ENGINEERING MATHEMATICS

1st sem

POLY STUDYGRAM

#SFI SSMPTC #HELLO SSM

Module - I

TRIGONOMETRY - I

1.1 ANGLES

INTRODUCTION

The word 'Trigonometry' is derived from two Greek words 'Trigon' and 'Metron'. The word 'Trigon' means a triangle and the word 'Metron' means a measure. Hence Trigonometry is the science of measuring triangles. It includes all manner of geometrical and algebraical investigations which are related to angles. The applications of Trigonometry are found in the field of navigation, surveying, astronomy and almost all parts of Engineering. Currently Trigonometry is used in many areas such as the science of Seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides of the ocean and in many other areas. Inaccessible distances that could not be measured directly can be predicted by using trigonometric ratios and identities.

DEFINITION OF AN ANGLE

Angle is a measure of rotation of a given ray about its initial point. The revolving ray is called the generating line of the angle.

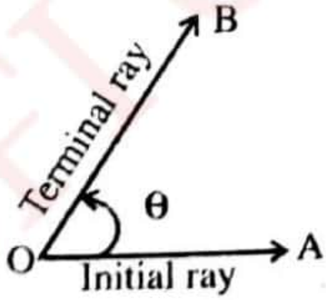


fig.1.1(a)

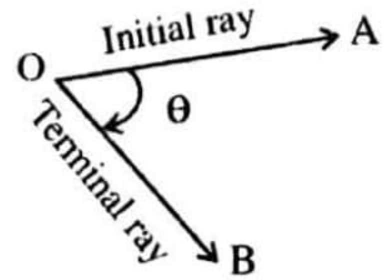


fig.1.1(b)

In fig 1.1(a) the initial ray, $\overrightarrow{OA}$ rotates in anticlockwise direction to reach the final position of $\overrightarrow{OB}$ and a positive angle AOB is traced out. In fig1.1(b) the initial ray, $\overrightarrow{OA}$ rotates in clockwise direction to reach the final position of $\overrightarrow{OB}$ and a negative angle AOB is traced out.

The sign of the angle is said to be positive or negative according to the direction of rotation of the initial ray at the centre. If the direction of rotation is anticlockwise, the angle traced out is positive and otherwise, the angle traced out is negative. In fig.1.1(a) the angle θ is positive and in fig.1.1(b) the angle θ is negative.

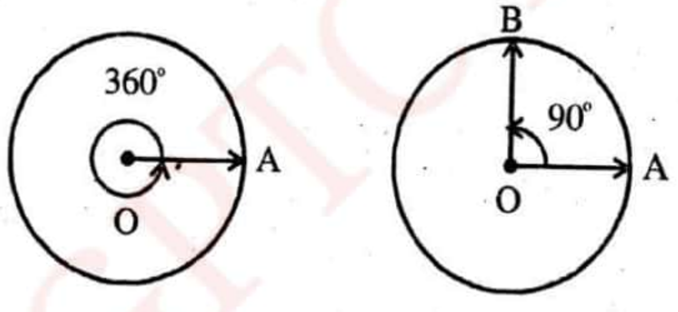
RIGHT ANGLE

If the revolving ray starting from its initial position to final position describes one quarter of a circle, the angle formed is a right angle.

Measurements of angles

The most common systems of measuring angles are sexagesimal system and circular system.

In sexagesimal system, the unit of measuring an angle is a degree. When a revolving ray makes one complete revolution, the angle traced out is divided in to 360 equal parts to get each part a degree.



$$1 \text{ right angle} = \frac{360}{4} = 90^\circ$$

$$1 \text{ degree} = 60 \text{ minutes } \left(1^\circ = 60'\right)$$

$$1 \text{ minute} = 60 \text{ seconds } \left(1' = 60''\right)$$

In circular system the unit of the measurement of an angle is a **radian**

One radian is the angle subtended at the centre of the circle by an arc of length equal to the radius of the circle

In figure 1.1(c) the length of $\widehat{AB}$ is equal to the radius of the circle and the angle subtended by this arc at the centre is called one radian. It is denoted by 1^c

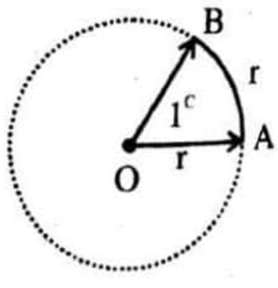
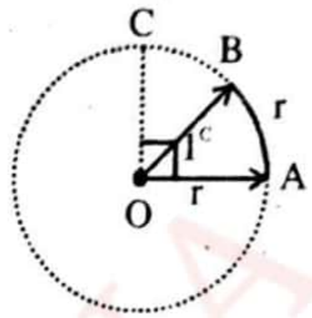


fig.1.1(c)

Consider a circle with centre 'O' and radius 'r'. The arc AB has length equal to the radius, so that it subtends an angle 1^C at the centre. $\overset{\frown}{AC}$ has length equal to one fourth of the circumference ($2\pi r$) and is $\frac{2\pi r}{4}$ or $\frac{\pi r}{2}$. Since the angles at the centre of the circle are proportional to the lengths of the arcs subtending them.



$$\text{ie; } \frac{r}{1^C} = \frac{\left(\frac{\pi r}{2}\right)}{\theta}$$

$$\frac{r}{1^C} = \frac{\pi r}{2\theta}$$

(θ is the angle subtended by the arc AC at the centre. It is evidently a right angle)

$\therefore\ \theta = \frac{\pi^C}{2}$ for a right angle. But in degrees $\theta = 90^o$

$$\boxed{\therefore 90^o = \frac{\pi^C}{2} = 1 \text{ right angle}}$$

THE RELATION BETWEEN DEGREES AND RADIANS

$$90^o = \frac{\pi^C}{2}$$

$$\pi^C = 180^o$$

$$1^C = \frac{180^o}{\pi} \text{ or } 1^o = \frac{\pi^C}{180}$$

$$\boxed{1^C = \frac{180^o}{\pi}, 1^o = \frac{\pi^C}{180}}$$

Qn. 1. Find the degree measure corresponding to the following radian measures

a) $\frac{\pi^C}{3}$ b) $\frac{\pi^C}{4}$

c) $\frac{\pi^C}{2}$ d) $2\pi^C$

e) $\dfrac{2\pi^C}{15}$ f) $\dfrac{1^C}{4}$

g) -3^C h) 8^C

Sol : We have $1^C = \dfrac{180^\circ}{\pi}$

a) $\dfrac{\pi}{3}^C = \dfrac{\pi}{3} \times \dfrac{180^\circ}{\pi} = 60^\circ$

b) $\dfrac{\pi}{4}^C = \dfrac{\pi}{4} \times \dfrac{180^\circ}{\pi} = 45^\circ$

c) $\dfrac{\pi}{2}^C = \dfrac{\pi}{2} \times \dfrac{180^\circ}{\pi} = 90^\circ$

d) $2\pi^C = 2\pi \times \dfrac{180^\circ}{\pi} = 360^\circ$

e) $\dfrac{2\pi^C}{15} = \dfrac{2\pi}{15} \times \dfrac{180^\circ}{\pi} = 24^\circ$

f) $\dfrac{1^C}{4} = \dfrac{1}{4} \times \dfrac{180^\circ}{\pi} = \dfrac{1}{4} \times \dfrac{180^\circ \times 7}{22} = (14.33)^\circ$ $\boxed{\pi = 22/7}$

g) $-3^C = -3 \times \dfrac{180^\circ}{\pi} = \dfrac{-3 \times 180^\circ \times 7}{22} = -(171.82)^\circ$

h) $8^C = 8 \times \dfrac{180^\circ}{\pi} = \dfrac{8 \times 180^\circ \times 7}{22} = (458.18)^\circ$

Qn. 2. Find the radian measures corresponding to the following degree measures

a) 30° b) 300°

c) 540° d) $40^\circ 20'$

e) $-37^\circ 30'$

Sol: We have, $1^\circ = \dfrac{\pi^C}{180}$

a) $30^\circ = 30 \times \dfrac{\pi^C}{180} = \dfrac{\pi^C}{6}$

b) $300^\circ = 300 \times \frac{\pi^C}{180} = \frac{5\pi^C}{3}$

c) $540^\circ = 540 \times \frac{\pi^C}{180} = 3\pi^C$

d) $40^\circ 20' = \left(40 + \frac{20}{60}\right)^\circ = 40\tfrac{1}{3}^\circ = \frac{121^\circ}{3}$

$\frac{121}{3} \times \frac{\pi}{180} = \frac{121\pi^C}{540}$

e) $-37^\circ 30' = -\left[37^\circ + \frac{30^\circ}{60}\right]$

$= -\left[37\tfrac{1}{2}^\circ\right] = \frac{-75^\circ}{2}$

$= \frac{-75}{2} \times \frac{\pi}{180}^C ; \qquad = -\left[\frac{5\pi}{24}\right]^C$

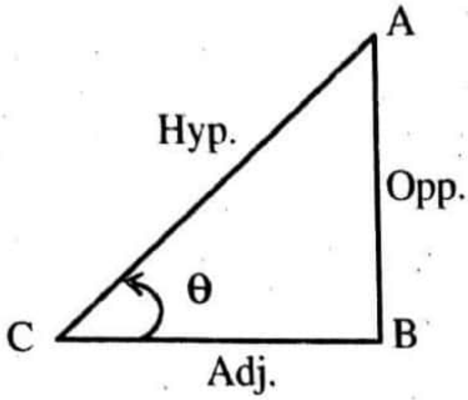
Remark : $1^\circ = 60' \qquad 1' = 60''$

1) $20^\circ 30' = \left(20 + \frac{30}{60}\right)^\circ = 20\tfrac{1}{2}^\circ$

2) $25^\circ 15' 30' = \left(25 + \frac{15}{60} + \frac{30}{60 \times 60}\right)^\circ$

1.2

TRIGONOMETRIC RATIOS FOR AN ACUTE ANGLE



Let ΔABC be a right angled triangle, which is right angled at 'B'. Let $\angle ACB$ be an acute angle in it and let it be 'θ'. With reference to θ, the side AB opposite to it is called opposite side and the side BC adjacent to it is called adjacent side. The side AC of the triangle being opposite to 90° is called hypotenuse. For the acute angle θ, six trigonometric functions are defined.

They are

1) $\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{AB}{AC}$ (written as $\sin\theta$)

2) $\text{cosine}\,\theta = \dfrac{\text{Adjacent side}}{\text{hypotenuse}} = \dfrac{BC}{AC}$ (written as $\cos\theta$)

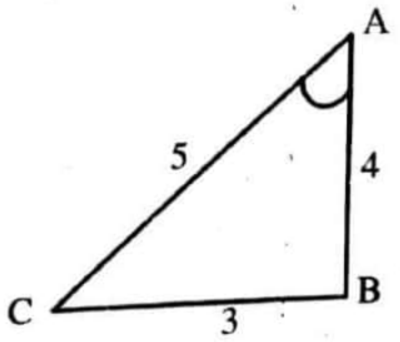
3) $\text{tangent}\,\theta = \dfrac{\text{Opposite side}}{\text{Adjacent side}} = \dfrac{AB}{BC}$ (written as $\tan\theta$)

4) $\text{cosecant}\,\theta = \dfrac{\text{hypotenuse}}{\text{opposite side}} = \dfrac{AC}{AB}$ (written as $\text{cosec}\,\theta$)

5) $\text{secant}\,\theta = \dfrac{\text{hypotenuse}}{\text{adjacent side}} = \dfrac{AC}{BC}$ (written as $\sec\theta$)

6) $\text{cotangent}\,\theta = \dfrac{\text{adjacent side}}{\text{opposite side}} = \dfrac{BC}{AB}$ (written as $\cot\theta$)

Qn. 1. Find the trigonometric ratios of $\angle A$ of a triangle ABC in which AB=4cm, BC=3cm and AC=5cm

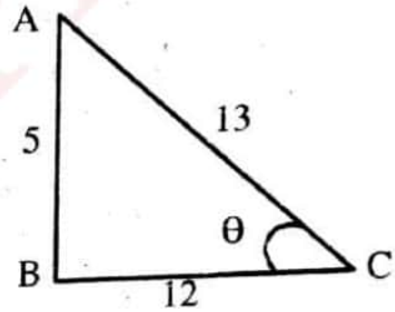


Sol: $\sin A = \frac{BC}{AC} = \frac{3}{5}$; $\cos A = \frac{AB}{AC} = \frac{4}{5}$; $\tan A = \frac{BC}{AB} = \frac{3}{4}$;

$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{5}{3}$; $\sec A = \frac{1}{\cos A} = \frac{5}{4}$; $\cot A = \frac{1}{\tan A} = \frac{4}{3}$

Qn. 2. If $\sin\theta = \frac{5}{13}$ 'θ' is acute,

find $\cos\theta$ and $\tan\theta$

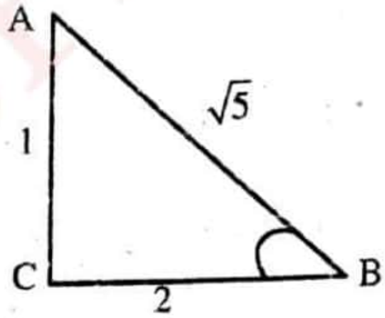


Sol: BC=12 by pythagorean theorem

$$\cos\theta = \frac{BC}{AC} = \frac{12}{13}$$

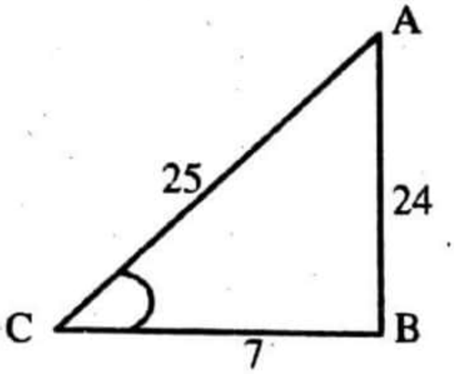
$$\tan\theta = \frac{AB}{BC} = \frac{5}{12}$$

Qn. 3. If $\tan B = \frac{1}{2}$ 'B' is acute in a right triangle ABC. Find SinB and CosB



Sol: $\sin B = \frac{AC}{AB} = \frac{1}{\sqrt{5}}$; $\cos B = \frac{BC}{AB} = \frac{2}{\sqrt{5}}$

Qn. 4. If $\cot C = \frac{7}{24}$ and C is acute in right triangle ABC, find the other trigonometric ratios



Sol: $\cot C = \frac{\text{Adj. side}}{\text{Opp. side}} = \frac{BC}{AB} = \frac{7}{24}$

$AC = 25$

$\sin C = \frac{24}{25}$; $\cos C = \frac{7}{25}$; $\tan C = \frac{24}{7}$; $\operatorname{cosec} C = \frac{25}{24}$; $\sec C = \frac{25}{7}$

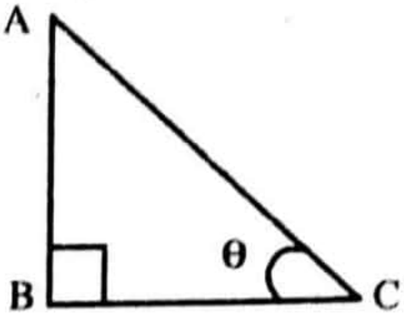
Remark: Trigonometric ratios are also called trigonometric functions.

TRIGONOMETRIC IDENTITIES

An equation involving trigonometric functions which is true for all those angles for which the functions are defined is known as trigonometric identity.

1) $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ or $\sin\theta = \frac{1}{\operatorname{cosec}\theta}$
2) $\sec\theta = \frac{1}{\cos\theta}$ or $\cos\theta = \frac{1}{\sec\theta}$
3) $\cot\theta = \frac{1}{\tan\theta}$ or $\tan\theta = \frac{1}{\cot\theta}$
4) $\tan\theta = \frac{\sin\theta}{\cos\theta}$
5) $\cot\theta = \frac{\cos\theta}{\sin\theta}$
6) $\sin^2\theta + \cos^2\theta = 1$
7) $1 + \tan^2\theta = \sec^2\theta$
8) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Pythagorean Relations



From the right angled triangle ABC,

$AB^2 + BC^2 = AC^2$

Dividing the whole equation by $(AC)^2$,

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2} \quad ....(1)$$

$$\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1$$

ie; $\sin^2\theta + \cos^2\theta = 1$

and we have $\sin^2\theta = 1 - \cos^2\theta$

ie, $\sin\theta = \sqrt{1-\cos^2\theta}$

and $\cos^2\theta = 1 - \sin^2\theta$

$\cos\theta = \sqrt{1-\sin^2\theta}$

Consider the equation $AB^2+BC^2=AC^2$

Dividing by $(BC)^2$

$$\frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} = \frac{AC^2}{BC^2} \quad \text{and} \quad \left(\frac{AB}{BC}\right)^2 + 1 = \left(\frac{AC}{BC}\right)^2$$

and is same as $1 + \tan^2\theta = \sec^2\theta$

ie, $\sec^2\theta - \tan^2\theta = 1$

or $\tan^2\theta = \sec^2\theta - 1$

Consider the first equation given by $AB^2+BC^2=AC^2$.

Divide the whole equation by AB^2

$$\frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$$

$$1 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

ie; $1+\cot^2\theta=\operatorname{cosec}^2\theta$

and $\operatorname{cosec}^2\theta-\cot^2\theta=1$ or $\cot^2\theta=\operatorname{cosec}^2\theta-1$

(1) $\sin^2\theta+\cos^2\theta=1$	(2) $1+\tan^2\theta=\sec^2\theta$
$\sin^2\theta=1-\cos^2\theta$	$\sec^2\theta-\tan^2\theta=1$
$\cos^2\theta=1-\sin^2\theta$	$\sec^2\theta-1=\tan^2\theta$
(3) $1+\cot^2\theta=\operatorname{cosec}^2\theta$	
$\operatorname{cosec}^2\theta-\cot^2\theta=1$	
$\cot^2\theta=\operatorname{cosec}^2\theta-1$	

Problems using Identities

Qn.5. Prove that $(1+\cos A)(1-\cos A)=\sin^2 A$

Sol: LHS $(1+\cos A)(1-\cos A)$

$=1-\cos^2 A \quad [\because (x+y)(x-y)=x^2-y^2]$

From a previous result we know that $1-\cos^2 A=\sin^2 A$

Qn.6. Prove that $\sin A.\cot A=\cos A$

Sol: LHS$=\sin A.\cot A=\sin A\dfrac{\cos A}{\sin A}=\cos A$

Qn.7. Prove that $\sin A.\sec A.\cot A=1$

Sol : LHS$=\sin A.\sec A.\cot A=\sin A\dfrac{1}{\cos A}.\dfrac{\cos A}{\sin A}=1$

Qn.8. Prove that $\tan x+\cot x=\sec x.\operatorname{cosec} x$

Sol : LHS $=\tan x+\cot x$

$\boxed{\tan A=\dfrac{\sin A}{\cos A};\ \cot A=\dfrac{\cos A}{\sin A}}$

$$=\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}=\frac{\sin^2 x+\cos^2 x}{\cos x.\sin x}$$

$$=\frac{1}{\cos x.\sin x}=\frac{1}{\cos x}.\frac{1}{\sin x}$$

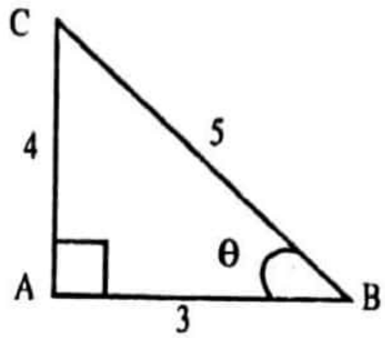
$=\sec x.\operatorname{cosec} x$

Qn.9. Prove that $\cos A.\tan A.\operatorname{cosec}A=1$

Sol: LHS, $\cos A.\tan A.\operatorname{cosec}A=\cos A\dfrac{\sin A}{\cos A}.\dfrac{1}{\sin A}=1$

Qn.10. In a right angled ΔABC

if, $\sin\theta = \frac{4}{5}$, verify that



(1) $\sin^2\theta + \cos^2\theta = 1$

(2) $1 + \tan^2\theta = \sec^2\theta$

(3) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$$\sin\theta = \frac{4}{5}, \operatorname{cosec}\theta = \frac{5}{4}$$

$$\cos\theta = \frac{3}{5}, \sec\theta = \frac{5}{3}$$

$$\tan\theta = \frac{4}{3}, \cot\theta = \frac{3}{4}$$

Sol: (1) $\sin^2\theta + \cos^2\theta = \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$

(2) LHS = $1 + \tan^2\theta = 1 + \left(\frac{4}{3}\right)^2 = 1 + \frac{16}{9} = \frac{25}{9}$

We have, $\sec\theta = \frac{5}{3}$

RHS = $\sec^2\theta = \frac{25}{9}$, LHS=RHS

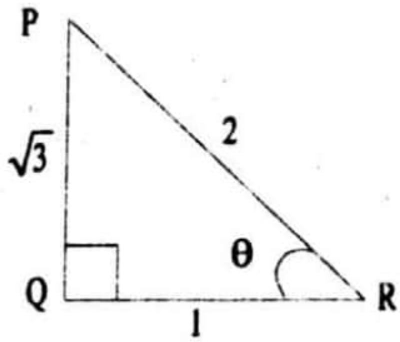
(3) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

LHS = $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

LHS = $1 + \cot^2\theta = 1 + \left(\frac{3}{4}\right)^2 = 1 + \frac{9}{16} = \frac{25}{16}$

$\operatorname{cosec}\theta = \frac{5}{4}$, $\operatorname{cosec}^2\theta = \frac{25}{16}$ =R.H.S.; $\therefore$ LHS=RHS

Qn.11. If $\cos\theta = \frac{1}{2}$ in a right triangle verify that



(1) $\tan\theta = \frac{\sin\theta}{\cos\theta}$

(2) $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Sol: From the figure $\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = \frac{1}{2}$, $\tan\theta = \sqrt{3}$ ....(1)

But $\frac{\sin\theta}{\cos\theta} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$ ....(2)

$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta}$ from (1) and (2)

From the figure, $\cot\theta = \frac{1}{\sqrt{3}}$ ....(1)

But $\frac{\cos\theta}{\sin\theta} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$ ....(2)

ie, $\cot\theta = \frac{\cos\theta}{\sin\theta}$ from (1) and (2)

Note:- We can also verify that $\cot\theta = \frac{1}{\tan\theta}$ and $\tan\theta = \frac{1}{\cot\theta}$

Qn.12. If $\sin\theta = 1/2$, θ is acute, find $\cos\theta$

Sol: $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$

Qn.13. If $\tan\theta = 2$ find $\sin\theta$

Sol: We know that $1+\tan^2\theta = \sec^2\theta$

$1+4 = \sec^2\theta$

ie, $\sec\theta = \sqrt{5}$, $\cos\theta = \frac{1}{\sqrt{5}}$

$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-1/5} = \sqrt{4/5} = 2/\sqrt{5}$

Qn.14. If $\cot\theta = 1/3$ find $\sin\theta$

Sol: We know that $1+\cot^2\theta = \operatorname{cosec}^2\theta$

$1+1/9 = \operatorname{cosec}^2\theta$

ie; $\operatorname{cosec}\theta = \sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3}$

$\therefore \sin\theta = \frac{3}{\sqrt{10}}$

Qn.15. If, $\sin\theta = \frac{3}{5}$ find $\cos\theta$ and $\tan\theta$

Sol: We know that $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3/5}{4/5} = \frac{3}{4}$

Qn.16. If $\tan\theta = 1$, find $\sin\theta$, and $\cos\theta$

Sol: $1+\tan^2\theta = \sec^2\theta,\ 1+1^2 = \sec^2\theta,\ \sec\theta = \sqrt{2}$

$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{2}}$

We know that $\tan\theta = 1,\ \cos\theta = \frac{1}{\sqrt{2}}$

$\frac{\sin\theta}{\cos\theta} = \frac{\tan\theta}{1} \Rightarrow \sin\theta = \cos\theta \times \tan\theta$

ie, $\sin\theta = 1\times\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

Qn.17. Prove that $(\sin A + \cos A)^2 = 1+2\sin A.\cos A$

Sol: LHS = $(\sin A+\cos A)^2 = \sin^2 A + 2\sin A.\cos A + \cos^2 A$

$= (\sin^2 A + \cos^2 A) + 2\sin A.\cos A$

$= 1 + 2\sin A.\cos A$

Qn.18. Prove that $\frac{\operatorname{cosec}\theta}{(\operatorname{cosec}\theta-1)} + \frac{\operatorname{cosec}\theta}{(\operatorname{cosec}\theta+1)} = 2\sec^2\theta$

Sol: LHS = $\frac{\operatorname{cosec}\theta}{(\operatorname{cosec}\theta-1)} + \frac{\operatorname{cosec}\theta}{(\operatorname{cosec}\theta+1)}$

$= \frac{\operatorname{cosec}\theta(\operatorname{cosec}\theta+1) + \operatorname{cosec}\theta(\operatorname{cosec}\theta-1)}{(\operatorname{cosec}\theta-1)(\operatorname{cosec}\theta+1)}$

$$= \frac{\operatorname{cosec}^2\theta + \operatorname{cosec}\theta + \operatorname{cosec}^2\theta - \operatorname{cosec}\theta}{\operatorname{cosec}^2\theta - 1} = \frac{2\operatorname{cosec}^2\theta}{\operatorname{cosec}^2\theta - 1}$$

$$= \frac{2\operatorname{cosec}^2\theta}{\cot^2\theta}$$

$$= 2\frac{1/\sin^2\theta}{\cos^2\theta/\sin^2\theta} = 2.\frac{1}{\cos^2\theta}$$

$$= 2\sec^2\theta$$

Equations used in this problem

$$\cot^2\theta = \operatorname{cosec}^2\theta - 1$$

$$\frac{1}{\sin\theta} = \operatorname{cosec}\theta; \qquad \frac{\cos\theta}{\sin\theta} = \cot\theta$$

Qn.19. Prove that $\frac{\sin\theta}{1+\cos\theta} + \frac{(1+\cos\theta)}{\sin\theta} = 2\operatorname{cosec}\theta$

Sol: LHS $= \frac{\sin\theta}{(1+\cos\theta)} + \frac{(1+\cos\theta)}{\sin\theta}$

$$= \frac{\sin\theta.\sin\theta + (1+\cos\theta)(1+\cos\theta)}{\sin\theta(1+\cos\theta)}$$

$$= \frac{\sin^2\theta + 1 + \cos^2\theta + 2\cos\theta}{\sin\theta(1+\cos\theta)} = \frac{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{2 + 2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = \frac{2}{\sin\theta} = 2\operatorname{cosec}\theta$$

Qn.20. Prove that $\sec^2 x + \operatorname{cosec}^2 x = \sec^2 x.\operatorname{cosec}^2 x$

Sol: LHS $= \sec^2 x + \operatorname{cosec}^2 x$

$$= \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x.\sin^2 x}$$

$$= \frac{1}{\cos^2 x.\sin^2 x} = \frac{1}{\cos^2 x} \times \frac{1}{\sin^2 x} = \sec^2 x \times \operatorname{cosec}^2 x$$

Qn.21. Prove that $(\cot A - 1)^2 + (\cot A + 1)^2 = 2\operatorname{cosec}^2 A$

Sol: $\text{LHS} = (\cot A - 1)^2 + (\cot A + 1)^2$

$$= \cot^2 A - 2\cot A + 1 + \cot^2 A + 2\cot A + 1$$

$$= 2\cot^2 A + 2 = 2\left(\cot^2 A + 1\right) = 2\left(\operatorname{cosec}^2 A\right)$$

Qn.22. Prove that, $\dfrac{1+\sin A}{\cos A} = \dfrac{\cos A}{1-\sin A}$

Sol: Take LHS $\dfrac{1+\sin A}{\cos A}$, Multiply Nr and Dr by (1–sinA)

$$\frac{(1-\sin A)(1+\sin A)}{(1-\sin A)\cos A} = \frac{1-\sin^2 A}{(1-\sin A)\cos A}$$

$$= \frac{\cos^2 A}{(1-\sin A)\cos A} = \frac{\cos A}{1-\sin A} = \text{R.H.S.}$$

Qn.23. Prove that $\sqrt{\dfrac{1+\sin A}{1-\sin A}} = \sec A + \tan A$

Sol: $\sqrt{\dfrac{1+\sin A}{1-\sin A}} = \sqrt{\dfrac{1+\sin A}{1-\sin A}} \cdot \sqrt{\dfrac{1+\sin A}{1+\sin A}}$

Multiplying Nr and Dr by $\sqrt{1+\sin A}$

$$= \frac{1+\sin A}{\sqrt{1-\sin^2 A}} = \frac{1+\sin A}{\cos A}$$

$$= \frac{1}{\cos A} + \frac{\sin A}{\cos A} = \sec A + \tan A$$

Qn.24. Prove that $\dfrac{\cos A}{1-\tan A} + \dfrac{\sin A}{1-\cot A} = \sin A + \cos A$

Sol: $\dfrac{\cos A}{\left(1 - \dfrac{\sin A}{\cos A}\right)} + \dfrac{\sin A}{\left(1 - \dfrac{\cos A}{\sin A}\right)}$

$$= \frac{\cos A}{\left(\dfrac{\cos A - \sin A}{\cos A}\right)} + \frac{\sin A}{\left(\dfrac{\sin A - \cos A}{\sin A}\right)} = \frac{\cos^2 A}{(\cos A - \sin A)} + \frac{\sin^2 A}{(\sin A - \cos A)}$$

$$= \frac{\cos^2 A}{\cos A - \sin A} - \frac{\sin^2 A}{(\cos A - \sin A)}$$

$$= \frac{\cos^2 A - \sin^2 A}{(\cos A - \sin A)}$$

$\boxed{\sin A - \cos A = -(\cos A - \sin A)}$

$$= \frac{(\cos A + \sin A)(\cos A - \sin A)}{(\cos A - \sin A)}$$

$$= \cos A + \sin A$$

Trigonometric Ratios of standard angles 0^o, 30^o, 45^o, 60^o and 90^o

See the table

T-function \ θ	0°	30°	45°	60°	90°
sin	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

Qn.25. Prove that cos60.cos30+sin60.sin30=cos30

Sol: LHS = cos60.cos30+sin60.sin30

$$= \frac{1}{2}.\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}.\frac{1}{2} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}$$

$$2\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2};$$ LHS = RHS

Qn.26. Show that cos45.cos60–sin45.sin60 = $\frac{1-\sqrt{3}}{2\sqrt{2}}$

Sol: LHS = cos45.cos60–sin45.sin60

$$= \frac{1}{\sqrt{2}}.\frac{1}{2} - \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2}$$

$= \frac{1-\sqrt{3}}{2\sqrt{2}}$, Hence the result

Qn.27. Verify that $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ if $\theta = 30^o$

Sol: $\tan 2\theta = \tan(2\times30) = \tan 60 = \sqrt{3}$ ....(1)

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\tan 30}{1-\tan^2 30} = \frac{2\times 1/\sqrt{3}}{1-\left(1/\sqrt{3}\right)^2}$$

$$= \frac{2/\sqrt{3}}{1-1/3} = 2/\sqrt{3}\times 3/2 = \sqrt{3} \text{ ....(2)}$$

From (1) and (2) it is clear that $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Qn.28. Find the value of sin60.cos30+cos60.sin30

Sol: sin60.cos30+cos60.sin30

$$= \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} + \frac{1}{2}.\frac{1}{2} = \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$$

Qn. 29. Find the value of $\tan^2 60+\tan^2 45$

Sol: $\tan^2 60 + \tan^2 45 = \left(\sqrt{3}\right)^2 + 1^2 = 3+1 = 4$

Qn.30. Prove that $\operatorname{cosec}^2 45.\cos^2 60.\sec^2 30 = \dfrac{2}{3}$

Sol: $\text{LHS} = \dfrac{1}{\sin^2 45}.\cos^2 60.\dfrac{1}{\cos^2 30}$

$$= \frac{1}{\left(1/\sqrt{2}\right)^2}\left(\frac{1}{2}\right)^2.\frac{1}{\left(\sqrt{3}/2\right)^2} = 2.\frac{1}{4}.\frac{4}{3} = \frac{2}{3}$$

Qn.31. Evaluate 4 $\tan^2 60 - 3\tan^2 30 + \tan^2 45$

Sol: $4\tan^2 60 - 3\tan^2 30+\tan^2 45$

$$= 4\times\left(\sqrt{3}\right)^2 - 3\times\left(\frac{1}{\sqrt{3}}\right)^2 + 1^2$$

$$= 4\times3 - 3\times 1/3 + 1 = 12-1+1 = 12$$

Qn.32. Evaluate $4\sin^3\left(\pi/3\right) - 3\cos\left(\pi/6\right)$

Sol: $4\sin^3\left(\pi/3\right) - 3\cos\left(\pi/6\right)$

$$= 4\sin^3\frac{180}{3} - 3\cos\frac{180}{6}$$

$$= 4\sin^3 60 - 3\cos 30 = 4.\left(\sqrt{3}/_2\right)^3 - 3.\sqrt{3}/_2$$

$$= 4.\frac{3\sqrt{3}}{8} - \frac{3\sqrt{3}}{2} = 0$$

Qn.33. Prove that $\dfrac{\tan 45 - \tan 30}{1 + \tan 45.\tan 30} = 2 - \sqrt{3}$

Sol: $\dfrac{\tan 45 - \tan 30}{1 + \tan 45.\tan 30} = \dfrac{1 - 1/\sqrt{3}}{1 + 1 \times 1/\sqrt{3}}$

$\dfrac{\sqrt{3} - 1/\sqrt{3}}{\sqrt{3} + 1/\sqrt{3}} = \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$. Multiply Nr and Dr by $\left(\sqrt{3} - 1\right)$

ie; $\dfrac{\left(\sqrt{3} - 1\right)\left(\sqrt{3} - 1\right)}{\left(\sqrt{3} + 1\right)\left(\sqrt{3} - 1\right)} = \dfrac{\left(\sqrt{3} - 1\right)^2}{3 - 1} = \dfrac{3 - 2\sqrt{3} + 1}{2}$

$$= \frac{4 - 2\sqrt{3}}{2} = \frac{2\left(2 - \sqrt{3}\right)}{2} = 2 - \sqrt{3}$$

Qn.34. Evaluate $\sin^3 \pi/_3 . \cos^2 \pi/_4 . \tan \pi/_6$

Sol: $\sin^3 \pi/_3 . \cos^2 \pi/_4 . \tan \pi/_6$

$$= \sin^3 60 . \cos^2 45 . \tan 30$$

$$= \left(\frac{\sqrt{3}}{2}\right)^3 \left(\frac{1}{\sqrt{2}}\right)^2 \frac{1}{\sqrt{3}}$$

$$= \frac{3\sqrt{3}}{8} . \frac{1}{2} . \frac{1}{\sqrt{3}} = \frac{3}{16}$$

Qn.35. Prove that $\dfrac{\tan 60 - \tan 45}{1 + \tan 60.\tan 45} = 2 - \sqrt{3}$

Sol: $\dfrac{\tan 60 - \tan 45}{1 + \tan 60.\tan 45} = \dfrac{\sqrt{3} - 1}{1 + \sqrt{3}}$

$$\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$ Multiplying Nr and Dr by $\sqrt{3}-1$

$$=\frac{(\sqrt{3}-1)^2}{3-1}=\frac{3-2\sqrt{3}+1}{2}$$

$$=\frac{4-2\sqrt{3}}{2}=\frac{2(2-\sqrt{3})}{2}=2-\sqrt{3}$$

EXERCISE 1(a)

1. If $\sin\theta=\frac{5}{13}$, 'θ' is acute, find other t-functions
2. If $\tan\theta=3$, 'θ' is acute find $\sin\theta$ and $\cos\theta$
3. If $\cot\theta=\frac{24}{7}$, 'θ' is acute find $\sec\theta$ and $\operatorname{cosec}\theta$
4. Show that $\frac{\sin\theta}{1-\cos\theta}+\frac{1-\cos\theta}{\sin\theta}=2\operatorname{cosec}\theta$
5. Show that $\frac{1+\sin\theta}{\cos\theta}+\frac{\cos\theta}{1+\sin\theta}=2\sec\theta$
6. Prove that $\frac{\sec\theta}{\sec\theta+1}+\frac{\sec\theta}{\sec\theta-1}=2\operatorname{cosec}^2\theta$
7. Evaluate cos60.cos30 + sin60.sin30
8. Evaluate $4\sin^3\frac{\pi}{3}-3\cos\frac{\pi}{6}$
9. If $\theta=30^\circ$ verify that $\sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}$
10. If $\theta=45^\circ$, Verify that $\cos 2\theta=2\cos^2\theta-1$
11. Evaluate $\sin\frac{\pi}{2}\times\cos\frac{\pi}{4}\times\tan\frac{\pi}{3}$
12. Prove that $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}=\sec\theta-\tan\theta$
13. Prove that $(\operatorname{cosec}-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)=1$
14. Prove that $\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\frac{1+\sin\theta}{\cos\theta}$ (Hint : take $1=\sec^2\theta-\tan^2\theta$ in 'Nr' only)

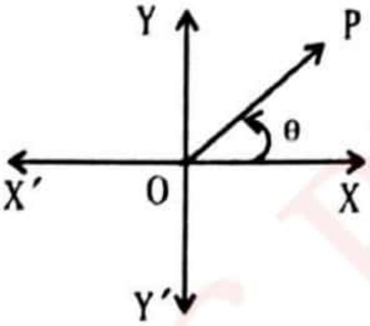
1.3

RELATED ANGLES

ANGLES IN DIFFERENT QUADRANTS

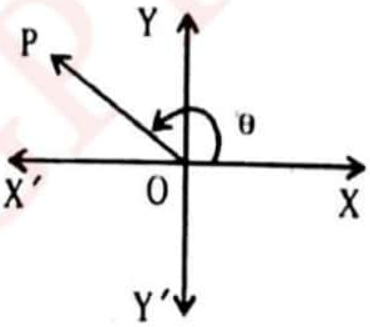
Let XOX′ and YOY′ be the co-ordinate axes and let 'O' be the origin. Let OP be the radius vector starting from the initial line OX and revolving around 'O' in anticlockwise direction making an angle 'θ' with the X axis.

fig.1.3(a)



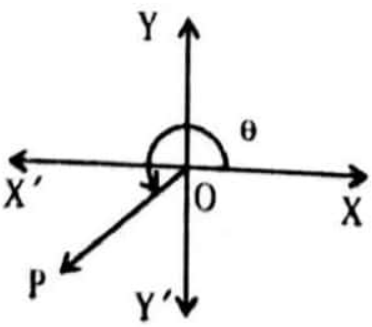
In fig.1.3(a) the radius vector OP lies in first quadrant and the angle 'θ' lies between 0° and 90°.

fig.1.3(b)



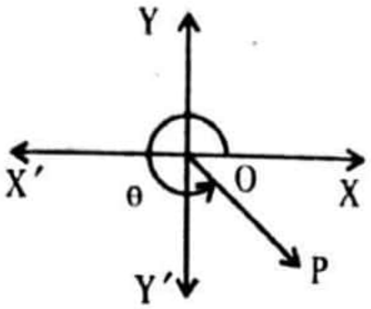
In fig.1.3(b) the radius vector OP lies in second quadrant and the angle 'θ' lies between 90° and 180°

fig.1.3(c)



In fig 1.3(c) the radius vector OP lies in third quadrant and the angle 'θ' lies between 180° and 270°

fig.1.3(d)

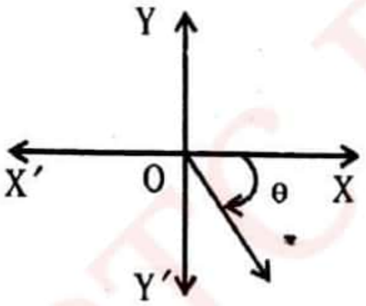


In fig.1.3(d) the radius vector lies in fourth quadrant and the angle 'θ' lies between 270° and 360°

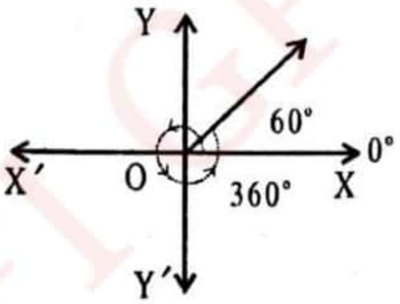
In all the above cases the initial ray rotates in anticlockwise direction to get positive angles in quadrants. If it rotates in clockwise direction the negative angles are formed in quadrants, see fig.1.3(e)

Examples of positive and negative angles

fig.1.3(e)

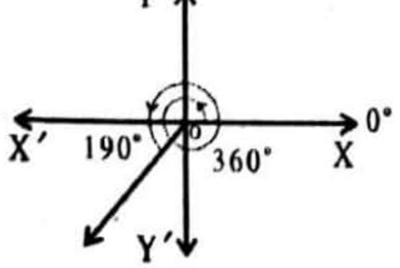


eg: 1) Draw the angle 420° and find the quadrant in which if lies.



