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Calculus I

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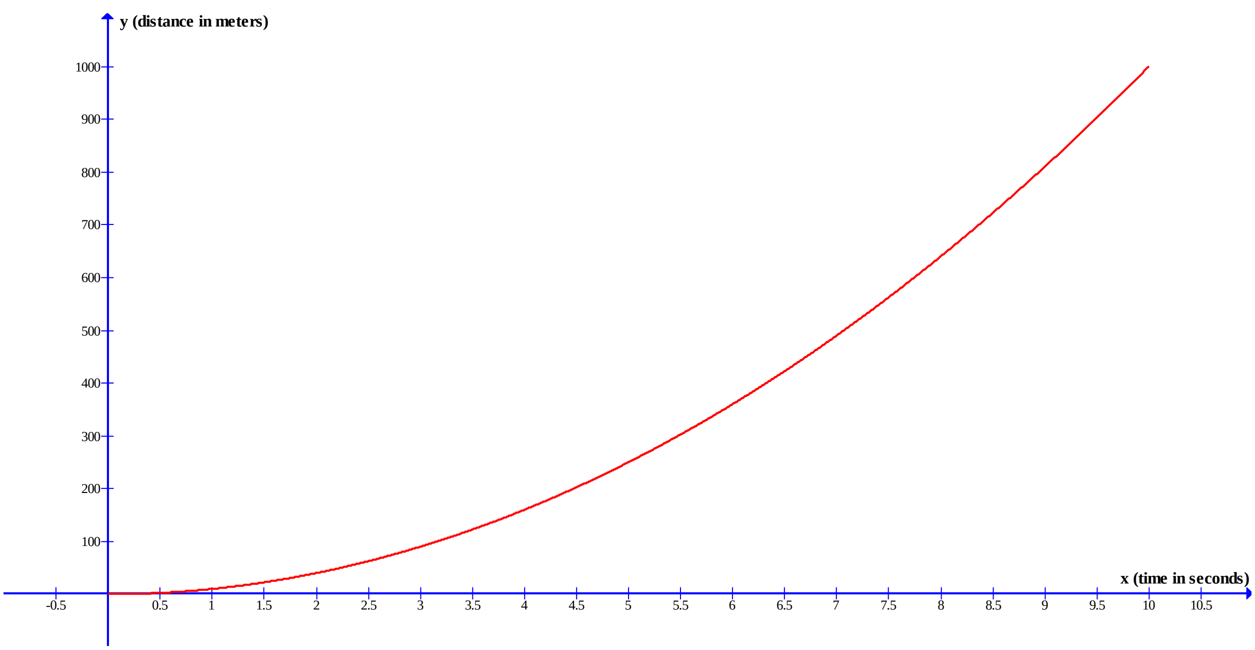
The Derivative and Superman

The derivative is one of the two fundamental concepts of calculus. But what application does it have in real life? Why should we care? Imagine that you are Superman. A man has just fallen out of a plane. He is falling, as physics tells us, at an accelerating speed. In order to avoid killing him, you need to catch him while traveling at the same speed that he is and slowly decelerate so that he is not squished. Unfortunately, you cannot simply wait until he reaches terminal velocity, because at his point, he will be too far away for you to catch.

