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Thomas Mikosch

Non-Life Insurance Mathematics

An Introduction with the
Poisson Process

Second Edition



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Preface to the Second Edition

The second edition of this book contains both basic and more advanced material on non-life insurance mathematics. Parts I and II of the book cover the basic course of the first edition; this text has changed very little. It aims at the undergraduate (bachelor) actuarial student as a first introduction to the topics of non-life insurance mathematics. Parts III and IV are new. They can serve as an independent course on stochastic models of non-life insurance mathematics at the graduate (master) level.

The basic themes in all parts of this book are point process theory, the Poisson and compound Poisson processes. Point processes constitute an important part of modern stochastic process theory. They are well understood models and have applications in a wide range of applied probability areas such as stochastic geometry, extreme value theory, queuing and large computer networks, insurance and finance.

The main idea behind a point process is counting. Counting is bread and butter in non-life insurance: the modeling of claim numbers is one of the major tasks of the actuary. Part I of this book extensively deals with counting processes on the real line, such as the Poisson, renewal and mixed Poisson processes. These processes can be studied in the point process framework as well, but such an approach requires more advanced theoretical tools. Parts I and II of this text are kept at a level which requires basic knowledge of probability theory and stochastic processes. Such knowledge is typically available in the 2nd or 3rd year of a bachelor program with a mathematical, statistical, actuarial or econometric orientation.

The new parts of this text tell the story of point processes in non-life insurance mathematics. The concept of a point process requires some knowledge of measure and advanced probability theory. The student who is familiar with the topics of Parts I and II will not have difficulties in continuing with Parts III and IV. Those who read from cover to cover will discover that many results in the first parts will become much more transparent and elegant in the framework of point processes. The advanced student may immediately start

with the parts on point and Poisson processes. I hope that not only the specialized actuarial student will benefit from this book, but anybody who wants to know about the wonderful world of counting points.

The main objective of writing this book was to produce lecture notes for my students. The material of this book grew out of courses I have been teaching in the bachelor and master actuarial programs at the University of Copenhagen. The interaction with the students kept me sufficiently realistic about the theoretical level which one may wish to approach. Through the years the success rate of my students has confirmed that the material of this book is accessible, both at the undergraduate and graduate levels.

Different ideas have inspired the process of writing the new chapters of this book. Norberg's [114] propagation of point process theory in non-life insurance mathematics has been a guide to Chapter 8. Since the beginning of the 1990s Norberg has pointed towards the enormous gains for actuarial science by using advanced point process theory.

Insurance mathematics is not a scientific island. Through its history actuarial science has interacted with mathematical, statistical and economic disciplines. The Cramér-Lundberg theory of collective risk is a fine example of how this applied stochastic theory gained from the ideas of renewal, queuing and large deviation theory. Various excursions will lead the reader of this book into different, but related fields of applied probability theory, including extreme value theory, teletraffic models and Lévy processes. Extreme value theory is a natural tool for analyzing the large claims in an insurance portfolio. In this text we will learn about the close relationship of extremes and point processes. We will also read about point process models which are used both in insurance mathematics and for modeling large computer networks. The compound Poisson process originates in Lundberg's thesis from 1903. This was the first use of a Lévy jump process. An excursion to Lévy processes will illustrate how a fundamental non-life insurance model was extended to a major class of stochastic processes.

In the process of writing the second edition of this book I have benefited from discussions with students, colleagues and friends. Jeffrey Collamore, Richard A. Davis, Anders Hedegaard Jessen, Jesper Lund Pedersen and Gennady Samorodnitsky have read parts of this book and proposed various improvements. My sincere thanks go to Sid Resnick for a long series of colorful and insightful comments. I have been blessed by an effort whose helpfulness went beyond what one would expect from a colleague and friend. I am indebted to Catriona Byrne, Marina Reizakis and Jef Boys from Springer-Verlag for efficient and competent professional editorial support.

Preface to the First Edition

To the outside world, insurance mathematics does not appear as a challenging topic. In fact, everyone has to deal with matters of insurance at various times of one's life. Hence this is quite an interesting perception of a field which constitutes one of the bases of modern society. There is no doubt that modern economies and states would not function without institutions which guarantee reimbursement to the individual, the company or the organization for its losses, which may occur due to natural or man-made catastrophes, fires, floods, accidents, riots, etc. The idea of insurance is part of our civilized world. It is based on the mutual trust of the insurer and the insured.

It was realized early on that this mutual trust must be based on science, not on belief and speculation. In the 20th century the necessary tools for dealing with matters of insurance were developed. These consist of probability theory, statistics and stochastic processes. The Swedish mathematicians Filip Lundberg and Harald Cramér were pioneers in these areas. They realized in the first half of the 20th century that the theory of stochastic processes provides the most appropriate framework for modeling the claims arriving in an insurance business. Nowadays, the Cramér-Lundberg model is one of the backbones of non-life insurance mathematics. It has been modified and extended in very different directions and, moreover, has motivated research in various other fields of applied probability theory, such as queuing theory, branching processes, renewal theory, reliability, dam and storage models, extreme value theory, and stochastic networks.

The aim of this book is to bring some of the standard stochastic models of non-life insurance mathematics to the attention of a wide audience which, hopefully, will include actuaries and also other applied scientists. The primary objective of this book is to provide the undergraduate actuarial student with an introduction to non-life insurance mathematics. I used parts of this text in the course on basic non-life insurance for 3rd year mathematics students at the Laboratory of Actuarial Mathematics of the University of Copenhagen. But I am convinced that the content of this book will also be of interest to others who have a background on probability theory and stochastic processes and would like to learn about applied stochastic processes. Insurance mathematics is a part of applied probability theory. Moreover, its mathematical tools are also used in other applied areas (usually under different names).

The idea of writing this book came in the spring of 2002, when I taught basic non-life insurance mathematics at the University of Copenhagen. My handwritten notes were not very much appreciated by the students, and so I decided to come up with some lecture notes for the next course given in spring, 2003. This book is an extended version of those notes and the associated weekly exercises. I have also added quite a few computer graphics to the text. Graphs help one to understand and digest the theory much easier than formulae and proofs. In particular, computer simulations illustrate where the limits of the theory actually are.

VIII Preface to the First Edition

When one writes a book, one uses the experience and knowledge of generations of mathematicians without being directly aware of it. Ole Hesselager's 1998 notes and exercises for the basic course on non-life insurance at the Laboratory of Actuarial Mathematics in Copenhagen were a guideline to the content of this book. I also benefitted from the collective experience of writing EKM [46]. The knowledgeable reader will see a few parallels between the two books. However, this book is an *introduction* to non-life insurance, whereas EKM assume that the reader is familiar with the basics of this theory and also explores various other topics of applied probability theory. After having read this book, the reader will be ready for EKM. Another influence has been Sid Resnick's enjoyable book about Happy Harry [123]. I admit that some of the mathematical taste of that book has infected mine; the interested reader will find a wealth of applied stochastic process theory in [123] which goes far beyond the scope of this book.

The choice of topics presented in this book has been dictated, on the one hand, by personal taste and, on the other hand, by some practical considerations. This course is the basis for other courses in the curriculum of the Danish actuarial education and therefore it has to cover a certain variety of topics. This education is in agreement with the Group Consultatif requirements, which are valid in most European countries.

As regards personal taste, I very much focused on methods and ideas which, in one way or other, are related to renewal theory and point processes. I am in favor of methods where one can see the underlying probabilistic structure without big machinery or analytical tools. This helps one to strengthen intuition. Analytical tools are like modern cars, whose functioning one cannot understand; one only finds out when they break down. Martingale and Markov process theory do not play an important role in this text. They are acting somewhere in the background and are not especially emphasized, since it is the author's opinion that they are not really needed for an introduction to non-life insurance mathematics. Clearly, one has to pay a price for this approach: lack of elegance in some proofs, but with elegance it is very much like with modern cars.

According to the maxim that non-Bayesians have more fun, Bayesian ideas do not play a major role in this text. Part II on experience rating is therefore rather short, but self-contained. Its inclusion is caused by the practical reasons mentioned above but it also pays respect to the influential contributions of Hans Bühlmann to modern insurance mathematics.

Some readers might miss a chapter on the interplay of insurance and finance, which has been an open subject of discussion for many years. There is no doubt that the modern actuary should be educated in modern financial mathematics, but that requires stochastic calculus and continuous-time martingale theory, which is far beyond the scope of this book. There exists a vast specialized literature on financial mathematics. This theory has dictated most of the research on financial products in insurance. To the best of the author's knowledge, there is no part of insurance mathematics which deals with

the pricing and hedging of insurance products by techniques and approaches genuinely different from those of financial mathematics.

It is a pleasure to thank my colleagues and students at the Laboratory of Actuarial Mathematics in Copenhagen for their support. Special thanks go to Jeffrey Collamore, who read much of this text and suggested numerous improvements upon my German way of writing English. I am indebted to Catriona Byrne from Springer-Verlag for professional editorial help.

If this book helps to change the perception that non-life insurance mathematics has nothing to offer but boring calculations, its author has achieved his objective.

Thomas Mikosch

Copenhagen, September 2003

Contents

Part I Collective Risk Models

1	The Basic Model	3
2	Models for the Claim Number Process	7
2.1	The Poisson Process	7
2.1.1	The Homogeneous Poisson Process, the Intensity Function, the Cramér-Lundberg Model	9
2.1.2	The Markov Property	12
2.1.3	Relations Between the Homogeneous and the Inhomogeneous Poisson Process	14
2.1.4	The Homogeneous Poisson Process as a Renewal Process	16
2.1.5	The Distribution of the Inter-Arrival Times	20
2.1.6	The Order Statistics Property	22
2.1.7	A Discussion of the Arrival Times of the Danish Fire Insurance Data 1980-1990	32
2.1.8	An Informal Discussion of Transformed and Generalized Poisson Processes	35
	Exercises	46
2.2	The Renewal Process	53
2.2.1	Basic Properties	53
2.2.2	An Informal Discussion of Renewal Theory	60
	Exercises	65
2.3	The Mixed Poisson Process	66
	Exercises	69

3	The Total Claim Amount	71
3.1	The Order of Magnitude of the Total Claim Amount	72
3.1.1	The Mean and the Variance in the Renewal Model.....	73
3.1.2	The Asymptotic Behavior in the Renewal Model	74
3.1.3	Classical Premium Calculation Principles	78
Exercises		80
3.2	Claim Size Distributions.....	82
3.2.1	An Exploratory Statistical Analysis: QQ-Plots	82
3.2.2	A Preliminary Discussion of Heavy- and Light-Tailed Distributions	86
3.2.3	An Exploratory Statistical Analysis: Mean Excess Plots	88
3.2.4	Standard Claim Size Distributions and Their Properties	94
3.2.5	Regularly Varying Claim Sizes and Their Aggregation..	99
3.2.6	Subexponential Distributions	103
Exercises		106
3.3	The Distribution of the Total Claim Amount.....	109
3.3.1	Mixture Distributions	110
3.3.2	Space-Time Decomposition of a Compound Poisson Process	115
3.3.3	An Exact Numerical Procedure for Calculating the Total Claim Amount Distribution	120
3.3.4	Approximation to the Distribution of the Total Claim Amount Using the Central Limit Theorem.....	125
3.3.5	Approximation to the Distribution of the Total Claim Amount by Monte Carlo Techniques	130
Exercises		138
3.4	Reinsurance Treaties	142
Exercises		149
4	Ruin Theory	151
4.1	Risk Process, Ruin Probability and Net Profit Condition	151
Exercises		156
4.2	Bounds for the Ruin Probability.....	157
4.2.1	Lundberg's Inequality	157
4.2.2	Exact Asymptotics for the Ruin Probability: the Small Claim Case	162
4.2.3	The Representation of the Ruin Probability as a Compound Geometric Probability	172
4.2.4	Exact Asymptotics for the Ruin Probability: the Large Claim Case	174
Exercises		177

Part II Experience Rating

5 Bayes Estimation	187
5.1 The Heterogeneity Model	187
5.2 Bayes Estimation in the Heterogeneity Model	189
Exercises	195
6 Linear Bayes Estimation	199
6.1 An Excursion to Minimum Linear Risk Estimation	200
6.2 The Bühlmann Model	204
6.3 Linear Bayes Estimation in the Bühlmann Model	206
6.4 The Bühlmann-Straub Model	209
Exercises	211

Part III A Point Process Approach to Collective Risk Theory

7 The General Poisson Process	215
7.1 The Notion of a Point Process	215
7.1.1 Definition and First Examples	215
7.1.2 Distribution and Laplace Functional	222
Exercises	224
7.2 Poisson Random Measures	226
7.2.1 Definition and First Examples	227
7.2.2 Laplace Functional and Non-Negative Poisson Integrals	232
7.2.3 Properties of General Poisson Integrals	236
Exercises	242
7.3 Construction of New Poisson Random Measures from Given Poisson Random Measures	244
7.3.1 Transformation of the Points of a Poisson Random Measure	244
7.3.2 Marked Poisson Random Measures	246
7.3.3 The Cramér-Lundberg and Related Models as Marked Poisson Random Measures	249
7.3.4 Aggregating Poisson Random Measures	254
Exercises	256
8 Poisson Random Measures in Collective Risk Theory	259
8.1 Decomposition of the Time-Claim Size Space	259
8.1.1 Decomposition by Claim Size	259
8.1.2 Decomposition by Year of Occurrence	261
8.1.3 Decomposition by Year of Reporting	263
8.1.4 Effects of Dependence Between Delay in Reporting Time and Claim Size	264

8.1.5	Effects of Inflation and Interest	266
	Exercises	267
8.2	A General Model with Delay in Reporting and Settlement of Claim Payments	268
8.2.1	The Basic Model and the Basic Decomposition of Time-Claim Size Space	268
8.2.2	The Basic Decomposition of the Claim Number Process	271
8.2.3	The Basic Decomposition of the Total Claim Amount ..	273
8.2.4	An Excursion to Teletraffic and Long Memory: The Stationary IBNR Claim Number Process	278
8.2.5	A Critique of the Basic Model	284
	Exercises	286
9	Weak Convergence of Point Processes	291
9.1	Definition and Basic Examples	292
9.1.1	Convergence of the Finite-Dimensional Distributions ..	292
9.1.2	Convergence of Laplace Functionals	294
	Exercises	299
9.2	Point Processes of Exceedances and Extremes	300
9.2.1	Convergence of the Point Processes of Exceedances ..	300
9.2.2	Convergence in Distribution of Maxima and Order Statistics Under Affine Transformations	305
9.2.3	Maximum Domains of Attraction	309
9.2.4	The Point Process of Exceedances at the Times of a Renewal Process	316
	Exercises	321
9.3	Asymptotic Theory for the Reinsurance Treaties of Extreme Value Type	324
	Exercises	331

Part IV Special Topics

10	An Excursion to Lévy Processes	335
10.1	Definition and First Examples of Lévy Processes	335
	Exercises	338
10.2	Some Basic Properties of Lévy Processes	338
	Exercises	340
10.3	Infinite Divisibility: The Lévy-Khintchine Formula	341
	Exercises	347
10.4	The Lévy-Itô Representation of a Lévy Process	348
	Exercises	355
10.5	Some Special Lévy Processes	355
	Exercises	361

11 Cluster Point Processes	363
11.1 The General Cluster Process	363
11.2 The Chain Ladder Method	365
11.2.1 The Chain Ladder Model.....	365
11.2.2 Mack's Model	366
11.2.3 Some Asymptotic Results in the Chain Ladder Model ..	369
11.2.4 Moments of the Chain Ladder Estimators	372
11.2.5 Prediction in Mack's Model	376
Exercises	381
11.3 An Informal Discussion of a Cluster Model with Poisson Arrivals	386
11.3.1 Specification of the Model	386
11.3.2 An Analysis of the First and Second Moments	389
11.3.3 A Model when Clusters are Poisson Processes	394
Exercises	402
References	405
Index	413
List of Abbreviations and Symbols	429

Part I

Collective Risk Models

The Basic Model

In 1903 the Swedish actuary Filip Lundberg [99] laid the foundations of modern risk theory. *Risk theory* is a synonym for non-life insurance mathematics, which deals with the modeling of claims that arrive in an insurance business and which gives advice on how much premium has to be charged in order to avoid bankruptcy (ruin) of the insurance company.

One of Lundberg's main contributions is the introduction of a simple model which is capable of describing the basic dynamics of a homogeneous insurance portfolio. By this we mean a portfolio of *contracts* or *policies* for similar risks such as car insurance for a particular kind of car, insurance against theft in households or insurance against water damage of one-family homes.

There are three assumptions in the model:

- Claims happen at the times T_i satisfying $0 \leq T_1 \leq T_2 \leq \dots$. We call them *claim arrivals* or *claim times* or *claim arrival times* or, simply, *arrivals*.
- The i th claim arriving at time T_i causes the *claim size* or *claim severity* X_i . The sequence (X_i) constitutes an iid sequence of non-negative random variables.
- The claim size process (X_i) and the claim arrival process (T_i) are *mutually independent*.

The iid property of the claim sizes, X_i , reflects the fact that there is a *homogeneous probabilistic structure* in the portfolio. The assumption that claim sizes and claim times be independent is very natural from an intuitive point of view. But the independence of claim sizes and claim arrivals also makes the life of the mathematician much easier, i.e., this assumption is made for mathematical convenience and tractability of the model.

Now we can define the *claim number process*

$$N(t) = \#\{i \geq 1 : T_i \leq t\}, \quad t \geq 0,$$

i.e., $N = (N(t))_{t \geq 0}$ is a counting process on $[0, \infty)$: $N(t)$ is the number of the claims which occurred by time t .

The object of main interest from the point of view of an insurance company is the *total claim amount process* or *aggregate claim amount process*:¹

$$S(t) = \sum_{i=1}^{N(t)} X_i = \sum_{i=1}^{\infty} X_i I_{[0,t]}(T_i), \quad t \geq 0.$$

The process $S = (S(t))_{t \geq 0}$ is a *random partial sum process* which refers to the fact that the deterministic index n of the partial sums $S_n = X_1 + \dots + X_n$ is replaced by the random variables $N(t)$:

$$S(t) = X_1 + \dots + X_{N(t)} = S_{N(t)}, \quad t \geq 0.$$

It is also often called a *compound (sum) process*. We will observe that the total claim amount process S shares various properties with the partial sum process. For example, asymptotic properties such as the central limit theorem and the strong law of large numbers are analogous for the two processes; see Section 3.1.2.

In Figure 1 we see a sample path of the process N and the corresponding sample path of the compound sum process S . Both paths jump at the same times T_i : by 1 for N and by X_i for S .

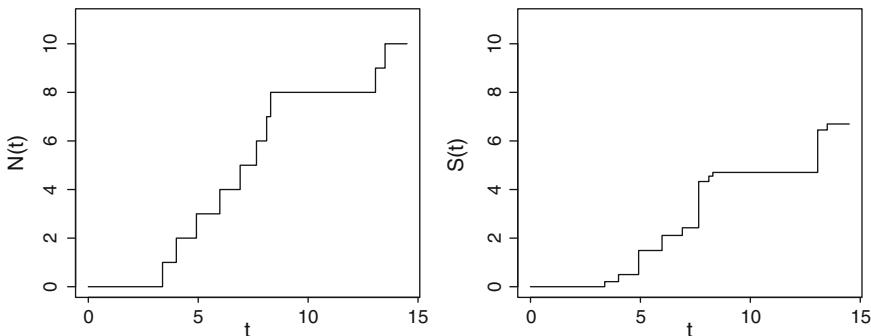


Figure 1.0.1 A sample path of the claim arrival process N (left) and of the corresponding total claim amount process S (right). Mind the difference of the jump sizes!

One would like to solve the following problems by means of insurance mathematical methods:

- Find sufficiently realistic, but simple,² probabilistic models for S and N . This means that we have to specify the distribution of the claim sizes X_i

¹ Here and in what follows, $\sum_{i=1}^0 a_i = 0$ for any real a_i and I_A is the indicator function of any set A : $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$.

² This requirement is in agreement with Einstein's maxim "as simple as possible, but not simpler".

and to introduce models for the claim arrival times T_i . The discrepancy between “realistic” and “simple” models is closely related to the question to which extent a mathematical model can describe the complicated dynamics of an insurance portfolio without being mathematically intractable.

- Determine the theoretical properties of the stochastic processes S and N . Among other things, we are interested in the distributions of S and N , their distributional characteristics such as the moments, the variance and the dependence structure. We will study the asymptotic behavior of $N(t)$ and $S(t)$ for large t and the average behavior of N and S in the interval $[0, t]$. To be more specific, we will give conditions under which the strong law of large numbers and the central limit theorem hold for S and N .
- Give simulation procedures for the processes N and S . Simulation methods have become more and more popular over the last few years. In many cases they have replaced rigorous probabilistic and/or statistical methods. The increasing power of modern computers allows one to simulate various scenarios of possible situations an insurance business might have to face in the future. This does not mean that no theory is needed any more. On the contrary, simulation generally must be based on probabilistic models for N and S ; the simulation procedure itself must exploit the theoretical properties of the processes to be simulated.
- Based on the theoretical properties of N and S , give advice how to choose a premium in order to cover the claims in the portfolio, how to build reserves, how to price insurance products, etc.

