Chapter | Eighteen

Multiple Objects and Interdependent Values

Our treatment of multiple-object auctions has been confined to the case of private values. Even in this relatively simple setting, we saw that the question of efficiency was a delicate one. On the other hand, we also saw that the question of efficiency in single-object auctions with interdependent values was delicate as well. In this chapter we study multiple-object auctions when buyers have interdependent values. As we will see, a combination of these two features only makes the attendant difficulties more acute, even insurmountable. In some sense we have reached the limits of what auction-like mechanisms can accomplish in terms of allocating efficiently, at least at a general level. It is perhaps fitting, therefore, that this is the last chapter.

It turns out that the problems associated with interdependence arise not from the multiplicity of the objects *per se* but rather from the multiplicity of signals that each buyer receives—that is, on the dimensionality of the available information. We begin by looking at situations in which there are multiple objects for sale but each buyer receives only a one-dimensional signal. We then look at the case of multidimensional information.

18.1 ONE-DIMENSIONAL SIGNALS

Our basic setup is the same as in Chapter 10. Specifically, there are K identical objects for sale and N potential buyers. Prior to the sale, each buyer receives a one-dimensional signal $x^i \in \mathcal{X}^i \equiv \left[0, \omega^i\right]$. Buyer i's valuations for the objects depend on the signals $\mathbf{x} = \left(x^1, x^2, \ldots, x^N\right)$ received by all the buyers and the *marginal value* of obtaining the kth unit is determined by the function

$$v_k^i(\mathbf{x}) = v_k^i\left(x^1, x^2, \ldots, x^N\right)$$

We suppose that the marginal values for successive units decline so that for all $k < K$,

$$v_k^i(\mathbf{x}) \geq v_{k+1}^i(\mathbf{x})$$

and that a buyer's valuations respond nonnegatively to all signals—that is, for all i,j, and k,

$$\frac{\partial v_k^i}{\partial x^j}(\mathbf{x}) \geq 0 \tag{18.1}$$

and positively to his own signal—that is, (18.1) holds with a strict inequality whenever $i = j$. We will write $\mathbf{v}^i(\mathbf{x}) = \left(v_1^i(\mathbf{x}), v_2^i(\mathbf{x}), \ldots, v_K^i(\mathbf{x})\right)$ to denote the vector of marginal valuations of buyer i when the signals are $\mathbf{x}$.

The vector valuations $\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^N$ are said to satisfy the multiunit *single crossing condition* if for all j, for all $i \neq j$, and for all k and l,

$$\frac{\partial v_k^j}{\partial x^j}(\mathbf{x}) > \frac{\partial v_l^i}{\partial x^j}(\mathbf{x}) \tag{18.2}$$

at every $\mathbf{x}$ such that $v_k^j(\mathbf{x})$ and $v_l^i(\mathbf{x})$ are equal and among the K highest of the $N \times K$ marginal values at $\mathbf{x}$.

We are interested in determining circumstances in which efficient allocations may be achieved. In this regard the importance of the single crossing condition has already been pointed out in the single object context.

18.1.1 An Efficient Direct Mechanism

Consider the following direct mechanism, obtained by combining elements from the generalized Vickrey-Clarke-Groves (VCG) mechanism for a single unit that was introduced in Chapter 10 and the Vickrey multiunit auction that was introduced in Chapter 12. Each buyer is asked to report his signal x^i. The K objects are then awarded efficiently relative to these reports—they are awarded to the K highest marginal values, when evaluated at the reported signals. Formally, given the signals $\mathbf{x}$, the $N \times K$ values $\left\{v_k^i(\mathbf{x}) : i = 1, 2, \ldots N;\ k = 1, 2, \ldots, K\right\}$ are computed and the K units are awarded to the K highest of these values—that is, if buyer i has $k^i \leq K$ of the K highest values, then i is awarded k^i units.

Fix the signals $\mathbf{x}^{-i}$ of the other buyers. For any signal z^i of buyer i define $\mathbf{c}^{-i}\left(z^i, \mathbf{x}^{-i}\right)$ to be the vector of *competing bids* facing buyer i. This is obtained by rearranging the $(N-1)K$ values $v_k^j\left(z^i, \mathbf{x}^{-i}\right)$ of buyers $j \neq i$ in decreasing order and selecting the first K of these. Thus, $c_1^{-i}\left(z^i, \mathbf{x}^{-i}\right)$ is the highest of the others' values, $c_2^{-i}\left(z^i, \mathbf{x}^{-i}\right)$ is the second-highest, and so on. Notice that the single-crossing condition (18.2) implies that for all i, k, and l the function $v_k^i\left(\cdot, \mathbf{x}^{-i}\right)$ crosses any $c_l^{-i}\left(\cdot, \mathbf{x}^{-i}\right)$ at most once and when it does, the former has a greater slope than does the latter.

