Spring 2017

ECON 504 GAME THEORY

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HOMEWORK 1

[1] a)- Yes, always. As $C(\cdot)$ is rational, there is a utility function $u: X \to R$ that represents the choice ($u(c(A)) \geq u(a)$ $\forall a \in A$; $c(\cdot)$ picks the u-maximal element in a menu). Suppose $A = A_1 \cup A_2$ and $c(A) \in A_1$.

Hence $u(c(A)) \geq u(a)$ $\forall a \in A_1$ hence $c(A_1) = c(A)$

$u(c(A)) \geq u(a)$ $\forall a \in A_2$ hence $c(c(A_1) \cup c(A_2)) = c(A)$

As $c(A)$ has the highest u-value, two-step or not it will be picked in any choice menu it is in.

b)- Yes, Consider $B \subset A$ and $c(A) \in B$

CASE i) $c(A)$ is a car (p, m) with $p \leq 20k$

then $c(A)$ is the cheapest car in A, hence in B; as there is a car cheaper than $20k$ in B ($c(A) \in B$) $c(A)$ is chosen from B as well; $C(B) = c(A)$.

CASE ii). $c(A)$ is a car (p, m) with $p > 20k$

Then there is no car cheaper than $20k$ in A; hence neither in B. So, in both A & B the choice is the least mileage car. In A, it was the car $c(A)$; as $B \subset A$ $c(A) \in B$, $c(A)$ is the least mileage car in B as well; $c(B) = c(A)$

ALTERNATIVE SOLUTION:

define directly the utility:

$$u(p,m) = \begin{cases} 20k - p & \text{if } p \leq 20k \\ -m & \text{if } p > 20k \end{cases}$$

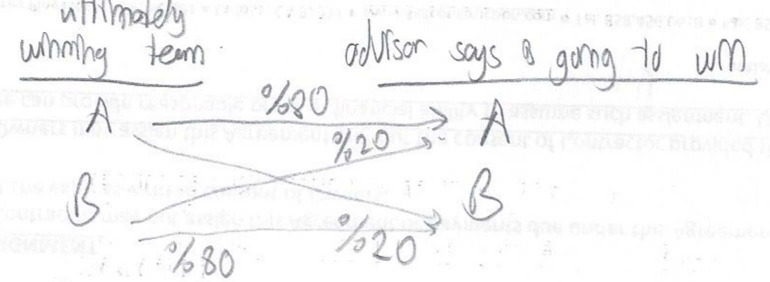
Notice we gave positive utility to all cars cheaper than $20k$ (and are ranked according to cheapness) and all expensive cars have negative utility and ranked according to mileage. As we have a utility repr.; the induced choice is automatically rational.

[2] a)- $EU_{no} = 0$ $EU_{bet\,A} = \%40 \cdot (20) + \%60 \cdot (-10) = 2$

$EU_{bet\,B} = \%40(-15) + \%60(10) = 0$

$\max\{EU_{no}, EU_A, EU_B\} = \boxed{2 = EU_A}$ bet A !

b)-



When advisor says A is gonna win; your posterior belief that A wins is;

$$pr(A \mid \text{advisor says } A) = \frac{\%40 \cdot \%80}{\%40\,\%80 + \%60\,\%20} = \frac{8}{11}$$

When he says B is going to win your post. belief that A is going to win is;

$$pr(A \mid \text{advisor says } B) = \frac{\%40 \cdot \%20}{\%40\,\%20 + \%60\,\%80} = \frac{1}{7}$$

Note that $EU_A(\text{belief} = \frac{8}{11}) = \frac{8}{11}(20) + \frac{3}{11}(-10) = \frac{130}{11} > 0$

$EU_B(\text{belief} = \frac{8}{11}) = \frac{8}{11}(-15) + \frac{3}{11}(10) = -\frac{90}{11} < 0$

$\rightarrow$ you'd bet A rather than not betting or betting B when you get "A is going to win" signal from the advisor.

$EU_A(\text{belief} = \frac{1}{7}) = \frac{1}{7}(20) + \frac{6}{7}(-10) = -\frac{40}{7} < 0$

$EU_B(\text{belief} = \frac{1}{7}) = \frac{1}{7}(-15) + \frac{6}{7}(10) = \frac{45}{7} > 0$

Indeed, you'd bet B after receiving B signal from advisor.