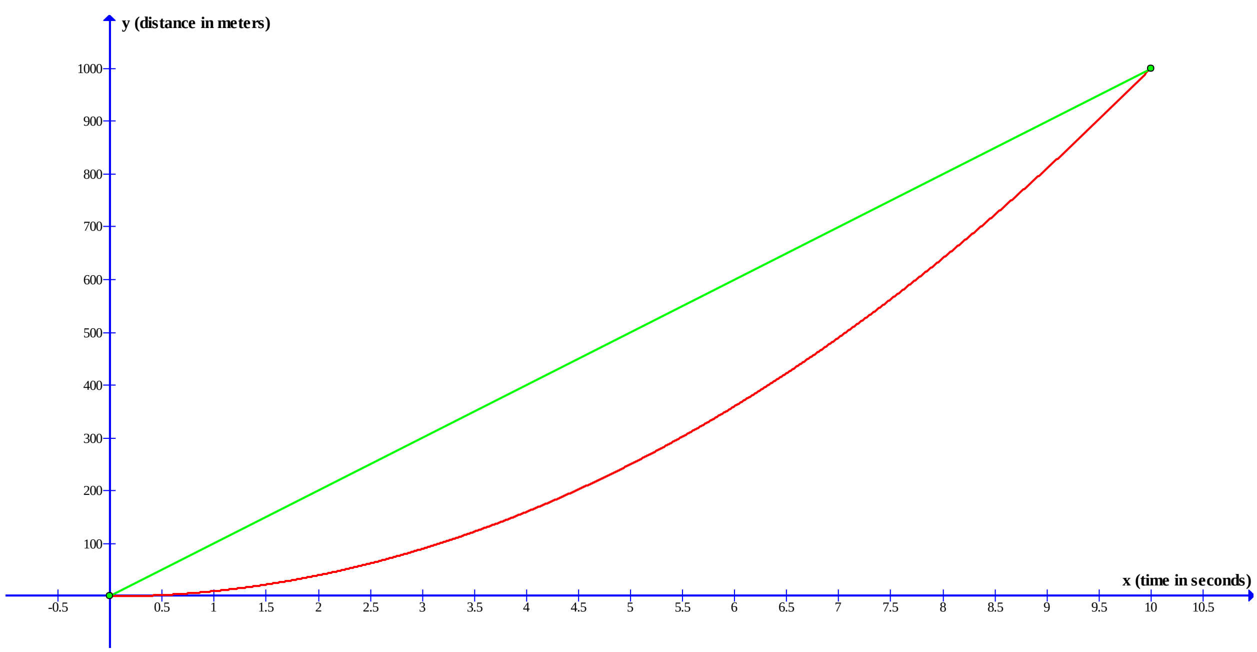
If we graph the man's position over time, it will look something like this.



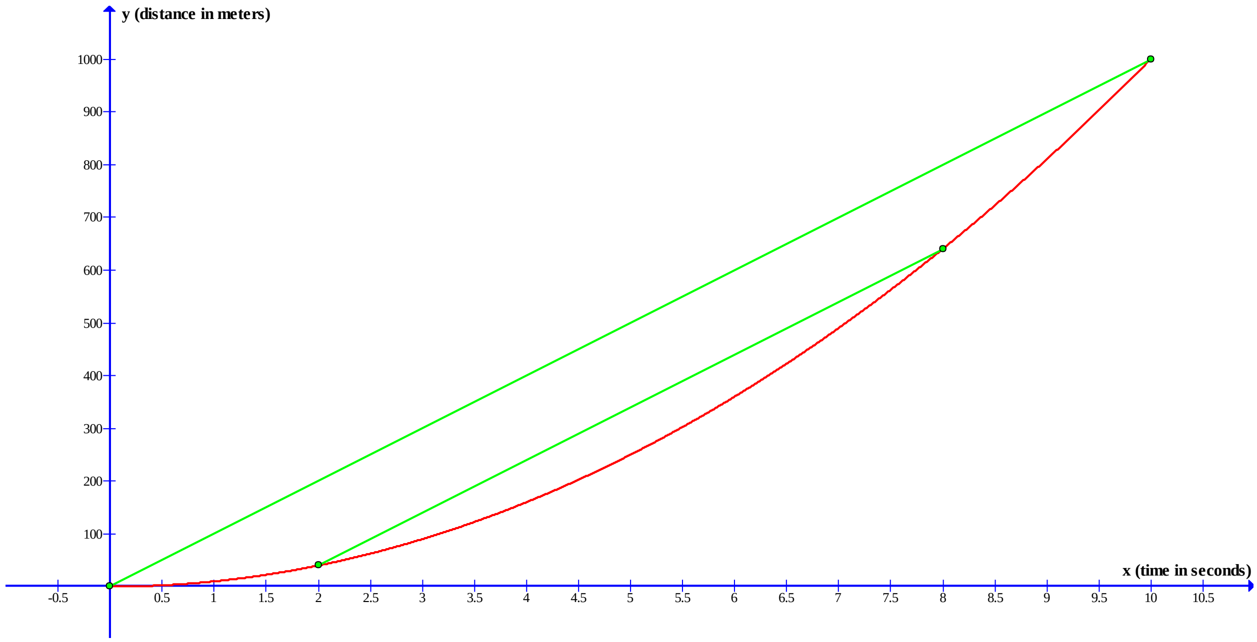
We can see that the x-axis is time in seconds, and the y-axis is distance in meters (in this case, meters fallen from the plane). Speed can also be thought of as distance over time, as in, for example, miles per hour, or in our problem, meters per second. Thus, it is reasonable to say that the *average* speed over any length of time is equal to the total distance traveled over the time interval. In other words it is equal to the change in the y-value over the change in the x-value, or $\frac{\Delta y}{\Delta x}$. This should look familiar, because it is also the slope of a line. So all we need to do to find the falling man's speed and allow Superman to save him is calculate the slope.

