I would appreciate if anyone could help me with this proof of the strong markov property for Brownian motion I'm trying to write. I have the following definitions and lemmas to help me with the proof.

Definiton 1 Let X be a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra. X is independent of $\mathcal{G}$ if, for every bounded continuous function f, the following holds:

$$\mathbb{E}\left[f(X) 1_G\right] = \mathbb{E}\left[f(X)\right] \mathbb{P}(G)$$

for every $G \in \mathcal{G}$.

Lemma 1 Let $\overline{X}, \overline{Y}$ be random vectors. If

$$\mathbb{E}\left[\prod_{i=1}^{n} f_i(X_i)\right] = \mathbb{E}\left[\prod_{i=1}^{n} f_i(Y_i)\right]$$

for all bounded continuous functions $f_1, \ldots, f_n$, then $\overline{X}$ and $\overline{Y}$ are identically distributed.

Definition 2 Let T be a stopping time. Then, the σ-algebra $\mathcal{F}_T$ is defined as

$$\mathcal{F}_T = \left(A \in \mathcal{F} \mid A \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t\right).$$

$\mathcal{F}_T$ is a summary of the events up to time T.

Theorem 1 Let $(B_t)_{t \geq 0}$ be a Brownian motion, and let T be a stopping time such that $\mathbb{P}(T < \infty) = 1$. Define the process $B'_s = B_{T+s} - B_T$. Then, (B'$_s)_{s \geq 0} isaBrownianmotion, independentof$ F_T.

Here's my proof, if there is a step which isn't clear,please point it out in the comments so I can edit it.

Proof Assume that T takes values in $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \}$ for some $n \in \mathbb{N}$. For $0 \leq s_1 < s_2 < \ldots < s_n < \infty$ and $G \in \mathcal{F}_T$, and for bounded continuous functions $f_1, \ldots, f_n$, we calculate

$$E\left[\prod_{k=1}^{n} f_k\left(B_{T+s_k} - B_{T+s_{k-1}}\right) 1_G\right] =$$

$$= E\left[\prod_{k=1}^{n} f_k\left(B_{T+s_k} - B_{T+s_{k-1}}\right) 1_G \sum_{m=0}^{\infty} 1(T = \tfrac{m}{n})\right]$$

$$= \text{ (Dominated Convergence Theorem)} =$$

$$= \sum_{m=0}^{\infty} E\left[\prod_{k=1}^{n} f_k\left(B_{T+s_k} - B_{T+s_{k-1}}\right) 1_G 1(T = \tfrac{m}{n})\right]$$

$$= \sum_{m=0}^{\infty} E\left[\prod_{k=1}^{n} f_k\left(B_{\frac{m}{n}+s_k} - B_{\frac{m}{n}+s_{k-1}}\right) 1_G 1(T = \tfrac{m}{n})\right]$$

$$= \sum_{m=0}^{\infty}\left(E\left[\prod_{k=1}^{n} f_k\left(B_{\frac{m}{n}+s_k} - B_{\frac{m}{n}+s_{k-1}}\right)\right] E\left[1_G 1(T = \tfrac{m}{n})\right]\right)$$

$$= \sum_{m=0}^{\infty}\left(E\left[\prod_{k=1}^{n} f_k\left(B_{s_k} - B_{s_{k-1}}\right)\right] E\left[1_G 1(T = \tfrac{m}{n})\right]\right)$$

$$= E\left[\prod_{k=1}^{n} f_k\left(B_{s_k} - B_{s_{k-1}}\right)\right] \sum_{m=0}^{\infty} E\left[1_G 1(T = \tfrac{m}{n})\right]$$

$$= E\left[\prod_{k=1}^{n} f_k\left(B_{s_k} - B_{s_{k-1}}\right)\right] E\left[1_G \sum_{m=0}^{\infty} 1(T = \tfrac{m}{n})\right]$$

$$= E\left[\prod_{k=1}^{n} f_k\left(B_{s_k} - B_{s_{k-1}}\right)\right] P\left[G\right]$$

We have shown that $B'_s = B - B_T$. The process $(B'_s)_{s\geq 0}$ inherits the following properties: Continuity of sample paths. By Lemma 3, the vectors $(B'_{s_2} - B'_{s_1}, \ldots, B'_{s_n} - B'_{s_{n-1}})$ and $(B_{s_2} - B_{s_1}, \ldots, B_{s_n} - B_{s_{n-1}})$ are identically distributed for any set $0 \leq s_1 < s_2 < \ldots < s_n < \infty$, and the first vector is independent of $\mathcal{F}_T$. Thus, we have shown that $(B'_s)_{s\geq 0}$ is a Brownian motion independent of $\mathcal{F}_T$.

What if T takes values in $[0, \infty)$? In this case, for T taking values in $\{0, \frac{1}{n}, \frac{2}{n}, \ldots\}$, we define $T_r = \frac{1}{r}\lceil rT \rceil$ for $r \in \mathbb{N}$. This is the largest multiple of $\frac{1}{r}$ that is $\geq T$.

It could be verified that T_r is indeed a stopping time. Since $T_r \geq T$, we have $\mathcal{F}_T \subseteq \mathcal{F}_{T_r}$. As $T_r \downarrow T$ as $r \to \infty$, for $G \in \mathcal{F}_T$, we have

$$E\left[\prod_{k=1}^{n} f_k\left(B^*_{s_k} - B^*_{s_{k-1}}\right) 1_G\right] = E\left[\prod_{k=1}^{n} f_k\left(B_{s_k} - B_{s_{k-1}}\right)\right] P(G)$$

$$E\left[\prod_{k=1}^{n} f_k\left(B^*_{s_k} - B^*_{s_{k-1}}\right) 1_G\right] = E\left[\prod_{k=1}^{n} f_k\left(B_{s_k} - B_{s_{k-1}}\right)\right] P(G)$$

As $r \to \infty$, by the continuity of realizations, we have $(B_{T_r+s_k} - B_{T_r+s_{k-1}}) \to (B_{T+s_k} - B_{T+s_{k-1}})$ $\mathbb{P}$-almost surely. By the Lebesgue Dominated Convergence Theorem, we can interchange the limit and the expectation, resulting in

$$\mathbb{E}\left[\prod_{k=1}^{n} f_k\left(B^*_{s_k} - B^*_{s_{k-1}}\right) 1_G\right] = \mathbb{E}\left[\prod_{k=1}^{n} f_k\left(B_{s_k} - B_{s_{k-1}}\right)\right] \mathbb{P}(G)$$