$P(H \mid p=0.5) = 0.5 \; p$

CS 189: Homework 5

Problem 1: Polynomial + Gaussian Kernels

1. $y=1$: $(x_1, x_2) = (8\cos(\theta) + w_1,\ 8\sin(\theta) + w_2)$

$P(x_1 \mid y=1) = 8\cos(\theta) + w_1 \qquad P(x_2 \mid y=1) = 8\sin(\theta) + w_2$

$\theta \sim \text{unif}(0, 2\pi) \qquad w_1, w_2 \sim \text{Normal}(0,1)$ iid

$y=-1$: $(x_1, x_2) = (v_1, v_2)$

$v_1, v_2 \sim \mathcal{N}(0,1)$ iid

$$\underbrace{\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ \vdots & \vdots \\ x_{200,1} & x_{200,2} \end{bmatrix}}_{200 \times 2} \rightarrow \underbrace{\begin{bmatrix} 1 \\ \vdots \\ -1 \\ -1 \\ -1 \end{bmatrix}}_{200 \times 1}$$

2. Using Regularized Least-Squares Classification, build classifier

$$\min_{\alpha} \|K\alpha - y\|^2 + \lambda \alpha^T K \alpha = J(\alpha)$$

using poly kernel, $k(x,z) = (1 + x^T z)^2$

$$\nabla_\alpha J = \frac{\partial}{\partial\alpha}(K\alpha - y)^T(K\alpha - y) + \lambda \frac{\partial}{\partial\alpha} \alpha^T K \alpha$$

$$= \frac{\partial}{\partial\alpha}(\alpha^T K^T - y^T)(K\alpha - y) + \lambda \frac{\partial}{\partial\alpha} \alpha^T \sum_{j=1}^{n} K_{ij}\alpha_j$$

$$= \frac{\partial}{\partial\alpha}(\alpha^T K^T K \alpha - 2y^T K\alpha + y^T y) + \lambda \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} K_{ij}\alpha_j$$

$$= 2K\alpha - 2y^T K + \lambda$$

$[\alpha_1\ \alpha_2\ \cdots\ \alpha_n] \begin{bmatrix} K_{11} & K_{12} & K_{1n} \\ & K_{ij} & \\ & & K_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

$K_{11}\alpha_1 + K_{12}\alpha_2 \ldots$

$$\alpha_1(K_{11}\alpha_1 + K_{12}\alpha_2 + \ldots + K_{1n}\alpha_n) + \alpha_2(K_{21}\alpha_1 + K_{22}\alpha_2 + \ldots + K_{2n}\alpha_n) + \ldots + \alpha_n(K_{n1}\alpha_1 + \ldots + K_{nn}\alpha_n)$$

$$\begin{matrix} \frac{\partial}{\partial\alpha_1}: \\ \frac{\partial}{\partial\alpha_2}: \\ \vdots \\ \frac{\partial}{\partial\alpha_n}: \end{matrix} \begin{bmatrix} 2K_{11}\alpha_1 + K_{21}\alpha_2 + \ldots + K_{n1}\alpha_n \\ K_{12}\alpha_1 + 2K_{22}\alpha_2 + \ldots + K_{n2}\alpha_n \\ \vdots \\ K_{1n}\alpha_1 + K_{2n}\alpha_2 + \ldots + 2K_{nn}\alpha_n \end{bmatrix} = \begin{bmatrix} 2K_{11} & K_{21} & \cdots & K_{n1} \\ K_{12} & 2K_{22} & \cdots & K_{n2} \\ \vdots & & & \vdots \\ K_{1n} & & \cdots & 2K_{nn} \end{bmatrix} \vec{\alpha}$$

2. Using regularized least squares classification:

$$\min_{\alpha} \|K\alpha - y\|^2 + \lambda \alpha^T K \alpha = J(\alpha)$$

$$\nabla_\alpha J(\alpha) = \frac{\partial}{\partial \alpha} (K\alpha - y)^T (K\alpha - y) + \lambda (K + K^T)\alpha$$

$$= \frac{\partial}{\partial \alpha} (\alpha^T K^T K \alpha - y^T K \alpha + y^T y) + \lambda (K + K^T)\alpha$$