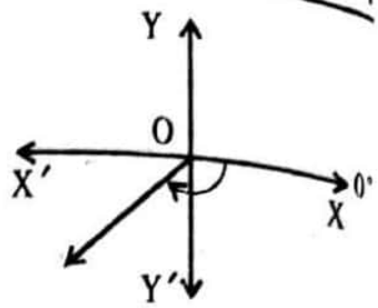
420° lies in first quadrant

eg: 2) Draw the angle 550° and state in which quadrant it lies



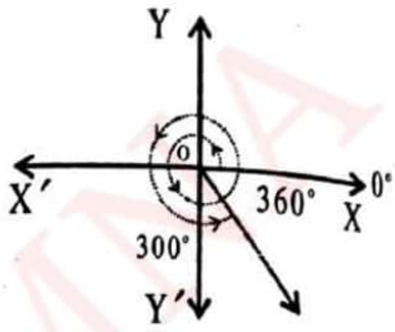
550° lies in third quadrant

eg: 3) Draw the angle -105° and find the quadrant in which it lies



For locating -105° in quadrant rotate $\overline{OX}$ in clockwise direction and the position is the third quadrant

eg: 4) Draw 660° and find the quadrant where it lies



a) Co-terminal angles

The angles having the same terminal ray are known as co-terminal angles

eg: 30°, 390°, 750° etc.

b) Quadrantal angles

If the terminal ray of an angle coinsides with one of the co-ordinate axes, the angle is said to be a quadrantal angle.

0°, 90°, 180°, 270°, 360°, 450° etc. are quadrant angles

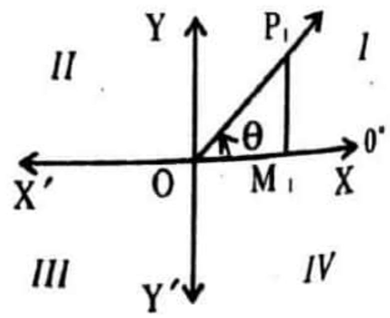
Table of results

T-function \ θ	0°	90°	180°	270°	360°
sin	0	1	0	-1	0
cos	1	0	-1	0	1
tan	0	∞	0	∞	0

Trigonometric functions of angles in quadrants

Case (1)

Suppose that 'θ' lies in first quadrant



$\sin\theta = \dfrac{P_1M_1}{OP_1}$ (+ve) [P_1M_1, OM_1 and OP_1 are positive]

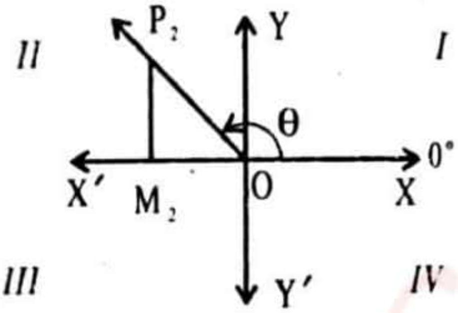
$\cos\theta = \dfrac{OM_1}{OP_1}$ (+ve)

$\tan\theta = \dfrac{P_1M_1}{OM_1}$ (+ve)

When 'θ' lies on first quadrant, all trigonometric functions are positive

Case (2)

Suppose that 'θ' lies in second quadrant .



$\sin\theta = \dfrac{P_2M_2}{OP_2}$ (+ve)

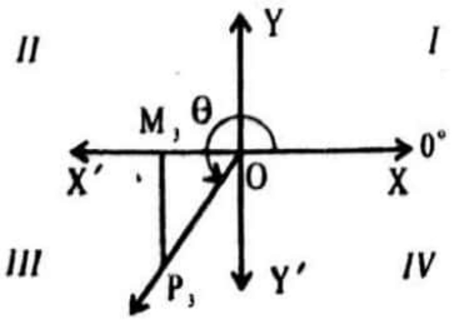
$\cos\theta = \dfrac{OM_2}{OP_2}$ (−ve) since OM_2 is negative

$\tan\theta = \dfrac{P_2M_2}{OM_2}$ (−ve)

When 'θ' lies in second quadrant $\sin\theta$ and $\operatorname{cosec}\theta$ are positive and all other t-fucntions are negative

Case (3)

Suppose that 'θ' lies in third quadrant.



$$\sin\theta = \frac{P_3M_3}{OP_3} \qquad (-ve) \qquad [P_3M_3, OM_3 \text{ are negative}]$$

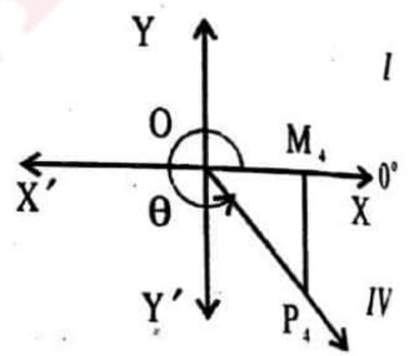
$$\cos\theta = \frac{OM_3}{OP_3} \qquad (-ve)$$

$$\tan\theta = \frac{P_3M_3}{OM_3} \qquad (+ve)$$

When 'θ' lies in third quadrant $\tan\theta$ and $\cot\theta$ are positive and all other trigonometric functions are negative.

Case (4)

Suppose that 'θ' lies in fourth quadrant



$$\sin\theta = \frac{P_4M_4}{OP_4} \qquad (-ve)$$

[P_4M_4 is -ve but OM_4 is +ve]

$$\cos\theta = \frac{OM_4}{OP_4} \qquad (+ve)$$

$$\tan\theta = \frac{P_4M_4}{OM_4} \qquad (-ve)$$

When 'θ' lies in fourth quadrant $\cos\theta$ and $\sec\theta$ are positive and all other trigonometric functions are negative

Remark: 1) The radius vector OP_1, OP_2, OP_3 and OP_4 are always positive.

2) In second, third and fourth cases we are extending the case of first quadrant. See the notes above

Qn. 1) If $\sin\theta = \frac{3}{5}$ 'θ' lies in second quadrants, find all other t-functions.

Sol: $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{9}{25}} = \pm\frac{4}{5}$

Since 'θ' lies in second quadrant, $\cos\theta$ is negative

$$\therefore \cos\theta = \frac{-4}{5}$$

We have, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3/5}{-4/5} = -3/4$

Now $\operatorname{cosec}\theta = \frac{5}{3}$, $\sec\theta = \frac{-5}{4}$, $\cot\theta = \frac{-4}{3}$

Qn. 2) If $\tan\theta = \frac{5}{12}$ 'θ' lies in third quadrant, find all other t-functions.

Sol: We have, $1+\tan^2\theta = \sec^2\theta$

ie, $\sec^2\theta = 1+\frac{5^2}{12^2} = \frac{169}{144}$

$\therefore \sec\theta = \pm\frac{13}{12}$

Here 'θ' lies in third quadrants and $\sec\theta$ is negative

$\therefore \sec\theta = \frac{-13}{12}$; $\therefore \cos\theta = \frac{-12}{13}$

Since $\tan\theta = \frac{5}{12}$, we can see that $\frac{\sin\theta}{\cos\theta} = \frac{5}{12}$

$\sin\theta = \frac{5\cos\theta}{12} = \frac{5\times -12/13}{12} = \frac{-5}{13}$

We can see that

$\operatorname{cosec}\theta = \frac{-13}{5}$ and $\cot\theta = \frac{12}{5}$

Qn. 3) If $\cos\theta = \frac{+1}{2}$ 'θ' lies in fourth quadrant, find all other t-functions

Sol: We have, $\sin\theta = \sqrt{1-\cos^2\theta}$

$= \sqrt{1-\frac{1}{4}}$

$\sqrt{3}$

Since 'θ' lies in fourth quadrant, $\sin\theta$ is negative ie; $\sin\theta = \frac{-\sqrt{3}}{2}$

$$\therefore \operatorname{cosec}\theta = \frac{-2}{\sqrt{3}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$$

$$\cot\theta = \frac{-1}{\sqrt{3}} \quad \text{Also } \sec\theta = 2$$

Conclusion

1) When 'θ' lies in first quadrant all the t-functions are positive

2) When 'θ' lies in second quadrant $\sin\theta$ and $\operatorname{cosec}\theta$ are positive and all other t-functions are negative

3) if 'θ' lies in third quadrant $\tan\theta$ and $\cot\theta$ are positive and all other t-functions are negative

4) If 'θ' lies in fourth quadrant, $\cos\theta$ and $\sec\theta$ are positive and all other t-functions are negative.

ASTC Rule

In first quadrant - All +ve
In second quadrant - sin +ve
In third quadrant - tan +ve
In fourth quadrant - cos +ve

$\therefore$ This rule is named as ASTC rule for easy remembrance.

Trigonometric Ratios of Allied Angles (Related Angles)

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90°. The angles $-\theta, 90\pm\theta, 180\pm\theta, 270\pm\theta, 360\pm\theta$ etc. are angles allied to the angle θ if θ is measured in degrees. The angles $-\theta, \pi/2\pm\theta, \pi\pm\theta, 3\pi/2\pm\theta, 2\pi\pm\theta$ etc. are allied angles to θ if θ is measured in radians.

A related angle of θ can be expressed in he form $n.90\pm\theta$ if θ is in degrees or $n.\pi/2\pm\theta$ if θ is in radians

Table of results

1) $\sin(-\theta) = -\sin\theta$
   $\cos(-\theta) = \cos\theta$
   $\tan(-\theta) = -\tan\theta$

2) $\sin(90-\theta) = \cos\theta$
   $\cos(90-\theta) = \sin\theta$
   $\tan(90-\theta) = \cot\theta$

3) $\sin(90+\theta) = \cos\theta$
   $\cos(90+\theta) = -\sin\theta$
   $\tan(90+\theta) = -\cot\theta$

4) $\sin(180-\theta) = \sin\theta$
   $\cos(180-\theta) = -\cos\theta$
   $\tan(180-\theta) = -\tan\theta$

5) $\sin(180+\theta) = -\sin\theta$
   $\cos(180+\theta) = -\cos\theta$
   $\tan(180+\theta) = \tan\theta$

6) $\sin(270-\theta) = -\cos\theta$
   $\cos(270-\theta) = -\sin\theta$
   $\tan(270-\theta) = \cot\theta$

7) $\sin(360-\theta) = -\sin\theta$
   $\cos(360-\theta) = \cos\theta$
   $\tan(360-\theta) = -\tan\theta$

8) $\sin(360+\theta) = \sin\theta$
   $\cos(360+\theta) = \cos\theta$
   $\tan(360+\theta) = \tan\theta$

When 'n' is an odd integer
$$\sin(n.90\pm\theta) = \pm\cos\theta$$
$$\cos(n.90\pm\theta) = \pm\sin\theta$$
$$\tan(n.90\pm\theta) = \pm\cot\theta$$

If 'n' is an even integer
$$\sin(n.90\pm\theta) = \pm\sin\theta$$
$$\cos(n.90\pm\theta) = \pm\cos\theta$$
$$\tan(n.90\pm\theta) = \pm\tan\theta$$

+ve or –sign depends on the quadrant in which the angle lies.

Evaluate the following.

1. $\sin 120 = \sin(180-60) = \sin 60 = \sqrt{3/2}$

2. $\cos 135 = \cos(180-45) = -\cos 45 = -1/\sqrt{2}$

3. $\sin 300 = \sin(360-60) = -\sin 60 = -\sqrt{3}/2$

4. $\cos 330 = \cos(360-30) = \cos 30 = \sqrt{3}/2$

5. $\tan 150 = \tan(180-30) = -\tan 30 = -1/\sqrt{3}$

6. $\operatorname{cosec} 120 = \dfrac{1}{\sin 120}$

$= \dfrac{1}{\sin(180-60)}$

$= \dfrac{1}{\sin 60} = \dfrac{1}{\sqrt{3}/2} = 2/\sqrt{3}$

7. $\cot 750 = \dfrac{1}{\tan 750} = \dfrac{1}{\tan(360+360+30)}$

$= \dfrac{1}{\tan 30} = \dfrac{1}{1/\sqrt{3}} = \sqrt{3}$

8. $\sin(510) = \sin(360+150) = \sin 150 = \sin 180-30)$

$= \sin 30 = 1/2$

9. $\cos(-840) = \cos 840 = \cos(360+360+120)$

$= \cos 120 = \cos(180-60) = -\cos 60$

$= -1/2$ $\boxed{\cos(-\theta) = \cos\theta}$

10. $\sin(-225) = -\sin 225 = -\sin(180+45)$

$= \sin 45 = 1/\sqrt{2}$ $\boxed{\sin(-\theta) = -\sin\theta}$

11. $\sin 390 = \sin(360+30) = \sin 30 = 1/2$

12. $\tan 135 = \tan(180-45) = -\tan 45 = -1$

13. $\cos 450 = \cos(360+90) = \cos 90 = 0$

14. $\operatorname{cosec} 480 = \dfrac{1}{\sin(360+120)} = \dfrac{1}{\sin 120}$

$= \dfrac{1}{\sin(180-60)} = \dfrac{1}{\sin 60} = 2/\sqrt{3}$

y - I

$$\sec 750 = \frac{1}{\cos(360+360+30)} = \frac{1}{\cos 30} = 2/\sqrt{3}$$

$\tan 225 = \tan(270-45) = \cot 45 = 1$ etc.

$$\sin(\theta-90) = \sin-(90-\theta) = -\sin(90-\theta) = -\cos\theta$$

$$\tan(\theta-360) = \tan-(360-\theta) = -\tan(360-\theta) = \tan\theta$$

Consider $\sin 1200 = \sin(360+360+360+120)$

Here it is enough to take $\sin(360+120)$ only

Qn. 4) Evaluate $\cos 570 \sin 510 - \sin 330 \cos 390$

Sol: $\cos 570 = \cos(360+210) = \cos 210$

$= \cos(180+30) = -\cos 30 = -\sqrt{3}/2$

$\sin 510 = \sin(360+150) = \sin 150 = \sin(180-30)$

$\sin 30 = 1/2$

$\sin 330 = \sin(360-30) = -\sin 30 = -1/2$

$\cos 390 = \cos(360+30) = \cos 30 = \sqrt{3}/2$

$$\text{LHS} \rightarrow -\sqrt{3}/2 \times 1/2 - {-1/2}\,\sqrt{3}/2 = -\sqrt{3/4} + \sqrt{3/4} = 0$$

Qn. 5) Prove that $\dfrac{\sin(180+A)\cos(90-A)\tan(270+A)}{\sec(540-A)\cos(360+A)} = -\sin A \cos A$

Sol: $\sin(180+A) = -\sin A$, $\cos(90-A) = \sin A$

$\tan(270+A) = -\cot A$, $\sec(540-A) = \dfrac{1}{\cos(540-A)}$

$\cos(540-A) = \cos(360+180-A)$

$= \cos(180-A) = -\cos A$

$$\therefore \sec(540-A) = \frac{1}{-\cos A} = -\sec A$$

cos (360+A) = cos A. Substituting the above results

$$\frac{\sin(180+A)\cos(90-A)\tan(270+A)}{\sec(540-A)\cos(360+A)} = \frac{-\sin A \sin A - \cot A}{-\sec A \cos A}$$

$$\frac{-\sin A \sin A}{-\left(\frac{1}{\cos A}\right)\cos A} \frac{-\cos A}{\sin A} = \frac{\sin A \cos A}{-1} = -\sin A \cos A$$

Qn. 6) Prove that $\frac{\cos(90+A)\sec(360+A)\tan(180-A)}{\sec(A-720)\sin(540+A)\cot(A-90)} = 1$

Sol: $\cos(90+A) = -\sin A$, $\sec(360+A) = \frac{1}{\cos(360+A)} = \frac{1}{\cos A} = \sec A$

$\tan(180-A) = -\tan A$

$\sec(A-720) = \sec^{-}(720-A) = \sec(720-A) = \sec A$

$\sin(540+A) = \sin(360+180+A) = \sin(180+A) = -\sin A$

$\cot(A-90) = \cot^{-}(90-A) = -\tan A$

Substituting, the above results, in the equation.

$$\frac{\cos(90+A)\sec(360+A)\tan(180-A)}{\sec(A-720)\sin(540+A)\cos(A-90)}$$

$$= \frac{-\sin A \sec A - \tan A}{\sec A - \sin A - \tan A} = 1$$

Remark :

1) $90-\theta$ and θ are complementary angles since the sum of those angles is 90°

2) $180-\theta$ and θ are supplimentary angles since the sum of those angles is 180° (see the above set of allied angles and collect complementary and supplimentary angles)

3) Function $(n.90 \pm \theta) = \pm$ function if 'n' is even

4) Function $(n.90 \pm \theta) = \pm$ co. function if 'n' is odd

+ or – sign depends on the quadrant in which the angle lies.

EXERCISE 1(b)

1) If $\tan\theta = \frac{24}{7}$, θ lies in third quadrant, find all other t-functions.

2) If $\sin\theta = \frac{-4}{5}$, θ lies in fourth quadrant find all other t-functions.

3) If $\cot\theta = -2$, θ lies in second quadrant find all other t-functions

4) If $\sec\theta = \frac{-13}{12}$, θ lies in third quadrant find all other t-functions

5) If $\cos\theta = \frac{1}{2}$, θ lies in first quadrant find $3\sin\theta - 4\tan\theta$

6) Prove that $\sin 420.\cos 390 + \sin^{-}(300)\cos 330 = \frac{3}{2}$

7) Prove that $\sin 120.\cos 330 + \cos 240.\sin 330 = 1$

8) Prove that $\sin 780.\sin 480 + \cos 120.\sin 30 = \frac{1}{2}$

9) Evaluate $3\tan\frac{2\pi}{3} - 2\sec\frac{3\pi}{4}$

10) Simplify $\dfrac{\cos(90+\theta)\sec(-\theta)\tan(180-\theta)}{\sec(360-\theta).\sin(180+\theta)\cot(90-\theta)}$

1.4

HEIGHTS AND DISTANCES

ANGLE OF ELEVATION AND ANGLE OF DEPRESSION

In fig.1.4(a), 'O' represents the observer and B represents the object. Suppose that the observer is watching the object above his eye level. In this case the angle between the ray of vision and the horizontal is called angle of elevation. In fig.1.4(a), 'θ' is the angle of elevation.

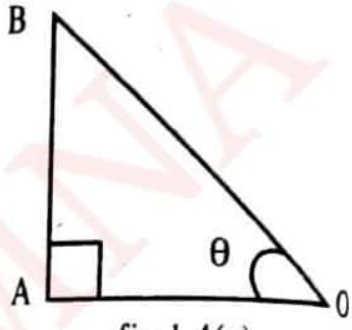


fig.1.4(a)

In fig.1.4(b), 'O' represents the observer and 'B' represents the object. Here the observer is watching the object below his eye level. In this case the angle between the ray of vision and the horizontal is called angle of depression. Here θ represents the angle of depression.

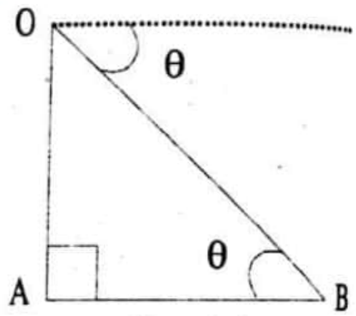


fig.1.4(b)

Ray of vision (line of sight)

The line joining the observer's eye and the object observed is known as line of sight

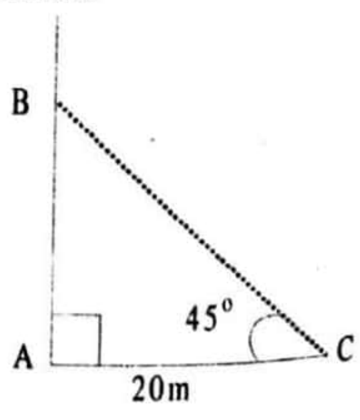
Angle of elevation

The angle between the horizontal line and the line of sight which is above the observer's eye is known as angle of elevation

Angle of depression

The angle between the horizontal line and the line of sight which is below the observer's eye is known as angle of depression.

Qn. 1. The rope supporting a flag post is fixed to the ground 20m away from the post making an angle of elevation 45° of the ground. Find the length of the rope.



Sol: In the figure AB represents the flag post and BC represents the rope.

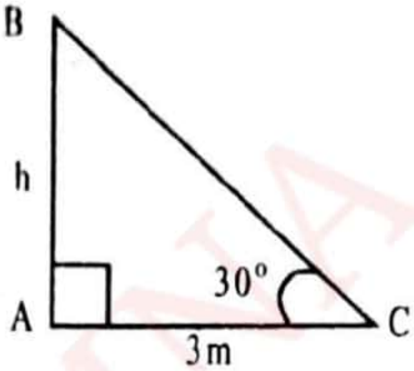
Consider $\cos 45 = \frac{AC}{BC} = \frac{20}{BC}$

ie: $\frac{1}{\sqrt{2}} = \frac{20}{BC}$

$BC = 20\sqrt{2}m$

Length of the rope = 28.280 m

Qn. 2. A man casts a shadow 3m long when the Sun's altitude is 30°. Find the height of the man.



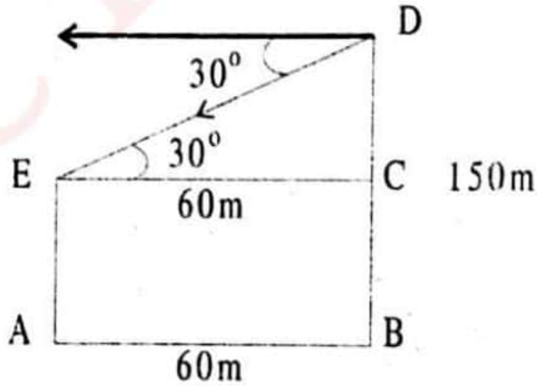
Sol: Let AB represents the man. Evidently AC represents his shadow

AC = 3m; $\angle C = 30^\circ$, h= ?

Consider $\tan 30 = \frac{h}{3}$ ie; h = 3 tan 30

$= 3\times {}^{1}\!/\!_{\sqrt{3}}\, m = 1.732m$

Qn. 3. The horizontal distance between two towers is 60m and the angle of depression of the first tower as seen from the second which is in 150m height is 30°. Find the height of the first tower?



Sol: Let AE represents the first tower and BD represents the second tower. AB = 60m = EC, $\angle DEC = 30^\circ$, BD = 150m.

First we have to find CD. For this take

$$\tan 30 = \frac{CD}{EC}$$

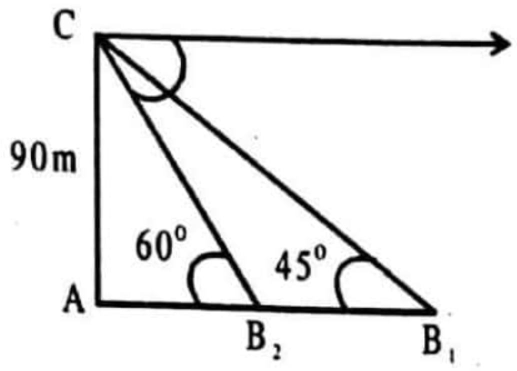
$$\frac{1}{\sqrt{3}} = \frac{CD}{60}$$

$$CD = \frac{60}{\sqrt{3}}\, m$$

The height of the first tower

$$AE = BD - CD = 150 - \frac{60}{\sqrt{3}} = 115.36m$$

Qn. 4. From the top of a light house 90m high, the angles of depression of two boats on the sea level are 45° and 60°. Find the distance between the boats.



Sol: Let AC represents the light house which is of height 90m

$\angle B_1 = 45^o$, $\angle B_2 = 60^o$. We have to find out B_1B_2 (distance)

From the figure, $\tan 45 = \dfrac{AC}{AB_1} = \dfrac{90}{AB_1}, 1 = \dfrac{90}{AB_1}$ ie; $AB_1 = 90m$

and consider $\tan 60 = \dfrac{AC}{AB_2} = \dfrac{90}{AB_2}$, $\sqrt{3} = \dfrac{90}{AB_2}$

$$AB_2 = \frac{90}{\sqrt{3}} m$$

$$B_1B_2 = AB_1 - AB_2 = 90 - 90/\sqrt{3} = 38.1m$$

EXERCISE 1(c)

1) If a force of 15 N makes a an angle of 45° with the horizontal. Find the horizontal and vertical components

2) A tower casts a shadow 90m long when the Sun's altitude is 45°. Find the height of the tower

3) An aeroplane starts from a place and flies 1000m along a straight line at 45° to the horizontal. Find the horizontal distance described.

4) A rope is stretched from the top of a vertical pole to a point 6 m from the foot of the pole. The rope makes an angle of 60° with the horizontal. Find the height of the pole.

1.5 COMPOUND ANGLES

We have already explained trigonometric functions in previous chapter. In this chapter we can see t–functions of compound angles.

Definition

An angle which is made up of sum or difference of two or more angles is called a compound angle.

For example A+B, A–B, A+B+C etc. are compound angles

(45+30), (60–45), (30+20+25) etc. are compound angles

Addition Formula

(i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(iii) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Subtraction formula

(i) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(ii) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(iii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

[Remember these equations. The proof is not included in the syllabus]

Simple problems on compound angles

Qn.1. If $\cos A = \frac{3}{5}$, $\tan B = \frac{5}{12}$, A and B are acute angles, find the values of $\sin(A+B)$ and $\cos(A-B)$.

Sol: If $\cos A = \frac{3}{5}$, $\sin A = \sqrt{1-\cos^2 A}$

ie; $\sin A = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}}$

$= \frac{4}{5}$ since 'A' is acute

We know that $\sec^2 B = 1 + \tan^2 B$

$$\text{ie; } \sec^2 B = 1 + \left(\frac{5}{12}\right)^2 = 1 + \frac{25}{144} = \frac{169}{144}$$

$$\text{ie; } \sec B = \frac{13}{12}; \qquad \Rightarrow \cos B = \frac{12}{13}$$

$$\text{and } \sin B = \sqrt{1-\cos^2 B} = \sqrt{1-\frac{144}{169}} = \frac{5}{13}$$

$$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$= \frac{4}{5}\times\frac{12}{13} + \frac{3}{5}\times\frac{5}{13} = \frac{63}{65}$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B = \frac{3}{5}\times\frac{12}{13} + \frac{4}{5}\times\frac{5}{13} = \frac{56}{65}$$

Qn.2. If $\tan A = \frac{3}{4}$, $\tan B = \frac{5}{12}$, A and B are actue angles, find tan (A–B)

Sol:

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B} = \frac{\frac{3}{4}-\frac{5}{12}}{1+\frac{3}{4}\,\frac{5}{12}}$$

$$= \frac{\frac{36-20}{48}}{\frac{48+15}{48}} = \frac{16}{63}$$

Qn.3. If $\tan A = \frac{3}{4}$, $\sin B = \frac{5}{13}$

(A lies in third quadrant and B lies in second quadrant).

Find (1) sin (A–B) (2) cos (A+B)

Sol:

$$\sec^2 A = 1 + \tan^2 A = 1 + \left(\frac{3}{4}\right)^2 = 1 + \frac{9}{16} = \frac{25}{16}$$

$$\sec A = \pm\frac{5}{4} \Rightarrow \cos A = \pm\frac{4}{5}$$

But $\cos A = \frac{-4}{5}$ ($\because$ A lies in 3rd quadrant) and $\sin A = \sqrt{1-\cos^2 A} = \sqrt{1-\frac{16}{25}} = \pm\frac{3}{5}$

But $\sin A = \frac{-3}{5}$ ($\because$ A lies in 3^{rd} quadrant)

$\sin B$ is given, $\cos B = \sqrt{1-\sin^2 B} = \sqrt{1-\left(\frac{5}{13}\right)^2}$

$= \sqrt{\frac{144}{169}} = \pm\frac{12}{13}$

and $\cos B = \frac{-12}{13}\left(\because B \text{ lies in } 2^{nd} \text{ quadrant}\right)$

Now $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$= \frac{-3}{5}\frac{-12}{13} - \frac{-4}{5}\frac{5}{13} = \frac{36}{65} + \frac{20}{65} = \frac{56}{65}$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$= \frac{-4}{5}\cdot\frac{-12}{13} - \frac{-3}{5}\cdot\frac{5}{13}$

$= \frac{48}{65} + \frac{15}{65} = \frac{63}{65}$

Qn.4. If $\tan A = 2$, $\tan B = 1$, A and B are acute angles. Find the value of $\tan(A-B)$ and $\cos(A-B)$

Sol: First find $\tan(A-B)$

We know $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$

$\frac{2-1}{1+2\times1} = \frac{1}{3}$

ie: $\tan(A-B) = \frac{1}{3}$

We know that $\sec^2(A-B) = 1+\tan^2(A-B)$

$= 1+\left(\frac{1}{3}\right)^2 = \frac{10}{9}$

$\therefore \sec(A-B) = \frac{\sqrt{10}}{3}$

ie; $\cos(A-B) = \dfrac{1}{\sec(A-B)} = \dfrac{3}{\sqrt{10}}$

Qn.5. If A and B are acute angles where $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, show that $A + B = \dfrac{\pi^c}{4}$

Sol: We have to show that $A + B = \dfrac{\pi^c}{4} = \dfrac{180^o}{4} = 45^o$

If $A + B = 45^o$, $\tan(A+B) = 1$

Here it is enough to show that $\tan(A+B) = 1$

$$\tan(A+B) = \frac{\tan A + \tan B}{1-(\tan A \tan B)}$$

$$= \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\,\frac{1}{3}}$$

$$= \frac{\frac{3+2}{6}}{\frac{6-1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1 \qquad \text{ie; } A + B = 45^o$$

Qn.6. If $\tan A = \dfrac{18}{17}, \tan B = \dfrac{1}{35}$, prove that $A - B = 45^o$

Sol: $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\frac{\frac{18}{17}-\frac{1}{35}}{1+\frac{18}{17}\times\frac{1}{35}} = \frac{\left(\frac{613}{595}\right)}{\left(\frac{613}{595}\right)} = 1$$

$\tan(A+B) = 1$ and $A + B = 45^o$

Qn.7. Find the value of tan 75 without using tables and show that $\tan 75 + \cot 75 = 4$

Sol: We have, $\tan 75 = \tan(45 + 30)$

$$= \frac{\tan 45 + \tan 30}{1 - \tan 45 \tan 30} = \frac{1+\frac{1}{\sqrt{3}}}{1-1\ \frac{1}{\sqrt{3}}}$$

$$= \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Multiply $\sqrt{3}+1$ on Numerator and deno min ator

$$\text{ie; } \tan 75 = \frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{(\sqrt{3}+1)^2}{3-1}$$

$$= \frac{3+2\sqrt{3}+1}{2} = \frac{4+2\sqrt{3}}{2}$$

$$= \frac{2(2+\sqrt{3})}{2} = 2+\sqrt{3}$$

Now $\cot 75 = \frac{1}{\tan 75} = \frac{1}{2+\sqrt{3}}$ (Multiply $2-\sqrt{3}$ on Nr and Dr)

$$= \frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$$

ie; $\tan 75 = 2+\sqrt{3}+2-\sqrt{3} = 4$

Qn.8. If $A + B = 45°$ show that $(1 + \tan A)(1 + \tan B) = 2$

Sol: We have $A + B = 45°$

$\tan(A + B) = \tan 45 = 1$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

$\tan A + \tan B = 1 - \tan A.\tan B$

$\tan A + \tan B + \tan A \tan B = 1$

Adding '1' on both sides,

$1 + \tan A + \tan B + \tan A \tan B = 1 + 1$

$1 + \tan A + \tan B(1 + \tan A) = 2$ [talking the common factors outside]

and $(1 + \tan A)(1 + \tan B) = 2$

Qn.9. Prove that $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$

Sol:
$\sin(A + B)\sin(A - B)$

$=(\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$

$=(\sin A\cos B)^2 - (\cos A\sin B)^2$

$=\sin^2 A.\cos^2 B - \cos^2 A\sin^2 B$

$=\sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B$

$=\sin^2 A - \sin^2 A.\sin^2 B - \sin^2 B + \sin^2 A.\sin^2 B$

$=\sin^2 A - \sin^2 B$

Qn.10. Prove that $\sin\left(\frac{\pi}{3} + A\right) - \sin\left(\frac{\pi}{3} - A\right) = \sin A$

Sol:

$$\sin\left(\pi/3 + A\right) - \sin\left(\pi/3 - A\right) = \left(\sin\pi/3\cos A + \cos\pi/3\sin A\right)$$

$$-\left(\sin\pi/3\cos A - \cos\pi/3\sin A\right)$$

$$= \sin\pi/3\cos A + \cos\pi/3.\sin A - \sin\pi/3\cos A + \cos\pi/3\sin A$$

$$= 2\cos\pi/3\sin A = 2\cos 60\sin A$$

$$= 2\,1/2\sin A = \sin A$$

In the above problem we have applied the following results.

