MATH 425b ASSIGNMENT 7 SOLUTIONS
SPRING 2016
Prof. Alexander

Chapter 9:

(29) If we interchange i_j and i_{j+1} for some $j \geq 1$, we get

$$D_{i_1 \ldots i_j i_{j+1} \ldots i_k} f = D_{i_1 \ldots i_{j-1}}(D_{i_j i_{j+1}}(D_{i_{j+2} \ldots i_k} f)). \tag{1}$$

Since $f \in \mathcal{C}^k$, we have $D_{i_{j+2} \ldots i_k} f \in \mathcal{C}^{j+1} \subset \mathcal{C}_2$ so $D_{i_j i_{j+1}}(D_{i_{j+2} \ldots i_k} f) = D_{i_{j+1} i_j}(D_{i_{j+2} \ldots i_k} f)$ by Theorem 9.41. As in (1),

$$\begin{aligned} D_{i_1 \ldots i_{j+1} i_j \ldots i_k} f &= D_{i_1 \ldots i_{j-1}}(D_{i_{j+1} i_j}(D_{i_{j+2} \ldots i_k} f)) \\ &= D_{i_1 \ldots i_{j-1}}(D_{i_j i_{j+1}}(D_{i_{j+2} \ldots i_k} f)) \\ &= D_{i_1 \ldots i_j i_{j+1} \ldots i_k} f, \end{aligned}$$

so we can switch adjacent indices i_j, i_{j+1}. Any permutation can be constructed out of such switches so indices can be arbitrarily permuted.

(A) Define the basepoint $\mathbf{x} = (2,3,4)$ and the increment $\delta = .01$. We want to approximate the matrix of $f'(\mathbf{x})$. The information $f(2.01,3,4) = (6.99, 6.03, 5.04)$ gives us an increment in the e_1 direction, so the first column of this matrix is

$$(*) \quad f'(\mathbf{x})e_1 \approx \frac{f(\mathbf{x}+\delta e_1) - f(\mathbf{x})}{\delta} = \frac{(6.99, 6.03, 5.04) - (7,6,5)}{.01} = (-1,3,4).$$

The next given information gives an increment in the $e_1 + e_2$ direction, so

$$(**) \quad f'(\mathbf{x})(e_1 + e_2) \approx \frac{f(\mathbf{x}+\delta(e_1+e_2) - f(\mathbf{x})}{\delta} = \frac{(7.01, 6.06, 5.05) - (7,6,5)}{.01} = (1,6,5).$$

Subtracting $(*)$ from $(**)$ gives the second column of the matrix:

$$f'(\mathbf{x})e_2 \approx (1,6,5) - (-1,3,4) = (2,3,1).$$

The last given information gives an increment in the $e_1 + e_2 + e_3$ direction, so

$$(***) \quad f'(\mathbf{x})(e_1+e_2+e_3) \approx \frac{f(\mathbf{x}+\delta(e_1+e_2+e_3) - f(\mathbf{x})}{\delta} = \frac{(7.01, 6.02, 5) - (7,6,5)}{.01} = (1,2,0),$$

and subtracting $(**)$ from $(***)$ gives the third column of the matrix:

$$f'(\mathbf{x})e_3 \approx (1,2,0) - (1,6,5) = (0,-4,-5).$$

Thus the matrix is

$$f'(\mathbf{x}) \approx \begin{bmatrix} -1 & 2 & 0 \\ 3 & 3 & -4 \\ 4 & 1 & -5 \end{bmatrix}.$$

From this we can estimate

$$f(2, 3.01, 4.01) \approx f(2,3,4) + f'(\mathbf{x})(0, .01, .01) \approx (7,6,5) + (.02, -.01, -.04) = (7.02, 5.99, 4.96).$$

(B)(a) Suppose (x, y_1) and (x, y_2) are points of U, with $y_1 < y_2$. Since U is convex, we have $(x, y) \in U$ for all $y \in (y_1, y_2)$. Therefore by the Fundamental Theorem of Calculus,

$$f(x, y_2) - f(x, y_1) = \int_{y_1}^{y_2} \frac{\partial f}{\partial y}(x, y)\, dy = 0.$$

This means that for fixed x, $f(x, y)$ has the same value, call it $g(x)$, for all y such that $(x, y) \in U$.

(b) Let U be the C-shaped region $((0,2) \times (0,3)) \backslash ([1,2) \times [1,2])$. Define f to be 0 on the left side $[(0,1) \times (0,3)$ and on the bottom $[1,2) \times (0,1)$, and define $f(x,y) = (x-1)^2$ on the top, $[1,2) \times (2,3)$. Since the derivatives of 0 and $(x-1)^2$ are both 0 at $x = 1$, the derivatives from the left side and right side match on the line $x = 1, 2 < y < 3$, so f is $\mathcal{C}'$. But f is not constant on any line $x = c$ with $1 < c < 2$, since $f(c, y) = 0$ for $c \in (0,1)$ and $f(c, y) = (c-1)^2$ for $2 < c < 3$.