$$0 = 2K\alpha - K^T y + \lambda (K + K^T)\alpha$$

$$0 = (2K + \lambda (K + K^T))\alpha = K^T y$$

$$\alpha = (2K + \lambda (K + K^T))^{-1} K^T y$$

$K_{ij} = k(x_i, x_j)$ where $k(x, z) = (x^T z + 1)^p$

$\alpha = K^T$

$$(2K + 2K\lambda)\alpha = K^T y$$

$$2K(\lambda + I)\alpha = K y$$

$2K^T K\alpha$

$$\alpha^T K^T K \alpha - 2K^T y + \lambda 2 K \alpha = 0$$

$$K^T K \alpha + \lambda K \alpha = K^T y$$

$$(K^T + \lambda I) K \alpha = K^T y$$

$$K = K^{-1} (K^T + \lambda)^{-1} K y$$

$$\alpha = (K^T + \lambda)^{-1} K^{-1} K y$$

$$K\alpha + \lambda K\alpha = Ky$$

$$K(K\alpha + \lambda\alpha)$$

$$K(K + \lambda I)\alpha = Ky$$

$$(K + \lambda I)\alpha = y$$

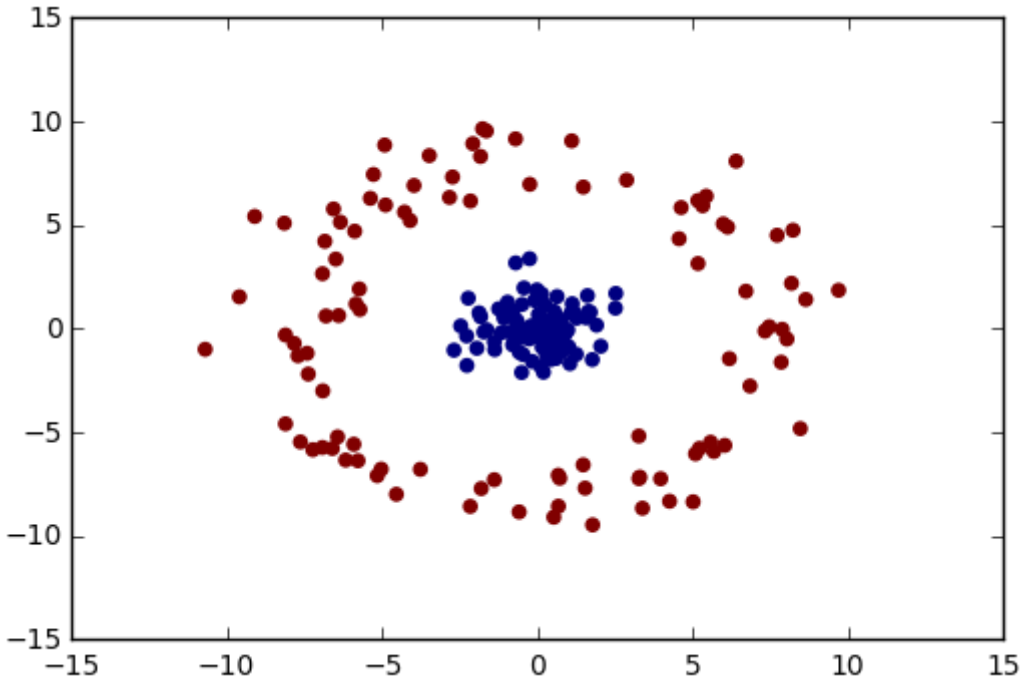
$$\alpha = (K + \lambda I)^{-1} y$$

In [14]:

```
import numpy as np
import matplotlib.pyplot as plt
import sklearn.metrics as metrics
from scipy import linalg
import pdb
%matplotlib inline

def generate_data():
    theta = np.random.uniform(0, 2*np.pi, 100)
    w1, w2 = np.random.normal(0, 1, 100), np.random.normal(0, 1, 100)
    first_x1, first_x2 = 8*np.cos(theta) + w1, 8*np.sin(theta) + w2
    second_x1, second_x2 = np.random.normal(0, 1, 100), np.random.normal(
, 1, 100)
    data = np.r_[np.c_[first_x1, first_x2, np.ones(100)], np.c_[second_x
1, second_x2, -1*np.ones(100)]]
    np.random.shuffle(data)
    X, y = data[:,:2], data[:,2]
    return X, y

def plot(X, y):
    fig = plt.figure()
    ax = fig.add_subplot(1, 1, 1)
    x1 = X[:,0]
    x2 = X[:,1]
    rgb = plt.get_cmap('jet')(y)
    ax.scatter(x1, x2, color=rgb)
    plt.show()
    # plt.savefig('/tmp/out.png')    # to save the figure to a file

def split_dataset(data, size=50000):