Suppose that when buyer i reports his signal as x^i and the others report $\mathbf{x}^{-i}$, he wins $k^i \leq K$ units. For any $k \leq k^i$, define

$$y_k^i\left(\mathbf{x}^{-i}\right) = \inf\left\{z^i : v_k^i\left(z^i, \mathbf{x}^{-i}\right) \geq c_{K-k+1}^{-i}\left(z^i, \mathbf{x}^{-i}\right)\right\}$$

to be the smallest signal for i that would result in his winning k units. By definition if $l < k$, then

$$y_l^i\left(\mathbf{x}^{-i}\right) \leq y_k^i\left(\mathbf{x}^{-i}\right)$$

Also, if buyer i wins k^i units when he reports his true signal x^i, then

$$x^i \leq y_{k^i+1}^i\left(\mathbf{x}^{-i}\right)$$

Now define

$$p_k^i = c_{K-k+1}^{-i}\left(y_k^i\left(\mathbf{x}^{-i}\right), \mathbf{x}^{-i}\right)$$

In the generalized VCG mechanism, i is asked to pay an amount

$$\sum_{k=1}^{k^i} p_k^i = \sum_{k=1}^{k^i} c_{K-k+1}^{-i}\left(y_k^i\left(\mathbf{x}^{-i}\right), \mathbf{x}^{-i}\right),$$

where as just defined, $y_k^i\left(\mathbf{x}^{-i}\right)$ is the smallest signal for i that would result in his winning k units. A buyer who does not win any units does not pay anything.

As an illustration, consider the situation in Figure 18.1 in which buyer i's valuations for the first and second unit are depicted as functions of his own signal. The competing bids c_K^{-i} and c_{K-1}^{-i} are also depicted, again as functions of i's signal. The signals of other buyers are fixed at some $\mathbf{x}^{-i}$. Suppose that buyer i

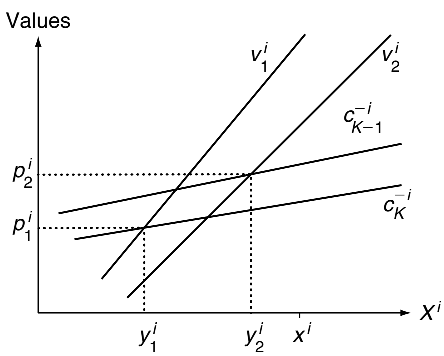

FIGURE 18.1 The generalized VCG mechanism for multiple units.

wins two units when he reports his signal as x^i. Thus, $v_2^i(x^i, \mathbf{x}^{-i}) > c_{K-1}^{-i}(x^i, \mathbf{x}^{-i})$, but $v_3^i(x^i, \mathbf{x}^{-i}) < c_{K-2}^{-i}(x^i, \mathbf{x}^{-i})$, so that he defeats two competing bids but not the third. To determine how much he should pay for the two units, first suppose his signal is 0. In that case, he would not win any units. Now raise i's signal until it reaches a level y_1^i such that he would just win a unit—that is, when $v_1^i = c_K^{-i}$. The price buyer i is asked to pay for the first unit is $p_1^i = c_1^{-i}(y_1^i, \mathbf{x}^{-i})$. Now raise his signal further until it reaches a level y_2^i such that he would just win two units—that is, when $v_2^i = c_{K-1}^{-i}$ and this occurs when i's signal is y_2^i. The price that buyer i is asked to pay for the second unit that he wins is $p_2^i = c_{K-1}^{-i}(y_2^i, \mathbf{x}^{-i})$.

Figure 18.2 portrays the same situation from a different perspective by emphasizing the relationship of the generalized VCG mechanism to the Vickrey multiunit auction. The left-hand panel shows buyer i's value vector—or equivalently, his true demand function—together with the vector of competing bids—or equivalently, the residual supply function facing buyer i. The demand function depends on i's signal but because values are interdependent, so does the residual supply function. As shown in the left-hand panel, buyer i wins two units. The price paid for the first unit is determined by finding the lowest signal such that his highest value v_1^i equals the lowest competing bid c_K^{-i}. As depicted in the middle panel of Figure 18.2, this occurs when $z^i = y_1^i$ and the buyer is asked to pay the amount $p_1^i = c_K^{-i}(y_1^i, \mathbf{x}^{-i})$ for the first unit. Now z^i is raised some more until it reaches a level such that he would win exactly two units. As depicted in the right-hand panel of the figure, this occurs when $z^i = y_2^i > y_1^i$ since we then have $v_2^i(y_2^i, \mathbf{x}^{-i}) = c_{K-1}^{-i}(y_2^i, \mathbf{x}^{-i})$ and, by the single crossing condition, $v_1^i(y_2^i, \mathbf{x}^{-i}) > c_K^{-i}(y_2^i, \mathbf{x}^{-i})$. The buyer is asked to pay an amount $p_2^i = c_{K-1}^{-i}(y_2^i, \mathbf{x}^{-i})$ for the second unit. Since $y_1^i < y_2^i < x^i$, at these prices, the buyer makes a positive surplus on each unit that he wins.

