CHAPTER 4

Riemannian manifolds and the Laplace-Beltrami operator

Our goal is to view measured data geometrically, so that geometric methods can be brought to bear on statistical questions. Namely, our prior is that data is sampled from a compact Riemannian manifold embedded in R^n for some n. We want to reconstruct the Riemannian metric of the manifold, and to obtain information about certain operators such as the Laplacian.

Knowing geometry that underlies data creates several opportunities for analysis. Knowledge of connected components would allow the decomposition of the manifold to isolate features of interest. Identification of coarse structure can be used to reduce model complexity. Symmetries in the underlying manifold could produce patterns in data, and discovery of these symmetries are important.

After establishing the basic geometric notions required, we can rigorously introduce the geometric prior, which is the assumption that a data set lies on a manifold embedded in an ambient Euclidean space. We then introduce local kernels as the fundamental tool which leverage the geometric prior, and tangible manifolds which are a large class of manifolds which can be learned from data using local kernels. Finally, we prove a key lemma which motivates the definitions of local kernels and tangible manifolds and will be used extensively in the following chapters.

4.1 Introduction to the Laplace-Beltrami operator

In the previous chapter we saw that Fourier basis on the unit circle is defined by the eigenfunctions of the Laplace-Beltrami operator $\Delta = \frac{d^2}{d\theta^2}$. In order to understand the Laplace-Beltrami operator on more general manifolds, we start by changing the geometry of the circle and explaining how and why the operator Δ must change.

Consider the ellipse defined by the embedding

$$\iota(\theta) = (a\cos\theta, b\sin\theta)^\top \in \mathbb{R}^2$$

for $\theta \in [0, 2\pi)$. The ellipse $E = \iota([0, 2\pi))$ is topologically the same as the unit circle for all $a, b \neq 0$, but is only isometric to the unit circle when $a = b = 1$. Suppose that we try to apply the same operator $\frac{d^2}{d\theta^2}$ to a function defined on the ellipse. A function $f : E \to \mathbb{R}$ is a function of two variables $f(x, y)$ defined on the range of ι, so one option would be to apply the operator $\frac{d^2}{d\theta^2}$ to the composition $f \circ \iota(\theta) = f(a\cos\theta, b\sin\theta)$. The problem with simply computing $\frac{d^2}{d\theta^2} f \circ \iota$ is that it puts equal weight on all points, whereas some points have significantly higher local arc-lengths than others. Instead, let $\tilde{\theta} = \gamma(\theta)$ be the arc-length to $\iota(\theta)$ starting from $\iota(0)$ which is given by,

$$\tilde{\theta} = \gamma(\theta) = \int_0^\theta \sqrt{D\iota(\theta)^\top D\iota(\theta)}\, d\theta = \int_0^\theta \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta. \tag{4.1}$$

To define the Laplace-Beltrami operator, we first rewrite a function in terms of the arc-length variable $\tilde{\theta}$ as

$$\Delta f(\iota(\theta)) \equiv \frac{d^2}{d\tilde{\theta}^2} f(\iota(\gamma^{-1}(\tilde{\theta}))) = \frac{d}{d\tilde{\theta}}\left(\frac{df(\iota(\theta))}{d\theta}\frac{d\gamma^{-1}}{d\tilde{\theta}}\right) = \alpha(\theta)\frac{d}{d\theta}\left(\alpha(\theta)\frac{df(\iota(\theta))}{d\theta}\right). \tag{4.2}$$

Notice that after applying the chain rule we only need to know $\alpha(\theta) \equiv \frac{d\theta}{d\tilde{\theta}}$, so computing the integral (4.1) is not actually required in order to define the Laplace-Beltrami operator. Since $d\tilde{\theta} = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}d\theta$ and $\frac{d\gamma^{-1}}{d\tilde{\theta}} = \frac{d\theta}{d\tilde{\theta}}$, we have