(C) Let $\mathbf{h} = (h_1, h_2)$ and $\mathbf{h}' = (0, h_2)$. If f is indeed differentiable at $\mathbf{a}$, then the entries of the 1×2 matrix of $f'(\mathbf{a})$ must be the partial derivatives, so $f'(\mathbf{a}) = T$ is then given by

$$Th = (D_1 f)(\mathbf{a})h_1 + (D_2 f)(\mathbf{a})h_2.$$

So we need to show this T really is the derivative. For this we view the increment from $\mathbf{a}$ to $\mathbf{a} + \mathbf{h}$ as a sum of two increments, from $\mathbf{a}$ to $\mathbf{a} + \mathbf{h}'$ and then from $\mathbf{a} + \mathbf{h}'$ to $\mathbf{a} + \mathbf{h}$. This gives

$$\begin{aligned} &|f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - T\mathbf{h}| \\ &\qquad = |f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a} + \mathbf{h}') + f(\mathbf{a} + \mathbf{h}') - f(\mathbf{a}) - (D_1 f)(\mathbf{a})h_1 - (D_2 f)(\mathbf{a})h_2| \\ &\qquad \le |f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a} + \mathbf{h}') - (D_1 f)(\mathbf{a})h_1| + |f(\mathbf{a} + \mathbf{h}') - f(\mathbf{a}) - (D_2 f)(\mathbf{a})h_2|. \end{aligned} \tag{2}$$

The second term in the last line of (2) is $o(|h_2|)$ by the definition of $D_2 f$. For the first term we can use the MVT, with h_2 fixed: for some $\xi \in (0, h_1)$, we have

$$f(\mathbf{a} + \mathbf{h}') - f(\mathbf{a}) = (D_1 f)(a_1 + \xi, a_2 + h_2)h_1,$$

so the first term on the right side of (2) is

$$|(D_1 f)(\mathbf{a} + (\xi, h_2)) - (D_1 f)(\mathbf{a})| \cdot |h_1|. \tag{3}$$

Since $D_1 f$ is continuous at $\mathbf{a}$, we have $|(D_1 f)(\mathbf{a}+(\xi, h_2)) - (D_1 f)(\mathbf{a})| \to 0$ as $\mathbf{h} \to 0$, which means that (3) is $o(|h_1|)$. Thus the right side of (2) is $o(|h_1|) + o(|h_2|) = o(|\mathbf{h}|)$, which proves that $f'(\mathbf{a}) = T$.

A central point here is that under the given assumptions, the intermediate point $\mathbf{a} + \mathbf{h}'$ must use $\mathbf{h}' = (0, h_2)$ and not $(h_1, 0)$, so the vertical increment occurs first and starts from $\mathbf{a}$, because we know nothing about $D_2 f$ at points other than $\mathbf{a}$. If we make the horizontal increment first, then the vertical increment does not start from $\mathbf{a}$ which is a problem.

(D)(a) By the triangle inequality, $\|A\| - \|B\| \leq \|A - B\|$ and $\|B\| - \|A\| \leq \|A - B\|$, so $\big|\|A\| - \|B\|\big| \leq \|A - B\|$. Therefore for $\delta = \epsilon$, we have $\|A - B\| < \delta$ implies $\big|\|A\| - \|B\|\big| < \epsilon$.

(b) By part (a), for $\delta = \epsilon$, we have $\|A - B\| < \delta$ implies $|\psi(A) - \psi(B)| = \big|\|A\| - \|B\|\big| < \epsilon$. This shows that ψ is continuous.

(c) By the Inverse Function Theorem, there exist open sets U_0, V_0 containing $\mathbf{a}$ and $f(\mathbf{a})$ such that f is a $\mathcal{C}'$ bijection of U_0 to V_0 and the inverse g is $\mathcal{C}'$. By (b) and the $\mathcal{C}'$ assumption, $\|g'(z)\|$ is a continuous function of z on V_0. Therefore if we take a ball G containing $f(\mathbf{a})$ with the closure $\overline{G} \subset V_0$, we have $\overline{G}$ compact so $\|g'(z)\|$ is bounded on $\overline{G}$, hence also on G. This means there exists M such that $\|g'(z)\| \leq M$ for all $z \in G$. Let $U = g(G) = f^{-1}(G)$, so U is an open set containing $\mathbf{a}$. By Theorem 9.19, for $w, z \in G$ we have $|g(w) - g(z)| \leq M|w - z|$. Taking $x, y \in U$ and $w = f(x), z = f(y)$, this says $|f(x) - f(y)| \geq M^{-1}|x - y|$.