Now, what is the value of this signal to you ??

Note that with ex-ante (unconditional) probability $\%40 \times \%80 + \%60\,\%20 = \%44$.

advisor will advise betting A (A signal)

and %56 will advise betting B.

$$EU_{\text{with advisor}} = \%44\left(\frac{130}{11}\right) + \%56\left(\frac{45}{7}\right) = 8.8$$ 3

(You could have calculated directly ; %40

$$= \%40 \cdot \%80\,(20) + \%40\,\%20\,(-15) + \%60\,\%80\,(10) + \%60\,\%20\,(-10)$$

$$= 6.4 + (-1.2) + 4.8 - 1.2 = 8.8$$

however you should show that after A signal, with the posterior belief it is indeed optimal to bet A as we did in the prev. page)

$$EU_{\text{with advisor}} - EU = 8.8 - 2 = \underline{6.8} = f$$

(c) If after advisor's signal, your decision does not change; you don't need his advice. Remember w/o advisor you were choosing to bet on A. If after B signal from advisor; with the posterior belief it is still optimal to bet on A ; the signal is useless : it is payoff-irrelevant.

$$pr(A \mid \text{advisor says } B) = \frac{\%40\,(1-p)}{\%40\,(1-p) + \%60\,p} = \frac{2-2p}{2+p}$$

$$EU_A\left(\text{belief} = \frac{2-2p}{2+p}\right) = \frac{2-2p}{2+p}(20) + \frac{3p}{2+p}(-10) = \frac{2-2p}{2+p}(-15) + \frac{3p}{2+p}(10) = EU_B(\text{belief})$$

$$(2-2p)\,35 = 3p \cdot 20 \Rightarrow 70 - 70p = 60p \qquad \frac{7}{13} = p$$

if $\frac{1}{2} \leq p < \frac{7}{13}$, I would still bet on A (optimally) after B signal; here I always bet on A ; signal is not worth anything.

(note that $EU_A\left(\text{belief} = \frac{2-2p}{2+p}\right) = \frac{40-70p}{2+p} > 0 = EU_{\text{no bet}}$ for $p \in \left(\frac{1}{2}, \frac{7}{13}\right)$)

so I'd bet.

For $p=1$ definitely signal would be informative & I will bet B after B signal.

$$EU_{p=1} = \%40 \cdot 20 + \%60 \cdot 10 = 14$$

$$EU_{p=1} - EU_{A,\text{no advisor}} = 14 - 2 = 12 = f$$

[3] T is str. dominated by X → C & D are str. dom. by A & B both.
→ Y & Z are both str. dom. by X ⟶ A is str. dom. by B.
(B, X) is the unique IESDS (iterated elimination eqm.) outcome.

[4] If any $x \in [0,100]$ is a bid with positive probability, we know (from prop 6.1 in textbook) that any bid should give the same expected utility;

$$EU_1(x,y) = pr(x>y)(1-x) + pr(x<y)(-x)$$

($pr(x=y)=0$ as we assume atomless distributions with well defined cdf $f(x)>0$ $\forall x$)

$$EU_1(x,y) = F(x)(100-x) + (1-F(x))(-x) = -x + F(x)$$

this should be the same for all $x \in [0,1]$ $\Rightarrow -x + 100F(x) = k$

$F(x) = \frac{x+k}{100}$ we know $F(100)=1$ as nobody bids over 100 as it is strictly dominated, hence not part of NE. $\Rightarrow k=0$

hence we have the uniform distribution $x \in U[0,100]$

see textbook problem 6.10 for further details & steps.

[5] (a) $\boxed{\alpha=1}$ Note that if agent i prefers $P \geq_i H$ then all $i' > i$ also prefer $P >_{i'} H$ ($4x-2+\alpha i \geq 0 \Rightarrow 4x-2+\alpha i' > 0$ for $i' > i$)

Hence we cannot have i plays P but i' plays H.