Unfortunately for the falling man, the graph is a curve. Calculating a single slope for a curve is impossible – the slope has a different value at every point on the graph. This is where the derivative comes in. One very simple way to think about the derivative is as *the way we take the slope of a curve at a single point*. In our example problem, this will also be the instantaneous velocity – as opposed to the average velocity – of the falling man.

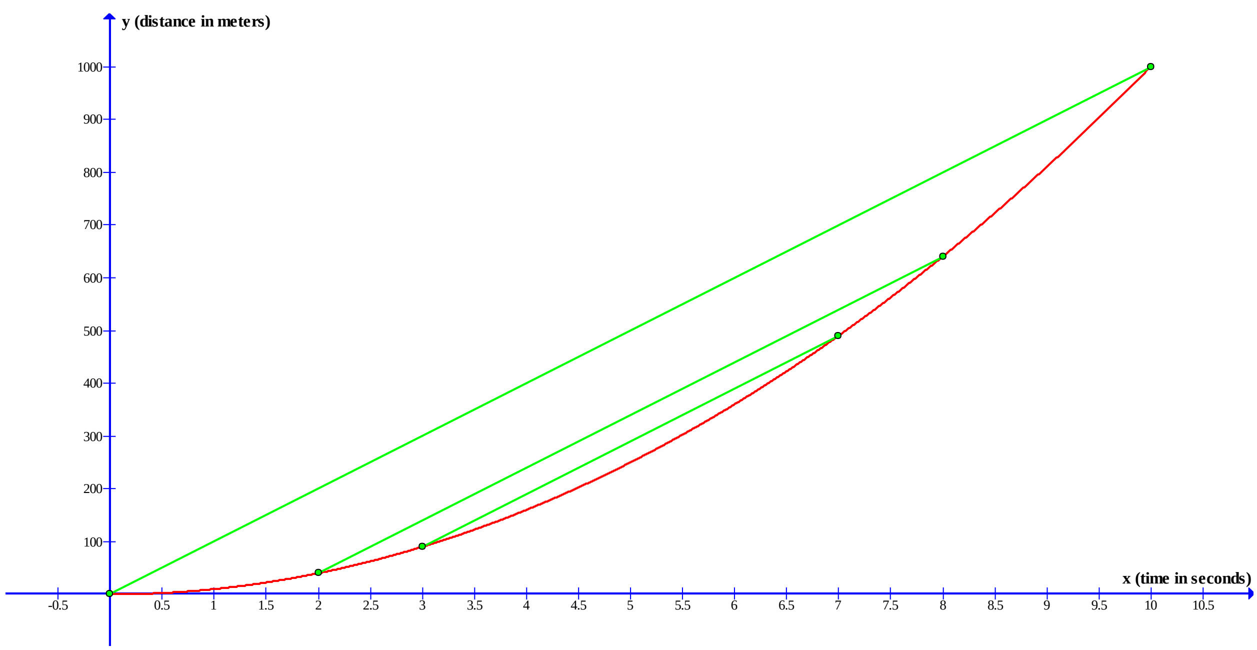
Here is our graph again, with a line whose slope represents the average speed of the man from time 0, when he first fell from the plane, to time 10.



