix, then the determinant becomes 0 because two columns of the matrix are both equal to $v^{(i)}$. This means the last sum also becomes 0, that is,

$$0 = \sum_{j=1}^n (\det S_j) v_j^{(i)}.$$

Since $\det S_j$ is the jth entry of $\mathcal{H}(\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n-1)})$, this sum is the dot product $\mathcal{H}(\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n-1)}) \cdot v^{(i)}$, so $v^{(i)}$ is orthogonal to $\mathcal{H}(\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(n-1)})$,

(b) From the formula for $\omega^{(F)}$,

$$\begin{aligned} d\omega^{(F)} &= \sum_{k=1}^{n} \epsilon_k \frac{\partial F_k}{\partial x_k}(\mathbf{x})\ dx_k \wedge dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \dots dx_n \\ &= \sum_{k=1}^{n} (-1)^{k-1} \epsilon_k \frac{\partial F_k}{\partial x_k}(\mathbf{x})\ dx_1 \wedge \cdots \wedge dx_n \end{aligned}$$

which is equal to $(\mathrm{div} F)(\mathbf{x})\ dx_1 \wedge \cdots \wedge dx_n$ provided we take $\epsilon_k = (-1)^{k-1}$.

(c) From part (b) and Stokes Theorem,

$$\begin{aligned} \int_D (\mathrm{div} F)(\mathbf{x})\ dx_1 \wedge \cdots \wedge dx_n &= \int_D d\omega^{(F)} = \int_{\partial D} \omega^{(F)} = \sum_{i=0}^{n} \int_{\tau_i} \omega^{(F)} \\ &= \sum_{i=0}^{n} \int_{\tau_i} \sum_{k=1}^{n} (-1)^{k-1} F_k(\mathbf{x})\ dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \dots dx_n \\ &= \sum_{i=0}^{n} \int_{Q^{n-1}} \sum_{k=1}^{n} (-1)^{k-1} F_k(\tau_i(\mathbf{u})) \frac{\partial(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)}{\partial(u_1, \dots, u_{n-1})}\ du_1 \cdots du_{n-1} \end{aligned}$$

The Jacobian in the last line is, by definition, $(-1)^{k-1} N_k(\tau_i(\mathbf{u}))$, so the last line is equal to

$$\sum_{i=0}^{n} \int_{Q^{n-1}} F(\tau_i(\mathbf{u})) \cdot N(\tau_i(\mathbf{u}))\ du_1 \cdots du_{n-1},$$

as desired.