1. expansions of sin(A+B) and sin(A-B)
2. $\cos\pi/3 = \cos\frac{180}{3} = \cos 60$
3. $\cos 60 = 1/2$

Qn.11. Express $\sqrt{3}\cos x + \sin x$ in the form $R\sin(x+\alpha)$ where 'α' is acute.

Sol: For expressing $\sqrt{3}\cos x + \sin x$ in the form $R\sin(x+\alpha)$, we have to find out R and α.

We have, $\sqrt{3}\cos x + \sin x = R\sin(x+\alpha)$

$= R[\sin x\cos\alpha + \cos x\sin\alpha)$

$\sqrt{3}\cos x + \sin x = R\sin x\cos\alpha + R\cos x\sin\alpha$

Equating the similar terms on both sides,

$\sqrt{3}\cos x = R\cos x\sin\alpha$ (termscontaining cos x)

$\sin x = R\sin x\cos\alpha$ (terms containing sin x)

ie; $\sqrt{3} = R\sin\alpha$ .... (1)

$1 = R\cos\alpha$ .... (2)

Squaring and adding (1) and (2)

$\sqrt{3^2} + 1^2 = R^2\sin^2\alpha + R^2\cos^2\alpha$

$3+1 = R^2(\sin^2\alpha + \cos^2\alpha)$

$4 = R^2 x\ 1$

$R = \pm 2$

Divide eq. (1) ny eq. (2)

ie; $\frac{\sqrt{3}}{1} = \frac{R\sin\alpha}{R\cos\alpha}$

$\sqrt{3} = \tan\alpha$

$\alpha = \tan^{-1}(\sqrt{3}) = 60^\circ$ (we know that $\tan 60 = \sqrt{3}$)

substitute R and α in the first equation to get, the expression

$\sqrt{3}\cos x + \sin x = R\sin(x+\alpha)$

$= \pm 2\sin(x + 60^\circ)$

Qn.12. If $x = 3\cos\theta + 4\sin\theta$ is written in the form of $x = r\sin(\theta+\alpha)$; find 'r'.

Sol: We have, $x = 3\cos\theta + 4\sin\theta = r\sin(\theta+\alpha)$

$= r(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$

$= r\sin\theta\cos\alpha + r\cos\theta\sin\alpha$

$3\cos\theta + 4\sin\theta = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha$

Equating the similar terms on both sides,

$3\cos\theta = r\cos\theta\sin\alpha$

$$4\sin\theta = r\sin\theta\cos\alpha$$

$$3 = r\sin\alpha\text{.........}(1)$$

$$4 = r\cos\alpha\text{......}(2)$$

Squaring and adding (1) and (2),

$$3^2+4^2 = r^2\sin^2\alpha + r^2\cos^2\alpha$$

$$= r^2\left(\sin^2\alpha+\cos^2\alpha\right)$$

$$= r^2$$

$$25 = r^2 \qquad \therefore r = \pm 5$$

Qn.13. Prove that $\dfrac{\cos A-\sin A}{\cos A+\sin A} = \tan(45-A)$

Sol: $\text{LHS} = \dfrac{\cos A-\sin A}{\cos A+\sin A}$

Dividing by cos A

$$\frac{\cos A-\sin A}{\cos A+\sin A} = \frac{\dfrac{\cos A-\sin A}{\cos A}}{\dfrac{\cos A+\sin A}{\cos A}}$$

$$= \frac{\dfrac{\cos A}{\cos A}-\dfrac{\sin A}{\cos A}}{\dfrac{\cos A}{\cos A}+\dfrac{\sin A}{\cos A}} = \frac{1-\tan A}{1+\tan A}\text{......}(1)$$

RHS = tan(45-A)

$$= \frac{\tan 45-\tan A}{1+\tan 45\tan A}$$

$$= \frac{1-\tan A}{1+\tan A}\text{......}(2)$$

(1) and (2) are equal

ie; $\dfrac{\cos A-\sin A}{\cos A+\sin A} = \tan(45-A)$

EXERCISE 1(d)

1) Prove that tan15+cot15=4

2) If $\sin A = \frac{-3}{5}$, $\sin B = \frac{12}{13}$ ('A' lies in third quadrant and 'B' lies in second quadrant). Find cos(A+B) and sin(A–B).

3) If $\cos A = \frac{-12}{13}$ and $\cot B = \frac{24}{7}$ ('A' lies in second quadrant and 'B' lies in first quadrant), find the values of $\sin(A+B)$ and $\cos(A+B)$

4) Express $4\cos x + 3\sin x$ in the form $K\sin(x+\alpha)$ where α is acute

5) If $\tan A = \frac{3}{4}$, $\tan B = \frac{5}{12}$ find the value of tan (A–B)

6) If $\tan A = \frac{3}{4}$, $\tan B = \frac{5}{12}$ find the value of $\tan(A-B)$

7) If tan A = 3, tan B = 1, A and B are acute angles, find cos (A – B)

8) Prove that $\sin(A+B).\sin(A-B) = \cos^2 B - \cos^2 A$

9) A current is given by $i = 11\sin\theta - 60\cos\theta$. Express it in the form $k\cos(\theta+\alpha)$

10) An alternating current current 'i' is given by i = 3sin wt + 4cos wt. Express this in the form $i = I\sin(wt+\alpha)$, α is acute.

11) If $\tan\alpha = \frac{1}{11}$, $\tan\beta = \frac{5}{6}$ prove that $\alpha + \beta = 45^\circ$

12) Prove that $\cos A + \cos\left(A + \frac{2\pi}{3}\right) + \cos\left(A - \frac{2\pi}{3}\right) = 0$

TRIGONOMETRY - II

2.1 MULTIPLE AND SUBMULTIPLE ANGLES

For a given angle θ; '$2\theta, 3\theta, 4\theta$ etc.' are multiple angles and '$\theta/2$, $\theta/4$ etc.' are submultiple angles.

MULTIPLE ANGLES (All derivations are excluded from the syllabus)

$\sin 2A = \sin(A+A) = \sin A \cos A + \cos A \sin A$

$= 2 \sin A \cos A$

$\therefore$ **$\sin 2A = 2 \sin A \cos A$**

$\cos 2A = \cos(A+A) = \cos A \cos A - \sin A \sin A$

$= \cos^2 A - \sin^2 A$

$\therefore$ **$\cos 2A = \cos^2 A - \sin^2 A$**

$$\tan 2A = \tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2\tan A}{1 - \tan^2 A}$$

$$\therefore \tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Now we can see some equations of sin 2A and in terms of 'tan A', we know that $\sin 2A = 2 \sin A \cos A$

$$= \frac{2\sin A \cos A}{1}$$

$$= \frac{2\sin A \cos A}{\sin^2 A + \cos^2 A}$$

Dividing the numerator and denominator by '$\cos^2 A$'.

$$\sin 2A = \frac{\dfrac{2\sin A \cos A}{\cos^2 A}}{\dfrac{\sin^2 A + \cos^2 A}{\cos^2 A}} = \frac{\dfrac{2\sin A}{\cos A}}{\dfrac{\sin^2 A}{\cos^2 A} + \dfrac{\cos^2 A}{\cos^2 A}}$$

$$= \frac{2\tan A}{\tan^2 A + 1} \text{ or } \frac{2\tan A}{1 + \tan^2 A}$$

$$\therefore \sin 2A = \frac{2\tan A}{1+\tan^2 A}$$

We know that $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{\cos^2 A - \sin^2 A}{1}$$

$$= \frac{\cos^2 A - \sin^2 A}{\sin^2 A + \cos^2 A}$$

Dividing the numerator and denominator by $\cos^2 A$,

$$\cos 2A = \frac{\dfrac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\dfrac{\sin^2 A + \cos^2 A}{\cos^2 A}} = \frac{\dfrac{\cos^2 A}{\cos^2 A} - \dfrac{\sin^2 A}{\cos^2 A}}{\dfrac{\sin^2 A}{\cos^2 A} + \dfrac{\cos^2 A}{\cos^2 A}}$$

$$= \frac{1-\tan^2 A}{1+\tan^2 A}$$

ie; $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

We have seen the equation of tan 2A in terms of tan A in the previous page.

$$\left(\tan 2A = \frac{2\tan A}{1-\tan^2 A}\right)$$

Now let us see some additional results for cos 2A which are very important.

We know that $\cos 2A = \cos^2 A - \sin^2 A$

$\cos 2A = 1 - 2\sin^2 A \qquad (\because \cos^2 A = 1 - \sin^2 A)$

and also $\cos 2A = \cos^2 A - (1 - \cos^2 A)$

$= 2\cos^2 A - 1 \quad (\because \sin^2 A = 1 - \cos^2 A)$

From these results we can deduce that,

$$\left[\begin{array}{l} \sin^2 A = \dfrac{1-\cos 2A}{2} \\[2ex] \text{and } \cos^2 A = \dfrac{1+\cos 2A}{2} \\[2ex] \text{and } \tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A} \end{array}\right]$$

Expressions for sin A, cosA, and tan A in terms of "A/2",

We know that $\sin 2A = 2 \sin A \cos A$

Replace '2A' by 'A' and 'A' by $A/2$

ie; $\sin A = 2\sin A/2 \cos A/2$

and also $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$

replacing '2A' by 'A' and 'A' by 'A/2'

$$\sin A = \frac{2\tan A/2}{1+\tan^2 A/2}$$

Similarly we have $\cos A = \cos^2 A/2 - \sin^2 A/2$

$= 2\cos^2 A/2 - 1$

$$= 1 - 2\sin^2 A/2 = \frac{1-\tan^2 A/2}{1+\tan^2 A/2}$$

and also $\tan A = \dfrac{2\tan A/2}{1-\tan^2 A/2}$

Also $\sin^2 A/2 = \dfrac{1-\cos A}{2}$ and $\cos^2 A/2 = \dfrac{1+\cos A}{2}$

$$\tan^2 A/2 = \frac{1-\cos A}{1+\cos A}$$

(Refer the similar equations of $\sin^2 A$, $\cos^2 A$ and $\tan^2 A$ in the previous pages)

Qn.1. If $\tan\theta = 2$ find $\sin\theta$, $\cos 2\theta$ and $\tan 2\theta$

Sol: $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta} = \dfrac{2\times 2}{1+2^2} = \dfrac{4}{5}$

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-4}{1+4} = \frac{-3}{5}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{4}{5}}{\frac{-3}{5}} = \frac{-4}{3}$$

(We shall also use $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ **here.)**

Qn.2. If $\tan A = \frac{3}{4}$ (A is acute)

and $\sin B = \frac{5}{13}$ (B lies in second quadrant)

Find sin 2A and sin2B

Sol: $\sin 2A = \frac{2\tan A}{1+\tan^2 A} = \frac{2\times \frac{3}{4}}{1+(\frac{3}{4})^2} = \frac{24}{25}$

$\sin B = 5/13$, $\cos^2 B = 1 - \sin^2 B$

ie; $\cos B = \sqrt{1-\sin^2 B} = \sqrt{1-\left(\frac{5}{13}\right)^2}$

$= \sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}}$

$= \frac{-12}{13}$

$\therefore \sin 2B = 2\sin B\cos B$

$= 2\times\frac{5}{13}\times\frac{-12}{13} = \frac{-120}{169}$

Qn.3. Prove that $\frac{1+\cos 2A}{\sin 2A}$ =cot A and deduce the value of cot 15

Sol: $\frac{1+\cos 2A}{2} = \cos^2 A$ and $1+\cos 2A = 2\cos^2 A$

$\sin 2A = 2\sin A\cos A$

$\therefore \frac{1+\cos 2A}{\sin 2A} = \frac{2\cos^2 A}{2\sin A\cos A} = \cot A$

ie; $\frac{1+\cos 2A}{\sin 2A} = \cot A$

put A = 15, $\cot 15 = \frac{1+\cos 30}{\sin 30} = \frac{1+\cos 30}{\sin 30} = \frac{1+\frac{\sqrt{3}}{2}}{\frac{1}{2}}$.

ie; $\cot 15 = \frac{2+\sqrt{3}}{2} 2 = 2+\sqrt{3}$

Qn.4. If $\sin A = 0.6$ 'A' is acute. Find $\sin 2A$

Sol: $\sin 2A = 2\sin A\cos A$

$\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-(0.6)^2}$

$= \sqrt{1-0.36} = \sqrt{0.64} = 0.8$

$\therefore\ \sin 2A = 2\sin A\cos A = 2\times 0.6\times 0.8 = 0.96$

Qn.5. Prove that $\cos 4\theta = 1-8\sin^2\theta\cos^2\theta$

Sol: We know that,

$\cos\theta = 1-2\sin^2\theta$

$\therefore \cos 4\theta = 1-2\sin^2(2\theta)$

$= 1-2\sin 2\theta\ \sin 2\theta$

$= 1-2\ 2\sin\theta\cos\theta\times 2\sin\theta\cos\theta$

$= 1-2\ 2\sin\theta\cos\theta\times 2\sin\theta\cos\theta$

$= 1-2\ 2\sin\theta\cos\theta\times\sin\theta\cos\theta$

$= 1-8\sin^2\theta\cos^2\theta$

Qn.6. Prove that $2\tan 10 + \tan 40 = \tan 50$

Sol: Consider $\tan(50-40) = \dfrac{\tan 50-\tan 40}{1+\tan 50\tan 40}$

$\tan 10 = \dfrac{\tan 50-\tan 40}{1+\tan 50\tan(90-50)}$

$= \dfrac{\tan 50-\tan 40}{1+\tan 50\cot 50}$

ie; $2\tan 10 = \tan 50-\tan 40.$

$\therefore\ 2\tan 10 + \tan 40 = \tan 50$

Qn.7. Prove that $1+\tan\theta\tan 2\theta = \sec 2\theta$

Sol: Consider $\tan(2\theta-\theta) = \dfrac{\tan 2\theta-\tan\theta}{1+\tan 2\theta\tan\theta}$

$\tan\theta = \dfrac{\tan 2\theta-\tan\theta}{1+\tan 2\theta\tan\theta}$

$(1+\tan 2\theta\ \tan\theta)\tan\theta = \tan 2\theta-\tan\theta$

$1+\tan 2\theta\ \tan\theta = \dfrac{\tan 2\theta-\tan\theta}{\tan\theta}$

$= \dfrac{\tan 2\theta}{\tan\theta}-1$

$$\frac{\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)}{\tan\theta}-1 = \frac{2}{1-\tan^2\theta}-1 = \frac{2-\left(1-\tan^2\theta\right)}{1-\tan^2\theta}$$

$$=\frac{1+\tan^2\theta}{1-\tan^2\theta}=\frac{1}{\cos 2\theta}=\sec 2\theta$$

Expressions of sin 3A, cos 3A (Derivations are excluded from the syallabus)

1. **sin 3A**

We know that $\sin 3A=\sin(2A+A)$

$=\sin 2A \cos A + \cos 2A \sin A$

$=2 \sin A \cos A \cos A + (1-2 \sin^2 A) \sin A$

$= 2\sin A\cos^2 A + (1-2\sin^2 A)\sin A$

$= 2\sin A(1-\sin^2 A) + (1-2\sin^2 A)\sin A$

$= 3\sin A-4\sin^3 A$

ie; $\sin 3A=3 \sin A-4 \sin^3 A$

2. **cos 3A**

We know that $\cos 3A=\cos(2A+A)$

$=\cos 2A\cos A-\sin 2A\sin A$

$=(2\cos^2 A-1)\cos A-2 \sin A \cos A \sin A$

$=(2\cos^2 A-1)\cos A-2 \sin^2 A\cos A$

$=(2\cos^2 A-1)\cos A-2(1-\cos^2 A) \cos A$

$=2 \cos^3 A-\cos A-2\cos A+2\cos^3 A$

$=4\cos^3 A-3 \cos A$

Qn.8. Prove that $\dfrac{\cos 3A+\cos A}{\sin 3A-\sin A}=\cot A$

Sol: $\dfrac{4\cos^3 A-3\cos A+\cos A}{3\sin A-4\sin^3 A-\sin A}=\dfrac{4\cos^3 A-2\cos A}{2\sin A-4\sin^3 A}$

$$=\frac{2\cos A\left(2\cos^2 A-1\right)}{2\sin A\left(1-2\sin^2 A\right)}$$

$$=\frac{2\cos A\cos 2A}{2\sin A\cos 2A}=\cot A$$

Qn.9. Prove that $\dfrac{\sin 3x}{\sin x}-\dfrac{\cos 3x}{\cos x}=2$

Sol: $\dfrac{3\sin x-4\sin^3 x}{\sin x}-\dfrac{4\cos^3 x-3\cos x}{\cos x}$

$=\dfrac{\sin x\left(3-4\sin^2 x\right)}{\sin x}-\dfrac{\cos x\left(4\cos^2 x-3\right)}{\cos x}$

$=3-4\sin^2 x-\left(4\cos^2 x-3\right)$

$=3-4\sin^2 x-4\cos^2 x+3$

$=6-\left(\sin^2 x+\cos^2 x\right)$

$=6-4=2$

Things to remember

1. $\sin 2\theta = 2\sin\theta\cos\theta$ $\qquad \sin\theta = 2\sin\theta/2 \,.\cos\theta/2$

$= \dfrac{2\tan\theta}{1+\tan^2\theta}$ $\qquad = \dfrac{2\tan\theta/2}{1+\tan^2\theta/2}$

2. $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\qquad \cos\theta = \cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2}$

$= 2\cos^2\theta - 1$ $\qquad = 2\cos^2\theta/2 - 1$

$= 1 - 2\sin^2\theta$ $\qquad = 1 - 2\sin^2\theta/2$

$= \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$ $\qquad = \dfrac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$

3. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ $\qquad \tan\theta = \dfrac{2\tan\theta/2}{1-\tan^2\theta/2}$

4. $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$ $\qquad \sin^2\theta/2 = \dfrac{1-\cos\theta}{2}$

5. $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ $\qquad \cos^2\theta/2 = \dfrac{1+\cos\theta}{2}$

6. $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$ $\qquad \tan^2\theta/2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

7. $\sin 3A = 3\sin A - 4\sin^3 A$

8. $\cos 3A = 4\cos^3 A - 3\cos A$

EXERCISE 2 (a)

1) If $\sin A = \frac{3}{5}$, 'A' is acute, find sin2A, cos2A, sin3A and cos3A

2) If $\cos\theta = \frac{1}{2}$, 'θ' is acute, find $\sin 2\theta$, $\cos 2\theta$, $\sin 3\theta$ and $\cos 3\theta$

3) If tanB = 1, 'B' lies in third quadrant, find sin2B, cos2B and tan2B

4) If $\tan\theta = 2$, find $\cos 2\theta$

5) Prove that $\frac{\sin 2A}{1+\cos 2A} = \tan A$ and deduce the value of tan 15

6) Calculate the possible values of $\cos\theta$ if $4\cos 2\theta + 2\cos\theta + 3 = 0$

(Hint: $\cos 2\theta = 2\cos^2\theta - 1$)

7) If $\cos 2\theta = \frac{-3}{5}$ find $\tan\theta$

8) Prove that $\frac{\sin 3A}{\sin A} + \frac{\cos 3A}{\cos A} = 4\cos 2A$

9) Prove that $\cos^4 A - \sin^4 A = \cos 2A$

10) If $\tan A = 0.38$ find tan 2A

11) If $\sin A = a$ find sin 3A

12) Prove that $\frac{\cot A - \tan A}{\cot A + \tan A} = \cos 2A$

2.2 PRODUCT FORMULAE AND CONVERSE

PRODUCT FORMULA (all derivations are excluded)

In the previous chapter we have learnt some formulae.

They are $\sin(A+B) = \sin A \cos B + \cos A \sin B$ ... (1)

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ ...(2)

$\sin(A-B) = \sin A \cos B - \cos A \sin B$ ...(3)

$\cos(A-B) = \cos A \cos B + \sin A \sin B$ ...(4)

Let $C = A+B$, $D = A-B$

$\sin C + \sin D = \sin(A+B) + \sin(A-B)$ form (1) and (3)

$= \sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$

$= 2\sin A \cos B$

$$= \sin\frac{A+B+A-B}{2}\cos\frac{A+B-(A-B)}{2}$$

$$= 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$ (where C and D are any two angles

ie; $\sin C + \sin D = 2\sin\frac{C-D}{2}\cos\frac{C-D}{2}$

Similarly we can prove that

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$$

eg: (1) $\sin 4\theta + \sin 2\theta = 2\sin\frac{4\theta+2\theta}{2}\cos\frac{4\theta-2\theta}{2}$

$= 2\sin 3\theta \cos\theta$

eg:(2) $\cos 65 + \cos 15 = 2\cos\frac{65+15}{2}\cos\frac{65-15}{2}$

$= 2\cos 40 \cos 25$

eg: (3) $\sin 5\theta - \sin 3\theta = 2\cos\frac{5\theta+3\theta}{2}\sin\frac{5\theta-3\theta}{2}$

$= 2\cos 4\theta \sin\theta$

eg: (4) $\cos 8\theta - \cos 4\theta = -2\sin\frac{8\theta+4\theta}{2}\sin\frac{8\theta-4\theta}{2}$

$= -2\sin 6\theta \sin 2\theta$

Qn.1. Show that cos 5–sin25 = sin 35

Sol: Given, cos 5=sin 35+sin25

consider RHS, sin 35+sin 25

$= 2\sin\frac{35+25}{2}\cos\frac{35-25}{2}$

=2sin 30 cos 5

$= 2\,\frac{1}{2}\cos 5$

=cos 5 = LHS

Qn.2. Prove that sin 33 + cos 63 = cos 3

Sol: Rewrite the above equation,

sin 33 = cos 3 – cos 63

$\text{RHS} = -2\sin\frac{3+63}{2}\sin\frac{3-63}{2}$

$= -2\sin 33\sin(-30)$

$= -2\sin 33 \times -\frac{1}{2}$

$= \sin 33 = \text{LHS}$

Qn.3. Prove that $\frac{\sin 4A + \sin 2A}{\cos 4A + \cos 2A} = \tan 3A$

Sol: $\text{LHS} = \frac{\sin 4A + \sin 2A}{\cos 4A + \cos 2A}$

$$= \frac{2\sin\frac{4A+2A}{2}\cos\frac{4A-2A}{2}}{2\cos\frac{4A+2A}{2}\cos\frac{4A-2A}{2}}$$

$$= \frac{2\sin 3A\cos A}{2\cos 3A\cos A} = \tan 3A$$

Qn.4. Prove that $\frac{\sin x + \sin 2x}{\cos x + \cos 2x} = \tan \frac{3x}{2}$

Sol: LHS = $\frac{\sin x + \sin 2x}{\cos x + \cos 2x} = \frac{2\sin\frac{3x}{2}\cos\frac{-x}{2}}{2\cos\frac{3x}{2}\cos\frac{-x}{2}} = \tan\frac{3x}{2}$

Qn.5. Prove that $\frac{\sin 2\alpha + \sin 5\alpha - \sin\alpha}{\cos 2\alpha + \cos 5\alpha + \cos\alpha} = \tan 2\alpha$

Sol: LHS, $\frac{\sin 2\alpha + \sin 5\alpha - \sin\alpha}{\cos 2\alpha + \cos 5\alpha + \cos\alpha}$

$= \frac{\sin 2\alpha + (\sin 5\alpha - \sin\alpha)}{\cos 2\alpha + (\cos 5\alpha + \cos\alpha)}$ ($\because$ the answer is in terms or 2α)

$= \frac{\sin 2\alpha + 2\cos 3\alpha \sin 2\alpha}{\cos 2\alpha + 2\cos 3\alpha\cos 2\alpha}$

$= \frac{\sin 2\alpha(1 + 2\cos 3\alpha)}{\cos 2\alpha(1 + 2\cos 3\alpha)} = \tan 2\alpha$

Qn.6. Prove that $\sin 50 - \sin 70 + \cos 80 = 0$

Sol: $\sin 50 - \sin 70 = 2\cos\frac{120}{2}\sin\frac{-20}{2}$

$= 2\cos 60 \sin - 10$ ($\because \sin(-\theta) = -\sin\theta$)

$= 2\cos 60 . -\sin 10 = -2\cos 60 \sin 10$

$= -2\ ^1\!/_2 \sin 10 = -\sin 10$

ie; LHS, $\sin 50 - \sin 70 + \cos 80$

$= -\sin 10 + \cos(90 - 10) = -\sin 10 + \sin 10 = 0$

Qn.7. Prove that $\cos 3A + \cos 5A + \cos 9A + \cos 17A = 4 \cos 4A \times \cos 6A \times \cos 7A$

Sol: $(\cos 3A + \cos 5A) + (\cos 9A + \cos 17A)$

$= 2\cos 4A \cos - A + 2\cos 13A \cos - 4A$

$= 2\cos 4A(\cos A + \cos 13A)$ $\boxed{\cos(-\theta) = \cos\theta}$

$= 2\cos 4A(2\cos 7A \cos - 6A)$

$= 4\cos 4A \cos 6A \cos 7A$

Qn.8. Prove that $\cos \pi/8 + \cos 3\pi/8 + \cos 5\pi/8 + \cos 7\pi/8 = 0$

Sol: $(\text{con}\ \pi/8 + \cos 3\pi/8) + (\cos 5\pi/8 + \cos 7\pi/8)$

$$= 2\cos\frac{4\pi}{16}.\cos\frac{-2\pi}{16} + 2\cos\frac{12\pi}{16}.\cos\frac{-2\pi}{16}$$

$$= 2\cos\frac{\pi}{4}\cos\frac{-\pi}{8} + 2\cos\frac{3\pi}{4}.\cos\frac{-\pi}{8} \qquad \boxed{\cos(-\theta) = \cos\theta}$$

$$= 2\cos\frac{\pi}{8}(\cos\frac{\pi}{4} + \cos\frac{3\pi}{4})$$

$$= 2\cos\frac{\pi}{8}2\cos\frac{4\pi}{8}\cos\frac{-2\pi}{8} = 0 \qquad (\because \cos\frac{\pi}{2} = \cos 90 = 0)$$

$$= 4\cos\frac{\pi}{8}.\cos\frac{\pi}{2}.\cos\frac{\pi}{4} = 0$$

CONVERSE OF PRODUCT FORMULA

We have seen four product formulae in the previous page.

Here we can see the converse of these formulae.

ie; $\sin A \cos B = \frac{1}{2}[\sin(A+B) + (\sin A - B)]$

$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

For the proof, RHS can be expanded and simplified to get LHS.

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

RHS= $\frac{1}{2}[\sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$

$= \frac{1}{2}[2\sin A \cos B] = \sin A \cos B$

eg: (1) Express $\sin 5x \cos 3x$ as a sum or difference.

ie; $\sin 5x \cos 3x = \frac{1}{2}[\sin 8x + \sin 2x]$

eg: (2) Express $\cos 3\theta \cos\theta$ as a sum or difference.

We have $\cos A \cos B = 1/2\,[\cos(A+B)+\cos(A-B)]$

$\cos 3\theta \cos\theta = 1/2\,[\cos 4\theta + \cos 2\theta]$

eg: (3) $\sin 5x \sin x$

We have $\sin A \sin B = -\frac{1}{2}[\cos(A+B)-\cos(A-B)]$

ie; $\sin 5x \sin x = -\frac{1}{2}[\cos 6x - \cos 4x]$

Qn.9. Show that $\sin 10 \sin 50 \sin 70 = 1/8$.

Sol: $\sin 10(\sin 50 \sin 70)$

$= \sin 10 \; -\frac{1}{2}(\cos(120) - \cos -20)$

$= -\frac{1}{2}\sin 10\,[\cos 120 - \cos 20]$

$= -\frac{1}{2}\sin 10\,[\cos(180-60) - \cos 20]$

$= -\frac{1}{2}\sin 10\,[-\cos 60 - \cos 20]$

$= -\frac{1}{2}\sin 10\,[-\frac{1}{2} - \cos 20]$

$= \frac{1}{4}\sin 10 + \frac{1}{2}\sin 10 \cos 20$

$= \frac{1}{4}\sin 10 + \frac{1}{2}\,\frac{1}{2}(\sin 30 + \sin -10)$

$= \frac{1}{4}\sin 10 + \frac{1}{4}\sin 30 - \frac{1}{4}\sin 10$

$= \frac{1}{4}\sin 10 = \frac{1}{4}\;\frac{1}{2} = \frac{1}{8}$

In the above problem, we have applied certain results.

These results are given below.

$\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\cos 120 = \cos(180-60) = -\cos 60 = -\frac{1}{2}$

$\sin 30 = \frac{1}{2}$

Qn.10. Show that $\cos 20 \cos 40 \cos 60 \cos 80 = \frac{1}{16}$

Sol: We have $\cos 60 = \frac{1}{2}$ ie; $\frac{1}{2} \cos 20 (\cos 40 \cos 80)$

$= \frac{1}{2} \cos 20 \frac{1}{2} [\cos (120) + \cos(-40)]$

$= \frac{1}{4} \cos 20(-\frac{1}{2} + \cos 40)$

$= -\frac{1}{8} \cos 20 + \frac{1}{4} \cos 20.\cos 40$

$= -\frac{1}{8} \cos 20 + \frac{1}{4} \frac{1}{2} (\cos 60 + \cos 20)$

$= -\frac{1}{8} \cos 20 + \frac{1}{8} \cos 60 + \frac{1}{8} \cos 20$

$= \frac{1}{8} \cos 60 = \frac{1}{8} \frac{1}{2} = \frac{1}{16}$

Things to remember

If C and D are any two angles,

1. $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

2. $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

3. $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

4. $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

If A and B are any two angles

1. $\sin A \cos B = \frac{1}{2} [\sin (A+B) + \sin (A-B)]$

2. $\cos A \sin B = \frac{1}{2} [\sin (A+B) - \sin (A-B)]$

3. $\cos A \cos B = \frac{1}{2} [\cos (A+B) + \cos(A-B)]$

4. $\sin A \sin B = -\frac{1}{2} [\cos (A+B) - \cos(A-B)]$

EXERCISE 2(b)

1. Prove that $\dfrac{\cos 3A - \cos A}{\sin A - \sin 3A} = \tan 2A$

2. Prove that $\dfrac{\sin A - \sin 4A}{\cos 4A - \cos A} = \cot \dfrac{5A}{2}$

3. Prove that $\dfrac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$

4. Prove that $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \tan \dfrac{A+B}{2} \cot \dfrac{A-B}{2}$

5. Show that $\cos 55 + \cos 65 + 175 = 0$

6. Prove that $\sin 50 - \sin 70 + \sin 10 = 0$

7. Prove that $\cos 21 - \sin 83 = -2 \sin 14 \sin 7$
   (consider RHS and apply converse of p-formula)

8. Prove that $\cos 80 + \cos 40 - \cos 20 = 0$
   (apply product formula)

9. Prove that $\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta$
   $= 4 \cos \theta \cos 2\theta \sin 4\theta$

10. Prove that $\sin 20 \sin 40 \sin 60 \sin 80 = \dfrac{3}{16}$

    (put $\sin 60 = \dfrac{\sqrt{3}}{2}$ and apply converse of product formula)

11. Prove that $\cos 20 \cos 40 \cos 80 = 1/8$

12. Prove that $\sin 20 \sin 40 \sin 80 = \sqrt{3}/8$

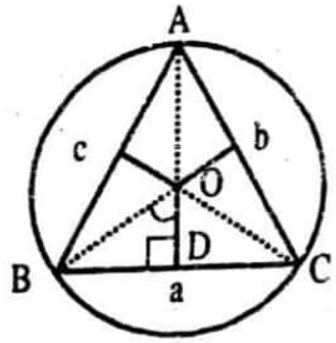
2.3

PROPERTIES AND SOLUTIONS OF TRIANGLES

In this chapter, let us study some important results connecting the sides and angles of a triangle. First of all we can see some common notations used in this chapter. For a triangle ABC, A, B and C denote the three angles and a, b and c denote the sides opposite to the angles A, B and C respectively. R is used for representing the circum radius of ΔABC (radius of the circum circle).

a) **THE SINE RULE** (The proof is excluded from the syllabus)

In any triangle ABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$



Consider the circumcircle of ΔABC. The perpendicular bisectors of the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ intersect at 'O'. Therefore 'O' is the circumcentre such that OA=OB=OC=R.

We have $\angle BOC = 2\ \angle BAC = 2A$

$\therefore \angle BOD = \angle COD = A$

In ΔODB, $\angle BOD = A$

Now, $\sin \angle BOD = \sin A = \dfrac{BD}{OB} = \dfrac{\frac{a}{2}}{R}$

$\sin A = \dfrac{a}{2R} \Rightarrow a = 2R \sin A$

Similarly,

$\sin B = \dfrac{b}{2R} \Rightarrow b = 2R \sin B$

$\sin C = \dfrac{c}{2R} \Rightarrow c = 2R \sin C$

Note:-

1) The circle which passes through the angular points of a ΔABC is called its circumcircle. The centre of this circle is the point of intersection of the perpendicular bisectors of the sides and is called the circum centre. Its radius is always denoted by 'R'.

2) The sine rule states that the sides of a triangle are proportional to the sines of angles opposite to them.

