MATH 106 HOMEWORK 3 SOLUTIONS

1. Using the Cauchy-Riemann equations, show that if f and $\bar{f}$ are both holomorphic then f is a constant.

Solution: Let $f = u+iv$, so $\bar{f} = u-iv$. Since they are holomorphic, we can use the Cauchy-Riemann equations:

$$u_x = v_y \quad \text{and} \quad u_x = -v_y \Rightarrow u_x = v_y = 0$$
$$u_y = -v_x \quad \text{and} \quad u_y = v_x \Rightarrow u_y = v_x = 0$$

Therefore $u_x = u_y = 0$ so u is constant, and similarly $v_x = v_y = 0$ so v is constant. Hence f is constant as well.

2. Determine the holomorphic functions f and g such that

$$\text{Re} f = x^2 - y^2 - 2y, \quad \text{Im} g = 2xy + y.$$

Solution: Let $f = u + iv$. Then,

$$u = x^2 - y^2 - 2y \Rightarrow u_x = 2x = v_y \Rightarrow v = 2xy + \phi(x) \Rightarrow$$
$$-v_x = -2y - \phi'(x) = u_y = -2y - 2 \Rightarrow \phi(x) = 2x + c \Rightarrow$$
$$f = x^2 - y^2 - 2y + i(2xy + 2x + c)$$

where c is a real constant.

Let $g = u + iv$. We have:

$$v = 2xy + y \Rightarrow v_y = 2x + 1 = u_x \Rightarrow u = x^2 + x + \phi(y) \Rightarrow$$
$$u_y = \phi'(y) = -v_x = -2y \Rightarrow \phi(y) = -y^2 + C \Rightarrow$$
$$g = x^2 - y^2 + x + c + i(2xy + y)$$

where as before c is a real constant.

3. Determine the domains where the following functions are holomorphic

(i) $f(z) = \frac{1}{z^3 - 8i}$

(ii) $f(z) = \frac{1}{z^2 + 2iz + 1}$

Solution:

(i) $z^3 = 8i = 8e^{\frac{\pi}{2}i + 2k\pi i} \Rightarrow z = 2e^{\frac{\pi i}{6} + \frac{2k\pi i}{3}}, k = 0, 1, 2$. The domain is

$$\mathbb{C} \setminus \left\{ \sqrt{3} + i, i - \sqrt{3}, -2i \right\}.$$

(ii) $z^2 + 2iz + 1 = (z+i)^2 + 2 \Rightarrow (z+i)^2 = -2 \Rightarrow z + i = \pm i\sqrt{2}$ so the domain is $\mathbb{C} \setminus \left\{ -i \pm i\sqrt{2} \right\}$.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $|f|$ is the constant function c. Check that $\bar{f}$ is holomorphic.

Solution: If $c = 0 \Rightarrow |f| = f = \bar{f} = 0$, hence $\bar{f}$ is holomorphic. Suppose $|f| = c \neq 0 \Rightarrow f(z) \neq 0 \quad \forall z \in \mathbb{C}$. Since f is holomorphic, we have $\frac{c^2}{f}$ is holomorphic. Now, $f\bar{f} = |f|^2 = c^2$ therefore $\bar{f} = \frac{c^2}{f}$, which we showed is holomorphic. So by Problem 1, f is a constant.

5. Consider the two functions

$$f(z) = e^{-y} \sin x - ie^{-y} \cos x, \quad \text{and} \quad g(z) = e^{y} \cos x + ie^{y} \sin x$$

Using the Cauchy-Riemann equations, prove that f is entire, but g is nowhere holomorphic.

Solution: f is defined everywhere on $\mathbb{C}$ and has continuous partial derivatives. To show it's holomorphic we need only to check the Cauchy-Riemann equations. We have

$$u_x = e^{-y} \cos x = v_y \quad \text{and} \quad u_y = -e^{-y} \sin x = -v_x$$

For g on the other hand we have

$$u_x = -e^{y} \sin x \text{ and } v_y = e^{y} \sin x.$$

Similarly

$$u_y = e^{y} \cos x \text{ and } v_x = e^{y} \cos x.$$

Hence $u_x = v_y$ implies $\sin x = 0$, while $u_y = -v_x$ implies $\cos x = 0$. These cannot be satisfied for the same values of x. Hence g is nowhere holomorphic.

6. Show that the function u is harmonic and determine its harmonic conjugate when
(i) $u(x, y) = 2x(1 - y)$
(ii) $u(x, y) = e^{-2x} \sin(2y)$

Solution: (i) We have $u(x, y) = 2x - 2xy$, so

$$u_{xx} = u_{yy} = 0 \Rightarrow u_{xx} + u_{yy} = 0.$$

We find the harmonic conjugate:

$$\begin{aligned}
&u_x = 2(1 - y) = v_y \Rightarrow v = 2y - y^2 + \phi(x) \\
&- v_x = -\phi'(x) = u_y = -2x \Rightarrow \phi(x) = x^2 + c \Rightarrow \\
&v = x^2 - y^2 + 2y + c
\end{aligned}$$

(ii) We have

$$\begin{aligned}
&u_x = -2e^{-2x} \sin(2y) \Rightarrow u_{xx} = 4e^{-2x} \sin(2y) \\
&u_y = 2e^{-2x} \cos(2y) \Rightarrow u_{yy} = -4e^{2x} \sin(2y) \Rightarrow u_{xx} + u_{yy} = 0
\end{aligned}$$

We can now determine the harmonic conjugate:

$$\begin{aligned}
&u_x = -2e^{-2x} \sin(2y) = v_y \Rightarrow v = e^{-2x} \cos(2y) + \phi(x) \\
&- v_x = 2e^{-2x} \cos(2y) - \phi'(x) = u_y = 2e^{-2x} \cos(2y) \Rightarrow \phi(x) = c \Rightarrow \\
&v = e^{-2x} \cos(2y) + c.
\end{aligned}$$

7. (i) Show that if v is a harmonic conjugate of u and if u is a harmonic conjugate of v then both u and v are constant.
(ii) Show that if v is a harmonic conjugate of u, then $-u$ is a haramonic conjugate of v.

Solution:
(i) By definition, the following four equations are satisfied:

$$u_x = v_y \text{ and } u_y = -v_x$$
$$v_x = u_y \text{ and } v_y = -u_x.$$

Together, they imply $u_x = u_y = v_x = v_y = 0$. Therefore u and v are constant functions.

(ii) By definition, $u_x = v_y$ and $u_y = -v_x$. Hence, $v_x = -u_y$ and $v_y = -(-u_x)$ showing that $-u$ is a harmonic conjugate of v.