Although statistical inference on the processes S and N is utterly important for the insurance business, we do not address this aspect in a rigorous way. The statistical analysis of insurance data is not different from standard statistical methods which have been developed for iid data and for counting processes. Whereas there exist numerous monographs dealing with the inference of iid data, books on the inference of counting processes are perhaps less known. We refer to the book by Andersen et al. [3] for a comprehensive treatment.

We start with the extensive Chapter 2 on the modeling of the claim number process N . The process of main interest is the *Poisson process*. It is treated in Section 2.1. The Poisson process has various attractive theoretical properties which have been collected for several decades. Therefore it is not surprising that it made its way into insurance mathematics from the very beginning, starting with Lundberg’s thesis [99]. Although the Poisson process is perhaps not the most realistic process when it comes to fitting real-life claim arrival times, it is kind of a benchmark process. Other models for N are modifications of the Poisson process which yield greater flexibility in one way or the other.

This concerns the *renewal process* which is considered in Section 2.2. It allows for more flexibility in choosing the distribution of the inter-arrival times $T_i - T_{i-1}$. But one has to pay a price: in contrast to the Poisson process when $N(t)$ has a Poisson distribution for every t , this property is in general not valid for a renewal process. Moreover, the distribution of $N(t)$ is in general

not known. Nevertheless, the study of the renewal process has led to a strong mathematical theory, the so-called *renewal theory*, which allows one to make quite precise statements about the expected claim number $EN(t)$ for large t . We sketch renewal theory in Section 2.2.2 and explain what its purpose is without giving all mathematical details, which would be beyond the scope of this text. We will see in Section 4.2.2 on ruin probabilities that the so-called *renewal equation* is a very powerful tool which gives us a hand on measuring the *probability of ruin* in an insurance portfolio. A third model for the claim number process N is considered in Section 2.3: the *mixed Poisson process*. It is another modification of the Poisson process. By randomization of the parameters of a Poisson process (“mixing”) one obtains a class of processes which exhibit a much larger variety of sample paths than for the Poisson or the renewal processes. We will see that the mixed Poisson process has some distributional properties which completely differ from the Poisson process.

After the extensive study of the claim number process we focus in Chapter 3 on the theoretical properties of the *total claim amount process* S . We start in Section 3.1 with a description of the order of magnitude of $S(t)$. Results include the mean and the variance of $S(t)$ (Section 3.1.1) and asymptotic properties such as the strong law of large numbers and the central limit theorem for $S(t)$ as $t \rightarrow \infty$ (Section 3.1.2). We also discuss classical premium calculation principles (Section 3.1.3) which are rules of thumb for how large the premium in a portfolio should be in order to avoid ruin. These principles are consequences of the theoretical results on the growth of $S(t)$ for large t . In Section 3.2 we hint at realistic claim size distributions. In particular, we focus on *heavy-tailed* claim size distributions and study some of their theoretical properties. Distributions with regularly varying tails and subexponential distributions are introduced as *the* natural classes of distributions which are capable of describing large claim sizes. Section 3.3 continues with a study of the distributional characteristics of $S(t)$. We show some nice closure properties which certain total claim amount models (“mixture distributions”) obey; see Section 3.3.1. We also show the surprising result that a disjoint decomposition of time and/or claim size space yields independent total claim amounts on the different pieces of the partition; see Section 3.3.2. Then various exact (numerical; see Section 3.3.3) and approximate (Monte Carlo, bootstrap, central limit theorem based; see Section 3.3.4) methods for determining the distribution of $S(t)$, their advantages and drawbacks are discussed. Finally, in Section 3.4 we give an introduction to reinsurance treaties and show the link to previous theory.

A major building block of classical risk theory is devoted to the *probability of ruin*; see Chapter 4. It is a global measure of the risk one encounters in a portfolio over a long time horizon. We deal with the classical small claim case and give the celebrated estimates of Cramér and Lundberg (Sections 4.2.1 and 4.2.2). These results basically say that ruin is very unlikely for small claim sizes. In contrast to the latter results, the large claim case yields completely different results: ruin is not unlikely; see Section 4.2.4.

Models for the Claim Number Process

2.1 The Poisson Process

In this section we consider the most common claim number process: the *Poisson process*. It has very desirable theoretical properties. For example, one can derive its finite-dimensional distributions explicitly. The Poisson process has a long tradition in applied probability and stochastic process theory. In his 1903 thesis, Filip Lundberg already exploited it as a model for the claim number process N . Later on in the 1930s, Harald Cramér, the famous Swedish statistician and probabilist, extensively developed collective risk theory by using the total claim amount process S with arrivals T_i which are generated by a Poisson process. For historical reasons, but also since it has very attractive mathematical properties, the Poisson process plays a central role in insurance mathematics.

Below we will give a definition of the Poisson process, and for this purpose we now introduce some notation. For any real-valued function f on $[0, \infty)$ we write

$$f(s, t] = f(t) - f(s), \quad 0 \leq s < t < \infty.$$

Recall that an integer-valued random variable M is said to have a Poisson distribution with parameter $\lambda > 0$ ($M \sim \text{Pois}(\lambda)$) if it has distribution

$$P(M = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \dots.$$

We say that the random variable $M = 0$ a.s. has a $\text{Pois}(0)$ distribution. Now we are ready to define the *Poisson process*.

Definition 2.1.1 (Poisson process)

A stochastic process $N = (N(t))_{t \geq 0}$ is said to be a Poisson process if the following conditions hold:

- (1) *The process starts at zero: $N(0) = 0$ a.s.*

- (2) *The process has independent increments: for any t_i , $i = 0, \dots, n$, and $n \geq 1$ such that $0 = t_0 < t_1 < \dots < t_n$, the increments $N(t_{i-1}, t_i)$, $i = 1, \dots, n$, are mutually independent.*
- (3) *There exists a non-decreasing right-continuous function $\mu : [0, \infty) \rightarrow [0, \infty)$ with $\mu(0) = 0$ such that the increments $N(s, t]$ for $0 < s < t < \infty$ have a Poisson distribution $\text{Pois}(\mu(s, t])$. We call μ the mean value function of N .*
- (4) *With probability 1, the sample paths $(N(t, \omega))_{t \geq 0}$ of the process N are right-continuous for $t \geq 0$ and have limits from the left for $t > 0$. We say that N has càdlàg (continue à droite, limites à gauche) sample paths.*

We continue with some comments on this definition and some immediate consequences.

We know that a Poisson random variable M has the rare property that

$$\lambda = EM = \text{var}(M),$$

i.e., it is determined only by its mean value (= variance) if the distribution is specified as Poisson. The definition of the Poisson process essentially says that, in order to determine the distribution of the Poisson process N , it suffices to know its mean value function. The mean value function μ can be considered as an inner clock or *operational time* of the counting process N . Depending on the magnitude of $\mu(s, t]$ in the interval $(s, t]$, $s < t$, it determines how large the random increment $N(s, t]$ is.

Since $N(0) = 0$ a.s. and $\mu(0) = 0$,

$$N(t) = N(t) - N(0) = N(0, t] \sim \text{Pois}(\mu(0, t]) = \text{Pois}(\mu(t)).$$

We know that the distribution of a stochastic process (in the sense of Kolmogorov's consistency or existence theorem¹) is determined by its finite-dimensional distributions. The finite-dimensional distributions of a Poisson process have a rather simple structure: for $0 = t_0 < t_1 < \dots < t_n < \infty$,

$$(N(t_1), N(t_2), \dots, N(t_n)) =$$

$$\left(N(t_1), N(t_1) + N(t_1, t_2], N(t_1) + N(t_1, t_2] + N(t_2, t_3], \dots, \sum_{i=1}^n N(t_{i-1}, t_i] \right).$$

where any of the random variables on the right-hand side is Poisson distributed. The independent increment property makes it easy to work with the finite-dimensional distributions of N : for any integers $k_i \geq 0$, $i = 1, \dots, n$,

¹ Two stochastic processes on the real line have the same distribution in the sense of Kolmogorov's consistency theorem (cf. Rogers and Williams [126], p. 123, or Billingsley [18], p. 510) if their finite-dimensional distributions coincide. Here one considers the processes as random elements with values in the product space $\mathbb{R}^{[0, \infty)}$ of real-valued functions on $[0, \infty)$, equipped with the σ -field generated by the cylinder sets of $\mathbb{R}^{[0, \infty)}$.

$$\begin{aligned}
& P(N(t_1) = k_1, N(t_2) = k_1 + k_2, \dots, N(t_n) = k_1 + \dots + k_n) \\
&= P(N(t_1) = k_1, N(t_1, t_2] = k_2, \dots, N(t_{n-1}, t_n] = k_n) \\
&= e^{-\mu(t_1)} \frac{(\mu(t_1))^{k_1}}{k_1!} e^{-\mu(t_1, t_2]} \frac{(\mu(t_1, t_2])^{k_2}}{k_2!} \dots e^{-\mu(t_{n-1}, t_n]} \frac{(\mu(t_{n-1}, t_n])^{k_n}}{k_n!} \\
&= e^{-\mu(t_n)} \frac{(\mu(t_1))^{k_1}}{k_1!} \frac{(\mu(t_1, t_2])^{k_2}}{k_2!} \dots \frac{(\mu(t_{n-1}, t_n])^{k_n}}{k_n!}.
\end{aligned}$$

The càdlàg property is nothing but a standardization property and of purely mathematical interest which, among other things, ensures the measurability property of the stochastic process N in certain function spaces.² As a matter of fact, it is possible to show that one can define a process N on $[0, \infty)$ satisfying properties (1)-(3) of the Poisson process and having sample paths which are left-continuous and have limits from the right.³ Later, in Section 2.1.4, we will give a constructive definition of the Poisson process. That version will automatically be càdlàg.

2.1.1 The Homogeneous Poisson Process, the Intensity Function, the Cramér-Lundberg Model

The most popular Poisson process corresponds to the case of a linear mean value function μ :

$$\mu(t) = \lambda t, \quad t \geq 0,$$

for some $\lambda > 0$. A process with such a mean value function is said to be *homogeneous*, *inhomogeneous* otherwise. The quantity λ is the *intensity* or *rate* of the homogeneous Poisson process. If $\lambda = 1$, N is called *standard homogeneous Poisson process*.

More generally, we say that N has an *intensity function* or *rate function* λ if μ is absolutely continuous, i.e., for any $s < t$ the increment $\mu(s, t]$ has representation

$$\mu(s, t] = \int_s^t \lambda(y) dy, \quad s < t,$$

for some non-negative measurable function λ . A particular consequence is that μ is a continuous function.

We mentioned that μ can be interpreted as operational time or inner clock of the Poisson process. If N is homogeneous, time evolves linearly: $\mu(s, t] = \mu(s + h, t + h]$ for any $h > 0$ and $0 \leq s < t < \infty$. Intuitively, this means that

² A suitable space is the Skorokhod space \mathbb{D} of càdlàg functions on $[0, \infty)$; cf. Billingsley [17].

³ See Chapter 2 in Sato [132].

claims arrive roughly uniformly over time. We will see later, in Section 2.1.6, that this intuition is supported by the so-called *order statistics property* of a Poisson process. If N has non-constant intensity function λ time “slows down” or “speeds up” according to the magnitude of $\lambda(t)$. In Figure 2.1.2 we illustrate this effect for different choices of λ . In an insurance context, non-constant λ may refer to seasonal effects or trends. For example, in Denmark more car accidents happen in winter than in summer due to bad weather conditions. Trends can, for example, refer to an increasing frequency of (in particular, large) claims over the last few years. Such an effect has been observed in windstorm insurance in Europe and is sometimes mentioned in the context of climate change. Table 3.2.18 contains the largest insurance losses occurring in the period 1970-2007: it is obvious that the arrivals of the largest claim sizes cluster towards the end of this time period. We also refer to Section 2.1.7 for an illustration of seasonal and trend effects in a real-life claim arrival sequence.

A homogeneous Poisson process with intensity λ has

- (1) càdlàg sample paths,
- (2) starts at zero,
- (3) has independent and *stationary* increments,
- (4) $N(t)$ is $\text{Pois}(\lambda t)$ distributed for every $t > 0$.

Stationarity of the increments refers to the fact that for any $0 \leq s < t$ and $h > 0$,

$$N(s, t] \stackrel{d}{=} N(s + h, t + h] \sim \text{Pois}(\lambda(t - s)),$$

i.e., the Poisson parameter of an increment only depends on the length of the interval, not on its location.

A process on $[0, \infty)$ with properties (1)-(3) is called a *Lévy process*.⁴ The homogeneous Poisson process is one of the prime examples of Lévy processes with applications in various areas such as queuing theory, finance, insurance, stochastic networks, to name a few. Another prime example of a Lévy process is *Brownian motion* B . In contrast to the Poisson process, which is a pure jump process, Brownian motion has continuous sample paths with probability 1 and its increments $B(s, t]$ are normally $N(0, \sigma^2(t - s))$ distributed for some $\sigma > 0$. Brownian motion has a multitude of applications in physics and finance, but also in insurance mathematics. Over the last 30 years, Brownian motion has been used to model prices of speculative assets (share prices, foreign exchange rates, composite stock indices, etc.).

Finance and insurance have been merging for many years. Among other things, insurance companies invest in financial derivatives (options, futures, etc.) which are commonly modeled by functions of Brownian motion such as solutions to stochastic differential equations. If one wants to take into account

⁴ We refer to Chapter 10 for an introduction to the theory of general Lévy processes and their relation with the Poisson process.

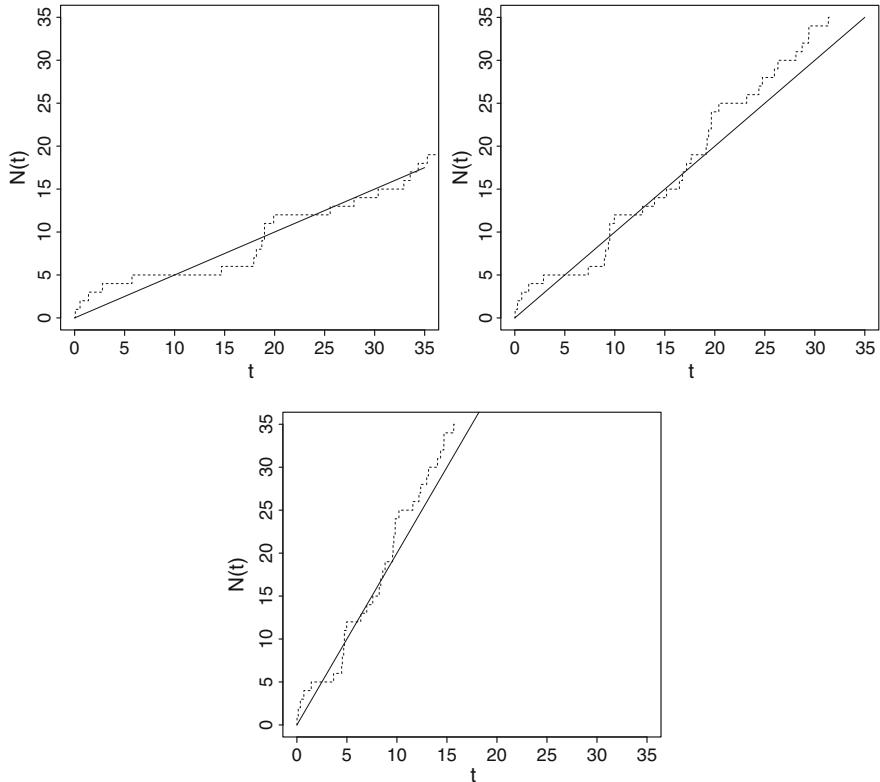


Figure 2.1.2 One sample path of a Poisson process with intensity 0.5 (top left), 1 (top right) and 2 (bottom). The straight lines indicate the corresponding mean value functions. For $\lambda = 0.5$ jumps occur less often than for the standard homogeneous Poisson process, whereas they occur more often when $\lambda = 2$.

jump characteristics of real-life financial/insurance phenomena, the Poisson process, or one of its many modifications, in combination with Brownian motion, offers the opportunity to model financial/insurance data more realistically. In this course, we follow the classical tradition of non-life insurance, where Brownian motion plays a less prominent role. This is in contrast to modern life insurance which deals with the inter-relationship of financial and insurance products.⁵ For example, unit-linked life insurance can be regarded as classical life insurance which is linked to a financial underlying such as a composite stock index (DAX, S&P 500, Nikkei, CAC40, etc.). Depending on

⁵ For a recent treatment of modern life insurance mathematics, see Møller and Steffensen [112].

the performance of the underlying, the policyholder can gain an additional bonus in excess of the cash amount which is guaranteed by the classical life insurance contracts.

Now we introduce one of the models which will be most relevant throughout this text.

Example 2.1.3 (The Cramér-Lundberg model)

The homogeneous Poisson process plays a major role in insurance mathematics. If we specify the claim number process as a homogeneous Poisson process, the resulting model which combines claim sizes and claim arrivals is called *Cramér-Lundberg model*:

- Claims happen at the arrival times $0 \leq T_1 \leq T_2 \leq \dots$ of a homogeneous Poisson process $N(t) = \#\{i \geq 1 : T_i \leq t\}$, $t \geq 0$.
- The i th claim arriving at time T_i causes the claim size X_i . The sequence (X_i) constitutes an iid sequence of non-negative random variables.
- The sequences (T_i) and (X_i) are independent. In particular, N and (X_i) are independent.

The total claim amount process S in the Cramér-Lundberg model is also called a *compound Poisson process*.

The Cramér-Lundberg model is one of the most popular and useful models in non-life insurance mathematics. Despite its simplicity it describes some of the essential features of the total claim amount process which is observed in reality.

We mention in passing that the total claim amount process S in the Cramér-Lundberg setting is a process with independent and stationary increments, starts at zero and has càdlàg sample paths. It is another important example of a Lévy process. Try to show these properties! \square

Comments

The reader who wants to learn about Lévy processes is referred to Sato's monograph [132] or the references given in Chapter 10. There we give a short introduction to this class of processes and explain the close relationship with general Poisson processes. For applications of Lévy processes in different areas, see the recent collection of papers edited by Barndorff-Nielsen et al. [12]. Rogers and Williams [126] can be recommended as an introduction to Brownian motion, its properties and related topics such as stochastic differential equations. For an elementary introduction, see Mikosch [107].

2.1.2 The Markov Property

Poisson processes constitute one particular class of *Markov processes* on $[0, \infty)$ with state space $\mathbb{N}_0 = \{0, 1, \dots\}$. This is a simple consequence of the independent increment property. It is left as an exercise to verify the *Markov*

property, i.e., for any $0 = t_0 < t_1 < \dots < t_n$ and non-decreasing natural numbers $k_i \geq 0$, $i = 1, \dots, n$, $n \geq 2$,

$$\begin{aligned} P(N(t_n) = k_n \mid N(t_1) = k_1, \dots, N(t_{n-1}) = k_{n-1}) \\ = P(N(t_n) = k_n \mid N(t_{n-1}) = k_{n-1}). \end{aligned}$$

Markov process theory does not play a prominent role in this course,⁶ in contrast to a course on modern life insurance mathematics, where Markov models are fundamental.⁷ However, the *intensity function of a Poisson process* N has a nice interpretation as the *intensity function of the Markov process* N . Before we make this statement precise, recall that the quantities

$$\begin{aligned} p_{k,k+h}(s, t) &= P(N(t) = k + h \mid N(s) = k) = P(N(t) - N(s) = h), \\ 0 \leq s < t, \quad k, h \in \mathbb{N}_0, \end{aligned}$$

are called the *transition probabilities* of the Markov process N with state space \mathbb{N}_0 . Since a.e. path $(N(t, \omega))_{t \geq 0}$ increases (verify this), one only needs to consider transitions of the Markov process N from k to $k+h$ for $h \geq 0$. The transition probabilities are closely related to the *intensities* which are given as the limits

$$\lambda_{k,k+h}(t) = \lim_{s \downarrow 0} \frac{p_{k,k+h}(t, t+s)}{s},$$

provided they and their analogs from the left exist, are finite and coincide. From the theory of stochastic processes, we know that the intensities and the initial distribution of a Markov process determine the distribution of this Markov process.⁸

Proposition 2.1.4 (Relation of the intensity function of the Poisson process and its Markov intensities)

Consider a Poisson process $N = (N(t))_{t \geq 0}$ which has a continuous intensity function λ on $[0, \infty)$. Then, for $k \geq 0$,

$$\lambda_{k,k+h}(t) = \begin{cases} \lambda(t) & \text{if } h = 1, \\ 0 & \text{if } h > 1. \end{cases}$$

In words, the intensity function $\lambda(t)$ of the Poisson process N is nothing but the intensity of the Markov process N for the transition from state k to state $k+1$. The proof of this result is left as an exercise.

⁶ It is, however, no contradiction to say that almost all stochastic models in this course have a Markov structure. But we do not emphasize this property.

⁷ See for example Koller [87] and Møller and Steffensen [112].

⁸ We leave this statement as vague as it is. The interested reader is, for example, referred to Resnick [123] or Rogers and Williams [126] for further reading on Markov processes.

