MATH 425b ASSIGNMENT 4 SOLUTIONS
SPRING 2016
Prof. Alexander

Chapter 8:

(13) Let $f(x) = x$ for $0 \le x \le 2\pi$. Using integration by parts, for $n \neq 0$ we get the Fourier coefficient

$$\begin{aligned} c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\ dx &= \frac{1}{2\pi}\left(\int_0^{2\pi} x\cos nx\ dx - i\int_0^{2\pi} x\sin nx\ dx\right) \\ &= \frac{1}{2\pi}\left(0 + \frac{2\pi}{n}i\right) \\ &= \frac{i}{n}, \end{aligned}$$

while

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} x\ dx = \pi.$$

Hence

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \pi^2 + 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

while

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)^2\ dx = \frac{1}{2\pi}\frac{x^3}{3}\Big|_0^{2\pi} = \frac{4\pi^2}{3}.$$

By Parseval's identity these are equal, that is,

$$\pi^2 + 2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{4\pi^2}{3}.$$

Therefore $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

(19) Suppose f is continuous with period 2π and α/π is irrational. Let $g_k(x) = e^{ikx}$. Consider first $k \neq 0$. Summing a finite geometric series we get that

$$\frac{1}{N}\sum_{n=1}^{N} g_k(x+n\alpha) = \frac{1}{N}\sum_{n=1}^{N} e^{ikx}e^{ikn\alpha} = \frac{1}{N}e^{ikx}\frac{1-e^{ik(N+1)\alpha}}{1-e^{ik\alpha}}. \tag{1}$$

Since α/π is irrational, we have for $k \neq 0$ that $e^{ik\alpha} = e^{i\pi(k\alpha/\pi)} \neq 1$, since $k\alpha/\pi$ is not an integer. Therefore the last denominator in (2) is not 0, so

$$\lim_N \frac{1}{N}\sum_{n=1}^{N} g_k(x+n\alpha) = \lim_N \frac{1}{N}e^{ikx}\frac{1-e^{ik(N+1)\alpha}}{1-e^{ik\alpha}} = 0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_k(t)\ dt.$$

Next, for $k = 0$ we have $g_k \equiv 1$ so

$$\lim_N \frac{1}{N} \sum_{n=1}^{N} g_k(x + n\alpha) = \lim_N \frac{1}{N} \cdot N = 1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_k(t)\ dt.$$

By Stone-Weierstrass, given $\epsilon > 0$ there exists a trigonometric polynomial

$$\varphi(x) = \sum_{k=-K}^{K} c_k g_k(x)$$

with $||f - \varphi||_\infty < \epsilon$. By the above results for g_k, we have

$$\begin{aligned}
\lim_N \frac{1}{N} \sum_{n=1}^{N} \varphi(x + n\alpha) &= \sum_{k=-K}^{K} c_k \lim_N \frac{1}{N} \sum_{n=1}^{N} g_k(x + n\alpha) \\
&= \sum_{k=-K}^{K} c_k \frac{1}{2\pi} \int_{-\pi}^{\pi} g_k(t)\ dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t)\ dt. \qquad (2)
\end{aligned}$$

In other words, the desired result is true for trigonometric polynomials like φ. To prove it for f, we compare the expressions for f and φ:

$$\begin{aligned}
\frac{1}{N} \sum_{n=1}^{N} f(x + n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ dx &= \frac{1}{N} \sum_{n=1}^{N} \varphi(x + n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(x)\ dx \\
&\quad + \frac{1}{N} \sum_{n=1}^{N} [f(x + n\alpha) - \varphi(x + n\alpha)] \\
&\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} [\varphi(x) - f(x)]\ dx \\
&= (I) + (II) + (III).
\end{aligned}$$

By (2), we have (I) $\to 0$ as $N \to \infty$, so $|(I)| < \epsilon$ if N is large. Also, since $||f - \varphi||_\infty < \epsilon$,

$$|(II)| \le \frac{1}{N} \sum_{n=1}^{N} \epsilon = \epsilon \quad \text{and} \quad |(III)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \epsilon\ dx = \epsilon,$$

so (I) + (II) + (III) $< 3\epsilon$. Since ϵ is arbitrary, this proves the desired result.

