Appendix | A

Continuous Distributions

Given a random variable X, which takes on values in $[0,\omega]$, we define its cumulative *distribution function* $F:[0,\omega] \to [0,1]$ by[1]

$$F(x) = \text{Prob}[X \leq x] \tag{A.1}$$

the probability that X takes on a value not exceeding x. By definition, the function F is nondecreasing and satisfies $F(0) = 0$ and $F(\omega) = 1$ (if $\omega = \infty$, then $\lim_{x\to\infty} F(x) = 1$). In this book we always suppose that F is increasing and continuously differentiable.

The derivative of F is called the associated probability *density function* and is usually denoted by the corresponding lowercase letter $f \equiv F'$. By assumption, f is continuous and we will suppose, in addition, that for all $x \in (0,\omega)$, $f(x)$ is positive. The interval $[0,\omega]$ is called the *support* of the distribution. When (A.1) holds we will say that X is distributed according to the distribution F or, equivalently, according to the density f.

If X is distributed according to F, then the *expectation* of X is

$$E[X] = \int_0^\omega x f(x)\, dx$$

and if $\gamma : [0,\omega] \to \mathbb{R}$ is some arbitrary function, then the expectation of $\gamma(X)$ is analogously defined as

$$E[\gamma(X)] = \int_0^\omega \gamma(x) f(x)\, dx$$

[1]We will allow for the possibility that X can take on any nonnegative real value. In that case, with a slight abuse of notation, we will write $\omega = \infty$.

Sometimes the expectation of $\gamma(X)$ is also written as

$$E[\gamma(X)] = \int_0^{\omega} \gamma(x)\, dF(x)$$

The *conditional expectation* of X given that $X < x$ is

$$E[X \mid X < x] = \frac{1}{F(x)} \int_0^x tf(t)\, dt$$

and so

$$\begin{aligned} F(x)E[X \mid X < x] &= \int_0^x tf(t)\, dt \\ &= xF(x) - \int_0^x F(t)\, dt \end{aligned} \tag{A.2}$$

which is obtained by integrating the right-hand side of the first equality by parts. The formula in (A.2) shows that $F(x)E[X \mid X < x]$ is the shaded area lying above the curve F in Figure A.1.

HAZARD RATES

Let F be a distribution function with support $[0, \omega]$. The *hazard rate* of F is the function $\lambda : [0, \omega) \to \mathbb{R}_+$ defined by

$$\lambda(x) \equiv \frac{f(x)}{1 - F(x)}$$

If F represents the probability that some event will happen before time x, then the hazard rate at x represents the instantaneous probability that the event will happen at x, given that it has not happened until time x. The event may be the

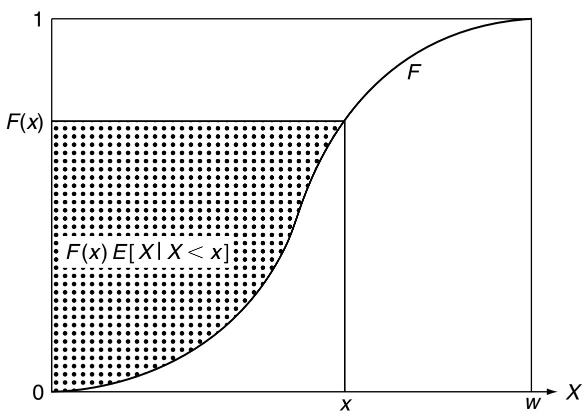

FIGURE A.1 Conditional expectation.

failure of some component—a lightbulb, for instance—and so it is sometimes also known as the "failure rate." Notice that as $x \to \omega$, $\lambda(x) \to \infty$.