The intensity function of a Markov process is a quantitative measure of the likelihood that the Markov process N jumps in a small time interval. An immediate consequence of Proposition 2.1.4 is that it is very unlikely that a Poisson process with continuous intensity function λ has jump sizes larger than 1. Indeed, consider the probability that N has a jump greater than 1 in the interval $(t, t + s]$ for some $t \geq 0, s > 0$:⁹

$$\begin{aligned} P(N(t, t + s] \geq 2) &= 1 - P(N(t, t + s] = 0) - P(N(t, t + s] = 1) \\ &= 1 - e^{-\mu(t, t+s]} - \mu(t, t + s] e^{-\mu(t, t+s]}. \end{aligned} \quad (2.1.1)$$

Since λ is continuous,

$$\mu(t, t + s] = \int_t^{t+s} \lambda(y) dy = s \lambda(t) (1 + o(1)) \rightarrow 0, \quad \text{as } s \downarrow 0.$$

Moreover, a Taylor expansion yields for $x \rightarrow 0$ that $e^x = 1 + x + o(x)$. Thus we may conclude from (2.1.1) that, as $s \downarrow 0$,

$$P(N(t, t + s] \geq 2) = o(\mu(t, t + s]) = o(s). \quad (2.1.2)$$

It is easily seen that

$$P(N(t, t + s] = 1) = \lambda(t) s (1 + o(1)). \quad (2.1.3)$$

Relations (2.1.2) and (2.1.3) ensure that a Poisson process N with continuous intensity function λ is very unlikely to have jump sizes larger than 1. Indeed, we will see in Section 2.1.4 that N has only upward jumps of size 1 with probability 1.

2.1.3 Relations Between the Homogeneous and the Inhomogeneous Poisson Process

The homogeneous and the inhomogeneous Poisson processes are very closely related: we will show in this section that a deterministic time change transforms a homogeneous Poisson process into an inhomogeneous Poisson process, and vice versa.

Let N be a Poisson process on $[0, \infty)$ with mean value function¹⁰ μ . We start with a standard homogeneous Poisson process \tilde{N} and define

⁹ Here and in what follows, we frequently use the o -notation. Recall that we write for any real-valued function h , $h(x) = o(1)$ as $x \rightarrow x_0 \in [-\infty, \infty]$ if $\lim_{x \rightarrow x_0} h(x) = 0$ and we write $h(x) = o(g(x))$ as $x \rightarrow x_0$ if $h(x) = g(x) o(1)$ for any real-valued function $g(x)$.

¹⁰ Recall that the mean value function of a Poisson process starts at zero, is non-decreasing, right-continuous and finite on $[0, \infty)$. In particular, it is a càdlàg function.

$$\widehat{N}(t) = \widetilde{N}(\mu(t)), \quad t \geq 0.$$

It is not difficult to see that \widehat{N} is again a Poisson process on $[0, \infty)$. (Verify this! Notice that the càdlàg property of μ is used to ensure the càdlàg property of the sample paths $\widehat{N}(t, \omega)$.) Since

$$\widehat{\mu}(t) = E\widehat{N}(t) = E\widetilde{N}(\mu(t)) = \mu(t), \quad t \geq 0,$$

and since the distribution of the Poisson process \widehat{N} is determined by its mean value function $\widehat{\mu}$, it follows that $N \stackrel{d}{=} \widehat{N}$, where $\stackrel{d}{=}$ refers to equality of the finite-dimensional distributions of the two processes. Hence the processes \widehat{N} and N are not distinguishable from a probabilistic point of view, in the sense of Kolmogorov's consistency theorem; see the remark on p. 8. Moreover, the sample paths of \widehat{N} are càdlàg as required in the definition of the Poisson process.

Now assume that N has a continuous and increasing mean value function μ . This property is satisfied if N has an a.e. positive intensity function λ . Then the inverse μ^{-1} of μ exists. It is left as an exercise to show that the process $\widetilde{N}(t) = N(\mu^{-1}(t))$ is a standard homogeneous Poisson process on $[0, \infty)$ if $\lim_{t \rightarrow \infty} \mu(t) = \infty$.¹¹

We summarize our findings.

Proposition 2.1.5 (The Poisson process under change of time)

Let μ be the mean value function of a Poisson process N and \widetilde{N} be a standard homogeneous Poisson process. Then the following statements hold:

- (1) The process $(\widetilde{N}(\mu(t)))_{t \geq 0}$ is Poisson with mean value function μ .
- (2) If μ is continuous, increasing and $\lim_{t \rightarrow \infty} \mu(t) = \infty$ then $(N(\mu^{-1}(t)))_{t \geq 0}$ is a standard homogeneous Poisson process.

This result, which immediately follows from the definition of a Poisson process, allows one in most cases of practical interest to switch from an inhomogeneous Poisson process to a homogeneous one by a simple time change. In particular, it suggests a straightforward way of simulating sample paths of an inhomogeneous Poisson process N from the paths of a homogeneous Poisson process. In an insurance context, one will usually be faced with inhomogeneous claim arrival processes. The above theory allows one to make an “operational time change” to a homogeneous model for which the theory is more accessible. See also Section 2.1.7 for a real-life example.

¹¹ If $\lim_{t \rightarrow \infty} \mu(t) = y_0 < \infty$ for some $y_0 > 0$, μ^{-1} is defined on $[0, y_0)$ and $\widetilde{N}(t) = N(\mu^{-1}(t))$ satisfies the properties of a standard homogeneous Poisson process restricted to the interval $[0, y_0)$. In Section 2.1.8 it is explained that such a process can be interpreted as a Poisson process on $[0, y_0)$.

2.1.4 The Homogeneous Poisson Process as a Renewal Process

In this section we study the sequence of the arrival times $0 \leq T_1 \leq T_2 \leq \dots$ of a homogeneous Poisson process with intensity $\lambda > 0$. It is our aim to find a constructive way for determining the sequence of arrivals, which in turn can be used as an alternative definition of the homogeneous Poisson process. This characterization is useful for studying the path properties of the Poisson process or for simulating sample paths.

We will show that any homogeneous Poisson process with intensity $\lambda > 0$ has representation

$$N(t) = \#\{i \geq 1 : T_i \leq t\}, \quad t \geq 0, \quad (2.1.4)$$

where

$$T_n = W_1 + \dots + W_n, \quad n \geq 1, \quad (2.1.5)$$

and (W_i) is an iid exponential $\text{Exp}(\lambda)$ sequence. In what follows, it will be convenient to write $T_0 = 0$. Since the random walk (T_n) with non-negative step sizes W_n is also referred to as *renewal sequence*, a process N with representation (2.1.4)-(2.1.5) for a general iid sequence (W_i) is called a *renewal (counting) process*. We will consider general renewal processes in Section 2.2.

Theorem 2.1.6 (The homogeneous Poisson process as a renewal process)

- (1) *The process N given by (2.1.4) and (2.1.5) with an iid exponential $\text{Exp}(\lambda)$ sequence (W_i) constitutes a homogeneous Poisson process with intensity $\lambda > 0$.*
- (2) *Let N be a homogeneous Poisson process with intensity λ and arrival times $0 \leq T_1 \leq T_2 \leq \dots$. Then N has representation (2.1.4), and (T_i) has representation (2.1.5) for an iid exponential $\text{Exp}(\lambda)$ sequence (W_i) .*

Proof. (1) We start with a renewal sequence (T_n) as in (2.1.5) and set $T_0 = 0$ for convenience. Recall the defining properties of a Poisson process from Definition 2.1.1. The property $N(0) = 0$ a.s. follows since $W_1 > 0$ a.s. By construction, a path $(N(t, \omega))_{t \geq 0}$ assumes the value i in $[T_i, T_{i+1})$ and jumps at T_{i+1} to level $i+1$. Hence the sample paths are càdlàg; cf. p. 8 for a definition.

Next we verify that $N(t)$ is $\text{Pois}(\lambda t)$ distributed. The crucial relationship is given by

$$\{N(t) = n\} = \{T_n \leq t < T_{n+1}\}, \quad n \geq 0. \quad (2.1.6)$$

Since $T_n = W_1 + \dots + W_n$ is the sum of n iid $\text{Exp}(\lambda)$ random variables it is a well-known property that T_n has a gamma $\Gamma(n, \lambda)$ distribution¹² for $n \geq 1$:

¹² You can easily verify that this is the distribution function of a $\Gamma(n, \lambda)$ distribution by taking the first derivative. The resulting probability density has the well-known gamma form $\lambda (\lambda x)^{n-1} e^{-\lambda x} / (n-1)!$. The $\Gamma(n, \lambda)$ distribution for $n \in \mathbb{N}$ is also known as the *Erlang distribution* with parameter (n, λ) .

$$P(T_n \leq x) = 1 - e^{-\lambda x} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!}, \quad x \geq 0.$$

Hence

$$P(N(t) = n) = P(T_n \leq t) - P(T_{n+1} \leq t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

This proves the Poisson property of $N(t)$.

Now we switch to the independent stationary increment property. We use a direct “brute force” method to prove this property. A more elegant way via point process techniques is indicated in Resnick [123], Proposition 4.8.1. Since the case of arbitrarily many increments becomes more involved, we focus on the case of two increments in order to illustrate the method. The general case is analogous but requires some bookkeeping. We focus on the adjacent increments $N(t) = N(0, t]$ and $N(t, t+h]$ for $t, h > 0$. We have to show that for any $k, l \in \mathbb{N}_0$,

$$\begin{aligned} q_{k,k+l}(t, t+h) &= P(N(t) = k, N(t, t+h] = l) \\ &= P(N(t) = k) P(N(t, t+h] = l) \\ &= P(N(t) = k) P(N(h) = l) \\ &= e^{-\lambda(t+h)} \frac{(\lambda t)^k (\lambda h)^l}{k! l!}. \end{aligned} \tag{2.1.7}$$

We start with the case $l = 0$, $k \geq 1$; the case $l = k = 0$ being trivial. We make use of the relation

$$\{N(t) = k, N(t, t+h] = l\} = \{N(t) = k, N(t+h) = k+l\}. \tag{2.1.8}$$

Then, by (2.1.6) and (2.1.8),

$$\begin{aligned} q_{k,k+l}(t, t+h) &= P(T_k \leq t < T_{k+1}, T_k \leq t+h < T_{k+1}) \\ &= P(T_k \leq t, t+h < T_k + W_{k+1}). \end{aligned}$$

Now we can use the facts that T_k is $\Gamma(k, \lambda)$ distributed with density $\lambda^k x^{k-1} e^{-\lambda x} / (k-1)!$ and W_{k+1} is $\text{Exp}(\lambda)$ distributed with density $\lambda e^{-\lambda x}$:

$$\begin{aligned} q_{k,k+l}(t, t+h) &= \int_0^t e^{-\lambda z} \frac{\lambda (\lambda z)^{k-1}}{(k-1)!} \int_{t+h-z}^{\infty} \lambda e^{-\lambda x} dx dz \\ &= \int_0^t e^{-\lambda z} \frac{\lambda (\lambda z)^{k-1}}{(k-1)!} e^{-\lambda(t+h-z)} dz \\ &= e^{-\lambda(t+h)} \frac{(\lambda t)^k}{k!}. \end{aligned}$$

For $l \geq 1$ we use another conditioning argument and (2.1.6):

$$\begin{aligned} & q_{k,k+l}(t, t+h) \\ &= P(T_k \leq t < T_{k+1}, T_{k+l} \leq t+h < T_{k+l+1}) \\ &= E[I_{\{T_k \leq t < T_{k+1} \leq t+h\}} \\ &\quad P(T_{k+l} - T_{k+1} \leq t+h - T_{k+1} < T_{k+l+1} - T_{k+1} \mid T_k, T_{k+1})]. \end{aligned}$$

Let N' be an independent copy of N , i.e., $N' \stackrel{d}{=} N$. Appealing to (2.1.6) and the independence of T_{k+1} and $(T_{k+l} - T_{k+1}, T_{k+l+1} - T_{k+1})$, we see that

$$\begin{aligned} & q_{k,k+l}(t, t+h) \\ &= E[I_{\{T_k \leq t < T_{k+1} \leq t+h\}} P(N'(t+h - T_{k+1}) = l-1 \mid T_{k+1})] \\ &= \int_0^t e^{-\lambda z} \frac{\lambda(\lambda z)^{k-1}}{(k-1)!} \int_{t-z}^{t+h-z} \lambda e^{-\lambda x} P(N(t+h-z-x) = l-1) dx dz \\ &= \int_0^t e^{-\lambda z} \frac{\lambda(\lambda z)^{k-1}}{(k-1)!} \int_{t-z}^{t+h-z} \lambda e^{-\lambda x} e^{-\lambda(t+h-z-x)} \frac{(\lambda(t+h-z-x))^{l-1}}{(l-1)!} \\ &\quad dx dz \\ &= e^{-\lambda(t+h)} \int_0^t \frac{\lambda(\lambda z)^{k-1}}{(k-1)!} dz \int_0^h \frac{\lambda(\lambda x)^{l-1}}{(l-1)!} dx \\ &= e^{-\lambda(t+h)} \frac{(\lambda t)^k}{k!} \frac{(\lambda h)^l}{l!}. \end{aligned}$$

This is the desired relationship (2.1.7). Since

$$P(N(t, t+h] = l) = \sum_{k=0}^{\infty} P(N(t) = k, N(t, t+h] = l),$$

it also follows from (2.1.7) that

$$P(N(t) = k, N(t, t+h] = l) = P(N(t) = k) P(N(h) = l).$$

If you have enough patience prove the analog to (2.1.7) for finitely many increments of N .

(2) Consider a homogeneous Poisson process with arrival times $0 \leq T_1 \leq T_2 \leq \dots$ and intensity $\lambda > 0$. We need to show that there exist iid exponential $\text{Exp}(\lambda)$ random variables W_i such that $T_n = W_1 + \dots + W_n$, i.e., we need to show that, for any $0 \leq x_1 \leq x_2 \leq \dots \leq x_n$, $n \geq 1$,

$$\begin{aligned} & P(T_1 \leq x_1, \dots, T_n \leq x_n) \\ &= P(W_1 \leq x_1, \dots, W_1 + \dots + W_n \leq x_n) \\ &= \int_{w_1=0}^{x_1} \lambda e^{-\lambda w_1} \int_{w_2=0}^{x_2-w_1} \lambda e^{-\lambda w_2} \dots \int_{w_n=0}^{x_n-w_1-\dots-w_{n-1}} \lambda e^{-\lambda w_n} dw_n \dots dw_1. \end{aligned}$$

The verification of this relation is left as an exercise. Hint: It is useful to exploit the relationship

$$\{T_1 \leq x_1, \dots, T_n \leq x_n\} = \{N(x_1) \geq 1, \dots, N(x_n) \geq n\}$$

for $0 \leq x_1 \leq \dots \leq x_n$, $n \geq 1$. \square

An important consequence of Theorem 2.1.6 is that the inter-arrival times

$$W_i = T_i - T_{i-1}, \quad i \geq 1,$$

of a homogeneous Poisson process with intensity λ are iid $\text{Exp}(\lambda)$. In particular, $T_i < T_{i+1}$ a.s. for $i \geq 1$, i.e., with probability 1 a homogeneous Poisson process does not have jump sizes larger than 1. Since by the strong law of large numbers $T_n/n \xrightarrow{\text{a.s.}} EW_1 = \lambda^{-1} > 0$, we may also conclude that T_n grows roughly like n/λ , and therefore there are no limit points in the sequence (T_n) at any finite instant of time. This means that the values $N(t)$ of a homogeneous Poisson process are finite on any finite time interval $[0, t]$.

The Poisson process has many amazing properties. One of them is the following phenomenon which runs in the literature under the name *inspection paradox*.

Example 2.1.7 (The inspection paradox)

Assume that you study claims which arrive in the portfolio according to a homogeneous Poisson process N with intensity λ . We have learned that the inter-arrival times $W_n = T_n - T_{n-1}$, $n \geq 1$, with $T_0 = 0$, constitute an iid $\text{Exp}(\lambda)$ sequence. Observe the portfolio at a fixed instant of time t . The last claim arrived at time $T_{N(t)}$ and the next claim will arrive at time $T_{N(t)+1}$. Three questions arise quite naturally:

- (1) What is the distribution of $B(t) = t - T_{N(t)}$, i.e., the length of the period $(T_{N(t)}, t]$ since the last claim occurred?
- (2) What is the distribution of $F(t) = T_{N(t)+1} - t$, i.e., the length of the period $(t, T_{N(t)+1}]$ until the next claim arrives?
- (3) What can be said about the joint distribution of $B(t)$ and $F(t)$?

The quantity $B(t)$ is often referred to as *backward recurrence time* or *age*, whereas $F(t)$ is called *forward recurrence time*, *excess life* or *residual life*.

Intuitively, since t lies somewhere between two claim arrivals and since the inter-arrival times are iid $\text{Exp}(\lambda)$, we would perhaps expect that $P(B(t) \leq x_1) < 1 - e^{-\lambda x_1}$, $x_1 < t$, and $P(F(t) \leq x_2) < 1 - e^{-\lambda x_2}$, $x_2 > 0$. However, these conjectures are not confirmed by calculation of the joint distribution function of $B(t)$ and $F(t)$ for $x_1, x_2 \geq 0$:

$$G_{B(t), F(t)}(x_1, x_2) = P(B(t) \leq x_1, F(t) \leq x_2).$$

Since $B(t) \leq t$ a.s. we consider the cases $x_1 < t$ and $x_1 \geq t$ separately. We observe for $x_1 < t$ and $x_2 > 0$,

$$\{B(t) \leq x_1\} = \{t - x_1 \leq T_{N(t)} \leq t\} = \{N(t - x_1, t] \geq 1\},$$

$$\{F(t) \leq x_2\} = \{t < T_{N(t)+1} \leq t + x_2\} = \{N(t, t + x_2] \geq 1\}.$$

Hence, by the independent stationary increments of N ,

$$\begin{aligned} G_{B(t), F(t)}(x_1, x_2) &= P(N(t - x_1, t] \geq 1, N(t, t + x_2] \geq 1) \\ &= P(N(t - x_1, t] \geq 1) P(N(t, t + x_2] \geq 1) \\ &= (1 - e^{-\lambda x_1}) (1 - e^{-\lambda x_2}). \end{aligned} \quad (2.1.9)$$

An analogous calculation for $x_1 \geq t$, $x_2 \geq 0$ and (2.1.9) yield

$$G_{B(t), F(t)}(x_1, x_2) = [(1 - e^{-\lambda x_1}) I_{[0, t)}(x_1) + I_{[t, \infty)}(x_1)] (1 - e^{-\lambda x_2}).$$

Hence $B(t)$ and $F(t)$ are independent, $F(t)$ is $\text{Exp}(\lambda)$ distributed and $B(t)$ has a truncated exponential distribution with a jump at t :

$$P(B(t) \leq x_1) = 1 - e^{-\lambda x_1}, \quad x_1 < t, \quad \text{and} \quad P(B(t) = t) = e^{-\lambda t}.$$

This means in particular that the forward recurrence time $F(t)$ has the same $\text{Exp}(\lambda)$ distribution as the inter-arrival times W_i of the Poisson process N . This property is closely related to the *forgetfulness property* of the exponential distribution:

$$P(W_1 > x + y \mid W_1 > x) = P(W_1 > y), \quad x, y \geq 0,$$

(Verify the correctness of this relation.) and is also reflected in the independent increment property of the Poisson process. It is interesting to observe that

$$\lim_{t \rightarrow \infty} P(B(t) \leq x_1) = 1 - e^{-\lambda x_1}, \quad x_1 > 0.$$

Thus, in an “asymptotic“ sense, both $B(t)$ and $F(t)$ become independent and are exponentially distributed with parameter λ .

We will return to the forward and backward recurrence times of a general renewal process, i.e., when W_i are not necessarily iid exponential random variables, in Example 2.2.14. \square

2.1.5 The Distribution of the Inter-Arrival Times

By virtue of Proposition 2.1.5, an inhomogeneous Poisson process N with mean value function μ can be interpreted as a time changed standard homogeneous Poisson process \tilde{N} :

$$(N(t))_{t \geq 0} \stackrel{d}{=} (\tilde{N}(\mu(t)))_{t \geq 0}.$$

In particular, let (\tilde{T}_i) be the arrival sequence of \tilde{N} and μ be increasing and continuous. Then the inverse μ^{-1} exists and

$$N'(t) = \#\{i \geq 1 : \tilde{T}_i \leq \mu(t)\} = \#\{i \geq 1 : \mu^{-1}(\tilde{T}_i) \leq t\}, \quad t \geq 0,$$

is a representation of N in the sense of identity of the finite-dimensional distributions, i.e., $N \stackrel{d}{=} N'$. Therefore and by virtue of Theorem 2.1.6 the arrival times of an inhomogeneous Poisson process with mean value function μ have representation

$$T_n = \mu^{-1}(\tilde{T}_n), \quad \tilde{T}_n = \tilde{W}_1 + \cdots + \tilde{W}_n, \quad n \geq 1, \quad \tilde{W}_i \text{ iid Exp}(1). \quad (2.1.10)$$

Proposition 2.1.8 (Joint distribution of arrival/inter-arrival times)

Assume N is a Poisson process on $[0, \infty)$ with a continuous a.e. positive intensity function λ . Then the following statements hold.

