ECEN 303: Random Signals and Systems

Chapter 2: Intuitive Probability and Combinatorics

References

- Chapter 2, Class Notes
- D. Bertsekas and J. Tsitsiklis, *Introduction to Probability*, Chapter 1.6.
- S. Ross, *A First Course in Probability*, Chapter 1.

Intuitive Probability

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- Experiment of measuring the temperature at 2PM in College Station in August

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- Experiment of flipping a coin and rolling a die, event of getting a tails and a prime number: $E = ?$

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Combinatorics

Counting Principle

Consider a process consists of k stages.

- There are n_1 possible outcomes at first stage
- There are n_2 possible outcomes at second stage
- $\vdots$
- There are n_k possible outcomes at k-th stage

Then, the total number of possible outcomes of the k stage process is

$$n_1 \cdot n_2 \cdots n_k$$

Counting Principle - Illustration

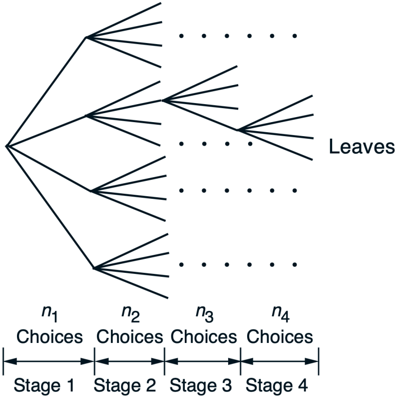

Figure 1.16: Illustration of the basic counting principle. The counting is carried out in r stages ($r = 4$ in the figure). The first stage has n_1 possible results. For every possible result of the first $i-1$ stages, there are n_i possible results at the ith stage. The number of leaves is $n_1 n_2 \cdots n_r$. This is the desired count.

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- Example: Three students - Amy, Bella, Chen - have to form a queue to get a coffee from Starbucks. How many different arrangements are possible?

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- Example: How many 3 letter words can be formed using the letters A, B, C, D, E, without using any letter more than once.

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  - The number of possibilities is equivalent to the number of k-permutations of n elements, which is given by $n!/(n-k)!$.

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- Number of k-permutations of a set with with n elements is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!}$$

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- Consider the set $S = \{a, b, c, d\}$. Number of 2-permutations of the elements of S? Number of 2-combinations of the elements of S?

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- Example (**Sampling without Replacement, without Ordering**): An urn contains n balls numbered one through n. A ball is drawn from the urn and placed in a separate jar. This process is repeated until the jar contains k balls, where $k \leq n$. We wish to compute the number of distinct combinations the jar can hold after the completion of this experiment.

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  - Because there is no ordering in the jar, this amounts to counting the number of k-element subsets of a given n-element set, which is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

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  - Using counting principle

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}$$

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  - The number of distinct sequences in which one, three and five each appear three times is equal to the number of partitions of $\{1, 2, \ldots, 9\}$ into three subsets of size three, namely

$$\frac{9!}{3!3!3!} = 1680.$$

Partition: Integer Solutions to Linear Equations

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$$\sum_{i=1}^{k} n_i = n.$$

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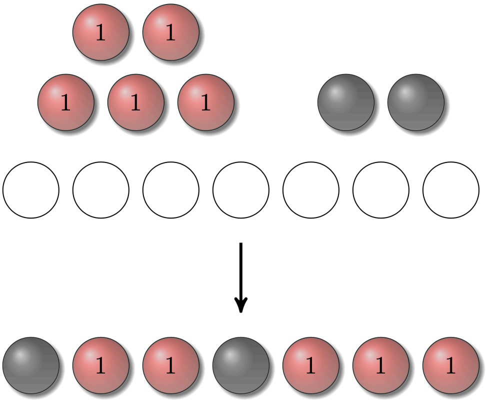

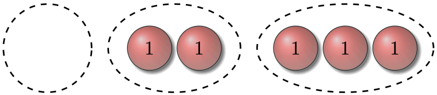

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$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

Partition: Integer Solutions to Linear Equations

- Example: **(Sampling with Replacement, without Ordering)** An urn contains k balls numbered one through k. A ball is drawn from the urn and its number is recorded. The ball is then returned to the urn. This procedure is repeated a total of n times. The outcome of this experiment is a table that contains the number of times each ball has come in sight. We are interested in computing the number of possible outcomes.

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- This is equivalent to counting the ways a set with n elements can be partitioned into k subsets. The number of possible outcomes is therefore given by

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

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   - $\binom{m+n}{n} = \binom{m+n}{m} = \frac{(m+n)!}{n!m!}$

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   - $3! \cdot 4!$

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   - $n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$

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   - Cardinality of the event $E = 4! \cdot \frac{12!}{3! \cdot 3! \cdot 3! \cdot 3!}$
   - Probability of the event

$$\frac{|E|}{|\Omega|} = \frac{\frac{4! \cdot 12!}{3! \cdot 3! \cdot 3! \cdot 3!}}{\frac{16!}{4! \cdot 4! \cdot 4! \cdot 4!}}$$