# DIFFERENSATION.

Let $y = f(x)$ be a given function, assume that $\Delta y$ is the incriment in $y$. Corresponding to a small incriment $\Delta x$ in $x$. Then the ratio $\frac{\Delta y}{\Delta x}$ is known as the incrimental ratio which gives average rate of change of $y$ with $x$

- The instantaneous rate of change of $y$ with respect to $x$ or derivative of w.r.t $x$ is given by

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}} \quad \text{— 1st principle.}$$

This method is known as method of 1st principle

*** Find the derivative of $y = x^n$ using the method of 1st principle.

A. $y = x^n$.

$\Delta y$ = incriment in $y$ / $\Delta x$ = incriment in $x$.

$$y + \Delta y = (x + \Delta x)^n.$$

$$\Delta y = (x + \Delta x)^n - y.$$

$$= (x + \Delta x)^n - x^n.$$

apply in 1st principle.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}.$$

as $\Delta x \to 0$, $x + \Delta x \to x$.

The above limit can be written as.

$$\lim_{x+\Delta x \to x} \frac{(x+\Delta x)^n - x^n}{x+\Delta x - x} = nx^{n-1}$$ (wher $n$ is rational no.)

eg: 1) $\frac{d(k)}{dx} = 0$, $k$ is constant.

$-2 \cdot x^{-3}$

$-2/x^3$

2) $\frac{d(x)}{dx} = 1$

3) $\frac{d(x^2)}{dx} = 2.x$.

$-2/x^3$

4) $\frac{d(x^{12})}{dx} = 12.x^{11}$.

5). $\frac{d(x^{-2})}{dx} = -2.x^{-2-1} = -2.x^{-3} = \frac{-2}{x^3}$

6). $\frac{d(x^{-5})}{dx} = \frac{-5.}{x^6}$

$\frac{3}{2} - \frac{1}{1} = \frac{3-2}{2} = \frac{1}{2}$

7). $\frac{d(x^{-n})}{dx} = \frac{-n}{x^{n+1}}$

8) $\frac{d(\sqrt{x})}{dx} = \frac{d x^{1/2}}{dx} = \frac{1}{2}.x^{1/2-1} = \frac{1}{2}.x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

8) $\frac{d(x^{3/2})}{dx} = \frac{3}{2}.x^{1/2}. = \frac{3\sqrt{x}}{2}$

9) $\frac{d(\frac{1}{x})}{dx} = \frac{-1}{x^2}$

10) $\frac{d}{dx}\cos x = -\sin x$

11) $\frac{d}{dx}\sin x = \cos x$

properties of derivatives

$x^n = nx^{n-1}$

$x^{-n} = \frac{-n}{x^{n+1}}$

a) $\frac{d}{dx}(Kf(x)) = K.\frac{d}{dx}(f(x))$.

eg: $\frac{d}{dx}(5x^3) = 5.\frac{d}{dx}x^3 = 5.\ 3x^2 = 15x^2$

: $\frac{d}{dx}(10x^{-2}) = 10.\frac{d}{dx}(x^{-2}) = 10.\frac{-2}{x^3} = \frac{-20}{x^3}$

b) $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$.

eg: $\frac{d}{dx}(x^3+\sqrt{x}) = \frac{d}{dx}x^3 + \frac{d}{dx}\sqrt{x}$

$= 3.x^2 + \frac{1}{2\sqrt{x}}$

: $\frac{d}{dx}(5x^2+2x^4) = \frac{d}{dx}(5x^2) + \frac{d}{dx}(2x^4)$

$= 5.2x + 2.4x^3$

$= 10x + 8x^3$

1) $y = 4x^5 + 10x^7$.

A. $\frac{d}{dx}(4x^5+10x^7) = \frac{d}{dx}(4x^5) + \frac{d}{dx}(10.x^7)$

$= 4.5x^4 + 10.7x^6$

$= 20x^4 + 70x^6$

2) $y = 5x^2 + \frac{7}{x} + 3\sqrt{x}$

$\frac{dy}{dx} = 5.2x + 7.\frac{-1}{x^2} + 3.\frac{1}{2\sqrt{x}}$

$= 10x + \frac{-7}{x^2} + \frac{3}{2\sqrt{x}}$ //

Imp $\sin x$, $\cos x$, $x^n$

3) $y = 7x^{-4} - 5x^{6}$

$$\frac{dy}{dx} = 7 \cdot \frac{-4}{x^5} - 5 \cdot 6 \cdot x^5$$

$$= \frac{-28}{x^5} - 30x^5$$

4) $y = \frac{10}{x^3} + 7x^{3/2}$

$$\frac{dy}{dx} = 10 \cdot \frac{1}{x^3} + 7 \cdot \frac{3}{2} x^{1/2}$$

$$= 10 \cdot \frac{-3}{x^4} + \frac{21\sqrt{x}}{2}$$

$$= \frac{-30}{x^4} + \frac{21\sqrt{x}}{2}$$

$\sin A - \sin B = 2\cos\frac{A+B}{2} \cdot \sin\frac{A-B}{2}$

• Find $\frac{d}{dx}$ derivative of $\sin x$, using 1st principle.

A. $y = \sin x$.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$y + \Delta y = \sin(x + \Delta x)$$

$$\Delta y = \sin(x + \Delta x) - y$$

$$\Delta y = \sin(x + \Delta x) - \sin x.$$

$$\sin A - \sin B = 2 \cdot \cos\frac{A+B}{2} \cdot \sin\frac{A-B}{2}$$

$$\Delta y = \cancel{\sin}\ 2 \cdot \cos\frac{x + \Delta x + x}{2} \cdot \sin\frac{\cancel{x} + \Delta x - \cancel{x}}{2}.$$

$$\Delta y = 2 \cdot \cos x + \frac{\Delta x}{2} \cdot \sin\frac{\Delta x}{2}.$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\cancel{2} \cdot \cos x + \frac{\Delta x}{2} \cdot \sin\frac{\Delta x}{2}}{\cancel{2} \cdot \frac{\Delta x}{2}}$$

$$= \lim_{\Delta x \to 0} \frac{\cos x + \frac{\Delta x}{2} \cdot \sin\frac{\Delta x}{2}}{\Delta x/2}$$

$$= \lim_{\Delta x \to 0} \cos x + \frac{\Delta x}{2} \cdot \boxed{\lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\Delta x/2}}$$

$$= \underline{\cos x}$$

- Find the derivative of $\cos x$. by using 1st principle.

A. $y = \cos x$.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

$A = x + \Delta x$
$B = x$

$$y + \Delta y = \cos(x + \Delta x).$$

$$\Delta y = \cos(x + \Delta x) - y.$$

$$= \cos(x + \Delta x) - \cos x.$$

$$\cos A - \cos B = -2 \cdot \sin\frac{A+B}{2} \cdot \sin\frac{A-B}{2}$$

$$\Delta y = -2 \cdot \sin\frac{x + \Delta x + x}{2} \cdot \sin\frac{\Delta x + x - x}{2}$$

$$= -2 \cdot \sin x + \frac{\Delta x}{2} \cdot \sin\frac{\Delta x}{2}.$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{-\cancel{2}\sin x + \frac{\Delta x}{2} \cdot \sin\frac{\Delta x}{2}}{\cancel{2}\Delta x/2}$$

$$= \lim_{\Delta x \to 0} \frac{-\sin x + \frac{\Delta x}{2} \cdot \sin\frac{\Delta x}{2}}{\Delta x/2}$$

$$= \lim_{\Delta x \to 0} -\sin x + \frac{\Delta x}{2} \cdot \lim_{\Delta x \to 0} \frac{\sin \Delta x/2}{\Delta x/2}$$

$$= \underline{-\sin x}$$

## product rule.

Let u, v are functions of $x$.

$$\boxed{\frac{d(uv)}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}}$$

eg: $y = x^2 \sin x$

$$\frac{d\, x^2 \sin x}{dx} = x^2 \cdot \frac{d \sin x}{dx} + \sin x \cdot \frac{d x^2}{dx}$$

$$= x^2 \cdot \cos x + \sin x \cdot 2x$$

- $y = x^3 \cos x$.

A. $$\frac{d\, x^3 \cos x}{dx} = x^3 \cdot \frac{d \cos x}{dx} + \cos x \cdot \frac{d x^3}{dx}$$

$$= -x^3 \cdot \sin x + \cos x \cdot 3x^2$$

- $y = (x^3 + 2x + 1) \sin x$

$$\frac{d}{dx} = (x^3 + 2x + 1) \frac{d}{dx} \sin x + \sin x \frac{d}{dx}(x^3 + 2x + 1)$$

$$= (x^3 + 2x + 1) \cos x + \sin x (3x^2 + 2)$$

## Quatient rule

$$\boxed{\frac{d(\frac{u}{v})}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}}$$

• Find derivative of $\tan x$ using quotient rule.

A. $y = \tan x$

$y = \dfrac{\sin x}{\cos x}$

$$\frac{d}{dx} = \frac{\cos x \cdot \frac{d \sin x}{dx} - \sin x \cdot \frac{d \cos x}{dx}}{(\cos x)^2}$$

Cosec x

Cot x

sec x

$$= \frac{\cos x \cdot \cos x - \sin x \cdot -\sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \cancel{\frac{\cos^2 x(1+\sin^2 x)}{\cos^2 x}} = 1 \cancel{+\sin^2 x}$$

$$= \cancel{\frac{\cos^2 x}{\cos^2 x}} + \cancel{\frac{\sin^2 x}{\cos^2 x}} = \cancel{1+\tan^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$\frac{\Delta y}{\Delta x}$

$\frac{dy}{dx} = \lim_{\Delta x \to 0}$

• $y = \operatorname{cosec} x$.

