Manuscript #2

The Calculus of Radar Guns and Speeding

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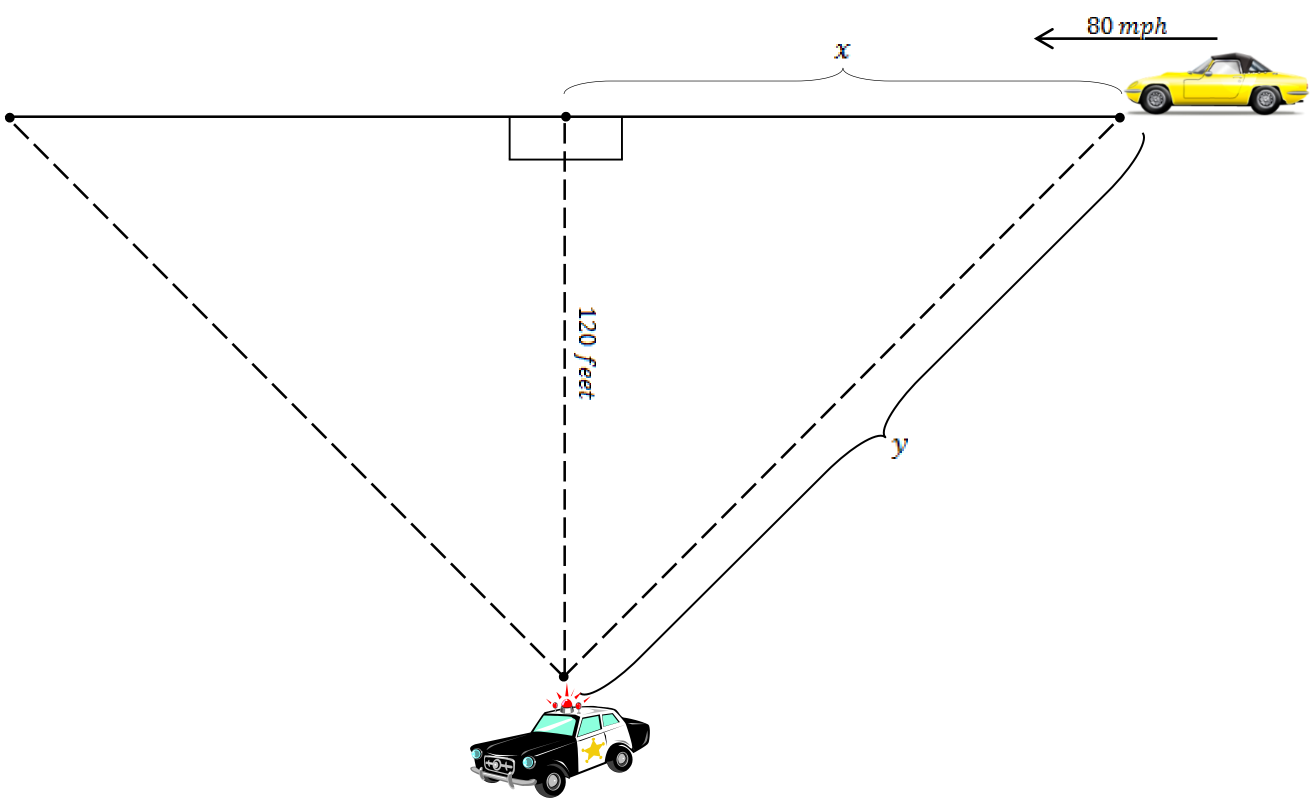
I. Introduction

Imagine that you are driving down a perfectly straight road at 80 miles per hour. Parked 120 feet to the side of the road is a cop car, and the officer is aiming a radar gun at your vehicle out of the open window. Due to the mechanics of radar guns, he reads the speed as approximated by the rate of change with respect to time of the distance between his gun and your car. This presents some interesting opportunities to apply calculus to a realistic situation.

II. Finding Displayed Speed at Different Points

First, let's determine what speed the radar gun will display at certain points, specifically, when you are 1 mile, 1/2 of a mile, 1/4 of a mile, and 1/10 of a mile away from the radar gun. To do this we must draw a picture of the situation.



We can see that y is the distance between your car and the radar gun, and that x is the distance between your car and a line perpendicular with the road intersecting the police car. The first important thing we must realize is that your *actual* speed can be thought of as the rate at which x changes with respect to time. Recall that this is the same as the derivative of x with respect to time, or $\frac{dx}{dt}$.

