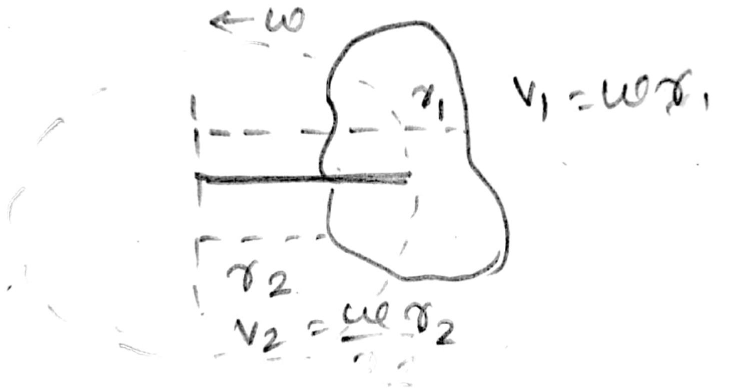
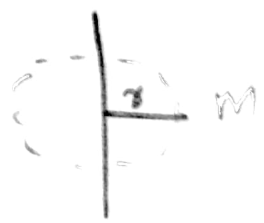
ROTATIONAL DYNAMICS

A rigid body is one it does not change its size of shape under the action of forces. If a rigid body is rotating about its axis all the particles in the body should have a common angular velocity ω. But ~~co~~ each with its own linear velocity



Moment of Inertia.

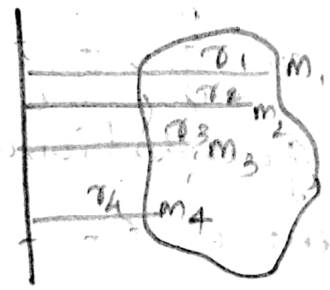
The rotational inertia is called moment of inertia. It depends on the mass of the particle and also their respective distances from the axis of rotation.



moment of inertia of a particle I about an axis of rotation is defined as the product of mass of the particle and the square of the distance of the particle from the axis of rotation.

From figure, $I = MR^2$

In the case of rigid body we can imagine that which is made up of large no. of particles m_1, m_2, m_3 .... etc which are at the distances r_1, r_2, r_3 ... etc.



Here the moment of inertia is the sum of inertia of all the particles. ie,

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots$$

$$I = \Sigma m_i r_i^2$$

unit $= kg\,m^2$

Dimension formula $= [ML^2]$

A torque(τ) is necessary to overcome rotational inertia / moment of inertia.

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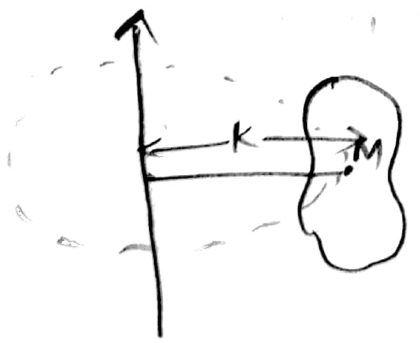
Centre of Mass & Centre of Gravity.

~~R p~~ The point of a body where all the masses of the particles of that body is concentrated is called centre of mass.

The force of gravity acting on the various particles of a rigid body can be represent by a single force mg acting at a point. This point is called centre of gravity.

~~If If~~ In the same gravitational field, centre of mass and centre of gravity coincides

Radius of gyration.



The radius of gyration is the distance whose square multiplied with the total mass 'M' gives moment of inertia of the body about the given axis.

ie, $I = Mk^2$

$$k^2 = \frac{I}{M}$$

$$k = \sqrt{I/M}$$

Theorams on moment of Inertia.

We can simply calculate the moment of inertia of various objects using two general theorams.