$y = \dfrac{1}{\sin x}$.

$$\frac{dy}{dx} = \frac{\sin x \cdot 0 - \cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$$= -\operatorname{cosec} x \cdot \cot x$$

• $y = \sec x$

$y = \dfrac{1}{\cos x}$

$$\frac{d}{dx} = \frac{\cos x \cdot 0 + \sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \sec x \cdot \tan x$$

- $y = \cot x$

$$y = \frac{\cos x}{\sin x}$$

$$\frac{d}{dx} = \frac{\sin x \times -\sin x - \cos x \cdot \cos x}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\operatorname{cosec}^2 x$$

Results

1) $\frac{d}{dx} \sin x = \cos x$

2). $\frac{d}{dx} \cos x = -\sin x$

3). $\frac{d}{dx} \tan x = \sec^2 x$

4) $\frac{d}{dx} \sec x = \sec x \cdot \tan x$

5) $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x$

c) $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$

- $y = (x^2+x+3)(3x^4-5x^3)$.

$$\frac{d}{dx} = (x^2+x+3)\frac{d}{dx} 3x^4-5x^3 + (3x^4-5x^3) \cdot \frac{d}{dx} x^2+x+3$$

$$= (x^2+x+3) \cdot (12x^3 - 15x^2) + (3x^4-5x^3)(2x+1+0)$$

$$= \cancel{x+}3.$$

• $y = (x^2+3\sqrt{x}+1)\tan x$

$$\frac{dy}{dx} = (x^2+3\sqrt{x}+1)\frac{d}{dx}\tan x + \tan x\,\frac{d}{dx}(x^2+3\sqrt{x}+1)$$

$$= (x^2+3\sqrt{x}+1)\sec^2 x + \tan x\left(2x + \frac{3}{2\sqrt{x}} + 0\right).$$

• $y = \sqrt{x}\cdot \operatorname{cosec} x.$

$$\frac{dy}{dx} = \sqrt{x}\cdot\frac{d}{dx}\operatorname{cosec} x + \operatorname{cosec} x\cdot\frac{d}{dx}\sqrt{x}$$

$$= \sqrt{x}\cdot -\operatorname{cosec} x\cdot\cot x + \operatorname{cosec} x\cdot\frac{1}{2\sqrt{x}}$$

• $y = \frac{8}{x^2} - 2\cot x$

$$\frac{dy}{dx} = 8\cdot x^{-2} - 2\cot x$$

$$= 8\cdot\frac{-2}{x^3} - \bar{2}\operatorname{cosec}^2 x$$

$$= 2\operatorname{cosec}^2 x - \frac{16}{x^3}$$

$(a+b)^2 =$

• $y = \frac{x^2-1}{x^2+1}$

$$\frac{d}{dx} = \frac{(x^2+1)\cdot\frac{d}{dx}(x^2-1) \cancel{+} - (x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\cdot 2x - (x^2-1)\cdot 2x}{(x^2+1)^2}$$

$$= \frac{2x\left(x^2+1-(x^2-1)\right)}{(x^2+1)^2} = \frac{2x\left(\cancel{x^2}+1-\cancel{x^2}+1\right)}{(x^2+1)^2}$$

$$= \frac{2x\cdot 2}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

• $y = (x^2+2x+3)(x^2-4)$.

$y = \frac{1-x^2}{1+x^2}$

$y = \frac{x^2+x+1}{x^2+2x+3}$

## Derivative of $\log x$ & $e^x$

1). $\frac{d}{dx}\log x = \frac{1}{x}$

2) $\frac{d}{dx}e^x = e^x$

• Find $\frac{d}{dx}$? $y = x^2\log x$.

A. $\frac{dy}{dx} = x^2.\frac{d}{dx}\log x + \log x.\frac{d}{dx}x^2$

$= x^2.\frac{1}{x} + \log x.2x$

$= x + \log x.2x$

• $y = x^3.e^x$

$\frac{dy}{dx} = x^3.\frac{d}{dx}e^x + e^x.\frac{d}{dx}x^3$

$= x^3.e^x + e^x.3x^2$

$= e^x(x^3+3x^2)$

• $y = \cos x.\log x$

$\frac{dy}{dx} = \cos x.\frac{1}{x} + \log x.-\sin x$

$$= \frac{\cos x}{x} + \sin x \cdot \log x$$

- $y = (x^2+x)\log x$

$$= (x^2+x)\cdot\frac{1}{x} + \log x\,(2x+1)$$

$$= \frac{x^2+x}{x} + \log x\,(2x+1)$$

$$= \frac{x(x+1)}{x} + \log x\,(2x+1)$$

$$= x+1 + \log x\,(2x+1)$$

- $y = x^2 \cdot \sec x$

$$= x^2(\sec x \cdot \tan x) + \sec x \cdot 2x$$

- $y = (x^2+e^x)\tan x$

$$= (x^2+e^x)\cdot\sec^2 x + \tan x\cdot(2x+e^x)$$

- $y = (x^2+2x+3)(x^2-4)$

$$= (x^2+2x+3)\cdot 2x + (x^2-4)\cdot(2x+2)$$

- $y = \frac{1-x^2}{1+x^2}$

$$= \frac{(1+x^2)(0-2x) - (1-x^2)\cdot(0+2x)}{(1+x^2)^2}$$

$$= \frac{-2x(1+x^2) - 2x(1-x^2)}{(1+x^2)^2} = \frac{-2x\left[1+\cancel{x^2}+1-\cancel{x^2}\right]}{(1+x^2)^2} = \frac{-2x \cdot 2}{(1+x^2)^2} = \frac{-4}{(1+x^2)^2}$$

$$\cancel{= \frac{2x(1+x^2) + 2x(1+x^2)}{(1+x^2)^2}}$$

• $y = \dfrac{x^2+x+1}{x^2+2x+3}$

$$= \frac{(x^2+2x+3)(2x+1) - x^2+x+1 \cdot (2x+2)}{(x^2+2x+3)^2}$$

• Find $\dfrac{dy}{dx}$ if $y = \dfrac{\cos x}{x+\sin x}$

A. $$\frac{dy}{dx} = \frac{(x+\sin x)\frac{d}{dx}\cos x - \cos x \frac{d}{dx}(x+\sin x)}{(x+\sin x)^2}$$

$$= \frac{-(x+\sin x)\sin x - \cos x(1+\cos x)}{(x+\sin x)^2}$$

$$= \frac{-x.\sin x - \sin^2 x - \cos x + \cos^2 x}{(x+\sin x)^2}$$

$$= \frac{-x\sin x - \cos x - (\sin^2 x + \cos^2 x)}{(x+\sin x)^2}$$

$$= \frac{-x\sin x - \cos x + 1}{(x+\sin x)^2}$$

• $y = \dfrac{x^3+2x+5}{x^3}$

$$\frac{dy}{dx} = \frac{x^3 \cdot \frac{d}{dx}(x^3-2x+5) - (x^3-2x+5)\frac{d}{dx}x^3}{(x^3)^2}$$

$$= \frac{x^3(3x^2-2) - (x^3-2x+5) \cdot 3x^2}{(x^3)^2}$$

$$= \frac{3x^5 - 2x^3 - (x^3 - 2x + 5)\cdot 3x^2}{x^6}$$

$$= \frac{\cancel{3x^5} - 2x^3 - \cancel{3x^5} + 6x^3 - 15x^2}{x^6}$$

$$= \frac{4x^3 - 15x^2}{x^6} = \frac{\cancel{x^2}(4x-15)}{x^{\cancel{6}\,4}} = \frac{4x-15}{x^4}$$

- $y = \dfrac{e^x}{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{\sqrt{x}\cdot e^x - e^x \cdot \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$= \frac{e^x\left(\sqrt{x} - \frac{1}{2\sqrt{x}}\right)}{x}$$

$$= \frac{e^x\left(\frac{2x-1}{2\sqrt{x}}\right)}{x}$$

$$= \frac{e^x(2x-1)}{2x^{3/2}}$$

$1 + \frac{1}{2} = \frac{3}{2}$

## Function of a function rule

Let $y = f(u)$, where $u = g(x)$. Then there will be function in $x$.

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}}$$

$\frac{d}{dx} = \frac{d}{dx}$ outer function $\cdot \frac{d}{dx}$ inner function.

eg: 1) $y = \sqrt{x^2+2x+1}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+2x+1}} .(2x+2) = \frac{2x+2}{2\sqrt{x^2+2x+1}}$

2) $y = \sin(x^2)$.

$\frac{dy}{dx} = \cos(x^2) . 2x$

3) $y = \frac{1}{x^2+2x+1}$

$= \frac{-1}{(x^2+2x+1)^2} 2x+2 . = \frac{-(2x+2)}{(x^2+2x+1)^2}$

• $y = \sin(x^2+3x+7)$. $\sin(u) \Rightarrow \cos u \cdot \frac{d}{dx}(u)$

$= \cos(x^2+3x-7).(2x+3)$

• $y = \log e^x$. $\log e^x = \log x = \frac{1}{x}$.

$\frac{dy}{dx} = \frac{1}{e^x} \cdot e^x = 1$

• $y = e^{\sin x}$ $e^{\sin x} \Rightarrow e^x$

$\frac{dy}{dx} = e^{\sin x} . \cos x$

• $y = (3x^2+4x+7)^{10}$.

$= 10.(3x^2+4x+7)^9.(6x+4)$

• $y = \frac{1}{(2x^2+3)^7}$ $\frac{1}{x^7} = x^{-7} \Rightarrow -7 . x^{-7-1} = -7 . x^{-8}$ $\frac{dy}{dx}$

$= \cancel{\frac{1}{2\sqrt{(2x^2+3)^7}}}$

$= \dfrac{-1}{(2x^2+3)^8} \cdot 4x$

$= \dfrac{-28x}{(2x^2+3)^8}$

- $y = \dfrac{1}{(2x^2+3x+5)}$

$= \dfrac{-1}{(2x^2+3x+5)^2}(4x+3) = \dfrac{-(4x+3)}{(2x^2+3x+5)^2}$

- $y = \log(\sec x + \tan x)$.

$= \dfrac{1}{(\sec x + \tan x)} \cdot \sec x \cdot \tan x + \sec^2 x$

$= \dfrac{1}{(\sec x + \tan x)} \cdot \sec x \cdot (\tan x + \sec x)$

$= \sec x$

- $y = \sin^2(5x)$.

$= (\sin(5x))^2$

$= 2(\sin(5x)) \cdot \cos(5x) \cdot 5$

$= 10(\sin(5x) \cdot \cos(5x))$

- $y = \sqrt{\tan(2x+1)}$

$= \dfrac{1}{2\sqrt{\tan(2x+1)}} \cdot \sec^2(2x+1) \cdot 2.$

$= \dfrac{2\sec^2(2x+1)}{2\sqrt{\tan(2x+1)}}$

- $y = \log(\log x)$

$$= \frac{1}{\log x} \cdot \frac{1}{x} = \frac{1}{x \log x}$$

- $y = e^{(ax+b)}$.

$$= e^{(ax+b)} \cdot a$$

- $y = (x^2+1)^{10} \cdot \sec 5x$.

$$= (x^2+1)^{10} \cdot \frac{d}{dx} \sec 5x + \sec 5x \cdot \frac{d}{dx}(x^2+1)^{10}$$

$$= (x^2+1)^{10} \cdot \sec 5x \cdot \tan(5x) 5 + \sec 5x \cdot 10(x^2+1)^9 \cdot 2x.$$

$$= 5(x^2+1)^{10} \sec 5x \cdot \tan 5x + 20x (x^2+1)^9 \cdot \sec 5x$$

- $y = \sin 3x \cdot \sin 5x$

$$= \sin 3x \cdot 5 \cos 5x + \sin 5x \cdot 3 \cos 3x$$

$$= 5 \cdot \sin 3x \cdot \cos 5x + 3 \sin 5x \cdot \cos 3x$$

$$= \cancel{\sin 3x \cdot \cos 5x (3+5)}.$$

- $y = x^2 \cdot \cos(x^2)$.

$$= x^2 \cdot 2x \cdot -\sin x^2 + \cos(x^2) \cdot 2x$$

$$= -2x^3 \cdot \sin x^2 + 2x \cos(x^2)$$

- ~~$y = x^2 \cdot \sec x$~~

• $y = \dfrac{\cot 11x}{(x^3-1)^2}$.

$$= \frac{(x^3-1)^2 \cdot \frac{d}{dx} \cot 11x - \cot 11x \cdot \frac{d}{dx} (x^3-1)^2}{(x^3-1)^4}.$$

$$= \frac{-(x^3-1)^2 \cdot \operatorname{cosec}^2(11x) \cdot 11 - \cot 11x \cdot 2(x^3-1) \cdot 3x^2}{(x^3-1)^4}$$

$$= \frac{-11 (x^3-1)^2 \operatorname{cosec}^2(11x) - \cancel{\cot 11x}\ 6x^2(x^3-1) \cdot \cot 11x}{(x^3-1)^4}.$$

$$= \frac{\cancel{(x^3-1)}\left[-11 \cdot (x^3-1) \cdot \operatorname{cosec}^2(11x) - 6x^2 \cot 11x\right]}{(x^3-1)^{\cancel{4}\,3}}$$

$=$

• $y = \cancel{y}\dfrac{e^{2x} \cdot \log x}{x^2}$.

$$\cancel{= \frac{x^2 \cdot 2 \cdot e^{2x} \cdot \frac{1}{x} - e^{2x} \cdot \log x \cdot 2x}{(x^2)^2}}$$

$=$ .

$$\bullet\ y = \frac{e^{2x} \cdot \log x}{x^2}$$

$$= \frac{x^2 \cdot \frac{d}{dx}(e^{2x} \cdot \log x) - e^{2x} \log x \cdot \frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2\left(e^{2x} \cdot \frac{1}{x} + \log x \cdot e^{2x} \cdot 2\right) - e^{2x} \log x \cdot 2x}{x^4}$$

~~Inverse of~~.