As in the Vickrey multiunit auction, the number of units that buyer i wins when he reports a signal z^i and the others report $\mathbf{x}^{-i}$ is equal to the number

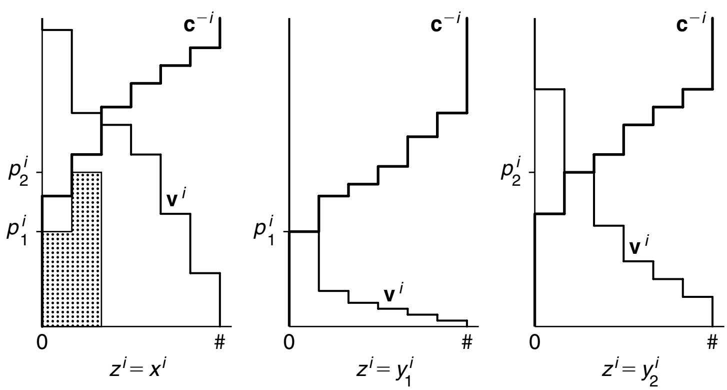

FIGURE 18.2 Prices in the generalized VCG mechanism.

of competing bids that he defeats. Likewise, the prices that buyer i pays are determined by the competing bids he defeats. But the manner in which prices are determined needs to be amended in order to account for the fact that values are now interdependent. The original Vickrey pricing rule of asking buyers to pay the competing bids $c_l^{-i}(x^i, \mathbf{x}^{-i})$ that they defeat would, however, give them the incentive to report lower signals in an attempt to lower the prices they would have to pay. The incentives for truth-telling are restored by making the prices paid by a buyer independent of his own reported signal. This reasoning leads to the following result.

Proposition 18.1. *Suppose that signals are one-dimensional and the valuations* $\mathbf{v}$ *satisfy the multiunit single crossing condition. Then truth-telling is an efficient* ex post *equilibrium of the generalized VCG mechanism.*

Proof. Suppose that when all buyers report their signals $\mathbf{x}$ truthfully, it is efficient for buyer i to be awarded k^i objects. In other words, when evaluated at $\mathbf{x}$, exactly k^i of his values $v_k^i(\mathbf{x})$ are among the K highest of all values.

By reporting his signal truthfully, buyer i pays for each of the k^i units an amount which is no greater than its true value, so makes a nonnegative profit. If he were to report a $z^i > x^i$, then he would win at least as many units as before. The prices for the first k^i units would remain the same as if he had reported $z^i = x^i$. For any additional units, however, the price paid would be too high, since for $k > k^i$,

$$y_k^i(\mathbf{x}^{-i}) > x^i$$

and the price paid would be

$$p_k^i = v_k^i\left(y_k^i(\mathbf{x}^{-i}), \mathbf{x}^{-i}\right) > v_k^i\left(x^i, \mathbf{x}^{-i}\right)$$

thus resulting in a loss. On the other hand, if he reports a $z^i < x^i$, then the number of units that he would win is at most what he would win by reporting $z^i = x^i$. For any of the units won the prices would be the same as before but he would forgo some surplus for units, say the k^ith unit, that he did not win. ■

In the case of private values, the generalized VCG mechanism reduces to the Vickrey multiunit auction and, as we have seen, in that case it is a dominant strategy to report truthfully.

18.1.2 Efficiency via Open Auctions

When values are interdependent, information necessary to determine the values is dispersed among the bidders. To achieve efficiency, this information must emerge during an auction. In the context of a single-object auction we saw that the English auction was quite remarkable in this regard—under relatively weak conditions the relevant information emerged in a way that resulted in efficiency. Specifically, in Chapter 9 we found that

1. With two bidders, the English auction had an efficient *ex post* equilibrium provided the single crossing condition was satisfied (see Proposition 9.1 on page 131).
2. With more than two bidders, the single crossing condition by itself was not enough. Stronger conditions—for instance, the average crossing condition—had to be invoked in order to guarantee that the English auction had an efficient *ex post* equilibrium (see Proposition 9.2 on page 135).

Here we explore the extent to which open ascending price auctions have analogous properties in the multiunit context.

TWO BIDDERS

We first consider situations in which there are only two bidders. As noted previously, when there was a single object for sale, the two-bidder English auction was efficient as long as the single crossing condition was satisfied (Proposition 9.1). In addition, with two bidders, the English auction was strategically equivalent to the sealed-bid second-price auction, so the latter was efficient as well. Now, with private values the multiunit analogs of the second-price and English auctions are the Vickrey and Ausubel auctions, respectively. The multiunit formats inherited all the relevant properties of their single unit counterparts and we wish to explore the extent to which the analogy can be pushed in the interdependent values environment.

It is instructive to begin by studying the sealed-bid Vickrey auction. When there are only two bidders, the K-unit Vickrey auction can be decomposed into K separate second-price auctions. This is because bidder 1 wins one unit if and only if his first bid, b_1^1, defeats the Kth bid of bidder 2, b_K^2. He wins a second unit if and only if his second bid, b_2^1, defeats the $K-1$st bid of bidder 2, b_{K-1}^2. In general, bidder 1 wins a kth unit if and only if his kth bid, b_k^1, defeats the $K-k+1$st bid of bidder 2, b_{K-k+1}^2. Moreover, the price bidder 1 pays for the first unit is just the defeated bid b_K^2, the price for the second unit is the defeated bid b_{K-1}^2, and so on. In particular, bidder 2's bids on units other than the $K-k+1$st unit do not affect the price that bidder 1 would pay were he to win a kth unit. A symmetric argument applies for bidder 2. Thus, we can think of the two-bidder Vickrey auction as K separate second-price auctions: in the kth auction, bidder 1 with value v_k^1 competes with bidder 2 with value v_{K-k+1}^2. This insight leads to the following generalization of Proposition 9.1.

