MATH 425b ASSIGNMENT 6
SPRING 2016
Prof. Alexander
Due Wednesday March 23.

Chapter 9 #7, 8 and:

(I) Show that $Tx = \sin x$ is not a contraction on $[-1, 1]$.

(II)(a) Give an example of a differentiable map $f : \mathbb{R} \to \mathbb{R}$ whose fixed points are exactly the integers.

(b) Show that such a map must have points x where $|f'(x)| > 1$.

(III) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$, f is differentiable at a point $x \in \mathbb{R}^n$, $h_n \to 0$ and $k_n \to 0$. Show that

$$f(x + h_n + k_n) - f(x) = \big[f(x + h_n) - f(x)\big] + \big[f(x + k_n) - f(x)\big] + o(|h_n| + |k_n|).$$

(IV) Let us associate a 2×2 matrix to each vector$(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ by putting the coordinates in the arrangement $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. For example, $(1, 0, 0, 1)$ corresponds to the identity matrix.

We can then define a function $f : \mathbb{R}^4 \to \mathbb{R}^4$ by specifying that the coordinates of $f(\mathbf{x})$ are the entries of the matrix $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}^2$, for $i = 1, 2, 3, 4$, in the order shown above. (Don't overlook the square on the matrix here!) For example, the first coordinate of $f(x)$ is $x_1^2 + x_2 x_3$.

Use this f to show that there are neighborhoods U, V of the identity matrix I such that for each matrix $B \in V$ there is a unique matrix $C \in U$ which is a square root of B, that is, $C^2 = B$, and this C is a continuously differentiable function of B.

(V) Let $E \subset \mathbb{R}^n$ be open and suppose $f : E \to \mathbb{R}^n$ is $\mathbb{C}'$ and $f'(\mathbf{a})$ is invertible.

(a) Show that there is a neighborhood W of $\mathbf{a}$ and $M < \infty$ such that $||f'(\mathbf{x})|| \leq M$ for all $\mathbf{x} \in W$.

(b) Show that there exists a linear transformation T such that

$$|\mathbf{h} - T(f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}))| = o(|\mathbf{h}|) \quad \text{as } \mathbf{h} \to 0.$$

(VI) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0, 0) = 0$ and

$$f(x, y) = \frac{4xy(x^2 - y^2)}{(x^2 + y^2)^2}, \quad (x, y) \neq (0, 0).$$

Show that the partial derivatives $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ exist, but f is not continuous at $(0, 0)$.

HINTS:

(7) Note the relevant idea in Theorem 9.21 is that you can decompose an increment in the plane, from $\mathbf{x}$ to $\mathbf{x}+\mathbf{h}$, as the sum of a horizontal increment and a vertical increment; similarly, in n dimensions you get a sum of n increments, one in the direction of each coordinate axis. To bound an increment of f when you want to show continuity, you can bound each of these n increments separately.

(8) If f has a local maximum at x, what does that say about the function $g(t) = f(x + te_i)$ for a given coordinate i? What does this tell you about one of the entries of the $1 \times n$ matrix $f'(x)$?

(I) Use the fact that $T'(0) = 1$. What does this say about the increment of T from 0 to x?

(II)(a) Think graphically–a fixed point is where the graph of f crosses what line?
(b) Apply the Mean Value Theorem to $g(x) = f(x) - x$.

(III) $f(x + h) - f(x)$ can be approximated—by what? Also, any quantity expressed as $o(|h_n + k_n|)$ can also be expressed as $o(|h_n| + |k_n|)$–why?

(V)(a) You can assume the fact that $||A||$ is a continuous function of A.
(b) Differentiability means, informally, that $f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) \approx A\mathbf{h}$, where $A = f'(\mathbf{a})$. Turn this into a statement that $\mathbf{h} \approx$ (something). This tells you what T needs to be. What major theorem can you now use? Also, use $\mathbf{h} = (\mathbf{x} + \mathbf{h}) - \mathbf{x}$ and (a).

(VI) To show the lack of continuity, consider what happens when $(x, y) \to (0, 0)$ along different lines $y = cx$.