Derivatives of inverse trignometric functions

1) $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

2) $\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$

3) $\frac{d}{dx} \tan^{-1} = \frac{1}{1+x^2}$.

$\bullet\ y = \sin^{-1}(x^2)$.

A. $\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}$

• $y = \tan^{-1}(\sqrt{x})$

$\frac{dy}{dx} = \cancel{\frac{1}{1+(x)}} \cdot \cancel{\frac{1}{2\sqrt{x}}} \cdot \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}}$

$= \frac{1}{1+x \cdot (2\sqrt{x})}$

$y = \log(\sin(x^2+a^2))$

$y = e^{\cos x}$

$y = e^{2x} \cdot \cos 3x$

$y = \frac{\sin(\log x)}{x}$

$y = (1+x^2)\tan^{-1}x$

$y = e^{x} \cdot \tan^{-1}x$

1) $y = \log(\sin(x^2+a^2))$

$= \frac{1}{\sin(x^2+a^2)} \cdot \cos(x^2+a^2) \cdot 2x$

$= \frac{2x \cdot \cos(x^2+a^2)}{\sin(x^2+a^2)}$

$= 2x \tan(x^2+a^2)$

2) $y = e^{\cos x}$

$= e^{\cos x} \cdot -\sin x$

$\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

$\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

$\tan^{-1}x = \frac{1}{1+x^2}$

3) $y = e^{2x} \cdot \cos(3x)$

$= e^{2x} \cdot \frac{d}{dx}\cos(3x) + \cos 3x \cdot \frac{d}{dx} e^{2x}$

$= e^{2x} \cdot -\sin(3x) \cdot 3 + \cos(3x) \cdot e^{2x} \cdot 2$

$= 2 \cdot e^{2x} \cdot \cos(3x) - 3 \cdot e^{2x} \sin(3x)$

4) $y = \frac{\sin(\log x)}{x}$

$= \frac{x \cdot \cos(\log x) \cdot \frac{1}{x} - \sin \log x \cdot \frac{d}{dx}x}{x^2}$

$$= \frac{\cos(\log x) - \sin(\log x)}{x^2}$$ //.

5) $y = (1+x^2) \cdot \tan^{-1} x$

$= (1+x^2) \frac{1}{1+x^2} + \tan^{-1}x \, (2x).$

$= 1 + \tan^{-1}x \cdot 2x$

6) $y = \frac{e^{2x} \tan^{-1} 3x}{\sqrt{x}}.$

$$= \frac{\sqrt{x} \cdot \frac{d}{dx}(e^{2x} \tan^{-1} 3x) - e^{2x} \tan^{-1} 3x \cdot \frac{d}{dx}\sqrt{x}}{(\sqrt{x})^2}$$

$$= \frac{\sqrt{x}\left[e^{2x} \cdot \frac{1}{1+(3x)^2} \cdot 3 + \tan^{-1} 3x \cdot e^{2x} \cdot 2\right] - e^{2x} \tan^{-1} 3x \frac{1}{2\sqrt{x}}}{x}$$

$$= \frac{\sqrt{x} \cdot \left[\frac{3e^{2x}}{1+9x^2} + 2 e^{2x} \tan^{-1} 3x\right) - \frac{e^{2x} \tan^{-1} 3x}{2\sqrt{x}}}{x}$$

• $y = \dfrac{\sin^{-1}\sqrt{x}}{x^3}$.

$$\frac{dy}{dx} = \frac{x^3 \frac{d}{dx}\sin^{-1}\sqrt{x} - \sin^{-1}\sqrt{x} \cdot \frac{d}{dx}x^3}{(x^3)^2}$$

$$= \frac{x^3 \cdot \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \cancel{2x} - \sin^{-1}\sqrt{x} \cdot 3x^2}{x^6}.$$

$$= \frac{\frac{x^3}{2\sqrt{x}\sqrt{1-x}} - \sin^{-1}\sqrt{x} \cdot 3x^2}{x^6}$$

• $y = \dfrac{x.\sec x}{3x+2}$

$$= \frac{3x+2 \cdot \frac{d}{dx} x\sec x - x.\sec x \cdot \frac{d}{dx} 3x+2}{(3x+2)^2}$$

$$= \frac{(3x+2)\left[x.\sec x.\tan x + \sec x.\right] - x.\sec x.3.}{(3x+2)^2}$$

$\frac{1}{2\sqrt{x}}$

$$= \frac{(3x+2)x\sec x\tan x + (3x+2)\sec x - 3x\sec x}{(3x+2)^2}$$

$$= \frac{(3x+2)x\sec x.\tan x + \cancel{3x\sec x} + 2\sec x - \cancel{3x\sec x}}{(3x+2)^2}$$

$$= \frac{(3x+2)x\sec x.\tan x + 2\sec x}{(3x+2)^2}$$

• $y = (1-x^2)\sin^{-1}(x)$.

$= (1-x^2) \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1}(x) \cdot -2x$

$= \frac{1-x^2}{\sqrt{1-x^2}} - 2x \cdot \sin^{-1}(x)$

• $y = \big((x+1)(x+2)\big)(2x^2+1)$.

$= (x+1)(x+2) \cdot \frac{d}{dx}(2x^2+1) + (2x^2+1)\frac{d}{dx}(x+1)(x+2)$.

$= (x+1)(x+2) \cdot 4x + (2x^2+1)\left((x+1) \cdot \frac{d}{dx}(x+2) + (x+2) \cdot \frac{d}{dx}(x+1)\right)$

$= (x+1)(x+2)\, 4x + (2x^2+1)\left[(x+1) \cdot 1 + (x+2) \cdot 1\right]$.

$= 4x \cdot (x+1)(x+2) + (2x^2+1)(2x+3)$.

$= 4x(x^2+3x+2) + 4x^3 + 6x^2 + 2x + 3$

$= 4x^3 + 12x^2 + 8x + 4x^3 + 6x^2 + 2x + 3$

$= 8x^3 + 18x^2 + 10x + 3$

• $x \log x \cdot \sin x$.

A. $y = (x \log x) \sin x$

$= (x \log x) \cdot \frac{d}{dx} \sin x + \sin x \cdot \frac{d}{dx}(x \log x)$

$= x \log x \cdot \cos x + \sin x \cdot \left[x \cdot \frac{d}{dx} \log x + \log x \cdot \frac{dx}{dx}\right]$

$= x \log x \cdot \cos x + \sin x \left[x \cdot \frac{1}{x} + \log x \cdot 1\right]$.

$= x \log x \cdot \cos x + \sin x \left[1 + \log x\right]$.

• $y = x.(x+2)(x+5)$.

~~Derivative of Implicite functions~~

$$\frac{dy}{dx} = x(x+2) \cdot \frac{d}{dx}(x+5) + (x+5) \cdot \frac{d}{dx} x(x+2).$$

$$= x(x+2)\cancel{(x+5)} + (x+5) \cdot \frac{d}{dx}(x^2+2x).$$

$$= x(x+2).\cancel{(x+5)} + (x+5)(2x+2).$$

$$= (x^2+2x) + (x+5)(2x+2)$$

$$\cancel{= 6x^2+12x + (x+5)2x + (x+5)2}.$$

$$\cancel{= 6x^2+12x + 2x^2+10x + 2x+10}$$

$$= \cancel{24x+8x^2+10}$$

$$= x^2+2x + 2x^2+12x+10$$

$$= 3x^2+14x+10$$

Derivative of Implicit function

A function of the form $f(x,y) = C$. is known as an Implicit function.

eg: $x^2+y^2=1$.

$x^2+2xy+y^2=8$

$$\boxed{\frac{d}{dx}F(y) = \frac{d}{dy}(F(y)) \cdot \frac{dy}{dx}}$$

10T
15C

eg: $\frac{d}{dx}y^2 = 2y \cdot \frac{dy}{dx}$

$\frac{d}{dx}\sin y = \cos y \cdot \frac{dy}{dx}$.

$\frac{d}{dx}(y^2+2y) = (2y+2)\frac{dy}{dx}$

• Find $\frac{d}{dx}$ if $x^2+y^2=1$.

A. ~~2x~~ Diff: wrt $x$.

$$2x + 2y \cdot \frac{dy}{dx} = 0.$$

$$-\cancel{2}x = \cancel{2}y \cdot \frac{dy}{dx}$$

$$\frac{-x}{y} = \frac{dy}{dx}$$

• $x^2+2xy+y^2=2$

$\cancel{2x + 2y \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = 0}$

$$2x + 2\left[x \cdot \frac{dy}{dx} + y \cdot \frac{dy}{dy}\right] + 2y \cdot \frac{dy}{dx} = 0.$$

$$2x + 2x \cdot \frac{dy}{dx} + 2y + 2y \cdot \frac{dy}{dx} = 0.$$

$$2x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = -2x - 2y.$$

$$\frac{dy}{dx}(2\cancel{x+2y}) = -(2\cancel{x+2y}).$$

$$\boxed{\frac{dy}{dx} = -1}$$

- $x^3+y^3=3axy$.
- $xy=c^2$.

$$x\cdot\frac{dy}{dx}+y=0$$

$$\frac{dy}{dx}=\frac{-y}{x}$$

A. $$3x^2+3y^2\frac{dy}{dx}=3a\left(x\cdot\frac{dy}{dx}+y\right)$$

$$3x^2+3y^2\cdot\frac{dy}{dx}=3a\cdot x\cdot\frac{dy}{dx}+3ay$$

$$3x^2+(3y^2-3ax)\cdot\frac{dy}{dx}=3ay.$$

$$\frac{dy}{dx}=\frac{3ay-3x^2}{(3y^2-3ax)}$$

- Find $\frac{d}{dx}$ if $x^2+y^2+2gx+2fy+c=0$ ($g, f$ & $c$ are constant)

A. diff. wrt $x$.

$$2x+2y\cdot\frac{dy}{dx}+2g+2f\frac{dy}{dx}+0=0.$$

$$2y\cdot\frac{dy}{dx}+2f\cdot\frac{dy}{dx}=-2x-2g$$

$$(2y+2f)\frac{dy}{dx}=-2(x+g).$$

$$\cancel{2}(y+f)\cdot\frac{dy}{dx}=-\cancel{2}(x+g)$$

$$\frac{dy}{dx}=\frac{-(x+g)}{y+f}$$

• $ax^2 + 2hxy + by^2 = 0$. ($a, h, b$ are Constant)

(prdt rule)

A. $\cancel{a.2x + 2h.1.\frac{dy}{dx} + b.2y.\frac{dy}{dx} = 0}$

$a.2x + 2h\left(x.\frac{dy}{dx} + y.1\right) + b2y.\frac{dy}{dx} = 0.$

(Dividing with 2)

$ax + hx.\frac{dy}{dx} + hy + by.\frac{dy}{dx} = 0$

$(hx + by)\frac{dy}{dx} = -ax - hy.$

$$\frac{dy}{dx} = \frac{-(ax + hy)}{hx + by}$$

• Find $\frac{dy}{dx}$, if $x^2y^2 = x^3 + y^3 + 3xy$.

