Bayesian Equilibrium and Auction Design

The analysis of auctions uses the same Bayesian Nash equilibrium concept we use in this course. Buyers do not know how other buyers value the object, but know their own value for the object for sale (private information). A bid contingent on your value is optimal if it maximizes a buyer's expected profit. The trade-off here is that higher bid raises the probability of winning the object, but lower the profit after winning the object.

In the auction context, the type of bidders or buyers is modelled as a distribution over an interval. In other words, there are a continuum of types of bidders. A type of bidder is distinguished by his or her valuation for the object. Let v be the buyer valuation of the object for sale in an auction. The value v can be any number in an interval $[a,b]$. There is a c.d.f. function $F(v)$ over the interval $[a,b]$. A buyer value is a sample taken from this distribution. Auctions are important because they are more familiar ways to price an object with uncertainty about its value. There are many ways auctions are designed or practiced.

Sealed bid first-price auction: Bids are submitted in sealed envelopes, then opened. If your bid is the highest, you are the winner. When you win, you pay your own bid. When bids are sealed, they are not known to other bidders. It is like a simultaneous game, in which you choose your bid without knowing other buyers' bid.

For simplicity, assume the seller has no value for the object (cost is sunk).

Bidders' value distribution $F(v)$ for the object.

We assume private-value, meaning that your value depends only on the information you receive, and is independent of other bidders' value (which depends on the information they receive). In other words, your value does not change when you get to know other's value for the object. This may apply when you place a high value on an art object even though you know other buyers have a lower valuation for it. In the real world, this assumption may be quite restrictive, especially when there is a resale opportunity. For example, you have low value for the object, but if you know someone is willing to pay a high price for it, you may bid differently after knowing this information. In this case, bidding may differ from the one based on private values. The private-value assumption is no longer valid if a buyer intends to resell the good to others after winning the auction. This is because the resale price depends on others' valuation, violating the private-value assumption. In fact, when we allow resale after auctions for a good which satisfies the private-value assumption, the bidding process often becomes a common-value auction.

We now give a simple example of equilibrium in the first-price auction. Suppose there are two bidders. Each buyer has valuation uniformly distributed over [0,1]. The bidding strategy b(v)=0.5v will be shown to be a Bayesian Nash equilibrium.

In this example we let $F(v) = v$. This is the uniform distribution over [0,1]. We want to verify that the bidding strategy $b(v) = \frac{v}{2}$ for each buyer is a Bayesian Nash Equilibrium. To do this, we consider a buyer with value v bidding b. The probability of winning for the buyer is $2b$. The profit after winning is $v - b$. Hence the expected profit is

$$(v-b)2b.$$

The buyer chooses b to maximize the above function of b (v is a constant here). Setting the (partial) derivative with respect to b equal 0, we get

$$-2b + 2v - 2b = 0,$$

and we get $b = 0.5v$. Hence the optimal bid is to bid half of your value. This proves that $b(v) = 0.5v$ is an equilibrium bidding strategy. More generally, equilibrium strategies can be found in the following way.

Two bidders competing for the object.
Strategies: $b_i : v \to b_i(v), i = 1,2$
The payoff after winning is

$$v - b.$$

If the probability of winning is $Q(b)$, then the expected payoff is

$$(v - b)Q(b).$$

We are assuming risk neutral behavior here. The probability that you win depends on how the rival buyer bids. The payoff matrix has infinitely many rows and columns.

If $b_2(v)$ is the bidding strategy of the rival buyer, and is increasing, let $\phi_2(b)$ be the inverse bidding strategy.

Then the probability of winning of the first bidder is $Q(b) = F(\phi_2(b)) = \phi_2(b)$. A buyer with value v_1 then chooses b to maximize the payoff

$$u(v_1, b) = (v_1 - b)\phi_2(b). \qquad \#$$

The first-order condition for the optimal bid b is given by

$$(v_1 - b)\phi_2'(b) - \phi_2(b) = 0,$$

or

$$\frac{\phi_2'(b)}{\phi_2(b)} = \frac{1}{v_1 - b} = \frac{1}{\phi_1(b) - b}.$$

Similarly, let $b_1(.), \phi_1(.)$ be the bidding and inverse bidding strategy of the first bidder. The first-order condition of the second bidder is

$$\frac{\phi_1'(b)}{\phi_1(b)} = \frac{1}{\phi_2(b) - b}.$$

Assume that both bidders bid the same way, so that $b_1(.) = b_2(.) = b(.), \phi_1(.) = \phi_2(.) = \phi(.)$. We are looking for the solution of the following differential equation