Proposition 18.2. *Suppose that there are two bidders and the valuations* $\mathbf{v}$ *satisfy the multiunit single crossing condition. Then there exists an* ex post *equilibrium of the Vickrey multiunit auction that is efficient.*

Proof. Fix a $k \leq K$ and define $l = K-k+1$. As in Chapter 9, the multiunit single crossing condition guarantees that there exist continuous and increasing functions ϕ_k^1 and ϕ_l^2 such that for all $p \leq \min\{\phi_k^1(\omega^1), \phi_l^2(\omega^2)\}$, they solve the following pair of equations

$$v_k^1\left(\phi_k^1(p), \phi_l^2(p)\right) = p$$

$$v_l^2\left(\phi_k^1(p), \phi_l^2(p)\right) = p \tag{18.3}$$

Define $\beta_k^1 : [0, \omega^1] \to \mathbb{R}_+$ by $\beta_k^1 = \left(\phi_k^1\right)^{-1}$ and $\beta_l^2 : [0, \omega^2] \to \mathbb{R}_+$ by $\beta_l^2 = \left(\phi_l^2\right)^{-1}$. Notice that since for all k, $v_{k+1}^1 \leq v_k^1$ and $v_{l-1}^2 \geq v_l^2$, the solutions satisfy the inequality $\beta_{k+1}^1 \leq \beta_k^1$ and $\beta_{l-1}^2 \geq \beta_l^2$.

We claim that it is an equilibrium for bidder 1 to bid a vector

$$\boldsymbol{\beta}^1(x^1) = \left(\beta_1^1(x^1), \beta_2^1(x^1), \ldots, \beta_K^1(x^1)\right)$$

when his signal is x^1 and for bidder 2 to bid a vector

$$\boldsymbol{\beta}^2(x^2) = \left(\beta_1^2(x^2), \beta_2^2(x^2), \ldots, \beta_K^2(x^2)\right)$$

when her signal is x^2.

Suppose that with these bids, bidder 1 wins k^1 units and bidder 2 wins $K - k^1$ units. Consider a $k \leq k^1$ and notice that we must have $p^1 \equiv \beta_k^1(x^1) > \beta_{K-k+1}^2(x^2) \equiv p^2$—that is, bidder 1's kth bid must have defeated bidder 2's $K-k+1$st bid, and by the rules of the Vickrey auction, he pays p^2 for the kth unit. Now (18.3) implies that

$$v_k^1\left(\phi_k^1(p^2), \phi_l^2(p^2)\right) = p^2,$$

where as usual, $l = K - k + 1$. Since $x^1 = \phi_k^1(p^1) > \phi_k^1(p^2)$ and $\phi_l^2(p^2) = x^2$,

$$v_k^1(x^1, x^2) > p^2$$

This implies that bidder 1 makes an *ex post* profit on the kth unit that he wins and since he cannot affect the price he pays, he cannot do better.

In addition, (18.3) also implies that

$$v_l^2\left(\phi_k^1(p^1), \phi_l^2(p^1)\right) = p^1$$

and since $\phi_l^2(p^1) > \phi_l^2(p^2) = x^2$ and $\phi_k^1(p^1) = x^1$,

$$v_l^2(x^1, x^2) < p^1$$

This implies that bidder 2 does not want to win the lth unit since the price would be too high. Thus, there exists an *ex post* equilibrium.

The equilibrium constructed here is efficient because from (18.3)

$$v_k^1\left(\phi_k^1(p^2),\phi_l^2(p^2)\right)=v_l^2\left(\phi_k^1(p^2),\phi_l^2(p^2)\right)$$

and again since $x^1=\phi_k^1(p^1)>\phi_k^1(p^2)$ and $\phi_l^2(p^2)=x^2$,

$$v_k^1(x^1,x^2)>v_l^2(x^1,x^2)$$

because of single crossing. This means that the unit in question—which is, simultaneously, the kth unit for bidder 1 and the lth unit for bidder 2—is indeed awarded to the bidder who values it more. ■

The reasoning in Proposition 18.2 applies to the Ausubel open ascending price auction as well. Recall that in the Ausubel auction the price rises and bidders indicate how many units they are willing to buy at the current price. A bidder is allocated a unit every time the residual supply from the other bidders increases. To find an equilibrium of the auction, let $\boldsymbol{\beta}^1$ and $\boldsymbol{\beta}^2$ be determined as above and consider the following pair of strategies. Both bidders' demands are K at a price of zero. For all $k\leq K$, bidder i reduces his demand from k to $k-1$ at the price $p_k^i=\beta_k^i\left(x^i\right)$. It is routine to verify that this constitutes an efficient *ex post* equilibrium of the Ausubel auction.

With two bidders, it appears that the single-unit results of Chapter 9 extend to the multiunit context in a relatively straightforward manner. The same is not true when the number of bidders is greater than two.

THREE OR MORE BIDDERS

There is no multiunit analog of Proposition 9.2. Once the number of bidders exceeds two, the Ausubel auction need not have an efficient equilibrium. Stringent restrictions on the valuation functions do not rectify the problem.