(1) The vector of the arrival times (T_1, \dots, T_n) has density

$$f_{T_1, \dots, T_n}(x_1, \dots, x_n) = e^{-\mu(x_n)} \prod_{i=1}^n \lambda(x_i) I_{\{0 < x_1 < \dots < x_n\}}. \quad (2.1.11)$$

(2) The vector of inter-arrival times $(W_1, \dots, W_n) = (T_1, T_2 - T_1, \dots, T_n - T_{n-1})$ has density

$$f_{W_1, \dots, W_n}(x_1, \dots, x_n) = e^{-\mu(x_1 + \dots + x_n)} \prod_{i=1}^n \lambda(x_1 + \dots + x_i), \quad x_i \geq 0. \quad (2.1.12)$$

Proof. Since the intensity function λ is a.e. positive and continuous, $\mu(t) = \int_0^t \lambda(s) ds$ is increasing and μ^{-1} exists. Moreover, μ is differentiable, and $\mu'(t) = \lambda(t)$. We make use of these two facts in what follows.

(1) We start with a standard homogeneous Poisson process. Then its arrivals \tilde{T}_n have representation $\tilde{T}_n = \tilde{W}_1 + \cdots + \tilde{W}_n$ for an iid standard exponential sequence (\tilde{W}_i) . The joint density of $(\tilde{T}_1, \dots, \tilde{T}_n)$ is obtained from the joint density of $(\tilde{W}_1, \dots, \tilde{W}_n)$ via the transformation:

$$(y_1, \dots, y_n) \xrightarrow{S} (y_1, y_1 + y_2, \dots, y_1 + \cdots + y_n),$$

$$(z_1, \dots, z_n) \xrightarrow{S^{-1}} (z_1, z_2 - z_1, \dots, z_n - z_{n-1}).$$

Note that $\det(\partial S(\mathbf{y})/\partial \mathbf{y}) = 1$. Standard techniques for density transformations (cf. Billingsley [18], p. 229) yield for $0 < x_1 < \cdots < x_n$,

$$\begin{aligned} f_{\tilde{T}_1, \dots, \tilde{T}_n}(x_1, \dots, x_n) &= f_{\tilde{W}_1, \dots, \tilde{W}_n}(x_1, x_2 - x_1, \dots, x_n - x_{n-1}) \\ &= e^{-x_1} e^{-(x_2 - x_1)} \cdots e^{-(x_n - x_{n-1})} = e^{-x_n}. \end{aligned}$$

Since μ^{-1} exists we conclude from (2.1.10) that for $0 < x_1 < \cdots < x_n$,

$$\begin{aligned}
P(T_1 \leq x_1, \dots, T_n \leq x_n) &= P(\mu^{-1}(\tilde{T}_1) \leq x_1, \dots, \mu^{-1}(\tilde{T}_n) \leq x_n) \\
&= P(\tilde{T}_1 \leq \mu(x_1), \dots, \tilde{T}_n \leq \mu(x_n)) \\
&= \int_0^{\mu(x_1)} \cdots \int_0^{\mu(x_n)} f_{\tilde{T}_1, \dots, \tilde{T}_n}(y_1, \dots, y_n) dy_n \cdots dy_1 \\
&= \int_0^{\mu(x_1)} \cdots \int_0^{\mu(x_n)} e^{-y_n} I_{\{y_1 < \dots < y_n\}} dy_n \cdots dy_1.
\end{aligned}$$

Taking partial derivatives with respect to the variables x_1, \dots, x_n and noticing that $\mu'(x_i) = \lambda(x_i)$, we obtain the desired density (2.1.11).

(2) Relation (2.1.12) follows by an application of the above transformations S and S^{-1} from the density of (T_1, \dots, T_n) :

$$f_{W_1, \dots, W_n}(w_1, \dots, w_n) = f_{T_1, \dots, T_n}(w_1, w_1 + w_2, \dots, w_1 + \dots + w_n).$$

□

From (2.1.12) we may conclude that the joint density of W_1, \dots, W_n can be written as the product of the densities of the W_i 's if and only if $\lambda(\cdot) \equiv \lambda$ for some positive constant λ . This means that only in the case of a homogeneous Poisson process are the inter-arrival times W_1, \dots, W_n independent (and identically distributed). This fact is another property which distinguishes the homogeneous Poisson process within the class of all Poisson processes on $[0, \infty)$.

2.1.6 The Order Statistics Property

In this section we study one of the most important properties of the Poisson process which in a sense characterizes the Poisson process. It is the *order statistics property* which it shares only with the mixed Poisson process to be considered in Section 2.3. In order to formulate this property we first give a well-known result on the distribution of the order statistics

$$X_{(1)} \leq \cdots \leq X_{(n)}$$

of an iid sample X_1, \dots, X_n .

Lemma 2.1.9 (Joint density of order statistics)

If the iid X_i 's have density f then the density of the vector $(X_{(1)}, \dots, X_{(n)})$ is given by

$$f_{X_{(1)}, \dots, X_{(n)}}(x_1, \dots, x_n) = n! \prod_{i=1}^n f(x_i) I_{\{x_1 < \dots < x_n\}}.$$

Remark 2.1.10 By construction of the order statistics, the support of the vector $(X_{(1)}, \dots, X_{(n)})$ is the set

$$C_n = \{(x_1, \dots, x_n) : x_1 \leq \dots \leq x_n\} \subset \mathbb{R}^n,$$

and therefore the density $f_{X_{(1)}, \dots, X_{(n)}}$ vanishes outside C_n . Since the existence of a density of X_i implies that all elements of the iid sample X_1, \dots, X_n are different a.s., the \leq 's in the definition of C_n could be replaced by $<$'s. \square

Proof. We start by recalling that the iid sample X_1, \dots, X_n with common density f has *no ties*. This means that the event

$$\tilde{\Omega} = \{X_{(1)} < \dots < X_{(n)}\} = \{X_i \neq X_j \text{ for } 1 \leq i < j \leq n\}$$

has probability 1. It is an immediate consequence of the fact that for $i \neq j$,

$$P(X_i = X_j) = E[P(X_i = X_j | X_j)] = \int_{\mathbb{R}} P(X_i = y) f(y) dy = 0,$$

since $P(X_i = y) = \int_{\{y\}} f(z) dz = 0$. Then

$$1 - P(\tilde{\Omega}) = P\left(\bigcup_{1 \leq i < j \leq n} \{X_i = X_j\}\right) \leq \sum_{1 \leq i < j \leq n} P(X_i = X_j) = 0.$$

Now we turn to the proof of the statement of the lemma. Let Π_n be the set of the permutations π of $\{1, \dots, n\}$. Fix the values $x_1 < \dots < x_n$. Then

$$P(X_{(1)} \leq x_1, \dots, X_{(n)} \leq x_n) = P\left(\bigcup_{\pi \in \Pi_n} A_{\pi}\right), \quad (2.1.13)$$

where

$$A_{\pi} = \{X_{\pi(i)} = X_{(i)}, i = 1, \dots, n\} \cap \tilde{\Omega} \cap \{X_{\pi(1)} \leq x_1, \dots, X_{\pi(n)} \leq x_n\}.$$

The identity (2.1.13) means that the ordered sample $X_{(1)} < \dots < X_{(n)}$ could have come from any of the ordered values $X_{\pi(1)} < \dots < X_{\pi(n)}$, $\pi \in \Pi_n$, where we also make use of the fact that there are no ties in the sample. Since the A_{π} 's are disjoint,

$$P\left(\bigcup_{\pi \in \Pi_n} A_{\pi}\right) = \sum_{\pi \in \Pi_n} P(A_{\pi}).$$

Moreover, since the X_i 's are iid,

$$\begin{aligned} P(A_{\pi}) &= P((X_{\pi(1)}, \dots, X_{\pi(n)}) \in C_n \cap (-\infty, x_1] \times \dots \times (-\infty, x_n]) \\ &= P((X_1, \dots, X_n) \in C_n \cap (-\infty, x_1] \times \dots \times (-\infty, x_n]) \\ &= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \prod_{i=1}^n f(y_i) I_{\{y_1 < \dots < y_n\}} dy_n \cdots dy_1. \end{aligned}$$

Therefore and since there are $n!$ elements in Π_n ,

$$\begin{aligned} & P(X_{(1)} \leq x_1, \dots, X_{(n)} \leq x_n) \\ &= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} n! \prod_{i=1}^n f(y_i) I_{\{y_1 < \dots < y_n\}} dy_n \cdots dy_1. \end{aligned} \quad (2.1.14)$$

By Remark 2.1.10 about the support of $(X_{(1)}, \dots, X_{(n)})$ and by virtue of the Radon-Nikodym theorem, we can read off the density of $(X_{(1)}, \dots, X_{(n)})$ as the integrand in (2.1.14). Indeed, the Radon-Nikodym theorem ensures that the integrand is the a.e. unique probability density of $(X_{(1)}, \dots, X_{(n)})$.¹³ \square

We are now ready to formulate one of the main results of this course.

Theorem 2.1.11 (Order statistics property of the Poisson process)

Consider the Poisson process $N = (N(t))_{t \geq 0}$ with continuous a.e. positive intensity function λ and arrival times $0 < T_1 < T_2 < \dots$ a.s. Then the conditional distribution of (T_1, \dots, T_n) given $\{N(t) = n\}$ is the distribution of the ordered sample $(X_{(1)}, \dots, X_{(n)})$ of an iid sample X_1, \dots, X_n with common density $\lambda(x)/\mu(t)$, $0 < x \leq t$:

$$(T_1, \dots, T_n \mid N(t) = n) \stackrel{d}{=} (X_{(1)}, \dots, X_{(n)}).$$

In other words, the left-hand vector has conditional density

$$f_{T_1, \dots, T_n}(x_1, \dots, x_n \mid N(t) = n) = \frac{n!}{(\mu(t))^n} \prod_{i=1}^n \lambda(x_i), \quad (2.1.15)$$

$$0 < x_1 < \dots < x_n < t.$$

Proof. We show that the limit

$$\lim_{h_i \downarrow 0, i=1, \dots, n} \frac{P(T_1 \in (x_1, x_1 + h_1], \dots, T_n \in (x_n, x_n + h_n] \mid N(t) = n)}{h_1 \cdots h_n} \quad (2.1.16)$$

exists and is a continuous function of the x_i 's. A similar argument (which we omit) proves the analogous statement for the intervals $(x_i - h_i, x_i]$ with the same limit function. The limit can be interpreted as a density for the conditional probability distribution of (T_1, \dots, T_n) , given $\{N(t) = n\}$.

¹³ Relation (2.1.14) means that for all rectangles $R = (-\infty, x_1] \times \cdots \times (-\infty, x_n]$ with $0 \leq x_1 < \cdots < x_n$ and for $\mathbf{X}_n = (X_{(1)}, \dots, X_{(n)})$, $P(\mathbf{X}_n \in R) = \int_R f_{\mathbf{X}_n}(\mathbf{x}) d\mathbf{x}$. By the particular form of the support of \mathbf{X}_n , the latter relation remains valid for any rectangles in \mathbb{R}^n . An extension argument (cf. Billingsley [18]) ensures that the distribution of \mathbf{X}_n is absolutely continuous with respect to Lebesgue measure with a density which coincides with $f_{\mathbf{X}_n}$ on the rectangles. The Radon-Nikodym theorem ensures the a.e. uniqueness of $f_{\mathbf{X}_n}$.

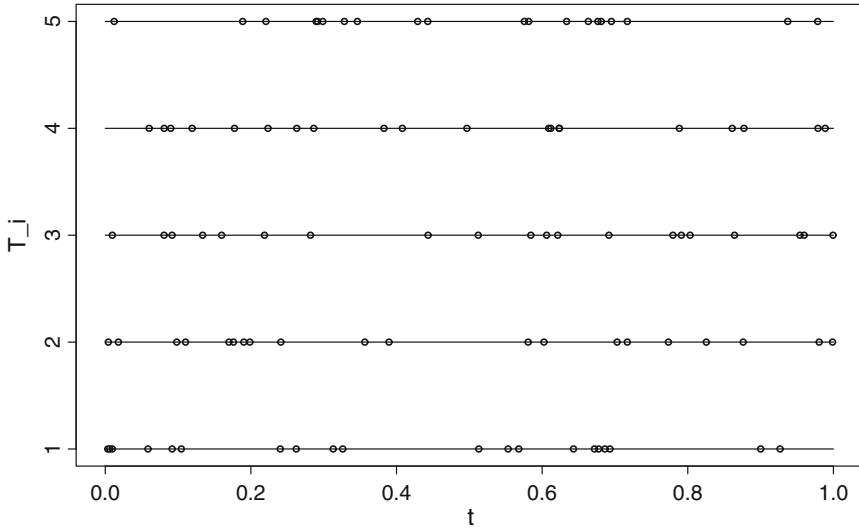


Figure 2.1.12 Five realizations of the arrival times T_i of a standard homogeneous Poisson process conditioned to have 20 arrivals in $[0, 1]$. The arrivals in each row can be interpreted as the ordered sample of an iid $U(0, 1)$ sequence.

Since $0 < x_1 < \dots < x_n < t$ we can choose the h_i 's so small that the intervals $(x_i, x_i + h_i] \subset [0, t]$, $i = 1, \dots, n$, become disjoint. Then the following identity is immediate:

$$\begin{aligned} & \{T_1 \in (x_1, x_1 + h_1], \dots, T_n \in (x_n, x_n + h_n], N(t) = n\} \\ &= \{N(0, x_1] = 0, N(x_1, x_1 + h_1] = 1, N(x_1 + h_1, x_2] = 0, \\ & \quad N(x_2, x_2 + h_2] = 1, \dots, N(x_{n-1} + h_{n-1}, x_n] = 0, \\ & \quad N(x_n, x_n + h_n] = 1, N(x_n + h_n, t] = 0\}. \end{aligned}$$

Taking probabilities on both sides and exploiting the independent increments of the Poisson process N , we obtain

$$\begin{aligned} & P(T_1 \in (x_1, x_1 + h_1], \dots, T_n \in (x_n, x_n + h_n], N(t) = n) \\ &= P(N(0, x_1] = 0) P(N(x_1, x_1 + h_1] = 1) P(N(x_1 + h_1, x_2] = 0) \\ & \quad P(N(x_2, x_2 + h_2] = 1) \cdots P(N(x_{n-1} + h_{n-1}, x_n] = 0) \\ & \quad P(N(x_n, x_n + h_n] = 1) P(N(x_n + h_n, t] = 0) \end{aligned}$$

$$\begin{aligned}
&= e^{-\mu(x_1)} \left[\mu(x_1, x_1 + h_1) e^{-\mu(x_1, x_1 + h_1)} \right] e^{-\mu(x_1 + h_1, x_2)} \\
&\quad \left[\mu(x_2, x_2 + h_2) e^{-\mu(x_2, x_2 + h_2)} \right] \dots e^{-\mu(x_{n-1} + h_{n-1}, x_n)} \\
&\quad \left[\mu(x_n, x_n + h_n) e^{-\mu(x_n, x_n + h_n)} \right] e^{-\mu(x_n + h_n, t)} \\
&= e^{-\mu(t)} \mu(x_1, x_1 + h_1) \dots \mu(x_n, x_n + h_n).
\end{aligned}$$

Dividing by $P(N(t) = n) = e^{-\mu(t)}(\mu(t))^n/n!$ and $h_1 \dots h_n$, we obtain the scaled conditional probability

$$\begin{aligned}
&\frac{P(T_1 \in (x_1, x_1 + h_1], \dots, T_n \in (x_n, x_n + h_n] \mid N(t) = n)}{h_1 \dots h_n} \\
&= \frac{n!}{(\mu(t))^n} \frac{\mu(x_1, x_1 + h_1)}{h_1} \dots \frac{\mu(x_n, x_n + h_n)}{h_n} \\
&\rightarrow \frac{n!}{(\mu(t))^n} \lambda(x_1) \dots \lambda(x_n), \quad \text{as } h_i \downarrow 0, \quad i = 1, \dots, n.
\end{aligned}$$

Keeping in mind (2.1.16), this is the desired relation (2.1.15). In the last step we used the continuity of λ to show that $\lambda'(x_i) = \lambda(x_i)$. \square

Example 2.1.13 (Order statistics property of the homogeneous Poisson process)

Consider a homogeneous Poisson process with intensity $\lambda > 0$. Then Theorem 2.1.11 yields the joint conditional density of the arrival times T_i :

$$f_{T_1, \dots, T_n}(x_1, \dots, x_n \mid N(t) = n) = n! t^{-n}, \quad 0 < x_1 < \dots < x_n < t.$$

A glance at Lemma 2.1.9 convinces one that this is the joint density of a uniform ordered sample $U_{(1)} < \dots < U_{(n)}$ of iid $U(0, t)$ distributed U_1, \dots, U_n . Thus, given there are n arrivals of a homogeneous Poisson process in the interval $[0, t]$, these arrivals constitute the points of a uniform ordered sample in $(0, t)$. In particular, this property is independent of the intensity λ ! \square

Example 2.1.14 (Symmetric function)

We consider a symmetric measurable function g on \mathbb{R}^n , i.e., for any permutation π of $\{1, \dots, n\}$ we have

$$g(x_1, \dots, x_n) = g(x_{\pi(1)}, \dots, x_{\pi(n)}).$$

Such functions include products and sums:

$$g_s(x_1, \dots, x_n) = \sum_{i=1}^n x_i, \quad g_p(x_1, \dots, x_n) = \prod_{i=1}^n x_i.$$

Under the conditions of Theorem 2.1.11 and with the same notation, we conclude that

$$(g(T_1, \dots, T_n) \mid N(t) = n) \stackrel{d}{=} g(X_{(1)}, \dots, X_{(n)}) = g(X_1, \dots, X_n).$$

For example, for any measurable function f on \mathbb{R} ,

$$\left(\sum_{i=1}^n f(T_i) \mid N(t) = n \right) \stackrel{d}{=} \sum_{i=1}^n f(X_{(i)}) = \sum_{i=1}^n f(X_i).$$

□

Example 2.1.15 (Shot noise)

This kind of stochastic process was used early on to model an electric current. Electrons arrive according to a homogeneous Poisson process N with rate λ at times T_i . An arriving electron produces an electric current whose time evolution of discharge is described as a deterministic function f with $f(t) = 0$ for $t < 0$. Shot noise describes the electric current at time t produced by all electrons arrived by time t as a superposition:

$$S(t) = \sum_{i=1}^{N(t)} f(t - T_i).$$

Typical choices for f are exponential functions $f(t) = e^{-\theta t} I_{[0, \infty)}(t)$, $\theta > 0$. An extension of classical shot noise processes with various applications is the process

$$S(t) = \sum_{i=1}^{N(t)} X_i f(t - T_i), \quad t \geq 0, \tag{2.1.17}$$

where

- (X_i) is an iid sequence, independent of (T_i) .
- f is a deterministic function with $f(t) = 0$ for $t < 0$.

For example, if we assume that the X_i 's are positive random variables, $S(t)$ is a generalization of the Cramér-Lundberg model, see Example 2.1.3. Indeed, choose $f = I_{[0, \infty)}$, then the shot noise process (2.1.17) is the total claim amount in the Cramér-Lundberg model. In an insurance context, f can also describe delay in claim settlement or some discount factor.

Delay in claim settlement is for example described by a function f satisfying

- $f(t) = 0$ for $t < 0$,
- $f(t)$ is non-decreasing,
- $\lim_{t \rightarrow \infty} f(t) = 1$.

In contrast to the Cramér-Lundberg model, where the claim size X_i is paid off at the time T_i when it occurs, a more general payoff function $f(t)$ allows one to delay the payment, and the speed at which this happens depends on the growth of the function f . Delay in claim settlement is advantageous from the point of view of the insurer. In the meantime the amount of money which was not paid for covering the claim could be invested and would perhaps bring some extra gain.

Suppose the amount Y_i is invested at time T_i in a riskless asset (savings account) with constant interest rate $r > 0$, (Y_i) is an iid sequence of positive random variables and the sequences (Y_i) and (T_i) are independent. Continuous compounding yields the amount $\exp\{r(t - T_i)\} Y_i$ at time $t > T_i$. For iid amounts Y_i which are invested at the arrival times T_i of a homogeneous Poisson process, the total value of all investments at time t is given by

$$S_1(t) = \sum_{i=1}^{N(t)} e^{r(t-T_i)} Y_i, \quad t \geq 0.$$

This is another shot noise process.

Alternatively, one may be interested in the present value of payments Y_i made at times T_i in the future. Then the present value with respect to the time frame $[0, t]$ is given as the *discounted sum*

$$S_2(t) = \sum_{i=1}^{N(t)} e^{-r(t-T_i)} Y_i, \quad t \geq 0.$$

A visualization of the sample paths of the processes S_1 and S_2 can be found in Figure 2.1.17. \square

The distributional properties of a shot noise process can be treated in the framework of the following general result.