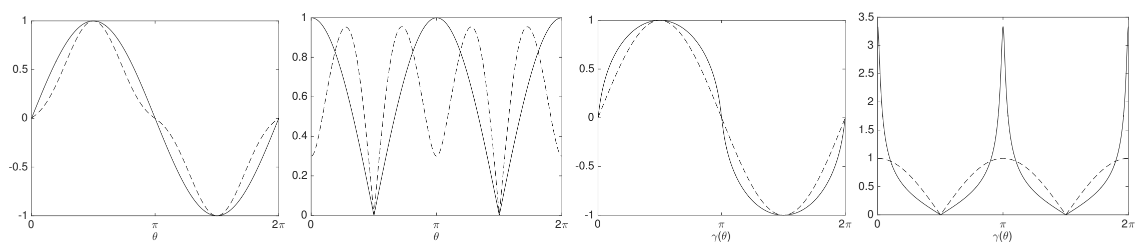
$$\Delta f(\iota(\theta)) = \frac{1}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \frac{d}{d\theta}\left(\frac{1}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \frac{df(\iota(\theta))}{d\theta}\right).$$

It is informative to compare the eigenfunctions of the Laplace-Beltrami operator on the circle and on the ellipse. Notice that on the ellipse we have $\Delta = \frac{d^2}{d\tilde{\theta}^2}$ so that the eigenfunctions are

$$\varphi_{2\ell}(\theta) = \cos(\ell\tilde{\theta}) = \cos(\ell\gamma(\theta)) \qquad \varphi_{2\ell+1}(\theta) = \sin(\ell\tilde{\theta}) = \sin(\ell\gamma(\theta))$$

Example 4.1 We show how smoothness on the ellipse compares to that on the circle due to arc-length difference.

In this example we consider the ellipse with $a = 1$ and $b = 1/10$ and we show how the eigenfunctions of the Laplace-Beltrami operator on the ellipse compare to the eigenfunctions of the operator $\frac{d^2}{d\theta^2}$ (which is the Laplace-Beltrami operator on the unit circle). In Fig. 4.1 we compare the first nontrivial eigenfunction of the respective operators and there derivatives with respect to θ and $\tilde{\theta}$. Notice that the eigenfunction of the circle $\sin(\theta)$, is smooth as a function of θ as shown in Fig. 4.1(b) but much rougher as a function of $\tilde{\theta}$ as shown in Fig. 4.1(d). The eigenfunction of the ellipse $\sin(\tilde{\theta})$ is much smoother than $\sin(\theta) = \sin(\gamma^{-1}(\tilde{\theta}))$ when both are considered as functions of the arc-length on the ellipse $\tilde{\theta}$.



(a) (b) (c) (d)

Fig. 4.1: (a) The first nontrivial eigenfunction of the circle $\sin(\theta)$ compared to the first nontrivial eigenfunction of the ellipse $\sin(\tilde{\theta}) = \sin(\gamma(\theta))$ as functions of θ. (b) Absolute value of the derivative of the eigenfunctions with respect to θ. (c) Eigenfunctions from plotted as function of $\tilde{\theta} = \gamma(\theta)$. (d) Absolute value of the derivative of the eigenfunctions with respect to $\tilde{\theta}$.

Changing coordinates to the arc-length parameterization gives a very natural definition of the Laplace-Beltrami operator for one-dimensional manifolds. Next, we would like to find the Laplace-Beltrami operator on m-dimensional manifolds as a natural generalization of the Laplacian on $\mathbb{R}^m$ which is given by

$$\sum_{i=1}^{m} \frac{d^2}{dx_i^2}. \tag{4.3}$$

At first glance this would appear difficult since it would be difficult to generalize the arc-length parameterization to a natural 'volumetric' parameterization for m-dimensional manifolds. However, as we saw in equation (4.2) the global arc-length parameterization is not required in the definition of the Laplace-Beltrami operator. Instead, we only need the Jacobian of the parameterization, which can be defined using a different local parameterization around each point of the manifold.

Since a m-dimensional manifold is locally equivalent to a ball in $\mathbb{R}^m$, we can use a diffeomorphism between a local region $U \ni x$ and a ball in $\mathbb{R}^m$ to 'lift' the standard Laplacian (4.3) to the manifold. The problem is that there are many possible parameterizations of a local region (corresponding to many possible diffeomorphism to the ball in $\mathbb{R}^m$). For a geometric space, the Laplace-Beltrami operator is uniquely defined by choosing the parameterization which is isometric to local region. In other words the Euclidean distances in the ball are exactly the intrinsic distances in between the cooresponding points in the local region. This is a direct generalization of the arc-length parameterization we used to define the Laplace-Beltrami operator on the ellipse. We now turn to the formal construction of the Laplace-Beltrami operator.