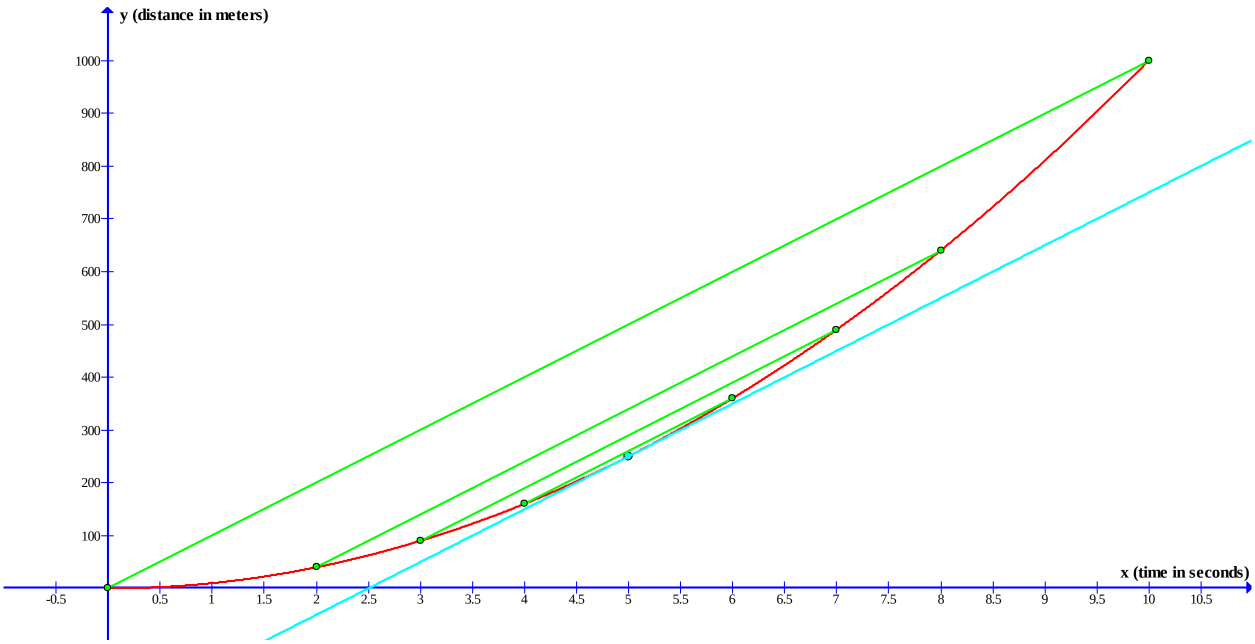
As we already determined, this is not enough information. Knowing the average speed is not going to allow Superman to catch the man properly – we must know his speed at a single point in time, his instantaneous speed. Assume that Superman wants to catch the falling man at time 5, or 5 seconds. If we pick two points much closer to this x-value, we get an average velocity that is *closer* to the instantaneous velocity:



If we pick two points closer still, we get an average velocity that is *even closer*:



If we continued to do this infinitely, we can imagine that we would eventually obtain a line that intersects the curve not at two points, but at only one.



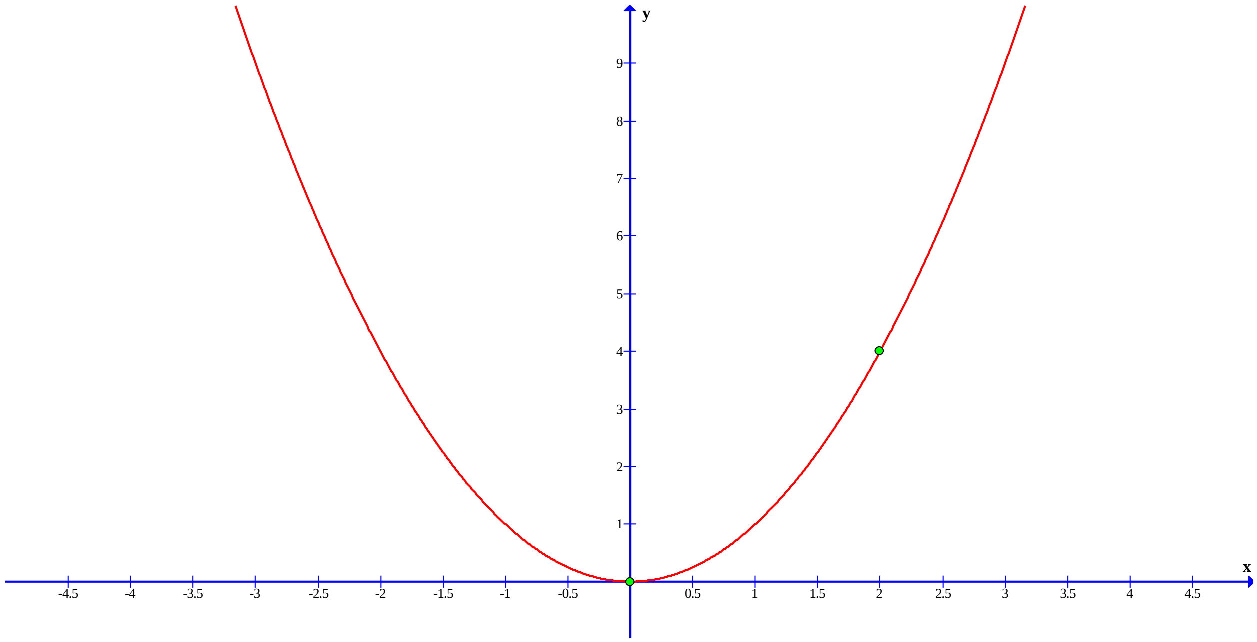
This line is referred to as the *tangent line*, and its slope is the instantaneous velocity.