Proposition 2.1.16 *Let (X_i) be an iid sequence, independent of the sequence (T_i) of arrival times of a homogeneous Poisson process N with intensity λ . Then for any measurable function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ the following identity in distribution holds*

$$S(t) = \sum_{i=1}^{N(t)} g(T_i, X_i) \stackrel{d}{=} \sum_{i=1}^{N(t)} g(t U_i, X_i),$$

where (U_i) is an iid $U(0, 1)$ sequence, independent of (X_i) and (T_i) .

Proof. A conditioning argument together with the order statistics property of Theorem 2.1.11 yields that for $x \in \mathbb{R}$,

$$P \left(\sum_{i=1}^{N(t)} g(T_i, X_i) \leq x \mid N(t) = n \right) = P \left(\sum_{i=1}^n g(t U_{(i)}, X_i) \leq x \right),$$

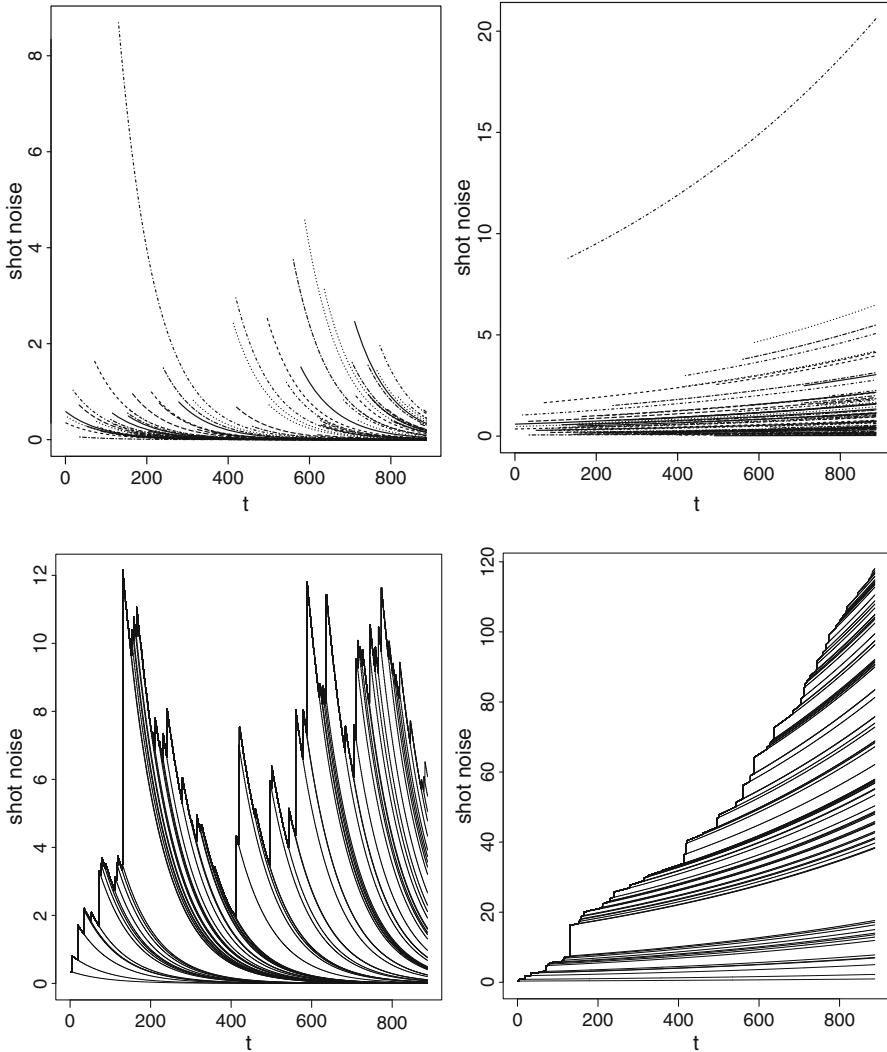


Figure 2.1.17 Visualization of the paths of a shot noise process. Top: 80 paths of the processes $Y_i e^{r(t-T_i)}$, $t \geq T_i$, where (T_i) are the point of a Poisson process with intensity 0.1, (Y_i) are iid standard exponential, $r = -0.01$ (left) and $r = 0.001$ (right). Bottom: The corresponding paths of the shot noise process $S(t) = \sum_{T_i \leq t} Y_i e^{r(t-T_i)}$ presented as a superposition of the paths in the corresponding top graphs. The graphs show nicely how the interest rate r influences the aggregated value of future claims or payments Y_i . We refer to Example 2.1.15 for a more detailed description of these processes.

where U_1, \dots, U_n is an iid $U(0, 1)$ sample, independent of (X_i) and (T_i) , and $U_{(1)}, \dots, U_{(n)}$ is the corresponding ordered sample. By the iid property of (X_i) and its independence of (U_i) , we can permute the order of the X_i 's arbitrarily without changing the distribution of $\sum_{i=1}^n g(t U_{(i)}, X_i)$:

$$\begin{aligned} & P\left(\sum_{i=1}^n g(t U_{(i)}, X_i) \leq x\right) \\ &= E\left[P\left(\sum_{i=1}^n g(t U_{(i)}, X_i) \leq x \mid U_1, \dots, U_n\right)\right] \\ &= E\left[P\left(\sum_{i=1}^n g(t U_{(i)}, X_{\pi(i)}) \leq x \mid U_1, \dots, U_n\right)\right], \end{aligned} \quad (2.1.18)$$

where π is any permutation of $\{1, \dots, n\}$. In particular, we can choose π such that for given U_1, \dots, U_n , $U_{(i)} = U_{\pi(i)}$, $i = 1, \dots, n$.¹⁴ Then (2.1.18) turns into

$$\begin{aligned} & E\left[P\left(\sum_{i=1}^n g(t U_{\pi(i)}, X_{\pi(i)}) \leq x \mid U_1, \dots, U_n\right)\right] \\ &= E\left[P\left(\sum_{i=1}^n g(t U_i, X_i) \leq x \mid U_1, \dots, U_n\right)\right] \\ &= P\left(\sum_{i=1}^n g(t U_i, X_i) \leq x\right) = P\left(\sum_{i=1}^{N(t)} g(t U_i, X_i) \leq x \mid N(t) = n\right). \end{aligned}$$

Now it remains to take expectations:

$$\begin{aligned} P(S(t) \leq x) &= E[P(S(t) \leq x \mid N(t))] \\ &= \sum_{n=0}^{\infty} P(N(t) = n) P\left(\sum_{i=1}^{N(t)} g(T_i, X_i) \leq x \mid N(t) = n\right) \end{aligned}$$

¹⁴ We give an argument to make this step in the proof more transparent. Since (U_i) and (X_i) are independent, it is possible to define $((U_i), (X_i))$ on the product space $\Omega_1 \times \Omega_2$ equipped with suitable σ -fields and probability measures, and such that (U_i) lives on Ω_1 and (X_i) on Ω_2 . While conditioning on $u_1 = U_1(\omega_1), \dots, u_n = U_n(\omega_1)$, $\omega_1 \in \Omega_1$, choose the permutation $\pi = \pi(\omega_1)$ of $\{1, \dots, n\}$ with $u_{\pi(1, \omega_1)} \leq \dots \leq u_{\pi(n, \omega_1)}$, and then with probability 1,

$$\begin{aligned} P(\{\omega_2 : (X_1(\omega_2), \dots, X_n(\omega_2)) \in A\}) &= \\ P(\{\omega_2 : (X_{\pi(1, \omega_1)}(\omega_2), \dots, X_{\pi(n, \omega_1)}(\omega_2)) \in A \mid U_1(\omega_1) = u_1, \dots, U_n(\omega_1) = u_n\}). \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} P(N(t) = n) \left. P\left(\sum_{i=1}^{N(t)} g(t U_i, X_i) \leq x \mid N(t) = n\right)\right. \\
&= P\left(\sum_{i=1}^{N(t)} g(t U_i, X_i) \leq x\right).
\end{aligned}$$

This proves the proposition. \square

It is clear that Proposition 2.1.16 can be extended to the case when (T_i) is the arrival sequence of an inhomogeneous Poisson process. The interested reader is encouraged to go through the steps of the proof in this more general case.

Proposition 2.1.16 has a multitude of applications. We give one of them and consider more in the exercises.

Example 2.1.18 (Continuation of the shot noise Example 2.1.15)

In Example 2.1.15 we considered the stochastically discounted random sums

$$S(t) = \sum_{i=1}^{N(t)} e^{-r(t-T_i)} X_i. \quad (2.1.19)$$

According to Proposition 2.1.16, we have

$$S(t) \stackrel{d}{=} \sum_{i=1}^{N(t)} e^{-r(t-tU_i)} X_i \stackrel{d}{=} \sum_{i=1}^{N(t)} e^{-r t U_i} X_i, \quad (2.1.20)$$

where (X_i) , (U_i) and N are mutually independent. Here we also used the fact that $(1 - U_i)$ and (U_i) have the same distribution. The structure of the random sum (2.1.19) is more complicated than the structure of the right-hand expression in (2.1.20) since in the latter sum the summands are independent of $N(t)$ and iid. For example, it is an easy matter to calculate the mean and variance of the expression on the right-hand side of (2.1.20) whereas it is a rather tedious procedure if one starts with (2.1.19). For example, we calculate

$$\begin{aligned}
ES(t) &= E\left(\sum_{i=1}^{N(t)} e^{-r t U_i} X_i\right) = E\left[E\left(\sum_{i=1}^{N(t)} e^{-r t U_i} X_i \mid N(t)\right)\right] \\
&= E[N(t)E(e^{-r t U_1} X_1)] \\
&= EN(t)Ee^{-r t U_1} EX_1 = \lambda r^{-1}(1 - e^{-r t}) EX_1.
\end{aligned}$$

Compare with the expectation in the Cramér-Lundberg model ($r = 0$): $ES(t) = \lambda t EX_1$. \square

Comments

The order statistics property of a Poisson process can be generalized to Poisson processes with points in abstract spaces. We give an informal discussion of these processes in Section 2.1.8. In Exercise 20 on p. 52 we indicate how the “order statistics property” can be implemented, for example, in a Poisson process with points in the unit cube of \mathbb{R}^d .

In Parts III and IV of this text we continue the discussion of generalized Poisson processes and their applications in a non-life insurance context. For example, in Section 11.3 we study payment processes which describe the settlement of claims arriving at the points of a homogeneous Poisson process on the real line. The combined process of the claim arrivals and payments is again a shot noise process.

2.1.7 A Discussion of the Arrival Times of the Danish Fire Insurance Data 1980-1990

In this section we want to illustrate the theoretical results of the Poisson process by means of the arrival process of a real-life data set: the *Danish fire insurance data* in the period from January 1, 1980, until December 31, 1990. The data were communicated to us by Mette Havning.¹⁵ There is a total of $n = 2\,167$ observations. Here we focus on the arrival process. In Section 3.2, and in particular in Example 3.2.11, we study the corresponding claim sizes.

The arrival and the corresponding inter-arrival times are plotted in Figure 2.1.19. Together with the arrival times we show the straight line $f(t) = 1.85t$. The value $\hat{\lambda} = n/T_n = 1/1.85$ is the maximum likelihood estimator of λ under the hypothesis that the inter-arrival times W_i are iid $\text{Exp}(\lambda)$.

In Table 2.1.21 we summarize some basic statistics of the inter-arrival times for each year and for the whole period. Since the reciprocal of the annual sample mean is an estimator of the intensity, the table gives one the impression that there is a tendency for increasing intensity when time goes by. This phenomenon is supported by the left graph in Figure 2.1.20 where the annual mean inter-arrival times are visualized together with moving average estimates of the intensity function $\lambda(t)$. The estimate of the mean inter-arrival time at $t = i$ is defined as the moving average¹⁶

¹⁵ In this text we consider two different versions of the Danish fire insurance data. Here we use the data which were reported by December 31, 1990. The claim sizes are expressed in terms of 1985 prices. If a claim was not completely settled on December 31, 1990, the size of this claim might possibly have changed after this date. For this reason the second data set (covering the period 1980-2002) often contains different reported sizes for claims incurred in 1980-1990.

¹⁶ Moving average estimates such as (2.1.21) are proposed in time series analysis in order to estimate a deterministic trend which perturbs a stationary time series. We refer to Brockwell and Davis [24] and Priestley [119] for some theory and properties of the estimator $(\hat{\lambda}(i))^{-1}$ and related estimates. More sophisticated

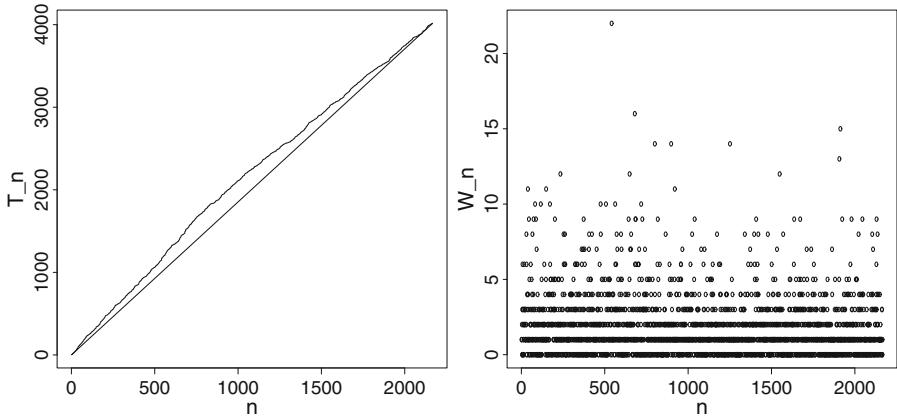


Figure 2.1.19 Left: The arrival times of the Danish fire insurance data 1980–1990. The solid straight line has slope 1.85 which is estimated as the overall sample mean of the inter-arrival times. Since the graph of (T_n) lies above the straight line an inhomogeneous Poisson process is more appropriate for modeling the claim number in this portfolio. Right: The corresponding inter-arrival times. There is a total of $n = 2\,167$ observations.

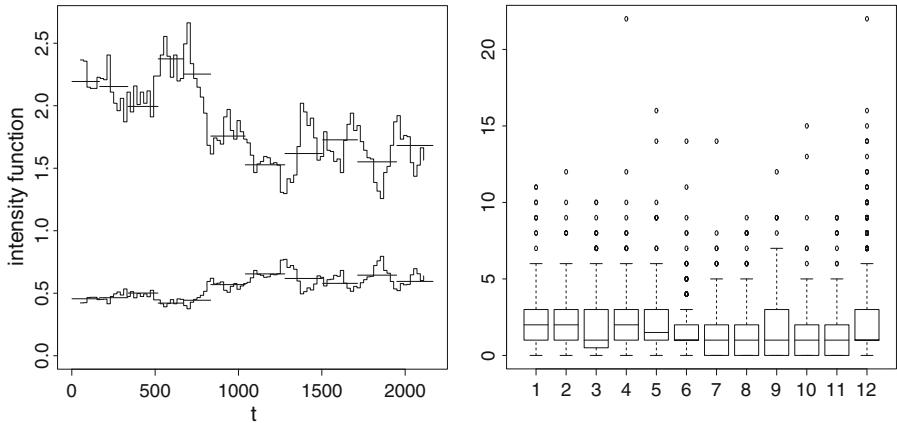


Figure 2.1.20 Left, upper graph: The piecewise constant function represents the annual expected inter-arrival time between 1980 and 1990. The length of each constant piece is the claim number in the corresponding year. The annual estimates are supplemented by a moving average estimate $(\hat{\lambda}(i))^{-1}$ defined in (2.1.21). Left, lower graph: The reciprocals of the values of the upper graph which can be interpreted as estimates of the Poisson intensity. There is a clear tendency for the intensity to increase over the last years. Right: Boxplots for the annual samples of the inter-arrival times (No 1-11) and the sample over 11 years (No 12).

$$(\widehat{\lambda}(i))^{-1} = (2m+1)^{-1} \sum_{j=\max(1,i-m)}^{\min(n,i+m)} W_j \quad \text{for } m=50. \quad (2.1.21)$$

The corresponding estimates for $\widehat{\lambda}(i)$ can be interpreted as estimates of the intensity function. There is a clear tendency for the intensity to increase over the last years. This tendency can also be seen in the right graph of Figure 2.1.20. Indeed, the boxplots¹⁷ of this figure indicate that the distribution of the inter-arrival times of the claims is less spread towards the end of the 1980s and concentrated around the value 1 in contrast to 2 at the beginning of the 1980s. Moreover, the annual claim number increases.

year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990	all
sample size	166	170	181	153	163	207	238	226	210	235	218	2 167
min	0	0	0	0	0	0	0	0	0	0	0	0
1st quartile	1	1	0.75	1	1	1	0	0	0	0	0	1
median	2	2	1	2	1.5	1	1	1	1	1	1	1
mean	2.19	2.15	1.99	2.37	2.25	1.76	1.53	1.62	1.73	1.55	1.68	1.85
$\widehat{\lambda} = 1/\text{mean}$	0.46	0.46	0.50	0.42	0.44	0.57	0.65	0.62	0.58	0.64	0.59	0.54
3rd quartile	3	3	3	3	3	2	2	2	3	2	2	3
max	11	12	10	22	16	14	14	9	12	15	9	22

Table 2.1.21 Basic statistics for the Danish fire inter-arrival times data.

Since we have gained statistical evidence that the intensity function of the Danish fire insurance data is not constant over 11 years, we assume in Figure 2.1.22 that the arrivals are modeled by an inhomogeneous Poisson process with continuous mean value function. We assume that the intensity is constant for every year, but it may change from year to year. Hence the mean value function $\mu(t)$ of the Poisson process is piecewise linear with possibly different slopes in different years; see the top left graph in Figure 2.1.22. We choose the estimated intensities presented in Table 2.1.21 and in the left graph of Figure 2.1.20. We transform the arrivals T_n into $\mu(T_n)$. According to the theory in Section 2.1.3, one can interpret the points $\mu(T_n)$ as arrivals of a standard homogeneous Poisson process. This is nicely illustrated in the top right graph of Figure 2.1.22, where the sequence $(\mu(T_n))$ is plotted against n . The graph is very close to a straight line, in contrast to the left graph in

estimators can be obtained by using kernel curve estimators in the regression model $W_i = (\lambda(i))^{-1} + \varepsilon_i$ for some smooth deterministic function λ and iid or weakly dependent stationary noise (ε_i) . We refer to Fan and Gijbels [49] and Gasser et al. [53] for some standard theory of kernel curve estimation; see also Müller and Stadtmüller [113].

¹⁷ The boxplot of a data set is a means to visualize the empirical distribution of the data. The middle part of the plot (box) indicates the median $x_{0.50}$, the 25% and 75% quantiles ($x_{0.25}$ and $x_{0.75}$) of the data. The “whiskers” of the data are the lines $x_{0.50} \pm 1.5(x_{0.75} - x_{0.25})$. Values outside the whiskers (“outliers”) are plotted as points.

Figure 2.1.19, where one can clearly see the deviations of the arrivals T_n from a straight line.

In the left middle graph we consider the histogram of the time changed arrival times $\mu(T_n)$. According to the theory in Section 2.1.6, the arrival times of a homogeneous Poisson can be interpreted as a uniform sample on any fixed interval, conditionally on the claim number in this interval. The histogram resembles the histogram of a uniform sample in contrast to the middle right graph, where the histogram of the Danish fire arrival times is presented. However, the left histogram is not perfect either. This is due to the fact that the data T_n are integers, hence the values $\mu(T_n)$ live on a particular discrete set.

The left bottom graph shows a moving average estimate of the intensity function of the arrivals $\mu(T_n)$. Although the function is close to 1 the estimates fluctuate wildly around 1. This is an indication that the process might not be Poisson and that other models for the arrival process could be more appropriate; see for example Section 2.2. The deviation of the distribution of the inter-arrival time $\mu(T_n) - \mu(T_{n-1})$, which according to the theory should be iid standard exponential, can also be seen in the right bottom graph in Figure 2.1.22, where a QQ-plot¹⁸ of these data against the standard exponential distribution is shown. The QQ-plot curves down at the right. This is a clear indication of a right tail of the underlying distribution which is heavier than the tail of the exponential distribution. These observations raise the question as to whether the Poisson process is a suitable model for the whole period of 11 years of claim arrivals.

A homogeneous Poisson process is a suitable model for the arrivals of the Danish fire insurance data for shorter periods of time such as one year. This is illustrated in Figure 2.1.23 for the 166 arrivals in the period January 1 - December 31, 1980.

As a matter of fact, the data show a clear seasonal component. This can be seen in Figure 2.1.24, where a histogram of all arrivals modulo 366 is given. Hence one receives a distribution on the integers between 1 and 366. Notice for example the peak around day 120 which corresponds to fires in April-May. There is also more activity in summer than in early spring and late fall, and one observes more fires in December and January with the exception of the last week of the year.

2.1.8 An Informal Discussion of Transformed and Generalized Poisson Processes

Consider a Poisson process N with claim arrival times T_i on $[0, \infty)$ and mean value function μ , independent of the iid positive claim sizes X_i with distribution function F . In this section we want to learn about a procedure which allows one to merge the Poisson claim arrival times T_i and the iid claim sizes X_i in one Poisson process with points in \mathbb{R}^2 .