$$\frac{\phi'(b)}{\phi(b)} = \frac{1}{\phi(b) - b}.$$

with the boundary condition $\phi(0) = 0$. It can be shown that we have the unique solution

$$\phi(b) = 2b.$$

This solution then gives us the equilibrium bidding strategy of each buyer

$$b(v) = \frac{v}{2}.$$

In auctions, we are interested in the expected revenue of the seller. If the equilibrium bid distribution is expressed by the cumulative distribution $G(v)$, then the expected revenue is given by

$$\int_0^{b^*} bdG(b),$$

where b^* is the highest equilibrium bid. The highest equilibrium bid in this example is 0.5, and $G(b) = (2b)^2 = 4b^2$. Hence the expected revenue is given by

$$\int_0^{0.5} bd(4b^2) = \int_0^{0.5} 8b^2 db = \frac{1}{3}.$$

Another way of computing the revenue is to compute the revenue contribution of each buyer. A buyer with value v pays $0.5v$ with probability v, hence the expected revenue contribution of a buyer is

$$\int_0^1 (0.5v * v)dF(v) = 0.5\int_0^1 v^2 dv = \frac{1}{6}.$$

There are two buyers, hence the total revenue in this computation is also $\frac{1}{6}$. This alternative way of computing the revenue is

$$2\int_0^1 b(v)F(v)dF(v),$$

More generally, the revenue formula is

$$n\int_\alpha^\beta b(v)F^{n-1}(v)dF(v) = \int_\alpha^\beta b(v)dF^n(v), \qquad \#$$

when n is the number of bidders, and the buyer value distribution is $F(v)$ over $[\alpha, \beta]$, and $b(v)$ is the equilibrium bidding strategy.

First-Price Auction with a reservation price ρ :

Any bid below ρ would not win. A buyer with value $v < \rho$ would not bid (or bid 0). For $v > \rho$, the payoff from bidding b is given by the same formula (ref: a1). The first-order condition is the same. The differential equation for the symmetric equilibrium strategy is the same. But the boundary condition is $b(\rho) = \rho$. The solution of the differential equation is given by,

$$b(v) = v - \frac{1}{F(v)}\int_\rho^v F(x)dx = v - \frac{1}{v}\int_\rho^v xdx, v \geq \rho$$

Hence we have the equilibrium bidding strategy strategy

$$b(v) = v - \frac{1}{v}\frac{1}{2}(v^2 - \rho^2) = \frac{1}{2}v + \frac{\rho^2}{2v}.$$

Thus higher reservation price increases the equilibrium bid of the buyers. The increase in bid is higher for buyers with a smaller value. The higher reservation price also can depress revenue because the buyer with value lower than ρ would not be able to contribute to the revenue of the seller. There is a trade-off.

Now we have $b^* = 0.5(1 + \rho)$. The revenue of the seller with the reservation price ρ is given by

$$\int_{\rho}^{1} b(v)dv^2 = 2\int_{\rho}^{1} b(v)vdv$$

$$= 2\int_{\rho}^{1}(\frac{1}{2}v + \frac{\rho^2}{2v})vdv = 2\int_{\rho}^{1}(\frac{1}{2}v^2 + \frac{\rho^2}{2})dv$$

$$= \int_{\rho}^{1}(v^2 + \rho^2)dv = \frac{1}{3}(1-\rho^3) + \rho^2(1-\rho).$$

To maximize the seller revenue, we can take the derivative

$$-4\rho^2 + 2\rho = 0,$$

We have the solution of the equation, $\rho = 0.5$. Hence the optimal reservation price is 0.5 for the maximization of revenue.

Second-Price Auctions (sealed bid): Bids are submitted in sealed envelopes. If your bid is the highest, you are the winner, you only pay the second highest bid, not your own bid. The winner's payment does not depend on the bidder's own bid (as long as s/he is the winner), but are affected by others' bids. This is unlike the case of first-price auctions.

It is easy to see that the bidding strategy b(v)=v is a Bayesian Nash equilibrium strategy. To see this, assume the other buyer bids truthfully. Note that if a buyer with value v bids x, then the probability of winning is x, but the buyer pays a random price uniformly distributed over [0,x].

Therefore the expected price to pay after winning is 0.5x. The expected profit of bidding x is then x(v-0.5x). Taking the derivative with respect to x, we have the first-order condition v-x=0, or v=x, hence the optimal bid is to bid truthfully. In other words, bidding truthfully by each buyer is a Bayesian Nash equilibrium strategy in the second-price sealed bid auction.

In fact it is also a dominant strategy for each buyer to bid truthfully. Given any bidding strategy of the other buyer, you don't want to bid higher than your value v, as any additional winning opportunity yields a loss (you pay a price higher than v).