So the pure NE has to be of the sort "$[\bar{i}, 1]$ plays P and $[0, \bar{i})$ plays H" for some $\bar{i} \in [0,1]$

(note that any 0-Lebesgue measure set of players do not change the measure of P-players, hence x; hence the eqm isn't responsive to 0-measure set of players; hence the agent $\bar{i}$ can play P or H, actually)

If $\bar{i} \in (0,1)$ players $[\bar{i}, 1]$ strictly prefer $P > H$ and players $[0, \bar{i})$ strictly prefer $H > P$

$\Rightarrow$ $\bar{i}$ should be indifferent between P & H $\Rightarrow$

$4(1-\bar{i})-2+1\cdot\bar{i} = u(1-\bar{i}, \bar{i}) =$ protesting utility $=$ home utility $= 0$

$2-3\bar{i} = 0 \quad \bar{i} = 2/3$

Hence $[0, \frac{2}{3}] \to H$ and $[\frac{2}{3}, 1] \to P$ is a pure NE!

For the endpoints, $\bar{i} = 0$ or 1,

$\bar{i} = 0 \Rightarrow$ all protest $P \Rightarrow 4\cdot 1 - 2 + 1\cdot i > 0 \quad \forall i \in [0,1]$

$\bar{i} = 1 \Rightarrow$ all home $H \Rightarrow 4\cdot 0 - 2 + 1\cdot i < 0 \quad \forall i \in [0,1]$

Hence there are 3 pure NE.

$\boxed{\alpha = 3}$ In this case the interior threshold $\bar{i}$ solves,

$4(1-\bar{i}) - 2 + 3\bar{i} = 0 \quad 2 - \bar{i} = 0 \to \leftarrow$ no interior solution.

For the endpoints,

$\bar{i} = 0$ all protest $P \Rightarrow 4\cdot 1 - 2 + 3i > 0 \quad \forall i$ "all P equilibrium"

$\bar{i} = 1$ all home $H \Rightarrow 4\cdot 0 - 2 + 3i > 0 \quad \forall i$ so no "all H" eqm.

In this case there is only "all P" equilibrium.

(b) $\boxed{\alpha = 1}$ For any given i $4x - 2 + i \gtreqless 0$ depending on x;

Hence any player i can justify his action, H or P, by a suitable belief x about how many players will protest. No strictly dominated strategies for anyone, i.e. all outcomes are rationalizable. (the whole set of all game outcomes; no prediction)

$\boxed{\alpha = 3}$ $\quad 4\cdot x - 2 + 3i > 0$ for all x if $4\cdot 0 - 2 + 3i > 0$

$\Rightarrow i > \frac{2}{3}$ Hence for $i > \frac{2}{3}$ P is a dominant strategy and H is strictly dominated. (They'd rather protest even if nobody else is protesting; $x = 0$)

Now as all players are rational, everybody knows that players $i > \frac{2}{3}$ will protest

Hence $x \geq 1 - \frac{2}{3} = \frac{1}{3}$

But if $x \geq \frac{1}{3}$ $\quad 4\cdot x - 2 + 3i > 4\cdot\frac{1}{3} - 2 + 3i > 0$ for all $i > \frac{2}{9}$ [6

Now, in this second round, $i \in \left(\frac{2}{9}, \frac{1}{3}\right)$ players find H str. dominated; hence they'd protest, too.

All players know that all players know that all are rational; hence all know $x \geq 1 - \frac{2}{9} = \frac{7}{9}$. But then; $4x - 2 + 3i > 4\cdot\frac{7}{9} - 2 + 3i > 0$ for all $i \in [0,1]$.

Hence for all $i \in [0,1]$ H is str. dominated.

The <u>unique rat. outcome</u> (IESDS) is "<u>all protest</u>"

[6] solution at the end $\Rightarrow$

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[7] (a)

A	3,	0,
B	0,	3,
C	1,	1,

C is not str. dominated (by A or B) but is str. dominated by $\frac{1}{2}A + \frac{1}{2}B$

(b) Cannot happen. If $u_1(s_1, \sigma_{-1}) > u_1(s_1', \sigma_{-1})$ $\forall \sigma_{-1}$ then $u_1(s_1, s_{-1}) > u_1(s_1', s_{-1})$ $\forall s_{-1}$ as $S_{-i} \subset \Delta(S_{-i})$ hence s_1' would be dominated by s_1 against pure strategies too.

(c)

A	3,	0,
B	0,	3,
C	2,	2,

A & B are not strictly dominated, but $\frac{1}{2}A + \frac{1}{2}B$ is.

(d) YES. s_2 is a BR to s_1 so is not a never-best-response in the first round, hence is not eliminated. As s_1 is not eliminated till the end, s_2 cannot be a never-best response on <u>any</u> round and makes it till the end. Hence is rationalizable.

