# ECEN 303: Random Signals and Systems

## Chapter 4: Conditional Probability

# References

- Chapter 4, Class Notes
- D. Bertsekas and J. Tsitsiklis, *Introduction to Probability*, Chapter 1.3-1.5
- M. Harchol-Balter, *Introduction to Probability for Computing*, Chapter
- S. Ross, *A First Course in Probability*, Chapter 3

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- Conditional probability of $A$ given $B$, denoted as $P(A|B)$

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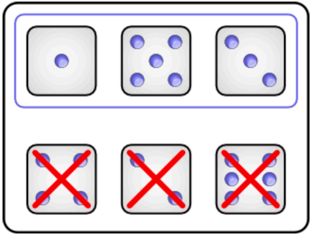

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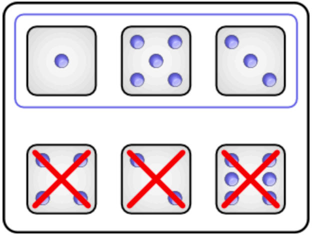

- In this example,

$$P(A|B) = \frac{\text{number of elements in } A \cap B}{\text{number of elements in } B}$$

# Conditional Probability

For any events $A, B$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

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$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3/8}{4/8} = \frac{3}{4}$$

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  $$\begin{aligned} P(A_1 \cup A_2|B) &= \frac{P((A_1 \cup A_2) \cap B)}{P(B)} = \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)} \\ &= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} = \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} \end{aligned}$$

# Conditional Probability: Examples

- A conservative design team, call it C, and an innovative design team, call it N, are asked to separately design a new product within a month. From past experience we know that:
  - The probability that team C is successful is $2/3$
  - The probability that team N is successful is $1/2$
  - The probability that at least one team is successful is $3/4$

  Assuming that exactly one successful design is produced, what is the probability that it was designed by team N?

# Conditional Probability: Examples

There are four possible outcomes here, corresponding to the four combinations of success and failure of the two teams:

$$SS: \text{ both succeed}, \qquad FF: \text{ both fail},$$
$$SF: \text{ C succeeds, N fails}, \qquad FS: \text{ C fails, N succeeds}.$$

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$$P(SS) + P(SF) = \frac{2}{3}, P(SS) + P(FS) = \frac{1}{2}, P(SS) + P(SF) + P(FS) = \frac{3}{4}.$$

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From these relations, together with the normalization equation $P(SS)+P(SF)+P(FS)+P(FF)=1$, we can obtain the probabilities of all the outcomes:

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The desired conditional probability is

$$P(\{FS\}|\{SF,FS\})=\frac{P(\{FS\})}{P(\{SF,FS\})}=\frac{\frac{1}{12}}{\frac{1}{4}+\frac{1}{12}}$$

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Let $A_1, A_2, \ldots, A_n$ be a collection of events. Then,

$$P\left(\bigcap_{k=1}^{n} A_k\right) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P\left(A_n \middle| \bigcap_{k=1}^{n-1} A_k\right).$$

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Then, the probability of getting three blue balls is

$$P(\cap_{i=1}^{3} B_i) = P(B_1)P(B_2|B_1)P(B_3|B_1 \cap B_2) = \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10}$$

# Conditional Probability: Examples

- If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of false alarm (a false indication of aircraft presence), and the probability of missed detection (nothing registers, even though an aircraft is present)?

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Consider the events

$$A = \{\text{aircraft is present}\}, \qquad B = \{\text{radar registers an aircraft presence}\}$$

$$A^{c} = \{\text{aircraft is not present}\}, \qquad B = \{\text{radar does not register an aircraft presence}\}$$

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Now, we can compute

$$P(\text{false alarm}) = P(A^c \cap B) = P(A^c)P(B|A^c) = 0.95 \cdot 0.1 = 0.095,$$
$$P(\text{missed detection}) = P(A \cap B^c) = P(A)P(B^c|A) = 0.05 \cdot 0.01 = 0.0005.$$