You don't want to bid below v either, as you can bid slightly higher and any additional winning opportunity yields a positive profit (you pay a price lower than v).

Therefore bidding truthfully is a dominant strategy (independent of how the other bidder bids in the second-price sealed bid auction).

A more formal verification of the equilibrium bidding strategy: $b(v) = v$,

The payoff from bidding b is

$$\int_{0}^{b}(v-x)dx = bv - \frac{b^2}{2}$$

Taking the derivative with respect to b, we get

$$v - b = 0,$$

or $b = v$. That is, the optimal bid for a buyer with value v is to bid v.

In a symmetric equilibrium (both buyers bid the same way) when a buyer bids x, the winning probability is $F(x) = x$. The winner pays an amount which is the random sample of the bid in $[0,x]$. Hence the expected payment after winning is $\frac{1}{2}x$. Contribution to revenue from one buyer is

$$\int_0^1 \frac{1}{2}x * xdF(x) = \frac{1}{2}\int_0^1 v^2 dv = \frac{1}{6}.$$

hence the total revenue is also $\frac{1}{3}$.

The revenue is the same as that of a first-price auction, even though the payment rules are not the same. In second-price auctions, you pay less after you win. But knowing this, buyers are willing to bid higher. In equilibrium, this higher bidding leads to the same expected amount of payment to the seller. The seller will get the same expected revenue from either the first-price auction or the second-price auction. In fact the revenue is the same if you use the third-price auction. There is a revenue equivalence result saying that:

Any auction is revenue equivalent as long as the lowest value buyer receives 0 payoff in equilibrium, and the winner is the highest bidder.

English auction is a dynamic open auction in which buyers bid until no one bids any further.

The one with the highest bid wins and pays the highest current bid which depends on the highest value of others rather than the buyer’s own valuation.

An open English auction is (strategically) equivalent to the sealed bid second-price auction when bidders’ valuation are independent from each other (called a private-value model).

In online auctions, a common practice is called a proxy bid by which a buyer chooses the maximum amount s/he will bid to the program. The computer automatically raises the bid for the buyer as long as the current highest bid is below the maximum.

A proxy bid makes it convenient for the buyers as there is no need to monitor the bidding process constantly.

The process of determining a maximum bid (which is hidden from other bidders just as in a sealed bid auction) is the same as submitting a single sealed bid.

Hence this online auction (most common ones online) is also strategically equivalent to a second-price sealed bid auction when buyers’ values are private.

Second-price auctions (without a minimum bid) are efficient, but may not give the seller the highest expected selling price. We have seen above that the revenue is higher with a reservation price. Efficiency objective is different from revenue objectives. There are trade-offs between the two objectives.

There are now auctions which allow you to win an iPad for less than $40 dollars. It is an all-pay auction meaning all bidders pay even if they don’t win the object. Such auctions often yield a higher revenue for the seller (without violating the revenue equivalence because the value 0 buyer gets a negative payoff).

Ideas of second-price auctions to reveal true preferences have been used in many other issues.

In school choice problems, parents may reveal their preferences for their kids to attend certain schools. Such preference revelations are considered important for school districts to evaluate the quality of their schools.

In order to have a true measure of evaluation of school quality, the system needs to induce the parents to reveal their true preferences.

The idea of a truthful revelation mechanism has useful applications to the school allocation problem. The way you allocate the kids to schools according to the reported preferences affect the willingness to report their true preferences to school authorities.

In voting models, or issues of the choice of public goods, there is also a need to design a mechanism so that people are willing to reveal their true value for a public project such a building a bridge.

Homework #5
due on Feb 15

1. (a) Assume that there are two buyers in a first-price auction, and each buyer value distribution is given by the uniform distribution $F(v) = 0.25v$ over the interval $[0,4]$, verify that $b(v) = 0.5v$ is also the equilibrium bidding strategy of each buyer.

(b) Compute the revenue by the formula (ref: a2).

2. Assume that there are two buyers, and each buyer has the value distribution $F(v) = 0.5v$ defined in $[0,2]$. Consider the sealed bid first-price auction .

(a) Verify that the bidding strategy $b(v) = 0.5v$ satisfies the first-order condition.

(b) Compute the seller's revenue from the auction.

3. Assume that each buyer has the value distribution $F(v) = 0.5v$ defined in $[0,2]$. Consider the sealed bid second-price auction with two buyers.

(a) Verify that the bidding strategy $b(v) = v$ is a Bayesian Nash equilibrium.

(b) Compute the seller's revenue from the auction, and compare with the revenue from the first-price auction in the problem above.