(e) Think s_1 = Don't cooperate for player 1 $\quad s_2$ = cooperate for player 2 in the Prisoner's Dilemma

s_2 is strictly dominated yet s_1 is a BR to s_2 too.

[7]

(f) No, str. dominant str. must be pure;

if $u_i\left(\frac{1}{2}A+\frac{1}{2}B, \sigma_{-i}\right) > u_i(s_i, \sigma_{-i})$ $\forall \sigma_{-i}$ & $\forall s_i$

as $u_i\left(\frac{1}{2}A+\frac{1}{2}B, \sigma_{-i}\right) = \frac{1}{2}u_i(A, \sigma_{-i}) + \frac{1}{2}u_i(B, \sigma_{-i})$

and $u_i\left(\frac{1}{2}A+\frac{1}{2}B, \sigma_{-i}\right) > u_i(A, \sigma_{-i})$

$u_i\left(\frac{1}{2}A+\frac{1}{2}, \sigma_{-i}\right) > u_i(B, \sigma_{-i})$ cannot hold.

(g)

	L	R
A	3,	0,
B	0,	3,
C	2,	2,

C is not a BR to L or R

but is a BR to $\frac{1}{2}L+\frac{1}{2}R$

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[8]

a)

2,1	0,0
3,1	$\boxed{1}$,2

has unique pure NE

player 1 gets $\boxed{1}$

3 decreased to 1

$\boxed{1}$ to $\boxed{-1}$

$\Longrightarrow$

2,1	0,0
1,1	$\boxed{-1}$,2

This game has unique pure NE payoff (2,1)

player 1 gets now 2 rather than 1 !!!

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b)

2,1	0,0
3,1	1,2

has unique pure NE where player 1 gets $\boxed{1}$

$\Longrightarrow$

2,1	0,0
~~3,1~~	~~1,2~~

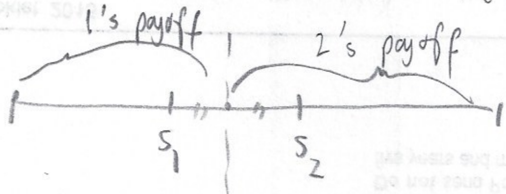
bottom strategy is eliminated

player 1 gets $\boxed{2}$ in the unique NE of the altered game

$\boxed{2} > \boxed{1}$

[6] a) Let s_1, s_2 vendor location choices; $u_1(s_1,s_2) = \begin{cases} \frac{s_1+s_2}{2} & s_1<s_2 \\ \frac{1}{2} & s_1=s_2 \\ 1-\frac{s_1+s_2}{2} & s_1>s_2 \end{cases}$

$u_1(s_1,s_2)$



Pure NE is $s_1 = s_2 = \frac{1}{2}$. Proof:

1) They choose the same location: as, if not, one could get closer to the other & grab more customers.

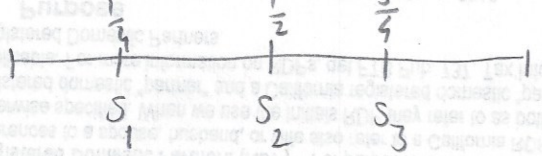
2) If $s_1 = s_2 \neq \frac{1}{2}$ then one could deviate to the direction of majority of customers, and get more than $\frac{1}{2}$ (and not share the market)

(b) 3 vendors: (1) applies; the outer player(s) can sneak closer to the others to increase share., hence $s_1 = s_2 = s_3$

But now, each get $\frac{1}{3}$. Any player can move in the direction of majority of customers and get almost $\frac{1}{2} > \frac{1}{3} \to \leftarrow$ No NE

(c) Now customers within $\frac{1}{4}$ can be grabbed. In (c), the move towards the majority does not apply because you only guarantee $\frac{1}{4}$ customers $\frac{1}{4} < \frac{1}{3}$.

$\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right)$ is the unique NE: because;



It is a NE: 1 or 3 wouldn't want to get closer to 2 (b.c of the distance limit of customers), similarly 2 wouldn't move right or left (he is indifferent!)

Uniqueness requires checking: $s_1 = s_2 = s_3$ cannot happen (see b), two at same location, $s_1 = s_2 < s_3$ cannot happen.