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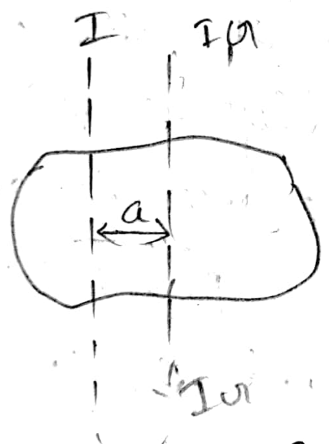
i) Parallel axis theoram.

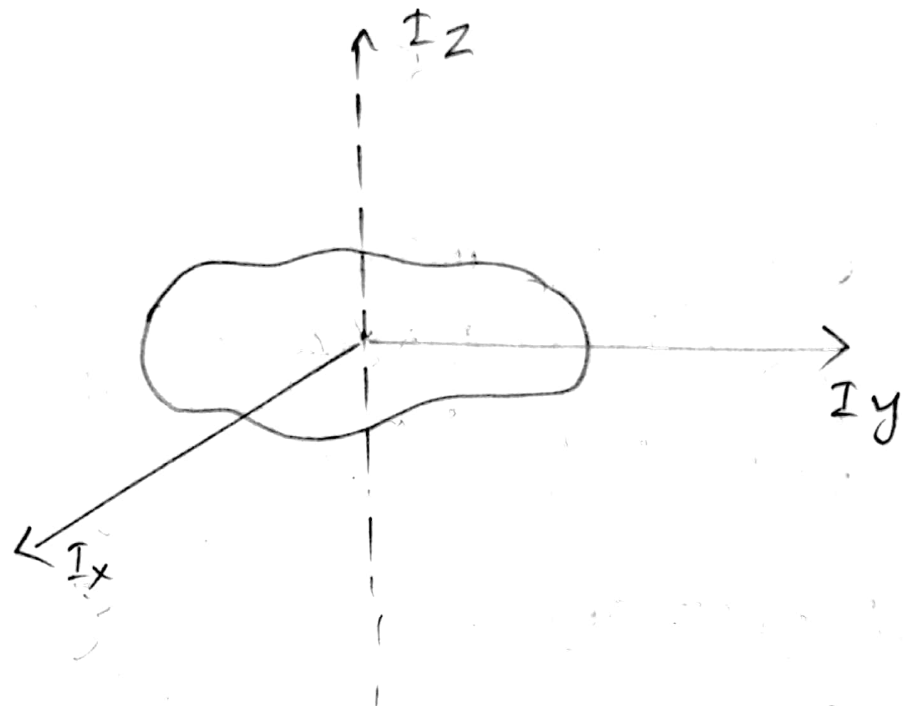


$$I = I_G + Ma^2$$

$$= I_G + Ma^2$$

The moment of inertia of any rigid body about a given axis is equal to the sum of its moment of inertia (I_G) about a ||el axis passing through the centre of mass and the product of mass of the body and square of the distance between the axis.

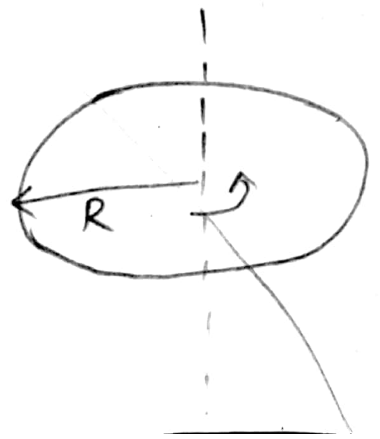
2) Theoram of ⊥r axis



The sum of the moments of inertia of a plane lamina about two mutually ⊥r axis lying in its plane is equal to the moment of inertia about an axis ⊥r to the plane of the lamina and passing through the point of intersection of the first two axis.

$$I_z = I_x + I_y$$

eg:- moment of inertia of a ring about an axis passing through its centre and ⊥r to its plane.



let M be the mass and 'R' be the radius. consider a small element of mass 'm' of the ring. Then the moment of inertia of this mass 'm'