Qn.1. Show that in $\Delta ABC, \sum a(\sin B-\sin C)=0$

Sol. In ΔABC, we know that $\sin B=\frac{b}{2R}$ and $\sin C=\frac{c}{2R}$

$$\sum a(\sin B-\sin C)=\sum a(b/_{2R}-c/_{2R})$$

$$=\frac{1}{2R}\sum a(b-c)$$

$$=\frac{1}{2R}[a(b-c)+b(c-a)+c(a-b)](\text{expanding}\sum a(b-c))$$

$$=\frac{1}{2R}[ab-ac+bc-ab+ac-bc]$$

$$=\frac{1}{2R}\times 0=0$$

Qn.2. Prove that in ΔABC, $(a+b)\sin\frac{C}{2}=c\cos\frac{A-B}{2}$

Sol. LHS, $(a+b)\sin\frac{C}{2}$

$$=(2R\sin A+2R\sin B)\sin\frac{C}{2}$$

$$=2R(\sin A+\sin B)\sin\frac{C}{2}$$

$$=2R2.\sin\frac{A+B}{2}.\cos\frac{A-B}{2}.\sin\frac{C}{2}$$

$$=\cos\frac{A-B}{2}\left(4R\sin\frac{A+B}{2}\sin\frac{C}{2}\right)$$

$$=\cos\frac{A-B}{2}\left(4R\sin\left(90-\frac{C}{2}\right)\sin\frac{C}{2}\right)$$

$$=\cos\frac{A-B}{2}\,4R\cos\frac{C}{2}\sin\frac{C}{2}$$

$$=\cos\frac{A-B}{2}\,2R\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)$$

$$=\cos\frac{A-B}{2}\,2R\sin C$$

$$=\cos\frac{A-B}{2}\,c$$

In the above problem we have applied the following results.

(1) sine rule

(2) $A+B+C=180,\quad \frac{A+B+C}{2}=90$

$$\sin\frac{A+B}{2}=\sin\left(90-\frac{C}{2}\right)=\cos\frac{C}{2}$$

(3) $\sin(90-\theta)=\cos\theta$

(4) $2\sin\frac{C}{2}\cos\frac{C}{2}=\sin C$

b) Napier's Formula (Tangent Rule)

(The proof is excluded from the syllabus)

In any ΔABC, $\quad \tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$

Consider $\frac{b-c}{b+c}\cot\frac{A}{2}$

$$=\frac{2R\sin B-2R\sin C}{2R\sin B+2R\sin C}\cot A/2$$

$$=\frac{2R(\sin B-\sin C)}{2R(\sin B+\sin C)}\cot A/2$$

$$=\frac{\sin B-\sin C}{\sin B+\sin C}\cot A/2$$

$$=\frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\cot A/2}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$= \cot\frac{B+C}{2}\tan\frac{B-C}{2}\cot A/2$$

$$= \tan\frac{B-C}{2}\cot\frac{B+C}{2}\cot A/2$$

$$= \tan\frac{B-C}{2}\cot\left(90 - A/2\right)\cot A/2$$

$$= \tan\frac{B-C}{2}\tan A/2 \cot A/2$$

$$= \tan\frac{B-C}{2}\times 1$$

In the above problem we have applied the following results.

(1) sine rule

(2) $\cot\frac{B+C}{2} = \cot(90 - A/2) = \tan A/2$ (refer the previous problem)

(3) $\tan A/2 \ \cot A/2 = 1$

Qn.3. In a ΔABC if a=6cm, b=8cm and $\sin B = 3/5$. Find sin A.

Sol. We know that $\frac{a}{\sin A} = \frac{b}{\sin B}$ ie; $\frac{6}{\sin A} = \frac{8}{(3/5)}$

$$\frac{6}{\sin A} = \frac{40}{3}$$

$$\sin A = \frac{18}{40} = \frac{9}{20}$$

Qn.4. In ΔABC, $\sin A = \frac{4}{5}$, $\sin B = \frac{1}{4}$, a=24cm, find b.

Sol. In a ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B}$

$$\frac{24}{(4/5)} = \frac{b}{(1/4)} \qquad \frac{24\times 5}{4} = 4b$$

$$b = \frac{120}{16} \ \therefore b = 7.5\text{cm}.$$

Qn.5. In a ΔABC, $A=30^\circ$, $C=45^\circ$, $a=2$cm. Find 'C'

Sol. We have $\dfrac{a}{\sin A}=\dfrac{c}{\sin C}$

$$\frac{2}{\sin 30}=\frac{c}{\sin 45}$$

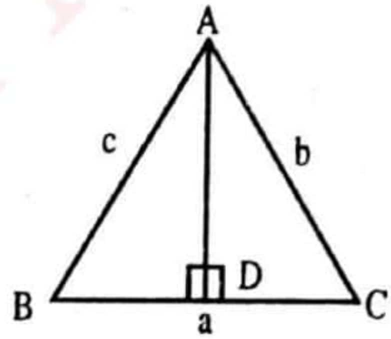
$$\frac{2}{\left(\frac{1}{2}\right)}=\frac{c}{\left(\frac{1}{\sqrt{2}}\right)}$$

$$4=\sqrt{2}c,\ c=\frac{4}{\sqrt{2}}\Rightarrow c=2\sqrt{2}\text{ cm.}$$

c) **Cosine Rule**

(Proof is excluded from the syllabus)

In any ΔABC; $c^2=a^2+b^2-2ab\cos C$



In the figure we can see

two right angled triangles.

From ΔABC; $AB^2=BD^2+AD^2$

ie; $AB^2=(BC-CD)^2+AD^2$

$=BC^2-2BC\,CD+CD^2+AD^2$

$=BC^2+(CD^2+AD^2)-2BC\,CD$

$=BC^2+AC^2-2BC\,CD$

ie; $c^2=a^2+b^2-2aCD$ ....(1)

From the ΔADC, $\cos C=\dfrac{CD}{b}$, $CD=b\cos C$

Substituting the value of CD in (1)

$\mathbf{c^2=a^2+b^2-2ab\cos C}$

Similarly we can show that $\mathbf{a^2=b^2+c^2-2bc\cos A}$

and also $\mathbf{b^2=a^2+c^2-2ac\cos B}$

We can also express the above formulae in the following ways

ie; $\cos A = \dfrac{b^2+c^2-a^2}{2bc}, \cos B = \dfrac{a^2+c^2-b^2}{2ac}$

$\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

Qn.6. $2\,[bc\cos A + ca\cos B + ab\cos C] = a^2 + b^2 + c^2$

Sol. LHS $= 2(bc\cos A + ca\cos B + ab\cos C)$

$$= 2\left[bc\left(\frac{b^2+c^2-a^2}{2bc}\right) + ca\left(\frac{a^2+c^2-b^2}{2ac}\right) + ab\left(\frac{a^2+b^2-c^2}{2ab}\right)\right]$$

$$= 2\left[\frac{b^2+c^2-a^2}{2} + \frac{a^2+c^2-b^2}{2} + \frac{a^2+b^2-c^2}{2}\right]$$

$$= \frac{2}{2}\left[b^2+c^2-a^2+a^2+c^2+-b^2+a^2+b^2-c^2\right]$$

$$= \frac{2}{2}\left[a^2+b^2+c^2\right] = a^2+b^2+c^2$$

Qn.7. Show that $a(b\cos C - c\cos B) = b^2 - c^2$

Sol. $$\text{LHS} = a\left[b\frac{a^2+b^2-c^2}{2ab} - c\left(\frac{a^2+c^2-b^2}{2ac}\right)\right]$$

$$= a\left[\frac{a^2+b^2-c^2}{2a} - \left(\frac{a^2+c^2-b^2}{2a}\right)\right]$$

$$= a\left[\frac{a^2+b^2-c^2-a^2-c^2+b^2}{2a}\right] = \frac{2b^2-2c^2}{2} = b^2-c^2 = \text{RHS}$$

Qn.8. Prove that, $R\left(a^2+b^2+c^2\right) = abc\left(\cot A + \cot B + \cot C\right)$

Sol. RHS, abc (cot A+cotB+cot C)

$$= abc\left(\frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C}\right) = \sin B$$

$$= \frac{abc\cos A}{\sin A} + \frac{abc\cos B}{\sin B} + \frac{abc\cos C}{\sin C}$$

$$= \frac{a}{\sin A} bc\cos A + \frac{b}{\sin B} ac\cos B + \frac{c}{\sin C} ab\cos C$$

$$= 2R\ bc\cos A + 2R\ ac\cos B + 2R\ ab\cos C$$

$$= 2R\left[bc\frac{b^2+c^2-a^2}{2bc} + ac\frac{a^2+c^2-b^2}{2ac} + ab\frac{a^2+b^2-c^2}{2ab}\right]$$

$$= 2R\left[\frac{b^2+c^2-a^2}{2} + \frac{a^2+c^2-b^2}{2} + \frac{a^2+b^2-c^2}{2}\right]$$

$$= 2R\left[\frac{b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2}{2}\right]$$

$$= 2R\left[\frac{a^2+b^2+c^2}{2}\right] = R\left(a^2+b^2+c^2\right)$$

d) **Projection Formula**

(Proof is excluded from the syllabus)

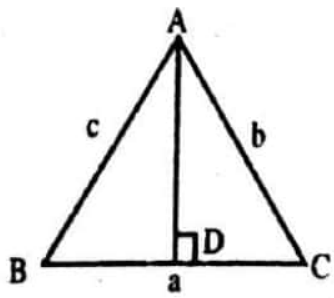
In any ΔABC, $a = b\cos C + c\cos B$ or

$b = a\cos C + c\cos A$ or $c = a\cos B + b\cos A$

Consider a triangle ABC. $\overline{AD}$ is perpendicular to $\overline{BC}$.

We can see two right angled triangles in the figure.

From ΔADB, $\cos B = \frac{BD}{AB} = \frac{BD}{c}$



$\therefore\ BD = c\cos B$ ...(1)

From the ΔADC,

$\cos C = \frac{CD}{AC} = \frac{CD}{b}$

$\therefore\ CD = b\cos C$ ....(2)

Now consider the side BC

$BC = BD + DC$

ie; $a = BD + DC$

$a = c\cos B + b\cos C$ from (1) and (2).

Similarly we can prove the other two results.

Qn.9. Show that $a(b^2+c^2)\cos A + b(c^2+a^2)\cos B + c(a^2+b^2)$ $\cos C = 3abc$

Sol. $a(b^2+c^2)\cos A + b(c^2+a^2)\cos B + c(a^2+b^2)\cos C$

$= ab^2\cos A + ac^2\cos A + bc^2\cos B + ba^2\cos B$

$+ca^2\cos C + cb^2\cos C$

$= (ab^2\cos A + ba^2\cos B) + (bc^2\cos B + cb^2\cos C)$

$+(ac^2\cos A + ca^2\cos C)$

$= ab(b\cos A + a\cos B) + bc(c\cos B + b\cos C)$

$+ac(c\cos A + a\cos C)$

$= abc + bca + acb$ (by projection formula)

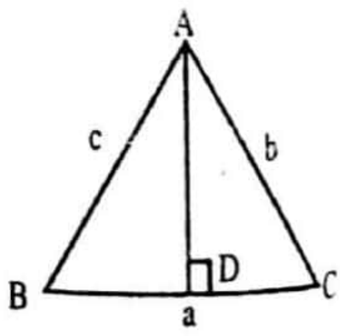
$= 3abc$

e) Area of a triangle (Δ)

Area of $\Delta ABC = \frac{1}{2} \times \text{base} \times \text{altitude}$

$= \frac{1}{2} BC.AD$

$= \frac{1}{2} a.c.\sin B$



Similarly $\Delta = \frac{1}{2} bc\sin A$

or $\Delta = \frac{1}{2} ab\sin C$

$\boxed{\sin B = \frac{AD}{c} \text{ from the figure}}$

If the three sides a, b and c are given, the area of ΔABC can be obtained by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

eg:1 Find the area of a triangle, Given b = 3 cm, c = 2 cm and A= 30°

$$A = \frac{1}{2} bc \sin A$$

$$= \frac{1}{2} \times 3 \times 2 \times \sin 30$$

$$= \frac{1}{2} \times 3 \times 2 \times \frac{1}{2}$$

$$= \frac{3}{2} \text{ cm}^2$$

eg:2. Find the area of a triangle for which a = 3 cm, b = 7 cm, and c = 8 cm.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$S = \frac{a+b+c}{2} = \frac{3+7+8}{2} = 9$$

$$A = \sqrt{9(9-3)(9-7)(9-8)}$$

$$= \sqrt{9 \times 6 \times 2 \times 1} = \sqrt{108} \text{ cm}^2$$

Things to remember

1. In any $\Delta ABC \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$ (sine rule)
2. In any $\Delta ABC \tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

   (Napier's formula or tangent rule)
3. In any $\Delta ABC\ a^2 = b^2 + c^2 - 2bc.\cos A$

   $b^2 = a^2 + c^2 - 2ac\cos B,\ c^2 = a^2 + b^2 - 2ab\cos C$ (cosine rule)
4. In any ΔABC, $a = b\cos C + c\cos B$

   $b = a \cos C + c \cos A, c = a \cos B + b \cos A$ (Projection formula)

EXERCISE 2(C)

marks

2 1. In a Δ ABC, $\sin A = \frac{4}{5}$, $\sin B = \frac{1}{4}$, a= 24 cm find 'b'. (Sine rule)

2 2. In a Δ ABC, a= 2 cm, c= 4cm, $C = 30^\circ$ find sin A (Sine rule)

2 3. In a Δ ABC, $A = 45^\circ$, $B = 60^\circ$
a = 5 cm find 'b'. (Sine rule)

5 4. Show that Δ ABC,

$$(a-b)\cos \frac{C}{2} = c \sin \frac{A-B}{2}$$ sine rule

5 5. In Δ ABC, a= $\sqrt{3}+1$, c=2, $B=30^\circ$ find 'b'. (use, cosine formula)

8. In any Δ ABC, show that

5 $$(b+c)\sin \frac{A}{2} = a \cos\left(\frac{B-C}{2}\right)$$ sine rule

9. In any Δ ABC, show that (projection formula)

3. $$b\left(a^2+c^2\right)\cos B + c\left(a^2+b^2\right)\cos C + a\left(b^2+c^2\right)\cos A = 3abc$$

10. Show that

abc = 4RΔ (we know that $\Delta = \frac{1}{2}$ bc sin A)

3 11. Find the area of a triangle having, a = 4cm, b = 5cm, c = 7cm

2 12. Find the area of a triangle having a = 4cm, b = 2cm, $C = 30^\circ$

Area of Δgle

SOLUTION OF TRIANGLES

Any triangle ABC has six, elements, the three angles A, B and C and the three sides a, b and c opposite to the three angles A, B and C respectively. If we are given any three independent elements, we can find the remaining three. This process is known as solution of a triangle.

We can see some of the cases where independent elements are given.

i) **CASE (1) Given the three sides.**

Suppose a, b and c are given. We have to find out A, B adn C. For this we shall use the cosine rule.

ie; $\cos A = \frac{b^2+c^2-a^2}{2bc} \Rightarrow A = \cos^{-1}\left[\frac{b^2+c^2-a^2}{2bc}\right]$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} \Rightarrow B = \cos^{-1}\left[\frac{a^2+c^2-b^2}{2ac}\right]$$

$$C = 180 - (A–B)$$

Qn. 10. Solve ΔABC. Given a = 4 cm, b= 5 cm, and c= 7 cm.

Sol: $A = \cos^{-1}\left[\frac{b^2+c^2-a^2}{2bc}\right] = \cos^{-1}\left[\frac{5^2+7^2-4^2}{2\times5\times7}\right] = \cos^{-1}\left(\frac{58}{70}\right)$

$$= \cos^{-1}(.8286) = 34^\circ\, 03'$$

$$B = \cos^{-1}\left[\frac{a^2+c^2-b^2}{2ac}\right] = \cos^{-1}\left[\frac{4^2+7^2-5^2}{2\times4\times7}\right]$$

$$= \cos^{-1}\left(\frac{40}{56}\right)$$

$$\cos^{-1}(.7143) = 44^\circ\, 25'$$

$$C = 180-(A+B) = 180\left(34^\circ\, 03' + 44^\circ\, 25'\right)$$

$$= 180 - \left(78^\circ\, 28'\right)$$

$$= 179^\circ\, 60' - 78^\circ\, 28' = 101^\circ\, 32'$$

ii) **CASE (2) Given two sides and included angle**

In ΔABC, suppose that the side a, b and angle C are given. The unknowns are A, B and c. For evaluating these elements, use **Napier's formula** and **Sine rule** together.

Napier's formula states that, in ΔABC, $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot C/2$

$$A - B = 2\tan^{-1}\left(\frac{a-b}{a+b}\cot C/2\right) \text{ .... (1)}$$

and $A + B = (180 - C)$ .... (2)

Solving the above equations (1) and(2) we can find A and B.

The side (c) can be obtained by using the sine rule.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Qn. 11. Solve ΔABC, Given a=87 cm, b = 53 cm. C=70°

Sol: $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot C/2$

$$A - B = 2\tan^{-1}\frac{a-b}{a+b}\cot C/2$$

$$A - B = 2\tan^{-1}\left[\frac{87-53}{87+53}\right]\cot 70/2$$

$$= 2\tan^{-1}\left[\frac{34}{140}\cot 35\right]$$

$$= 2\tan^{-1}(.3469) = 2\times 19^{\circ}\,08' = 38^{\circ}\,16'$$

$A + B = 180 - 70 = 110°$

$2A = 148°\ \ 16'$

$$A = \frac{148^{\circ}\,16'}{2} = 74^{0}\,08'$$

A + B = 110

B = 110 – A = 110 – 74°08′ =35°52′

Now we have to find c

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{87}{\sin 74^\circ 08'} = \frac{c}{\sin 70}$$

$$c = \frac{\sin 70 \times 87}{\sin 74^\circ 08'}$$

$$= 84.99 \text{ cm}$$

Qn.12. In ΔABC, Given a = 18 m, b = 13 m, C= $73^\circ\ 23'$. Find 'A'.

Sol: We know that $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot C/2$

$$= \frac{18-13}{18+13}\cot\left(\frac{73^\circ 23'}{2}\right)$$

$$= \frac{5}{31}\cot\left(36^\circ 42'\right)$$

$$\frac{A-B}{2} = \tan^{-1}\left(\frac{5}{31}\cot 36^\circ 42'\right)$$

$$A - B = 2\times 12^\circ 20' = 24^\circ 40' \text{ .... (1)}$$

$$A + B = 180 - C = 106^\circ 37' \text{ ....(2)}$$

Adding (1) and (2)

$$2A = 131^{\circ 2} 17'; \quad A = \frac{131^\circ 17'}{2} = 65^\circ 39'$$

Note: If the sides are very small quanatities we shall use cosine rule and sine rule together to get the unknown elements)

eg: If b = 2 cm, c = 3 cm, A= 60°

The unknown elements are a, B and C

By using cosine rule,

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$= 2^2 + 3^2 - 2\times 2\times 3\cos 60$$

$$= 4 + 9 - 12\frac{1}{2}$$

$$= 13 - 6 = 7 \qquad \therefore a = \sqrt{7}$$

By using the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{\sqrt{7}}{\sin 60} = \frac{2}{\sin B}$$

$$\sin B = \frac{2\sin 60}{\sqrt{7}}, \quad B = \sin^{-1}\left[\frac{2\sin 60}{\sqrt{7}}\right]$$

$$\therefore B = 40^o\,53'$$

$$C = 180 - (A + B)$$

$$= 180 - (60 + 40^o\,53') = 79^o\,07'$$

iii) **CASE (3) Given any one side and two angles**

Let 'a' be the given side and B and C be the angles

The unknowns are A, b and c

A = 180 – (B+C).

The sides 'b' and 'c' can be obtained by the Sine rule.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \text{ and } \frac{a}{\sin A} = \frac{c}{\sin C}$$

Qn. 13. In a triangle ABC, $C = 38^o 20'$, $B = 45^o$, b = 64 cm. Find the unknown elements.

Sol: the unknowns are A, a, and c

A = 180–(B+C)

$$= 180 - (38^o 20' + 45^o)$$

$$= 96^o 40'$$

We have, $\frac{a}{\sin A} = \frac{b}{\sin B}$

$$\therefore \frac{a}{\sin 96^o 40'} = \frac{64}{\sin 45} \qquad \text{ie; } a = \frac{64.\sin 96^o 40'}{\sin 45^o}$$

89.89 cm

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{64}{\sin 45} = \frac{c}{\sin 38^\circ 20'}$$

$$\therefore c = \frac{64 \times \sin 38^\circ 20'}{\sin 45} = 56.13$$

Qn. 14. Solve ΔABC, given $A = 35^\circ$, $B = 68^\circ$, $c = 25$ cm

Sol: We have $C = 180 - (A + B)$

$= 180 - (35+68) = 77^\circ$

Also $\frac{a}{\sin A} = \frac{c}{\sin C}$

$$\frac{a}{\sin 35} = \frac{25}{\sin 77} \qquad \therefore a = \frac{25 \sin 35}{\sin 77} = 14.71$$

and $\frac{b}{\sin B} = \frac{c}{\sin C}$

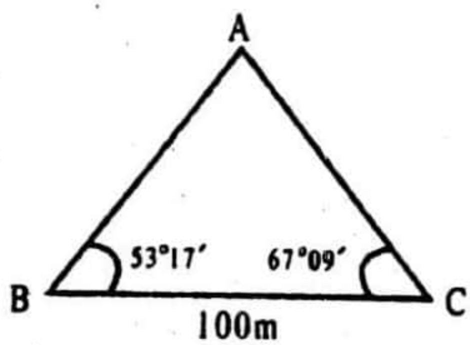
$$\frac{b}{\sin 68} = \frac{25}{\sin 77}$$

$$b = \frac{25 \times \sin 68}{\sin 77} = 23.78$$

Qn. 15. Two angles of a triangular plot of land are $53^\circ 17'$ and $67^\circ 09'$ and the side between them is measured to be 100 cm. How many metres of fencing is required to fence the plot

Sol: $A = 180 - \left(53^\circ 17' + 67^\circ 09'\right)$

$= 180 - 120^\circ 26' = 59^\circ 34'$



We have to find side 'b' and 'c'

for that, $\frac{a}{\sin A} = \frac{b}{\sin B}$

$$\frac{100}{\sin 59^\circ 34'} = \frac{b}{\sin 53^\circ 17'}$$

$$\therefore b = \frac{100 \times \sin 53^{\circ}17'}{\sin 59^{\circ}34'} = 92.97'$$

Similarly $\frac{a}{\sin A} = \frac{c}{\sin C}$

$$\frac{100}{\sin 59^{\circ}34'} = \frac{c}{\sin 67^{\circ}09'}$$

$$\therefore c = \frac{100 \times \sin 67^{\circ}09'}{\sin 59^{\circ}34'}$$

$= 106.88$

$a + b + c = 100 + 92.97 + 106.88$

$= 299.85$ m

$\therefore$ The length of the fencing = 299.85 m.

Things to remember

1. If a, b and c are given, A, B and C can be obtained by the equations of cos A, cos B or cos C.

2. In case (2), if two sides and included angle are given, we shall use any of the three equations of Napier's formula and sine rule together to find the solution.

   ie; $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot C/2$

   $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot A/2$ $\quad \tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot B/2$

3. If two angles and any one side are given, first we can find the third angle, and the sides are obtained by the sine rule.

EXERCISE 2(d)

I Solve ΔABC, Given

1) $a=\sqrt{2}$, $b=\sqrt{3}$, $c=\sqrt{5}$
2) $a = 22.3$, $b = 16.9$, $c = 12.5$
3) $a = 2$, $b = 3$, $c = 4$
4) $a = 15$, $b = 21.5$, $c = 27.5$

II 1) Solve ΔPQR if $P = 31.9$ m, $q = 56.31$ m and $4 = 40.27$ m.

2) Find the greatest angle of ΔXYZ if $x = 51$, $y = 81$, $z = 108$

(Hint: the greatest angle is the angle opposite to the greatest side)

3) Find the smallest angle of ΔABC if $a = 2$ cm, $b = 3$ cm, $c = 4$ cm

(The least angle is the angle opposite to the smallest side)

III Solve ΔABC

1) Given $a = 87$, $b = 53$, $C = 110°$
2) Given $a = 3$ cm, $b = 7$ cm, $C = 38°$

(Hint: take $\tan\frac{B-A}{2}=\frac{b-a}{b+a}\cot\frac{C}{2}$ for avoiding – sign)

3) Given $a = 5$ cm, $c = 8$ cm, $B = 30°$
4) Given $b = 17.5$ m, $c = 12.5$ m, $A = 73°40'$
5) Given $b=\sqrt{3}$, $c=1$, $A = 30°$ (use cosine rule and sine rule)

IV 1) Solve ΔABC given $B = 60°$, $A = 30°$, $c = 13$ cm

2) Solve ΔABC given $A = 80°$, $B = 53°$, $a = 152$ cm

3) Solve ΔABC given $B = 88°36'$, $C = 31°54'$, $a = 63$

4) 'A' and 'B' are two points on opposite banks of a river. From A, a line AC = 275 ft. is laid off and the angles $CAB = 125°40'$ and $ACB = 48°50'$ are measured. Find the length of AB

Module - III

DIFFERENTIAL CALCULUS

3.1 LIMITS

Calculus is a branch of mathematics which deals with the science of calculations; especially the calculations involving very small numbers. In calculus a new operation that of taking a limit is introduced. Differential calculus is based on the limit of a certain ratio and Integral calculus is based on the limit of a certain sum.

Variables, Constants and Functions

A variable is a varying quantity, whose value changes during any mathematical investigation. ie; The value of a variable changes from time to time.

For example, the angle between the hour hand and minute hand of a working clock, the atmospheric temperature, the length of the shadow of a fixed pole etc. are variables. We can find different variables in our day to day life.

Variables are represented by the symbols x, y, z, s, t etc.

Dependent and Independent variables

A variable whose value is chosen arbitrarily is called independent variable. A dependent variable depends on independent variable. For each value of the independent variable say 'x', we can find a definite value for the dependent variable say 'y'.

Consider the equation $y = x^3$. The value of 'x' can be chosen arbitrarily so that for each value of 'x', there may be a definite 'y'. 'x' is the independent variable and 'y' is the dependent variable, since it depends on 'x'

x	0	1	2	3	4
y	0	1	8	27	64

Constant is a fixed quantity whose value remains unchanged through out a mathematical investigation.

$\pi = 3.14$ or $22/7$, $e = 2.71828$, the sum of the three angles of a triangle etc. are examples of constants.

The constants are usually represented by k, a, b, c etc.

Functions

If a variable 'y' depends on a variable 'x' in such a way that each value of 'x' determines exactly one value of 'y' then we say that 'y' is a function of 'x'. This can be written as y=f(x)

Consider y = sin x

x	0	90	180	270	360
y	0	1	0	-1	0

'x' can attain any value and is the independent variable and 'y' depends on 'x'.

A function 'f' is a rule that gives a unique output with each input. If the input is denoted by 'x', the output is denoted by f(x)

Remark: A function can't attain two or more different outputs to the same input

Functions of the form y = f(x) are called explicit funtions

$y = e^x$, $y = x^2+x+1$, $y = \cos x$ etc. are examples of explicit functions.

If a function in 'x' and 'y' is given in such a way that 'x' and 'y' can not be seperated easily, then the function is called implicit function.

For example, $x^2+y^2=25$, $x^2+xy+y^2=0$, $x^3+y^3+3xy=15$, $x^{2/3} + y^{2/3} = 1$ etc. are implicit functions in 'x' and 'y'

Parametric function

If two variables 'x' and 'y' are expressed in terms of a third variable, (θ or t), then the third variable is called the parameter and the function containing the parameter is known as parametric function

eg: 1) $x = at^2$, $y = 2at$
(represents a parametric function with parameter 't')

2) $x = a\cos\theta$, $y = b\sin\theta$
(represents a parametric function with parameter θ)

3) $x = t^2+1$, $y = t^3$
(represents a parametric function with parameter is 't')

Limits (The concept of x tends to a)

Consider a variable 'x' taking an infinite number of values according to a certain rule, As 'x' approaches 'a' after some time, we say that 'x' tends to 'a' or Limit of 'x' is equal to 'a.'

Symbolically , Lt x = a

for example If, x : 2.1 2.3 2.5 2.7 2.9 2.91 2.99 2.999

x approaches '3' ie; Lt x = 3

Limits of a function

Let us consider the sequence of numbers 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ ...... $\frac{1}{x}$. As 'x' increases, the value of $\frac{1}{x}$ becomes smaller and smaller. The value of $\frac{1}{x}$ will never become equal to zero, however large the value of 'x' may be. As the number of terms of the sequence increases indefinitely, the value approaches to zero. ie; $\lim\limits_{x \to \infty}\left[\frac{1}{x}\right] = 0$, ie $\frac{1}{x}$ approaches a value nearer and nearer to zero.

At the same time consider the sequence of numbers 1, $\frac{1}{.01}$, $\frac{1}{.001}$, $\frac{1}{.0001}$, $\frac{1}{.00001}$, ... $\frac{1}{x}$. As 'x' decreases, the value of $\frac{1}{x}$ becomes larger and larger and when the number of terms of the sequence increases indefinitely, the value approaches to infinity. ie; $\lim\limits_{x \to 0}\left[\frac{1}{x}\right] = \infty$

Let us see the function $\left[\frac{x^3 - 27}{x - 3}\right]$. When we put x = 3 the function is undefined; ∴ We have no idea about the function at x = 3. Now it is essential to study the function at the neighbourhood of x = 3. There comes the concept of limits.

For the function y = f(x), we can calculate the values of y corresponding to the values of 'x'. As the values of 'x' approaches the value 'a', the corresponding values of 'y' may approach another value. In this case we say that f(x) approaches a limit.

The necessary and sufficient condition for f(x) tending a limit 'ℓ' is that, given any positive number 'ε' however small, we can always find a corresponding positive number 'δ' such that $|f(x) - \ell| < \varepsilon$ when $|x - a| < \delta$

We express this as $\lim_{x \to a} f(x) = \ell$

Now let us see a function $\frac{x^2-4}{x-2}$

x :	2.5	2.4	2.3	2.2	2.1	2.05	2.04	2.02	2.01	2.099
y :	4.5	4.4	4.3	4.2	4.1	4.05	4.04	4.02	4.01	4.009

As 'x' approaches 2, 'y' approaches 4 ie; $\lim_{x \to 2}\left[\frac{x^2-4}{x-2}\right] = 4$.

Here we can see that as the difference between x and 2 is little, the difference between y and 4 also becomes little.

Properties of limits

If u, v are any two functions:

1) Lt (u+v) = Lt u+Lt v
2) Lt (u−v) = Lt u − Lt v
3) Lt (uv) = Lt u . Lt v
4) $\text{Lt}\left(\frac{u}{v}\right) = \frac{\text{Lt } u}{\text{Lt } v}$ where $\text{Lt } v \neq 0$

Now let us study a function, $y = \frac{1}{x}$

x :	10	100	1000	10000	100000	1000000	etc.
$\frac{1}{x}$	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10000}$	$\frac{1}{100000}$	$\frac{1}{1000000}$	etc.

As $x \to \infty$

$\frac{1}{x} \to 0$ (a negligible quantity)

This can be written as $\lim_{x \to \infty}\left(\frac{1}{x}\right) = 0$

The limits of the functions in the case of infinity can be obtained by taking the highest power of 'x' outside of the brackets and putting the term consisting of $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$ etc. 'O' *eg:* Calculate $\lim_{x\to\infty}\left(\dfrac{x^2+2x+1}{x^2+x-3}\right)$

$$\lim_{x\to\infty}\frac{x^2\left(1+\frac{2x}{x^2}+\frac{1}{x^2}\right)}{x^2\left(1+\frac{x}{x^2}-\frac{3}{x^2}\right)}$$

$$\lim_{x\to\infty}\left(\frac{1+\frac{2}{x}+\frac{1}{x^2}}{1+\frac{1}{x}-\frac{3}{x^2}}\right)=1 \text{ since } \frac{1}{x},\frac{1}{x^2}\to 0$$

Qn. 1. Evaluate $\lim_{x\to\infty}\left(\dfrac{x^2-2x+8}{4x^3-3}\right)$

Sol: $\lim_{x\to\infty}\left(\dfrac{x^2-2x+8}{4x^3-3}\right)$

$$=\lim_{x\to\infty}\frac{x^2\left(1-\frac{2x}{x^2}+\frac{8}{x^2}\right)}{x^3\left(4-\frac{3}{x^3}\right)}=\lim_{x\to\infty}\frac{1}{x}\left(\frac{1-\frac{2}{x}+\frac{8}{x^2}}{4-\frac{3}{x^3}}\right)$$

As $x\to\infty$, $\frac{1}{x}\to 0$, $\frac{1}{x^2}\to 0$, $\frac{1}{x^3}\to 0$

ie; $\lim_{x\to\infty}\frac{1}{x}\left(\dfrac{1-\frac{2}{x}+\frac{8}{x^2}}{4-\frac{3}{x^3}}\right)=0$ (as there is a '$1/x$' in the outside of the bracket, the whole function tends to zero)

Qn. 2. Evaluate $\lim_{x\to\infty}\left(\dfrac{3x^2+5}{x^2-2}\right)$

Sol: $\lim_{x\to\infty}\left[\dfrac{3x^2+5}{x^2-2}\right]=\lim_{x\to\infty}\dfrac{x^2\left(3+5/x^2\right)}{x^2\left(1-2/x^2\right)}=\lim_{x\to\infty}\left(\dfrac{3+5/x^2}{1-2/x^2}\right)=3$

($\because \frac{1}{x^2}\to 0$ as $x\to\infty$)

More about Limits

Now let us see the function $\lim_{x \to 2} \frac{x^2-4}{x-2}$ again.

Put x =2 + h in the function

where, 'h' is a very small quantity

$$\frac{x^2-4}{x-2} = \frac{(2+h)^2-4}{(2+h)-2}$$

$$= \frac{4+4h+h^2-4}{h} = \frac{h^2+4h}{h}$$

$$\frac{h(h+4)}{h} = h+4$$

As x tends to '2', h tends to '0'

$$\text{ie; } \lim_{x \to 2}\left[\frac{x^2-4}{x-2}\right] = 4$$

Thus we have obtained the limit of the function $\frac{x^2-4}{x-2}$ as $x \to 2$ in two ways (see previous pages). Now we can see some general methods used to calculate different types of limits

Finding limits by simple substitution

Let us see a function x^2-3x+4. Suppose we have to see limit as $x \to 3$.

$$\lim_{x \to 3}\left(x^2-3x+4\right) = 3^2-3\times3+4 = 4$$

(Here the value of the function at x=3 is same as the limit)

eg: 1) Calculate $\lim_{x \to 0}\left(\frac{ax+b}{cx+d}\right)$

$$= \left(\frac{a.0+b}{c.0+d}\right) = \frac{b}{d}$$

2) Calculate $\lim_{x \to 1}\left(\frac{2x+3}{4x-1}\right)$

$$= \frac{2\times1+3}{4\times1-1} = \frac{5}{3}$$

Finding Limits by factorization, simplification and substitution

We have learnt to calculate the limit by substitution. Now let us see the limit

$$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)=\lim_{x\to 2}\frac{(x+2)(x-2)}{(x-2)}$$

$$=\lim_{x\to 2}(x+2)=2+2=4$$

ie; This value is similar to the previous value, that we have calculated in the earlier pages.