¹⁸ The reader who is unfamiliar with QQ-plots is referred to Section 3.2.1.

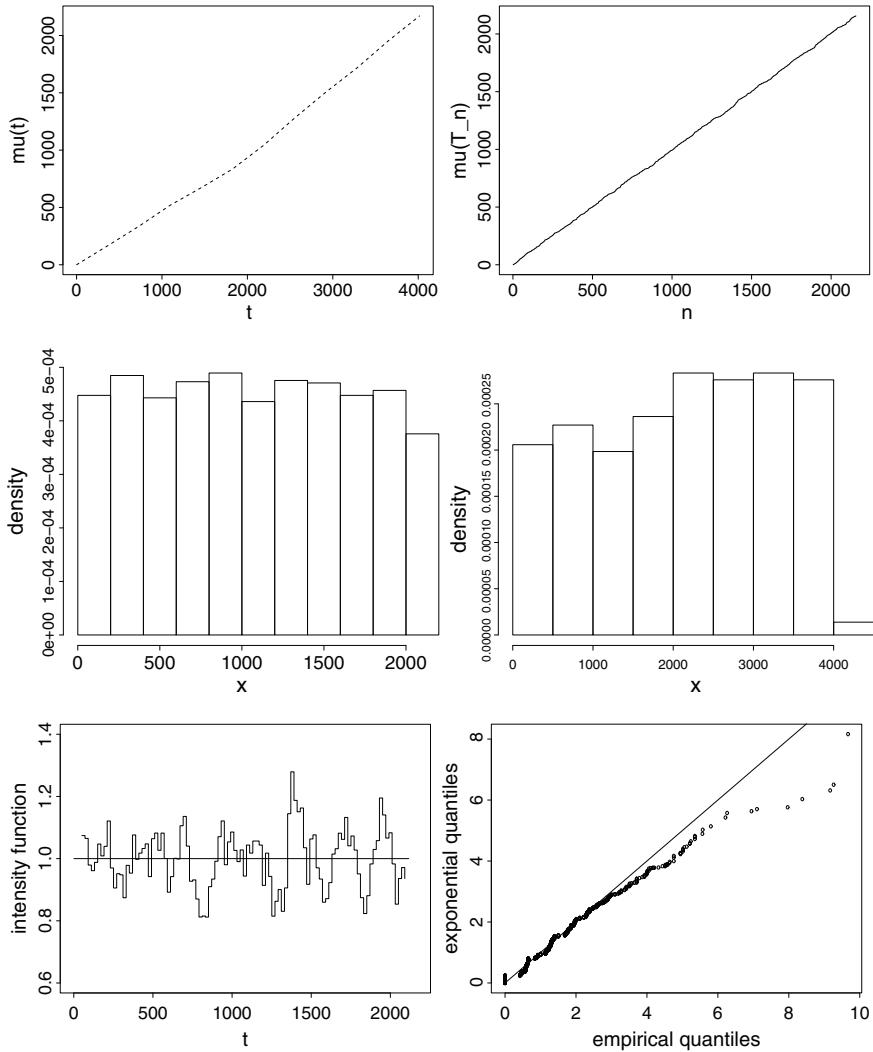


Figure 2.1.22 Top left: The estimated mean value function $\mu(t)$ of the Danish fire insurance arrivals. The function is piecewise linear. The slopes are the estimated intensities from Table 2.1.21. Top right: The transformed arrivals $\mu(T_n)$. Compare with Figure 2.1.19. The histogram of the values $\mu(T_n)$ (middle left) resembles a uniform density, whereas the histogram of the T_n 's shows clear deviations from it (middle right). Bottom left: Moving average estimate of the intensity function corresponding to the transformed sequence $(\mu(T_n))$. The estimates fluctuate around the value 1. Bottom right: QQ-plot of the values $\mu(T_n) - \mu(T_{n-1})$ against the standard exponential distribution. The plot curves down at the right end indicating that the values come from a distribution with tails heavier than exponential.

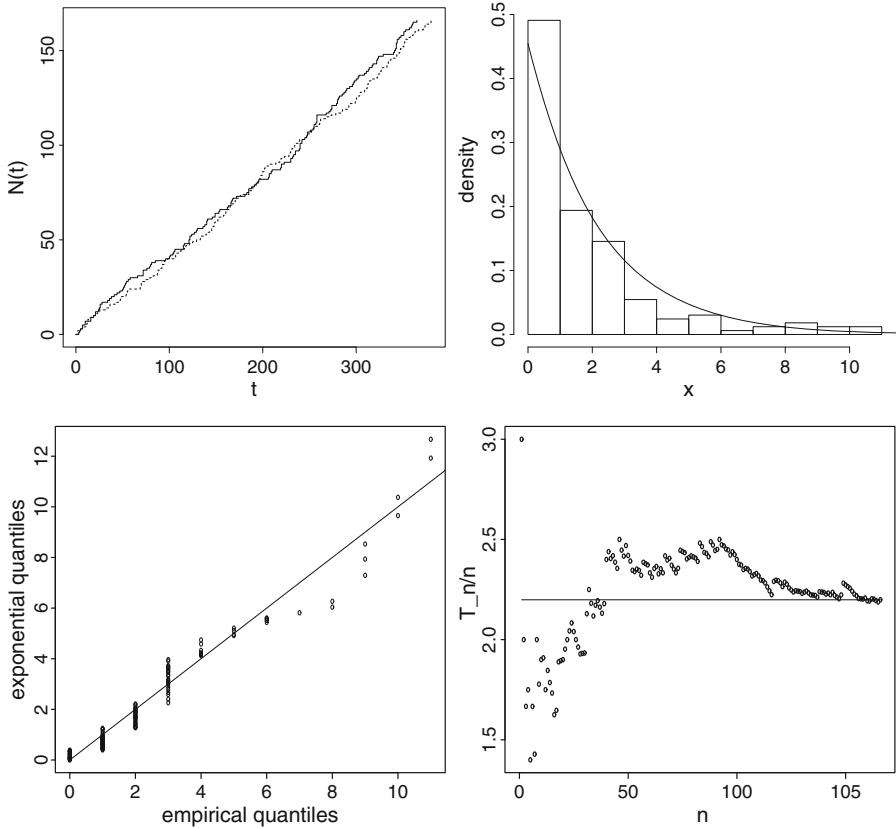


Figure 2.1.23 The Danish fire insurance arrivals from January 1, 1980, until December 31, 1980. The inter-arrival times have sample mean $\hat{\lambda}^{-1} = 2.19$. Top left: The renewal process $N(t)$ generated by the arrivals (solid boldface curve). For comparison, one sample path of a homogeneous Poisson process with intensity $\lambda = (2.19)^{-1}$ is drawn. Top right: The histogram of the inter-arrival times. For comparison, the density of the $\text{Exp}(\lambda)$ distribution is drawn. Bottom left: QQ-plot for the inter-arrival sample against the quantiles of the $\text{Exp}(\lambda)$ distribution. The fit of the data by an exponential $\text{Exp}(\lambda)$ is not unreasonable. However, the QQ-plot indicates a clear difference to exponential inter-arrival times: the data come from an integer-valued distribution. This deficiency could be overcome if one knew the exact claim times. Bottom right: The ratio T_n/n as a function of time. The values cluster around $\hat{\lambda}^{-1} = 2.19$ which is indicated by the constant line. For a homogeneous Poisson process, $T_n/n \xrightarrow{\text{a.s.}} \lambda^{-1}$ by virtue of the strong law of large numbers. For an iid $\text{Exp}(\lambda)$ sample W_1, \dots, W_n , $\hat{\lambda} = n/T_n$ is the maximum likelihood estimator of λ . If one accepts the hypothesis that the arrivals in 1980 come from a homogeneous Poisson process with intensity $\lambda = (2.19)^{-1}$, one would have an expected inter-arrival time of 2.19, i.e., roughly every second day a claim occurs.

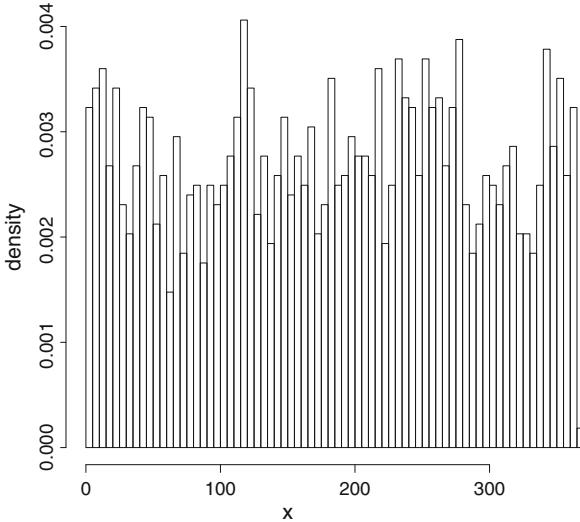


Figure 2.1.24 Histogram of all arrival times of the Danish fire insurance claims considered as a distribution on the integers between 1 and 366. The bars of the histogram correspond to periods of 5 days. There is a clear indication of seasonality in the data.

Define the counting process

$$M(a, b) = \#\{i \geq 1 : X_i \leq a, T_i \leq b\} = \sum_{i=1}^{N(b)} I_{(0,a]}(X_i), \quad a, b \geq 0.$$

We want to determine the distribution of $M(a, b)$. For this reason, recall the characteristic function¹⁹ of a Poisson random variable $M \sim \text{Pois}(\gamma)$:

$$E e^{itM} = \sum_{n=0}^{\infty} e^{itn} P(M = n) = \sum_{n=0}^{\infty} e^{itn} e^{-\gamma} \frac{\gamma^n}{n!} = e^{-\gamma(1-e^{it})}, \quad t \in \mathbb{R}. \quad (2.1.22)$$

We know that the characteristic function of a random variable M determines its distribution and vice versa. Therefore we calculate the characteristic function of $M(a, b)$. A similar argument as the one leading to (2.1.22) yields

¹⁹ In what follows we work with characteristic functions because this notion is defined for all distributions on \mathbb{R} . Alternatively, we could replace the characteristic functions by moment generating functions. However, the moment generating function of a random variable is well-defined only if this random variable has certain finite exponential moments. This would restrict the class of distributions we consider.

$$\begin{aligned}
E e^{itM(a,b)} &= E \left[E \exp \left\{ i t \sum_{j=1}^{N(b)} I_{(0,a]}(X_j) \right\} \middle| N(b) \right] \\
&= E \left[(E \exp \{ i t I_{(0,a]}(X_1) \})^{N(b)} \right] \\
&= E \left([1 - F(a) + F(a) e^{it}]^{N(b)} \right) \\
&= e^{-\mu(b) F(a) (1 - e^{it})}.
\end{aligned} \tag{2.1.23}$$

We conclude from (2.1.22) and (2.1.23) that $M(a, b) \sim \text{Pois}(F(a)\mu(b))$. Using similar characteristic function arguments, one can show that

- The increments

$$\begin{aligned}
M((x, x+h] \times (t, t+s]) \\
= \#\{i \geq 1 : (X_i, T_i) \in (x, x+h] \times (t, t+s]\}, \quad x, t \geq 0, h, s > 0,
\end{aligned}$$

are $\text{Pois}(F(x, x+h] \mu(t, t+s))$ distributed.

- For disjoint intervals $\Delta_i = (x_i, x_i + h_i] \times (t_i, t_i + s_i]$, $i = 1, \dots, n$, the increments $M(\Delta_i)$, $i = 1, \dots, n$, are independent.

From measure theory, we know that the quantities $F(x, x+h] \mu(t, t+s)$ determine the product measure $\gamma = F \times \mu$ on the Borel σ -field of $[0, \infty)^2$, where F denotes the distribution function as well as the distribution of X_i and μ is the measure generated by the values $\mu(a, b]$, $0 \leq a < b < \infty$. This is a consequence of the extension theorem for measures; cf. Billingsley [18]. In the case of a homogeneous Poisson process, $\mu = \lambda \text{Leb}$, where Leb denotes Lebesgue measure on $[0, \infty)$.

In analogy to the extension theorem for deterministic measures, one can find an extension M of the random counting variables $M(\Delta)$, $\Delta = (x, x+h] \times (t, t+s]$, such that for any Borel set²⁰ $A \subset [0, \infty)^2$,

$$M(A) = \#\{i \geq 1 : (X_i, T_i) \in A\} \sim \text{Pois}(\gamma(A)),$$

and for disjoint Borel sets $A_1, \dots, A_n \subset [0, \infty)^2$, $M(A_1), \dots, M(A_n)$ are independent. We call $\gamma = F \times \mu$ the *mean measure* of M , and M is called a *Poisson process* or a *Poisson random measure with mean measure γ* , denoted $M \sim \text{PRM}(\gamma)$. Notice that M is indeed a random counting measure on the Borel σ -field of $[0, \infty)^2$.

The embedding of the claim arrival times and the claim sizes in a Poisson process with two-dimensional points gives one a precise answer as to how many claim sizes of a given magnitude occur in a fixed time interval. For example, the number of claims exceeding a high threshold u , say, in the period $(a, b]$ of time is given by

²⁰ For A with mean measure $\gamma(A) = \infty$, we write $M(A) = \infty$.

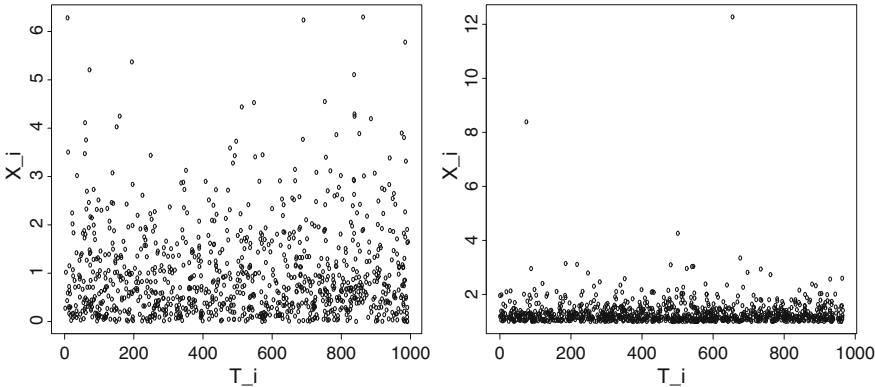


Figure 2.1.25 1000 points (T_i, X_i) of a two-dimensional Poisson process, where (T_i) is the sequence of the arrival times of a homogeneous Poisson process with intensity 1 and (X_i) is a sequence of iid claim sizes, independent of (T_i) . Left: Standard exponential claim sizes. Right: Pareto distributed claim sizes with $P(X_i > x) = x^{-4}$, $x \geq 1$. Notice the difference in scale of the claim sizes!

$$M((u, \infty) \times (a, b]) = \#\{i \geq 1 : X_i > u, T_i \in (a, b]\}.$$

This is a $\text{Pois}((1 - F(u)) \mu(a, b])$ distributed random variable. It is independent of the number of claims below the threshold u occurring in the same time interval. Indeed, the sets $(u, \infty) \times (a, b]$ and $[0, u] \times (a, b]$ are disjoint and therefore $M((u, \infty) \times (a, b])$ and $M([0, u] \times (a, b])$ are independent Poisson distributed random variables.

In the previous sections²¹ we used various transformations of the arrival times T_i of a Poisson process N on $[0, \infty)$ with mean measure ν , say, to derive other Poisson processes on the interval $[0, \infty)$. The restriction of processes to $[0, \infty)$ can be relaxed. Consider a measurable set $E \subset \mathbb{R}$ and equip E with the σ -field \mathcal{E} of the Borel sets. Then

$$N(A) = \#\{i \geq 1 : T_i \in A\}, \quad A \in \mathcal{E},$$

defines a *random measure* on the measurable space (E, \mathcal{E}) . Indeed, $N(A) = N(A, \omega)$ depends on $\omega \in \Omega$ and for fixed ω , $N(\cdot, \omega)$ is a counting measure on \mathcal{E} . The set E is called the *state space* of the random measure N . It is again called a *Poisson random measure* or *Poisson process* with mean measure ν restricted to E since one can show that $N(A) \sim \text{Pois}(\nu(A))$ for $A \in \mathcal{E}$, and $N(A_i), i = 1, \dots, n$, are mutually independent for disjoint $A_i \in \mathcal{E}$. The notion of Poisson random measure is very general and can be extended to abstract state spaces E . At the beginning of the section we considered a particular

²¹ See, for example, Section 2.1.3.

example in $E = [0, \infty)^2$. The Poisson processes we considered in the previous sections are examples of Poisson processes with state space $E = [0, \infty)$.

One of the strengths of this general notion of Poisson process is the fact that *Poisson random measures remain Poisson random measures under measurable transformations*. Indeed, let $\psi : E \rightarrow \tilde{E}$ be such a transformation and \tilde{E} be equipped with the σ -field $\tilde{\mathcal{E}}$. Assume N is PRM(ν) on E with points T_i . Then the points $\psi(T_i)$ are in \tilde{E} and, for $A \in \tilde{\mathcal{E}}$,

$$N_\psi(A) = \#\{i \geq 1 : \psi(T_i) \in A\} = \#\{i \geq 1 : T_i \in \psi^{-1}(A)\} = N(\psi^{-1}(A)),$$

where $\psi^{-1}(A) = \{x \in E : \psi(x) \in A\}$ denotes the inverse image of A which belongs to \mathcal{E} since ψ is measurable. Then we also have that $N_\psi(A) \sim \text{Pois}(\nu(\psi^{-1}(A)))$ since $EN_\psi(A) = EN(\psi^{-1}(A)) = \nu(\psi^{-1}(A))$. Moreover, since disjointness of A_1, \dots, A_n in \mathcal{E} implies disjointness of $\psi^{-1}(A_1), \dots, \psi^{-1}(A_n)$ in \mathcal{E} , it follows that $N_\psi(A_1), \dots, N_\psi(A_n)$ are independent, by the corresponding property of the PRM N . We conclude that $N_\psi \sim \text{PRM}(\nu(\psi^{-1}))$.

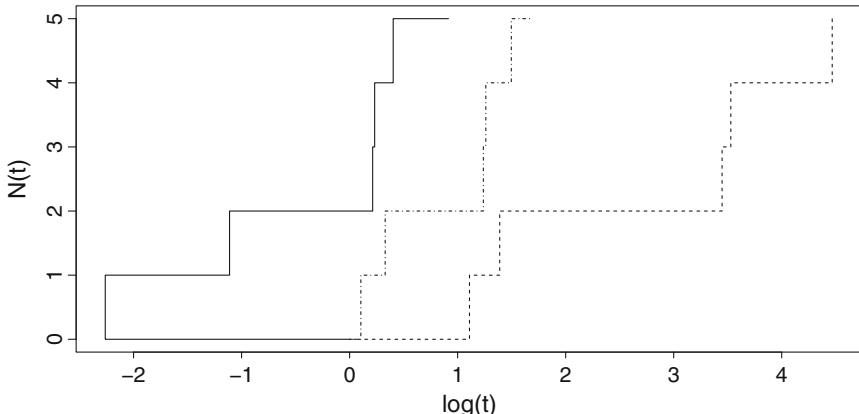


Figure 2.1.26 Sample paths of the Poisson processes with arrival times $\exp\{T_i\}$ (bottom dashed curve), T_i (middle dashed curve) and $\log T_i$ (top solid curve). The T_i 's are the arrival times of a standard homogeneous Poisson process. Time is on logarithmic scale in order to visualize the three paths in one graph.

Example 2.1.27 (Measurable transformations of Poisson processes remain Poisson processes)

- (1) Let \tilde{N} be a Poisson process on $[0, \infty)$ with mean value function $\tilde{\mu}$ and arrival times $0 < T_1 < T_2 < \dots$. Consider the transformed process

$$N(t) = \#\{i \geq 1 : 0 \leq T_i - a \leq t\}, \quad 0 \leq t \leq b - a,$$

for some interval $[a, b] \subset [0, \infty)$, where $\psi(x) = x - a$ is clearly measurable. This construction implies that $N(A) = \#\{i \geq 1 : \psi(T_i) \in A\} = 0$ for $A \subset [0, b-a]^c$, the complement of $[0, b-a]$. Therefore it suffices to consider N on the Borel sets of $[0, b-a]$. This defines a *Poisson process on $[a, b]$* with mean value function $\mu(t) = \tilde{\mu}(t) - \tilde{\mu}(a)$, $t \in [a, b]$.

(2) Consider a standard homogeneous Poisson process on $[0, \infty)$ with arrival times $0 < T_1 < T_2 < \dots$. We transform the arrival times with the measurable function $\psi(x) = \log x$. Then the points $(\log T_i)$ constitute a Poisson process N on \mathbb{R} . The Poisson measure of the interval $(a, b]$ for $a < b$ is given by

$$N(a, b] = \#\{i \geq 1 : \log(T_i) \in (a, b]\} = \#\{i \geq 1 : T_i \in (e^a, e^b]\}.$$

This is a $\text{Pois}(e^b - e^a)$ distributed random variable, i.e., the mean measure of the interval $(a, b]$ is given by $e^b - e^a$.

