Andrew Zito

Courtney Thatcher

Calculus I

11 December 2013

Fun With Hyperbolic Trig Functions

Introduction

I will discuss a three part problem. In the first part, I will demonstrate why $\sinh(x)$ and $\cosh(x)$ are the derivatives of one another. In the second part, I will find a formula for the inverse of $\sinh(x)$. In the third part will calculate an arbitrary integral.

Part One: Hyperbolic Trig Function Derivatives

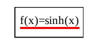
This section deals with the functions $\sinh(x)$ and $\cosh(x)$ and their derivatives. First, it is necessary to define both functions:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

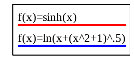
Let's take the derivative of $\sinh(x)$ – or rather, its expanded form – first. To start, pull out the constant $\frac{1}{2}$ obtaining $\frac{1}{2}\frac{d}{dx}(e^x - e^{-x})$. Now we can take the derivative of each piece. e^x is its own derivative. By using the chain rule on e^{-x} we obtain $\frac{d}{dx}e^x \cdot \frac{d}{dx}(-x)$, which simplifies to $-e^{-x}$. Thus we obtain $\frac{1}{2}(e^x + e^{-x})$ or $\frac{e^x+e^{-x}}{2}$. Notice that this is also the expanded form of $\cosh(x)$. The reverse of this property is also true. If we take the derivative of $\cosh(x)$, we will obtain $\frac{e^x - e^{-x}}{2}$, the expanded form of $\sinh(x)$, since there is no negative in the numerator of $\cosh(x)$ to cancel the negative from $\frac{d}{dx}e^{-x}$.

Part Two: Inverse of Hyperbolic Sine

First it is necessary to recognize that $\sinh(\mathrm{x})$ has an inverse. Consider the graph of this function:



This graph passes the horizontal line test, meaning that $\sinh(x)$ has complete one-to-one correspondence. Every x-value corresponds to only one y-value and every y-value corresponds to only one x-value. Thus we can deduce that there will be an inverse function, which will look like $\sinh(x)$ flipped over the line $y = x$:



To find an equation for $\operatorname{arcsinh}(x)$ we will solve for x and then swap x and y. If we multiply both sides of the expanded form of $\sinh(x)$ by two, thus removing the denominator, we obtain: $e^x - e^{-x} = 2y$. At this point, one might be tempted to use the natural logarithm to eliminate the "e"s and bring the exponents down. However, this will not work because you will end up taking the natural logarithm of a negative number. Instead, we must use the quadratic equation. To do this, we must convert our equation into a quadratic by multiplying through by e^x, obtaining: $e^x(e^x - e^{-x}) = 2e^x y$ which can be simplified to $(e^x)^2 - \frac{e^x}{e^x}) = 2e^x y$ then $(e^x)^2 - 1 = 2e^x y$ then $(e^x)^2 - 1 - 2e^x y = 0$. We can rewrite this in a way that looks more like a quadratic: $(e^x)^2 - 2y(e^x) - 1 = 0$. Now we can apply the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ In our problem, $a = 1$, $b = -2y$, and $c = -1$, and $x = e^x$. If we plug all this in we obtain: $e^x = \frac{-(-2y) \pm \sqrt{(-2y)^2 - 4(1)(-1)}}{2(1)} = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = \frac{2y \pm \sqrt{4}\sqrt{y^2 + 1}}{2} = y \pm \sqrt{y^2 + 1}$. Now we must take the natural log of both sides to isolate x, and so we obtain: $x = \ln(y \pm \sqrt{y^2 + 1})$. If we swap x and y, we get $\ln(x \pm \sqrt{x^2+1})$. Because $\sqrt{x^2+1}$ will always be more than x, we can deduce that $\ln(x - \sqrt{x^2+1})$, since you cannot take the natural logarithm of a negative number. Therefore, the final answer is $\sinh(x) = \ln(x + \sqrt{x^2+1})$.

Part Three: What About Integrals?

The integral we will be taking is $\int_0^1 \frac{1}{\sqrt{x^2+1}}$. To take this integral will we work in reverse order and take the derivative of $\operatorname{arcsinh}(x)$. Remember that the derivative $\frac{d}{dx}\ln f(x) = \frac{f\prime(x)}{f(x)}$ In our case, $f(x) = x + \sqrt{x^2+1}$, and so $\frac{d}{dx}\ln f(x) = \frac{1+\frac{1}{2}(x^2+1)^{-1/2}\cdot(2x)}{x+(x^2+1)^{1/2}} = \frac{1+x(x^2+1)^{-1/2}}{x+(x^2+1)^{1/2}}$. Now if we factor out $(x^2+1)^{-1/2}$, we obtain: $\frac{(x^2+1)^{-1/2}(x+\frac{1}{(x^2+1)^{-1/2}})}{x+(x^2+1)^{1/2}} = \frac{(x^2+1)^{-1/2}(x+(x^2+1)^{1/2})}{(x+(x^2+1)^{1/2})}$ We can cancel the denominator with the right-hand term in the numerator, leaving us with: $(x^2+1)^{-1/2} = \frac{1}{\sqrt{x^2+1}}$ This is, in fact, our integrand. Since we now know that $\operatorname{arcsinh}(x)$ is the anti-derivative of our integrand, we can use the Fundamental Theorem of Calculus to rewrite our integral as: $\operatorname{arcsinh}(1) + \operatorname{arcsinh}(0)$ and solve: $.881 - 0 = .881$.

Conclusion

We found that $\sinh(x)$ and $\cosh(x)$ are the derivatives of one another; we found the inverse of $\sinh(x)$; and finally we took the integral of $\operatorname{arcsinh}(x)$ as a way to find an integral. Hopefully these three sub-problems have demonstrated some of the properties of hyperbolic trigonometric functions.