Since

$$-\lambda(x) = \frac{d}{dx} \ln(1 - F(x))$$

if we write

$$F(x) = 1 - \exp\left(-\int_0^x \lambda(t)\,dt\right) \tag{A.3}$$

then this shows that any arbitrary function $\lambda : [0, \omega) \to \mathbb{R}_+$ such that for all $x < \omega$,

$$\int_0^x \lambda(t)\,dt < \infty$$

and

$$\lim_{x \to \omega} \int_0^x \lambda(t)\,dt = \infty \tag{A.4}$$

is the hazard rate of some distribution; in particular, that defined by (A.3). The fact that $\lambda(x) \geq 0$ ensures that F is nondecreasing and (A.4) ensures that $F(\omega) = 1$. Using the formula in (A.2), it may be verified that $E[X] = E[1/\lambda(X)]$.

If for all $x \geq 0$, $\lambda(x)$ is a constant, say $\lambda(x) \equiv \lambda > 0$, then (A.3) results in the *exponential distribution*

$$F(x) = 1 - \exp(-\lambda x)$$

whose expectation $E[X] = 1/\lambda$.

Closely related to the hazard rate is the function $\sigma : (0, \omega] \to \mathbb{R}_+$ defined by

$$\sigma(x) \equiv \frac{f(x)}{F(x)}$$

sometimes known as the *reverse hazard rate*.[2] Since

$$\sigma(x) = \frac{d}{dx} \ln F(x)$$

if we write

$$F(x) = \exp\left(-\int_x^\omega \sigma(t)\,dt\right) \tag{A.5}$$

[2]In some applications this is also referred to as the inverse of the *Mills' ratio*.

then this shows that any arbitrary function $\sigma : (0,\omega) \to \mathbb{R}_+$ such that for all $x > 0$,

$$\int_x^\omega \sigma(t)\,dt < \infty$$

and

$$\lim_{x\to 0} \int_x^\omega \sigma(t)\,dt = \infty \tag{A.6}$$

is the reverse hazard rate of some distribution; in particular, that defined by (A.5). The fact that $\sigma(x) \geq 0$ ensures that F is nondecreasing and (A.6) ensures that $F(0) = 0$.

JOINTLY DISTRIBUTED RANDOM VARIABLES

Let X and Y be two random variables taking on values in $[0,\omega_X]$ and $[0,\omega_Y]$, respectively. We will say that X and Y have the *joint density* $f : [0,\omega_X] \times [0,\omega_Y] \to \mathbb{R}_+$ if for all $x' < x''$ and $y' < y''$

$$\text{Prob}\left[x' \leq X \leq x'' \text{ and } y' \leq Y \leq y''\right] = \int_{y'}^{y''} \int_{x'}^{x''} f(x,y)\,dx\,dy$$

We will then say that X and Y are jointly distributed according to f. We will assume that f is continuous and positive on $(0,\omega_X) \times (0,\omega_Y)$.

The *marginal density* of X is

$$f_X(x) = \int_0^{\omega_Y} f(x,y)\,dy$$

and the marginal density of Y is similarly defined. The random variables X and Y are *independent* if and only if

$$f(x,y) = f_X(x) \times f_Y(y)$$

For any $x > 0$, the *conditional density* of Y given that $X = x$ is

$$f_Y(y \mid X = x) = \frac{f(x,y)}{f_X(x)}$$

and for any $x > 0$, the *conditional expectation* of Y given that $X = x$ is defined as

$$E[Y \mid X = x] = \int_0^{\omega_Y} y f_Y(y \mid X = x)\,dy$$

Let us denote by $E[Y \mid X] : [0, \omega_X] \to \mathbb{R}_+$ the function of X whose value at $X = x$ is $E[Y \mid X = x]$. The function $E[Y \mid X]$ is then also a random variable and it is meaningful to speak of its expectation. Using the preceding definitions, it can be verified that

$$E_X[E_Y[Y \mid X]] = E_Y[Y]$$

This identity is sometimes known as the "law of iterated expectation."

Extensions to an arbitrary finite number of random variables are straightforward.

NOTES ON APPENDIX A

The material on continuous random variables is quite standard and can be found in any reasonable book on probability theory. Ross (1989) has presented a concise treatment of the relevant concepts and results.