Example 18.1. *When there are three or more bidders and values are interdependent, the Ausubel open ascending price auction need not allocate efficiently. This may happen even if bidders have unit demands.*

Suppose that there are two units for sale ($K=2$) and three bidders ($N=3$), each of whom wants at most one unit of the good; this is the case of unit demand. The valuations are

$$\begin{aligned}v^1(x^1,x^2,x^3)&=x^1+\alpha x^2\\ v^2(x^1,x^2,x^3)&=x^2\\ v^3(x^1,x^2,x^3)&=x^3\end{aligned}$$

and the signals X^i all lie in $[0,1]$ and $\alpha\in(0,1)$ is a parameter. Clearly, the single crossing condition is satisfied.

Consider an open ascending price auction. Since each bidder wants at most one unit of the goods, the workings of the Ausubel auction are the same as the workings of the multiunit English auction. In other words, the price rises until one of the bidders drops out. At that stage, the total supply is equal to the total demand, so the auction is over. Thus, the two units are sold to the two remaining bidders at the price at which the first bidder drops out.

Each bidder need only decide when to drop out. Bidder 2 has private values, so it is weakly dominant for him to drop out when the price reaches his private value. The same is true for bidder 3. Suppose that there is an equilibrium in which bidder 1 follows an increasing and continuous strategy β^1, so he drops out when the price reaches $\beta^1(x^1)$.

First, suppose $\beta^1(0) = 0$. Now if the signals are such that $\alpha x^2 > x^3$, then it is efficient for bidders 1 and 2 to get one unit each. But if x^1 is close to zero, the continuity of β^1 implies that bidder 1 will drop out first, thereby leading to an inefficient outcome. Second, suppose $\beta^1(0) > 0$. Now if the signals are such that $x^2 < x^3$ and x^1 is small, it is efficient for bidders 2 and 3 to get one unit each. But if $x^2 < \beta^1(0)$, then bidder 2 will drop out first, again leading to an inefficient outcome. ▲

In a single-unit English auction the object is not awarded until all but one of the bidders have dropped out. This means that the winning bidder knows the signals of all bidders and hence his own value. This is not true in the multiunit analog of the English auction. Some units are awarded *before* all the information has been revealed and it is this feature that leads to inefficiency. Thus, it seems that, except for the case of two bidders, the common auction formats are ill equipped to handle interdependent values in the multiunit context. All this is still under the somewhat uncomfortable assumption that while bidders have multiunit demands, their signals are one-dimensional. The state of affairs is even worse, however, if buyers' signals are multidimensional.

18.2 MULTIDIMENSIONAL SIGNALS

Now suppose that each buyer's information consists of an L-dimensional signal $\mathbf{x}^i = (x_1^i, x_2^i, \ldots, x_L^i)$ and that buyer i's valuations for the K objects depend on the $L \times N$ signals $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N)$ received by all the buyers. For the moment we make no assumptions on L and K. The *marginal value* of obtaining the kth unit is determined by the function

$$v_k^i(\mathbf{x}) = v_k^i\left(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right)$$

All signals are assumed to have a nonnegative effect on all values—that is, for all i,j and k, l,

$$\frac{\partial v_k^i}{\partial x_l^j} \geq 0$$

18.2.1 Single Object

Although we are concerned with multiple-object auctions, the problems resulting from multidimensional signals are already apparent when only a single object is for sale. Thus, while buyers receive L dimensional signals $\mathbf{x}^i$, there is only a single object for sale whose value to i can be written as $v^i(\mathbf{x}) = v^i(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N)$. The following example illustrates the fundamental nature of the difficulty in the simplest possible setting.

Example 18.2. *Suppose there is a single object for sale to one of two buyers. Buyer* 1 *receives a two-dimensional signal* $\mathbf{x}^1 = (x_1^1, x_2^1)$, *whereas buyer* 2 *is completely uninformed. Buyer* 1*'s first signal determines his own value for the object, whereas his second signal determines buyer* 2*'s value. Thus,*

$$
\begin{aligned}
v^1(\mathbf{x}^1) &= x_1^1 \\
v^2(\mathbf{x}^1) &= x_2^1
\end{aligned}
$$

Clearly, efficiency requires that the object go to buyer 1 only if $x_1^1 \geq x_2^1$. We claim that there does not exist an incentive compatible mechanism that is efficient. The revelation principle ensures that it is enough to consider direct mechanisms.

Consider an efficient mechanism whose payment rule asks buyer 1 to make the payment $M^1(\mathbf{z}^1)$ when he reports $\mathbf{z}^1 = (z_2^1, z_2^1)$ (buyer 2 is completely uninformed). First, notice that M^1 must be constant for all $\mathbf{z}^1$ such that $z_1^1 > z_2^1$. Otherwise, then buyer 1 will report a signal $\mathbf{z}^1$ that minimizes his payment $M^1(\mathbf{z}^1)$ over all $\mathbf{z}^1$ such that $z_1^1 > z_2^1$. In the same way M^1 must be constant for all $\mathbf{z}^1$ such that $z_1^1 < z_2^1$. If not, then buyer 1 will report a signal $\mathbf{z}^1$ that minimizes his payment $M^1(\mathbf{z}^1)$ over all $\mathbf{z}^1$ such that $z_1^1 < z_2^1$. Thus, we conclude that there exist two payments m' and m'' (these could be positive or negative) such that

$$
M^1\left(\mathbf{z}^1\right) = \begin{cases} m' & \text{if } z_1^1 > z_2^1 \\ m'' & \text{if } z_1^1 < z_2^1 \end{cases}
$$