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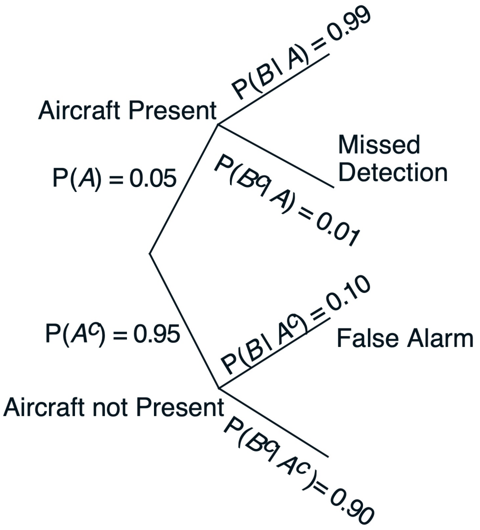
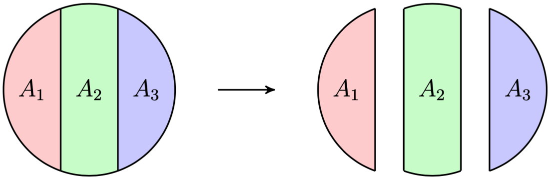


# Total Probability Theorem

- Consider the set $A$ and its partition $A_1, A_2, A_3$
  - Since $A_i$s are disjoint and $A = \cup_{i=1}^{3} A_i$, we get $P(A) = \sum_{i=1}^{3} P(A_i)$



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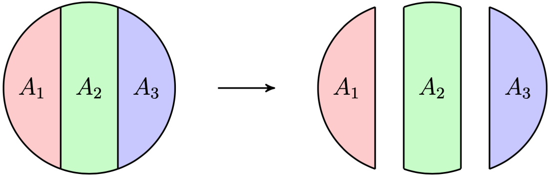
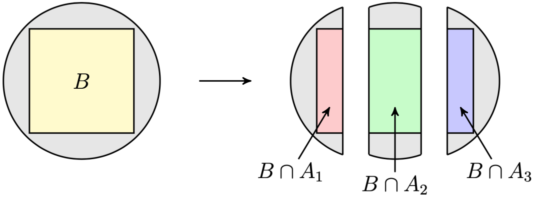



- Now consider the set $B \subset A$ and its partitions $B_1, B_2, B_3$, where $B_i = B \cap A_i$



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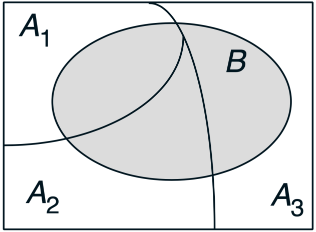

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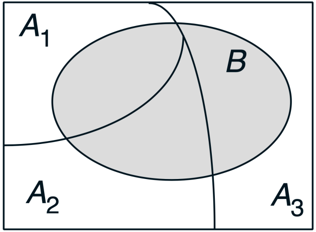


- From the previous figures, it is easy to see that

$$P(B) = \sum_{i=1}^{3} P(B_i) = \sum_{i=1}^{3} P(B \cap A_i) = \sum_{i=1}^{3} P(B|A_i)P(A_i)$$

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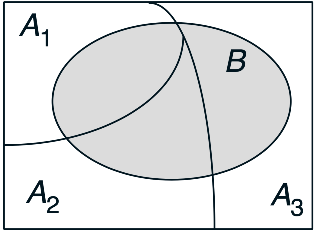


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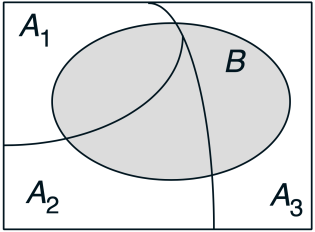


## Total Probability Theorem

Let $A_1, A_2, \ldots, A_n$ be a collection of events that forms a partition of the sample space $\Omega$. Suppose that $P(A_k) > 0$ for all $k$. Then, for any event $B$, we can write

$$\begin{aligned} P(B) &= P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B) \\ &= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n). \end{aligned}$$

# Total Probability Theorem

Proof: The collection of events $A_1, A_2, \ldots, A_n$ forms a partition of the sample space $\Omega$. We can therefore write

$$B = B \cap \Omega = B \cap \left( \bigcup_{k=1}^{n} A_k \right).$$

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Since $A_1, A_2, \ldots, A_n$ are disjoint sets, the events $A_1 \cap B, A_2 \cap B, \ldots, A_n \cap B$ are also disjoint.