Qn. 3. Evaluate $\lim_{x\to 1}\left[\frac{x^2+4x-5}{x^2+x-2}\right]$

Sol: $\lim_{x\to 1}\frac{(x+5)(x-1)}{(x+2)(x-1)}$

$$=\lim_{x\to 1}\frac{(x+5)}{(x+2)}=\frac{1+5}{1+2}=2$$

Qn. 4. Evaluate $\lim_{x\to 2}\left[\frac{x^2-5x+6}{x^2+x-6}\right]$

Sol: $\lim_{x\to 2}\frac{(x-3)(x-2)}{(x+3)(x-2)}=\lim_{x\to 2}\left(\frac{x-3}{x+3}\right)$

$$=\frac{2-3}{2+3}=\frac{-1}{5}$$

Qn. 5. Evaluate $\lim_{x\to 1}\left[\frac{x^2+x-2}{x^2+2x-3}\right]$

Sol: $=\lim_{x\to 1}\frac{(x+2).(x-1)}{(x+3).(x-1)}=\lim_{x\to 1}\left(\frac{x+2}{x+3}\right)=\frac{1+2}{1+3}=\frac{3}{4}$

Qn. 6. Evaluate $\lim_{x\to -6}\left[\frac{x^2+5x-6}{x+6}\right]$

Sol: $\lim_{x\to -6}\left[\frac{x^2+5x-6}{x+6}\right]$

$$=\lim_{x\to -6}\frac{(x+6)(x-1)}{(x+6)}=\lim_{x\to -6}(x-1)=-6-1=-7$$

Algebraical Limit

$$\lim_{x \to a}\left(\frac{x^n - a^n}{x - a}\right) = na^{n-1}$$ for all rational values of 'n'

Proof: Put x=a+h As $x \to a$, $h \to 0$

$$\frac{x^n - a^n}{x - a} = \frac{(a+h)^n - a^n}{a + h - a} = \frac{(a+h)^n - a^n}{h}$$

$$= \frac{a^n + nC_1a^{n-1}h + nC_2a^{n-2}h^2 + ... + h^n - a^n}{h}$$

Using the binomial expression of $(x+a)^n$

$$= \frac{nC_1a^{n-1}h + nC_2a^{n-2}h^2 + ...}{h}$$

$$= \frac{h\left[nC_1a^{n-1} + nC_2a^{n-2}h + nC_3a^{n-3}h^2....\right]}{h}$$

$$= nC_1a^{n-1} + nC_2a^{n-2}h + \infty$$

$$\lim_{x \to a}\left(\frac{x^n - a^n}{x - a}\right)$$

$$= \lim_{h \to 0}\left[nC_1a^{n-1} + nC_2a^{n-2}h + ...\right]$$

$$= na^{n-1} \qquad [\because nC_1 = n \ \text{since } h \to 0]$$

$$\boxed{\text{ie; } \lim_{x \to a}\left(\frac{x^n - a^n}{x - a}\right) = na^{n-1}}$$

eg: 1) Evaluate $\lim_{x \to 2}\left[\frac{x^4 - 16}{x - 2}\right]$

$$\lim_{x \to 2}\left[\frac{x^4 - 2^4}{x - 2}\right] = 4.2^{4-1} = 4.2^3 = 32 \qquad (\because n=4, a=2)$$

Qn. 7. Evaluate $\lim_{x\to 3}\left[\frac{x^3-27}{x^2-9}\right]$

Sol: The limit is not algebraical. We have to change it into this form. For this, divide the numerator and denominator by (x–3).

$$\text{ie; } \lim_{n\to 3}\left(\frac{\frac{x^3-27}{x-3}}{\frac{x^2-9}{x-3}}\right)$$

$$=\lim_{x\to 3}\left(\frac{x^3-27}{x-3}\right)\div\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)$$

$$=\lim_{x\to 3}\left(\frac{x^3-3^3}{x-3}\right)\div\lim_{x\to 3}\left(\frac{x^2-3^2}{x-3}\right)$$

$$=3\times 3^{3-1}\div 2\times 3^{2-1}$$

$$=3\times 3^2\div 2\times 3=\frac{27}{6}=\frac{9}{2}$$

Qn. 8. Evaluate $\lim_{x\to 4}\left(\frac{x^3-64}{x^2-16}\right)$

Sol: $\lim_{x\to 4}\left(\frac{x^3-64}{x^2-16}\right)=\lim_{x\to 4}\left(\frac{x^3-4^3}{x^2-4^2}\right)$

$$=\lim_{x\to 4}\frac{\left(\frac{x^3-4^3}{x-4}\right)}{\left(\frac{x^2-4^2}{x-4}\right)} \quad \text{[dividing by (x–4)]}$$

$$\lim_{x\to 4}\left(\frac{x^3-4^3}{x-4}\right)\div\lim_{x\to 4}\left(\frac{x^2-4^2}{x-4}\right)$$

$$=3\times 4^{3-1}\div 2\times 4^{2-1}$$

$$=3\times 4^2\div 2\times 4=\frac{3.4^2}{2\times 4}=6$$

Qn. 9. Evaluate $\lim\limits_{x\to 2}\left[\dfrac{x\sqrt{x}-2\sqrt{2}}{x-2}\right]$

Sol: $\lim\limits_{x\to 2}\left[\dfrac{x\sqrt{x}-2\sqrt{2}}{x-2}\right]=\lim\limits_{x\to 2}\left[\dfrac{x^1.x^{1/2}-2^1.2^{1/2}}{x-2}\right]$

$\lim\limits_{x\to 2}\left[\dfrac{x^{3/2}-2^{3/2}}{x-2}\right]\left(\because x^m.x^n=x^{m+n}\right)$

$=\dfrac{3}{2}\times 2^{3/2-1}=\dfrac{3}{2}\times 2^{1/2}=\dfrac{3}{\sqrt{2}}$

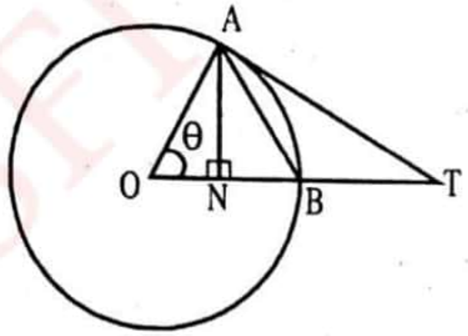
Qn. 10. Evaluate $\lim\limits_{t\to 0}\left[\dfrac{(2+t)^2-4}{t}\right]$

Sol: $=\lim\limits_{2+t\to 2}\left[\dfrac{(2+t)^2-2^2}{2+t-2}\right]=2\times 2^{2-1}=2\times 2=4$ (Using algebraical limit)

Trigonometrical Limit

$\lim\limits_{\theta\to 0}\left(\dfrac{\sin\theta}{\theta}\right)=1$ (θ is in radians)

Proof : Let AB be an arc of a circle of radius 'r', whose radian measure is 'θ' ie; $\angle BOA=\theta$. Let $\overline{AT}$ be a tangent to the circle which cuts $\overline{OB}$ at 'T' externally. Join 'A' and 'B'. It is clear that



Area of ΔAOB < Area of sector AOB < Area ΔAOT

Area of ΔAOB

$=\dfrac{1}{2}\times \text{base}\times\text{altitude}$

$= \frac{1}{2} \times OB \times AN$

$= \frac{1}{2} \times r \times r\sin\theta$ $\left[\sin\theta = \frac{AN}{OA} = \frac{AN}{r}\right]$

$= \frac{1}{2} r^2 \sin\theta$

Area of sector OAB is $\frac{1}{2} r^2\theta$ (area of a sector)

Area of $\Delta AOT = \frac{1}{2} \times \text{base} \times \text{altitude}$

$= \frac{1}{2} \times OT \times AN$

$= \frac{1}{2} \times \frac{r}{\cos\theta} \times r\sin\theta$ $(AN = r\sin\theta, \text{ from } \Delta AON)$

From ΔAOT , $\angle A = 90^\circ$ and $\cos\theta = \frac{OA}{OT} = \frac{r}{OT}$

$= \frac{1}{2} r^2 \frac{\sin\theta}{\cos\theta}$

From the previous relation,

$\frac{1}{2} r^2 \sin\theta < \frac{1}{2} r^2\theta < \frac{1}{2} r^2 \frac{\sin\theta}{\cos\theta}$

$\sin\theta < \theta < \frac{\sin\theta}{\cos\theta}$

$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

$1 > \frac{\sin\theta}{\theta} > \cos\theta$

ie; $\frac{\sin\theta}{\theta}$ lies between 1 and $\cos\theta$

But as $\theta \to 0$ $\cos\theta \to 1$

ie; It is clear that

$\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1$ **(Proof is excluded from the syllabus)**

Qn. 11. Evaluate $\lim_{\theta\to 0}\left(\frac{\sin m\theta}{\sin n\theta}\right)$

Sol: $$\lim_{\theta\to 0}\left(\frac{\sin m\theta}{\sin n\theta}\right) = \lim_{\theta\to 0}\left(\frac{\sin m\theta}{\theta}\right)+\lim_{\theta\to 0}\left(\frac{\sin n\theta}{\theta}\right)$$

$$=\lim_{\theta\to 0}\left(\frac{\sin m\theta}{m\theta}\right)m+\lim_{\theta\to 0}\left(\frac{\sin n\theta}{n\theta}\right)n$$

$$=m\lim_{\theta\to 0}\left(\frac{\sin m\theta}{m\theta}\right)+n\lim_{\theta\to 0}\left(\frac{\sin n\theta}{n\theta}\right)$$

$$=m\times 1+n\times 1=\frac{m}{n}$$

Qn. 12. Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)$

Sol: $$\tan x=\frac{\sin x}{\cos x}$$

$$\therefore \lim_{x\to 0}\left(\frac{\tan x}{x}\right)=\lim_{x\to 0}\left(\frac{\frac{\sin x}{\cos x}}{x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{x.\cos x}\right)=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\times\frac{1}{\cos x}$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\times\lim_{x\to 0}\left(\frac{1}{\cos x}\right)=1\times\frac{1}{\cos x}=1 \quad (\because \cos 0=1)$$

Qn. 13. Evaluate $=\lim_{x\to \pi/2}\frac{(\cos\theta)}{\left(\pi/2-\theta\right)}$

Sol: We know that, $\cos\theta=\sin(90-\theta)=\sin\left(\pi/2-\theta\right)$

(from reduction formulae)

ie; $$\lim_{\theta\to \pi/2}\left(\frac{\cos\theta}{\pi/2-\theta}\right)=\lim_{\theta\to \pi/2}\frac{\sin\left(\pi/2-\theta\right)}{\left(\pi/2-\theta\right)}=\lim_{\pi/2-\theta\to 0}\left[\frac{\sin\left(\pi/2-\theta\right)}{\left(\pi/2-\theta\right)}\right]=1$$

(by trigonometrical limit)

Qn. 14. Evaluate $\lim_{\theta\to 0}\left[\dfrac{\sin 4\theta+\sin 2\theta}{6\theta}\right]$

Sol: $\lim_{\theta\to 0}\left[\dfrac{\sin 4\theta}{6\theta}+\dfrac{\sin 2\theta}{6\theta}\right]$

$$=\frac{1}{6}\lim_{\theta\to 0}\left[\frac{\sin 4\theta}{\theta}+\frac{\sin 2\theta}{\theta}\right]$$

$$=\frac{1}{6}\lim_{\theta\to 0}\left[\frac{\sin 4\theta.4}{4\theta}+\frac{\sin 2\theta.2}{2\theta}\right]$$

$$=\frac{1}{6}\left[4\underset{4\theta\to 0}{\mathrm{Lt}}\left(\frac{\sin 4\theta}{4\theta}\right)+2\underset{2\theta\to 0}{\mathrm{Lt}}\left(\frac{\sin 2\theta}{2\theta}\right)\right]$$

$$=\frac{1}{6}.[4\times 1+2\times 1]=1$$

Note : In the above case we shall also use the product formula

$$\sin C+\sin D=2\sin\frac{C+D}{2}.\cos\frac{C-D}{2}$$

Qn. 15. Evaluate $\lim_{x\to 0}\left(\dfrac{1-\cos 2x}{x^2}\right)$

Sol: We know that $1-\cos 2x=2\sin^2 x$ from 'Trigonometry'

ie; $\lim_{x\to 0}\left(\dfrac{1-\cos 2x}{x^2}\right)=\lim_{x\to 0}\left(\dfrac{2\sin^2 x}{x^2}\right)$

$$=2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{x}\right)$$

$$=2\times\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\times\lim_{x\to 0}\left(\frac{\sin x}{x}\right)$$

$$=2\times 1\times 1=2$$

Qn. 16. Evaluate $\lim_{\theta\to 0}\left(\dfrac{\sin 3\theta.\cos\theta}{\theta}\right)$

Sol: $\lim_{\theta\to 0}\left(\dfrac{\sin 3\theta.\cos\theta}{\theta}\right)=\lim_{\theta\to 0}\left(\dfrac{\sin 3\theta}{3\theta}.3\cos\theta\right)$

(Multiply '3' on Nr and Dr)

$$= \lim_{\theta\to 0}\left(\frac{3\sin 3\theta}{3\theta}\right)\times \lim_{\theta\to 0}(\cos\theta)$$

$$= 3\underset{3\theta\to 0}{\mathrm{Lt}}\left(\frac{\sin 3\theta}{3\theta}\right)\times \underset{\theta\to 0}{\mathrm{Lt}}(\cos\theta)$$

$$= 3\times 1\times \cos 0 = 3 \quad (\because \cos 0 = 1)$$

Things to remember

Procedure for calculating limits:

If the limit of a function is given in the form $\lim_{x\to a}\frac{x^n - a^n}{x-a}$, we shall apply the value na^{n-1} to get the limit. Some times we have to change a function into this form by certain procedure.

If a limit is given as $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$, the value is always '1'. Sometimes we have to change a function into this form by certain steps.

If a variable x tends to '∞' is given, the limit of the function is obtained by taking **the highest power of 'x' outside** the brackets, cancelling common terms and applying the rule ie; as $x\to\infty$, $1/x \to 0$.

(If we get an indeterminate $\left(\frac{0}{0}\right)$ form this method is not suitable). In such cases factorize the numerator and denominator, cancel the common terms and substitute the limit of the variable to get the limit.

Continuity of a function

We have already learnt about limits. From the concept of limits we shall define an important property of functions called "continuity".

A function is said to be continuous if its graph is a continuous curve. Whenever there is a break or jump from one point to another on a curve, the function is discontinuous (not continuous) ie; A function 'f(x)' is said to be continuous at a point x = a if $\lim_{x\to a} f(x) = f(a)$. A function is said to be continuous in an interval if it is continuous at each point of an interval.

EXERCISE 3(a)

Evaluate the following

1. $\lim\limits_{x \to 2} (2x+3)$

2. $\lim\limits_{x \to 0} \left(\dfrac{cx+d}{ax+b}\right)$

3. $\lim\limits_{x \to 2} \left(\dfrac{x^2-2x}{x-2}\right)$

4. $\lim\limits_{x \to 1} \left(\dfrac{x^2-1}{x^2-x}\right)$ Hint: factorize

5. $\lim\limits_{x \to 3} \left(\dfrac{x3-27}{x-3}\right)$

6. $\lim\limits_{x \to 2} \left(\dfrac{x3-8}{x^2-4}\right)$

7. $\lim\limits_{x \to 4} \left(\dfrac{x3-64}{x^2-16}\right)$

8. $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x}-1}{x}$

9. $\lim\limits_{x \to a} \dfrac{x^m-a^m}{x^n-a^n}$

10. $\lim\limits_{x \to \infty} \left(\dfrac{x^3-2x+3}{2x^3-4x+6}\right)$

11. $\lim\limits_{x \to \infty} \left(\dfrac{x^2-2x+3}{3x^2-2x}\right)$

12. $\lim\limits_{x \to \infty} \left(\dfrac{4x^2-3x+1}{3x^2+x+5}\right)$

13. $\lim\limits_{\theta \to 0} \left(\dfrac{\sin 5\theta}{\theta}\right)$

14. $\lim\limits_{\theta \to 0} \left(\dfrac{\sin 2\theta . \cos\theta}{\theta}\right)$

15. $\lim\limits_{\theta \to 0} \left(\dfrac{\tan 3\theta}{\theta}\right)$ $\left(\text{put } \tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta}\right)$

16. $\lim\limits_{x \to \infty} \left(\dfrac{2x^2+3x}{5x^2+4x+1}\right)$

17. $\lim\limits_{x \to 3} \left(\dfrac{x^2-3x}{x^2-9}\right)$ Hint: factorize

18. $\lim\limits_{x \to \infty} \left(\dfrac{x^2-2x+3}{2x^2-3x}\right)$

19. $\lim\limits_{x \to \infty} \left(\dfrac{2x-3}{x^2-8x+4}\right)$

20. $\lim\limits_{q \to 0} \left(\dfrac{\sin a\theta}{\tan b\theta}\right)$

21. $\lim\limits_{x \to \infty} \dfrac{(3x+1)(2x+4)}{(x+1)(x-7)}$

3.2

DIFFERENTIATION - I

The differential calculus provides the method of investigating the rate at which one variable changes with respect to the other. In other words, rate of change of one variable with respect to the unit change in the other.

The ratio of differences of varying quantities of a function f(x) would simply provide an average rate. In order to know what happens to the ratio at any point or value or instant, we have frist to take the average rate and then find the limit when the change in the variable 'x' approaching to zero. This limit plays an important role in establishing the ultimate result called derivative (differential coefficient).

Increment and incremental ratio

Let 'y' be a continuous function of 'x'. A small change in the value of 'x' is called an increment in 'x' and it is denoted by Δx (read 'delta x'). This change in 'x' will produce a corresponding change in 'y' denoted by Δy or increment in 'y'.

eg: Consider a function $y = x^2$

If $x = 2$, $y = 4$

If $x = 2.1$ $y=(2.1)^2=4.41$ when the value of 'x' changes from 2 to 2.1. the value of 'y' changes from 4 to 4.41

Remark: Increment may be positive or negative

if $y = f(x)$ is a function of 'x' and the increments in 'x' and 'y' are Δx and Δy respectively, then the ratio of the increments $\frac{\Delta y}{\Delta x}$ is called the incremental ratio.

In the above example $\Delta x = 2.1 - 2 = 0.1$. $\Delta y = 4.41 - 4 = .41$

Differential coefficient (Derivative)

Consider a continuous function $y = f(x)$. Let Δx be a small increment in x and Δy be the corresponding increment in 'y'. **The limit of the incremental ratio as Δx tends to zero is called the differential coefficient of 'y' with respect to 'x'. The differential coefficient or derivative of 'y' with respect to 'x' is denoted by $\frac{dy}{dx}$.**

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{\Delta y}{\Delta x}\right]$$

The process of finding $\frac{dy}{dx}$ is called differentiation.

Remark : $\frac{\Delta y}{\Delta x}$ represents the average rate of change of 'y' w.r.t. 'x'. To know the rate at a point, we have first to take the average rate and then take the limit as the change in the variable x approaching to zero. It is denoted by $\frac{dy}{dx}$.

Let y=f(x) be a continuous function of 'x'. Take Δx as a small increment in 'x' and Δy be the corresponding increment in 'y'.

$$y + \Delta y = f(x + \Delta x)$$

$$\Delta y = f(x + \Delta x) - y$$

$$= f(x + \Delta x) - f(x)$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{\Delta y}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right]$$

$\frac{dy}{dx}$ is the differential coefficient (derivative) of 'y' w.r.t. 'x'

$\frac{dy}{dx} = \frac{d}{dx}(y)$. $\frac{d}{dx}$ is the differential operator of 'y'. The process of finding the differential coefficient by the above method is called differentiation by **first principles.**

Remark: 1) $\frac{dy}{dx} = \frac{d}{dx}(y)$ where $\frac{d}{dx}$ is the differential operator of y

2) Derivative of 'y' w.r.t. 't' is denoted by $\frac{dy}{dt}$ for a function $y = f(t)$

3) Derivative of 'z' w.r.t 'x' is denoted by $\frac{dz}{dx}$ for a function $z = f(x)$

ϕ, f, ψ etc. are using only to represent functions of different types.

Qn. 1. Differentiate x^n by the method of first principles.

Sol: Let Δx be a small increment in 'x' and

Δy be the corresponding increment in 'y'.

$$y = x^n$$

$$y + \Delta y = (x + \Delta x)^n$$

$$\Delta y = (x + \Delta x)^n - y$$

$$= (x + \Delta x)^n - x^n$$

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

$$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left[\frac{(x + \Delta x)^n - x^n}{\Delta x}\right]$$

$$= \lim_{(x+\Delta x) \to x}\left[\frac{(x + \Delta x)^n - x^n}{x + \Delta x - x}\right]$$

$$\therefore \frac{dy}{dx} = nx^{n-1} \qquad \left[\text{by the algebraical limit } \lim_{x \to a}\left(\frac{x^n - a^n}{x - a}\right) = nx^{n-1}\right]$$

Note: We have seen that, $\frac{d}{dx}x^n = nx^{n-1}$

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2}$$

$$= \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

$$\therefore \boxed{\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}}$$

Similarly $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right) = -1x^{-2} = \frac{-1}{x^2}$

We shall find the derivatives of all functions of the above type by using the result, $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

eg: 1) Find the derivatives of the following

a) x b) x^4 c) $\frac{1}{\sqrt{x}}$ d) $\frac{1}{x^3}$ e) $\frac{1}{x^{10}}$ f) 5

a) $\frac{d}{dx}(x)=\frac{d}{dx}\left(x^1\right)=1x^{1-1}=1x^0=1$ ($x^0=1$ always)

b) $\frac{d}{dx}\left(x^4\right)=4x^3$

c) $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)=\frac{d}{dx}\left(x^{\frac{-1}{2}}\right)=\frac{-1}{2}x^{\frac{-1}{2}-1}=\frac{-1}{2}x^{\frac{-3}{2}}$

d) $\frac{d}{dx}\left(\frac{1}{x^3}\right)=\frac{d}{dx}\left(x^{-3}\right)=-3x^{-3-1}=-3x^{-4}$

e) $\frac{d}{dx}\left(\frac{1}{x^{10}}\right)=\frac{d}{dx}\left(x^{-10}\right)=-10x^{-10-1}=-10x^{-11}$

f) $\frac{d}{dx}(5)=\frac{d}{dx}\left(5x^0\right)=5\times 0\times x^{0-1}=0$

$\boxed{\frac{d}{dx}(k)=0}$ where k is a constant.

Rules of differentiation

If u and v are any two functions of 'x'

$$\frac{d}{dx}(u\pm v)=\frac{d}{dx}(u)\pm\frac{d}{dx}(v)$$

$\frac{d}{dx}(ku)=k\frac{du}{dx}$ where 'k is a constant.

eg: 1) $\frac{d}{dx}\left(x^2+2x+1\right)$

$$=\frac{d}{dx}x^2+2\frac{d}{dx}x+\frac{d}{dx}(1)$$

$$=2x^1+2.1+0=2x+2$$

eg: 2) $\frac{d}{dx}\left(3x^6+\frac{1}{x}-5\right)$

$$3\frac{d}{x}x^6+\frac{d}{dx}\left(x^{-1}\right)-\frac{d}{dx}(5)$$

$= 3 \times 6x^5 + -1x^{-1-1} - 0$

$= 18x^5 - 1x^{-2} = 18x^5 - \frac{1}{x^2}$

eg: 3) $\frac{d}{dx}\left(\frac{4}{x^3} + 5x^{10} - \frac{1}{\sqrt{x}}\right)$

$= 4\frac{d}{dx}x^{-3} + 5\frac{d}{dx}x^{10} - \frac{d}{dx}x^{\frac{-1}{2}}$

$= 4 \times -3x^{-3-1} + 5 \times 10 \times x^9 - \frac{-1}{2}x^{\frac{-1}{2}-1}$

$= -12x^{-4} + 50x^9 + \frac{1}{2}x^{\frac{-3}{2}}$

$= \frac{-12}{x^4} + 50x^9 + \frac{1}{2}\frac{1}{x^{\frac{3}{2}}}$

eg: 4) $\frac{d}{dx}(2x+5)$

$= 2\frac{d}{dx}(x) + \frac{d}{dx}5 = 2 \times 1 + 0 = 2$

eg: 5) $\frac{d}{dx}(4x^{10} - 3x^5 - 2)$

$= 4\frac{d}{dx}x^{10} - 3\frac{d}{dx}x^5 - \frac{d}{dx}2$

$= 4 \times 10x^9 - 3 \times 5x^4 - 0 = 40x^9 - 15x^4$

Results

1. Derivative of x^n w.r.t 'x' is nx^{n-1} ie; $\frac{d}{dx}(x^n) = nx^{n-1}$
2. Derivative of x w.r.t 'x' is 1 ie; $\frac{d}{dx}(x) = 1$
3. Derivative of a constant (K) w.r.t any variable is '0'. ie; $\frac{d}{dx}(K) = 0$
4. Derivative of $\sqrt{x}$ w.r.t 'x' is $\frac{1}{2\sqrt{x}}$ ie; $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
5. Derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$ ie; $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

The above results are most frequently in use and remember always.

Qn. 2. Differentiate 'sin x' by the method of first principles.

Sol: Let Δx be a small increment in 'x' and

Δy be the corresponding increment in 'y'.

Let $y = \sin x$

$$y + \Delta y = \sin(x + \Delta x)$$

$$\Delta y = \sin(x + \Delta x) - y$$

$$= \sin(x + \Delta x) - \sin x$$

$$= 2\cos\frac{x + \Delta x + x}{2}.\sin\frac{x + \Delta x - x}{2}$$

$$= 2\cos\frac{2x + \Delta x}{2}.\sin\frac{\Delta x}{2}$$

$$= 2\cos\left(x + \frac{\Delta x}{2}\right).\sin\frac{\Delta x}{2}$$

$$\frac{\Delta y}{\Delta x} = \frac{2\cos\left(x + \frac{\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right)}{\Delta x}$$

$$= \frac{2\cos\left(x + \frac{\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right).\frac{1}{2}}{\frac{\Delta x}{2}}$$ [Multiplying $\frac{1}{2}$ on 'Nr' and 'Dr']

$$= \cos\left(x + \frac{\Delta x}{2}\right).\left(\frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}\right)$$

$$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left[\cos\left(x + \frac{\Delta x}{2}\right).\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right]$$

$$= \lim_{\Delta x \to 0}\cos\left(x + \frac{\Delta x}{2}\right).\lim_{\Delta x \to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$= \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) \times 1 \qquad \left[\text{by} \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= \cos x$$

ie; $\frac{d}{dx}(\sin x) = \cos x$

Qn. 3. Differentiate 'cos x' by the method of first principles.

Sol: Let Δx be a small increment in 'x' and

Δy be the corresponding increment in 'y'.

$$y = \cos x$$

$$y + \Delta y = \cos(x + \Delta x)$$

$$\Delta y = \cos(x + \Delta x) - y$$

$$= \cos(x + \Delta x) - \cos x$$

$$\Delta y = -2\sin\left(\frac{x + \Delta x + x}{2}\right).\sin\left(\frac{x + \Delta x - x}{2}\right)$$

$$\therefore \Delta y = -2\sin\left(\frac{2x + \Delta x}{2}\right).\sin\frac{\Delta x}{2}$$

ie; $\Delta y = -2\sin\left(x + \frac{\Delta x}{2}\right).\sin\frac{\Delta x}{2}$

$$\frac{\Delta y}{\Delta x} = \frac{-2\sin\left(x + \frac{\Delta x}{2}\right).\sin\frac{\Delta x}{2}}{\Delta x}$$

$$= \frac{-2\sin\left(x + \frac{\Delta x}{2}\right).\sin\frac{\Delta x}{2}.\frac{1}{2}}{\frac{\Delta x}{2}}$$

$$= \frac{-\sin\left(x + \frac{\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$$

$$\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right)=\lim_{\Delta x\to 0} -\sin\left(x+\frac{\Delta x}{2}\right).\lim_{\Delta x\to 0}\left(\frac{\sin \Delta x/2}{\Delta x/2}\right)$$

$$=-\sin x\times 1$$

ie; $\frac{dy}{dx}=-\sin x$

ie; $\frac{d}{dx}(\cos x)=-\sin x$

Qn. 4. Differentiate 'tan x' by the method of first principles (The proof is excluded from the syllabus)

Sol: $y=\tan x$

$$y+\Delta y=\tan(x+\Delta x)=\frac{\sin(x+\delta x)}{\cos(x+\delta x)}$$

$$\Delta y=\frac{\sin(x+\Delta x)}{\cos(x+\Delta x)}-y$$

$$=\frac{\sin(x+\Delta x)}{\cos(x+\Delta x)}-\frac{\sin x}{\cos x}$$

$$=\frac{\sin(x+\Delta x).\cos x-\cos(x+\Delta x).\sin x}{\cos(x+\Delta x).\cos x}$$

[$\because$ sinA cosB − cosA sinB = sin (A−B)]

$$=\frac{\sin(x+\Delta x-x)}{\cos(x+\Delta x).\cos x}=\frac{\sin \Delta x}{\cos(x+\Delta x)\cos x}$$

$$\frac{\Delta y}{\Delta x}=\frac{\sin \Delta x}{\Delta x.\cos(x+\Delta x)\cos x}$$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right)=\lim_{\Delta x\to 0}\left[\frac{\sin \Delta x}{\Delta x.\cos(x+\Delta x)\cos x}\right]$$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\left(\frac{\sin \Delta x}{\Delta x}\right).\lim_{\Delta x\to 0}\left(\frac{1}{\cos(x+\Delta x)\cos x}\right)=\frac{1}{\cos^2 x}=\sec^2 x$$

$$\boxed{\frac{d}{dx}\tan x=\sec^2 x}$$

Results

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$

eg: 1) Find the derivative of 5 cos x + 3 tan x w.r.t. 'x'

$$\frac{d}{dx}(5\cos x + 3\tan x)$$

$$= 5\frac{d}{dx}(\cos x) + 3\frac{d}{dx}(\tan x)$$

$$= -5\sin x + 3\sec^2 x = -5\sin x + 3\sec^2 x$$

eg: 2) Find the derivative of $3\sin x + 2\sqrt{x} - \frac{1}{\sqrt{x}}$

$$\frac{d}{dx}\left(3\sin x + 2\sqrt{x} - \frac{1}{\sqrt{x}}\right)$$

$$= \frac{d}{dx}(3\sin x) + \frac{d}{dx}\left(2\sqrt{x}\right) - \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)$$

$$= 3\frac{d}{dx}(\sin x) + 2\frac{d}{dx}\left(\sqrt{x}\right) - \frac{d}{dx}\left(x^{-1/2}\right)$$

$$= 3\cos x + 2\frac{1}{2\sqrt{x}} - \frac{-1}{2}x^{\frac{-1}{2}-1}$$

$$= 3\cos x + \frac{1}{\sqrt{x}} + \frac{1}{2}x^{-3/2}$$

eg: 3) If $y = 3x - 2\sqrt{x} + 5\sec x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x - 2\sqrt{x} + 5\sec x\right)$$

$$= 3 - 2\frac{1}{2\sqrt{x}} + 5\sec x\ \tan x$$

$$= 3 - \frac{1}{\sqrt{x}} + 5\sec x\ \tan x$$

eg: 4) If $y = x^{3/2} - \frac{4}{\sqrt{x}} + 7x^3$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{3/2} - \frac{4}{\sqrt{x}} + 7x^3\right)$$

$$= \frac{d}{dx}\left(x^{3/2}\right) - 4\frac{d}{dx}\left(x^{-1/2}\right) + 7\frac{d}{dx}\left(x^3\right)$$

$$= \frac{3}{2}x^{1/2} - 4 \times \frac{-1}{2} \times x^{-3/2} + 7 \times 3x^2$$

$$= \frac{3}{2}x^{1/2} + 2x^{-3/2} + 21x^2$$

Product Rule

If u and v are any two functions of 'x' then

$$\frac{d}{dx}(u \times v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}(u \times v) = \text{first} \times \text{derivative of second} + \text{second} \times \text{derivative of first}$$

eg: 1) $\frac{d}{dx}\left(x^2 \sin x\right) = x^2 \frac{d}{dx}\sin x + \sin x \frac{d}{x}\left(x^2\right)$

$x^2 \cos x + \sin x.2x$ $\qquad \left(\because \frac{d}{dx}\sin x = \cos x, \frac{d}{dx}x^2 = 2x\right)$

eg : 2) $\frac{d}{dx}\left(\sqrt{x}\tan x\right)$

$$= \sqrt{x}.\frac{d}{dx}(\tan x) + \tan x.\frac{d}{dx}\left(\sqrt{x}\right)$$

$$= \sqrt{x}.\sec^2 x + \tan x.\frac{1}{2\sqrt{x}}$$

eg : 3) $\frac{d}{dx}(\sin x.\cos x) = \sin x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(\sin x)$

$$= \sin x - \sin x + \cos x.\cos x$$

$$= \cos^2 x - \sin^2 x$$

Quotient Rule

If u and v are any two functions of 'x' then

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

eg: 1) $\frac{d}{dx}\left(\frac{x^2}{\sin x}\right)$

$$=\frac{\sin x\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$=\frac{\sin x.2x-x^2\cos x}{\sin^2 x}$$

eg: 2) $\frac{d}{dx}\left(\frac{\cos x}{\sqrt{x}}\right)=\frac{\sqrt{x}\frac{d}{dx}(\cos x)-\cos x\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$

$$=\frac{\sqrt{x}.-\sin x-\cos x\frac{1}{2\sqrt{x}}}{x}$$

3) $\frac{d}{dx}\left(\frac{x-1}{x+1}\right)=\frac{(x+1)\frac{d}{dx}(x-1)-(x-1)\frac{d}{dx}(x+1)}{(x+1)^2}$

$$=\frac{(x+1)1-(x-1)1}{(x+1)^2}=\frac{2}{(x+1)^2}$$

Differential coefficient of cosec x, sec x and cot x

Qn. 5. $\frac{d}{dx}(\text{cosec } x)$

Sol: $=\frac{d}{dx}\left(\frac{1}{\sin x}\right)$ $\qquad \frac{d}{dx}k=0, \frac{d}{dx}1=0$

$$=\frac{\sin x\frac{d}{dx}(1)-1\times\frac{d}{dx}\sin x}{\sin^2 x}$$

$$= \frac{\sin x \times 0 - 1 \times \cos x}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin^2 x} = \frac{-\cos x}{\sin x} \frac{1}{\sin x}$$

$$= -\cot x . \operatorname{cosec} x$$

ie; $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x . \cot x$

Qn. 6. $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$

Sol: $= \dfrac{\cos x \times \frac{d}{dx}(1) - 1 \times \frac{d}{dx}\cos x}{\cos^2 x}$

$$= \frac{\cos x \times 0 - 1 \times -\sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} . \frac{1}{\cos x} = \tan x . \sec x$$

ie; $\frac{d}{dx}(\sec x) = \sec x . \tan x$

Qn. 7. $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$

Sol: $= \dfrac{\sin x \frac{d}{dx} \cos x - \cos x \frac{d}{dx} \sin x}{\sin^2 x}$

$$= \frac{\sin x . -\sin x - \cos x . \cos x}{\sin^2 x}$$

$$= \frac{-\left(\sin^2 x + \cos^2 x\right)}{\sin^2 x} = \frac{-1}{\sin^2 x}$$

ie; $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

Results

1) $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$

2) $\frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}}$

3) $\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{-1}{x^2}$

4) $\frac{d}{dx}(\sin x)=\cos x$

5) $\frac{d}{dx}(\cos x)=-\sin x$

6) $\frac{d}{dx}(\tan x)=\sec^2 x$

7) $\frac{d}{dx}\operatorname{cosec} x=-\operatorname{cosec} x.\cot x$

8) $\frac{d}{dx}\sec x=\sec x.\tan x$

9) $\frac{d}{dx}\cot x=-\operatorname{cosec}^2 x$

Examples

1) Find $\frac{d}{dx}\left(x^2\sec x\right)$

$$\frac{d}{dx}\left(x^2\sec x\right)=x^2\frac{d}{dx}(\sec x)+\sec x\frac{d}{dx}\left(x^2\right)$$

$$=x^2\sec x.\tan x+\sec x.2x$$

2) Find $\frac{d}{dx}\left(\sqrt{x}.\cot x\right)$

$$=\sqrt{x}\frac{d}{dx}(\cot x)+\cot x\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=\sqrt{x}.-\operatorname{cosec}^2 x+\cot x.\frac{1}{2\sqrt{x}}$$

$$=-\sqrt{x}.\operatorname{cosec}^2 x+\frac{\cot x}{2\sqrt{x}}$$

3) Find $\frac{d}{dx}\left(\frac{\sqrt{x}}{\operatorname{cosec} x}\right)$

$$\frac{d}{dx}\left(\frac{\sqrt{x}}{\operatorname{cosec} x}\right)=\frac{\operatorname{cosec} x\frac{d}{dx}\left(\sqrt{x}\right)-\sqrt{x}\frac{d}{dx}(\operatorname{cosec} x)}{\operatorname{cosec}^2 x}$$

$$= \frac{\operatorname{cosec} x \cdot \frac{1}{2\sqrt{x}} - \sqrt{x} \cdot -\operatorname{cosec} x . \cot x}{\operatorname{cosec}^2 x}$$

$$= \frac{\frac{\operatorname{cosec} x}{2\sqrt{x}} + \sqrt{x} . \operatorname{cosec} x . \cot x}{\operatorname{cosec}^2 x}$$