(prdt rule)

A. $x^2.\frac{d}{dx}y^2 + y^2.\frac{d}{dx}x^2 = 3x^2 + 3y^2.\frac{dy}{dx} + 3\left[x.\frac{dy}{dx} + y.1\right]$

$x^2.2y.\frac{dy}{dx} + y^2.2x = 3x^2 + 3y^2\frac{dy}{dx} + 3x.\frac{dy}{dx} + 3y.$

$2x^2y.\frac{dy}{dx} - 3y^2.\frac{dy}{dx} - 3x\frac{dy}{dx} = 3x^2 + 3y - 2xy^2$

$\frac{dy}{dx}(2x^2y - 3y^2 - 3x) = 3x^2 + 3y - 2xy^2$

$$\frac{dy}{dx} = \frac{3x^2 + 3y - 2xy^2}{2x^2y - 3y^2 - 3x}$$

## Parametric Equations

An equn of the form $x = f(t)$, $y = g(t)$ is known as a parametric equn with parameter 't'.

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

- Find $\frac{dy}{dx}$ if $x = at^2$, $y = 2at$.

A. $\frac{dy}{dt} = 2a \cdot \frac{dt}{dt} = 2a \cdot 1 = 2a$.

$\frac{dx}{dt} = a \cdot 2t = 2at$

$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

- Find $\frac{dy}{dx}$ if $x = a\sin^3 t$, $y = b\cos^3 t$.

$\frac{dy}{dt} = b \cdot 3\cos^2 t \cdot -\sin t$

$= -3b\cos^2 t \cdot \sin t$

$\frac{dx}{dt} = a \cdot 3\sin^2 t \cdot \cos t$

$= 3a\sin^2 t \cdot \cos t$

$\frac{dy}{dx} = \frac{-3b\cos^2 t \cdot \sin t}{3a\sin^2 t \cdot \cos t}$

$= \frac{-b}{a} \cot t$

• $\frac{dy}{dx} = \cot \theta/2$ ; if $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$

A. $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$.

$\frac{dy}{d\theta} = \frac{d}{d\theta} a(1-\cos\theta)$.

$= a(0+\sin\theta) = a\sin\theta$.

$\frac{dx}{d\theta} = \frac{d}{d\theta} a(\theta - \sin\theta) = a(1-\cos\theta)$.

$\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos a)} = \frac{\sin\theta}{1-\cos\theta}$

$\sin\theta = \sin 2(\theta/2) = 2\cdot\sin\theta/2 \cdot \cos\theta/2$

$\sin 2x = 2\sin x \cdot \cos x$

$1-\cos\theta = 1-\cos 2(\theta/2) = 2\sin^2\theta/2$

$1-\cos 2x = 2\sin^2 x$

$\frac{dy}{dx} = \frac{\cancel{2}\sin\theta/2 \cdot \cos\theta/2}{\cancel{2}\sin^2\theta/2} = \cot\theta/2$

• Find $\frac{dy}{dx}$ if $x = a\sec\theta$, $y = b\tan\theta$.

A. $\frac{dx}{d\theta} = \frac{d}{d\theta} a\sec\theta = a\cdot\sec\theta\cdot\tan\theta$

$\frac{dy}{d\theta} = \frac{d}{d\theta} b\tan\theta = b\cdot\sec^2\theta$

$$\frac{dy}{dx} = \frac{\cancel{a\sec\theta\cdot\tan\theta}}{\cancel{a}\,\cancel{b}\,\cancel{\sec^2\theta}} \quad \frac{b\sec^2\theta}{a\sec\theta\cdot\tan\theta}$$

$$= \frac{\cancel{a\tan\theta}}{\cancel{b\sec\theta}} \neq \;=\; \frac{b\sec\theta}{a\tan\theta} =$$

$$\frac{b\cdot\frac{1}{\cos\theta}}{a\cdot\frac{\sin\theta}{\cos\theta}} = \frac{b}{a}\operatorname{cosec}\theta$$

- $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$ find $\frac{dy}{dx}$

A. $$\frac{dx}{dt} = \frac{(1+t^2)\cdot\frac{d}{dt}(1-t^2) - (1-t^2)\cdot\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)\cdot 2t - (1-t^2)\cdot 2t}{(1+t^2)^2}$$

$$= \frac{-2t\,(1+\cancel{t^2}+1-\cancel{t^2})}{(1+t^2)^2} = \frac{-4t}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{(1+t^2)\cdot 2 - 2t\cdot 2t}{(1+t^2)^2}$$

$$= \frac{2(1+t^2) - 4t^2}{(1+t^2)^2} = \frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$$= \frac{2-2t^2}{(1+t^2)^2} = \frac{2(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{\frac{2(1-t^2)}{(1+t^2)^2}}{\frac{-4t}{(1+t^2)^2}} = \frac{\cancel{2}(1-t^2)}{\cancel{4}t} = \frac{-(1-t^2)}{t}$$

HW

1) $x = 3\cos t - \cos^3 t$
$y = 3\sin t - \sin^3 t$

2) $x = e^t \cdot \cos t$
$y = e^t \cdot \sin t$

1) $\boxed{x = 3\cos t - \cos^3 t}$

$$\frac{dx}{dt} = -3\sin t - 3\cos^2 t \cdot -\sin t$$

$$= 3\cos^2 t \cdot \sin t - 3\sin t$$

$$= 3\sin t(\cos^2 t - 1) = 3\sin t \cdot -\sin^2 t$$

$$= -3\sin^3 t$$

$$\frac{dy}{dt} = 3\cos t - 3\sin^2 t \cdot \cos t$$

$$= 3\cos t(1-\sin^2 t) = 3\cos^3 t$$

$$\frac{dy}{dx} = \frac{\cancel{3}\cos^3 t}{-\cancel{3}\sin^3 t} = \frac{-\cos^3 t}{\sin^3 t} = -\cot^3 t$$

Ans:
2) $x = e^t \cos t$, $y = e^t \sin t$

$$\frac{dy}{dx} = e^t \cos t + \sin t \cdot e^t$$
$$= e^t(\cos t + \sin t)$$

$$\frac{dx}{dt} = e^t \cdot -\sin t + \cos t \cdot e^t$$
$$= e^t(\cos t - \sin t).$$

$$\frac{dy}{dx} = \frac{e^t(\cos t + \sin t)}{e^t(\cos t - \sin t)} = \frac{\cos t + \sin t}{\cos t - \sin t}$$

## Successive differentiation

Given a function $y = f(x)$, then $\frac{dy}{dx}$ is the 1st order derivative of $y$. The second order derivative $\frac{d^2y}{dx^2}$ is given by $\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)}$ It is also denote $\frac{dy}{dx} = y'$ and $\frac{d^2y}{dx^2} = y''$

- Find $\frac{d^2y}{dx^2}$ if $y = x^3 + 7x^2 + 5$

A. $\frac{dy}{dx} = 3x^2 + 14x$

$$\frac{d^2y}{dx^2} = 6x + 14.$$

- Find $y''$, $y = 2\sin x + 3\cos x$

A. $y' = 2.\cos x - 3\sin x$

$$y'' = -2\sin x - 3\cos x$$
$$= -(2\sin x + 3\cos x) = -y$$

- $y = \cos 3x + \sin 7x.$

$y' = -\sin 3x \cdot 3 + \cos 7x \cdot 7$

$= 7 \cdot \cos 7x - 3\sin 3x$

$y'' = 7 \cdot -\sin 7x \cdot 7 - 3 \cdot \cos 3x \cdot 3.$

$= -49 \cdot \sin 7x - 9\cos 3x$

- $y = e^{7x}$

$y' = e^{7x} \cdot 7.$

$y'' = 7 \cdot e^{7x} \cdot 7.$

$= 49\, e^{7x}$

- $y = e^x \cdot \cos x.$

$y' = e^x \cdot -\sin x + \cancel{\cos x}\ \cos x \cdot e^x$

$= \cos x \cdot e^x - \sin x \cdot e^x.$

$y'' = (\cos x \cdot e^x + e^x - \sin x) - (\sin x \cdot e^x + e^x \cos x)$

$= e^x \cancel{\cos x} + e^x \sin x - \sin x\, e^x - e^x \cancel{\cos x}$

$= -2e^x \sin x$

- $y = x^2 \cdot \sin x$

$y' = 2 \cdot x^2 \cdot \cos x + \sin x \cdot 2x$

$y'' = (x^2 \cdot -\sin x + \cos x \cdot 2x) + (\sin x \cdot 2 + 2x \cdot \cos x)$

$= (2x\cos x - x^2 \cdot \sin x) + (2 \cdot \sin x + 2x \cos x).$

$= 4x \cos x + 2\sin x (2 - x^2)$ //

• $y = e^x + e^{-x}$. ST $y'' - y = 0$

A. $y' = e^x - e^{-x}$

$y'' = e^x + e^{-x}$

$y'' - y = 0$

$e^x - e^{-x}$

$y'' = e^x + e^{-x}$ (-1)

• $y = e^{2x} + e^{3x}$ : ST $y'' - 5y' + 6y = 0$

A. $y' = e^{2x} \cdot 2 + e^{3x} \cdot 3$

$y'' = 2e^{2x} \cdot 2 + 3e^{3x} \cdot 3$

$= 4e^{2x} + 9e^{3x}$

$y'' - 5y' + 6y = 0 \Rightarrow 4e^{2x} + 9e^{3x} - 5(e^{2x} \cdot 2 + e^{3x} \cdot 3) + 6(e^{2x} + e^{3x})$

$\Rightarrow 4\cancel{e^{2x}} + 9\cancel{e^{3x}} - 10\cancel{e^{2x}} - 15\cancel{e^{3x}} + \cancel{6e^{2x}} + 6e^{3x}$

$\Rightarrow 0$

• $y = 2\cos px + 3\sin px$, ST $y'' + p^2 y = 0$ (were $p$ = Constant)

~~y' = 2 sin px . p + 3 cos~~

$y' = -2\sin px \cdot p + 3\cos px \cdot p$

$= p(3\cos px - 2\sin px)$

$y'' = p(3 \cdot -\sin px \cdot p - 2 \cdot \cos px \cdot p)$

$= p^2(-3\sin px - 2\cos px)$

$= -p^2(2\cos px + 3\sin px)$ $= -p^2 y$

$\therefore y'' - p^2 y = y'' + p^2 y = 0$ //

- $y = y = A\cos px + B\sin px$, S.T. ~~y'' is propo~~ $y'' \alpha y$.

