2.3.2 *Show that* $u = \sin(x)\sinh(y)$ *and* $v = \cos(x)\cosh(y)$ *satisfy the Cauchy-Riemann equations. What analytic function* $f = u + iv$*?*

We want to verify that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

In this example

$$\frac{\partial u}{\partial x} = \cos(x)\sinh(y) = \frac{\partial v}{\partial y},$$

and

$$\frac{\partial u}{\partial y} = \sin(x)\cosh(y) = -\frac{\partial v}{\partial x},$$

as desired.

Elementary manipulations gives

$$u + iv = \sin(x)\sinh(y) + i\cos(x)\cosh(y) = \frac{i}{2}[e^{ix+y} + e^{-ix+y}].$$

Since $iz = ix - y$ this gives

$$u + iv = \frac{i}{2}[e^{ix-y} + e^{-ix+y}] = \frac{i}{2}[e^{iz} + e^{-iz}] = i\cos(z).$$

2.3.3 *Show that if* f *and* $\overline{f}$ *are both analytic on a domain* D*, then* f *is constant.*

The functions $u = (f + \overline{f})/2$ and $v = (f - \overline{f})/(2i)$ would then be analytic and real valued, so constant by the Theorem on page 50.

2.3.8 *Derive the polar form of the Cauchy-Riemann equations,*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

By the chain rule

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta).$$

and

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial f}{\partial x}r\sin(\theta) + \frac{\partial f}{\partial y}r\cos(\theta).$$

Rewrite this as

$$r\frac{\partial f}{\partial r} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y},$$

$$\frac{\partial f}{\partial \theta} = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

The Cauchy-Riemann equations give

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = r\frac{\partial u}{\partial r} = x\frac{\partial v}{\partial y} - y\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \theta}$$

and

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = r\frac{\partial v}{\partial r} = -x\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial \theta}.$$

2.4.4

$$\frac{d}{dz}Tan^{-1}(z) = \frac{d}{dz}\frac{1}{2i}Log(\frac{1+iz}{1-iz}) = \frac{1}{2i}\frac{1-iz}{1+iz}\frac{i(1-iz)+i(1+iz)}{(1-iz)^2} = \frac{1}{1+z^2}.$$

Changing the branch alters the logarithm by adding a constant, so does not affect the derivative.

2.4.5

$$\frac{d}{dz}cos^{-1}(z) = -i\frac{d}{dz}log(z \pm \sqrt{z^2-1}$$

$$= -i\frac{1}{z \pm \sqrt{z^2-1}}\frac{\sqrt{z^2-1} \pm z}{\sqrt{z^2-1}} = \pm\frac{1}{\sqrt{1-z^2}}.$$

Even as a real valued function of a real variable the function $y = \cos^{-1}(x)$ has derivatives of opposite sign for $0 < y < \pi/2$ versus $-\pi/2 < y < 0$. Thus changing the branch changes the derivative by a sign.

2.5.2 *Show that if v is a harmonic conjugate for u, then $-u$ is a harmonic conjugate for v.*

The definition says v is a harmonic conjugate for u if both u and v are harmonic and $f = u + iv$ is analytic. We're given that $f = u + iv$ is analytic. Simply note that $-if = v - iu$ is still analytic, so $-u$ is a harmonic conjugate for v.

2.5.5 *Show that Laplace's equation in polar coordinates is*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Actually, the result is a bit stronger than stated. In their respective coordinates we have

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

First multiply by r^2 to rewrite the equation as

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0.$$

From the solution to problem 2.3.8 we have

$$r\frac{\partial u}{\partial r}=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial \theta}=-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}.$$

Now use the product rule to expand

$$r\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}=r^2\frac{\partial^2 u}{\partial r}+r\frac{\partial u}{\partial r}=[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}][x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}]$$

$$=x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}+xy\frac{\partial^2 u}{\partial x\partial y}.$$

Similarly,

$$\frac{\partial^2 u}{\partial \theta^2}=[-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}][-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}]$$

$$=y^2\frac{\partial^2 u}{\partial x^2}+-y\frac{\partial u}{\partial y}-xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}+-x\frac{\partial u}{\partial x}-xy\frac{\partial^2 u}{\partial x\partial y}.$$

Addition gives

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=(x^2+y^2)\frac{\partial^2 u}{\partial x^2}+(x^2+y^2)\frac{\partial^2 u}{\partial y^2}$$

as desired.