4) Find $\frac{d}{dx}\left(\frac{2x-1}{4x+3}\right)$

$$\frac{d}{dx}\left(\frac{2x-1}{4x+3}\right) = \frac{(4x+3)\frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}(4x+3)}{(4x+3)^2}$$

$$= \frac{(4x+3)2-(2x-1)4}{(4x+3)^2} = \frac{10}{(4x+3)^2}$$

Differential Coefficients of e^x and log x

(The derivation is excluded from the syllabus)

1. Consider the function $y = e^x$

$$y + \Delta y = e^{x+\Delta x}$$

$$\Delta y = e^{x+\Delta x} - y = e^{x+\Delta x} - e^x$$

$$\frac{\Delta y}{\Delta x} = \frac{e^{x+\Delta x} - e^x}{\Delta x}$$

$$= \frac{e^x . e^{\Delta x} - e^x}{\Delta x} = e^x\left(\frac{e^{\Delta x} - 1}{\Delta x}\right)$$

But $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

ie; $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} e^x\left(\frac{e^{\Delta x} - 1}{\Delta x}\right)$

$$= \lim_{\Delta x \to 0} e^x\left(\frac{1 + \frac{\Delta x}{1!} + \frac{(\Delta x)^2}{2!} + \ldots\ldots \infty - 1}{\Delta x}\right)$$

$$= \lim_{\Delta x \to 0} e^x\left(\frac{\Delta x}{\Delta x}+\frac{(\Delta x)^2}{2!\,\Delta x}+,....\infty\right)$$

$$= e^x \lim_{\Delta x \to 0}\left(1+\frac{\Delta x}{2!}+\frac{(\Delta x)^2}{3!}+.....\right)$$

$$= e^x.(1+0)=e^x$$

(Since, terms containing Δx becomes zero)

ie: $\frac{d}{dx}\left(e^x\right)=e^x$

$$\boxed{\frac{d}{dx}\left(e^x\right)=e^x}$$

For the above proof we have applied a result,

ie, $e^{\Delta x}=1+\frac{\Delta x}{1!}+\frac{(\Delta x)^2}{2!}+.......\infty$

2. Consider the function $y = \log_e x$. (The derivation is not expected)

Let $y = \log_e x$

By the definition of logarithm the above equation can be re-written in the form

$x = e^y$ .... (1)

take $\frac{d}{dy}x = \frac{d}{dy}e^y$

$\frac{dx}{dy} = e^y$ $\qquad \left(\because \frac{d}{dx}e^x = e^x\right)$

$\therefore \frac{dy}{dx} = \frac{1}{e^y}$ $\qquad \frac{dy}{dx} = \frac{1}{x}$ from (1)

Therefore, we have $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\boxed{\frac{d}{dx}(\log x) = \frac{1}{x}}$$

Examples

1. $$\frac{d}{dx}\left(e^x \log x\right) = e^x \frac{d}{dx}\log x + \log x \frac{d}{dx} e^x$$
$$= e^x \frac{1}{x} + \log x . e^x$$

2. $$\frac{d}{dx}\left(e^x \sin x\right) = e^x \frac{d}{dx}\sin x + \sin x \frac{d}{dx} e^x$$
$$= e^x . \cos x + \sin x e^x$$

3. $$\frac{d}{dx}\left(\sqrt{x}.\log x\right) = \sqrt{x}.\frac{d}{dx}\log x + \log x \frac{d}{dx}\sqrt{x} = \sqrt{x}\frac{1}{x} + \log x . \frac{1}{2\sqrt{x}}$$

4. $$\frac{d}{dx}\left(e^x . \tan x\right) = e^x \frac{d}{dx}\tan x + \tan x \frac{d}{dx} e^x$$
$$= e^x \sec^2 x + \tan x . e^x$$

5. $$\frac{d}{dx}\left(\frac{\log x}{x}\right) = \frac{x.\frac{1}{x} - \log x . 1}{x^2} = \frac{1 - \log x}{x^2}$$

6. $$\frac{d}{dx}\left(\frac{e^x}{\sqrt{x}}\right) = \frac{\sqrt{x}.e^x - e^x . \frac{1}{2\sqrt{x}}}{\left(\sqrt{x}\right)^2}$$

Things to remember

1. $\frac{d}{dx} x^n = nx^{n-1}$
2. $\frac{d}{dx}\sin x = \cos x$
3. $\frac{d}{dx}\cos x = -\sin x$
4. $\frac{d}{dx}\tan x = \sec^2 x$
5. $\frac{d}{dx}\text{cosec}x = -\text{cosec}x.\cot x$
6. $\frac{d}{dx}\sec x = \sec x . \tan x$
7. $\frac{d}{dx}\cot x = -\text{cosec}^2 x$
8. $\frac{d}{dx}\log x = \frac{1}{x}$
9. $\frac{d}{dx} e^x = e^x$
10. $\frac{d}{dx}(x) = 1$
11. $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$
12. $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$
13. $\frac{d}{dx}(k) = 0$ where 'k' is a constant

14. If u and v are any two functions of 'x'

i) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$ ii) $\frac{d}{dx}(ku) = k\frac{du}{dx}$

iii) $\frac{d}{dx}(u \times v) = u\frac{dv}{dx} + v\frac{du}{dx}$ iv) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Differential coefficients of inverse t-functions

1. Let $y = \sin^{-1} x$ we have to find $\frac{dy}{dx}$

$x = \sin y$ $x = f(y)$ form

differentiating w.r.t 'y', $\therefore \frac{dx}{dy} = \frac{d}{dy}(\sin y) = \cos y$

and $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

2. Let $y = \cos^{-1} x$

$x = \cos y$ $x = f(y)$ form

$$\frac{dx}{dy} = \frac{d}{dy}\cos y = -\sin y$$

$$\frac{dy}{dx} = \frac{1}{-\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$$

3. Let $y = \tan^{-1} x$

$x = \tan y$ $x = f(y)$ form

$$\frac{dx}{dy} = \frac{d}{dy}(\tan y) = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

If $y = \sin^{-1} x$, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

If $y = \cos^{-1} x$, $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$

If $y = \tan^{-1} x$, $\frac{dy}{dx} = \frac{1}{1+x^2}$

Extension of product rule

We know that $\frac{d}{dx}(u \times v) = u.\frac{dv}{dx} + v.\frac{du}{dx}$ if 'u' and 'v' are any two functions of 'x'. Extending this result,

$$\frac{d}{dx}(uvw) = u\frac{d}{dx}(vw) + vw\frac{du}{dx}$$

$$= u\left[v\frac{dw}{dx} + w\frac{dv}{dx}\right] + vw\frac{du}{dx}$$

$$= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

eg : 1) $\frac{d}{dx}\left[x(x+1)(x+2)\right]$

$$= x(x+1)\frac{d}{dx}(x+2) + (x+2)\frac{d}{dx}x(x+1)$$

$$= x(x+1)1 + (x+2)\left[(x)\frac{d}{dx}(x+1) + (x+1)\frac{d}{dx}(x)\right]$$

$$= x(x+1) + (x+2)(x.1 + (x+1)1)$$

$$= x(x+1) + (x+2)(2x+1)$$

also, $\frac{d}{dx}\left(\frac{uv}{w}\right) = \dfrac{w\frac{d}{dx}(uv) - uv\frac{d}{dx}(w)}{w^2}$

eg : 2) $\frac{d}{dx}\left(\frac{x.\sin x}{(x^2+1)}\right) = \dfrac{(x^2+1)\frac{d}{dx}(x\sin x) - x\sin x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$

$$= \frac{(x^2+1)\left[x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x\right] - x\sin x 2x}{(x^2+1)^2}$$

$$= \frac{(x^2+1)[x\cos x + \sin x.1] - x\sin x.2x}{(x^2+1)^2}$$

EXERCISE 3(b)

Find the derivatives of the following w.r.t 'x'

1. $x + \sin x + 5 \tan x$
2. $3 \cos x - 4 \tan x$
3. $2\sqrt{x} - \frac{3}{x} - \frac{1}{\sqrt{x}}$
4. $1 - \frac{1}{x} + \frac{2}{x^2}$
5. $4e^x - 3\operatorname{cosec} x$
6. $\frac{8}{x^2} - 2\cot x$
7. $\frac{\sqrt{x}}{\sin x}$
8. $\frac{\tan x}{x}$
9. $x^3.e^x$
10. $\frac{1-x^2}{1+x^2}$
11. $x^n \log x$
12. $\frac{x^2+x+1}{x^2+2x+3}$
13. $\frac{x+\sqrt{x}}{1+\sqrt{x}}$
14. $\left(5x^2+3x-1\right)\left(3x^2+2\right)$
15. $\left(2x^2+3\right)\sin x$
16. $(1-\cos x)\tan x$
17. $\frac{\cos x}{(x+\sin x)}$
18. $\frac{x \sec x}{(3x+2)}$
19. $(x-1)\left(x^2-1\right)\left(x^3-5\right)$
20. $x^2(3-2x)\left(x^2+4x\right)$
21. $\frac{\sin^{-1} x}{x}$
22. $e^x.\tan^{-1} x$
23. $\frac{x \sin^{-1} x}{\left(1+x^2\right)}$
24. $\left(x^2+3\right)\tan^{-1} x$
25. $xe^x \sin^{-1} x$

3.3

DIFFERENTIATION - II

Function of a Function Rule

We have already learnt about differential coefficients of some standard functions; $(x^n, \sin x, \cos x, e^x, \text{etc.})$ In this portion we can see differential coefficients of composite functions.

If $y = f(u)$ where $u = \phi(x)$.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Extension

If $y = f(u)$, where $u = \phi(v)$

and $v = \psi(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

eg: 1) Let $y = (x^2 + 3x + 1)^5$

Take $x^2 + 3x + 1$ as another variable 'u'

ie; $u = x^2 + 3x + 1$, $\frac{du}{dx} = 2x + 3$ ....(1)

Now, $y = u^5$, $\frac{dy}{du} = 5u^4$ ....(2) $\left(\because \frac{d}{dx}(x^n) = nx^{n-1}\right)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 5u^4 (2x + 3)$$

$$= 5(x^2 + 3x + 1)^4 (2x + 3)$$

eg: 2) If $y = \sec^2(x^2 + 1)$

Let $u = \sec(x^2 + 1)$

$y = u^2 \Rightarrow \frac{dy}{du} = 2u$ ....(1)

Let $x^2 + 1 = v$

$\frac{dv}{dx} = 2x$ ....(2)

$u = \sec v$ from the above equation

$\frac{du}{dv} = \sec v.\tan v$ ....(3)

From (1), (2) and (3) $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$

$= 2\sec\left(x^2+1\right)\sec\left(x^2+1\right)\tan\left(x^2+1\right)2x$

Differentiation by the above procedure is difficult. Using the above procedure we shall derive some other equations which are easily understandable. For this we shall start with $(ax+b)^n$

Let $y = (ax+b)^n$

put $u = ax + b$

$\frac{du}{dx} = a$ ...(1)

$y=(ax+b)^n \Rightarrow y=u^n$

$\therefore \frac{dy}{du} = nu^{n-1}\left(\because \frac{d}{dx}x^n = nx^{n-1}\right)$

ie; $\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = nu^{n-1}\times a$

$= n(ax+b)^{n-1}\times a$

From the above problem it is clear that if $y = (ax+b)^n$

$\frac{dy}{dx} = n(ax+b)^{n-1}\times\frac{d}{dx}(ax+b)$ $\left(\because \frac{d}{dx}(ax+b) = a\right)$

ie; if $y = \left[f(x)\right]^n$, $\frac{dy}{dx} = n\left(f(x)\right)^{n-1}\frac{d}{dx}f(x)$

$$\boxed{\text{If } y = \left[f(x)\right]^n,\quad \frac{dy}{dx} = n\left(f(x)\right)^{n-1}\frac{d}{dx}f(x)}$$

Examples

1) If $y = (2x+3)^5$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = 5(2x+3)^4 .2 = 10(2x+3)^4$$

2) If $y = (4x-3)^7$

$$\frac{dy}{dx} = 7(4x-3)^6 \frac{d}{dx}(4x-3)$$

$$= 7(4x-3)^6 .4 = 28(4x-3)^6 \qquad \left[\because \frac{dy}{dx} = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$$

3) If $y = \sin^2 x = (\sin x)^2$

$$\frac{dy}{dx} = 2\sin x \frac{d}{dx}(\sin x) = 2\sin x \cos x$$

4) If $y = \tan^5 x = (\tan x)^5$

$$\frac{dy}{dx} = 5\tan^4 x \frac{d}{dx}(\tan x)$$

$$= 5\tan^4 x \sec^2 x$$

5) If $y = \sec^3 x = (\sec x)^3$

$$= \frac{dy}{dx} = 3\sec^2 x \frac{d}{dx}(\sec x)$$

$$= 3\sec^2 x \sec x \tan x$$

$$= 3\sec^3 x \tan x$$

6) If $y = \sqrt{2x-3}$

$$= (2x-3)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}(2x-3)^{1/2-1}\frac{d}{dx}(2x-3)$$

$$= \frac{1}{2}(2x-3)^{-1/2}\,2$$

$$= \frac{1}{2\sqrt{2x-3}}2 \qquad = \frac{1}{\sqrt{2x-3}}$$

$$\boxed{\begin{array}{l}\text{If } y=\sqrt{f(x)} \\ \dfrac{dy}{dx}=\dfrac{1}{2\sqrt{f(x)}}.\dfrac{d}{dx}(f(x))\end{array}}$$

7) $$\frac{d}{dx}=\left(\frac{1}{x^2+x+1}\right)=\frac{d}{dx}\left(x^2+x+1\right)^{-1}$$

$$=-1\left(x^2+x+1\right)^{-1-1}\frac{d}{dx}\left(x^2+x+1\right)$$

$$=-1\left(x^2+x+1\right)^{-2}(2x+1)$$

$$=\frac{-1}{\left(x^2+x+1\right)^2}(2x+1)$$

$$\boxed{\begin{array}{l}\text{If } y=\dfrac{1}{f(x)} \\ \dfrac{dy}{dx}=\dfrac{-1}{\left[f(x)\right]^2}\dfrac{d}{dx}f(x)\end{array}}$$

In this part we are going to extend the function of function rule for all functions including t–functions, logarithmic and exponential functions. For example, suppose we have to find $\frac{d}{dx}\sin(2x)$

$y=\sin 2x$ Let $u=2x$

$y=\sin u,$ $\frac{dy}{du}=\cos u$

$u=2x,$ $\frac{du}{dx}=2$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=2\cos 2x$$

ie; $$\frac{d}{dx}\sin f(x)=\cos f(x)\frac{d}{dx}f(x)$$

From this example we shall deduce the following results.

Results

1. $\frac{d}{dx}\sin f(x) = \cos f(x) \times \frac{d}{dx} f(x)$

2. $\frac{d}{dx}\cos f(x) = -\sin f(x) \frac{d}{dx} f(x)$

3. $\frac{d}{dx}\tan f(x) = \sec^2 f(x) \frac{d}{dx} f(x)$

4. $\frac{d}{dx}\operatorname{cosec} f(x) = -\operatorname{cosec} f(x) \cot f(x) \frac{d}{dx} f(x)$

5. $\frac{d}{dx}\sec f(x) = \sec f(x) \tan f(x) \frac{d}{dx} f(x)$

6. $\frac{d}{dx}\cot f(x) = -\operatorname{cosec}^2 f(x) \frac{d}{dx} f(x)$

7. $\frac{d}{dx}\log f(x) = \frac{1}{f(x)} \frac{d}{dx} f(x)$

8. $\frac{d}{dx} e^{f(x)} = e^{f(x)} . \frac{d}{dx} f(x)$

9. $\frac{d}{dx}\sin^{-1} f(x) = \frac{1}{\sqrt{1-[f(x)]^2}} \frac{d}{dx} f(x)$

10. $\frac{d}{dx}\tan^{-1} f(x) = \frac{1}{1+(f(x))^2} \frac{d}{dx} f(x)$

Examples

1) $\frac{d}{dx}(\sin 2x) = \cos 2x \, \frac{d}{dx} 2x = \cos 2x \, 2$

2) $\frac{d}{dx}(\tan 3x) = \sec^2(3x) \frac{d}{dx} 3x = \sec^2 3x \, 3$

3) $\frac{d}{dx}\left(\sec\sqrt{x}\right) = \sec\sqrt{x} \tan\sqrt{x} \frac{d}{dx}\sqrt{x}$

$= \sec\sqrt{x} \tan\sqrt{x} \frac{1}{2\sqrt{x}}$

4) $\frac{d}{dx}\operatorname{cosec}(x^2) = -\operatorname{cosec}(x^2)\cot(x^2)\frac{d}{dx}(x^2)$

$= -\operatorname{cosec}(x^2)\cot(x^2)2x$

5) $\frac{d}{dx}\cot(x^4) = -\operatorname{cosec}^2(x^4)\frac{d}{dx}(x^4)$

$= -\operatorname{cosec}^2(x^4)4x^3$

$= -4x^3.\operatorname{cosec}^2(x^4)$

6) $\frac{d}{dx}\log(\log x) = \frac{1}{\log x}\frac{d}{dx}(\log x)$

$= \frac{1}{\log x}.\frac{1}{x} = \frac{1}{x\log x}$

7) $\frac{d}{dx}(e^{3x}) = e^{3x}\frac{d}{dx}(3x) = e^{3x}3$

8) $\frac{d}{dx}\sin^{-1}(x^2) = \frac{1}{\sqrt{1-(x^2)^2}}\frac{d}{dx}(x^2) = \frac{1}{\sqrt{1-x^4}}2x = \frac{2x}{\sqrt{1-x^4}}$

9) $\frac{d}{dx}\tan^{-1}(2x) = \frac{1}{1+(2x)^2}\frac{d}{dx}(2x)$

$= \frac{1}{1+4x^2}\times 2 = \frac{2}{1+4x^2}$

10) $\frac{d}{dx}\cos(x^3) = -\sin(x^3).\frac{d}{dx}(x^3)$

$= -\sin(x^3)3x^2$

$= -3x^2\sin(x^3)$

Differentiate the following.

1. e^{ax+b}
2. $e^{\sin 2x}$
3. $\log(\sec x + \tan x)$
4. $\log\sin(x^2 + a^2)$
5. $\log\sin\sqrt{x}$
6. $\sin(e^x)$

7. $e^{(\sin^2 x)}$

8. $\cos\sqrt{x}$

9. $\cot^5(x^2)$

10. $\sin^3(\sqrt{x})$

11. $\sec^2(x^2)$

12. $\cos^2 3x$

13. $e^{\cos x}$

14. $e^{\sqrt{x}}$

15. $\log(\operatorname{cosec} x - \cot x)$

Answers

1. $\frac{d}{dx}\left(e^{ax+b}\right) = e^{(ax+b)}\frac{d}{dx}(ax+b) = e^{ax+b}a$

2. $\frac{d}{dx}\left(e^{\sin 2x}\right) = e^{\sin 2x}\frac{d}{dx}(\sin 2x) = e^{\sin 2x}\cos 2x\, 2$

$= 2\cos 2x e^{\sin 2x}$

3. $\frac{d}{dx}\log(\sec x + \tan x)$

$= \frac{1}{\sec x + \tan x}\left(\sec x.\tan x + \sec^2 x\right)$

$= \frac{\sec x(\tan x + \sec x)}{(\sec x + \tan x)} = \sec x$

4. $\frac{d}{dx}\log\sin\left(x^2+a^2\right) = \frac{1}{\sin\left(x^2+a^2\right)}\frac{d}{dx}\sin\left(x^2+a^2\right)$

$= \frac{\cos\left(x^2+a^2\right)2x}{\sin\left(x^2+a^2\right)}\left(\because \frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right)$

$= 2x\cot\left(x^2+a^2\right)$

5. $\frac{d}{dx}\log\left(\sin\sqrt{x}\right) = \frac{1}{\sin\sqrt{x}}\frac{d}{dx}\left(\sin\sqrt{x}\right)$

$$\left(\because \frac{d}{dx}\log f(x)=\frac{1}{f(x)}\frac{d}{dx}f(x)\right)$$

$$=\frac{1}{\sin\sqrt{x}}\cos\sqrt{x}\,\frac{1}{2\sqrt{x}}$$

$$=\frac{\cos\sqrt{x}}{\sin\sqrt{x}}\frac{1}{2\sqrt{x}}=\frac{\cot\sqrt{x}}{2\sqrt{x}}$$

6. $$\frac{d}{dx}\sin\left(e^x\right)=\cos\left(e^x\right)\frac{d}{dx}e^x$$

$$=\cos\left(e^x\right)e^x$$

7. $$\frac{d}{dx}e^{\sin^2 x}=e^{\sin^2 x}\frac{d}{dx}\left(\sin^2 x\right)$$

$$=e^{\sin^2 x}2\sin x\cos x$$

8. $$\frac{d}{dx}\left(\cos\sqrt{x}\right)=-\sin\sqrt{x}\,\frac{1}{2\sqrt{x}}$$

9. $$\frac{d}{dx}\cot^5\left(x^2\right)=\frac{d}{dx}\left[\cot\left(x^2\right)\right]^5=5\cot^4\left(x^2\right)\frac{d}{dx}\cot\left(x^2\right)$$

$$=5\cot^4\left(x^2\right)\times-\operatorname{cosec}^2\left(x^2\right)2x$$

$$=-10x\cot^4(x^2).\operatorname{cosec}^2(x^2)$$

10. $$\frac{d}{dx}\left(\sin^3\sqrt{x}\right)=\frac{d}{dx}\left(\sin\sqrt{x}\right)^3=3\sin^2\sqrt{x}\frac{d}{dx}\sin\sqrt{x}$$

$$=3\sin^2\sqrt{x}\cos\sqrt{x}\,\frac{1}{2\sqrt{x}}$$

11. $$\frac{d}{dx}\sec^2\left(x^2\right)=2\sec\left(x^2\right)\frac{d}{dx}\sec\left(x^2\right)$$

$$=2\sec\left(x^2\right)\sec\left(x^2\right)\tan\left(x^2\right)2x$$

$$=2\sec^2\left(x^2\right)\tan\left(x^2\right)2x$$

$$=4x.\sec^2\left(x^2\right).\tan\left(x^2\right)$$

12. $$\frac{d}{dx}\left(\cos^2(3x)\right)=\frac{d}{dx}(\cos 3x)^2=2\cos(3x)\frac{d}{dx}\cos 3x$$

$$=2\cos 3x.-\sin 3x3$$

$$=-6\sin 3x\cos 3x$$

13. $$\frac{d}{dx}e^{\cos x}=e^{\cos x}\frac{d}{dx}\cos x=e^{\cos x}\times-\sin x$$

14. $$\frac{d}{dx}e^{\sqrt{x}}=e^{\sqrt{x}}\frac{d}{dx}\sqrt{x}=e^{\sqrt{x}}.\frac{1}{2\sqrt{x}}$$

15. $$\frac{d}{dx}\log(\operatorname{cosec}x-\cot x)=\frac{1}{\operatorname{cosec}x-\cot x}\frac{d}{dx}(\operatorname{cosec}x-\cot x)$$

$$=\frac{\left(-\operatorname{cosec}x\cot x-^{-}\operatorname{cosec}^2x\right)}{(\operatorname{cosec}x-\cot x)}=\frac{\left(\operatorname{cosec}^2x-\operatorname{cosec}\cot x\right)}{(\operatorname{cosec}x-\cot x)}$$

$$=\frac{\operatorname{cosec}x(\operatorname{cosec}x-\cot x)}{(\operatorname{cosec}x-\cot x)}=\operatorname{cosec}x$$

Problems involving Product and Ratio

Differentiate the following

1. $x^2\sec 3x$
2. $e^{2x}\cos 3x$
3. $\sqrt{x}\tan^2 x$
4. $\left(x^2+1\right)^{10}\sec 5x$
5. $\left(1+x^2\right)\sin^2 x$
6. $\sin 3x\sin 5x$
7. $\dfrac{\sin 2x}{1+\cos 2x}$
8. $\dfrac{\cot 11x}{\left(x^3-1\right)^2}$
9. $e^{2x}\log x$
10. $x^2\cos\left(x^2\right)$

Answers

1. $$\frac{d}{dx}\left(x^2\sec 3x\right)=x^2\frac{d}{dx}(\sec 3x)+\sec 3x\frac{d}{dx}\left(x^2\right)$$

(applying product rule after taking $u = x^2$ and $v = \sec 3x$)

$=x^2\sec 3x\tan 3x\ 3+\sec 3x\ 2x$

$$=3x^2\sec 3x.\tan 3x+2x.\sec 3x$$

2. $$\frac{d}{dx}\left(e^{2x}\cos 3x\right)=e^{2x}\frac{d}{dx}(\cos 3x)+\cos 3x\frac{d}{dx}\left(e^{2x}\right)$$

$$=e^{2x}\times -\sin 3x3+\cos 3x\times e^{2x}\times 2$$

$$=-3e^{2x}\sin 3x+2e^{2x}\cos 3x$$

3. $$\frac{d}{dx}\left(\sqrt{x}\tan^2 x\right)=\sqrt{x}\frac{d}{dx}\left(\tan^2 x\right)+\tan^2 x\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\sqrt{x}.2\tan x.\sec^2 x+\tan^2 x.\frac{1}{2\sqrt{x}}$$

4. $$\frac{d}{dx}\left(x^2+1\right)^{10}\sec 5x=\left(x^2+1\right)^{10}\frac{d}{dx}(\sec 5x)+\sec 5x\frac{d}{dx}\left(x^2+1\right)^{10}$$

$$=\left(x^2+1\right)^{10}.\sec 5x.\tan 5x.5+\sec 5x.10\left(x^2+1\right)^9.2x$$

$$=5\left(x^2+1\right)^{10}\sec 5x\tan 5x+20\sec 5x\left(x^2+1\right)^9 x$$

5. $$\frac{d}{dx}\left(1+x^2\right)\sin^2 x=\left(1+x^2\right)\frac{d}{dx}\left(\sin^2 x\right)+\sin^2 x\frac{d}{dx}\left(1+x^2\right)$$

$$=\left(1+x^2\right)2\sin x\cos x+\sin^2 x.2x$$

$$=\left(1+x^2\right)2\sin x\cos x+2x.\sin^2 x$$

6. $$\frac{d}{dx}(\sin 3x\sin 5x)=\sin 3x\frac{d}{dx}(\sin 5x)+\sin 5x\frac{d}{dx}(\sin 3x)$$

$$=\sin 3x\cos 5x.5+\sin 5x.\cos 3x.3$$

$$=5\sin 3x.\cos 5x+3\sin 5x.\cos 3x$$

7. $$\frac{d}{dx}\left(\frac{\sin 2x}{1+\cos 2x}\right)=\frac{(1+\cos 2x)\frac{d}{dx}(\sin 2x)-\sin 2x\frac{d}{dx}(1+\cos 2x)}{(1+\cos 2x)^2}$$

(Using Quotient Rule after taking u = sin 2x, v=1+cos2x)

$$=\frac{(1+\cos 2x)\cos 2x.2-\sin 2x^{-}\sin 2x2}{(1+\cos 2x)^2}$$

8. $$\frac{d}{dx}\frac{\cot 11x}{\left(x^3-1\right)^2}=\frac{\left(x^3-1\right)^2\frac{d}{dx}(\cot 11x)-\cot 11x\frac{d}{dx}\left(x^3-1\right)^2}{\left(x^3-1\right)^4}$$

$$=\frac{(x^3-1)^2.-\operatorname{cosec}^2 11x.11-\cot 11x.2(x^3-1)3x^2}{(x^3-1)^4}$$

$$=\frac{-11(x^3-1)^2\operatorname{cosec}^2 11x-6x^2\cot 11x(x^3-1)}{(x^3-1)^4}$$

9. $\frac{d}{dx}\left(e^{2x}\log x\right)=e^{2x}\frac{d}{dx}(\log x)+\log x\frac{d}{dx}\left(e^{2x}\right)$

$=e^{2x}\frac{1}{x}+\log xe^{2x}2$

10. $\frac{d}{dx}\left(x^2\cos\left(x^2\right)\right)=x^2\frac{d}{dx}\cos\left(x^2\right)+\cos\left(x^2\right)\frac{d}{dx}\left(x^2\right)$

$=x^2.-\sin\left(x^2\right)2x+\cos\left(x^2\right)2x$

$=-2x^3\sin\left(x^2\right)+2x\cos\left(x^2\right)$

Qn. 1. If $y=\frac{1}{\sec\sqrt{x}}$ find $\frac{dy}{dx}$

Sol: $\frac{dy}{dx}=\frac{\sec\sqrt{x}\,(1)-1\frac{d}{dx}\sec\sqrt{x}}{\sec^2\sqrt{x}}$ $\left(u=1 \text{ and } v=\sec\sqrt{x}\right)$

$$=\frac{\sec\sqrt{x}.0-\sec\sqrt{x}\tan\sqrt{x}\frac{1}{2\sqrt{x}}}{\sec^2\sqrt{x}}$$

$$=\frac{-\sec\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}\sec^2\sqrt{x}}=\frac{-\tan\sqrt{x}}{2\sqrt{x}\sec\sqrt{x}}$$

Qn. 2. If $y=\frac{\sin(\log x)}{x}$ fid $\frac{dy}{dx}$

Sol: $\frac{dy}{dx}=\frac{x\frac{d}{dx}\sin(\log x)-\sin(\log x)\frac{d}{dx}(x)}{x^2}$

$$=\frac{x\cos(\log x)\times\frac{1}{x}-\sin(\log x)1}{x^2}=\frac{\cos(\log x)-\sin(\log x)}{x^2}$$

Qn. 3. If $y = \frac{x^2 \sec x}{(x^2+3)}$ find $\frac{dy}{dx}$

Sol: Take $u = x^2 \sec x$, $v = x^2+3$ and use quotient rule.

$$\frac{dy}{dx}\left(\frac{x^2 \sec x}{(x^2+3)}\right) = \frac{(x^2+3)\left[x^2 \sec x \tan x + \sec x\, 2x\right] - x^2 \sec x\, 2x}{(x^2+3)^2}$$

Qn. 4. If $y = \frac{\sin 2x}{x}$; find $\frac{dy}{dx}$

Sol:
$$\frac{dy}{dx} = \frac{x\frac{d}{dx}(\sin 2x) - \sin 2x \frac{d}{dx}(x)}{x^2}$$

$$\frac{x \cos 2x.2 - \sin 2x}{x^2} = \frac{2x \cos 2x - \sin 2x}{x^2}$$

EXERCISE - 3(c)

Differentiate the following

I

1. $(2x - x^2)^4$
2. $\frac{1}{(x^3+1)^3}$
3. $\frac{1}{\sqrt{1-4x}}$
4. $(4x^2+5x+6)^5$
5. $\frac{1}{(x^2+\sin x)}$
6. $\sin^7(x^3)$
7. $\tan^{10}(x^2)$
8. $\sec^3 \sqrt{x}$
9. $\frac{1}{(2x+3)^6}$
10. $\sqrt{1+4x}$
11. $\frac{1}{(8x-3)^2}$
12. $(x^2+2x+1)^{10}$

II

1. $\sin(7x-8)$
2. $\cos(\cot x)$
3. $\operatorname{cosec}(5x+1)$
4. $\tan(3x-4)$

5. $\cot\left(x^2\right)$ 6. $\sec(3x+5)$

7. $\log\left(\tan\sqrt{x}\right)$ 8. $\log(\log(x+2))$

9. $e^{\tan x}$ 10. $e^{1/x}$

11. $e^{\frac{x^2}{x^2+1}}$ 12. $\log\left(x+\sqrt{1+x^2}\right)$

13. $\cot(5x+1)$ 14. $e^{\sin\sqrt{x}}$

15. e^{x^3+1} 16. $\sin\left(e^{3x}\right)$

17. $e^{(x\sin x)}$

$\frac{d}{dx}\sin f(x)$, $\frac{d}{dx}\cos f(x)$, $\frac{d}{dx}\log f(x)$, $\frac{d}{dx}e^{f(x)}$, etc. refer the previous note.

III

1. $x^2\cos\left(x^2\right)$ 2. $\sin 3x.\cos 3x$

3. $\frac{\cos 2x}{\cos x}$ 4. $\frac{\left(2x^2+3\right)^6}{\tan 9x}$

5. $\frac{\sqrt{x^2+1}}{\sec 5x}$ 6. $x^2.\cos 3x$

7. $x^3.\sqrt{\operatorname{cosec} 5x}$ 8. $\left(1+x^2\right)\tan^{-1}x$

9. $e^{3x}.\log x$ 10. $x^2.\sec 3x$

11. $\frac{x\operatorname{cosec}x}{3x-2}$ 12. $\frac{e^{2x}.\log x}{x^2}$

13. $\frac{e^x.\sin x}{(1+\log x)}$ 14. $e^x.\log(\sin x)$

15. $\sec x.\sqrt{x}.e^x$

For the above apply product rule or quotient rule and also the results of function of function rule given in earlier pages.

IV

1. $\left(1+x^2\right)\sin^{-1}(2x)$

2. $\dfrac{e^{2x}.\tan^{-1}3x}{\sqrt{x}}$

3. $\dfrac{\sin^{-1}\left(\sqrt{x}\right)}{x^3}$

Hint: $\dfrac{d}{dx}\sin^{-1}f(x)=\dfrac{1}{\sqrt{1-f(x)^2}}.\dfrac{d}{dx}f(x)$

$$\frac{d}{dx}\tan^{-1}f(x)=\frac{1}{1+(f(x))^2}\frac{d}{dx}f(x)$$

Differentiation of Implicit functions

We have considered so far only functions of the type $y = f(x)$. If two variables x and y are connected by the relation $f(x, y) = 0$, we differentiate the given equation term by term and solve the resulting equation for the differential coefficient.