Suppose buyer 1's signal is $\mathbf{x}^1$. If he reports $z_1^1 > z_2^1$, he gets the object and pays m' so that his payoff is $x_1^1 - m'$. On the other hand, if he reports $z_1^1 < z_2^1$, he does not obtain the object, so his payoff is $-m''$. It is better to report $z_1^1 > z_2^1$ if and only if

$$
x_1^1 \geq m' - m''
$$

But this means that buyer 1's incentives to report $z_1^1 > z_2^1$ versus $z_1^1 < z_2^1$ do not depend on the signal x_2^1. But efficiency hinges precisely on a comparison of x_1^1 and x_2^1, so it is impossible to achieve. ▲

The reader may wonder—quite legitimately—whether the impossibility of achieving efficiency in the preceding example is not due to the failure of some sort of single crossing condition. In situations in which signals were one-dimensional, the single crossing condition—requiring that a buyer's signal have a greater impact on his own value than on others' values—was vital to the question of efficiency. In Example 18.2, however, buyer 1's second signal, x_2^1, has a greater impact on buyer 2's value than on his own. Consider the following, only slightly more complicated, example.

Suppose that both buyers receive two-dimensional signals $\mathbf{x}^i = (x_1^i, x_2^i)$ and that their respective values are

$$\begin{aligned} v^1(\mathbf{x}^1) &= x_1^1 + x_2^1 \\ v^2(\mathbf{x}^1) &= x_1^2 + \alpha x_2^1, \end{aligned}$$

where $\alpha \in (0,1)$ is a parameter. Now each buyer's signals have a greater impact on his own value than on the value of the other buyer. But virtually the same argument as the preceding one shows that efficiency is impossible to achieve. Buyer 1 cares only about the sum of his own signals $x_1^1 + x_2^1$, so he cannot be induced to reveal more than that. Buyer 2's value, however, depends on buyer 1's second signal, x_2^1, alone. To determine who should win the object it is necessary to extract information regarding x_2^1 separately from that regarding the sum $x_1^1 + x_2^1$. Buyer 1, however, cannot be provided with the incentives to reveal x_2^1 separately from $x_1^1 + x_2^1$. As a result, efficiency cannot be attained.

More generally, consider a direct mechanism with a payment rule $\mathbf{M}$ in which truth-telling is an *ex post* equilibrium. Consider the signals $\mathbf{x}^{-i}$ of all buyers other than i. As in Chapter 10, we will say that buyer i is *pivotal at* $\mathbf{x}^{-i}$ if there exist signal vectors $\mathbf{y}^i$ and $\mathbf{z}^i$ such that $v^i(\mathbf{y}^i, \mathbf{x}^{-i}) > \max_{j \neq i} v^j(\mathbf{y}^i, \mathbf{x}^{-i})$ and $v^i(\mathbf{z}^i, \mathbf{x}^{-i}) < \max_{j \neq i} v^j(\mathbf{z}^i, \mathbf{x}^{-i})$. In other words, when the others' signals are $\mathbf{x}^{-i}$, i's signal determines whether or not it is efficient for him to get the object. Incentive compatibility requires that when his signal is $\mathbf{y}^i$, it is optimal for i to report $\mathbf{y}^i$ rather than $\mathbf{z}^i$, so that

$$v^i(\mathbf{y}^i, \mathbf{x}^{-i}) - M^i(\mathbf{y}^i, \mathbf{x}^{-i}) \geq -M^i(\mathbf{z}^i, \mathbf{x}^{-i})$$

Likewise, when his signal is $\mathbf{z}^i$, it is optimal to report $\mathbf{z}^i$ rather than $\mathbf{y}^i$, so that

$$-M^i(\mathbf{z}^i, \mathbf{x}^{-i}) \geq v^i(\mathbf{z}^i, \mathbf{x}^{-i}) - M^i(\mathbf{y}^i, \mathbf{x}^{-i})$$

Combining the two conditions results in

$$v^i(\mathbf{y}^i, \mathbf{x}^{-i}) \geq M^i(\mathbf{y}^i, \mathbf{x}^{-i}) - M^i(\mathbf{z}^i, \mathbf{x}^{-i}) \geq v^i(\mathbf{z}^i, \mathbf{x}^{-i})$$

and so a necessary condition for incentive compatibility is

$$v^i(\mathbf{y}^i, \mathbf{x}^{-i}) \geq v^i(\mathbf{z}^i, \mathbf{x}^{-i})$$

that is, buyer i's value when he wins the object must be at least as high as when he does not. Put another way, keeping others' signals fixed, an increase in buyer i's value that results from a change in his own signal cannot cause him to lose if he were winning earlier. Once again we see that *ex post* incentive compatible mechanisms must be *monotonic* in values—an increase in the value resulting from a change in his own signal must increase the chances that a buyer wins. In Chapter 10, we saw that when signals were one-dimensional this monotonicity (see (10.1)), together with the requirement of efficiency, led to the single crossing condition. When signals are multidimensional, however, monotonicity in values can be reconciled with efficiency only in very exceptional circumstances. Let us see why.

