ECEN 303: Random Signals and Systems

Chapter 3: Basic Concepts of Probability

References

- Chapter 3, Class Notes
- D. Bertsekas and J. Tsitsiklis, *Introduction to Probability*, Chapter 1.2
- M. Harchol-Balter, *Introduction to Probability for Computing*, Chapter 2
- S. Ross, *A First Course in Probability*, Chapter 2

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- Experiment of measuring the temperature at 2PM in College Station in August

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- Experiment of flipping a coin and rolling a die: $\Omega = ?$

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- Experiment of flipping a coin and rolling a die, event of getting a tails and a prime number: $E = ?$

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$$
\begin{array}{cccccc}
 & E_1 & & & E_2 & \\
\end{array}
$$

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\Omega = \left\{
\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
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- What is E_1^c?

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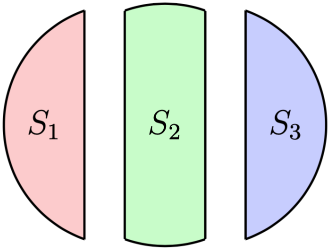

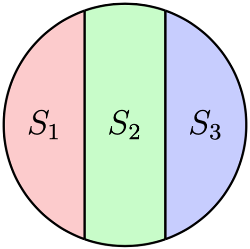

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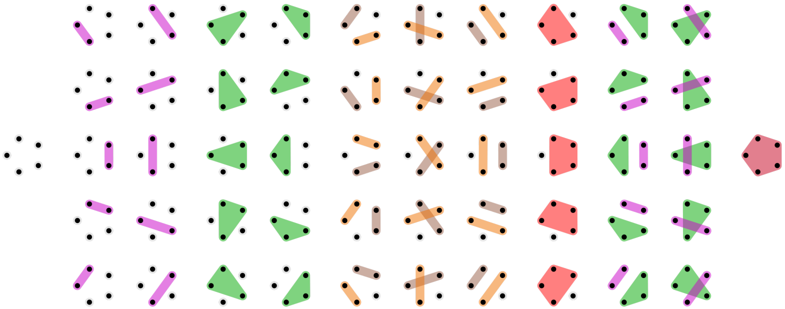

Set Difference

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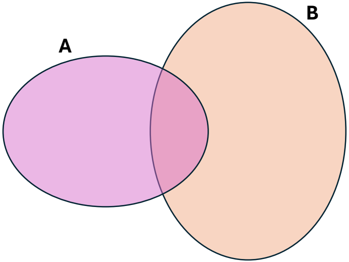

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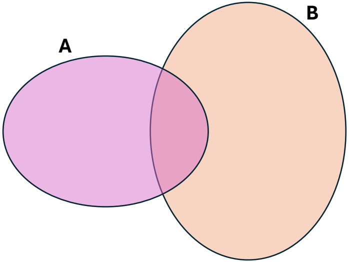

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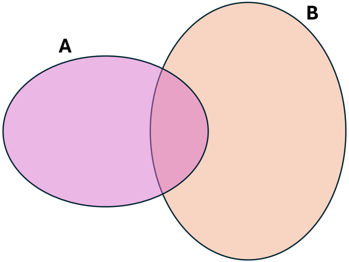

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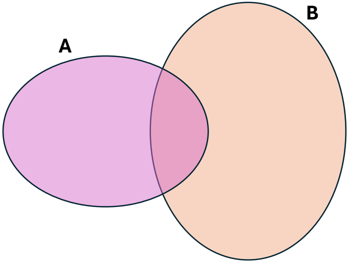

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- B and $A - B$ form a partition of the set $A \cup B$

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2. **(Normalization)** The probability of the sample space Ω is equal to one,
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3. **(Countable Additivity)** If A and B are disjoint events, $A \cap B = \emptyset$, then the probability of their union satisfies
$$P(A \cup B) = P(A) + P(B).$$

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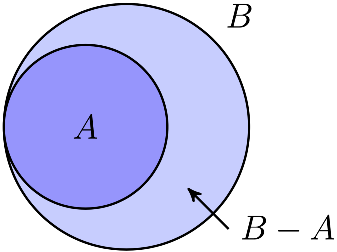

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$$P(B) = P(A) + P(B - A) \geq P(A),$$

where the inequality follows from the nonnegativity of probability laws. $\square$

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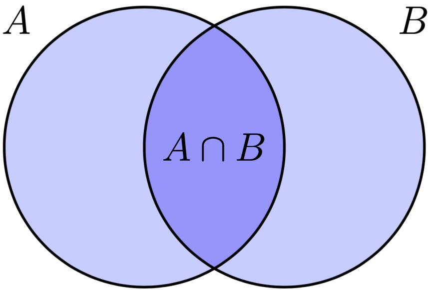

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Combining these two equations yields the desired result. $\square$

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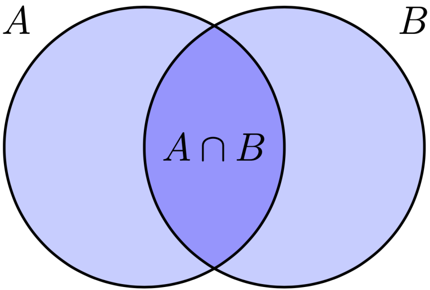

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Union Bound

Let $A_1, A_2, \ldots, A_n$ be a collection of events, then

$$P\left(\bigcup_{k=1}^{n} A_k\right) \le \sum_{k=1}^{n} P(A_k).$$

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Probability that he likes neither book is

$$P(B_1^c \cap B_2^c) = P((B_1 \cup B_2)^c) = 1 - P((B_1 \cup B_2)) = 1 - 0.6 = 0.4$$

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The probability of the number of coin tosses being even can be computed as

$$P(\{2, 4, 6, \ldots\}) = \sum_{k=1}^{\infty} P(2k) = \sum_{k=1}^{\infty} \frac{1}{2^{2k}} = \frac{1}{4} \frac{1}{\left(1 - \frac{1}{4}\right)} = \frac{1}{3}.$$

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- R and J plans to meet a coffee shop at a given time. Each will arrive at the meeting place with a delay between 0 and 1 hour, with all pairs of delays being equally likely. The first to arrive will wait for 15 minutes and will leave if the other has not yet arrived. What is the probability that they will meet?

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  This probability law satisfies the three probability axioms.

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