(A)(a)

$$\langle \varphi_1, \varphi_3 \rangle = \int_{-\infty}^{\infty} \varphi_1(x)\varphi_3(x)\ dx = \int_{-\infty}^{\infty} (16x^4 - 24x^2)e^{-x^2}\ dx.$$

Integration by parts gives

$$\int_{-\infty}^{\infty} 24x^2 e^{-x^2}\,dx = \frac{1}{3}x^3 e^{-x^2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} 16x^4 e^{-x^2}\,dx = \int_{-\infty}^{\infty} 16x^4 e^{-x^2}\,dx, \tag{3}$$

since

$$\lim_{s\to\infty} s^\alpha e^{-s} = 0 \quad \text{for all } \alpha. \tag{4}$$

Therefore $\langle \varphi_1, \varphi_3\rangle = 0$.

(b) We have

$$\|\varphi_1\|_{L^2}^2 = \int_{-\infty}^{\infty} |\varphi_1(x)|^2\,dx = \int_{-\infty}^{\infty} 4x^2 e^{-x^2}\,dx.$$

By (4), the integrand $|\varphi_1(x)|^2 = 4x^2 e^{-x^2/2}$ is bounded by some finite M and satisfies $x^2|\varphi_1(x)|^2 \to 0$ as $|x| \to \infty$, so there exists A such that

$$4x^2 e^{-x^2} \leq \begin{cases} M & \text{if } |x| \leq A, \\ \frac{1}{x^2} & \text{if } |x| > A. \end{cases}$$

Since $1/x^2$ has a finite integral on $[A, \infty)$, it follows that $0 < \|\varphi_1\|_{L^2}^2 < \infty$. Therefore we can choose $c_1 = 1/\|\varphi_1\|_{L^2}$ to make $\psi_1 = c_1\varphi_1$ satisfy $\|\psi_1\|_{L^2} = c_1\|\varphi_1\|_{L^2} = 1$. Similarly we take $c_3 = 1/\|\varphi_3\|_{L^2}$. Then $\langle \psi_1, \psi_3\rangle = c_1 c_3 \langle \varphi_1, \varphi_3\rangle = 0$, so ψ_1, ψ_3 are orthonormal.

(c) This problem is incorrectly stated since the function g is not in L^2, because $\int_{-\infty}^{\infty} g(x)^2\,dx = \infty$. (It was supposed to be $g(x) = xe^{-x^2/2}$.) We can nonetheless find the "formally" correct coefficients, which are $a_i = \langle g, \psi_i\rangle$. In fact, integrating by parts,

$$\begin{aligned} a_1 = \langle g, \psi_1\rangle &= c_1\langle g, \varphi_1\rangle \\ &= c_1 \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,dx \\ &= c_1 \int_{-\infty}^{\infty} (-x)(-xe^{-x^2/2})\,dx \\ &= c_1 \left[-xe^{-x^2/2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \right] \\ &= c_1 \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \end{aligned} \tag{5}$$

From the hint (given in email), this last integral is $\sqrt{2\pi}$, so $a_1 = c_1\sqrt{2\pi} = 2\pi^{1/4}$. Next we have

$$a_3 = c_3\langle g, \varphi_3\rangle = c_3 \int_{-\infty}^{\infty} (8x^4 - 12x^2)e^{-x^2/2}\,dx.$$

Similarly to (3) we get

$$\int_{-\infty}^{\infty} 8x^4 e^{-x^2/2}\,dx = \int_{-\infty}^{\infty} 24x^2 e^{-x^2/2}\,dx,$$

and the last 4 lines of (5) show that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\ dx = \int_{-\infty}^{\infty} e^{-x^2/2}\ dx = \sqrt{2\pi},$$

so

$$a_3 = c_3 \int_{-\infty}^{\infty} (24x^2 - 12x^2) e^{-x^2/2}\ dx = 12c_3\sqrt{2\pi} = 4\sqrt{6}\pi^{1/4}.$$