Suppose that there exists an $\mathbf{x}^i$ such that for some buyer j, $v^i(\mathbf{x}^i,\mathbf{x}^{-i}) = v^j(\mathbf{x}^i,\mathbf{x}^{-i}) \equiv p$, say, and these values are the highest among all buyers. Consider, as in Figure 18.3, the level curves of v^i and v^j in the space of i's signals. If these intersect, and the slopes of the level curves at the point of intersection are different, then as depicted, we can find two signals $\mathbf{y}^i$ and $\mathbf{z}^i$ such that

$$p > v^i(\mathbf{y}^i,\mathbf{x}^{-i}) > v^j(\mathbf{y}^i,\mathbf{x}^{-i})$$

but

$$v^j(\mathbf{z}^i,\mathbf{x}^{-i}) > v^i(\mathbf{z}^i,\mathbf{x}^{-i}) > p$$

Efficiency requires that i win when the signal is $\mathbf{y}^i$ and lose when the signal is $\mathbf{z}^i$. But since $v^i(\mathbf{z}^i,\mathbf{x}^{-i}) > v^i(\mathbf{y}^i,\mathbf{x}^{-i})$, it is the case that buyer i wins when his value is $v^i(\mathbf{y}^i,\mathbf{x}^{-i})$ but loses when his value increases to $v^i(\mathbf{z}^i,\mathbf{x}^{-i})$, thereby violating the monotonicity property of incentive compatible mechanisms. Thus, we have argued that as long as the level curves $v^i = p$ and $v^j = p$ intersect, efficiency is impossible. The following result summarizes our findings.

Proposition 18.3. *Suppose there exists an efficient direct mechanism in which truth-telling is an* ex post *equilibrium. If buyers' signals are multidimensional,*

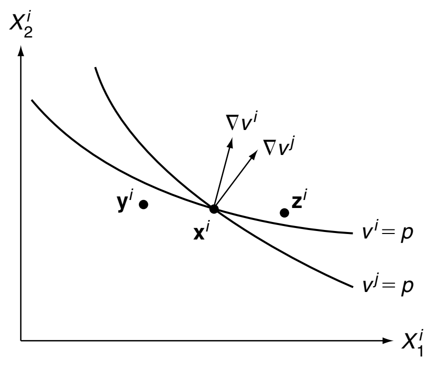

FIGURE 18.3 Efficiency with multidimensional signals.

then at any $\mathbf{x}^i$ *such that* $v^i(\mathbf{x}^i, \mathbf{x}^{-i}) = v^j(\mathbf{x}^i, \mathbf{x}^{-i})$ *and these are maximal, it must be that*

$$\nabla v^i(\mathbf{x}^i, \mathbf{x}^{-i}) = \nabla v^j(\mathbf{x}^i, \mathbf{x}^{-i})$$

where ∇v^i *and* ∇v^j *denote the L-vectors of partial derivatives of* v^i *and* v^j *with respect to i's signals (the gradients of* v^i *and* v^j *with respect to* $\mathbf{x}^i$*).*

The necessary condition in the preceding proposition—the equality of the gradients—is a very strong requirement that is almost never satisfied; it is non-generic in the sense that a slight perturbation in the valuation functions will cause it to be violated. The essential difficulty is that a buyer's payment—a one-dimensional instrument—can only be used to extract one-dimensional information from the buyer. The information that can be extracted concerns the *value* of the buyer, rather than his *signal* and information concerning i's value alone is, in general, insufficient to decide the efficiency question when values are interdependent. The one exception occurs when buyer i's signals affect all buyers' values in the same manner so the relevant information can effectively be reduced to a one-dimensional variable. Otherwise, such a reduction is impossible, and as a result, so is efficiency.

Another way to understand Proposition 18.3 is to note that in the present context monotonicity requires that the following multidimensional version of single crossing holds. Suppose $v^i(\mathbf{x}^i, \mathbf{x}^{-i}) = v^j(\mathbf{x}^i, \mathbf{x}^{-i})$ and $\mathbf{t}^i \neq \mathbf{0}$ is such that the directional derivative of v^i with respect to $\mathbf{x}^i$ in the direction $\mathbf{t}^i$ is positive—that is,

$$\nabla v^i(\mathbf{x}^i, \mathbf{x}^{-i}) \cdot \mathbf{t}^i > 0$$

In other words, i's utility increases if his signal $\mathbf{x}^i$ increases in the direction $\mathbf{t}^i$. Monotonicity now requires that for any such $\mathbf{t}^i$,

$$\nabla v^i(\mathbf{x}^i, \mathbf{x}^{-i}) \cdot \mathbf{t}^i > \nabla v^j(\mathbf{x}^i, \mathbf{x}^{-i}) \cdot \mathbf{t}^i \tag{18.4}$$

that is, the change in i's value must be greater than the change in j's value. But as is clear from Figure 18.3, this cannot hold if the level curves of v^i and v^j intersect: if we let $\mathbf{t}^i = \mathbf{z}^i - \mathbf{x}^i$, then (18.4) is violated. In this sense, the appropriate single crossing condition cannot be generically satisfied once signals are multidimensional.

18.2.2 Multiple Objects

It is not surprising that the impossibility result from the previous section extends to the multiple-object setting. But there is an important exception—the case of additively and informationally *separable* valuations.