4.2 Riemannian manifolds

To begin, we summarize facts about differentiable manifolds that we will need below. For a d-dimensional differentiable manifold $\mathscr{M}$, there is a neighborhood U_x of each point x for which $H_x : U_x \to \mathbb{R}^d$ is a diffeomorphism onto its image D. For each u in U_x, there is a vector

$$H_x(u) = \begin{bmatrix} x_1(u) \\ \vdots \\ x_d(u) \end{bmatrix}$$

and we call $x_1, \ldots, x_d : U_x \to \mathbb{R}$ the *local coordinate functions*.

Let $C_x^1(\mathscr{M}, \mathbb{R})$ denote the differentiable functions defined in a neighborhood of x. A tangent vector in $T_x(\mathscr{M})$ is a linear function mapping $C_x^1(\mathscr{M}, \mathbb{R}) \to \mathbb{R}$ that obeys the product rule. In particular, the x_i lie in $C_x^1(\mathscr{M}, \mathbb{R})$, and we define the tangent vector $\dfrac{\partial}{\partial x_i}$ to be the linear function that satisfies

$$\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}.$$

Given local coordinate basis $\{x_1, \ldots, x_d\}$ at a point x, the tangent space $T_x(\mathscr{M})$ is spanned by the basis $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\}$.

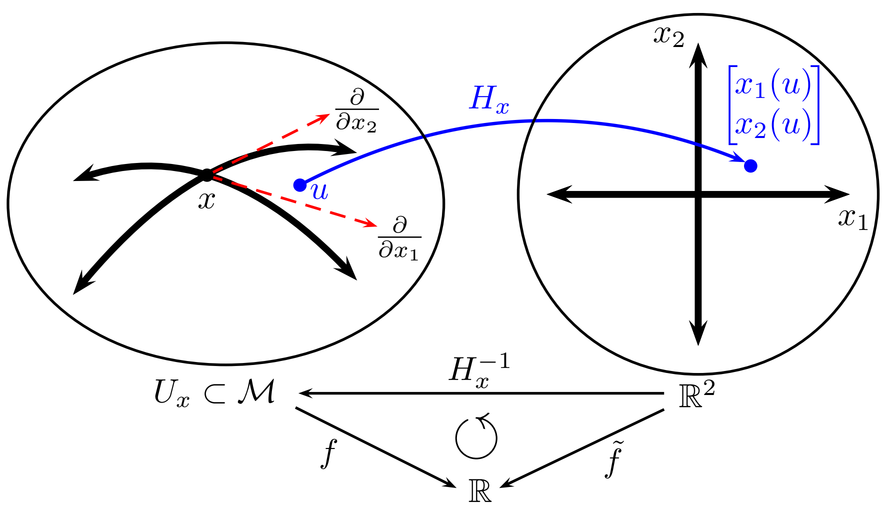


Fig. 4.2: The derivative of $f : \mathscr{M} \to \mathbb{R}$ is defined in a region U_x near $x \in \mathscr{M}$ by forming a map from $\tilde{f} : \mathbb{R}^d \to \mathbb{R}$ so that the standard partial derivative can be applied.

Any function $f \in C^1(U_x, \mathbb{R})$ can be represented in local coordinates by

$$f(u) = fH_x^{-1}H_x(u) = \tilde{f}\begin{bmatrix} x_1(u) \\ \vdots \\ x_d(u) \end{bmatrix}$$

where $\tilde{f} \equiv fH_x^{-1} : D \to \mathbb{R}$. Using calculus on $\mathbb{R}^d$, we find for each i the partial derivative $\dfrac{\partial \tilde{f}}{\partial x_i}$, and define the *differential* $df : T_x\mathscr{M} \to \mathbb{R}$ by showing where it maps the basis elements

$$df\left(\frac{\partial}{\partial x_i}\right) \equiv \frac{\partial \tilde{f}}{\partial x_i}.$$

A *vector field* $V(x)$ on $\mathscr{M}$ is an assignment of a vector at each point of $\mathscr{M}$, or in other words, a function V on $\mathscr{M}$ where $V(x) \in T_x\mathscr{M}$.