$= I = mR^2$

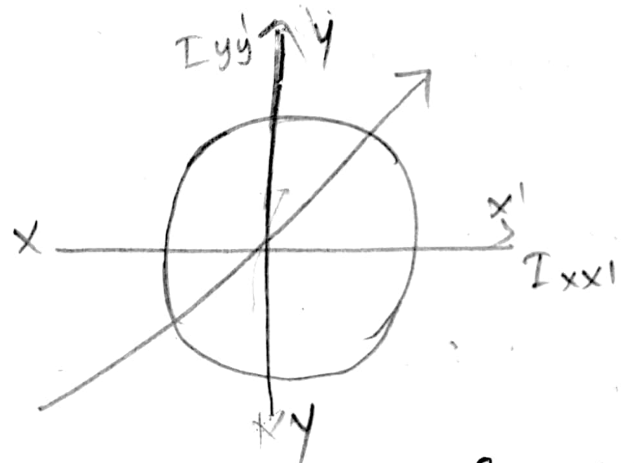
hence the moment of inertia of the complete ring $I = \Sigma mR^2$

$= R^2 \Sigma m$

$= MR^2$

Case-I :

Moment of inertia of a ring about a diameter.



Here we can apply to axis theorem. i.e,

$I_z = I_y + I_x$

here I_x & I_y are same. that is the moment of inertia about the diameter xx' & yy'

$I_{xx'} + I_{yy'} = I$

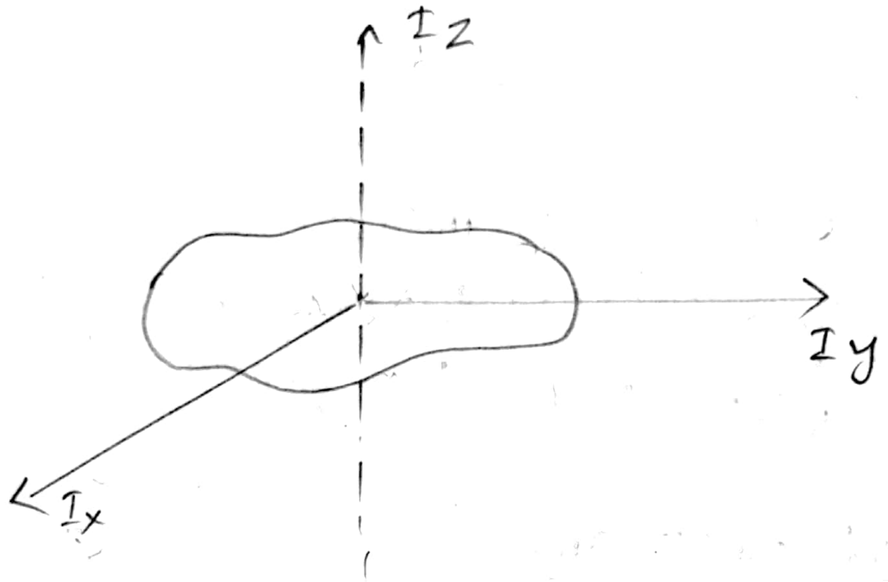
$I_z = I + I$

$MR^2 = 2I$

$I = \frac{MR^2}{2}$

But we know that $I_z = MR^2$

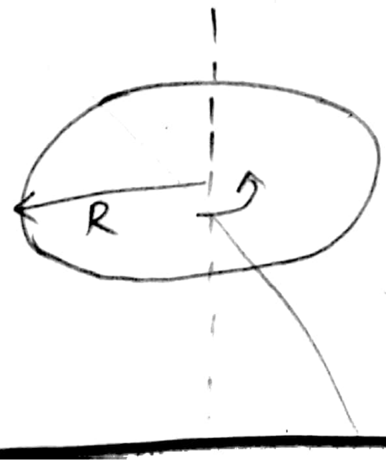
2) Theoram of ⊥r axis



The sum of the moments of inertia of a plane lamina about two mutually ⊥r axis lying in its plane is equal to the moment of inertia about an axis ⊥r to the plane of the lamina and passing through the point of intersection of the first two axis.

$$I_z = I_x + I_y$$

eg:- moment of inertia of a ring about an axis passing through its centre and ⊥r to its plane.