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Since $A_1, A_2, \ldots, A_n$ are disjoint sets, the events $A_1 \cap B, A_2 \cap B, \ldots, A_n \cap B$ are also disjoint. Combining these two facts, we get

$$\begin{aligned} P(B) &= P\left( B \cap \left( \bigcup_{k=1}^{n} A_k \right) \right) = P\left( \bigcup_{k=1}^{n} (B \cap A_k) \right) \\ &= \sum_{k=1}^{n} P\left( B \cap A_k \right) = \sum_{k=1}^{n} P(A_k) P\left( B | A_k \right), \end{aligned}$$

where the fourth equality follows from the third axiom of probability. $\square$

# Total Probability Theorem: Examples

- An urn contains five green balls and three red balls. A second urn contains three green balls and nine red balls. One of the two urns is picked at random, with equal probabilities, and a ball is drawn from the selected urn. We wish to compute the probability of obtaining a green ball.

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$$P(G|U_1) = \frac{5}{8}, \quad P(G|U_2) = \frac{3}{12}, \quad P(U_i) = \frac{1}{2}, i = 1, 2$$

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Using this, we can compute

$$\begin{aligned} P(G) &= P(G \cap U_1) + P(G \cap U_2) \\ &= P(G|U_1)P(U_1) + P(G|U_2)P(U_2) \\ &= \frac{5}{8}\frac{1}{2} + \frac{3}{12}\frac{1}{2} = \frac{7}{16} \end{aligned}$$

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Since Alice starts her class up-to-date, we have $P(U_1) = 0.8, \quad P(B_1) = 0.2$
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Two events are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

# Independence: Example

- Suppose that two dice are rolled at the same time, a red die and a blue die. We observe the numbers that appear on the upper faces of the two dice. The sample space for this experiment is composed of thirty-six equally likely outcomes. Verify if the following events are independent:
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$$P(\{R=4\}\cap\{R+B=11\})=0,\quad P(\{R=4\}=\frac{6}{36}\quad P(\{R+B=11\}=\frac{2}{36}$$

In this case, we conclude that getting a four on the red die and a sum total of eleven are not independent events.

# Independence: Example

- If events $A$ and $B$ are independent, then $A^c$ and $B^c$ are independent.

# Independence of Multiple Events

The events $A_1, A_2, \ldots, A_n$ are independent provided that

$$P\left(\bigcap_{i \in \mathbb{I}} A_i\right) = \prod_{i \in \mathbb{I}} P(A_i),$$

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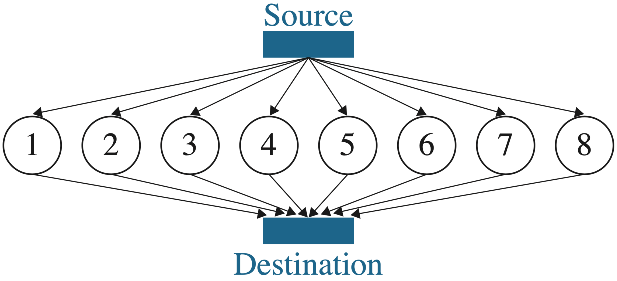
The three equalities in (1) assert that $A$, $B$ and $C$ are pairwise independent. Note that the fourth equation does not follow from the first three conditions, nor does it imply any of them. Pairwise independence does not necessarily imply independence.

# Independence: Example

- Suppose you are routing a packet from the source node to the destination node, as shown in the Figure below. On the plus side, there are 8 possible paths on which the packet can be routed. On the minus side, each of the 16 edges in the network independently only works with probability $p$. What is the probability that the transmitted packet will be received successfully at the destination?



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$P(\text{at least one path works}) = 1 - P(\text{no (zero) path works}) = 1 - P(E^c)$
$P(E^c) = P(E_1^c \cap E_2^c \cap \cdots \cap E_8^c) = P(E_1^c) \cdot P(E_2^c) \cdots P(E_8^c) = (1 - p^2)^8$

# Independence: Example

- A fair coin is flipped twice. Let $A$ denote the event that heads is observed on the first toss. Let $B$ be the event that heads is obtained on the second toss. Finally, let $C$ be the event that the two coins show distinct sides. Verify if $A, B, C$ are independent.

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and, therefore, these events are *pairwise independent*. However, we can verify that

$$P(A \cap B \cap C) = 0 \neq \frac{1}{8} = P(A)P(B)P(C).$$

This shows that events $A$, $B$ and $C$ are not independent.