It is possible to *estimate* the slope of the tangent line without resorting to calculus. Make a table of values representing the ever decreasing changes in y-value over changes in x-value and you can hazard a guess as to where the slope is headed:

Δy	600	500	400	300	200
Δx	6	5	4	3	2

While the answer may seem obvious in this case, it is hardly a precise way to go about solving the problem, and it will not stand up to more complex functions.

To find this slope more accurately, we *take a limit*. A limit is essentially the following. We pick a variable and postulate that it “goes to”, or approaches infinitely closely, a value. We then look at what another variable does as our original variable gets closer and closer to the chosen value. In most cases, these variables are x and y, or some equivalent. For example, in this parabola:

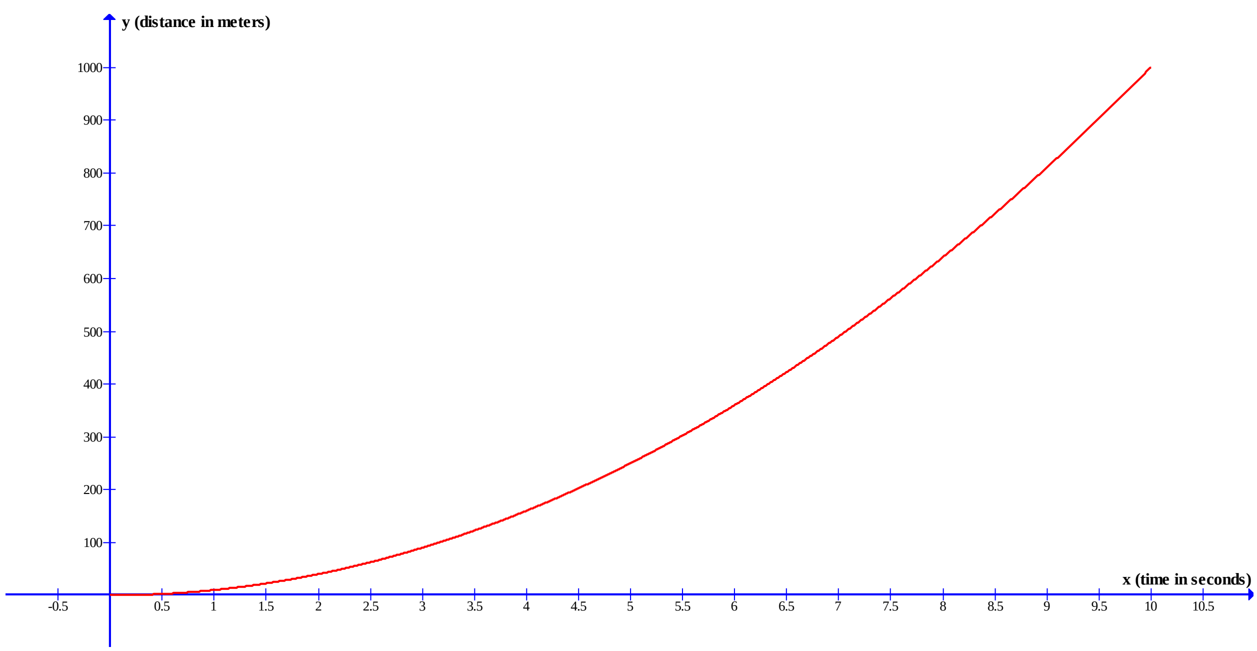


we could say that the limit as x approaches 0 of y is 0, because as x gets closer and closer to 0 from both directions, y also gets closer and closer to 0. This would be notated as: $\lim_{x \to 0} y = 0$ or, more commonly, $\lim_{x \to 0} f(x) = 0$ Similarly, we could say that the limit as x approaches 2 of y is 4 ($\lim_{x \to 2} f(x) = 4$), because as x gets closer and closer to 2 from both directions, y gets closer and closer to 4.