Alternatively, transform the arrival times T_i by the exponential function. The resulting Poisson process M is defined on $[1, \infty)$. The Poisson measure of the interval $(a, b] \subset [1, \infty)$ is given by

$$M(a, b] = \#\{i \geq 1 : e^{T_i} \in (a, b]\} = \#\{i \geq 1 : T_i \in (\log a, \log b]\}.$$

This is a $\text{Pois}(\log(b/a))$ distributed random variable, i.e., the mean measure of the interval $(a, b]$ is given by $\log(b/a)$. Notice that this Poisson process has the remarkable property that $M(ca, cb]$ for any $c \geq 1$ has the same $\text{Pois}(\log(b/a))$ distribution as $M(a, b]$. In particular, the expected number of points $\exp\{T_i\}$ falling into the interval $(ca, cb]$ is independent of the value $c \geq 1$. This is somewhat counterintuitive since the length of the interval $(ca, cb]$ can be arbitrarily large. However, the larger the value c the higher the threshold ca which prevents sufficiently many points $\exp\{T_i\}$ from falling into the interval $(ca, cb]$, and on average there are as many points in $(ca, cb]$ as in $(a, b]$. \square

Example 2.1.28 (Construction of transformed planar PRM)

Let (T_i) be the arrival sequence of a standard homogeneous Poisson process on $[0, \infty)$, independent of the iid sequence (X_i) with common distribution function F . Then the points (T_i, X_i) constitute a $\text{PRM}(\nu)$ N with state space $E = [0, \infty) \times \mathbb{R}$ and mean measure $\nu = \text{Leb} \times F$; see the discussion on p. 39.

After a measurable transformation $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the points $\psi(T_i, X_i)$ constitute a $\text{PRM } N_\psi$ with state space $E_\psi = \{\psi(t, x) : (t, x) \in E\}$ and mean measure $\nu_\psi(A) = \nu(\psi^{-1}(A))$ for any Borel set $A \subset E_\psi$. We choose $\tilde{\psi}(t, x) = t^{-1/\alpha}(\cos(2\pi x), \sin(2\pi x))$ for some $\alpha \neq 0$, i.e., the PRM $N_{\tilde{\psi}}$ has points $\mathbf{Y}_i = T_i^{-1/\alpha}(\cos(2\pi X_i), \sin(2\pi X_i))$. In Figure 2.1.30 we visualize the points \mathbf{Y}_i of the resulting PRM for different choices of α and distribution functions F of X_1 .

Planar PRMs such as the ones described above are used, among others, in spatial statistics (see Cressie [37]) in order to describe the distribution of random configurations of points in the plane such as the distribution of minerals, locations of highly polluted spots or trees in a forest. The particular

PRM $N_{\tilde{\psi}}$ and its modifications are major models in multivariate extreme value theory. It describes the dependence of extremes in the plane and in space. In particular, it is suitable for modeling clustering behavior of points \mathbf{Y}_i far away from the origin. See Resnick [122] for the theoretical background on multivariate extreme value theory and Mikosch [108] for a recent attempt to use $N_{\tilde{\psi}}$ for modeling multivariate financial time series. \square

Example 2.1.29 (Modeling arrivals of Incurred But Not Reported (IBNR) claims)

In a portfolio, the claims are not reported at their arrival times T_i , but with a certain delay. This delay may be due to the fact that the policyholder is not aware of the claim and only realizes it later (for example, a damage in his/her house), or that the policyholder was injured in a car accident and did not have the opportunity to call his agent immediately, or the policyholder's flat burnt down over Christmas, but the agent was on a skiing vacation in Switzerland and could not receive the report about the fire, etc.

We consider a simple model for the reporting times of IBNR claims: the arrival times T_i of the claims are modeled by a Poisson process N with mean value function μ and the delays in reporting by an iid sequence (V_i) of positive random variables with common distribution F . Then the sequence $(T_i + V_i)$ constitutes the reporting times of the claims to the insurance business. We assume that (V_i) and (T_i) are independent. Then the points (T_i, V_i) constitute a PRM(ν) with mean measure $\nu = \mu \times F$. By time t , $N(t)$ claims have occurred, but only

$$N_{\text{IBNR}}(t) = \sum_{i=1}^{N(t)} I_{[0,t]}(T_i + V_i) = \#\{i \geq 1 : T_i + V_i \leq t\}$$

have been reported. The mapping $\psi(t, v) = t + v$ is measurable. It transforms the points (T_i, V_i) of the PRM(ν) into the points $T_i + V_i$ of the PRM N_{ψ} with mean measure of a set A given by $\nu_{\psi}(A) = \nu(\psi^{-1}(A))$. In particular, $N_{\text{IBNR}}(s) = N_{\psi}([0, s])$ is Pois($\nu_{\psi}([0, s])$) distributed. We calculate the mean value $\nu_{\psi}([0, s])$ in Example 7.3.9 below. There we further discuss this IBNR model in the context of point processes. \square

Comments

The Poisson process is one of the most important stochastic processes. For the abstract understanding of this process one would have to consider it as a *point process*, i.e., as a random counting measure. We have indicated in Section 2.1.8 how one has to approach this problem. In Chapters 7 and 8 we give a more advanced treatment of the theory of point processes. There we focus on *generalized Poisson processes* or *Poisson random measures* and their use in non-life insurance applications.

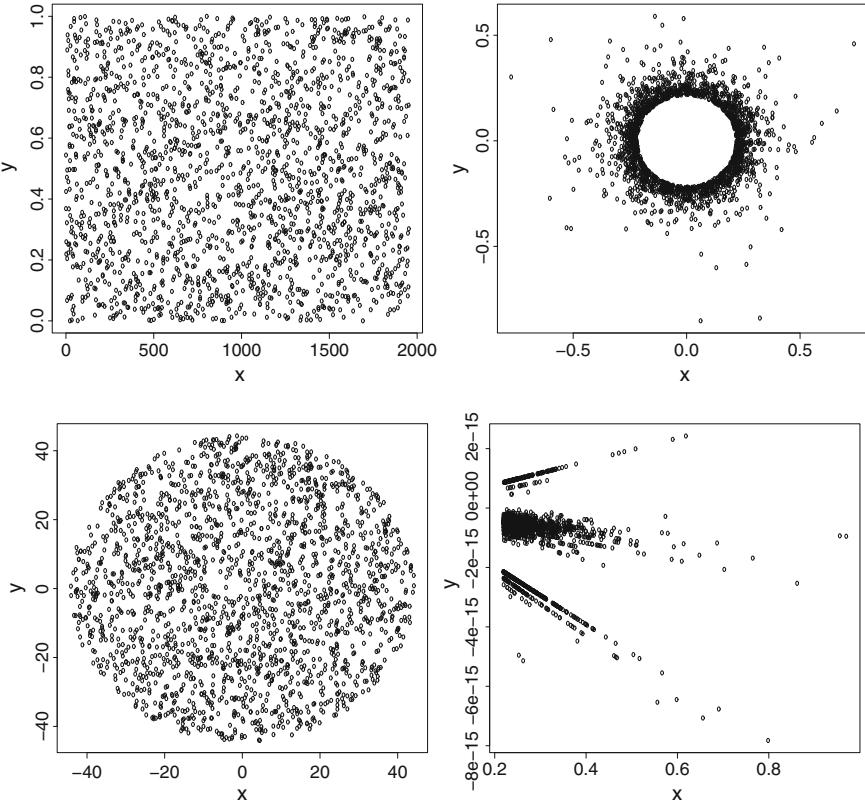


Figure 2.1.30 Poisson random measures in the plane.

Top left: 2 000 points of a Poisson random measure with points (T_i, X_i) , where (T_i) is the arrival sequence of a standard homogeneous Poisson process on $[0, \infty)$, independent of the iid sequence (X_i) with $X_1 \sim U(0, 1)$. The PRM has mean measure $\nu = \text{Leb} \times \text{Leb}$ on $[0, \infty) \times (0, 1)$.

After the measurable transformation $\tilde{\psi}(t, x) = t^{-1/\alpha} (\cos(2\pi x), \sin(2\pi x))$ for some $\alpha \neq 0$ the resulting PRM $N_{\tilde{\psi}}$ has points $\mathbf{Y}_i = T_i^{-1/\alpha} (\cos(2\pi X_i), \sin(2\pi X_i))$.

Top right: The points of the process $N_{\tilde{\psi}}$ for $\alpha = 5$ and iid $U(0, 1)$ uniform X_i 's. Notice that the spherical part $(\cos(2\pi X_i), \sin(2\pi X_i))$ of \mathbf{Y}_i is uniformly distributed on the unit circle.

Bottom left: The points of the process $N_{\tilde{\psi}}$ with $\alpha = -5$ and iid $U(0, 1)$ uniform X_i 's.

Bottom right: The points of the process $N_{\tilde{\psi}}$ for $\alpha = 5$ with iid $X_i \sim \text{Pois}(10)$.

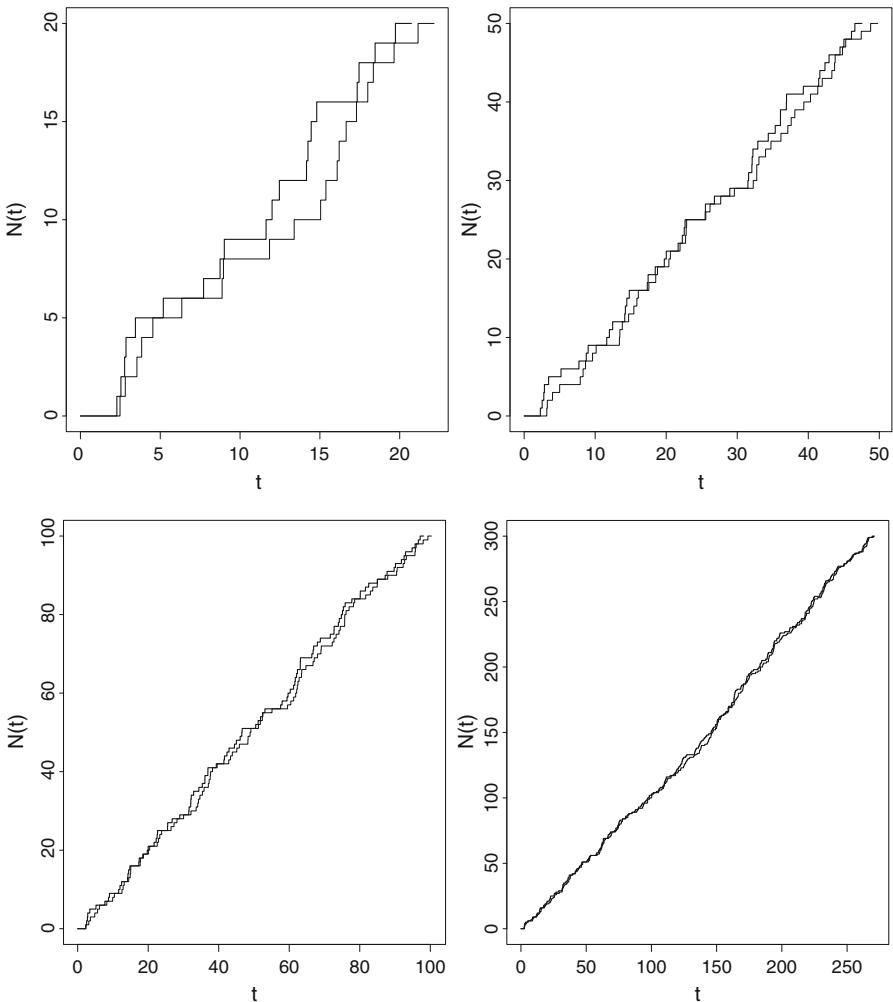


Figure 2.1.31 *Inurred But Not Reported claims.* We visualize one sample of a standard homogeneous Poisson process with n arrivals T_i (top boldface graph) and the corresponding claim number process for the delayed process with arrivals $T_i + V_i$, where the V_i 's are iid Pareto distributed with distribution $P(V_1 > x) = x^{-2}$, $x \geq 1$, independent of (T_i) . Top: $n = 30$ (left) and $n = 50$ (right). Bottom: $n = 100$ (left) and $n = 300$ (right). As explained in Example 2.1.29, the sample paths of the claim number processes differ from each other approximately by the constant value EV_1 . For sufficiently large t , the difference is negligible compared to the expected claim number.

As a matter of fact, various other counting processes such as the renewal process treated in Section 2.2 are approximated by suitable Poisson processes in the sense of convergence in distribution. Therefore the Poisson process with nice mathematical properties is also a good approximation to various real-life counting processes such as the claim number process in an insurance portfolio. In Chapter 9 we develop the theory of convergence in distribution of point processes. The convergence to a Poisson process is of particular interest. We show how these asymptotic relations can be used to determine the distribution of extremely large claim sizes.

The treatment of general Poisson processes requires more sophisticated tools and techniques from the theory of stochastic processes. For a gentle introduction to point processes and generalized Poisson processes we refer to Embrechts et al. [46], Chapter 5; for a rigorous treatment at a moderate level, Resnick's monograph [123] or Kingman's book [85] are good references. Resnick [122] is an advanced text on the Poisson process with applications to extreme value theory. See also Daley and Vere-Jones [38] or Kallenberg [79] for rigorous treatments of the general point process theory.

Exercises

Sections 2.1.1-2.1.2

- (1) Let $N = (N(t))_{t \geq 0}$ be a Poisson process with continuous intensity function $(\lambda(t))_{t \geq 0}$.
 - (a) Show that the intensities $\lambda_{n,n+k}(t)$, $n \geq 0$, $k \geq 1$ and $t > 0$, of the Markov process N with transition probabilities $p_{n,n+k}(s, t)$ exist, i.e.,

$$\lambda_{n,n+k}(t) = \lim_{h \downarrow 0} \frac{p_{n,n+k}(t, t+h)}{h}, \quad n \geq 0, k \geq 1,$$

and that they are given by

$$\lambda_{n,n+k}(t) = \begin{cases} \lambda(t), & k = 1, \\ 0, & k \geq 2. \end{cases} \quad (2.1.24)$$

- (b) What can you conclude from $p_{n,n+k}(t, t+h)$ for h small about the short term jump behavior of the Markov process N ?
 - (c) Show by counterexample that (2.1.24) is in general not valid if one gives up the assumption of continuity of the intensity function $\lambda(t)$.
- (2) Let $N = (N(t))_{t \geq 0}$ be a Poisson process with continuous intensity function $(\lambda(t))_{t \geq 0}$. By using the properties of N given in Definition 2.1.1, show that the following properties hold:
 - (a) The sample paths of N are non-decreasing.
 - (b) The process N does not have a jump at zero with probability 1.
 - (c) For every fixed t , the process N does not have a jump at t with probability 1. Does this mean that the sample paths do not have jumps?

- (3) Let N be a homogeneous Poisson process on $[0, \infty)$ with intensity $\lambda > 0$. Show that for $0 < t_1 < t < t_2$,

$$\lim_{h \downarrow 0} P(N(t_1 - h, t - h] = 0, N(t - h, t] = 1, N(t, t_2] = 0 \mid N(t - h, t] > 0) \\ = e^{-\lambda(t-t_1)} e^{-\lambda(t_2-t)}.$$

Give an intuitive interpretation of this property.

- (4) Let N_1, \dots, N_n be independent Poisson processes on $[0, \infty)$ defined on the same probability space. Show that $N_1 + \dots + N_n$ is a Poisson process and determine its mean value function.

This property extends the well-known property that the sum $M_1 + M_2$ of two independent Poisson random variables $M_1 \sim \text{Pois}(\lambda_1)$ and $M_2 \sim \text{Pois}(\lambda_2)$ is $\text{Pois}(\lambda_1 + \lambda_2)$. We also mention that a converse to this result holds. Indeed, suppose $M = M_1 + M_2$, $M \sim \text{Pois}(\lambda)$ for some $\lambda > 0$ and M_1, M_2 are independent non-negative random variables. Then both M_1 and M_2 are necessarily Poisson random variables. This phenomenon is referred to as *Raikov's theorem*; see Lukacs [97], Theorem 8.2.2. An analogous theorem can be shown for so-called *point processes* which are counting processes on $[0, \infty)$, including the Poisson process and the renewal process, see Chapter 7 for an introduction to the theory of point processes. Indeed, if the Poisson process N has representation $N \stackrel{d}{=} N_1 + N_2$ for independent point processes N_1, N_2 , then N_1 and N_2 are necessarily Poisson processes.

- (5) Consider the total claim amount process S in the Cramér-Lundberg model.
- Show that the total claim amount $S(s, t]$ in $(s, t]$ for $s < t$, i.e., $S(s, t] = S(t) - S(s)$, has the same distribution as the total claim amount in $[0, t-s]$, i.e., $S(t-s)$.
 - Show that, for every $0 = t_0 < t_1 < \dots < t_n$ and $n \geq 1$, the random variables $S(t_1), S(t_1, t_2], \dots, S(t_{n-1}, t_n]$ are independent. Hint: Calculate the joint characteristic function of the latter random variables.
- (6) For a homogeneous Poisson process N on $[0, \infty)$ show that for $0 < s < t$,

$$P(N(s) = k \mid N(t)) = \begin{cases} \binom{N(t)}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{N(t)-k} & \text{if } k \leq N(t), \\ 0 & \text{if } k > N(t). \end{cases}$$

Section 2.1.3

- (7) Let \tilde{N} be a standard homogeneous Poisson process on $[0, \infty)$ and N a Poisson process on $[0, \infty)$ with mean value function μ .
- Show that $N_1 = (\tilde{N}(\mu(t)))_{t \geq 0}$ is a Poisson process on $[0, \infty)$ with mean value function μ .
 - Assume that the inverse μ^{-1} of μ exists, is continuous and $\lim_{t \rightarrow \infty} \mu(t) = \infty$. Show that $\tilde{N}_1(t) = N(\mu^{-1}(t))$ defines a standard homogeneous Poisson process on $[0, \infty)$.
 - Assume that the Poisson process N has an intensity function λ . Which condition on λ ensures that $\mu^{-1}(t)$ exists for $t \geq 0$?
 - Let $f : [0, \infty) \rightarrow [0, \infty)$ be a non-decreasing continuous function with $f(0) = 0$. Show that

$$N_f(t) = N(f(t)), \quad t \geq 0,$$

is again a Poisson process on $[0, \infty)$. Determine its mean value function.

Sections 2.1.4-2.1.5

- (8) Recall from Theorem 2.1.6 that the homogeneous Poisson process \tilde{N} with intensity $\tilde{\lambda} > 0$ can be written as a renewal process

$$\tilde{N}(t) = \#\{i \geq 1 : \tilde{T}_i \leq t\}, \quad t \geq 0,$$

where $\tilde{T}_i = \tilde{W}_1 + \dots + \tilde{W}_i$ and (\tilde{W}_n) is an iid $\text{Exp}(\tilde{\lambda})$ sequence.

Let N be a Poisson process with mean value function μ which has an a.e. positive continuous intensity function λ . Let $0 \leq T_1 \leq T_2 \leq \dots$ be the arrival times of the process N .

- (a) Show that the random variables $\int_{T_n}^{T_{n+1}} \lambda(s) ds$ are iid exponentially distributed.
 - (b) Show that, with probability 1, no multiple claims can occur, i.e., at an arrival time T_i of a claim, $N(T_i) - N(T_i^-) = 1$ a.s. and $P(N(T_i) - N(T_i^-) > 1 \text{ for some } i) = 0$.
- (9) Consider a homogeneous Poisson process N with intensity $\lambda > 0$ and arrival times T_i .
- (a) Assume the renewal representation $N(t) = \#\{i \geq 1 : T_i \leq t\}$, $t \geq 0$, for N , i.e., $T_0 = 0$, $W_i = T_i - T_{i-1}$ are iid $\text{Exp}(\lambda)$ inter-arrival times. Calculate for $0 \leq t_1 < t_2$,

$$P(T_1 \leq t_1) \quad \text{and} \quad P(T_1 \leq t_1, T_2 \leq t_2). \quad (2.1.25)$$

- (b) Assume the properties of Definition 2.1.1 for N . Calculate for $0 \leq t_1 < t_2$,

$$P(N(t_1) \geq 1) \quad \text{and} \quad P(N(t_1) \geq 1, N(t_2) \geq 2). \quad (2.1.26)$$

(c) Give reasons why you get the same probabilities in (2.1.25) and (2.1.26).

- (10) Consider a homogeneous Poisson process on $[0, \infty)$ with arrival time sequence (T_i) and set $T_0 = 0$. The inter-arrival times are defined as $W_i = T_i - T_{i-1}$, $i \geq 1$.
- (a) Show that T_1 has the *forgetfulness property*, i.e., $P(T_1 > t+s | T_1 > t) = P(T_1 > s)$, $t, s \geq 0$.
 - (b) Another version of the forgetfulness property is as follows. Let $Y \geq 0$ be independent of T_1 and Z be a random variable whose distribution is given by

$$P(Z > z) = P(T_1 > Y+z | T_1 > Y), \quad z \geq 0.$$

Then Z and T_1 have the same distribution. Verify this.