# Conditional Independence

We say that events $A_1$ and $A_2$ are conditionally independent, given event $B$, if

$$P(A_1 \cap A_2 | B) = P(A_1 | B) P(A_2 | B).$$

# Conditional Independence

- By definition of conditional independence,

$$P(A_1 \cap A_2 | B) = \frac{P(A_1 \cap A_2 \cap B)}{P(B)} = \frac{P(B)P(A_1|B)P(A_2|A_1 \cap B)}{P(B)}$$
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- If $A_1$ and $A_2$ are conditionally independent, given event $B$, then

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- If $A_1$ and $A_2$ are conditionally independent, given event $B$, then

$$P(A_1|B)P(A_2|A_1 \cap B) = P(A_1|B)P(A_2|B)$$

- Under the assumption that $P(A_1|B) > 0$, we then get

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- Under the assumption that $P(A_1|B) > 0$, we then get

$$P(A_2|A_1 \cap B) = P(A_2|B).$$

- This implies that, given event $B$ has taken place, the additional information that $A_1$ has also occurred does not affect the likelihood of $A_2$.

# Conditional Independence: Example

- Two dice are rolled at the same time, a red die and a blue die. Let $A_1$ be the event of getting two on red die and $A_2$ be the event of getting 6 on blue die. Let $B$ be the event of getting the an odd number as the sum of the two dice.
  - Verify if $A_1$ and $A_2$ are independent

# Conditional Independence: Example

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- Two events that are independent with respect to an unconditional probability law may not be conditionally independent.

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Proof: Bayes' rule is easily verified. We can write the probability of $A \cap B$ as $P(A \cap B) = P(A|B)P(B)$. Rearranging the terms yields the first equality. $\square$

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Let $A_1, A_2, \ldots, A_n$ be a collection of events that forms a partition of the sample space $\Omega$. Suppose that $P(A_k) > 0$ for all $k$. Then, for any event $B$ such that $P(B) > 0$, we can write

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)}$$

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$$P(A_i \cap B) = P(A_i|B)P(B) = P(B|A_i)P(A_i).$$

Rearranging the terms yields the first equality. The second equality is obtained by applying 'Total Probability Theorem' to the denominator $P(B)$. □

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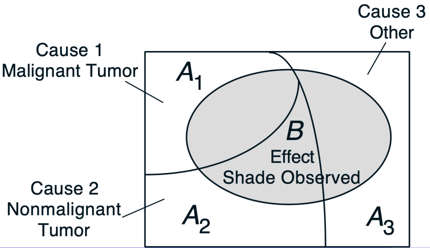
- We observe a shade in a person's X-ray (this is event $B$, the *effect*) and we want to estimate the likelihood of three mutually exclusive and collectively exhaustive potential *causes*: cause 1 (event $A_1$) is that there is a malignant tumor, cause 2 (event $A_2$) is that there is a nonmalignant tumor, and cause 3 (event $A_3$) corresponds to reasons other than a tumor. We assume that we know the probabilities $P(A_i)$ and $P(B|A_i)$, $i = 1, 2, 3$. We want to compute $P(A_1|B)$



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- (Example) If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05. Compute $P(\text{aircraft present} \mid \text{radar registers})$.

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$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{0.05 \cdot 0.99}{0.05 \cdot 0.99 + 0.95 \cdot 0.1} \approx 0.3426.$$

# Bayes' Rule: Example

- April, a biochemist, designs a test for a latent disease. If a subject has the disease, the probability that the test results turn out positive is $0.95$. Similarly, if a subject does not have the disease, the probability that the test results come up negative is $0.95$. Suppose that one percent of the population is infected by the disease. We wish to find the probability that a person who tested positive has the disease.

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Let $D$ denote the event that a person has the disease, and let $A$ be the event that the test results are positive. Using Bayes' rule,

$$\begin{aligned} P(D|A) &= \frac{P(D)P(A|D)}{P(D)P(A|D) + P(D^c)P(A|D^c)} \\ &= \frac{0.01 \cdot 0.95}{0.01 \cdot 0.95 + 0.99 \cdot 0.05} \\ &\approx 0.1610. \end{aligned}$$

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Although the test may initially appear fairly accurate, the probability that a person with a positive test carries the disease remains small.