# ECEN 303: Random Signals and Systems

## Chapter 1: Mathematical Review

# References

- Chapter 1, Class Notes

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- $S \neq T$ expresses the fact that $S$ and $T$ are different

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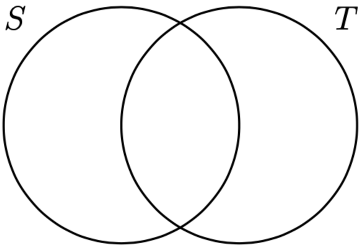


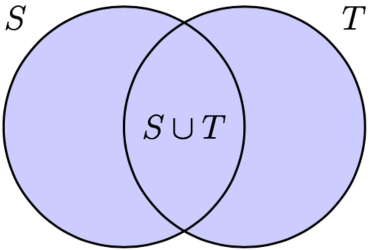

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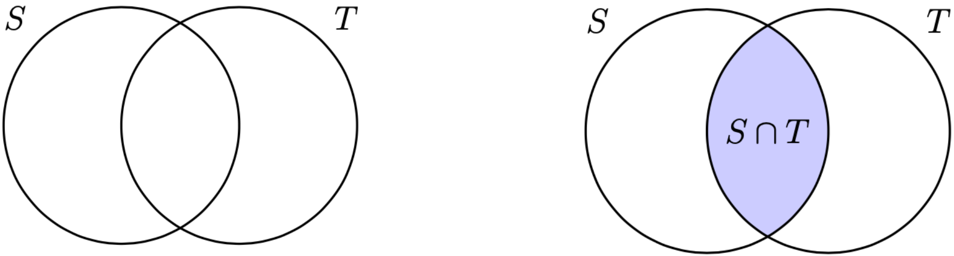

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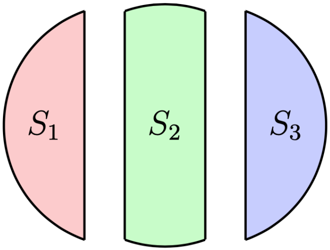


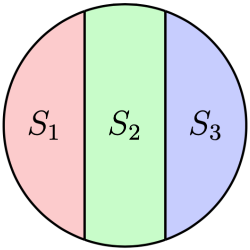

# Algebra of Sets: Distributive laws

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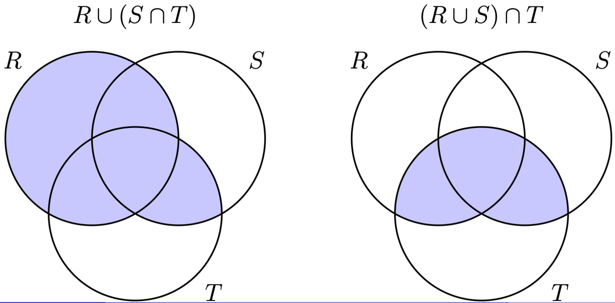

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# Proof of De Morgan's laws

# Cartesian Products

- Given sets $S$ and $T$, the cartesian product $S \times T$ is the set of all ordered pairs $(x, y)$ such that $x$ is an element of $S$ and $y$ is an element of $T$, $S \times T = \{(x, y) | x \in S \text{ and } y \in T\}$.

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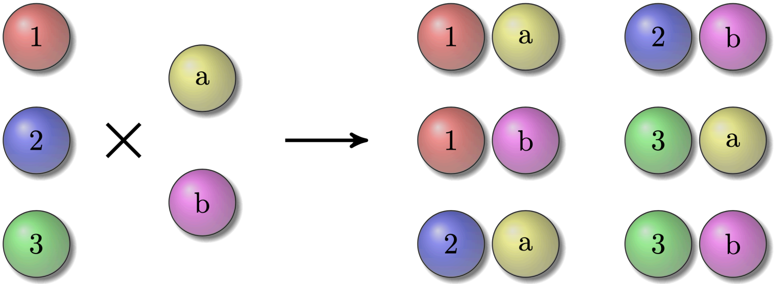

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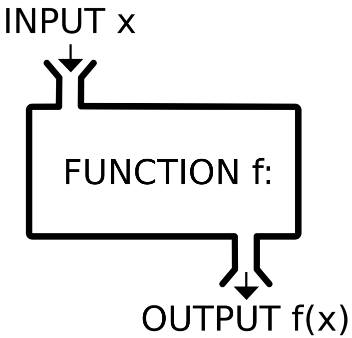