    return data[:size], data[size:]

def kernelize_poly(x, z, p=2):
    return np.power(x.T.dot(z) + 1, p)

def kernelize_gauss(x, z, gamma=10):
    return np.exp(-gamma*(x-z).T.dot(x-z))

def train(X_train, y_train, reg=0, kernelize=kernelize_poly):
    ''' Build a model from X_train -> y_train '''
    K = np.zeros((X_train.shape[0], X_train.shape[0]))
    for i in range(X_train.shape[0]):
        for j in range(X_train.shape[0]):
            K[i][j] = kernelize(X_train[i], X_train[j])
    return linalg.solve(K + reg*np.eye(K.shape[0]), y_train, sym_pos=Tru
e)

def predict(model, X_train, X_test, kernelize=kernelize_poly):
    ''' From model and data points, output prediction vectors '''
    pred = np.zeros(X_test.shape[0])
    for i in range(X_test.shape[0]):
        h = 0
        for j in range(X_train.shape[0]):
            h += model[j]*kernelize(X_train[j], X_test[i])
        pred[i] = h
    return np.where(pred>0.0, 1.0, -1.0)
```

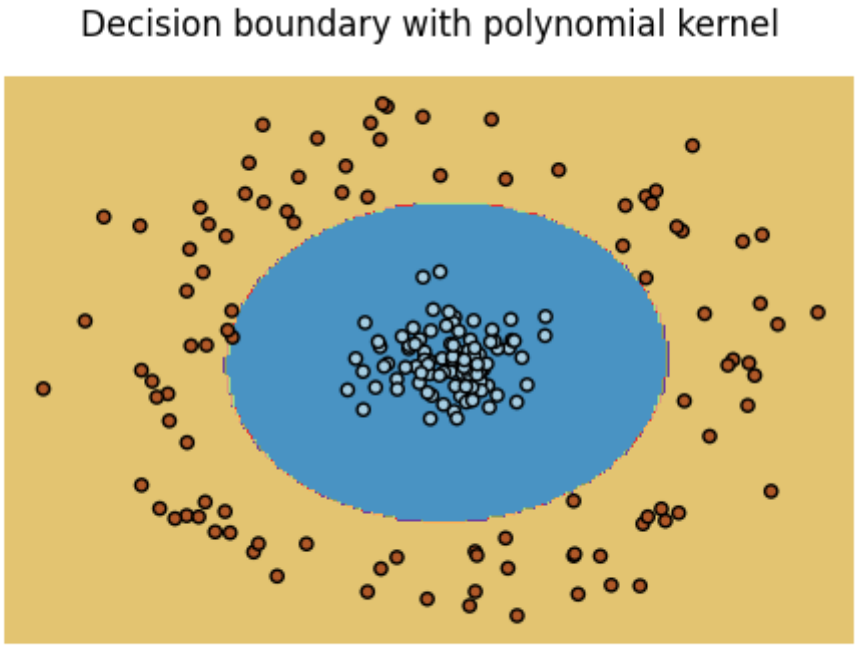
```
def plot_contour(X_train, y_train, kernelize=kernelize_poly, reg=1e-6, h=
, title='polynomial kernel'):
    model = train(X_train, y_train, reg, kernelize)
    x_min, x_max = X_train[:,0].min() - 1, X_train[:,0].max() + 1
    y_min, y_max = X_train[:,1].min() - 1, X_train[:,1].max() + 1

    xx, yy = np.meshgrid(np.arange(x_min, x_max, h), np.arange(y_min, y_
max, h))

    grid = np.c_[xx.ravel(), yy.ravel()]
    pred_grid = predict(model, X_train, grid, kernelize).reshape(xx.shap
e)

    plt.contourf(xx, yy, pred_grid, cmap=plt.cm.Paired)
    plt.axis('off')
    plt.title('Decision boundary with ' + title)

    plt.scatter(X_train[:,0], X_train[:,1], c=y_train, cmap=plt.cm.Paire
d)
```