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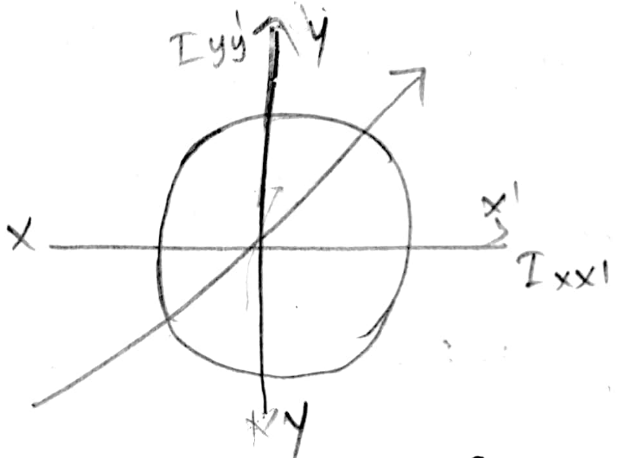

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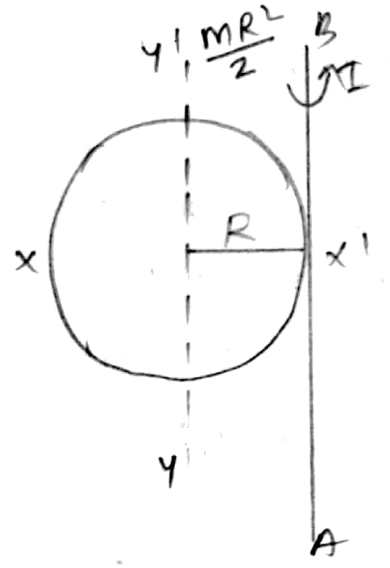
$I = \frac{MR^2}{2}$

But we know that $I_z = MR^2$

2. Moment of inertia of a ring about a tangent



From fig let I be the moment of inertia of the ring about the tangent AB. We can apply ||el axis theoram here.

$$I_{AB} = I_{YY'} + MR^2$$

But we have $I_{YY'} = \frac{MR^2}{2}$

$$So \quad I = \frac{MR^2}{2} + MR^2$$

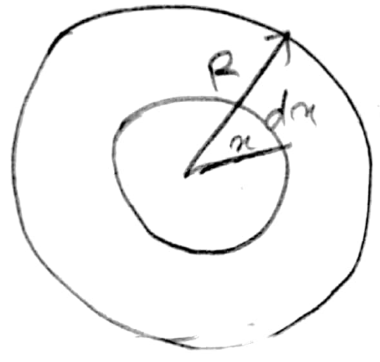
$$= \frac{3MR^2}{2}$$

Moment of inertia of a circular disc.

Case - 1

~~an~~ about an axis passing through the centre and $\perp$r to its plane.

Let M be the mass and 'R' be the radius, The disc can be imagine to be made up of large no of rings of very small width and gradually increasing radius from zero to R.



Consider such a ring of radius 'x' and width dx. Total mass of disc 'M'

$\therefore$ mass / unit area of the disk $= \frac{M}{\pi R^2}$.

$\therefore$ Area of the ring of radius 'x' and width dx $= 2\pi x \, dx$.