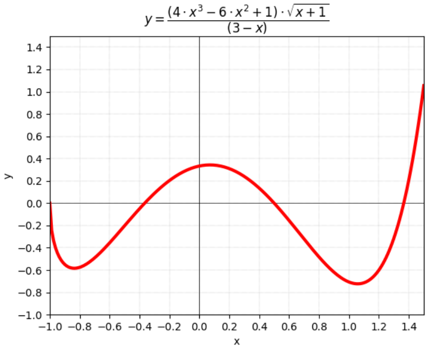


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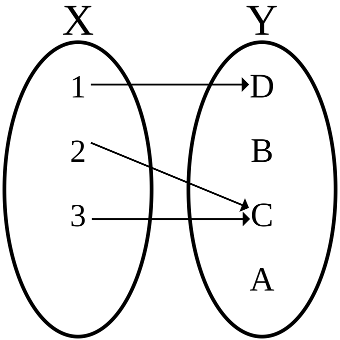


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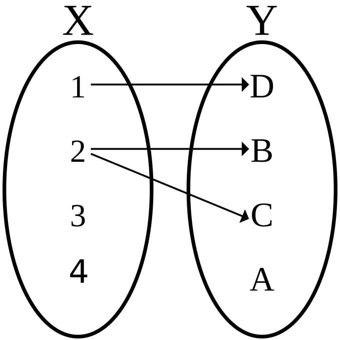


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- To specify these sets explicitly, the notation $f : X \to Y$ is frequently used, where $X$ indicates the domain and $Y$ is the codomain.
- The image of $X$ under function $f : X \to Y$ is the set of all objects of the form $f(x)$, where $x$ ranges over the elements of $X$: $\{f(x) \in Y | x \in X\}$

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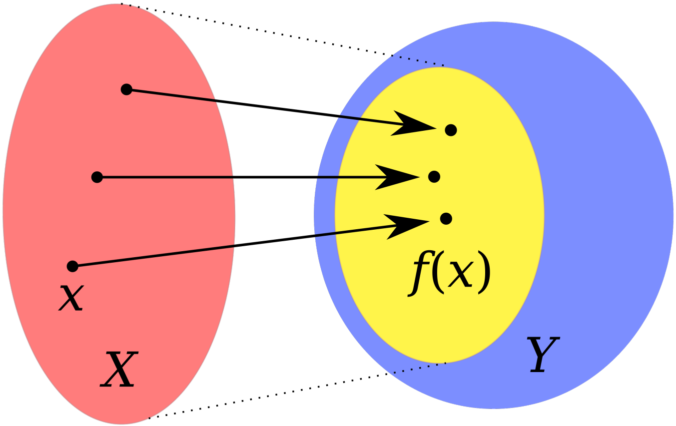


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# Function: Examples

- Example: Consider the function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$. In this case, the domain $\mathbb{R}$ and the codomain $\mathbb{R}$ are identical. The rule of correspondence for this function is $x \mapsto x^2$, which should be read "$x$ maps to $x^2$."

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- Example: Suppose $S$ is a subset of the real numbers. We define the indicator function of set $S$, denoted $\mathbf{1}_S : \mathbb{R} \to \{0, 1\}$, by

$$\mathbf{1}_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S. \end{cases}$$

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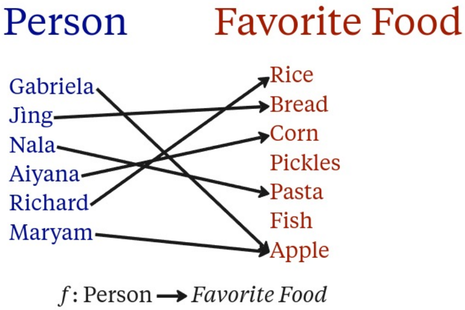


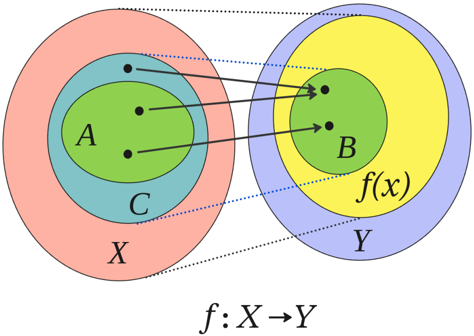


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# Onto (Surjective) function

- The function $f$ is surjective or onto if its image is equal to the codomain. More specifically, a function $f : X \to Y$ is surjective if and only if, for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

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# Examples