Qn. 5. find $\dfrac{dy}{dx}$ when x and y are connected by $ax^2+2hxy+by^2=0$

Sol: Consider $ax^2+2hxy+by^2=0$, [a, b, h are constants] differentiating term by term.

$$\frac{d}{dx}\left(ax^2\right)+\frac{d}{dx}(2hxy)+\frac{d}{dx}\left(by^2\right)=0$$

$$a\frac{d}{dx}\left(x^2\right)+2h\frac{d}{dx}(xy)+b\frac{d}{dx}\left(y^2\right)=0$$

$$a\times 2x+2h\left[x\frac{dy}{dx}+y\times 1\right]+b\times 2y\frac{dy}{dx}=0$$

$$2ax+2hx\frac{dy}{dx}+2hy+2by\frac{dy}{dx}=0$$

$$\frac{dy}{dx}[2hx+2by]=-2ax-2hy$$

$$\frac{dy}{dx}=\frac{-2ax-2hy}{2hx+2by}=\frac{-(ax+hy)}{(hx+by)}$$

$$\boxed{\begin{aligned}\frac{d}{dx}\left(y^2\right)&=\frac{d}{dy}\left(y^2\right)\times\frac{dy}{dx}\\&=2y\times\frac{dy}{dx}\end{aligned}}$$

Qn. 6. $x^3 + y^3 = 3axy$.Find 3axy. Find $\frac{dy}{dx}$

Sol: $\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}(3axy)$

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\frac{d}{dx}(xy) = 3a\left[x\frac{dy}{dx} + y\times 1\right]$$

$$3x^2 + 3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay$$

$$\frac{dy}{dx}\left[3y^2 - 3ax\right] = 3ay - 3x^2$$

$$\frac{dy}{dx} = \frac{3ay - 3x^2}{3y^2 - 3ax}$$

$$\boxed{\frac{d}{dx}\left(y^3\right) = 3y^2\frac{dy}{dx}}$$

Qn. 7. Find $\frac{dy}{dx}$ if $x^2 + y^2 + 2gx + 2fy + c = 0$

Sol: $x^2 + y^2 + 2gx + 2fy + c = 0$, differentiating term by term;

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) + \frac{d}{dx}(2gx) + \frac{d}{dx}(2fy) + \frac{d}{dx}(c) = 0$$

$$2x + 2y\frac{dy}{dx} + 2g\times 1 + 2f\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y + 2f) = -2x - 2g$$

g, f are constants $\boxed{\frac{d}{dx}c = 0}$

$$\frac{dy}{dx}2(y + f) = -2(x + g)$$

$$\frac{dy}{dx} = \frac{-2(x + g)}{2(y + f)} = \frac{-(x + g)}{(y + f)}$$

Qn. 8. If $x^{2/3} + y^{2/3} = a^{2/3}$ Find $\frac{dy}{dx}$

Sol: $\frac{d}{dx}x^{2/3} + \frac{d}{dx}y^{2/3} = \frac{d}{dx}a^{2/3}$ we know that 'a' is a constant

$$\frac{2}{3}x^{2/3-1}+\frac{2}{3}y^{2/3-1}\frac{dy}{dx}=0$$

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0$$

$$\frac{2}{3}y^{-1/3}\frac{dy}{dx}=\frac{-2}{3}x^{-1/3}$$

$$\frac{dy}{dx}=\frac{-2}{3}\frac{x^{-1/3}}{\frac{2}{3}y^{-1/3}}=\frac{x^{-1/3}}{y^{-1/3}}$$

$$\boxed{\frac{d}{dx}\left(y^{2/3}\right)=\frac{2}{3}y^{2/3-1}\frac{dy}{dx}}$$

Qn. 9. If $x^2+xy+y^2=0$, find $\frac{dy}{dx}$

Sol: $\frac{d}{dx}\left(x^2+xy+y^2\right)=\frac{d}{dx}0$

$$2x+x\frac{dy}{dx}+y\times1+2y\frac{dy}{dx}=0$$

$$x\frac{dy}{dx}+2y\frac{dy}{dx}=-2x-y$$

$$\frac{dy}{dx}(x+2y)=-2x-y$$

$$\frac{dy}{dx}=\frac{-2x-y}{x+2y}$$

0

Qn. 10. Find $\frac{dy}{dx}$ from $xy=c^2$

Sol: $xy=c^2$

$$\frac{d}{dx}(xy)=\frac{d}{dx}\left(c^2\right)(c \text{ is a constant})$$

$$x.\frac{dy}{dx}+y\times1=0$$

$$x\frac{dy}{dx}=-y,$$

$$\frac{dy}{dx}=\frac{-y}{x}$$

Qn. 11. Find $\frac{dy}{dx}$ if $x^2y^2 = x^3 + y^3 + 3xy$

Sol: $x^2y^2 = x^3 + y^3 + 3xy$

$$\frac{d}{dx}\left(x^2y^2\right) = \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) + \frac{d}{dx}(3xy)$$

$$x^2\frac{d}{dx}\left(y^2\right) + y^2\frac{d}{dx}\left(x^2\right) = \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) + 3\frac{d}{dx}(xy)$$

$$x^2 2y\frac{dy}{dx} + y^2 2x = 3x^2 + 3y^2\frac{dy}{dx} + 3\left[x\frac{dy}{dx} + y \times 1\right]$$

$$= 3x^2 + 3y^2\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y$$

$$2x^2y\frac{dy}{dx} - 3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} = 3x^2 + 3y - 2xy^2$$

$$\frac{dy}{dx}\left[2x^2y - 3y^2 - 3x\right] = 3x^2 + 3y - 2xy^2$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 + 3y - 2xy^2}{2x^2y - 3y^2 - 3x}$$

EXERCISE 3(d)

Find $\frac{dy}{dx}$

1. $x^2y^3 - y^2x^3 = a^2$
2. $2x^3 + 6xy + 2y^3 = 16$
3. $y^4 = \left(x^2 + 3\right)^3$
4. $x^2y^2 + 10y = 1$
5. $x^3 + y^3 = 25$
6. $ax^2 + by^2 + 2gx + 2fy + c = 0$
7. $x^3y - x^2 + xy - 1 = 0$

$$\boxed{\frac{d}{dx}f(y) = \frac{df(y)}{dy} \times \frac{dy}{dx}}$$

Remember this always!

PARAMETRIC EQUATIONS

Sometimes both 'x' and 'y' are expressed in terms of a third variable called a parameter.

Let $x = f(t)$ and $y = \phi(t)$. Here the parameter is 't'

Here $\dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)}$

Qn. 12. If $x = at^2, y = 2at$, find $\dfrac{dy}{dx}$

Sol: $\dfrac{dx}{dt} = \dfrac{d}{dt}\left(at^2\right) = a\dfrac{d}{dt}t^2 \left(\because \text{we know that} \dfrac{d}{dx}\left(x^2\right) = 2x\right)$

$= a \times 2t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(2at) = 2a\dfrac{d}{dt}t = 2a$

$\dfrac{dy}{dt} = \dfrac{\left(dy/dt\right)}{\left(dx/dt\right)} = \dfrac{2a}{2at} = \dfrac{2a}{2at} = \dfrac{1}{t}$

Qn. 13. If $x = a(\theta - \sin\theta)$, and $y = a(1-\cos\theta)$, and $y = a(1-\cos\theta)$. Find $\dfrac{dy}{dx}$.

Sol: $\dfrac{dx}{d\theta} = \dfrac{d}{d\theta}a(\theta - \sin\theta)$

$= a\dfrac{d}{d\theta}(\theta - \sin\theta)$

$= a(1-\cos\theta)$

$\dfrac{dy}{d\theta} = \dfrac{d}{d\theta}a(1-\cos\theta)$

$= a\dfrac{d}{d\theta}(1-\cos\theta)$

$= a \times -^{-} \sin\theta = a\sin\theta$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a \sin\theta}{a(1-\cos\theta)}$$

$$= \frac{\sin\theta}{1-\cos\theta}$$

$$\text{ie; } \frac{dy}{dx} = \frac{\sin\theta}{2\sin^2 \theta/2} = \frac{2\sin \theta/2 \cos \theta/2}{2\sin^2 \theta/2} \left(\because 1-\cos\theta = 2\sin^2 \theta/2\right)$$

$$= \frac{\cos \theta/2}{\sin \theta/2} = \cot \theta/2$$

$$\boxed{\sin\theta = 2\sin \theta/2 \cos \theta/2}$$

Qn. 14. If $x = a \sec \theta$

$y = b\tan\theta$. Find $\frac{dy}{dx}$

Sol: $\frac{dx}{d\theta} = \frac{d}{d\theta}(a \sec\theta)$

$$= a\frac{d}{d\theta}\sec\theta = a\sec\theta\tan\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\tan\theta)$$

$$= b\frac{d}{d\theta}\tan\theta = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

$$= \frac{b}{a}\frac{\sec\theta}{\tan\theta} = \frac{b}{a}\frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}$$

$$= \frac{b}{a}\frac{1}{\cos\theta}\frac{\cos\theta}{\sin\theta}$$

$$= \frac{b}{a}\operatorname{cosec}\theta$$

Qn. 15. Find $\frac{dy}{dx}$ when $x = 3\cos t - \cos^3 t$

$y = 3\sin t - \sin^3 t$

Sol: $\frac{dx}{dt} = \frac{d}{dt}\left(3\cos t - \cos^3 t\right)$

$= 3\frac{d}{dt}\cos t - \frac{d}{dt}\cos^3 t$

$= 3^- \sin t - 3\cos^2 t^- \sin t$

$= -3\sin t + 3\cos^2 t \sin t$

$\frac{dy}{dt} = \frac{d}{dt}\left(3\sin t - \sin^3 t\right)$

$= 3\frac{d}{dt}\sin t - \frac{d}{dt}\sin^3 t$

$= 3\cos t - 3\sin^2 t\cos t$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{3\cos t - 3\sin^2 t\cos t}{-3\sin t + 3\cos^2 t\sin t} = \frac{3\cos t\left(1-\sin^2 t\right)}{-3\sin t\left(1-\cos^2 t\right)}$$

$$= \frac{3\cos t\cos^2 t}{-3\sin t\sin^2 t} = \frac{-\cos^3 t}{\sin^3 t} = -\cot^3 t$$

Qn. 16. If, x= a(cos t +t sin t)

$y = a\left(\sin t - t\cos t\right)$find $\frac{dy}{dx}$

Sol: $$\frac{dx}{dt}=\frac{d}{dt}a(\cos t+t\sin t)$$

$$=a\frac{d}{dt}(\cos t+t\sin t)$$

$$=a\left[-\sin t+(t\cos t+\sin t\times 1)\right]$$

$$=a\,t\cos t$$

$$\frac{dy}{dt}=\frac{d}{dt}a(\sin t-t\cos t)$$

$$=a\frac{d}{dt}(\sin t-t\cos t)$$

$$=a\left[\left(\cos t-\left(t\times^{-}\sin t+\cos t\times 1\right)\right)\right]$$

$$=a(\cos t+t\sin t-\cos t)$$

$$=at\sin t$$

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{at\sin t}{at\cos t}=\tan t$$

Qn. 17. If $x=a\cos^3 t,\ y=b\sin^3 t$. Find $\frac{dy}{dx}$?

Sol: $$\frac{dx}{dt}=\frac{d}{dt}\left(a\cos^3 t\right)=a3\cos^2 t^{-}\sin t$$

$$\frac{dy}{dt}=\frac{d}{dt}\left(b\sin^3 t\right)=b.3\sin^2 t\cos t$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{b.3\sin^2 t\cos t}{a.3\cos^2 t^{-}\sin t}=\frac{-b}{a}\tan t$$

EXERCISE 3(e)

Find $\frac{dy}{dx}$

1. $x = a\sin\theta, y = b\cos\theta$
2. $x = a\sec\theta, y = b\tan\theta$
3. $x = at^2, y = 2at$
4. $x = 2\sin t, y = \cos 2t$
5. $x = 3\cos\theta - \cos^3\theta, y = 3\sin\theta - \sin^3\theta$
6. $x = \frac{a}{t}, y = bt$
7. $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$
8. $x = 6t - 3t^4, y = 8t^3$
9. $x = e^t\cos t,\ y = e^t\sin t$
10. $x = \frac{1-t^2}{1+t^2} \quad y = \frac{2t}{1+t^2}$

SUCCESSIVE DIFFERENTIATION

Let us see a function y = f(x). If we differentiate this w.r.t 'x' we will get the **first derivative** of this function. The derivative of the first derivative is the **second derivative** of the function.

$$\frac{dy}{dx} = \frac{d}{dx}(f(x)) = f'(x) \rightarrow \text{first derivative}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}f'(x) = f''(x) \rightarrow \text{second derivative and so on}$$

Examples

1) Find the second derivative of sin x

$y = \sin x$

$\frac{dy}{dx} = \frac{d}{dx}(\sin x)$

$= \cos x$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(\cos x)$

$= -\sin x$ is the second derivative.

2) Find the second derivative of x cos x

$y = x \cos x$

$$\frac{dy}{dx} = \frac{d}{dx}(x \cos x)$$

$= x - \sin x + \cos x 1$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-x \sin x + \cos x)$$

$$= \frac{d}{dx}(-x \sin x) + \frac{d}{dx}\cos x$$

$= -[x \cos x + \sin x 1] + -\sin x$

$= -x \cos x - 2 \sin x$

3) Find the second derivative of $x^2 \log x$.

$y = x^2 \log x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 \log x\right)$$

$$= x^2 \frac{1}{x} + \log x 2x \quad = x + 2x \log x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(x + 2x \log x)$$

$$= \frac{d}{dx}(x) + \frac{d}{dx}(2x \log x)$$

$$= 1 + 2\left[x \times 1/x + \log x \times 1\right]$$

$= 1 + 2[1 + \log x] = 3 + 2 \log x$

Qn. 18. If $y = e^x + e^{-x}$ prove that $\frac{d^2y}{dx^2} = y$

Sol: $$\frac{dy}{dx} = \frac{d}{dx}\left(e^x + e^{-x}\right)$$

$$= \frac{d}{dx}e^x + \frac{d}{dx}e^{-x}$$

$= e^x + e^{-x} - 1 = e^x - e^{-x}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(e^x - e^{-x}\right)$$

$$= \frac{d}{dx}e^{x} - \frac{d}{dx}e^{-x}$$

$$= e^{x} - e^{-x} - 1 = e^{x} + e^{-x} = y$$

Qn. 19. If $y = x\sin x$, prove that $\frac{d^2y}{dx^2} + y = 2\cos x$

Sol: $\frac{dy}{dx} = \frac{d}{dx}(x\sin x)$

$= x\cos x + \sin x \times 1$ (By Product rule)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(x\cos x + \sin x)$$

$$= \frac{d}{dx}(x\cos x) + \frac{d}{dx}(\sin x)$$

$$= x^{-}\sin x + \cos x + \cos x$$

$$= -x\sin x + 2\cos x$$

$$\frac{d^2y}{dx^2} + x\sin x = 2\cos x$$

ie; $\frac{d^2y}{dx^2} + y = 2\cos x$

Qn. 20. If $y = a\sin x + b\cos x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Sol: $\frac{dy}{dx} = a\cos x + b^{-}\sin x$

$$\frac{d^2y}{dx^2} = a^{-}\sin x - b\cos x$$

$$\frac{d^2y}{dx^2} + y = {}^{-}a\sin x - b\cos x + a\sin x + b\cos x = 0$$

Qn. 21. If $y = ae^{x} + be^{2x}$, prove that $y'' - 3y' + 2y = 0$

Sol: $y' = \frac{dy}{dx} = ae^{x} + be^{2x}2$

$= ae^{x} + 2be^{2x}$

$$y'' = \frac{d^2y}{dx^2} = ae^x + 2be^{2x}2 = ae^{2x} + 4be^{2x}$$

$$y'' - 3y' + 2y$$

$$= ae^x + 4be^{2x} - 3\left(ae^x + 2be^{2x}\right) + 2\left(ae^x + be^{2x}\right)$$

$$= ae^x + 4be^{2x} - 3ae^x - 6be^{2x} + 2ae^x + 2be^{2x}$$

$$= 0$$

Qn. 22. If $y = x^2 \sin x$, prove that $x^2 y'' - 4xy' + \left(x^2 + 6\right)y = 0$

***Sol:** $y' = \frac{dy}{dx} = \frac{d}{dx}\left(x^2 \sin x\right)$

$$= x^2 \cos x + \sin x 2x$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(x^2 \cos x + 2x \sin x\right)$$

$$= \frac{d}{dx}\left(x^2 \cos x\right) + 2\frac{d}{dx}\left(x \sin x\right)$$

$$= x^2 - \sin x + \cos x 2x + 2\left[x \cos x + \sin x 1\right]$$

$$= -x^2 \sin x + 2x \cos x + 2x \cos x + 2 \sin x$$

$$= -x^2 \sin x + 4x \cos x + 2 \sin x$$

$$x^2 y'' - 4xy' + \left(x^2 + 6\right)y = x^2\left(-x^2 \sin x + 4x \cos x + 2 \sin x\right)$$

$$-4x\left(x^2 \cos x + 2x.\sin x\right) + \left(x^2 + 6\right)x^2 \sin x$$

$$= -x^4 \sin x + 4x^3 \cos x + 2x^2 \sin x$$

$$-4x^3 \cos x - 8x^2 \sin x + x^4 \sin x + 6x^2 \sin x$$

$$= 0$$

$$y' = y_1 = \frac{dy}{dx}, \text{ and } y'' = y_2 = \frac{d^2y}{dx^2}$$

Qn. 23. Find $\frac{dy}{dx}$ if $e^y = \sin\left(x + y\right)$

Sol: $\frac{d}{dx}e^y = \frac{d}{dx}\sin\left(x + y\right)$

$$e^y \frac{dy}{dx} = \cos(x+y)\left[1+\frac{dy}{dx}\right]$$

$$e^y \frac{dy}{dx} = \cos(x+y)+\cos(x+y)\frac{dy}{dx}$$

$$e^y \frac{dy}{dx}-\cos(x+y)\frac{dy}{dx}=\cos(x+y)$$

$$\frac{dy}{dx}\left[e^y-\cos(x+y)\right]=\cos(x+y)$$

$$\frac{dy}{dx}=\frac{\cos(x+y)}{e^y-\cos(x+y)}$$

Qn. 24. If $y=\sqrt{\sin x+y}$ Prove that $\frac{dy}{dx}=\frac{\cos x}{2y-1}$

Sol: $y=\sqrt{\sin x+y}$

$$y^2=\sin x+y \qquad \frac{d}{dx}y^2=\frac{d}{dx}(\sin x+y)$$

$$2y\frac{dy}{dx}=\cos x+\frac{dy}{dx}$$

$$2y\frac{dy}{dx}-\frac{dy}{dx}=\cos x$$

$$\frac{dy}{dx}(2y-1)=\cos x$$

$$\frac{dy}{dx}=\frac{\cos x}{2y-1}$$

Qn. 25. If $x=t^4-5, y=t^7+6.$ Find ${dy}/{dx}$ at $t={1}/{2}$

Sol: $\frac{dx}{dt}=4t^3,\ \frac{dy}{dt}=7t^6$

$$\frac{dy}{dx}=\frac{{dy}/{dt}}{{dx}/{dt}}=\frac{7t^6}{4t^3}=\frac{7t^3}{4}$$

$$\text{at } t={1}/{2},\ \frac{dy}{dx}={7}/{4}\times\left({1}/{2}\right)^3=\frac{7}{32}$$

EXERCISE 3(f)

1. Find the second derivative of

(a) $\sin^2 x$ (b) $x^2 + \frac{2}{x}$

(c) $\frac{x}{(x-2)}$ (d) $\frac{x}{4} + \frac{4}{x}$

2. If $y = a \sin mx$ show that

$$\frac{d^2y}{dx^2} + m^2 y = 0$$

3. If $xy = ax^2 + \frac{b}{x}$ prove that

$$x^2 y'' + 2(xy' - y) = 0$$

$$\left[\text{Hint} : y = \frac{ax^2 + b/x}{x} = ax + \frac{b}{x^2}, \text{ find } y' \text{ and } y''\right]$$

4. If $y = \sin^{-1} x$ prove that

$$(1 - x^2) y'' - xy' = 0$$

5. $y = A \cos px + B \sin px$ show that $\frac{d^2y}{dx^2}$ is proportional to 'y'.

$$\left[\text{Hint} : - \text{We have to show that } \frac{d^2y}{dx^2} = ky\right]$$

6. Find the second derivative of e^{-5x}

7. If $y = a \cos(\log x) + b \sin(\log x)$ show that

$$x^2 y'' + xy' + y = 0$$

8. If $y = x + \frac{1}{x}$, prove that $x^2 y'' + xy' = y$

9. If $y = x \cos x$, prove that $y'' + y + 2\sin x = 0$

10. If $y = x^2 \cos x$, prove that $x^2 y'' - 4xy' + (x^2 6) y = 0$

Module - IV

APPLICATIONS OF DIFFERENTIATION

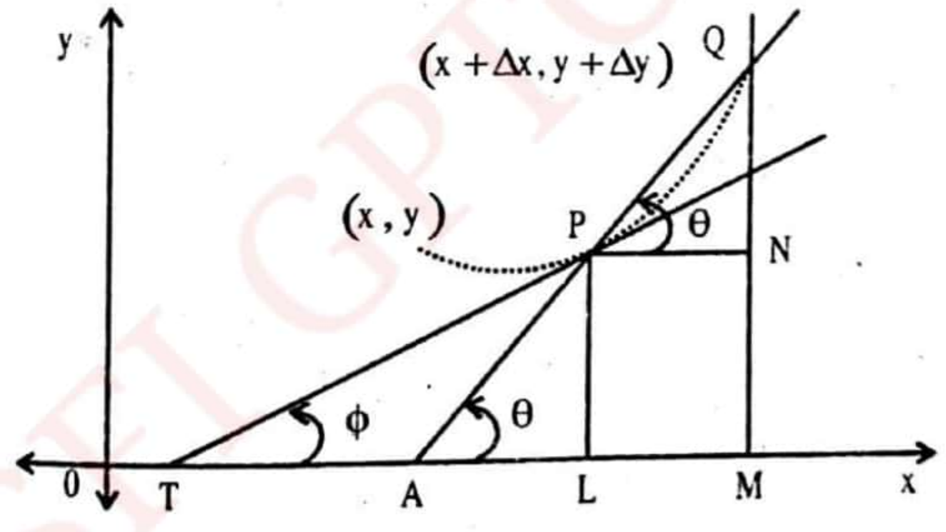
4.1 TANGENTS AND NORMALS

The different methods of differentiation so far studied by us would now be used in solving various types of problems under the name "applications of derivatives".

Prominent among them are (1) tangents and normals (2) Rate of change (velocity, acceleration and rates) (3) maxima and minima

Tangents and Normals

A tangent to any curve at a point 'p' is defined to be the limiting position of a chord PQ of the curve when Q approaches 'p' along the curve



From the figure we can see that $\overline{PQ}$ is a chord having end points (x,y) and $(x+\Delta x, y+\Delta y)$. $\overline{AQ}$ is making an angle 'θ' with the X-axis. $\therefore \angle P = \theta$. ($\angle A$ and $\angle P$ are corresponding angles).

Now $\tan\theta = \dfrac{QN}{PN} = \dfrac{y+\Delta y - y}{x+\Delta x - x} = \dfrac{\Delta y}{\Delta x}$

ie; slope of chord $\overline{PQ} = \dfrac{\Delta y}{\Delta x}$ ($\because$ slope $m = \tan\theta$)

In the case of a tangent, P and Q should coincide at the same point. Δx will tend to zero there.

$\therefore$ Slope of the tangent

$$= \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{dy}{dx} \quad \text{(same as } \tan\phi \text{ in the figure)}$$

Then, $\frac{dy}{dx}$ is the slope of the tangent to the curve $y = f(x)$ at the p (x, y). [This derivation is not included in the syllabus)

Note : 1) Normal to a curve is a straight line $\perp r$ to its tangent ie; produc slope of the tangent and normal is -1. (see perpendicularity and parallelism straight lines) $\therefore$ If m_1 is the slope of the tangent and m_2 is the slope of normal.

$m_1 \times m_2 = -1$, $m_2 = \frac{-1}{m_1}$ ie; We can see that slope of the norm

$= \frac{-1}{\left(\frac{dy}{dx}\right)}$. In fig.4.1(a) we can see a tangent and a normal.

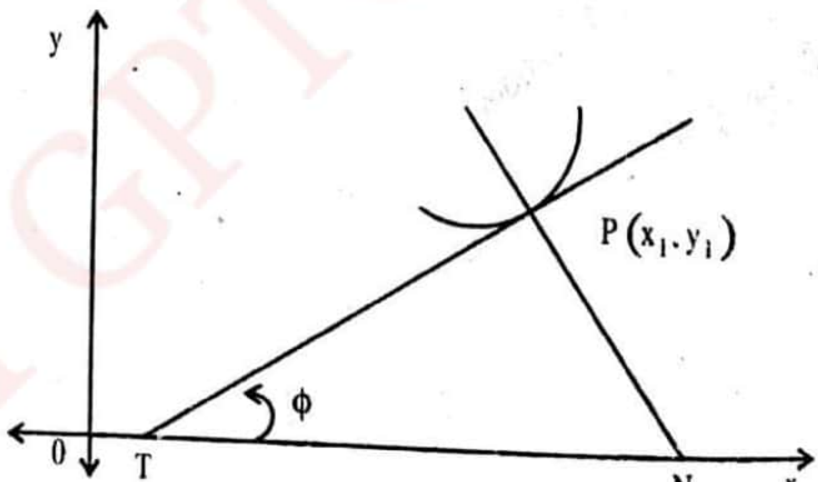


fig.4.1(a)

(2) Slope of a curve is same as the slope of the tangent to a curve.

Slope of the tangent to a curve $= \frac{dy}{dx}$ at (x, y) and

Slope of the normal to a curve $= \frac{-1}{\left(\frac{dy}{dx}\right)}$ at (x, y)

Qn.1. Find the slope of the curve (slope of the tangent) and slope of the normal to the following curves:-

i) $y = x^2 - 3x = 2$ at (3, 2)

Sol: Slope of the curve $= \frac{dy}{dx}$ at $(3,2)$

$= 2x - 3$ at $(3, 2) = = 2\times 3 - 3$

$= 3$ (putting $x = 3$ there)

Slope of the normal $= \frac{-1}{dy/dx} = \frac{-1}{3}$

ii) $y = \tan x$ at $x = \pi/3$

Sol: Slope of the curve $= \frac{dy}{dx} = \frac{d}{dx}(\tan x)$

$= \sec^2 x$ at $x = \frac{\pi}{3}$

$= \sec^2\left(\pi/3\right) = \sec^2\left(\frac{180}{3}\right) = \sec^2 60 = 4$

Slope of the normal $= \frac{-1}{\left(dy/dx\right)} = \frac{-1}{4}$

iii) $x^2 + y^2 = 25$ at $(3, -4)$

Sol: Here the function is in implicit form

$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}25$ (refer "Implicit Functions")

$2x + 2y\frac{dy}{dx} = 0\left(\because \frac{d}{dx}k = 0\right)$

$2y\frac{dy}{dx} = -2x$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$ at $(3, -4)$

ie; $\frac{dy}{dx} = \frac{-3}{-4} = \frac{3}{4}$

$$\text{Slope of the tangent} = \frac{dy}{dx} = \frac{3}{4}$$

$$\text{Slope of the normal} = \frac{-1}{dy/dx} = \frac{-4}{3}$$

iv) $x = 2t,\ y = 2/t$ at $t = 1$

Sol: Here the function is in parametric form.

For finding dy/dx we have to find dx/dt and dy/dt.

$$\frac{dx}{dt} = \frac{d}{dt}(2t) = 2$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{2}{t}\right) = 2\frac{d}{dt}\left(\frac{1}{t}\right) = 2\times\frac{-1}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\left(dy/dt\right)}{\left(dx/dt\right)} = \frac{-2/t^2}{2} = \frac{-1}{t^2} \text{ at } t = 1$$

$$\therefore \frac{dy}{dx} = -1$$

Slope of the tangent $= -1$ and

$$\text{Slope of the normal} = \frac{-1}{\left(dy/dx\right)} = 1$$

v) $y = \sqrt{25 - x^2}$ at $(4, 3)$

Sol:
$$\frac{dy}{dx} = \frac{1}{2\sqrt{25-x^2}}.-2x = \frac{-x}{\sqrt{25-x^2}}$$

$$= \frac{-4}{\sqrt{25-16}} = \frac{-4}{3}$$

$$\text{Slope of the tangent} = \frac{-4}{3}$$

$$\text{Slope of the normal} = \frac{3}{4}$$

Equations of Tangents and Normals

Tangent and normal are straight lines. If $(x_1 y_1)$ is a given point on a tangent and (x, y) is any point on that line, the equation is given by $y - y_1 = \frac{dy}{dx}(x - x_1)$. Since $\frac{dy}{dx}$ is the slope of the tangent.

[refer slope point form of straight lines $y - y_1 = m(x - x_1)$]

Since the slope of the normal is $\frac{-1}{dy/dx}$, the equation to a normal passing through a given point (x_1, y_1) is $(y - y_1) = \frac{-1}{\left(\frac{dy}{dx}\right)}(x - x_1)$ if (x, y) is any point on the line.

Equation of the tangent to a curve $\Rightarrow (y - y_1) = \frac{dy}{dx}(x - x_1)$

Equation of the normal to a curve $\Rightarrow (y - y_1) = \frac{-1}{\left(dy/dx\right)}(x - x_1)$

Qn.2. Find the equation of the tangent and normal to the following curves.

i) $y = 3x^2 + x - 2$ at (1, 2)

Sol: $\frac{dy}{dx} = 3.2x+1$ at (1, 2)

$= 3.2 . 1 + 1 = 7$

Slope of the tangent = 7

Equation of the tangent passing through (1, 2) is $(y_1 - y_1) = \frac{dy}{dx}(x - x_1)$

$y - 2 = 7(x - 1)$

$y - 2 = 7x - 7$

$7x - y - 5 = 0$

Equation of the normal

$\Rightarrow (y - y_1) = \frac{-1}{\left(dy/dx\right)}(x - x_1)$

$$y-2=\frac{-1}{7}(x-1)$$

$$7(y-2)=-1(x-1)$$

$$7y-14=-x+1$$

$$x+7y-15=0$$

ii) $y^2 = 4ax$ at $(a, 2a)$

Sol: $$\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}(4ax)$$

$$2y\frac{dy}{dx}=4a$$

$$\frac{dy}{dx}=\frac{4a}{2y}=\frac{2a}{y} \text{ at the point } (a, 2a) = \frac{2a}{2a}=1$$

$\therefore$ Slope of the tangent = 1

Equation of the tangent is,

$$y-y_1=\frac{dy}{dx}(x-x_1)$$

$$y-2a=1(x-a)$$

$$y-2a=x-a$$

$$-x+y-a=0 \quad \text{or} \quad x-y+a=0$$

Equation of the normal is $y-y_1=\dfrac{-1}{\left(dy/dx\right)}(x-x_1)$

$$y-2a=\frac{-1}{1}(x-a)$$

ie; $y-2a=-x+a$

$$x+y-3a=0$$

iii) $y=\cos x$ at $x=\dfrac{\pi}{6}$

Sol: Since $y=\cos x$, $\dfrac{dy}{dx}=-\sin x$

$$=-\sin\frac{\pi}{6}=-\tfrac{1}{2} \text{ at } x=\frac{\pi}{6}$$

ie; slope of the tangent is

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$x_1 = \frac{\pi}{6} \text{ (given)}$$

$$y_1 = \cos\frac{\pi}{6} = \cos 30 = \frac{\sqrt{3}}{2} \text{ (since } y = \cos x \text{ is the curve)}$$

$$\therefore y - \frac{\sqrt{3}}{2} = \frac{-1}{2}\left(x - \frac{\pi}{6}\right)$$

ie: $2y - \sqrt{3} = -\left(x - \frac{\pi}{6}\right)$

$x + 2y - \sqrt{3} - \frac{\pi}{6} = 0$ is the equation to the tangent.

Equation of the normal is $y - y_1 = \frac{-1}{dy/dx}(x - x_1)$

$$\Rightarrow y - \frac{\sqrt{3}}{2} = \frac{-1}{-1/2}\left(x - \pi/6\right)$$

$$y - \frac{\sqrt{3}}{2} = 2\left(x - \frac{\pi}{6}\right)$$

$-2x + y - \frac{\sqrt{3}}{2} + \frac{\pi}{3} = 0$ is the required equation to the normal.

Note : We know that slope of x axis is zero and slope of y axis is ∞

Qn.3. For what values of 'x' is the tangent to the curve $\frac{x}{x^2+1}$ parallel to the x axis

Sol: Since the tangent to the curve is parallel to the x axis, slope of the tangent is equal to zero (see the above note. Also slopes of two parallel lines are equal)

$\therefore$ Slope of the tangent $= \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{x^2+1}\right) = \frac{(x^2+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1) - x(2x)}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{1-x^2}{(x^2+1)^2} = 0$$

ie; $1-x^2=0$ $\quad 1=x^2 \quad x=\pm 1$

Qn.4. Find the value of x for which the tangent to the curve $y=x^3-2x^2+x+1$ are parallel to the x-axis

Sol: $y = x^3 - 2x^2 + x + 1$

$$\frac{dy}{dx} = 3x^2 - 4x + 1$$

Given that the tangent of the curve is parallel to the x-axis

$$\therefore \frac{dy}{dx} = 0$$

$3x^2-4x+1=0$

$$x = \frac{4 \pm \sqrt{16-12}}{6} = \frac{4 \pm 2}{6} = \left(1, \tfrac{1}{3}\right)$$

Qn.5. Find the values of 'x' for which the tangent to the curve $y = \frac{x}{(1-x)^2}$ will be parallel to the (1) x-axis (2) y-axis

Sol: (1) $y = \frac{x}{(1-x)^2}$

$$\frac{dy}{dx} = \frac{(1-x)^2 \frac{d}{dx}x - x\frac{d}{dx}(1-x)^2}{(1-x)^4}$$

$$= \frac{(1-x)^2.1 - x.2(1-x)\times -1}{(1-x)^4}$$

$$= \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4}$$

$$= \frac{1-2x+x^2+2x-2x^2}{(1-x)^4} = \frac{1-x^2}{(1-x)^4}$$

If the tangent is parallel to the x axis, $\frac{dy}{dx}=0$ ie; $\frac{1-x^2}{(1-x)^4}=0$

$$\frac{(1+x)(1-x)}{(1-x)^4}=0 \Rightarrow \frac{1+x}{(1-x)^3}=0 \Rightarrow 1+x=0$$ and $x=-1$

Sol: (2) If the tangent is parallel to the y axis $\frac{dy}{dx}=\infty$

ie; $\frac{1-x^2}{(1-x)^4}=\infty$ ie; $\frac{(1+x)(1-x)}{(1-x)^4}=\infty$

$\Rightarrow \frac{(1+x)}{(1-x)^3}=\infty$ means $(1-x)^3=0$ or $x=1$

Qn.6. Find the slope of the curve at $t=1$ where $x=t^2+1$ and $y=t^3-3t^2+3$

Sol: Slope of the curve is given by ${dy}/{dx}$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3t^2-6t}{2t}$$ at $t=1$ ie; $\frac{dy}{dx}=\frac{3\times1^2-6\times1}{2\times1}=\frac{-3}{2}$

Qn.7. Find the tangent and normal to the curve $y=3x^2-1$ at $(1, 2)$

Sol: Slope of the tangent $=6x=6$ at $(1, 2)$

Slope of the normal $= {-1}/{6}$

Equation to the tangent is $y-y_1=\frac{dy}{dx}(x-x_1)$

$y-2=6(x-1)$

$y-2=6x-6$

$6x-y-4=0$

Equation to the normal is

$$y-y_1=\frac{-1}{dy/dx}(x-x_1)$$

$$y-2=\frac{-1}{6}(x-1)$$

$6y-12=-x+1$

$-x-6y+13=0$ or $x+6y-13=0$

Things to remember

1. Slope of the tangent to the curve $y = f(x)$ is $\frac{dy}{dx}$ and the slope of the normal to the curve is $\frac{-1}{dy/dx}$

2. Equation to the tangent to the curve $y=f(x)$ at (x_1, y_1) is $y - y_1 = \frac{dy}{dx}(x - x_1)$ and equation to the normal is $y - y_1 = \frac{-1}{dy/dx}(x - x_1)$.