(E) Let $g(w, x, y) = x^3 + y^3 + w^3 - 3xyw$. We want to show that we can solve $g(w, x, y) = 0$ for w as $w = f(x, y)$ in a neighborhood of $(0, -1, 1)$, with f differentiable. Since g is $\mathcal{C}'$, by the Implicit Function Theorem it is enough to show that the left part A_w of the matrix A of g' is invertible at $(w, x, y) = (0, -1, 1)$. This matrix is $A = \begin{bmatrix} 3w^2 - 3xy & 3x^2 - 3wy & 3w^2 - 3xw \end{bmatrix}$ so A_w is just the 1×1 matrix $3w^2 - 3xy$, which is equal to 3 at $(0, -1, 1)$. Since this value is nonzero, we have the necessary invertibility.

(F) By definition, $g(y)$ is the value for which $f(g(y), y) = 0$. By the Chain Rule we have

$$0 = \frac{d}{dy} f(g(y), y) = f'(g(y), y) \begin{bmatrix} g'(y) \\ 1 \end{bmatrix}.$$

For $y = b$ we have $g(y) = a$ so this becomes $f'(a, b) \begin{bmatrix} g'(b) \\ 1 \end{bmatrix} = 0$. Since f is real-valued, $f'(a, b)$ is just the gradient ∇f (given as a 1×2 matrix), and the matrix product is the dot product of ∇f and $(g'(b), 1)$. Hence the 0 value of this dot product says $\nabla f \perp (g'(b), 1)$.

(G)(a) Let $f = (f_1, f_2)$. In the notation of the Implicit Function Theorem, the matrices at $\mathbf{x} = (3, 2), \mathbf{y} = (1, 1, 2)$ are

$$A_x = \begin{bmatrix} -2x_1 & 2x_2 \\ 2x_1 & 4x_2 \end{bmatrix} = \begin{bmatrix} -6 & 4 \\ 6 & 8 \end{bmatrix}, \qquad A_y = \begin{bmatrix} 2y_1 & 2y_2 & 2y_3 \\ 2y_1 & -2y_2 & 2y_3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ 2 & -2 & 4 \end{bmatrix}.$$

Since A_x is invertible, we can locally solve for $\mathbf{x} = g(\mathbf{y})$ with

$$g'((1,1,2)) = -A_x^{-1}A_y = \frac{1}{72}\begin{bmatrix} 8 & -4 \\ -6 & -6 \end{bmatrix}\begin{bmatrix} 2 & 2 & 4 \\ 2 & -2 & 4 \end{bmatrix} = \frac{1}{72}\begin{bmatrix} 8 & 24 & 16 \\ -24 & 0 & -48 \end{bmatrix}.$$

This means that if we move $\mathbf{y}$ in direction $\Delta\mathbf{y} = (0,1,1)$, $\mathbf{x}$ must move in direction

$$\Delta\mathbf{x} = g'((1,1,2))\Delta\mathbf{y} = \frac{1}{72}\begin{bmatrix} 8 & 24 & 16 \\ -24 & 0 & -48 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/9 \\ -2/3 \end{bmatrix}.$$

(b) We need

$$f'(x,y)\begin{bmatrix} \mathbf{h} \\ \mathbf{k} \end{bmatrix} = 0,$$

that is, $A_y\mathbf{k} = -A_x\mathbf{h}$. This can be solved for $\mathbf{k}$ if and only if $A_x\mathbf{h}$ is in the range of A_y, which is the span of the columns $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$. In other words, $\mathbf{h}$ must be in the span of $A_x^{-1}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/9 \\ 1/3 \end{bmatrix}$ and $A_x^{-1}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$. But this span is two-dimensional so $\mathbf{h}$ can be arbitrary.

(c) A_x and A_y are still given by the formulas in (a) as functions of $\mathbf{x}$ and $\mathbf{y}$, which at the new values $\mathbf{x} = (3,2), \mathbf{y} = (1,0,2)$ gives

$$A_x = \begin{bmatrix} -6 & 4 \\ 6 & 8 \end{bmatrix}, \qquad A_y = \begin{bmatrix} 2 & 0 & 4 \\ 2 & 0 & 4 \end{bmatrix}.$$

The range of A_y now consists only of multiples of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ so $\mathbf{h}$ must be a multiple of $A_x^{-1}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/9 \\ 1/3 \end{bmatrix}$.