Definition 4.1 A *Riemannian manifold* is a differentiable manifold with an inner product $g_x : T_x\mathscr{M} \times T_x\mathscr{M} \to \mathbb{R}$ on each tangent space, such that for any smooth vector fields $V(x)$ and $W(x)$ on $\mathscr{M}$, the map $x \mapsto g_x(V(x), W(x))$ is smooth. The inner product is called the *Riemannian metric*.

Because the Riemannian metric is an inner product at each x, given a local coordinate basis it can be represented by a symmetric positive-definite matrix $\mathbf{g} = \mathbf{g_x}$, and by abuse of notation, we can write $g(u,v) = u^T \cdot \mathbf{g} \cdot v$ for tangent vectors $u, v \in T_x\mathscr{M}$.

The advantage of a Riemannian manifold (compared to a differentiable manifold, for example) is that the distance between two points on the manifold can be uniquely defined. This is the basis of geometry. First of all, the length of tangent vectors v can be measured as $||v||^2 = g_x(v,v)$. Then, given a differentiable curve $\gamma : [0,1] \to \mathscr{M}$, the *length* of the curve can be defined by the one-dimensional Riemann integral

$$L(\gamma) \equiv \int_0^1 ||\gamma'(t)||\, dt.$$

Finally, given two points x and y on the manifold $\mathscr{M}$, the *geodesic distance* between the points is defined to be

$$d_g(x,y) \equiv \inf_{\gamma} L(\gamma)$$

where the infimum is taken over all curves with $\gamma(0) = x$ and $\gamma(1) = y$.

4.3 Laplace-Beltrami operator

We can use the metric on a Riemannian manifold to define the concept of gradient. For a function $f \in C^\infty(\mathscr{M}, \mathbb{R})$, the *gradient* ∇f is a vector field on $\mathscr{M}$ that satisfies

$$g(\nabla f, X) = (df)(X)$$

for any vector field X on $\mathscr{M}$, where $g(u,v)$ represents the Riemannian metric.

In the local basis coordinates $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\}$ at x,

$$\nabla f = \mathbf{g}^{-1}\begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_d} \end{bmatrix}.$$

where we have replaced $\tilde{f}$ with f by abuse of notation. To justify this definition, note that the matrix inverse $\mathbf{g^{-1}}$ is needed to ensure

$$g(\frac{\partial}{\partial x_j}, \nabla f) = [0\cdots 0\ 1\ 0\cdots 0]\ \mathbf{g}\ \mathbf{g^{-1}} \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_d} \end{bmatrix} = \frac{\partial f}{\partial x_j} = df\left(\frac{\partial}{\partial x_j}\right).$$

The *divergence* of a vector field X on $\mathscr{M}$ is a scalar that is written in local basis $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\}$ at x as

$$\mathrm{div} X = \mathrm{div}\left(\begin{bmatrix} c_1(x) \\ \vdots \\ c_d(x) \end{bmatrix}\right) = \frac{1}{\sqrt{|\mathbf{g}|}} \begin{bmatrix} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_d} \end{bmatrix} \left[\sqrt{|\mathbf{g}|} \begin{bmatrix} c_1(x) \\ \vdots \\ c_d(x) \end{bmatrix}\right]$$

where $X = \sum_{i=1}^{d} c_i(x) \frac{\partial}{\partial x_i}$. Here we have denoted the determinant of the Riemannian metric by $|\mathbf{g}|$.