But we are not trying to find the *actual* speed; we are trying to find the speed that the radar gun displays. Through the same logic we applied to x we can see that the displayed speed is the rate at which y changes with respect to time, or $\frac{dy}{dt}$. So we need to find $\frac{dy}{dt}$ for certain values of y. This means that we need an equation for $\frac{dy}{dt}$ in terms of y alone, and *that* means that we need an equation for y, which we will take the derivative of. Applying the Pythagorean Theorem, we obtain (remembering to convert feet to miles) $x^2 + .023^2 = y^2$, or $y^2 = x^2 + .00053$. If we take the derivative of this with respect to t (time) we get $2y\frac{dy}{dt} = 2x\frac{dx}{dt}$. Remember that the $\frac{dy}{dt}$ and the $\frac{dx}{dt}$ are there because we used the Chain Rule; since we are taking the derivative with respect to t we must add in these extra terms for both x and y. Performing some basic algebra we can obtain $\frac{dy}{dt} = \frac{x\frac{dx}{dt}}{y}$. If we plug in 80, which we know is equal to $\frac{dx}{dt}$, we obtain $\frac{dy}{dt} = \frac{80x}{y}$. One problem remains: how can we use this equation to find $\frac{dy}{dt}$ for any y when it still has an x in it? We must eliminate the x. Using the Pythagorean Theorem again, we can see that $x = \sqrt{y^2 - .00053}$. If we plug in this expression, we obtain an equation for $\frac{dy}{dt}$ that is in terms of y alone: $\frac{dy}{dt} = \frac{80\sqrt{y^2 - .00053}}{y}$.

Now we can plug in our specific y-values. The following table displays the values that this gives us for $\frac{dy}{dt}$, and thus the displayed speed:

y	$\frac{dy}{dt}$
1 mile	79.98 mph
1/2 mile	79.92 mph
1/4 mile	79.66 mph
1/10 mile	77.85 mph

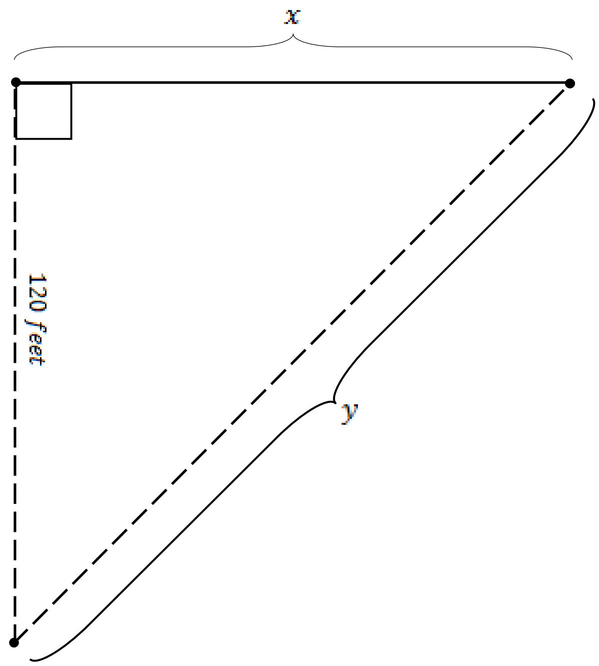
III. The Perpendicular and Passing Cases

You may notice that that the displayed speed is decreasing. This makes sense because both x and y are decreasing. It is also the case that $\frac{dy}{dt}$ is approaching zero, and that when you are directly perpendicular to the cop it will equal zero. In terms of physics, this is because the radio waves being emitted by the gun no longer bounce off the car at a different speed than they were emitted; it is as if the cop is pointing his radar gun at a brick wall. But mathematically speaking, why should this be so? Notice that when you are perpendicular to the cop, y=.023. This means that if we plug y into our equation we get $\frac{dy}{dt} = \frac{80\sqrt{.00053-.00053}}{(.0053)}=\frac{80\sqrt{0}}{(.0053)}=\frac{0}{2(.0053)}=0$. It may help to think of this as the *critical point* of this function, where the derivative is briefly zero before changing concavity.

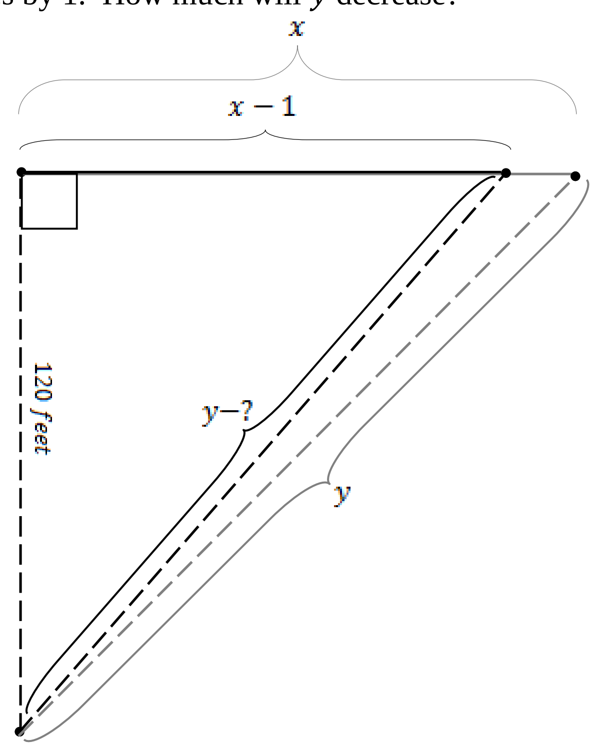
When you pass the cop, nothing changes mathematically. Since we can't have negative distance, all that happens is that we are working with the left side of our imaginary triangle instead of the right side. The equation and its results remain the same. In graphical terms, this means that the graph of the derivative is symmetrical around the y-axis.