So taking the limit just means finding what y approaches as x approaches some value. Notice that in the above parabola, the limit of y at any x-value is equal to the y-value at that x-value. This is because a parabola is a continuous function, a function defined by the above characteristic – that at any given point, the limit of y equals the y-value. A more intuitive, though less exact, way to think of continuous functions is this: if you can draw it without picking up your pencil or pen, it is continuous. Importantly, this means that when taking the limit of any continuous function, all we have to do is plug in the value that x is approaching for x in the function we are taking the limit of.

How does all this contribute to finding the instantaneous velocity, a.k.a. the slope of the tangent line, a.k.a. the derivative, and allowing Superman to save the falling man? Well, recall that taking the limit was a precise method of finding the slope of the tangent line. Consider the limit as *the change in x-value* approaches 0 of *the change in y-value* over *the change in x-value*, or $\lim_{\Delta x \to 0} f(x) = \frac{\Delta y}{\Delta x}$ We know that $\frac{\Delta y}{\Delta x}$ in this case is the slope of the tangent line – remember that as we approached the slope of the tangent line in our table of values, the change in the x-value grew smaller and smaller (approached 0) because the points we chose were closer and closer together – and so we know that this limit is the answer we have been looking for, the instantaneous velocity of the falling man. But there is more work to do. Taking the limit has provided us with an equation to work with, so that it is *possible* to determine the exact slope – however, as it is currently written, we cannot do so.

Back to our original graph:



Assume the function that gives this graph is $10x^2$. We need to fit this into our limit equation so that we can actually *find* the change in the y-value and thus the limit. Δy can also be written as f(x + Δx) – f(x) In other words, we define the larger y-value as f({the smaller x-value} + {the difference between the smaller and larger x-values}) and then subtract from that the smaller y-value, thus yielding the change in y-value. We can now rewrite the limit equation as $\lim_{\Delta x \to 0} f(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x}$ This equation is one general definition of the derivative.

Recall that earlier, we said that for any continuous function we can just plug in the value x is approaching to find the limit. In our case, "Δx" has replaced "x" from our parabola example as the variable that approaches a value. Our falling man function, $10x^2$, is indeed continuous (at least where we have defined it). We know that Superman wants to catch the falling man at time 5 (5 seconds), or x = 5. Simple algebra is now all that is required to take the limit. Before plugging in our numbers, though, we must rearrange the equation so that the Δx on the bottom is cancelled – otherwise, when we plug in the value that Δx is approaching, our numerator will be divided by 0, an impossibility. First, rewrite our general derivative definition with our falling man function replacing the general f(x): $\lim_{\Delta x \to 0} 10x^2 = \frac{10(x+\Delta x)^2 - 10x^2}{\Delta x}$ Next, multiply out and combine like terms in the numerator to obtain: $\lim_{\Delta x \to 0} 10x^2 = \frac{20x\Delta x + 10\Delta x^2}{\Delta x}$ Then cancel the $\Delta x'$s: $\lim_{\Delta x \to 0} 10x^2 = 20x + 10\Delta x$ Finally we have an equation we can plug numbers into. If we plug our x, or time, value of 5 in for x, and 0 in for Δx, we get: $\lim_{\Delta x \to 0} 10x^2 = 20(5) + 10(0) = 100 + 0 = \mathbf{100.}$ In our problem, this is 100 meters per second. It is the instantaneous velocity of the falling man. It is the slope of the tangent line. In a way, it is the slope of a curve.

We started by asking "What is the slope of a curve?" We learned that the derivative is how we find the slope of a curve at a single point, which is the best anyone can do. While searching for the derivative, we encountered two ways of thinking about it, one graphical – the slope of the tangent line – and one algebraic – limits. We also helped Superman save the life of an innocent traveler.