The *Laplace-Beltrami* operator $\Delta_{\mathscr{M}}$ is defined by

$$\Delta_{\mathscr{M}} f = \mathrm{div}(\nabla f)$$

for a function $f \in C^\infty(\mathscr{M}, \mathbb{R})$. In local coordinates, we compute

$$\begin{aligned} \Delta_{\mathscr{M}} f &= \mathrm{div}\left(\mathbf{g^{-1}} \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_d} \end{bmatrix}\right) = \frac{1}{\sqrt{|\mathbf{g}|}} \begin{bmatrix} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_d} \end{bmatrix} \left[\sqrt{|\mathbf{g}|}\ \mathbf{g^{-1}} \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_d} \end{bmatrix}\right] \\ &= \frac{1}{\sqrt{|\mathbf{g}|}} \left(\frac{\partial}{\partial x}\right)^T \left(\sqrt{|\mathbf{g}|}\ \ \mathbf{g^{-1}}\ \frac{\partial f}{\partial x}\right). \end{aligned} \tag{4.4}$$

On the Riemannian manifold $\mathscr{M} = \mathbb{R}^d$, there is a global basis of local coordinates $\{x_1, \ldots, x_d\}$, $T_x\mathbb{R}^d = \mathbb{R}^d$, and the Riemannian metric is represented by the identity matrix $\mathbf{g} = I_d$ at each x. In the global basis, the Laplace-Beltrami operator is

$$\Delta f = \frac{1}{\sqrt{|\mathbf{g}|}} \left(\frac{\partial}{\partial x}\right)^T \left(\sqrt{|\mathbf{g}|}\ \ \mathbf{g^{-1}}\ \frac{\partial f}{\partial x}\right) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}.$$

In general, for an embedded manifold with embedding $\iota : \mathscr{M} \to \mathbb{R}^n$, the induced Riemannian metric is

$$g_x(u, v) = g(\mathbf{D}\iota(\theta)u, \mathbf{D}\iota(\theta)v) = u^T \mathbf{g}(x) v$$

where $\mathbf{g}(x)$ is the $d \times d$ matrix $\mathbf{g}(x) = \mathbf{D}\iota(x)^T \mathbf{D}\iota(x)$. In the next section, we exhibit several examples of embedded manifolds, and calculate their Laplace-Beltrami operators.

4.4 Examples

Example 4.2 Circle in $\mathbb{R}^2$.

Consider the unit circle $\mathscr{M} = S^1$ with global coordinate $\theta \in [0, 2\pi)$. Let $\iota : S^1 \to \mathbb{R}^2$ be the function

$$\iota(\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

Then $\iota(\mathscr{M})$ is the embedded circle in $\mathbb{R}^2$.

At any point $\theta \in \mathscr{M}$, the tangent space $T_\theta \mathscr{M}$ is spanned by $\frac{\partial}{\partial\theta}$. The induced Riemannian metric, or pullback metric, is

$$g_\theta(u,v) = \langle D\iota(\theta)u, D\iota(\theta)v\rangle = u^T g(\theta) v$$

where $g(\theta)$ is the $m \times m$ matrix $g(\theta) = D\iota(\theta)^T D\iota(\theta)$. Here m is the dimension of the manifold and for the embedded circle,

$$D\iota(\theta) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

and

$$g(\theta) = [-\sin\theta, \cos\theta] \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = [1].$$

The Laplace-Beltrami operator is

$$\Delta f = \frac{1}{\sqrt{|g|}} \left(\frac{\partial}{\partial x}\right)^T \left(\sqrt{|g|}\ \ g^{-1} \frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial\theta}\right) = \frac{\partial^2 f}{\partial\theta^2},$$

as expected. Eigenfunctions of the Laplace-Beltrami operator on the circle include the constant function with eigenvalue 0, and for each integer k, the functions $\sin k\theta$ and $\cos k\theta$, with eigenvalue $-k^2$.

Example 4.3 Ellipse in $\mathbb{R}^2$.

This example is another embedding of the circle S^1, but the pullback metric will differ from the trivial metric on S^1. Let $\iota : S^1 \to \mathbb{R}^2$ be the function

$$\iota(\theta) = \begin{bmatrix} a\cos\theta \\ b\sin\theta \end{bmatrix}$$

where $\theta \in [0, 2\pi]$ and consider the embedded manifold $\iota(\mathscr{M})$, an ellipse in $\mathbb{R}^2$. At any point $\iota(\theta) \in \mathscr{M}$, the tangent space $T_\theta \mathscr{M}$ is spanned by $\frac{\partial}{\partial\theta}$, as in the previous example. The pullback metric on $\mathscr{M}$ from the ambient metric in $\mathbb{R}^2$ imposed on $\iota(\mathscr{M})$ is the 1×1 matrix

$$g(\theta) = D\iota(\theta)^T D\iota(\theta) = [-a\sin\theta, b\cos\theta] \begin{bmatrix} -a\sin\theta \\ b\cos\theta \end{bmatrix} = a^2\sin^2\theta + b^2\cos^2\theta.$$

