1. (a) Define the function f such that

$$f(x) = 9 \sum_{n=1}^{20} \frac{(-1)^n}{2^n} \cos(\pi n x)\ .$$

Print the result for $f(0.1)$. `(9 points)`

(b) Define the first derivative of $f(x)$ using numerical differentiation. Print its value at $x = 0.1$. `(8 points)`

(c) Use the Newton method to find the zero of $f(x)$. Use $x = 0.5$ as the starting point. `(8 points)`

2. (a) Implement the mid-point integration method. Use it to compute $\int_0^2 \cos(x)dx$ and print the result. `(9 points)`

(b) Check experimentally that the error of the mid-point method scales as h^p, find the value of p. `(8 points)`

(c) Apply the Richardson extrapolation to the problem in a). Demonstrate that you get better precision in comparison with a). `(8 points)`

3. (a) Define the function $h(k) = \frac{1+(-1)^k}{k+1}$. Print $h(10)$. `(9 points)`

(b) Solve the linear system for c_i

$$\sum_{i=0}^{n} h_{i+k} c_i = -h_{n+1+k}$$

take $n = 3$. Print your result for c_0, c_1, c_2, c_3. `(8 points)`

(c) Find the zeros of the polynomial $\phi(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + x^4$. Print the result. (You can use function “roots”). `(8 points)`

4. (a) Find the volume of the intersection of the ellipse $x^2 + y^2/4 < 1$ and the domain $y > -1$ using the Monte Carlo method with 1000 points. (9 points)

(b) Compare your result with the analytic result for the area $\frac{\sqrt{3}}{2} + \frac{4\pi}{3}$. Print the actual error and the theoretical estimate for the error of the Monte Carlo integration. (8 points)

(c) Make a plot of the points used in a) outside the domain of integration in red and the points inside the domain in yellow. (8 points)

5. Given 4 functions $f_0(x) = 1,\ f_1(x) = \sin(x),\ f_2(x) = \cos(x),\ f_3(x) = \sin(2x)$, find their linear combination which gives the least square approximation (fit) for the points

x_i	0.1	0.2	0.3	0.4	0.5
y_i	10	5	-4	-7	-1

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In order to do this, complete the following steps:

(a) Build a 4×4 matrix A and 4-dimensional vector b (9 points)

$$A_{kj} = \sum_{i=0}^{4} f_k(x_i) f_j(x_i) \quad , \quad b_k = \sum_{i=0}^{4} f_k(x_i) y_i \ .$$

(b) Find vector a from the equation $A.a = b$. Define the function $F(x) = a_0 f_0(x) + \cdots + a_3 f_3(x)$. Print $F(0.25)$. (8 points)

(c) Plot $F(x)$ as a smooth curve and the data points as dots. (8 points)

6. (a) Use the Euler method to solve the equation $y'(x) = \frac{1}{y(x)} + y(x)\sin(2x)$ in the interval $x \in [0, 10]$ with the initial data $y(0) = 1/4$. Use 200 nodal points. Print your result for $y(10)$. (9 points)

(b) Plot the resulting function $y(x)$ for $x \in [0, 1]$. (8 points)

(c) Update the Euler method in a) to the 2nd order Runge-Kutta method. Print $y(10)$. You may need to use the formula: (8 points)

$$y_{n+1} = y_n + \frac{h}{2} F(x_n, y_n) + \frac{h}{2} F\Big(x_n + h, y_n + hF(x_n, y_n)\Big) \ .$$

Final Page