A. $y \alpha y$; if and only if $y'' = ky$, $k$ is constant

$y' = -A \cdot \sin px \cdot p + B \cdot \cos px \cdot p$.

$= p(B\cos px - A\sin px)$

$y'' = p(B - \sin px \cdot p - A \cdot \cos px \cdot p)$.

$= p \cdot p(-B\sin px - A \cdot \cos px)$.

$= -p^2 (B\sin px + A\cos px)$.

$= -p^2 y$.

$\frac{y''}{y} = -p^2 \Rightarrow$ is a constant

$\therefore y'' \alpha y$

- $y = x^2 \sin x$, P.T $x^2 y'' - 4xy' + (x^2+6)y = 0$.

A: $y' = x^2 \cdot \cos x + \sin x \cdot 2x$

$y'' = (-x^2 \cdot \sin x + \cos x \cdot 2x) + (\sin x \cdot 2 + 2x \cdot \cos x)$

$= 2x\cos x - x^2 \sin x + (2\sin x + 2x\cos x)$

$= 4x\cos x + \sin x(2 - x^2)$

$x^2 y'' = x^2(4x\cos x + \sin x(2 - x^2))$

$= 4x^3 \cos x + \sin x(2x^2 - x^4)$. — ①

$$4xy' = 4x(x^2.\cos x + 2x\sin x).$$

$$= 4x^3\cos x + 8x^2.\sin x \text{ --- } ②$$

$$(x^2+6)y = (x^2+6).x^2\sin x$$

$$= x^4\sin x + 6x^2\sin x \text{ --- } ③$$

$$x^2y'' - 4xy' + (6+x^2)y \Rightarrow ① - ② + ③$$

$$= [4x^3\cos x + \sin x(2x^2 - x^4)] - [4x^3\cos x + 8x^2\sin x] + [x^4\sin x + 6x^2\sin x]$$

$$= -x^4\sin x + x^4\sin x + 4x^3\cos x - 4x^3\cos x + 2x^2\sin x - 8x^2\sin x + 6x^2\sin x$$

$$= 0$$

HW

- $y = x^2\cos x$, S.T $x^2y'' - 4xy' + (x^2+6)y = 0$.
- $y = a\cos(\log x) + b\sin(\log x)$. ST $x^2.y'' + xy' + y = 0$

$$y' = -a\sin(\log x).\frac{1}{x} + b\cos(\log x)\frac{1}{x}$$

$$= \frac{1}{x}(b\cos(\log x) - a\sin(\log x))$$

~~$y''$~~ multiply by $x$

$$xy' = b\cos(\log x) - a\sin(\log x)$$

Diff. wrt $x$

$$x \cdot y'' + y' = \frac{1}{x} b\cos(\log x) - a\sin x(\log x) \cdot \frac{1}{x}$$

multiply by $x$

$$x^2y'' + xy' = b\cos(\log x) - a\sin x(\log x).$$

$$= -(b\cos(\log x)) + (a\sin(\log x))$$

$$= -y.$$

$$x^2 \cdot y'' + xy' - y = 0.$$

• $y = x^2\cos x$, ST $x^2y'' - 4xy' + (x^2+6)y = 0$.

$y' = \cancel{2x \cdot x^2\cos x}$

$y' = x^2 \cdot -\sin x + \cos x \cdot 2x$

$= 2x \cdot \cos x - x^2 \cdot \sin x.$

$y'' = 2(x \cdot -\sin x + \cos x) - (2x\sin x + x^2\cos x)$

$= 2\cos x - 2x\sin x - 2x \cdot \sin x - x^2\cos x$

$= (2 + x^2) \cdot \cos x - 4x \cdot \sin x.$

$x^2y'' = x^2[(2+x^2)\cos x - 4x\sin x]$

$=$

$y=\sin^{-1}x$, ST $(1-x^2)y''-xy'=0$.

A $y' = \dfrac{1}{\sqrt{1-x^2}}$.

$\sqrt{1-x^2}\cdot y' = 1$, squaring all,

$(1-x^2)(y')^2 = 1$. now diff. wrt.

$\frac{d}{dx}(1-x^2)(y')^2 = \frac{d}{dx}1$

$(1-x^2)\cdot 2y'y'' + (y')^2\cdot(0-2x) = 0$.

$2y'(1-x^2)y'' + (y')^2\cdot -2x = 0$.

(dividing ~~2y'~~ 2y')

$(1-x^2)y'' - xy' = 0$

- $y = \sqrt{\sin(x) + y}$

$$y' = \frac{1}{2\sqrt{\sin x + y}} \left(\cos x + \frac{dy}{dx}\right).$$

$$\frac{y'}{\frac{1}{2\sqrt{\sin x}}}$$

$$\frac{1}{2\sqrt{\sin x + y}} \cdot y' = \cos x + \frac{dy}{dx}$$

$$\frac{y' 2\sqrt{y}}{y' 2y}$$

$$\Rightarrow 2y \cdot y' = \cos x + \frac{dy}{dx}$$

$$2y \cdot y' - y' = \cos x$$

$$(2y - 1)\, y' = \cos x$$

$$y' = \frac{\cos x}{2y - 1}$$

- $e^y = \sin(x + y)$.

diff. wrt. $x$

$$e^y \cdot \frac{dy}{dx} = \cos(x+y) \cdot \left(1 + \frac{dy}{dx}\right)$$

$$e^y \frac{dy}{dx} = \cos(x+y) \cdot 1 + \cos(x+y) \cdot \frac{dy}{dx}$$

$$e^y \frac{dy}{dx} - \cos(x+y) \frac{dy}{dx} = \cos(x+y)$$

$$\frac{dy}{dx}\left(e^y - \cos(x+y)\right) = \cos(x+y).$$

$$\frac{dy}{dx} = \frac{\cos(x+y)}{e^y - \cos(x+y)}$$

# Applications of Differensiation.

chap.1

## Tangents & Normals.

slope of a straight line is given by $m = \tan\theta$ were $\theta$ is the angle made by the straight line with x-axis.

Given a curve $y = f(x)$; & Let $P(x_0, y_0)$ be a given point on this curve then the slope of tangent ~~though~~ at $P(x', y')$ is given by $\left(\frac{dy}{dx}\right)_{(x_0, y_0)}$

$$\boxed{m = \left(\frac{dy}{dx}\right)_{(x_0, y_0)}}$$

- Find the Slope of tangent to the Curve $y = 3x^2 + 4x - 5$ at $(1, 2)$

A) slope of tangent at $(1,2) = \left(\frac{dy}{dx}\right)_{(1,2)}$.

$$\frac{dy}{dx} = 6x + 4.$$

$$m = 10$$

- $y^2 = 4ax$ at $(a, 2a)$, Slop?

A. $\cancel{\frac{dy}{dx} = 4a}$

$\left(\frac{dy}{dx}\right)_{(a, 2a)} =$

$\frac{dy}{dx} \Rightarrow ?$

Diff. w.r.t $x$

$$2y \cdot \frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\left(\frac{dy}{dx}\right)_{(a,2a)} = \frac{2a}{2a} = 1$$

- Find the slope of circle $x^2+y^2=25$ at $(3,4)$.

A slope of tangent at $(3,4) = \left(\frac{dy}{dx}\right)_{(3,4)}$.

$$x^2+y^2=25$$

Diff. w.r.t $x$

$$2x + 2y \cdot y' = 0$$

$$y' = \frac{-2x}{2y} = \frac{-x}{y}$$

$$\left(\frac{dy}{dx}\right)_{(3,4)} = \frac{-3}{4}$$

- Find slope of tangent $x = 2t$, $y = \frac{2}{t}$ at $t = 3$.

slope at $(t=3) \Rightarrow \left(\frac{dy}{dx}\right)_{(t=3)}$

$x = 2t, y = 2/t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2/t^2}{2} = \frac{-2}{2t^2} = \frac{-1}{t^2}$$

$$\left(\frac{dy}{dx}\right)_{(t=3)} = \frac{-1}{9}$$

• ~~$y = \tan^{-1}x$ at $x = \pi/3$.~~

~~A. $\frac{dy}{dx} = \frac{1}{1+x^2}$.~~

~~$\left(\frac{dy}{dx}\right)_{(x=\pi/3)} = \frac{1}{1+(\pi/3)^2} = \frac{1}{1+\frac{\pi^2}{9}} = \frac{9}{9+\pi^2}$~~

• $y = \tan x$

$\frac{dy}{dx} = \sec^2 x$

$\left(\frac{dy}{dx}\right)_{(x=\pi/3)} = \sec^2(\pi/3) = \frac{1}{\cos^2(\pi/3)} = \frac{1}{(1/2)^2} = 4$ //

**Normal**

slop of normal to the curve $y = f(x)$ at $P(x_0, y_0)$.

$= \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_0,y_0)}}$.

$\boxed{P(x_0, y_0) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_0,y_0)}}}$

• Find the slope at normal at curve $y = \tan x$ at $x = \pi/3$

A. we know that $\left(\frac{dy}{dx}\right)_{(x=\pi/4)} = 4$

$\therefore$ slope $= \frac{-1}{4}$ //

• Find slope of tangent & normal the curve $y^2 = 4x$ at $(4,4)$

A. $\frac{dy}{dx} \Rightarrow 2y \cdot \frac{dy}{dx} = 4$. $-1 \times \frac{4}{2}$

$\cancel{\frac{dy}{dx}} \cancel{4}$

$$\frac{dy}{dx} = \frac{\cancel{4}}{\cancel{2}y} = \frac{2}{y}$$

$$\frac{-1}{1/2} = \frac{-2}{}$$

$$\left(\frac{dy}{dx}\right)_{(4,4)} = \frac{2}{4} = 1/2$$

$$P(4,4) = \frac{-1}{2/y}$$

$$= \frac{-1}{2/4} = \frac{-4}{2} = \underline{\underline{-2}}$$

## Equ: of Tangent & Normal

Given a curve, $y = f(x)$ and let $P(x_1, y_1)$ be a given point on this curve. then equ: of tangent

$$P(x_1, y_1) = y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} (x - x_1).$$

Equ: of normal at $P(x_1, y_1) = y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} (x - x_1)$.

- Find equ: of tangent & normal to curve. $y = 3x^2 + 4x - 3$ at $(1, 4)$.

A. $(x_1, y_1) = (1, 4)$

Equ: of tangent at $(1,4) = y - 4 = \left(\frac{dy}{dx}\right)_{(1,4)} (x - 1)$

$$\frac{dy}{dx} = 6x + 4$$

$\left(\frac{dy}{dx}\right)_{(1,4)} = 6 \times 1 + 4 = 10.$

* Equ: of tangent : $y - 4 = 10(x-1)$.

$\Rightarrow y - 4 = 10x - 10$.