$\therefore$ mass of the ring $= \frac{M}{\pi R^2} \times$ area of the ring.

$$= \frac{M}{\pi R^2} \times 2\pi x \, dx.$$

$$= \frac{M \, 2x \, dx}{R^2}$$

$\therefore$ moment of inertia of the ring about the axis passing through the centre $\perp$ to its plane. $= \cancel{M \, 2x \, dx}$

$$= \frac{M \, 2x \, dx}{R^2} x^2$$

(moment of inertia of a ring is mass $\times$ radius^2)

$$= \frac{M \, 2x^3 \, dx}{R^2} \,//$$

moment of inertia of the disk

we have to integrate btw the limits

$x = 0$ to $x = R$

$$I = \int_0^R \frac{2Mx^3}{R^2}\,dx$$

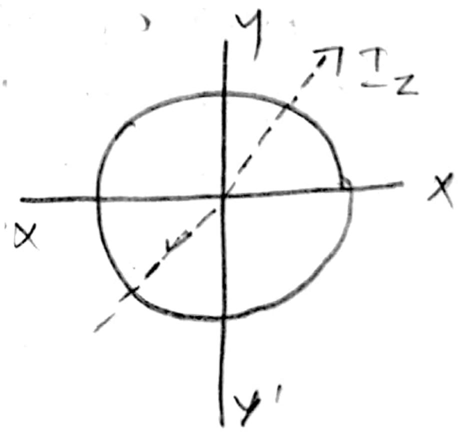
$$= \frac{2M}{R^2}\int_0^R x^3\,dx$$

$$= \frac{\cancel{2}M}{R^2}\,\frac{R^4}{\cancel{4}}$$

$$I = \frac{MR^2}{2}$$

<u>case - 2</u>

moment of inertia of a disc about a diameter



on applying $\perp r$ axis theoram we can write,

$$I_Z = I_{xx'} + I_{yy'}$$

$$\cancel{I_z}\ I_{xx'} = I_{yy'} = I$$

$$I_Z = I + I$$

$$I_Z = 2I$$

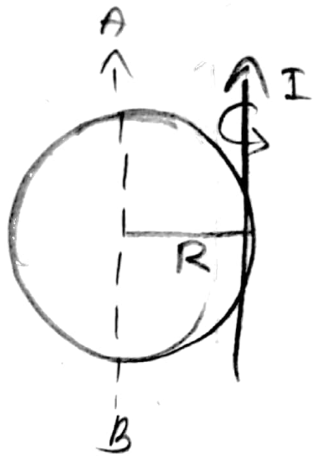
here $I_Z = \frac{MR^2}{2}$ ⇒ moment of inertia of disk about the axis $\perp r$ to the plane of the disc.

$$\frac{MR^2}{2} = 2I$$

$$\boxed{I = \frac{MR^2}{4}}$$

~~Ms.~~

Case-3 :- moment of Inertia of the disc about a tangent.



~~The~~ on applying $\parallel^{el}$ axis theoram the moment of inertia about the tangent I should be equal to $I_{AB} + MR^2$

$I = I_{AB} + MR^2$ (from the figure and appy $\parallel^{el}$ axis theoram)

But we know that I_{AB} is the m. of ineal. of the disk about the diameter.

ie, $I_{AB} = \frac{1}{4} MR^2$

$$I = \frac{MR^2}{4} + MR^2$$

$$I = \frac{5MR^2}{4}$$

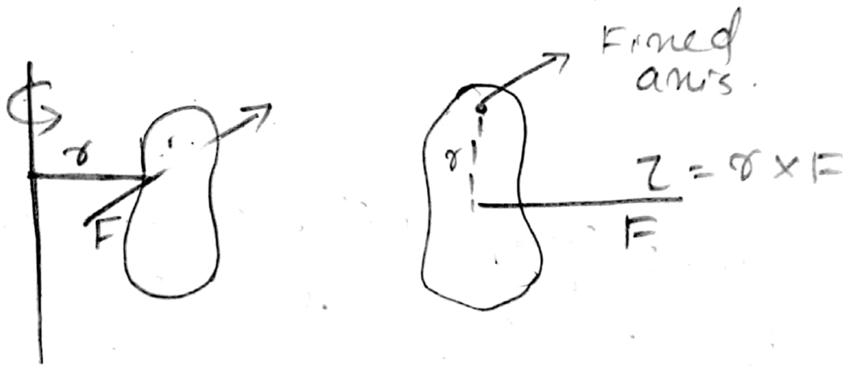
Torque

The rotational effect of force is called torque

$\tau = r \times F$

unit = Nm   Dimension Formula = ML^2T^{-2}

τ is defined as product of force 'F' and the ⊥r distance 'r' btw the line of action of the force and the axis of rotation.