The Laplacian in the new metric is

$$\Delta f = \frac{1}{\sqrt{|g(\theta)|}} \left(\frac{\partial}{\partial x}\right)^T \left(\sqrt{|g(\theta)|}\ \ g(\theta)^{-1} \frac{\partial f}{\partial x}\right) = \frac{1}{\sqrt{g(\theta)}} \frac{\partial}{\partial\theta} \left(\frac{1}{\sqrt{g(\theta)}} \frac{\partial f}{\partial\theta}\right).$$

It is straightforward to check that along with the constant eigenfunction with eigenvalue 0, there are linearly independent pairs of eigenfunctions for each positive integer k of form

$$s_k(\theta) = \sin 2\pi k I \int_0^\theta \sqrt{g(s)}\, ds, \qquad c_k(\theta) = \cos 2\pi k I \int_0^\theta \sqrt{g(s)}\, ds \tag{4.5}$$

where $I = \left(\int_0^{2\pi} \sqrt{g(s)}\, ds\right)^{-1}$. The corresponding eigenvalues are $-(2\pi k I)^2$.

Note that there are now two separate metrics we can consider on the topological circle S^1, the trivial circle metric $g(\theta) = 1$, and the metric $g(\theta) = a^2\sin^2\theta + b^2\cos^2\theta$ induced from the ellipse embedding.

Example 4.4 Flat torus in $\mathbb{R}^4$.

The torus $T^2 = S^1 \times S^1$ can be globally parameterized by two circle coordinates (θ, ϕ). For constants $a, b > 0$, an embedding of the flat torus into $\mathbb{R}^4$ can be made via the function

$$\iota(\theta,\phi) = \begin{bmatrix} a\cos\theta \\ a\sin\theta \\ b\cos\phi \\ b\sin\phi \end{bmatrix}$$

where $\theta,\phi \in [0,2\pi]$. At a point $(\theta,\phi) \in \mathscr{M}$, the two-dimensional tangent space $T_{(\theta,\phi)}\mathscr{M}$ is spanned by $\left\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\right\}$, and the tangent space $T_{\iota(\theta,\phi)}\iota(\mathscr{M})$ of the embedded manifold $\iota(\mathscr{M})$ is spanned by the two vectors $\left\{D\iota(\theta,\phi)\frac{\partial}{\partial\theta}, D\iota(\theta,\phi)\frac{\partial}{\partial\phi}\right\}$. The pullback metric on $\mathscr{M}$ from $\iota(\mathscr{M})$ is

$$g(\theta,\phi) = D\iota(\theta,\phi)^T D\iota(\theta,\phi) = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

since

$$D\iota(\theta,\phi) = \begin{bmatrix} -a\sin\theta & 0 \\ a\cos\theta & 0 \\ 0 & -b\sin\phi \\ 0 & b\cos\phi \end{bmatrix}.$$

The Laplacian on the flat torus is therefore

$$\Delta f = \frac{1}{\sqrt{|g|}} \begin{bmatrix} \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \end{bmatrix} \left(\sqrt{|g|}\;\; g^{-1} \begin{bmatrix} \frac{\partial f}{\partial\theta} \\ \frac{\partial f}{\partial\phi} \end{bmatrix} \right) = \frac{1}{a^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{b^2}\frac{\partial^2 f}{\partial\phi^2}.$$

Along with the constant eigenfunction with eigenvalue 0, the Laplace-Beltrami operator has pairs of eigenfunctions for each positive integer k of form $\sin k\theta$ and $\cos k\theta$ with eigenvalues $-k^2/a^2$, and pairs $\sin k\phi$ and $\cos k\phi$ with eigenvalues $-k^2/b^2$. In addition, note that the four functions $\sin k\theta \sin k\phi, \sin k\theta \cos k\phi, \cos k\theta \sin k\phi$, and $\cos k\theta \cos k\phi$ are eigenfunctions with eigenvalues $-k^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$.