$\Rightarrow 10x - y - 6 = 0$

Equ: of normal $= y - 4 = \frac{-1}{10}(x-1)$

$\Rightarrow 10(y-4) = -(x-1)$.

$\Rightarrow 10y - 40 + x + 1 = 0$.

$\Rightarrow x + 10y - 41 = 0$

- Find the equ: of tangent and normal to the curve $y^2 = 4ax$ at $(a, 2a)$

A. $y^2 = 4ax$

$2y \cdot \frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{\cancel{4}^{2}a}{\cancel{2}y}$

$\frac{dy}{dx} = \frac{2a}{y}$.

$\left(\frac{dy}{dx}\right)_{(a,2a)} = \frac{2a}{2a} = 1$

Equ: of tangent $= y - 2a = 1(x-a)$

$\Rightarrow x - a - y + 2a = 0$.

$\Rightarrow x + a - y = 0$.

Equ: of normal $= y - 2a = -1(x-a)$

$\Rightarrow y - 2a + x - a = 0$

$\Rightarrow x + y - 3a = 0$

$y = \cos x$ at $x = \pi/6$.

A. Equ: of tangent at $(x_1, y_1)$.

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x,y_1)} (x - x_1).$$

$$x_1 = \pi/6 \, , \quad y_1 = \cos \pi/6 = \sqrt{3}/2 .$$

Equ of tangent is $= y - \frac{\sqrt{3}}{2} = \left(\frac{dy}{dx}\right)_{(x = \pi/6)} (x - \pi/6).$

$$\frac{dy}{dx} = -\sin x$$

$$\left(\frac{dy}{dx}\right)_{(x = \pi/6)} = -\sin \pi/6 = -1/2$$

Equ: of tangent $= y - \frac{\sqrt{3}}{2} = \frac{-1}{2}(x - \pi/6)$

multiply with 2.

$\Rightarrow 2y - \sqrt{3} = -1(x - \pi/6)$

$\Rightarrow 2y - \sqrt{3} + x - \pi/6 = 0.$

$\Rightarrow x + 2y - \sqrt{3} - \pi/6 = 0$

Equ: of normal

$y - \frac{\sqrt{3}}{2} = 2(x - \pi/6).$

$\Rightarrow y - \frac{\sqrt{3}}{2} = 2x - \pi/3.$

$\Rightarrow 2x - y - \pi/3 + \sqrt{3}/2 = 0$

$y - \sqrt{3}/2 = \frac{-1}{-1/2} = 2x - \pi/3$

$P(x, y_1) \quad y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$

• Equ: Of tangent & normal to the curve $y = 3x^2 - 1$ at $(1,2)$.

A. Equ: Of tangent $y - y_1 = \left(\frac{dy}{dx}\right)_{(1,2)} (x - x_1)$

$\left(\frac{dy}{dx}\right) = 6x$

$\left(\frac{dy}{dx}\right)_{(1,2)} = 6$

$\Rightarrow y - 2 = 6(x-1)$

$\Rightarrow y - 2 = 6x - 6.$

$\Rightarrow y - 2 + 6 - 6x = 0$

$\Rightarrow y - 6x + 4 = 0.$

$y - 2 = \frac{-1}{6}(x-1).$

$6(y-2) = -x + 1.$

$6y - 12 = -x + 1$

$6y - 13 + x = 0$

• For what values of $x$ is the tangent to curve $y = x^3 - 2x^2 + x + 1$ ||el to x-axis

A. we have find to the values of $x$ ~~at~~ where $\frac{dy}{dx} = 0.$

$\frac{dy}{dx} = 3x^2 - 4x + 1.$

$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 4x + 1 = 0.$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 4 \times 3 \times 1}}{2 \times 3}$

$= 1, \frac{1}{3}$

• Find the values of $x$ at which tangent to curve $y = \frac{x}{x^2+1}$ $\parallel^{el}$ to $x$-axis.

A. $y = \frac{x}{x^2+1}$

$$\frac{dy}{dx} = \frac{(x^2+1)\cdot 1 - x\cdot 2x}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(1+x^2)^2}$$

$$\frac{dy}{dx} = 0.$$

$$\frac{1-x^2}{(1+x^2)^2} = 0 \Rightarrow 1-x^2 = 0.$$

$$x^2 = 1$$

$$\underline{x = \pm 1}$$

• Find the values of $x$ where tangent to curve $y = \frac{x}{(1-x)^2}$

a) $\parallel^{el}$ to $x$-axis

b) $\parallel^{el}$ to $y$-axis

A a) $$\frac{dy}{dx} = \frac{(1-x)^2\cdot 1 - x\cdot(2\cdot(1-x))(-1)}{(1-x^2)^{2\times 1}}$$

$$= \frac{(1-x)^2 + 2x - \cancel{2x^2}}{(1-x)^4}$$

[illegible] $(1-x)$ common. [illegible] Cancel

$$= (1-x)^2 \ldots$$

$$= \frac{1-x+2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}$$

$$= \frac{1+x}{(1-x)^3} = 0 \Rightarrow 1+x = 0$$

$$\Rightarrow \underline{x = -1}$$

b) we have find value of $x$, where $\frac{dy}{dx} = \infty$

$$\frac{dy}{dx} = \frac{1+x}{(1-x)^3} = \infty$$

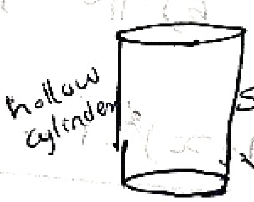
$$(1-x)^3 = 0 \Rightarrow \underline{x = 1}$$

Area of Circle $= \pi r^2$

Volume of cube $= a^3$.

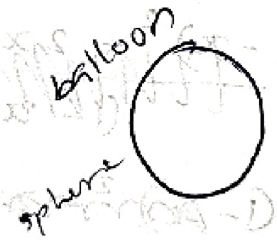
Volume of Sphere $= 4/3 \pi r^3$

Surface area of Sphe $= 4\pi r^2$




$S = \pi r^2 + 2\pi r h$

$V = \pi r^2 h$




$V = \frac{4}{3}\pi r^3$

$S = 4\pi r^2$

RATES & MOTION.

Let $y = f(x)$ be a given function, then $\frac{dy}{dx}$ is represent rate of change of y. wrt x.

For eg: Consider the area of circle of radius 'r' which is given by, $a = \pi r^2$

Then the rate of change of area wrt radius

$$\frac{dA}{dr} = \pi \cdot 2r = \underline{2\pi r}.$$

eg: Volume ~~with~~ of cube with side = a

$V = a^3$.

$$\frac{dv}{da} = \underline{3a^2}.$$

eg: Volume of sphere radius r

$v = \frac{4}{3}\pi r^3$

$$\frac{dv}{dr} = \frac{4}{\cancel{3}}\pi \cancel{3} r^2$$

$$= \underline{4\pi r^2}$$

Let S be the displacement of particle in time 't'. Then

velocity, $v = \frac{ds}{dt}$

accelaration, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

• Displacement of a particle in time 't' is given by $s = t^3 + 3t^2 + 4t + 2$, find velocity & acceleration when $t = 2$ sec.

A. $s = t^3 + 3t^2 + 4t + 2$.

$v = \frac{ds}{dt}$

$= \frac{d}{dt}(t^3 + 3t^2 + 4t + 2)$

$= 3t^2 + 6t + 4$.

$a = \frac{d^2s}{dt^2} = \frac{d}{dt}(3t^2 + 6t + 4)$

$= 6t + 6$.

when $t = 2$ sec

$v = 3t^2 + 6t + 4$

$= 3 \times (2)^2 + 6 \times 2 + 4, = 28\ m/s$

$a = 6t + 6 = 6 \times 2 + 6 = 18\ m/s^2$

• displacement of a particle in time 't' is given by $s = 2t + \frac{3}{t}$ Find velocity & acceleration, when $t = 3$ sec

A. $v = \frac{ds}{dt} = \frac{d}{dt}(2t + \frac{3}{t})$

$= 2 + 3 \cdot \frac{-1}{t^2} = 2 - \frac{3}{t^2}$

$$a = \frac{d^2s}{dt^2} = \frac{d}{dt}\left(2 - 3/t^2\right)$$

$$= 0 - 3 \cdot \frac{-2}{t^3} = \frac{6}{t^3}$$

when $t = 3$ sec.

$$v = 2 - \frac{3}{9} = \frac{15}{9}\ m/s$$

$$a = \frac{6}{3^3} = \frac{6}{27}\ m/s^2$$

- A particle moves such that the displacement from a fixed point 'O' is given by

$$S = 5\cos nt + 4\sin nt$$

where $n$ = constant; S.T acceleration varies as ~~this~~ it's displacement.

A. we have S.T $a \propto s$.

$a = k.s$, where $k$ = constant

$$\frac{ds}{dt} = 5 - \sin nt + 4\cos nt . n.$$

$$= n(4\cos nt - 5\sin nt)$$

$$\frac{d^2}{dt^2} = n(-4\sin nt . n - 5\cos nt . n)$$

$$= -n^2(5\cos nt + 4\sin nt).$$

$\therefore$ $n^2$ = constant

$a = k.s$

$\therefore a \propto s$

- displacement $s = ae^{nt} + be^{-nt}$; ST accelaration Varies as it's displacement

A. we S.T $a \propto s$

$a = k.s.$

$$\frac{ds}{dt} = a.e^{nt}.n + b.e^{-nt}.-n$$

$$\frac{d^2s}{dt} = n(a.e^{nt} - b.e^{-nt})$$

$$\frac{d^2s}{dt} = n(a.e^{nt}n + b.e^{-nt}.-n)$$

$$= n^2(a.e^{nt} + be^{-nt})$$

$$= n^2.s$$

$$\Rightarrow n^2 \propto s$$

- displacement $s = ae^{t} + be^{-t}$ S.T $a = s$.

A. we S.T $a = s$.

$$\frac{ds}{dt} = a.e^{t} + be^{-t}.-1.$$

$$= a.e^{t} - b.e^{-t}$$

$$\frac{d^2s}{dt^2} = a.e^{t} - b.e^{-t}.-1$$

$$= a.e^{t} + b.e^{-t} = s$$

$\therefore a = s$

A dist. traveled by a particle it time 't' is given by $S = t^3 - 6t^2 + 8t - 4$.

i) Find the time 't' when $a = 12 cm/s^2$

ii) Find velocity at this time.