(B)(a) Integrating by parts and using $\cos n\pi = (-1)^n$ we get for $n \neq 0$

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos nx\ dx &= \int_{-\pi}^{0} x\cos nx\ dx \\ &= x\frac{\sin nx}{n}\Big|_{-\pi}^{0} - \int_{-\pi}^{0} \frac{\sin nx}{n}\ dx \\ &= \frac{\cos nx}{n^2}\Big|_{-\pi}^{0} \\ &= \frac{1-(-1)^n}{n^2} \\ &= \begin{cases} \frac{2}{n^2}, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases}\end{aligned}$$

and similarly

$$\int_{-\pi}^{\pi} f(x)\sin nx\ dx = \frac{(-1)^n\pi}{n}.$$

Hence for $n \neq 0$

$$\begin{aligned} c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\ dx &= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} f(x)\cos nx\ dx - i\int_{-\pi}^{\pi} f(x)\sin nx\ dx\right] \\ &= \begin{cases} \frac{1}{\pi n^2} + \frac{(-1)^n}{2n} i, & n \text{ odd}, \\ \frac{(-1)^n}{2n} i, & n \text{ even}, \end{cases}\end{aligned}$$

while

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{2\pi}\int_{-\pi}^{0} x\ dx = -\frac{\pi}{4}.$$

(b) By Theorem 8.14, the Fourier series of f converges pointwise in $(-\pi, \pi)$. In particular, at $x = 0$ we have

$$0 = f(0) = \sum_{n\in\mathbb{Z}} c_n.$$

The imaginary parts of c_{-n} and c_n cancel for $n \neq 0$, and the real part $1/\pi n^2$ is the same for c_{-n} and c_n, so this becomes

$$0 = c_0 + 2\sum_{n\geq 1, n \text{ odd}} \frac{1}{\pi n^2} \quad \text{so} \quad \sum_{n\geq 1, n \text{ odd}} \frac{1}{n^2} = -\frac{\pi}{2}c_0 = \frac{\pi^2}{8}.$$

(C) Let $f'(x) \sim \sum_{n\in\mathbb{Z}} b_n e^{inx}$ be the Fourier series of f'; we want to show $b_n = inc_n$. Let

$$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx, \quad \beta_n = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx,$$

so $c_n = \alpha_n + i\beta_n$. To calculate b_n for $n \neq 0$, we can integrate by parts:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\ dx = \frac{1}{2\pi} f(x)\cos nx\Big|_{-\pi}^{\pi} + \frac{1}{2\pi}\int_{-\pi}^{\pi} nf(x)\sin nx\ dx = 0 - n\beta_n,$$

where the 0 is because $f(\pi) = f(-\pi)$ by periodicity, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\ dx = \frac{1}{2\pi} f(x)\sin nx\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} nf(x)\cos nx\ dx = 0 - n\alpha_n.$$

Combining these we get

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}\ dx = -n\beta_n + in\alpha_n = in(\alpha_n + i\beta_n) = inc_n.$$

(D) Fix $z_0 \in D$ and expand f around z_0: in a neighborhood of z_0,

$$\begin{aligned} f(z) = f(z_0) + \sum_{n=0}^{\infty} c_n(z-z_0)^n &= f(z_0) + \sum_{n=N}^{\infty} c_n(z-z_0)^n \\ &= f(z_0) + c_N(z-z_0)^N\left(1 + \sum_{n=N+1}^{\infty} \frac{c_n}{c_N}(z-z_0)^{n-N}\right). \quad (6) \end{aligned}$$

where c_N is the first nonzero coefficient in the series. The last series

$$g(z) = \sum_{n=N+1}^{\infty} \frac{c_n}{c_N}(z-z_0)^{n-N}$$

is continuous at z_0 (since it's analytic) and satisfies $g(z_0) = 0$, so there is a neighborhood of z_0 where

$$\mathrm{Re}(1 + g(z)) > 0. \quad (7)$$

Letting $z = z_0 + re^{i\theta}$, we can choose θ so that $c_N(z-z_0)^N = c_N r^N e^{iN\theta}$ is a positive real number, which with (6) and (7) shows that, provided r is small (so z is close to z_0),

$$A(z) - A(z_0) = \mathrm{Re}(f(z) - f(z_0)) = \mathrm{Re}[c_N(z-z_0)^N(1+g(z)] > 0.$$

But this means A does not have a local maximum at z_0, and z_0 is arbitrary.