Example 4.5 Curved torus in $\mathbb{R}^3$.

The torus T^2 can be also be embedded into $\mathbb{R}^3$ via the function

$$\iota(\theta,\phi) = \begin{bmatrix} (R+r\cos\theta)\sin\phi \\ (R+r\cos\theta)\cos\phi \\ r\sin\theta \end{bmatrix},$$

with derivative

$$D\iota(\theta,\phi) = \begin{bmatrix} -r\sin\theta\sin\phi & (R+r\cos\theta)\cos\phi \\ -r\sin\theta\cos\phi & -(R+r\cos\theta)\sin\phi \\ r\cos\theta & 0 \end{bmatrix}.$$

Here $r < R$ are the minor and major radii of the torus, respectively. The pullback metric on $\mathscr{M}$ from $\iota(\mathscr{M})$ is

$$g(\theta,\phi) = D\iota(\theta,\phi)^T D\iota(\theta,\phi) = \begin{bmatrix} r^2 & 0 \\ 0 & (R+r\cos\theta)^2 \end{bmatrix}.$$

The Laplacian on the curved torus in local coordinates is therefore

$$
\begin{aligned}
\Delta f &= \frac{1}{\sqrt{|g|}} \begin{bmatrix} \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \end{bmatrix} \left(\sqrt{|g|}\ g^{-1} \begin{bmatrix} \frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial \phi} \end{bmatrix} \right) \\
&= \frac{1}{r(R+r\cos\theta)} \begin{bmatrix} \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \end{bmatrix} \left(r(R+r\cos\theta) \begin{bmatrix} r^{-2} & 0 \\ 0 & (R+r\cos\theta)^{-2} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial \phi} \end{bmatrix} \right) \\
&= \frac{1}{r(R+r\cos\theta)} \left[\frac{\partial}{\partial \theta} \left(\frac{R+r\cos\theta}{r} \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left(\frac{r}{R+r\cos\theta} \frac{\partial f}{\partial \phi} \right) \right] \\
&= \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{(R+r\cos\theta)^2} \frac{\partial^2 f}{\partial \phi^2} - \frac{\sin\theta}{r(R+r\cos\theta)} \frac{\partial f}{\partial \phi}
\end{aligned}
$$

Example 4.6 Sphere in $\mathbb{R}^3$.

The sphere S^2 can be globally parameterized by two circle coordinates (θ, ϕ). The sphere is the image of the function $\iota : [0,\pi] \times [0,2\pi] \to \mathbb{R}^3$ defined by

$$
\iota(\theta,\phi) = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}
$$

where θ is the colatitude and ϕ is the azimuthal angle. Since

$$
D\iota(\theta,\phi) = \begin{bmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{bmatrix},
$$

the pullback metric in these coordinates is

$$
g(x) = D\iota^T D\iota = \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta & 0 \\ 0 & \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}.
$$

The Laplacian is

$$
\begin{aligned}
\Delta f &= \frac{1}{\sqrt{|g|}} \begin{bmatrix} \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \end{bmatrix} \left(\sqrt{|g|}\ g^{-1} \begin{bmatrix} \frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial \phi} \end{bmatrix} \right) \\
&= \frac{1}{\sin\theta} \begin{bmatrix} \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \end{bmatrix} \left[\begin{bmatrix} \sin\theta & 0 \\ 0 & \frac{1}{\sin\theta} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial \phi} \end{bmatrix} \right] \\
&= \frac{1}{\sin\theta} \left[\frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left(\frac{1}{\sin\theta} \frac{\partial f}{\partial \phi} \right) \right] \\
&= \cot\theta \frac{\partial f}{\partial \theta} + \frac{\partial^2 f}{\partial \theta^2} + \csc^2\theta \frac{\partial^2 f}{\partial \phi^2}.
\end{aligned}
$$

The eigenvalues are of the form $k(k+1)$ for $k = 0,1,2,\ldots$ and the eigenfunctions are called spherical harmonics [REFERENCE].

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