A. $S = t^3 - 6t^2 + 8t - 4$

$$\frac{ds}{dt} = 3t^2 - 12t + 8$$

$$\frac{d^2s}{dt^2} = 6t - 12.$$

i) when $a = 12 \ cm/s^2$

$$6t - 12 = 12$$

$$6t = 24$$

$$\underline{t = 4 \ sec}$$

ii). $V = \frac{ds}{dt} = 3t^2 - 12t + 8$ (when $t = 4$)

$$= 3 \times 4^2 - 12 \times 4 + 8$$

$$= \underline{8 \ cm/s}$$

- The displacement of a particle from a mean position 'O' in time 't' given by $S = 2t^3 - 9t^2 + 12t + 6$

i) Find 't' when $a = 0$.

ii) also find velocity at this time

A. $S = 2t^3 - 9t^2 + 12t + 6$

$$\frac{ds}{dt} = 6t^2 - 18t + 12$$

$$\frac{d^2s}{dt^2} = 12t - 18.$$

(when $a = 0$)

i) $\frac{d^2s}{dt^2} = 0 \Rightarrow 12t - 18 = 0.$

$\Rightarrow 12t = 18$

$t = 18/12$

$t = 3/2$ sec

ii) Velocity $\frac{ds}{dt}$ at $t = 3/2$

$$\frac{ds}{dt} = 6t^2 - 18t + 12$$

$$= 6 \times \left(\frac{3}{2}\right)^2 - 18 \times \frac{3}{2} + 12$$

$$= \frac{27}{2} - 27 + 12 = \frac{-3}{2} \text{ cm/s}$$

- work done by a moving body is given by $W = 2t - \frac{3}{t}$ units over an interval of $t$ seconds.

i) Find power of Force Creating this motion. when $t = 2$ sec

A. $W = 2t - \frac{3}{t}$

$P = \frac{W}{t}$ (we know)

$P = \frac{dW}{dt}$

$\therefore \frac{dW}{dt} = 2 + \frac{3}{t^2} = 2 + \frac{3}{t^2}$

when $t = 2$ sec $\Rightarrow 2 + \frac{3}{(2)^2} = \frac{11}{4}$ units

## Related rates :

→ If 2 variables are related then their rate of change also related. we can find the rate of change of one variable, if the rate of change of the other variable is given.

eg: The radius of a circle increasing at a rate of 2cm per second. Find the rate at which the area is increasing when the radius is 6m.

A. $A = \pi r^2$

$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

$\frac{dy}{dx} = 2\pi \cdot 2r \cdot$

given $r = 6.$

$$\frac{dr}{dt} = 2.$$

$$\Rightarrow \frac{dA}{dt} = 2\pi \cdot 6 \cdot 2$$

$$= 24\pi \ cm^2/sec$$

• A circular disc expanding when heated at a rate of $12 \ cm^2/s^2$. find the rate at which the area is increasing. when the radius is 4 cm.

A. $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

given $\frac{dA}{dt} = 12 \text{ cm}^2/\text{se}$ and $r = 4$ cm

$12 = 2\pi \cdot 4 \cdot \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{12}{8\pi} = \frac{3}{2\pi}$ //

- gas is excaping from a spherical balloon at a rate of 20 cc/sec. Find the radius at which the radius is decreasing when radius is 4 cm.

A. Volume of Spherical balloon, $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3}r^2 \cdot \frac{dr}{dt}$$

$$= 4\pi r^2 \cdot \frac{dr}{dt}$$

given $\frac{dV}{dt} = -20$. and $r = 4$

$\Rightarrow -20 = 4\pi \cdot 4^2 \cdot \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{-5}{16\pi}$

Hence the radius is $\downarrow$ at the rate of $\frac{5}{16\pi}$ cm/s

- A spherical balloon inflated, by pumping 15 cc/sec find the rate at which the radius is increasing when the radius is 3 cm.

A. $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$$

$$= 4\pi r^2 \frac{dr}{dt}$$

given $\frac{dV}{dt} = 15\,cc/sec$ and $r = 3$

$$\Rightarrow 15 = 4\pi \cdot 3^2 \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{15}{4\pi \cdot 9}$$

$$= \frac{15}{36\pi}$$

• Air is pumped into a spherical rubber bladder of radius 3. If the radius increase at uniform rate of 1 cm/minute. Find the rate at which the volume is increases at the end of 3 minute

A. $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$$

$$= 4\pi r^2 \cdot \frac{dr}{dt}$$

$\frac{dr}{dt} = 1\,cm/minute.$

radius after 3 minute $= 3 + (3 \times 1) = 6$

$$\frac{dV}{dt} = 4\pi \cdot 6^2 \cdot 1 = 4\pi \times 36 = 144\pi \ cm^3/min$$

- A circular plate of radius 3 inches expands when heated at the rate of .2 inch / sec. Find the rate at which area of the plate increasing at the end of 4 sec.

A. $A = \pi r^2$

$$\frac{dA}{dt} = \pi . 2r . \frac{dr}{dt}$$

after 4 sec. radius = $3 + (2 \times 4) = 11$.

$$\frac{dr}{dt} = 2$$

$$\frac{dr}{dt} = \pi . 2 \times 11 \times 2$$

$$= 44 \pi \text{ inch}^2/\text{sec}$$

- A balloon is spherical in shape gas is escaping from it at the rate of 10 cc / sec. How fast is the surface area shrinking when the radius is 15cm.

A. surface area : $S = 4\pi r^2$

$r = 15cm.$

$\frac{dv}{dt} = 10cc/sec$

$$\frac{ds}{dt} = 4\pi . 2r . \frac{dr}{dt}$$

$$= 8\pi . r . \frac{dr}{dt} \quad \text{①}$$

$r = 15$ cm.

To find $\frac{dv}{dt}$ - given that $\frac{dv}{dt} = -10$.

$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$. $V = \frac{4}{3}\pi r^3$

$$\frac{dv}{dt} = \frac{4}{3}\pi 3 . r^2 . \frac{dr}{dt}$$

$$\frac{dv}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow -10 = 4\pi \cdot 15^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{-10}{4\pi \cdot 15^2}$$

From ①,

$$\frac{ds}{dt} = \cancel{8}^{2}\cancel{\pi}\,\cancel{15} \cdot \frac{-10}{\cancel{4}\cancel{\pi} \cdot 15^{\cancel{2}}}$$

$$= \frac{-20}{15} = \frac{-4}{3}.$$

- Gas is escaped from a spherical balloon at the rate of 20 cc/sec. Find the rate at which the surface area shrinking when the radius is 10 cm.

A- $S = 4\pi r^2$

$$\frac{ds}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

$r = 10$ cm.

To find $\frac{dv}{dt}$, given that $\frac{dv}{dt} = -20$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dv}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow -20 = 4\pi \cdot 10^2 \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-20}{4\pi \cdot 10^2}$$

from ① $\Rightarrow \frac{ds}{dt} = \cancel{8}^{2}\cancel{\pi}(\cancel{10}) \cdot \frac{-20}{\cancel{4}\cancel{\pi} \cdot 10^{\cancel{2}}} = \frac{-4\cancel{0}}{1\cancel{0}} = -4$

# MAXIMA & MINIMUM.

Given a function $y = f(x)$ the function is increasing at the point $x = a$, If $\left(\frac{dy}{dx}\right)_{(x=a)} > 0$.

The function is decreasing at the point $x = b$.

$$\left(\frac{dy}{dx}\right)_{(x=b)} < 0.$$

eg: check whether the function ~~th~~ is increase/decrease at the given points.

1) $y = 4x^2 - 2x + 1$ at $x = 3, -2$.

A. $\frac{dy}{dx} = 8x - 2$

$\left(\frac{dy}{dx}\right)_{(x=3)} = 8.3 - 2 = 22 > 0$. (Function is increasing)

$\left(\frac{dy}{dx}\right)_{(x=-2)} = 8.-2 - 2 = -18 < 0$ (Function is decreasing)

- $y = x^3 - 6x^2 + 5x + 2$ at $x = -2, 2$.

A. $\frac{dy}{dx} = 3x^2 - 12x + 5$

$\left(\frac{dy}{dx}\right)_{(x=-2)} = 3.(-2)^2 - 12(-2) + 5.$

$= 12 + 24 + 5 = 41 > 0$ (increasing)

$\left(\frac{dy}{dx}\right)_{(x=2)} = 3.(2)^2 - 12.(2) + 5$

$= 12 - 24 + 5 = -7 < 0$. (decreasing).

- Find the range of values of $x$ for which the function $y = x^2 - 3x + 4$ is increasing.

A. we have to find the range of values of $x$ for which $\frac{dy}{dx} > 0$.

$$\frac{dy}{dx} = 2x - 3.$$

$$2x - 3 > 0.$$

$$2x > 3.$$

$$x > 3/2$$

i.e the function is increasing ~~at~~ $x > 3/2$

- Find the range of values of $x$ for which $y = 4x^2 - 3x + 10$ is decreasing

A. $y = 4x^2 - 3x + 10.$

$$\frac{dy}{dx} = 8x - 3.$$

we have to find $\left(\frac{dy}{dx}\right) < 0.$

$$8x - 3 < 0.$$

$$8x < 3.$$

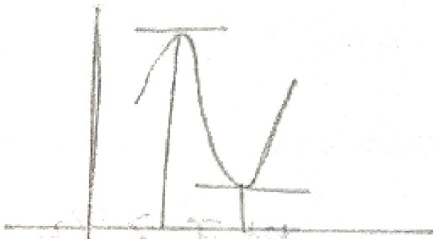
$$x < 3/8.$$

i.e function decreasing at $x < 3/8$

## Stationary points / turning points

given a function is $y = f(x)$, A point $x = a$ is a stationary point of $y = f(x)$, if the function neither increases nor decreases at that point.

A stationary point is also called "turning point" since the function changes its nature at that point.



$x=a$ $x=b$

($a, b$ = stationary point)

* $x = a$ is stationary point of $y = f(x)$, if and only if.

$$\left(\frac{dy}{dx}\right)_{(x=a)} = 0.$$

* Find the stationary points of following functions.

1) $y = x^2 - 4x + 10.$

$$\frac{dy}{dx} = 2x - 4.$$

$$\frac{dy}{dx} = 0. \Rightarrow 2x - 4 = 0.$$

$$\Rightarrow 2x = 4$$

$$x = 2$$

* $y = 2x^3 - 9x^2 + 12x + 2$

A $\frac{dy}{dx} = 6x^2 - 18x + 12$

$\frac{dy}{dx} = 0 \Rightarrow 6x^2 - 18x + 12 = 0$

add = -3
prdt = 2

~~$6x^2 - 18x = 12$~~

$6(x^2 - 3x + 2) = 0$

$\Rightarrow (x^2 - 3x + 2) = 0 \longrightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

ഉത്തരം 1, 2

$\Rightarrow$ ~~$(x-1)(x-2) = 0$~~

$\Rightarrow x = 1, 2.$

- $y = x^3 - 3x^2 - 9x + 5$

$\frac{dy}{dx} = 3x^2 - 6x + 9.$

$\left(\frac{dy}{dx}\right) = 0 \Rightarrow 3x^2 - 6x - 9 = 0$

$\Rightarrow 3(x^2 - 2x - 3) = 0.$

$\Rightarrow x^2 - 2x - 3 = 0 \longrightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow x = 3, -1$

Maximum & ~~point~~ minimum point

given a function $y = f(x)$, Then $x = a$ is the max. point ~~and~~ of $y = f(x)$.

1) $\frac{dy}{dx} = 0$

2) $\frac{d^2y}{dx^2} < 0.$ at $x = a$

| Max. point | Minimum. |
| --- | --- |
| 1) $\frac{dy}{dx} = 0$ | 1) $\frac{dy}{dx} = 0$ |
| 2) $\frac{d^2y}{dx^2} < 0$ | 2) $\frac{d^2y}{dx^2} > 0$ |

- a point $x = b$ is the minimum point of $y = f(x)$.