In [6]:
```
X, y = generate_data()
X_train, y_train = X, y#X[:-50], y[:-50]
X_test, y_test = X, y#X[-50:], y[-50:]

plot(X, y)

model_poly = train(X_train, y_train, 1e-6)
pred_poly_train = predict(model_poly, X_train, X_train)
pred_poly_test = predict(model_poly, X_train, X_test)

model_gauss = train(X_train, y_train, reg=1e-4, kernelize=kernelize_gaus
s)
pred_gauss_train = predict(model_gauss, X_train, X_train, kernelize=kern
elize_gauss)
pred_gauss_test = predict(model_gauss, X_train, X_test, kernelize=kernel
ize_gauss)

print("Train accuracy: {0}".format(metrics.accuracy_score(y_train, pred_
poly_train)))
print("Test accuracy: {0}".format(metrics.accuracy_score(y_test, pred_po
ly_test)))

print("Train accuracy: {0}".format(metrics.accuracy_score(y_train, pred_
gauss_train)))
print("Test accuracy: {0}".format(metrics.accuracy_score(y_test, pred_ga
uss_test)))
```

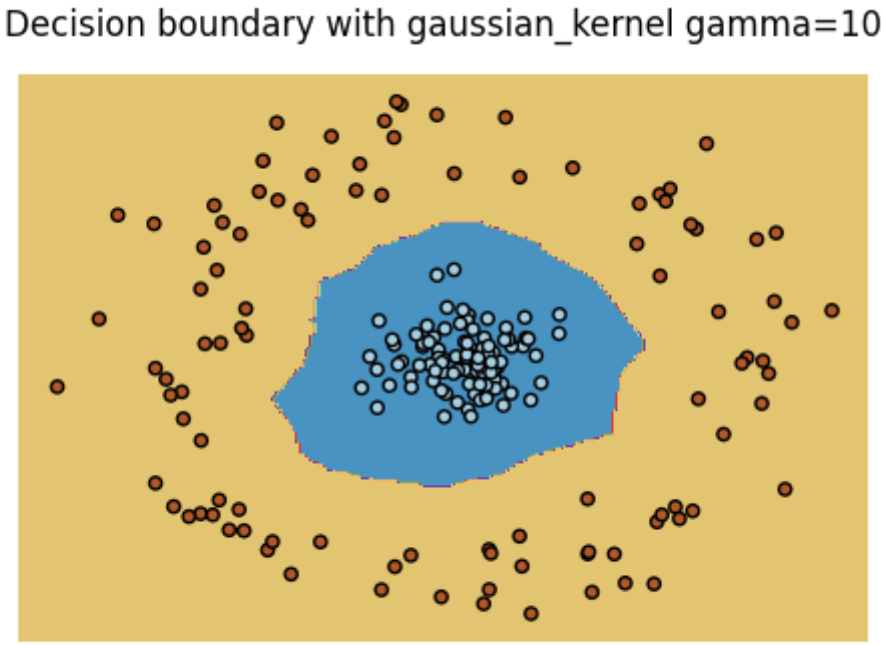



```
Train accuracy: 0.995
Test accuracy: 0.995
Train accuracy: 1.0
Test accuracy: 1.0
```

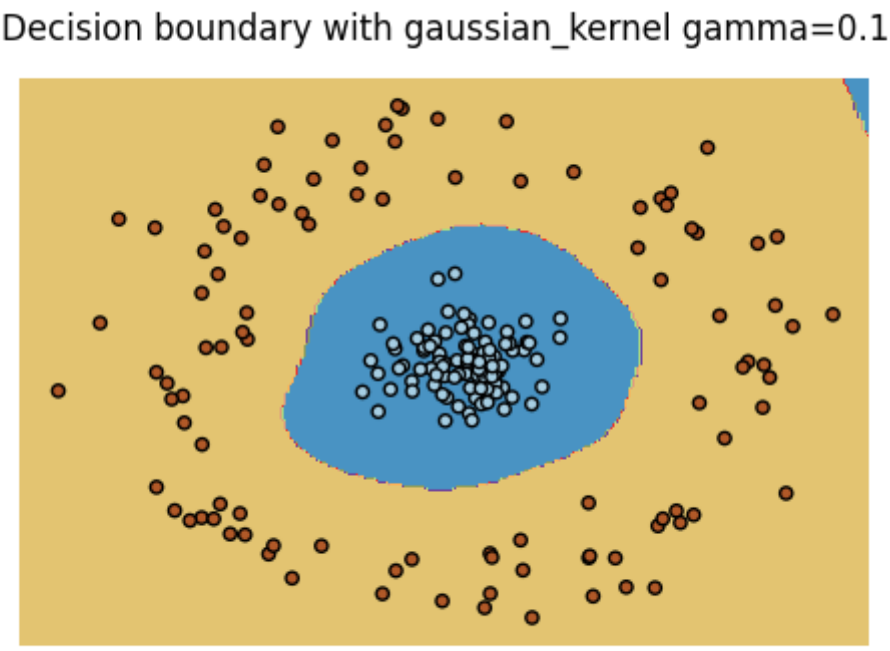
In [7]:
```
# Polynomial Kernel with reg=1e-6
plot_contour(X_train, y_train)
```



