ECEN 303: Homework 4
Fall 2024

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1. (3 points) For what values of the constant C do the following define the PMF on the positive integers $1, 2, \ldots$?

   (a) $p_X(k) = C \cdot 2^{-k}$

   (b) $p_X(k) = C \cdot 2^k/k!$

2. (3 points) A soccer team has 2 games scheduled for one weekend. It has a 0.4 probability of not losing the first game and a 0.7 probability of not losing the second game, independent of the first. If it does not lose a particular game, the team is equally likely to win or tie, independent of what happens in the other game. The team will receive 2 points for a win, 1 for a tie and 0 for a loss. Find the PMF of the number of points that the team earns over the weekend.

3. (3 points) We toss n coins, and each one shows heads with probability p, independently of each of the others. Each coin which shows heads is tossed again. What is the PMF of the number of heads resulting from the second round of tosses?

4. (4 points) An internet service provider uses 50 modems to serve the needs of 1000 customers. It is estimated that at a given time, each customer will need a connection with probability 0.01, independent of the other customers.

   (a) What is the PMF of the number of modems in use at the given time?

   (b) Repeat part (a) by approximating the PMF of the number of customers that need a connection with a Poisson PMF.

   (c) What is the probability that there are more customers needing a connection than there are modems? Provide an exact as well as an approximate formula based on the Poisson approximation of part (b).

5. (3 points) Is it generally true that $\mathrm{E}[\frac{1}{X}] = \frac{1}{\mathrm{E}[X]}$? Give a proof or a counter example.

6. (4 points) Suppose that two teams play a series of games that ends when one of them has won M games. Suppose that each game played is, independently, won by team A with probability p. Find the expected number of games that are played when (i) $M = 2$ and (ii) $M = 3$. Also, show in both cases that this number is maximized when $p = \frac{1}{2}$.

7. (4 points) There are two possible causes for a breakdown of a machine. To check the first possibility would cost C_1 dollars, and, if that were the cause of the breakdown, the trouble could be repaired at a cost of R_1 dollars. Similarly, there are costs C_2 and R_2 associated with the second possibility. Let p and $1-p$ denote, respectively, the probabilities that the breakdown is caused by the first and second possibilities. Under what conditions on $p, C_i, R_i, i = 1, 2$, should we check the first possible cause of breakdown and then the second, as opposed to reversing the checking order, so as to minimize the expected cost involved in returning the machine to working order?

   A slightly different answer can be obtained if we assume that there is no need to have the testing cost in the second round. In that case, we get

$$\mathrm{E}[X_1] = C_1 + pR_1 + (1-p)R_2, \qquad \mathrm{E}[X_2] = C_2 + (1-p)R_2 + pR_1$$

   and $\mathrm{E}[X_1] < \mathrm{E}[X_2]$, if $C_1 < C_2$.

8. (4 points) For a nonnegative integer-valued random variable N, show that

$$\mathrm{E}[N] = \sum_{k=0}^{\infty} P(N > k)$$

9. (3 points) Let X be a random variable having expected value μ and variance σ^2. Find the expected value and variance of the random variable $Y = (X - \mu)/\sigma$

10. (3 points) If X is Poisson random variable with parameter λ, compute $\mathrm{E}[1/(X+1)]$

11. (3 points) Let X be a random variable with mean μ and variance σ^2. Find the constant c that best approximates the random variable X in the sense that c minimizes the *mean squared error* $\mathrm{E}[(X-c)^2]$

12. (3 points) A computer reserves a path in a network for 10 minutes. To extend the reservation the computer must successfully send a "refresh" message before the expiry time. However, messages are lost with probability 1/2. Suppose that it takes 10 seconds to send a refresh request and receive an acknowledgment. When should the computer start sending refresh messages in order to have a 99% chance of successfully extending the reservation time?