1) $\frac{dy}{dx} = 0$

2) $\frac{d^2y}{dx^2} > 0.$

• Find the max. value of function $y = 2x^3 - 3x^2 - 36x + 10$.

A. $y = 2x^3 - 3x^2 - 36x + 10$

$\frac{dy}{dx} = 6x^2 - 6x - 36$.

where $\frac{dy}{dx} = 0$, (Find stationary points)

$\Rightarrow 6x^2 - 6x - 36 = 0$.

$6(x^2 - x - 6) = 0$

$\therefore x = 3, -2$

$= \frac{+1 \pm \sqrt{1+24}}{2}$

$= \frac{1 \pm \sqrt{25}}{2} = \frac{1 \pm 5}{2} = 3, -2$

The stationary points are $3, -2$

$\frac{d^2y}{dx^2} = \frac{d}{dx} 6x^2 - 6x - 36$.

$= 12x - 6$.

where $\frac{d^2y}{dx^2} < 0$ (at max. point)

when $x = 3$, $\frac{d^2y}{dx^2} = 12 \cdot 3 - 6 = 30 > 0$.

when $x = -2$, $\frac{d^2y}{dx^2} = 12 \cdot -2 - 6 = -30 < 0$.

$\therefore x = -2$ is the max. point

$$\begin{array}{r} 16 \\ 12 \\ \hline 28 \\ \hline 44 \end{array}$$

Max. value at $y = 2 \cdot (-2)^3 - (3(-2)^2) - 36(-2) + 10$

$= 2 - 16 - 12 + 72 + 10$

$= -28 + 72 + 10 = 44 + 10$

$= \underline{54}$

• Deflection of a beam is given by $y = 2x^3 - 9x^2 + 12x$, find the max. deflection.

A. $y = 2x^3 - 9x^2 + 12x$

$$\frac{dy}{dx} = 6x^2 - 18x + 12$$

Find stationary point

$$\frac{dy}{dx} = 0.$$

$\Rightarrow 6x^2 - 18x + 12 = 0$

$\Rightarrow 6(x^2 - 3x + 2) = 0.$

$(x^2 - 3x + 2) = 0.$

$\frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2}$

$= 2, 1$

$x = 2, 1 \Rightarrow$ stationary points

$$\frac{d^2y}{dx^2} = \frac{d}{dx} 6x^2 - 18x + 12$$

$$= 12x - 18$$

at $x = 1$; $\frac{d^2y}{dx^2} = 12 \cdot 1 - 18 = -6 < 0$.

at $x = 2$ $\frac{d^2y}{dx^2} = 12 \cdot 2 - 18 = 6 > 0$

$\therefore x = 1$ is the max. point

$$\text{max. deflection} = 2(1)^3 - 9(1)^2 + 12(1)$$

$$= 2 - 9 + 12 = -7 + 12 = 5$$

• Bending moment of a rod of 10 m long and weighting 40 kg & resting at its ends at a dist. of $x$ m from one end is given by $M = 2(10x - x^2)$. find the max. bending moment

A. $m = 2(10x - x^2)$.

Here we have to find the max. value.

Finding stationary points.

$$\frac{dm}{dx} = 2(10-2x).$$

$$\frac{dm}{dx} = 0.$$

$$2(10-2x) = 0.$$

$$10 - 2x = 0$$

$$+2x = +10$$

$x = 5$ (it is the stationary point)

$$\frac{d^2m}{dx} = \frac{d}{dx}(2(10-2x))$$

$$= 2(0-2) = -4$$

when $x=5$ ; $\frac{d^2m}{dx^2} = -4 < 0.$

$\therefore$ $x=5$ is the max. point

Max. value of $m = 2(10 \cdot 5 - (5)^2)$

$$= 2(50-25) = 2 \times 25$$

$$= 50 \text{ kgm}$$

Hw• Deflection of a ~~wave~~ beam is given $y = 4x^3 + 9x^2 - 12x + 2$

find the max. deflection.

PT a rectangular of fixed perimetere has its max. area when it becomes a square.

A. $x$ : length
$y$ = breadth

$A = xy$ — ①

$2x + 2y = l$

$2y = l - 2x$

$y = \frac{l}{2} - x$

sub ① ⇒

$A = x\left(\frac{l}{2} - x\right)$

$= \frac{l}{2}x - x^2$

we have to find value of $x$, A is max.

Finding stationary point

$$\frac{dA}{dx} = \frac{l}{2} - 2x$$

$$\frac{dA}{dx} = 0.$$

$$\frac{l}{2} - 2x = 0,$$

$$\frac{l}{2} = 2x.$$

$$x = l/4$$

$x$ ന് വില ആവശ്യമില്ല.
$\therefore -2 < 0$

$$\frac{d^2A}{dx^2} = -2 \quad , \text{when } x = l/4, \ \frac{d^2A}{dx^2} = -2 < 0.$$

when $x = l/4$ is the max. point

i.e Area A is max. when $x = l/4$.

when $x = l/4$, $y = l/2 - l/4 = l/4$

i.e A is max. when $x = y = l/4$.

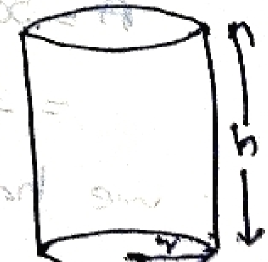
when rectangular is square

The hollow of cylinder can 100 cc of water. is to be made so that the area of the metal used in minimum. P.T the radius which will give minimum area is $3\sqrt{\frac{100}{\pi}}$ cm.

A. Surface area $= \pi r^2 + 2\pi r h$ — ①.

Given $V = \pi r^2 h = 100.$

$$h = \frac{100}{\pi r^2}$$



Sub ① $\Rightarrow$

$$S = \pi r^2 + 2\pi r \cdot \frac{100}{\pi r^2}$$

$$S = \pi r^2 + \frac{200}{r}$$

we have to find the value of $r$, $S$ is minimum

finding stationary points

$$\frac{dS}{dr} = 2\pi r - \frac{200}{r^2}$$

$$\frac{dS}{dr} = 0 \Rightarrow 2\pi r - \frac{200}{r^2} = 0$$

$$\Rightarrow \quad \Rightarrow \cancel{2}\pi r = \frac{\cancel{200}}{r^2}$$

$$r^3 = \frac{100}{\pi}$$

$$r = 3\sqrt{\frac{100}{\pi}}$$ (It is stationary point)

$$\frac{d^2S}{dr^2} = 2\pi - 200 \cdot \frac{-2}{r^3}$$

$$= 2\pi + \frac{400}{r^3}$$

when $r = 3\sqrt{\frac{100}{\pi}}$, $\therefore r^3 = \frac{100}{\pi}$.

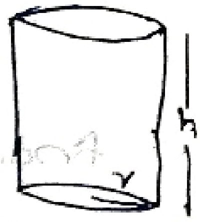
$$\therefore \frac{d^2S}{dr^2} = 2\pi + \frac{400}{100/\pi}$$

$$= 2\pi + 4\pi$$

$$= 6\pi > 0$$

$\therefore$ when $r = 3\sqrt{\frac{100}{\pi}}$ the $S$ is minimum

1. The sum of the diameter and length of an open cylindrical vessel, is 40 cm. P.T max. volume obtained when the radius = length.




A. Volume of cylinder $\Rightarrow V = \pi r^2 h$.

Here we have to find the value of $r$ and $h$, at volume is max.

$$V = \pi r^2 h \quad —① $$

given that $2r + h = 40$.

$$\Rightarrow h = 40 - 2r$$

sub ① $\Rightarrow V = \pi r^2 (40 - 2r)$

$$= 40\pi r^2 - 2\pi r^3.$$

Stationary points finding

$$\frac{dv}{dr} = 80\pi r - 6\pi r^2$$

$$\frac{dv}{dr} = 0 \Rightarrow 80\pi r - 6\pi r^2 = 0$$

$$80\cancel{\pi}\cancel{r} = 6\cancel{\pi}r^{\cancel{2}}$$

$$6r = 80$$

$$r = 80/6 = \frac{40}{3}$$

$r = 40/3$ is stationary

$$\frac{d^2v}{dr^2} = 80\pi - 12\pi r$$

when $r = 40/3$

$$\frac{d^2v}{dr^2} = 80\pi - \cancel{12}^{4}\pi \cdot 40/\cancel{3}$$

$$= 80\pi - 160\pi.$$

$$= -80\pi < 0$$

∴ i.e V max. when $r = 40/3$

when $r = 40/3$

$$h = 40 - 2 \cdot 40/3$$

$$= \underline{40/3}$$

∴ V is max. at $r = h = 40/3$

• Find 2 numbers whose sum is 14 and sum of squares is minimum.

A. no. $= x$

no. $= y$.

$$S = x^2 + y^2 \quad ①$$

$$x + y = 14$$

$$y = 14 - x$$

In ① $\Rightarrow S = x^2 + (14 - x)^2$

finding point of stationarity.

$$\frac{ds}{dx} = 2x + 2(14 - x) \cdot (-1)$$

$$= 2x - 28 + 2x$$

$$= -28 + 4x$$

$4\overline{)28}$ $>0$

$$\frac{ds}{dx} = 0 \Rightarrow 4x = 28$$

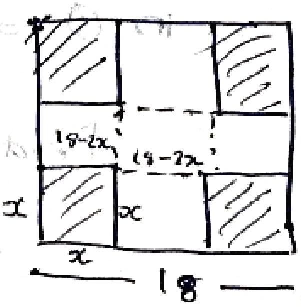
$$\underline{x = 7}$$

$$\frac{d^2s}{dx^2} = 4 > 0$$

$x = 7$, ∴ $y = 14 - 7 = \underline{7}$

• A open box is made ~~of~~ out of a square sheet of side 18 cm. by cutting of equal squares ~~and~~ at each corners and turning up this sides what size of the square should be cuts inoder that volume of the box is maximum.

A. Let $x$ be the side of the square which are cut off. Then volume of box:




$$V = (18-2x)(18-2x)x$$

Now we have find that the value of $x$, $V$ is max.

$$V = (324 - \cancel{36} - 36x + 4x^2)x$$

$$= (324 - 72x + 4x^2)x$$

$$= 4x^3 - 72x^2 + 324x$$

$$= 4x^3 - 72x^2 + 324x$$

Finding the Stationary points.

$$\frac{dV}{dx} = 12x^2 - 144x + 324$$
$$= 12(x^2 - 12x + 27).$$

$$\frac{dV}{dx} = 0 \Rightarrow 12(x^2 - 12x + 27) = 0$$

$\rightarrow \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\Rightarrow x^2 - 12x + 27 = 0.$$
$$\Rightarrow x = 3, 9 \text{ (are the st. points)}$$

$x = 9$ is not possible $\because$ bcz (total length $= 18$)

$$\Rightarrow \therefore x = 3.$$

$$\frac{d^2V}{dx^2} = \cancel{144}\ 24x - 144.$$

$24 \times 3 = 72$

$144 - 72 = 72$

when $x = 3$.

$$\frac{d^2V}{dx^2} = 24 \times 3 - 144.$$
$$= 72 - 144$$
$$= -72 < 0.$$

$\therefore$ Volume of the box is max. at $x = 3$.

Volume of box ~~is~~ will be max. If the remove Square of side 3.