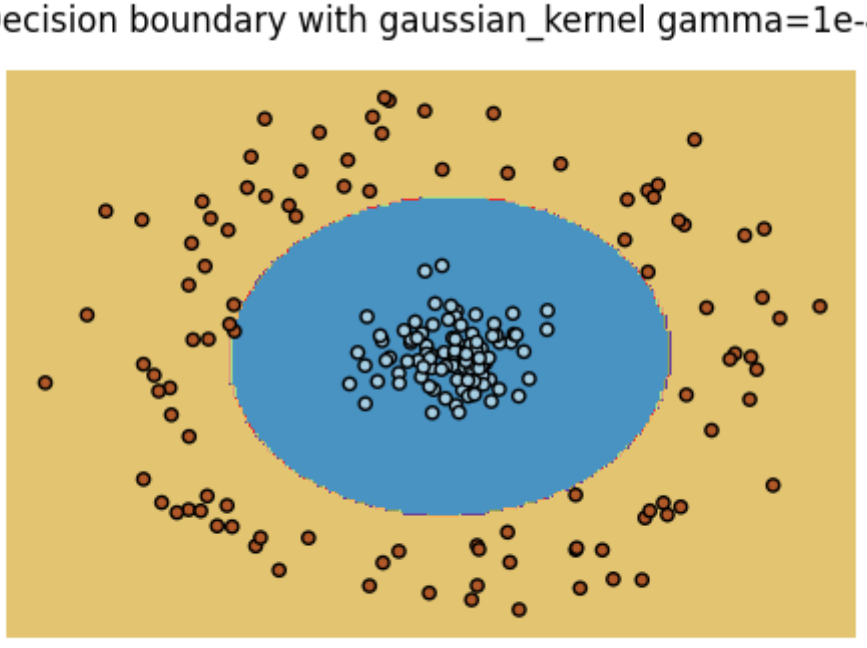
In [19]:
```
# Gaussian Kernel with reg=1e-4, gamma=10
plot_contour(X_train, y_train, kernelize=kernelize_gauss, reg=1e-4, titl
e='gaussian_kernel gamma=10')
```



In [20]:
```
# Gaussian Kernel with reg=1e-4, gamma=0.1
kernelize_gauss_1 = lambda x,z: kernelize_gauss(x, z, gamma=0.1)
plot_contour(X_train, y_train, kernelize=kernelize_gauss_1, reg=1e-4, ti
tle='gaussian_kernel gamma=0.1')
```



In [21]:
```
# Gaussian Kernel with reg=1e-4, gamma=1e-4
kernelize_gauss_2 = lambda x,z: kernelize_gauss(x, z, gamma=1e-4)
plot_contour(X_train, y_train, kernelize=kernelize_gauss_2, reg=1e-4, ti
tle='gaussian_kernel gamma=1e-4')
```