IV. Why the Displayed Speed is Always Slower

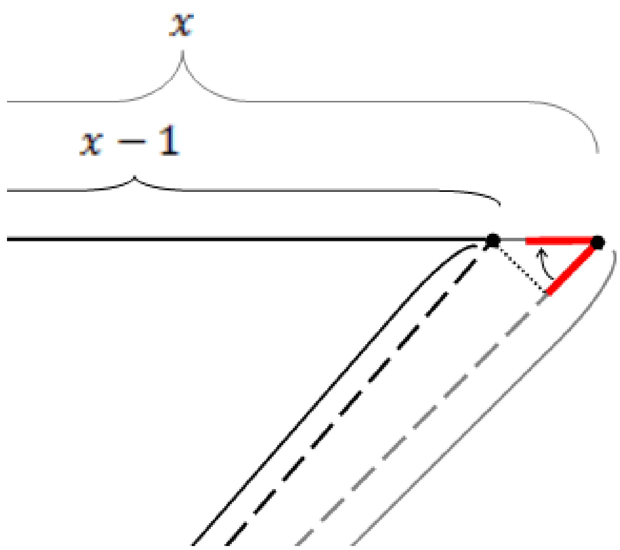
Don't ever tell this to a cop if you are pulled over, but the speed his radar gun displays is always going to be slower than your actual speed. This is because x always changes at a faster rate than y. Consider the right half of our imaginary triangle:



Suppose that x decreases by 1. How much will y decrease?



One way to think about this is geometrically. The amount y is decreasing by is roughly equal to the red line in the following diagram:



It certainly appears to be the case that this distance is shorter than the distance x is changing.

To take a more exact approach, consider our earlier equation of the derivative: $\frac{dy}{dt} = \frac{80x}{y}$. This is the equation we had before we plugged in an expression for x in order to make the equation in terms of y alone. Notice that we could also write this as $\frac{dy}{dt} = 80 \times \frac{x}{y}$. This means that if $\frac{x}{y} \geq 1$ then $\frac{dy}{dt} \geq 80$, and that if $\frac{x}{y} < 1$ then $\frac{dy}{dt} < 80$. Now consider the nature of a right triangle where x is a leg and y is the hypotenuse. Remember that $y^2 = x^2 + .00053$. We can plainly see from this that y will always be greater than x. Thus, $\frac{x}{y}$ will always be less than 1, and $\frac{dy}{dt}$ or the displayed speed will always be less than 80, the actual speed.

V. 10% Error

When will the displayed speed be off by 10%? In this case the error is $(displayed\ speed) - (actual\ speed)$. The percentage error will be $\frac{|(displayed\ speed)-(actual\ speed)|}{(actual\ speed)} \times 100$. We need to determine when this equation is equal to 10%, or in other words when $\frac{|(displayed\ speed)-(actual\ speed)|}{(actual\ speed)} \times 100 = 10$. First, we must plug in our actual speed, which is 80 miles per hour. $\frac{|(displayed\ speed)-80|}{80} \times 100 = 10$. Now we must solve for the displayed speed. Performing some algebraic manipulation yields $|(displayed\ speed) - 80| = 8$. Remember that when solving for an absolute value, we must take into account that the answer could be positive *or* negative, and so we must split this equation into two separate equations: $(displayed\ speed) - 80 = 8$ and $(displayed\ speed) - 80 = -8$. Solving for the displayed speed in each of these, we obtain answers of 88 and 72 respectively. Given that we know the displayed speed can never be higher than the actual speed, we can conclude that when the displayed speed is 72, the percent error will be 10%.

Now, at what distance will this happen? In other words, at what y-value will the percent error of the displayed speed be 10%? Recall that the displayed speed is equal to $\frac{dy}{dt}$. We solved for this earlier in the problem: $\frac{dy}{dt} = \frac{80\sqrt{y^2-.00053}}{y}$. So now we just need to solve for y when $\frac{dy}{dt} = 72$. First, multiply through by y: $72y = 80\sqrt{y^2 - .00053}$. Next divide by 80: $0.9y = \sqrt{y^2 - .00053}$. Then square both sides: $0.81y^2 = y^2 - .00053$. Now subtract y^2: $-0.19y^2 = -.00053$. Finally divide by -0.19 and take the square root: $y = \sqrt{.0027} =$ .052. So when the distance between you and the cop car is .053 of a mile, the speed displayed on his radar gun will be off by 10%.

VI. Conclusion

We saw how to find the speed displayed by the radar gun for any distance between you and the cop car. Next we saw that when you draw parallel to the cop the radar gun will display zero, and we considered what will happen once you drive past the cop. Then we saw why the displayed speed will always be lower than the actual speed. Finally, we deduced at what point the displayed speed will be off by 10% from the actual speed.