SEPARABILITY AND EFFICIENCY

Suppose that set K of distinct objects, labeled $a, b, c, \ldots$, and so on, are for sale. Each buyer receives a multidimensional signal

$$\mathbf{x}^i = \left(x_a^i\right)_{a \in K}$$

with the interpretation that x_a^i is the one-dimensional information received by buyer i that pertains to object a. Buyers' valuations for the different objects are *informationally separable*—that is, the value of an object $a \in K$ to buyer i is of the form

$$v_a^i\left(x_a^1, x_a^2, \ldots, x_a^N\right)$$

In other words, the value to i of obtaining object a depends on the signals of all buyers that pertain only to object a, and not on signals x_b^j, say.

We also need that for each $a \in K$, the valuations $v_a^1, v_a^2, \ldots, v_a^N$ satisfy the *single crossing* condition.

Suppose further that there are no complementarities and that the valuations are *additively separable*—the value derived from obtaining particular bundle of objects is simply the sum of the values of the individual objects in that bundle. Thus, for instance, the value of the bundle $\{a, b\}$ to buyer i is simply

$$v_a^i\left(x_a^1, x_a^2, \ldots, x_a^N\right) + v_b^i\left(x_b^1, x_b^2, \ldots, x_b^N\right)$$

The separable model, albeit special, may be quite natural in some settings. Suppose that the objects being sold are the rights to conduct business in different regions of a country. For instance, object a may be a license awarding exclusive rights to supply local telephone services in area a. Each buyer may have some information x_a^i concerning demand conditions in area a. In that case, as a first approximation, it is not unnatural to suppose that a buyer's value for license a depends only on information concerning demand conditions in region a and not on demand conditions in another region, say b.

The virtue of the separable specification is that the problem of allocating the objects efficiently neatly decomposes into K separate problems: the objects can be efficiently allocated one at a time. Object a should be allocated to the buyer who derives the largest benefit from it—that is, to the buyer with the largest value of $v_a^1, v_a^2, \ldots, v_a^N$—and this allocation does not depend in any way on other values v_b^i. Similarly, object b should be allocated to the buyer with the largest value of $v_b^1, v_b^2, \ldots, v_b^N$ regardless of the values v_a^i. This in turn means that the allocation of object a depends only on the signals x_a^i that pertain to a, the allocation of object b depends only on the signals x_b^i that pertain to b, and so on.

The separable specification also implies that the mechanism design problem can likewise be decomposed into K separate problems, each with one-dimensional signals. We can then use K generalized VCG mechanisms for allocating single objects as in Chapter 10. In particular, each buyer is asked to report his signal $\mathbf{x}^i$. Object a is awarded to the buyer with the highest value v_a^i for a when evaluated at the reported signals $x_a^1, x_a^2, \ldots, x_a^N$ of all buyers that pertain to a. If buyer i is awarded object a, then he pays $v_a^i\left(y_a^i\left(\mathbf{x}_a^{-i}\right), \mathbf{x}_a^{-i}\right)$ where $y_a^i\left(\mathbf{x}_a^{-i}\right)$ is defined by

$$y_a^i(\mathbf{x}_a^{-i}) = \inf\left\{z_a^i : v_a^i\left(z_a^i, \mathbf{x}_a^{-i}\right) \geq \max_{j \neq i} v_a^j\left(z_a^i, \mathbf{x}_a^{-i}\right)\right\}$$

Truth-telling is an *ex post* equilibrium in each mechanism in isolation and the separable specification guarantees that no buyer can gain from simultaneously misreporting more than one signal. Thus, in the separable model an efficient allocation of multiple objects can be attained via the generalized VCG mechanism.

Proposition 18.4. *Suppose buyers' valuations for K different objects are additively and informationally separable. Then truth-telling is an efficient* ex post *equilibrium of the generalized VCG mechanism.*

In the separable model the K objects would also be efficiently allocated in a sequence of English auctions. This assumes, of course, that the valuations for each object a are such that an efficient equilibrium exists. This would be satisfied, for instance, if for each a, the valuations satisfied the average crossing condition introduced in Chapter 9.

In general, the valuations need not be separable and buyers' valuations for object a, say, may depend on both x_a^i and x_b^i. In that case, the arguments leading up to Proposition 18.3 imply that efficiency cannot be attained.

CHAPTER NOTES

Much of the material in this chapter originates in the paper by Maskin (1992) who recognized that the possibility of attaining efficiency hinged on the dimensionality of buyers' signals. Maskin (2003) is a very nice survey of the area.

Proposition 18.2, demonstrating the efficiency of the Vickrey auction when there are only two bidders with one-dimensional signals, was derived by Perry and Reny (2002). They also observed that this result does not hold once there are three or more bidders with multiunit demand. Example 18.1, showing that with three or more bidders open auctions are inefficient even with single-unit demand, is due to Morgan (2001, private communication).

Maskin (1992) was the first to show the impossibility of allocating efficiently if buyers have multidimensional information. Proposition 18.3 is based on an impossibility result in Dasgupta and Maskin (2000). Jehiel and Moldovanu (2001) have extended these results in many directions. The interpretation of the impossibility result as a necessary failure of single crossing is due to Reny (2001, private communication).