In [ ]:

```
import numpy as np
import scipy.stats
import scipy.io
import sklearn.metrics as metrics
import pdb
import pandas as pd
import sys
import csv
import math
import random

class RandomForest:
    def __init__(self, size=10, depth=6, subset=0.4):
        self.size = size
        self.depth = depth
        self.subset = subset
        self.trees = []

    def train(self, train_data, train_labels, categorical=[], bag_attrib
utes=True):
        for i in range(self.size):
            print("Training tree: ", i)
            data = np.c_[train_data, train_labels]
            rows = random.sample(range(data.shape[0]), int(self.subset*d
ata.shape[0]))
            tree = DecisionTree(self.depth, bag_attributes)
            tree.train(data[rows][:,:-1], data[rows][:,-1], categorical)
            if i%50 == 0:
                print(tree)
            self.trees.append(tree)
        return self.trees

    def predict(self, test_data):
        pred = np.zeros(test_data.shape[0])
        for i, tree in enumerate(self.trees):
            print("Predicting from tree: ", i)
            pred += tree.predict(test_data)
        return np.round(pred/self.size)

class DecisionTree:
    def __init__(self, depth=10, bag_attributes=False):
        self.root = Node()
        self.depth = depth
        self.bag_attributes = bag_attributes

    def entropy(self, hist):
        if np.count_nonzero(hist) < 2:
            return 0
        prob = hist/np.sum(hist)
        return -(prob).dot(np.log2(prob))

    def impurity(self, left_label_hist, right_label_hist):
        left_entropy, left_tot = self.entropy(left_label_hist), np.sum(l
eft_label_hist)
        right_entropy, right_tot = self.entropy(right_label_hist), np.su
m(right_label_hist)
        return (left_entropy*left_tot + right_entropy*right_tot)/(left_t
```

```
ot+right_tot)

    def counter(self, combined):
        sort = combined[np.argsort(combined[:,0])]
        freq = np.array([int(sort[0][0]), 0, 0]).reshape((1, 3))
        for i in range(sort.shape[0]):
            val = int(sort[i][0])
            label = int(sort[i][1])
            if val != freq[-1][0]:
                freq = np.r_[freq, np.array([val, 0, 0]).reshape((1,
3))]
            freq[-1][label+1] += 1
        return freq

    def segmenter(self, data, labels, categorical):
        ''' Finds split rule (feature and threshold) with lowest impurit
y '''
        split_rule = (0, 0, '')
        min_impurity = sys.maxsize

        if self.bag_attributes:
            subset = random.sample(range(data.shape[1]), int(np.sqrt(dat
a.shape[1])))
        else:
            subset = np.arange(data.shape[1])
        data = data[:,subset]
        for i, col in enumerate(data.T):
            freq = self.counter(np.c_[col, labels])
            if i in categorical:
                for j, row in enumerate(freq):
                    in_set = freq[j,1:]
                    not_in_set = np.sum(freq[:j,1:], axis=0) + np.sum(fr
eq[j+1:,1:],axis=0)
                    imp = self.impurity(in_set, not_in_set)
                    if imp < min_impurity:
                        min_impurity = imp
                        split_rule = (subset[i], row[0], 'categorical')
            else:
                for j, row in enumerate(freq):
                    left = np.sum(freq[:j,1:], axis=0)
                    right = np.sum(freq[j:,1:], axis=0)
                    imp = self.impurity(left, right)
                    if imp < min_impurity:
                        min_impurity = imp
                        split_rule = (subset[i], row[0], 'numerical') #
 (feature i, threshold_value row[0]) [can add left in to indicate index
 of split]
        return split_rule

    def train(self, train_data, train_labels, categorical=[]):
        ''' Grows decision tree to find best splits of input data '''
        def trainer(train_data, train_labels, depth):
            node = Node()
            if depth == 0:
                node.label =
np.round(np.sum(train_labels)/train_labels.shape[0])
                return node
```

```
            feature, threshold, var_type = self.segmenter(train_data, tr
ain_labels, categorical)

            data = np.c_[train_data, train_labels]
            sort = data[np.argsort(data[:,feature])]
            threshold_ind = np.where(sort[:,feature]==threshold)[0] #if
 too slow, can return index from segmenter

            if threshold_ind[0] == 0:
                node.label =
np.round(np.sum(train_labels)/train_labels.shape[0])
                return node
            if var_type == 'categorical':
                left_data = sort[threshold_ind[0]:threshold_ind[-1]+1] #
 left contains category
                right_data = np.r_[sort[:threshold_ind[0]],sort[threshol
d_ind[-1]+1:]] # right does not contain category
            else:
                left_data = sort[:threshold_ind[0]] # discludes threshol
d val and less