~~Newton's eqn for linear~~

we know that F = ma, ie newtons eqn for linear case so analogously we can write ||| ly.

$\tau = I\alpha$

a = α angular acc

m = I

τ = F

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Translational motion.	Rotational motion.
1. Linear displacement (x)	angular displacement (θ)
2. Linear velocity (v)	angular velocity (ω)
3. Linear acceleration (a)	angular acceleration (α)
4. mass (m)	Inertia (I)
5. Force $F = ma$	Torque $\tau = I\alpha$

we have linear momentum $P = mv$.
analogous to that we have angular momentum,

$$L = I\omega$$

we have the relation btw Force & linear momentum,

ie, $F = \frac{dP}{dt}$

Relation btw τ & angular momentum (L)

we have $\tau = I\alpha$

I = moment of inertia
α - ang. acceleration

$\therefore$ α = ~~$\omega_2 -$~~ rate of change of angula. velocity

$$= \frac{\omega_2 - \omega_1}{t}$$

$$\tau = I \times \frac{\omega_2 - \omega_1}{t}$$

$$\tau = \frac{I\omega_2 - I\omega_1}{t}$$

$\therefore I\omega_2 = L_2$

$I\omega_1 = L_1$

$L = I\omega$

$$\tau = \frac{L_2 - L_1}{t}$$

$$\tau = \frac{dL}{dt}$$

This eqn is analogous to Newton's formula $F = \frac{dP}{dt}$

Torque is the rate of change of angular momentum.

Rotational kinetic energy

We have the translational kinetic energy $\frac{1}{2}mv^2$. The energy of a body by virtue of its rotational motion is called rotational kinetic energy.

$$E = \frac{1}{2} I \omega^2$$

But we have $L = I\omega$

$$L^2 = I^2 \omega^2$$

$$\frac{L^2}{2} = \frac{I^2 \omega^2}{2}$$

$$\frac{L^2}{2I} = \frac{I^{\cancel{2}} \omega^2}{2 \cancel{I}}$$

$$\frac{L^2}{2I} = \frac{I\omega^2}{2}$$

$$\frac{L^2}{2I} = E_{rot}$$

It is analogous to translational eqn

$$K.E = \frac{P^2}{2M}$$

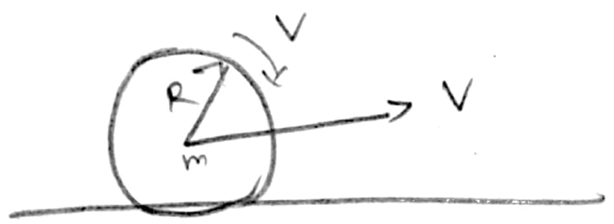
$$P^2 = 2ME$$

$$P = \sqrt{2ME}$$

||ely: $L^2 = 2EI$

$$L = \sqrt{2IE}$$

K.E of a disc rolling on a horizontal plane



Let m and R be the mass & radius of the disc. If centre of mass advances forward while the disc is rotating. So it has two types of motion. Translation and rotation. So it has two types of energy. Translation K.E & Rotational K.E

So, Total energy = Translational K.E + Rotational K.E

$$= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

for a disc $I = \frac{mR^2}{2}$

$$\omega = v/R.$$

$$\text{Total energy} = \frac{1}{2}mv^2 + \frac{1}{2}\frac{mR^2}{2}\frac{v^2}{R^2}$$

$$= \frac{1}{2}mv^2 + \frac{1}{4}mv^2$$

$$= \frac{3}{4}mv^2$$