3. If he tangent to the curve $y=f(x)$ is $\boxed{\text{parallel to the x axis } \frac{dy}{dx} = 0}$ and if it is $\boxed{\text{parallel to the y axis } \frac{dy}{dx} = \infty}$

EXERCISE 4(a)

1. Find the slope of the tangent and normal of the following curves.

a) $y = \tan x$ at $x = \frac{\pi}{3}$

b) $y = e^{x^2}$ at $x = \frac{1}{2}$

c) $y = \log(\cos x)$ at $x = \frac{\pi}{4}$

d) $y = 3x^2 + x + 2$ at $(1, 2)$

e) $x = \cos 2t$, $y = \sin 2t$ at $t = \frac{\pi}{8}$

f) $xy = 2$ at $(1, 2)$

Find the equations of tangents and normals for the following curves

2\. a) $y=\sqrt{25-x^2}$ at (3, 2)

b) $y=\cos x$ at $x=\frac{\pi}{3}$

c) $x^2+y^2=25$ at (3, –4)

d) $y=\frac{1}{3+x}$ at (–4, –1)

e) $xy=2$ at (1, 2)

f) $x=ct,\ y=\frac{c}{t}$ at $\left(ct, \frac{c}{t}\right)$

g) $y=x^2+x-1$ at (2, 7)

3\. a) For what values of 'x' is the tangent to the curve $y=2x^3-9x^2+12x-3$ parallel to the x axis

b) Show that all the points on the curve $x^3+y^3=3axy$ at which the tangents are parallel to the x-axis lie on the curve, $ay=x^2$

c) Find the points on the curve $y=x^3-2x^2+x+1$ at which the tangents are parallel to the x axis

d) Find the slope of the curve $y=\log_e(\cos x)$ at $x=\frac{\pi}{4}$

(hint: slope of the curve = $\frac{dy}{dx}$)

e) Find the equation to the curve having slope $\sqrt{1-y^2}$ at (x, y) and passing through (2, 0)

(Hint: $\frac{dy}{dx}=\sqrt{1-y^2}$ after finding the value use $y-y_1=\frac{dy}{dx}(x-x_1)$)

4.2

RATES AND MOTION

In the previous portion we have seen that $\frac{dy}{dx}$ is the slope of the tangent to the curve $y = f(x)$. In this part we are introducing $\frac{dy}{dx}$ as the rate of change of the function 'y' w.r.t. 'x' at the value 'x'

We know that $\frac{dy}{dx} = \underset{\Delta x \to 0}{Lt}\left(\frac{\Delta y}{\Delta x}\right)$,

$\left(\frac{\Delta y}{\Delta x}\right)$ gives the average rate of change of 'y' w.r.t. 'x'. $\underset{\Delta x \to 0}{Lt}\left(\frac{\Delta y}{\Delta x}\right)$ is therefore the rate of change of 'y' w.r.t. 'x' at the value 'x'.

Velocity and Acceleration

If 'S' is the distance travelled by a body at a time 't', $\frac{ds}{dt}$ represents the rate of change of displacement w.r.t 't'. ie; the velocity of the particle.

Velocity $(v) = \frac{ds}{dt}$

The acceleration of a body is the rate of change of velocity at the time 't'

ie; $\frac{dv}{dt}$

ie; Acceleration $= \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$

$$\boxed{v = \frac{ds}{dt},\ a = \frac{dv}{dt}}$$

Qn.1. Find the rate of change of area of a circle with respect to the radius.

Sol: We know that $A = \pi r^2$

$\frac{dA}{dr}$ is the rate of change of area w.r.t radius

$\frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right)$

$= \pi\frac{d}{dr}r^2 = \pi.2r = 2\pi r$

Qn.2. Find the rate of change of the volume of a cube w.r.t. its side.

Sol: $V = a^3$ where 'a' is the side. Rate of change of volume w.r.t. its side is represented by $\frac{dV}{da} = \frac{d}{da}(a^3) = 3a^2$

Qn.3. The work done by a moving body is expressed as $w = 2t - \frac{3}{t}$ units over an interval of t secs. Find the power of the force creating the motion at $t = 2$ secs. [power $= \frac{dw}{dt}$ (rate of change of work done w.r.t. the time)]

Sol: $\frac{dw}{dt} = \frac{d}{dt}\left(2t - \frac{3}{t}\right)$

$= 2\frac{d}{dt}(t) - 3\frac{d}{dt}\left(\frac{1}{t}\right)$

$= 2\frac{d}{dt}(t) - 3\frac{d}{dt}(t^{-1})$

$= 2 \times 1 - 3 \times -1t^{-2} = 2 + \frac{3}{t^2}$

At $t = 2$ secs

$\frac{dw}{dt} = 2 + \frac{3}{t^2}$

$= 2 + \frac{3}{2^2} = 2 + \frac{3}{4}$

$= \frac{11}{4}$ units.

Qn.4. A particle moves such that the displacement from a fixed point "O" is always given by $S = 5 \cos nt + 4 \sin nt$, where 'n' is a constant. Prove that the acceleration varies as its displacement S at the instant.

Sol: $S = 5 \cos nt + 4 \sin nt$

Velocity $= \frac{ds}{dt}$

$v = \frac{d}{dt}(5\cos nt + 4\sin nt)$

$= 5\frac{d}{dt}(\cos nt) + 4\frac{d}{dt}(\sin nt)$

$= 5.\,{}^{-}\sin nt.n + 4\cos nt.n$

$= -5n\sin nt + 4n\cos nt$ $\left(\because \frac{d}{dt}\sin f(x) = \cos f(x)\frac{d}{dx}f(x)\right)$

acceleration $(a) = \frac{dv}{dt}$

$= \frac{d}{dt}(-5n\sin nt + 4n\cos nt)$

$= -5n\frac{d}{dt}\sin nt + 4n.\frac{d}{dt}\cos nt$

$= -5n\cos nt.n + 4n.{}^{-}\sin nt.n$

$= 5n^2\cos nt - 4n^2\sin nt$

$= n^2[5\cos nt + 4\sin nt]$ ie, $a = -n^2 S$ where 'n' is a constant

ie; Acceleration α displacement

Qn.5. The distance S metres travelled by a particle is given by $S = ae^{nt} + be^{-nt}$ where 't' represents the time. Show that acceleration varies as the distance.

Sol: $S = ae^{nt} + be^{-nt}$

Velocity $= \frac{ds}{dt} = \frac{d}{dt}\left(ae^{nt} + be^{-nt}\right)$

$= a\frac{d}{dt}\left(e^{nt}\right) + b\frac{d}{dt}\left(e^{-nt}\right)$

$= a.e^{nt}.n + be^{-nt}.-n$

$= ane^{nt} - bne^{-nt}$

Acceleration $= \frac{d}{dt}\left(ane^{nt}\right) - \frac{d}{dt}\left(bne^{-nt}\right)$

$= an\frac{d}{dt}\left(e^{nt}\right) - bn.\frac{d}{dt}\left(e^{-nt}\right)$

$= ane^{nt}.n - bne^{-nt}.-n$

$= an^2e^{nt} + bn^2e^{-nt}$

$= n^2\left(ae^{nt} + be^{-nt}\right)$

$= n^2S$

ie, $a = n^2S$. $\therefore$ Acceleration varies as distance.

Qn.6. The distance described by a particle in 't' secs is given by $S = ae^{t} + be^{-t}$. Show that the acceleration of the body is always equal to the distance passed over.

Sol: We have $S = ae^{t} + be^{-t}$

Velocity $(v) = \dfrac{ds}{dt} = \dfrac{d}{dt}\left(ae^{t} + be^{-t}\right) = ae^{t} + be^{-t} \times -1$ ie, $v = ae^{t} - be^{-t}$

Acceleration $(a) = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(ae^{t} - be^{-t}\right) = ae^{t} - be^{-t}.-1$

ie, $a = ae^{t} + be^{-t}$ which is same as 'S'.

Qn.7. If $S = t^2 - 4t + 3$. Find the velocity at $t = 4$ secs.

Sol: $S = t^2 - 4t + 3$

Velocity $= \dfrac{d}{dt}\left(t^2 - 4t + 3\right)$

$= 2t - 4 + 0 \quad (\because \dfrac{d}{dt}k = 0)$

$= 2t - 4$

At $t = 4$ secs, Velocity $= 2 \times 4 - 4 = 4$ units.

Qn.8. If 'S' denotes the displacement of a particle at the time 't' secs and $S = t^3 - 6t^2 + 8t - 4$.

(1) Find the time when the acceleration is 12 cm/sec^2.

(2) The velocity at that time.

Sol: (1) $S = t^3 - 6t^2 + 8t - 4$

Velocity, $v = \dfrac{ds}{dt} = \dfrac{d}{dt}\left(t^3 - 6t^2 + 8t - 4\right)$

$= 3t^2 - 6 \times 2t + 8$

$= 3t^2 - 12t + 8$

Acceleration, $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(3t^2 - 12t + 8\right)$

$= 3\dfrac{d}{dt}\left(t^2\right) - 12\dfrac{d}{dt}(t) + \dfrac{d}{dt}(8)$

$= 3 \times 2t - 12 \times 1 + 0$

$= 6t - 12$

When the acceleration is 12 cm/sec^2, $6t - 12 = 12$

$6t = 24$

$t = \dfrac{24}{6} = 4$ secs.

Sol: (2) We have to find, velocity at $t = 4$ secs.

$v = 3t^2 - 12t + 8$

$= 3\times4^2 - 12\times4 + 8$

$= 48 - 48 + 8 = 8cm/sec.$

Qn.9. Find the velocity and acceleration of a particle at $t = 3$ secs whose displacement is given by $S = 3t^3 - t^2 + 9t + 1$

Sol: $S = 3t^3 - t^2 + 9t + 1$

Velocity, $v = \frac{ds}{dt} = \frac{d}{dt}\left(3t^3 - t^2 + 9t + 1\right)$

$= 9t^2 - 2t + 9$

At $t = 3$ secs, $v = 9\times3^2 - 2\times3 + 9$

$= 84$ units

Acceleration, $a = \frac{dv}{dt}$

$= \frac{d}{dt}\left(9t^2 - 2t + 9\right) = 18t - 2$

At $t = 3$ secs, $a = 52$ units.

> If 'S' is the displacement of a body at a time 't',
> the velocity is given by $\frac{dS}{dt}$ and acceleration is given by $\frac{d^2S}{dt^2}$ or $\frac{dv}{dt}$

EXERCISE 4(b)

1. Find the velocity and acceleration at time $t = 4$ secs of a body whose displacement S metres related to time 't' seconds is given by the equation

   $S = \frac{1}{2}t^2 + \sqrt{t}$

2. A particle moves in a straight line in such a way that its displacement is given by $s = 9t^2 - t^3$ in t seconds. Find the displacement is given by velocity is zero. (Hint:- Find $\frac{ds}{dt}$ and equate it to zero and find the time and substitute 't' in the equation)

3. If $s = \frac{2}{3}t + \cos t$. Find the expressions of velocity and acceleration.

4. A particle is projected vertically upwards and its height 'h' and time 't' are connected by $h = 60t - 16t^2$. Find the greatest height attained.

5. Assuming that force is proportional to acceleration, show that a particle moving along X-axis so that its displacement $x = ae^{kt} + be^{-kt}$ has a force acting on it which is proportional to the displacement.

[Hint:- Find $\frac{d^2x}{dt^2}$ and show that it is proportional to 'x'

ie; $\frac{d^2x}{dt^2} = kx$]

6. The displacement of a body is given by $x = 4 \cos 3t + 5 \sin 3t$. Show that the acceleration of the body is always proportional to the displacement.

7. The distance travelled by a particle moving along a straight line is given by $s = 2t^3 - 9t^2 + 12t + 6$.

Find the value of 't' when the acceleration is zero.

[Hint:- $\frac{d^2s}{dt^2} = 0$ and find 't']

8. Find the rate of change of volume of a sphere w.r.t the radius

[Hint: $\frac{dV}{dr}$ when $V = \frac{4}{3}\pi r^3$]

9. Find the rate of change of volume of a cone w.r.t the radius, if its radius is always equal to the height (ie; $r = h$).

10. Find the rate of change of volume of a cube w.r.t the side 'x'.

[Hint: Find $\frac{dV}{dx}$, where $V = x^3$]

Related Rates

In this portion we can see the methods of finding the rates of change of certain variables, if the rates of change of related variables are given. For this, functional relations connecting the variables should be known. For example, consider the area of a circle $A = \pi r^2$. Rate of change of area w.r.t the time 't' is given by $\frac{dA}{dt}$

$$\frac{dA}{dt} = \frac{d}{dt}\left(\pi r^2\right) = \pi \frac{d}{dt}\left(r^2\right) = \pi.2r\frac{dr}{dt}$$

If $\frac{dr}{dt}$ is known, we shall find $\frac{dA}{dt}$ and vice versa.

Similarly volume of a sphere is $V = \frac{4}{3}\pi r^3$, Rate of change of volume w.r.t the time 't' is given by $\frac{dV}{dt}$.

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi\frac{d}{dt}\left(r^3\right)$$

$$= \frac{4}{3}\pi.3r^2\frac{dr}{dt}$$

If $\frac{dr}{dt}$ is known, we can find out $\frac{dV}{dt}$ and vice versa.

Remark: If the rate of change of a variable w.r.t the time is increasing, put the positive sign and if it is decreasing put the negative sign.

Qn.10. A circular patch of oil spreads out on water, the area growing at the rate of 6 sq.cm/mt. How fast is the radius increasing when the radius is 2cms.

Sol: Area of a circle $A = \pi r^2$

Differentiate w.r.t time

$$\frac{dA}{dt} = \frac{d}{dt}\pi r^2 \qquad \frac{dA}{dt} = 6\text{sq.cm / mt}$$

$$= \pi\frac{d}{dt}r^2 \qquad r = 2\text{cms.}$$

$$= \pi 2r\frac{dr}{dt}$$

$$6=\pi\times 2\times 2\left(\frac{dr}{dt}\right)=4\pi\left(\frac{dr}{dt}\right)$$

$$\frac{dr}{dt}=\frac{6}{4\pi}\text{cm / mt.}=\frac{3}{2\pi}\text{cm / min.}$$

Qn.11. A spherical balloon is inflated by pumping 25cc of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15cm.

Sol: Volume of a spherical balloon $V=\frac{4}{3}\pi r^3$

$$\frac{d}{dt}V=\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \qquad \frac{dV}{dt}=25\text{cc / sec.}$$

$$=\frac{4}{3}\pi\frac{d}{dt}r^3 \qquad r = 15\text{ cm.}$$

$$=\frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right)$$

$$25=\frac{4}{3}\pi\times 3\times 15^2\left(\frac{dr}{dt}\right)$$

$$\frac{dr}{dt}=\frac{25\times 3}{4\pi\times 3\times 15^2}\text{cm / sec.}$$

$$=\frac{5}{180\pi}\text{cm / sec.}=\frac{1}{36\pi}\text{cm / sec.}$$

Qn.12. A balloon is spherical in shape. Gas is escaping from it at the rate of 10 cc/sec. How fast is the surface area shrinking when the radius is 15cm.

Sol: $V=\frac{4}{3}\pi r^3$, $\frac{dV}{dt}=-10\text{cc / sec.}$ ($\therefore$ the volume is decreasing)

$$\frac{d(V)}{dt}=\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \qquad r = 15\text{ cm}$$

$$=\frac{4}{3}\pi\frac{d}{dt}(r^3)$$

$$= \frac{4}{3}\pi \times 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\text{ie, } -10 = 4\pi \times 15^2 \left(\frac{dr}{dt}\right)$$

$$\frac{dr}{dt} = \frac{-10}{4\pi \times 15^2} \text{ cm/sec.}$$

$$= \frac{-1}{90\pi} \text{ cm/sec.}$$

Surface area of a spherical balloon $S = 4\pi r^2$

$$\frac{d(S)}{dt} = \frac{d}{dt}\left(4\pi r^2\right)$$

$$= 4\pi \frac{d}{dt} r^2 = 4\pi \times 2r \frac{dr}{dt}$$

$$= 8\pi r \times \frac{-1}{90\pi} = 8\pi \times 15 \times \frac{-1}{90\pi} = \frac{-4}{3} \text{ sq.cm/sec.}$$

$\therefore$ Surface area is shrinking at the rate of $\frac{4}{3}\text{cm}^2/\text{sec}$

Qn.13. Air is pumped in to a spherical rubber bladder of radius 3". If the radius increases at a uniform rate of 1" per minute, find the rate at which the volume is increasing at the end of 3 minutes.

Sol: $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$

$$= \frac{4}{3}\pi \frac{d}{dt}\left(r^3\right)$$

$$= \frac{4}{3}\pi \times 3r^2 \frac{dr}{dt}$$

$$= 4\pi r^2 \frac{dr}{dt}$$

We know that $\frac{dr}{dt} = 1$ inch / minute

First, we have to find the radius at the end of 3 minute.

ie; $r = 3 + 3 \times \frac{dr}{dt}$

$= 3 + 3 \times 1 = 6$ inches

$$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi \times 6^2 \times 1$$

$= 144\pi$ cubic inch / mt.

EXERCISE 4(c)

1. The radius of a circular plate is increasing in length at 0.1 cm/sec., when heated. What is the rate at which the area is increasing when the radius is 12cm.

2. A circular plate of radius 3 inches expands when heated at the rate of 2 inch/ second. Find the rate at which the area of the plate increasing at the end of 3 seconds.

   [$A = \pi r^2$ find $\frac{dA}{dt}$]

3. A baloon is spherical in shape. Gas is escaping from it at the rate of 20 cc/second. How fast is the surface area shrinking when the radius is 15 cm

4. A spherical balloon is inflated with air such that its volume increases at the rate of 5cc/second. Find the rate at which its curved surface is increasing when its radius is 7cm.

   [Hint:- surface area $A = 4\pi r^2$, find dA/dt]

4.3 MAXIMA & MINIMA

Increasing and decreasing Functions

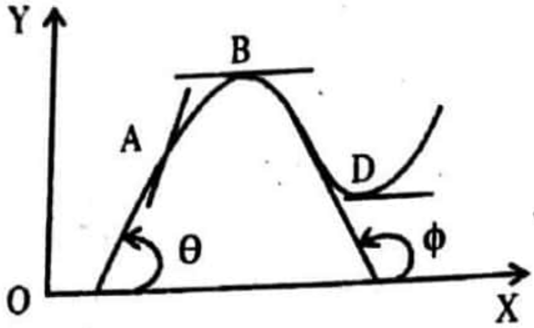


fig.4(a)

Consider the function $y=f(x)$, if y increases as x increases, it is called an increasing function of 'x'.

On the contrary, if y decreases as x increases, it is called a decreasing function of 'x'.

We can also see that the tangent at any point on the curve of an increasing function makes an acute angle with the x-axis, so that

$\frac{dy}{dx} = \tan\theta$ is positive.

The tangent at any point on the curve of a decreasing function makes an obtuse angle with the x-axis showing that $\frac{dy}{dx} = \tan\phi$ is negative.

See fig.4(a)

For increasing functions, $\frac{dy}{dx}$ is positive (ie, $\frac{dy}{dx} > 0$) and for decreasing functions, $\frac{dy}{dx}$ is negative (ie, $\frac{dy}{dx} < 0$). See fig.4(a) and identify the points where the function increases and decreases.

Qn. 1. Find whether the function $x^3 - 6x^2 + 5x - 2$ is increasing or decreasing when $x = -2$ and when $x = 2$.

Sol: Let $y = x^3 - 6x^2 + 5x - 2$

$$\frac{dy}{dx} = 3x^2 - 6.2x + 5 = 3x^2 - 12x + 5$$

At $x = -2$, $\frac{dy}{dx} = 3. - 2^2 - 12\times -2 + 5 = 41$

ie; since $\frac{dy}{dx} > 0$ at $x = -2$ the function is increasing at $x = -2$

Now Let $x = 2$

$$\frac{dy}{dx} = 3x^2 - 12x + 5$$

$= 3.2^2 - 12.2 + 5$

$= -7 < 0$ and the function is decreasing at $x = -7$

Qn. 2. Find the range of values of 'x' for which $(x^2 - 3x + 4)$ is

(1) increasing (2) decreasing

Sol: $y = x^2 - 3x + 4;\ \frac{dy}{dx} = 2x - 3$

If te function is increasing $\frac{dy}{dx} > 0$

ie; $2x - 3 > 0$

$2x > 3$

ie; $x > \frac{3}{2}$

If the function is decreasing $\frac{dy}{dx} < 0$

$2x - 3 < 0 \Rightarrow 2x < 3$

ie; $x < \frac{3}{2}$

Turning Points and Turning Values

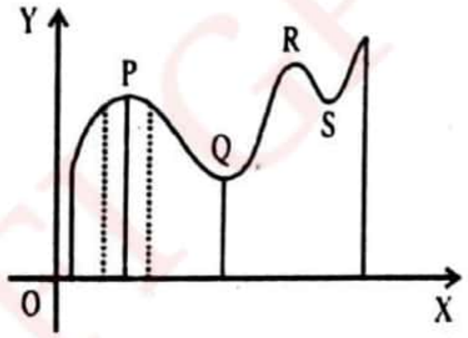


fig.4(b)

We can see that at the points P, Q, R and S, the function neither increases nor decreases. Such points are called turning points (or stationary points). The values of the function at these points are turning values. It may be noted that the tangents to the curve at turning points are parallel to the x-axis and therefore, at these points $\frac{dy}{dx} = 0$ (we have explained this earlier in "Tangents and Normals")

For finding the turning values, we can first find the turning points by taking $\frac{dy}{dx} = 0$ and substitute the values of x in the function, y=f(x).

Qn. 3. Find the turning values $2x^3 - 9x^2 + 12x + 2$

Sol: $2x^3 - 9x^2 + 12x + 2$

$$\frac{dy}{dx} = 6x^2 - 18x + 12$$

At the turning points $\frac{dy}{dx} = 0$

$6x^2 - 18x + 12 = 0$

$6(x^2 - 3x + 2) = 0$

$(x - 2)(x - 1) = 0$

or $x = 2$ or 1 are the turning points.

If we substitute these values in 'y', we will obtain the turning values.

$y = 2x^3 - 9x^2 + 12x + 2$

when $x=1$, $y = 2 \times 1^3 - 9 \times 1^2 + 12 \times 1 + 2 = 7$

when $x = 2$, $y = 2 \times 2^3 - 9 \times 2^2 + 12 \times 2 + 2 = 6$

Maxima and Minima

A continuous function $y = f(x)$ is said to have a maximum value at a point, if $y = f(x)$ ceases to increase and begins to decrease at that point as x increases. At the neighbourhood of this point, the slope of the curve changes from positive to negative ie, $\frac{dy}{dx}$ changes from positive to negative (slope is given by $\frac{dy}{dx}$) $\therefore \frac{dy}{dx}$ becomes a decreasing function and therefore $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is negative ie; at a maximum point, $\frac{d^2y}{dx^2}$ is negative. (see the points 'p' and 'R' in fig.4(b).

A continuous function $y=f(x)$ is said to have a minimum value at a point if $y = f(x)$ ceases to decrease and begins to increase at that point as 'x' increases. At the neighbourhood of a minimum piont, the slope of the curve changes from negative to positive. ie, $\frac{dy}{dx}$ becomes an increasing function and therefore $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is positive ie, $\frac{d^2y}{dx^2}$ is positive (see the points 'Q' and 'S' in fig.4(b)).

Conditions of Maxima and Minima

1. At a maximum point $\frac{dy}{dx}=0, \frac{d^2y}{dx^2}<0$

2. At a minimum point $\frac{dy}{dx}=0, \frac{d^2y}{dx^2}>0$

Qn. 4. The deflection of a beam is given by $y = 2x^3 - 9x^2 + 12x$. Find the maximum deflection.

Sol: $y = 2x^3 - 9x^2 + 12x$

$$\frac{dy}{dx} = 2.3x^2 - 9.2x + 12$$

$= 6x^2 - 18x + 12$

At a maxima or a minima $\frac{dy}{dx}=0$

$6x^2 - 18x + 12 = 0$ $\qquad$ $6(x^2 - 3x + 2) = 0$

$x^2 - 3x + 2 = 0$ $\qquad$ $(x - 2)(x - 1) = 0$

$x = 1, 2$

Check whether the function is maximum or minimum at $x = 1$ and 2

$$\frac{d^2y}{dx^2} = 12x - 18$$

when $x = 1$, $12.1 - 18 < 0 = -6 < 0$

when $x = 2$, $12.2 - 18 > 0 = 6 > 0$

$\therefore$ The function is maximum when $x=1$ because $\frac{d^2y}{dx^2}<0$ at $x = 1$

For finding the maximum value put $x = 1$ in the first equation.

$y = 2.1^3 - 9.1^2 + 12.1$

$y = 5$

The maximum value of y is '5'

Qn. 5. The bending moment of a rod of 10m long and weighing 40kg and resting at its ends at a distance of x m from one end is given by $M=2(10x - x^2)$ kgm. Find the maximum bending moment.

Sol: $M = 2(10x - x^2)$

At a maximum or a minimum, $\frac{dM}{dx}=0$

ie; $\frac{d}{dx} 2\left(10x - x^2\right) = 0$

$2\frac{d}{dx}\left(10x - x^2\right) = 0$

$2(10 - 2x) = 0$

$10 - 2x = 0 \Rightarrow x = 5$

$\frac{d^2M}{dx^2} = \frac{d}{dx}\left(\frac{dM}{dx}\right)$

$= \frac{d}{dx} 2(10 - 2x)$

$= 2(-2) = -4 < 0$

$\therefore \frac{d^2M}{dx^2} < 0$ and 'M' is maximum at $x = 5$

The maximum bending moment $= 2(10x - x^2)$ at $x = 5$
$= 50$ kg.m.

Procedure for finding maxima or minima

1. Read the problem carefully and find out the variable or quantity which is to be made maximum or minimum. Express it as a function of a single independent variable, Say 'x' in the form $y = f(x)$.
2. Differentiating w.r.t. 'x' obtain $\frac{dy}{dx}$.
3. Solve the equation, $\frac{dy}{dx} = 0$ and find the value of 'x' say x_1 and x_2.
4. Find $\frac{d^2y}{dx^2}$
5. Put each of the values of 'x' in $\frac{d^2y}{dx^2}$ and check whether $\frac{d^2y}{dx^2}$ is negative or positive. If $\frac{d^2y}{dx^2}$ is negative, y is maximum at the value and if $\frac{d^2y}{dx^2}$ is positive y is minimum at the value.

Maxima & Minima (Type 2)

If a function is not in terms of a single variable, we have to change it into a single variable by using the data given in the question. For this see the following examples

Qn. 6. Prove that a rectangle of fixed perimeter has its maximum area when it becomes a square.

Sol: Let x and y be the length and breadth of the rectangle of fixed perimeter 'P'

ie; $2x + 2y = p$

$2y = p - 2x$

$y = \frac{p-2x}{2}$ ....(1)

Area of a rectangle $= x \times y$

ie; $A = x \times y$

$A = x\left(\frac{p-2x}{2}\right) = \frac{1}{2}\left(px - 2x^2\right)$

At a maximum or a minimum $\frac{dA}{dx} = 0$

$\frac{dA}{dx} = \frac{d}{dx}\frac{1}{2}\left(px - 2x^2\right)$

$\Rightarrow \frac{1}{2}(p - 4x) = 0$

$p - 4x = 0, \quad x = \frac{p}{4} \qquad \therefore y = \frac{p-2x}{2} = \frac{p - 2p/4}{2}$

$= \frac{p - p/2}{2} = \frac{p}{4}$

$x = p/4, \; y = p/4$

Check whether the function is maximum or minimum at these values of 'x' and 'y'.

$\frac{d^2A}{dx^2} = \frac{d}{dx}\frac{1}{2}(p - 4x) = \frac{1}{2}(-4) < 0$

since $\frac{d^2A}{dx^2} < 0$ 'A' is maximum at $x = \frac{p}{4}$ and $y = \frac{p}{4}$

Qn. 7. A hollow cylindrical can to hold 100cc of water is to be made so that the area of the metal used is a minimum. prove that the radius which will give minimum area is $\sqrt[3]{\frac{100}{\pi}}$ cm.

Sol: Volume of the can $= \pi r^2 h = 100cc$ ($\because$ for a cylinder $V = \pi r^2 h$)

$h = \frac{100}{\pi r^2}$ ....(1)

Surface area of a cylinder $= \pi r^2 + 2\pi rh$

$S = \pi r^2 + 2\pi rh$

$= \pi r^2 + 2\pi r.\frac{100}{\pi r^2}$ from (1)

$S = \pi r^2 + \frac{200}{r}$ ....(2)

If the surface area is minimum,

(1) $\frac{ds}{dr} = 0$ (2) $\frac{d^2s}{dr^2} > 0$

$\frac{ds}{dr} = 0 \Rightarrow \frac{d}{dr}\left(\pi r^2 + \frac{200}{r}\right) = 0$

$\pi 2r + 200\frac{-1}{r^2} = 0$ $2\pi r = \frac{200}{r^2}$

$2\pi r^3 = 200 \Rightarrow r^3 = \frac{100}{\pi}$ and $r = \sqrt[3]{\frac{100}{\pi}}$ cm.

Check whether the surface area is minimum at this value.

For that, take $\frac{d^2s}{dr^2} = \frac{d}{dr}\left(\frac{ds}{dr}\right)$

$= \frac{d}{dr}\left(2\pi r - \frac{200}{r^2}\right)$

$= 2\pi\frac{d}{dr}(r) - 200\frac{d}{dr}\left(r^{-2}\right)$

$= 2\pi + \frac{400}{r^3} > 0$ where $r = \sqrt[3]{\frac{100}{\pi}}$

$\therefore$ The radius which will give minimum surface area is $\sqrt[3]{\frac{100}{\pi}}$ cm

Qn. 8. A big water tank is to be constructed having a square base, an open top and vertical walls, to contain 256 cubic metres of water. Find the side of the base and the height if the sum of the area of the bottom and the walls is to be a minimum.

Sol: The water tank has a square base having side 'x' and height 'h'

The volume of the tank = x^2h ($\because$ base area $\times$ height = V)

$x^2h = 256$

$$h = \frac{256}{x^2} \quad \text{..............(1)}$$

Sum of the area of the bottom and the walls = x^2+4xh

$S = x^2 + 4xh$ ($\because$ the base is a square and the walls are 4 rectangles)

$$= x^2 + 4x.\frac{256}{x^2}$$

$$= x^2 + \frac{1024}{x}$$

If the area is minimum $\frac{ds}{dx} = 0$ and $\frac{d^2s}{dx^2} > 0$

$$\frac{ds}{dx} = 0 \Rightarrow \frac{d}{dx}\left(x^2 + \frac{1024}{x}\right) = 0$$

$$2x + 1024 \times \frac{-1}{x^2} = 0$$

$$2x = \frac{1024}{x^2}, \quad 2x^3 = 1024$$

$x^3 = 512, \quad x = 8m$

If $x = 8m$, $h = \frac{256}{x^2} = \frac{256}{8^2} = 4m$

Now check whether the area is minimum or not.

For that, evaluate $\frac{d^2s}{dx^2}$

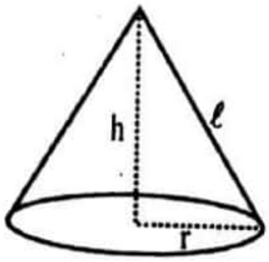
$$\frac{d^2s}{dx^2} = \frac{d}{dx}\left(\frac{ds}{dx}\right) = \frac{d}{dx}\left(2x - \frac{1024}{x^2}\right)$$

$$= 2 - 1024\frac{d}{dx}\left(x^{-2}\right) = 2 - 1024 \times \frac{-2}{x^3}$$

$$= 2 + \frac{2048}{x^3} > 0$$ ie; 'S' is minimum at $x = 8$m and $h = 4$m

Qn. 9. Find the maximum volume of a cone whose slant height is 'ℓ'cm

Sol: We know that $r^2 = \ell^2 - h^2$



Volume of a cone, $V = \frac{1}{3}\pi r^2 h$

We have to change V in terms of a single variable,

$V = \frac{1}{3}\pi\left(\ell^2 - h^2\right)h$ ....(1) ['ℓ' is a constant (given) 'h' is considered as the variable]

$$V = \frac{1}{3}\pi\ell^2 h - \frac{1}{3}\pi h^3$$

$$\frac{dV}{dh} = \frac{1}{3}\pi\ell^2 - \frac{1}{3}\pi.3h^2$$

$$= \frac{1}{3}\pi\ell^2 - \pi h^2$$

At a maxima or a minima $\frac{dV}{dh} = 0$ ie, $\frac{1}{3}\pi\ell^2 - \pi h^2 = 0$

$$\frac{1}{3}\pi\ell^2 = \pi h^2$$

$$h^2 = \frac{\ell^2}{3}$$

$$h = \frac{\ell}{\sqrt{3}}$$

$$\frac{d^2v}{dh^2} = \frac{d}{dh}\left[\frac{dv}{dh}\right] = \frac{d}{dh}\left[\frac{1}{3}\pi\ell^2 - \pi h^2\right]$$

$0-2\pi h$

$= -2\pi \frac{\ell}{\sqrt{3}} < 0$

$\therefore$ The cone has a maximum volume at $h = \frac{\ell}{\sqrt{3}}$

Maximum volume is given by substituting $h = \frac{\ell}{\sqrt{3}}$ in eq. (1)

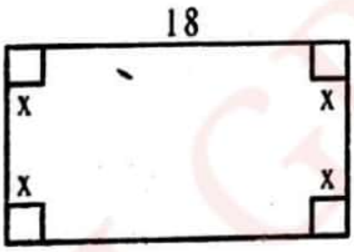
ie, $V = \frac{1}{3}\pi\left(\ell^2 - h^2\right)h$

$= \frac{1}{3}\pi\left(\ell^2 - \frac{\ell^2}{3}\right) \cdot \frac{\ell}{\sqrt{3}}$

$= \frac{2\pi\ell^3}{9\sqrt{3}}$

Qn. 10. An open box is to be made out of a square sheet of side 18 cm by cutting off equal squares at each corner and turning up the sides. What size of the squares should be cut in order that the volume of the box may be maximum.

Sol: Let x be the side of the square sheet which is cut from each of the corners.



The length of the open box = 18 – 2x

The breadth of the open box = 18 – 2x

The height of the open box = x

$\therefore$ Volume of the open box, $V = \ell bh$

$V = (18 - 2x)(18 - 2x)x$

$= 4x^3 - 72x^2 + 324x \qquad \therefore \frac{dV}{dx} = 12x^2 - 144x + 324$

$= 12(x^2 - 12x + 27)$

At a maxima or a minima

$\frac{dV}{dx} = 0 \Rightarrow 12\left(x^2 - 12x + 27\right) = 0$ ie; $x^2 - 12x + 27 = 0$

$(x-9)(x-3) = 0$

$x = 3$ or 9

$x = 9$ cm. is in admissible

(Since the length of the original sheet is only 18 cm)

$\therefore$ The side of the square sheet cut off = 3 cm

EXERCISE 4(d)

1. Find the stationery points of the curve $y = x^3 - 3x^2 - 9x + 5$
2. The deflection of a beam is given by $y = 4x^3 + 9x^2 - 12x + 2$. Find the maximum deflection (see the condition for maxima)
3. Find the minimum value of $2x^3 - 3x^2 - 36x + 10$
4. A cylindrical can open at one end is to have a volume of $64\pi\,cm^3$. Find the radius and height such that the metal used is a minimum.

   (Hint: $\pi r^2 h = 64\pi$ and $A = \pi r^2 + 2\pi rh$)
5. The perimeterof a rectangle is 100m. Find the sides when the area is maximum. (Hint: $2x + 2y = 100$ and $A = xy$)
6. Find the maximum area of a rectangle whose perimeter is 'ℓ' units. (Hint: take $2x + 2y = \ell$, find 'y' in terms of 'x' and consider $A = xy$)
7. The sum of the diameter and length of an open cylindrical vessel is 40 cm. Prove that the maximum volume is obtained when the radius is equal to the length.
8. An open box is to be made out of a square sheet of side 8 cm by cutting off equal squares at each corner and turning up the sides. What size of the squares should be cut in order that the volume of the box may be maximum.

   (Hint: $V = lbh$, $V = (8-2x)(8-2x)x$, expand this and find $\frac{dV}{dx}$ and $\frac{d^2V}{dx^2}$ etc.)