                right_data = sort[threshold_ind[0]:] # includes threshol
d val and greater

            if left_data.size == 0 or right_data.size == 0:
                pdb.set_trace()

            num_ones = np.sum(sort[:,-1])
            num_zeros = sort.shape[0] - num_ones
            node.split_rule = (feature, threshold, var_type, num_zeros,
num_ones)

            node.left = trainer(left_data[:,:-1], left_data[:,-1],
depth-1)
            node.right = trainer(right_data[:,:-1], right_data[:,-1], de
pth-1)
            return node
        self.root = trainer(train_data, train_labels, self.depth)
        return self.root

    def predict(self, test_data):
        ''' Traverse tree to find best label to classify test data'''
        pred = np.zeros(test_data.shape[0])
        for i, row in enumerate(test_data):
            node = self.root
            while node.split_rule != None:
                feature, threshold, var_type, __, __ = node.split_rule
                if var_type == 'categorical':
                    if row[feature] == threshold:
                        node = node.left
                    else:
                        node = node.right
                else:
                    if row[feature] < threshold:
                        node = node.left
                    else:
                        node = node.right
```

```
            pred[i] = node.label
        return pred

    def __str__(self):
        return str(self.root)

class Node:
    def __init__(self, split_rule=None, left=None, right=None, label=Non
e):
        self.split_rule = split_rule # (feature_index, threshold_value,
 var_type)
        self.left = left
        self.right = right
        self.label = label # set iff leaf node

    def __str__(self, level=0):
        ret = "\t"*level+repr(self)+"\n"
        if self.label != None:
            return ret
        # ret = "\t"*level+repr(self.value)+"\n"
        ret += self.left.__str__(level+1)
        ret += self.right.__str__(level+1)
        # for child in self.children:
            # ret += child.__str__(level+1)
        return ret

    def __repr__(self):
        return '<tree node representation>'

    def __repr__(self):
        if self.split_rule != None:
            return "Branch: " + str(self.split_rule)
        else:
            return "Leaf: " + str(self.label)

# Pre-processing
def load_dataset(filename):
    mat = scipy.io.loadmat(filename)
    return mat['training_data'], mat['test_data'],
mat['training_labels']

def split_dataset(data, prop=0.5):
    size = int(data.shape[0]*prop)
    return data[:size], data[size:]

def classify(input_file, output_file, depth,  size=10, forest=False, cat
egories=[]):
    X_train, X_test, y_train = load_dataset(input_file)
    data = np.c_[X_train, y_train.T]
    np.random.shuffle(data)
    X_train, X_validate = split_dataset(data[:,:-1])
    y_train, y_validate = split_dataset(data[:,-1:])

    if forest:
        classifier = RandomForest(size, depth)
    else:
        classifier = DecisionTree(depth)
```

```
    classifier.train(X_train, y_train, categories)
    print(classifier)

    pred_y_train = classifier.predict(X_train)
    pred_y_validate = classifier.predict(X_validate)

    train_accuracy = metrics.accuracy_score(y_train, pred_y_train)
    validate_accuracy = metrics.accuracy_score(y_validate, pred_y_valida
te)
    print("Train accuracy: {0}".format(train_accuracy))
    print("Validation accuracy: {0}".format(validate_accuracy))

    pred_y_test = classifier.predict(X_test)

    c = csv.writer(open(output_file, "wt"))
    c.writerow(['Id', 'Category'])
    for i in range(len(pred_y_test)):
        c.writerow((i+1, int(pred_y_test[i])))

    return train_accuracy, validate_accuracy

if __name__ == "__main__":
    spam_train, spam_validate = classify('spam_data/spam_data','spam_dat
a/kaggle.csv', 8, 200, forest=True)

    categories = [1, 3, 5, 6, 7, 8 ,9 ,13]
    census_train, census_validate =
classify('census_data/census_data','census_data/kaggle.csv', 40, 7, cate
gories)
```

3. Spam

a) I added features porn, single, earn, click, and subscribe to my feature set and used a bag-of-words model. Also for the forests I implemented feature bagging in order to decorrelate variables as well as data bagging in order to reduce variance

b) Tree: Train accuracy: 0.8329466357308585, Validation accuracy: 0.8205723124516628 Forest: Train accuracy: 0.8437741686001546 Validation accuracy: 0.8159319412219644 Kaggle: 0.78518 (Using Forest with 100 trees of depth 40)

c) Tree exclamation > 1 meter > 1 para > 1 pain > 1 volumes > 1 bracket > 1 semicolon > 1 and > 1

d) Forest root splits frequencies (pain>1) 4 (private>1) 1 (bank>1) 10 (spam>1) 5 (prescription>1) 6 (featured>1) 8 (differ>1) 2 (other>1) 1 (energy>1) 1 (message>1) 11 (path>1) 16 (semicolon>1) 1 (semicolon>2) 2 (dollar>1) 1, (dollar>2) 1 (sharp>1) 15 (exclamation>1) 2 (bracket>1) 9 (click>1) 3 (subscribe>1) 3