- (c) Show that the events $\{W_1 < W_2\}$ and $\{\min(W_1, W_2) > x\}$ are independent.
 - (d) Determine the distribution of $m_n = \min(T_1, T_2 - T_1, \dots, T_n - T_{n-1})$.
- (11) Suppose you want to simulate sample paths of a Poisson process.
- (a) How can you exploit the renewal representation to simulate paths of a homogeneous Poisson process?
 - (b) How can you use the renewal representation of a homogeneous Poisson N to simulate paths of an inhomogeneous Poisson process?

Sections 2.1.6

- (12) Let U_1, \dots, U_n be an iid $U(0, 1)$ sample with the corresponding order statistics $U_{(1)} < \dots < U_{(n)}$ a.s. Let (\tilde{W}_i) be an iid sequence of $\text{Exp}(\lambda)$ distributed random variables and $\tilde{T}_n = \tilde{W}_1 + \dots + \tilde{W}_n$ the corresponding arrival times of a homogeneous Poisson process with intensity λ .

- (a) Show that the following identity in distribution holds for every fixed $n \geq 1$:

$$(U_{(1)}, \dots, U_{(n)}) \stackrel{d}{=} \left(\frac{\tilde{T}_1}{\tilde{T}_{n+1}}, \dots, \frac{\tilde{T}_n}{\tilde{T}_{n+1}} \right). \quad (2.1.27)$$

Hint: Calculate the densities of the vectors on both sides of (2.1.27). The density of the vector

$$[(\tilde{T}_1, \dots, \tilde{T}_n)/\tilde{T}_{n+1}, \tilde{T}_{n+1}]$$

can be obtained from the known density of the vector $(\tilde{T}_1, \dots, \tilde{T}_{n+1})$.

- (b) Why is the distribution of the right-hand vector in (2.1.27) independent of λ ?
(c) Let T_i be the arrivals of a Poisson process on $[0, \infty)$ with a.e. positive intensity function λ and mean value function μ . Show that the following identity in distribution holds for every fixed $n \geq 1$:

$$(U_{(1)}, \dots, U_{(n)}) \stackrel{d}{=} \left(\frac{\mu(T_1)}{\mu(T_{n+1})}, \dots, \frac{\mu(T_n)}{\mu(T_{n+1})} \right).$$

- (13) Let W_1, \dots, W_n be an iid $\text{Exp}(\lambda)$ sample for some $\lambda > 0$. Show that the ordered sample $W_{(1)} < \dots < W_{(n)}$ has representation in distribution:

$$\begin{aligned} & (W_{(1)}, \dots, W_{(n)}) \\ & \stackrel{d}{=} \left(\frac{W_n}{n}, \frac{W_n}{n} + \frac{W_{n-1}}{n-1}, \dots, \frac{W_n}{n} + \frac{W_{n-1}}{n-1} + \dots + \frac{W_2}{2}, \right. \\ & \quad \left. \frac{W_n}{n} + \frac{W_{n-1}}{n-1} + \dots + \frac{W_1}{1} \right). \end{aligned}$$

Hint: Use a density transformation starting with the joint density of W_1, \dots, W_n to determine the density of the right-hand expression.

- (14) Consider the stochastically discounted total claim amount

$$S(t) = \sum_{i=1}^{N(t)} e^{-rT_i} X_i,$$

where $r > 0$ is an interest rate, $0 < T_1 < T_2 < \dots$ are the claim arrival times, defining the homogeneous Poisson process $N(t) = \#\{i \geq 1 : T_i \leq t\}$, $t \geq 0$, with intensity $\lambda > 0$, and (X_i) is an iid sequence of positive claim sizes, independent of (T_i) .

- (a) Calculate the mean and the variance of $S(t)$ by using the order statistics property of the Poisson process N . Specify the mean and the variance in the case when $r = 0$ (Cramér-Lundberg model).

- (b) Show that $S(t)$ has the same distribution as

$$e^{-rt} \sum_{i=1}^{N(t)} e^{rT_i} X_i.$$

- (15) Suppose you want to simulate sample paths of a Poisson process on $[0, T]$ for $T > 0$ and a given continuous intensity function λ , by using the order statistics property.
- (a) How should you proceed if you are interested in one path with exactly n jumps in $[0, T]$?
 - (b) How would you simulate several paths of a homogeneous Poisson process with (possibly) different jump numbers in $[0, T]$?
 - (c) How could you use the simulated paths of a homogeneous Poisson process to obtain the paths of an inhomogeneous one with given intensity function?
- (16) Let (T_i) be the arrival sequence of a standard homogeneous Poisson process N and $\alpha \in (0, 1)$.
- (a) Show that the infinite series

$$X_\alpha = \sum_{i=1}^{\infty} T_i^{-1/\alpha} \quad (2.1.28)$$

converges a.s. Hint: Use the strong law of large numbers for (T_n) .

- (b) Show that

$$X_{N(t)} = \sum_{i=1}^{N(t)} T_i^{-1/\alpha} \xrightarrow{\text{a.s.}} X_\alpha \quad \text{as } t \rightarrow \infty.$$

Hint: Use Lemma 2.2.6.

- (c) It follows from standard limit theory for sums of iid random variables (see Feller [51], Theorem 1 in Chapter XVII.5) that for iid $U(0, 1)$ random variables U_i ,

$$n^{-1/\alpha} \sum_{i=1}^n U_i^{-1/\alpha} \xrightarrow{d} Z_\alpha, \quad (2.1.29)$$

where Z_α is a positive random variable with an α -stable distribution determined by its Laplace-Stieltjes transform $E \exp\{-s Z_\alpha\} = \exp\{-c s^\alpha\}$ for some $c > 0$, all $s \geq 0$. See p. 178 for some information about Laplace-Stieltjes transforms. Show that $X_\alpha \stackrel{d}{=} c' Z_\alpha$ for some positive constant $c' > 0$.

Hints: (i) Apply the order statistics property of the homogeneous Poisson process to $X_{N(t)}$ to conclude that

$$X_{N(t)} \stackrel{d}{=} t^{-1/\alpha} \sum_{i=1}^{N(t)} U_i^{-1/\alpha},$$

where (U_i) is an iid $U(0, 1)$ sequence, independent of $N(t)$.

(ii) Prove that

$$(N(t))^{-1/\alpha} \sum_{i=1}^{N(t)} U_i^{-1/\alpha} \xrightarrow{d} Z_\alpha \quad \text{as } t \rightarrow \infty.$$

Hint: Condition on $N(t)$ and exploit (2.1.29).

(iii) Use the strong law of large numbers $N(t)/t \xrightarrow{\text{a.s.}} 1$ as $t \rightarrow \infty$ (Theorem 2.2.5) and the continuous mapping theorem to conclude the proof.

- (d) Show that $EX_\alpha = \infty$.
- (e) Let Z_1, \dots, Z_n be iid copies of the α -stable random variable Z_α with Laplace-Stieltjes transform $Ee^{-sZ_\alpha} = e^{-cs^\alpha}$, $s \geq 0$, for some $\alpha \in (0, 1)$ and $c > 0$. Show that for every $n \geq 1$ the relation

$$Z_1 + \dots + Z_n \stackrel{d}{=} n^{1/\alpha} Z_\alpha$$

holds. It is due to this “stability condition” that the distribution gained its name.

Hint: Use the properties of Laplace-Stieltjes transforms (see p. 178) to show this property.

- (f) Consider Z_α from (e) for some $\alpha \in (0, 1)$.
- (i) Show the relation

$$Ee^{itA} Z_\alpha^{1/2} = e^{-c|t|^{2\alpha}}, \quad t \in \mathbb{R}, \quad (2.1.30)$$

where $A \sim N(0, 2)$ is independent of Z_α . A random variable Y with characteristic function given by the right-hand side of (2.1.30) and its distribution are said to be *symmetric 2α-stable*.

(ii) Let Y_1, \dots, Y_n be iid copies of Y from (i). Show the stability relation

$$Y_1 + \dots + Y_n \stackrel{d}{=} n^{1/(2\alpha)} Y.$$

(iii) Conclude that Y must have infinite variance. Hint: Suppose that Y has finite variance and try to apply the central limit theorem.

The interested reader who wants to learn more about the exciting class of stable distributions and stable processes is referred to Samorodnitsky and Taqqu [131].

Section 2.1.8

- (17) Let $(N(t))_{t \geq 0}$ be a standard homogeneous Poisson process with claim arrival times T_i .
 - (a) Show that the sequences of arrival times $(\sqrt{T_i})$ and (T_i^2) define two Poisson processes N_1 and N_2 , respectively, on $[0, \infty)$. Determine their mean measures by calculating $EN_i(s, t)$ for any $s < t$, $i = 1, 2$.
 - (b) Let N_3 and N_4 be Poisson processes on $[0, \infty)$ with mean value functions $\mu_3(t) = \sqrt{t}$ and $\mu_4(t) = t^2$ and arrival time sequences $(T_i^{(3)})$ and $(T_i^{(4)})$, respectively. Show that the processes $(N_3(t^2))_{t \geq 0}$ and $(N_4(\sqrt{t}))_{t \geq 0}$ are Poisson on $[0, \infty)$ and have the same distribution.
 - (c) Show that the process

$$N_5(t) = \#\{i \geq 1 : e^{T_i} \leq t + 1\}, \quad t \geq 0,$$

is a Poisson process and determine its mean value function.

- (d) Let N_6 be a Poisson process on $[0, \infty)$ with mean value function $\mu_6(t) = \log(1+t)$. Show that N_6 has the property that, for $1 \leq s < t$ and $a \geq 1$, the distribution of $N_6(at-1) - N_6(as-1)$ does not depend on a .
- (18) Let (T_i) be the arrival times of a homogeneous Poisson process N on $[0, \infty)$ with intensity $\lambda > 0$, independent of the iid claim size sequence (X_i) with $X_i > 0$ and distribution function F .
- (a) Show that for $s < t$ and $a < b$ the counting random variable

$$M((s, t] \times (a, b]) = \#\{i \geq 1 : T_i \in (s, t], X_i \in (a, b]\}$$

- is $\text{Pois}(\lambda(t-s)F(a, b])$ distributed.
- (b) Let $\Delta_i = (s_i, t_i] \times (a_i, b_i]$ for $s_i < t_i$ and $a_i < b_i$, $i = 1, 2$, be disjoint. Show that $M(\Delta_1)$ and $M(\Delta_2)$ are independent.
- (19) Consider the two-dimensional PRM $N_{\tilde{\psi}}$ from Figure 2.1.30 with $\alpha > 0$.
- (a) Calculate the mean measure of the set $A(r, S) = \{\mathbf{x} : |\mathbf{x}| > r, \mathbf{x}/|\mathbf{x}| \in S\}$, where $r > 0$ and S is any Borel subset of the unit circle.
- (b) Show that $EN_{\tilde{\psi}}(A(rt, S)) = t^{-\alpha} EN_{\tilde{\psi}}(A(r, S))$ for any $t > 0$.
- (c) Let $\mathbf{Y} = R(\cos(2\pi X), \sin(2\pi X))$, where $P(R > x) = x^{-\alpha}$, $x \geq 1$, X is uniformly distributed on $(0, 1)$ and independent of R . Show that for $r \geq 1$,

$$EN_{\tilde{\psi}}(A(r, S)) = P(\mathbf{Y} \in A(r, S)).$$

- (20) Let (E, \mathcal{E}, μ) be a measure space such that $0 < \mu(E) < \infty$ and τ be $\text{Pois}(\mu(E))$ distributed. Assume that τ is independent of the iid sequence (X_i) with distribution given by

$$F_{X_1}(A) = P(X_1 \in A) = \mu(A)/\mu(E), \quad A \in \mathcal{E}.$$

- (a) Show that the counting process

$$N(A) = \sum_{i=1}^{\tau} I_A(X_i), \quad A \in \mathcal{E},$$

- is $\text{PRM}(\mu)$ on E . Hint: Calculate the joint characteristic function of the random variables $N(A_1), \dots, N(A_m)$ for any disjoint $A_1, \dots, A_m \in \mathcal{E}$.
- (b) Specify the construction of (a) in the case that $E = [0, 1]$ equipped with the Borel σ -field, when μ has an a.e. positive density λ . What is the relation with the order statistics property of the Poisson process N ?
- (c) Specify the construction of (a) in the case that $E = [0, 1]^d$ equipped with the Borel σ -field for some integer $d \geq 1$ when $\mu = \lambda \text{Leb}$ for some constant $\lambda > 0$. Propose how one could define an “order statistics property” for this (homogeneous) Poisson process with points in E .
- (21) Let τ be a $\text{Pois}(1)$ random variable, independent of the iid sequence (X_i) with common distribution function F and a positive density on $(0, \infty)$.
- (a) Show that

$$N(t) = \sum_{i=1}^{\tau} I_{(0, t]}(X_i), \quad t \geq 0,$$

- defines a Poisson process on $[0, \infty)$ in the sense of Definition 2.1.1.
- (b) Determine the mean value function of N .

- (c) Find a function $f : [0, \infty) \rightarrow [0, \infty)$ such that the time changed process $(N(f(t)))_{t \geq 0}$ becomes a standard homogeneous Poisson process.
- (22) For an iid sequence (X_i) with common continuous distribution function F define the sequence of partial maxima $M_n = \max(X_1, \dots, X_n)$, $n \geq 1$. Define $L(1) = 1$ and, for $n \geq 1$,

$$L(n+1) = \inf\{k > L(n) : X_k > X_{L(n)}\}.$$

The sequence $(X_{L(n)})$ is called the *record value sequence* and $(L(n))$ is the sequence of the *record times*.

It is well-known that for an iid standard exponential sequence (W_i) with record time sequence $(\tilde{L}(n))$, $(W_{\tilde{L}(n)})$ constitute the arrivals of a standard homogeneous Poisson process on $[0, \infty)$; see Example 7.2.4.

- (a) Let $R(x) = -\log \bar{F}(x)$, where $\bar{F} = 1 - F$ and $x \in (x_l, x_r)$, $x_l = \inf\{x : F(x) > 0\}$ and $x_r = \sup\{x : F(x) < 1\}$. Show that $(X_{L(n)}) \stackrel{d}{=} (R^{-}(W_{\tilde{L}(n)}))$, where $R^{-}(t) = \inf\{x \in (x_l, x_r) : R(x) \geq t\}$ is the *generalized inverse of R* .
- (b) Conclude from (a) that $(X_{L(n)})$ is the arrival sequence of a Poisson process on (x_l, x_r) with mean measure of $(a, b] \subset (x_l, x_r)$ given by $R(a, b]$.

2.2 The Renewal Process

2.2.1 Basic Properties

In Section 2.1.4 we learned that the homogeneous Poisson process is a particular renewal process. In this section we want to study this model. We start with a formal definition.

Definition 2.2.1 (Renewal process)

Let (W_i) be an iid sequence of a.s. positive random variables. Then the random walk

$$T_0 = 0, \quad T_n = W_1 + \cdots + W_n, \quad n \geq 1,$$

is said to be a *renewal sequence* and the counting process

$$N(t) = \#\{i \geq 1 : T_i \leq t\} \quad t \geq 0,$$

is the corresponding *renewal (counting) process*.

We also refer to (T_n) and (W_n) as the sequences of the arrival and inter-arrival times of the renewal process N , respectively.

Example 2.2.2 (Homogeneous Poisson process)

It follows from Theorem 2.1.6 that a homogeneous Poisson process with intensity λ is a renewal process with iid exponential $\text{Exp}(\lambda)$ inter-arrival times W_i . \square

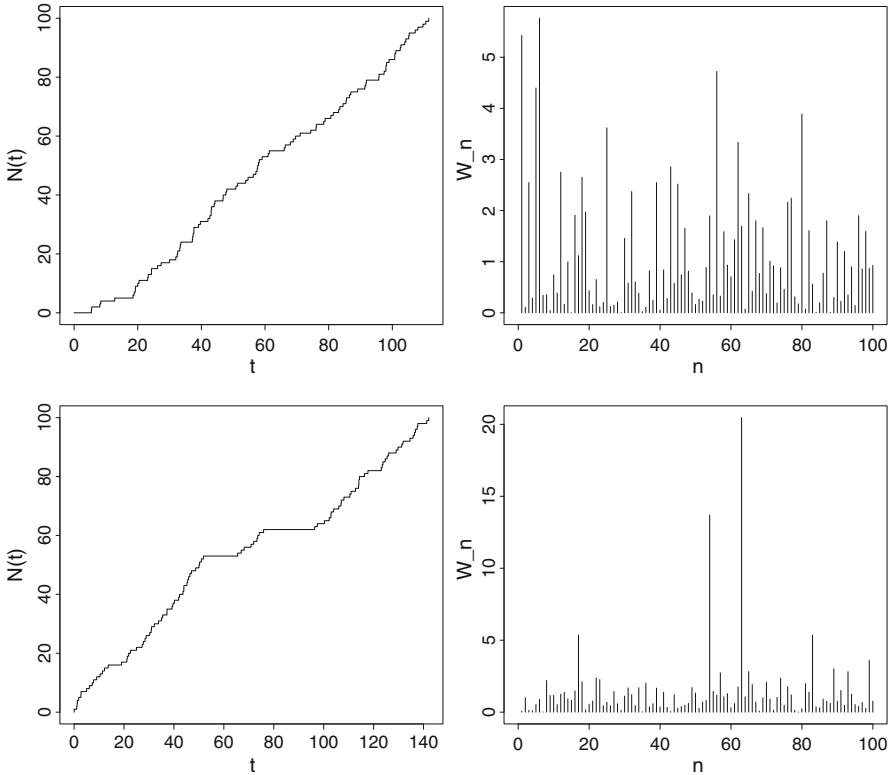


Figure 2.2.3 One path of a renewal process (left graphs) and the corresponding inter-arrival times (right graphs). Top: Standard homogeneous Poisson process with iid standard exponential inter-arrival times. Bottom: The renewal process has iid Pareto distributed inter-arrival times with $P(W_i > x) = x^{-4}$, $x \geq 1$. Both renewal paths have 100 jumps. Notice the extreme lengths of some inter-arrival times in the bottom graph; they are atypical for a homogeneous Poisson process.

A main motivation for introducing the renewal process is that the (homogeneous) Poisson process does not always describe claim arrivals in an adequate way. There can be large gaps between arrivals of claims. For example, it is unlikely that windstorm claims arrive according to a homogeneous Poisson process. They happen now and then, sometimes with years in between. In this case it is more natural to assume that the inter-arrival times have a distribution which allows for modeling these large time intervals. The log-normal or the Pareto distributions would do this job since their tails are much heavier than those of the exponential distribution; see Section 3.2. We have also seen

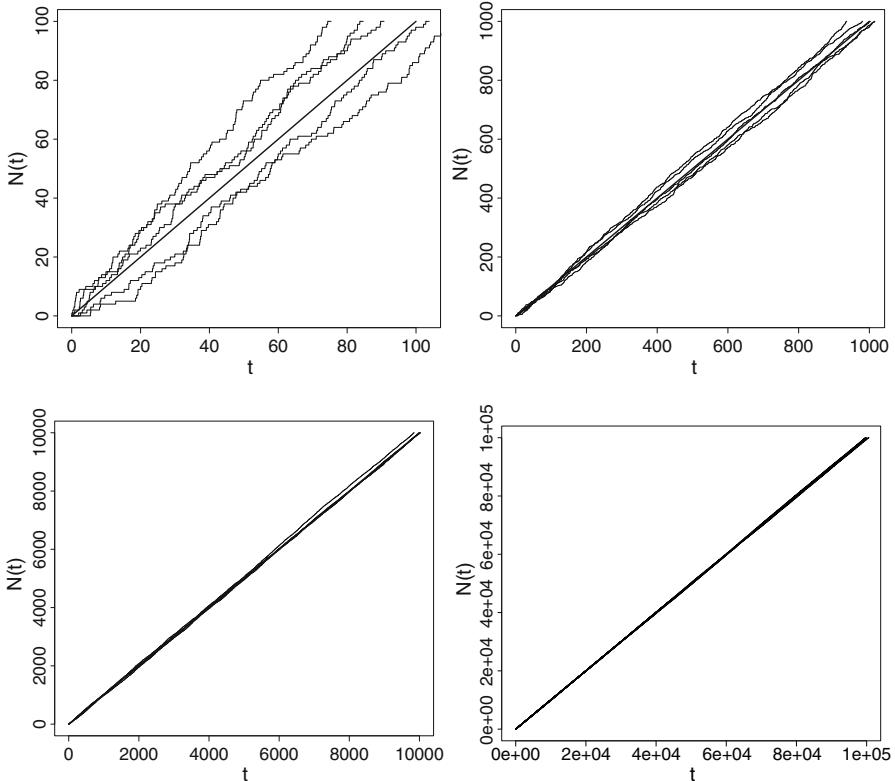


Figure 2.2.4 Five paths of a renewal process with $\lambda = 1$ and $n = 10^i$ jumps, $i = 2, 3, 4, 5$. The mean value function $EN(t) = t$ is also indicated (solid straight line). The approximation of $N(t)$ by $EN(t)$ for increasing t is nicely illustrated; on a large time scale $N(t)$ and $EN(t)$ can hardly be distinguished.

in Section 2.1.7 that the Poisson process is not always a realistic model for real-life claim arrivals, in particular if one considers long periods of time.

On the other hand, if we give up the hypothesis of a Poisson process we lose most of the nice properties of this process which are closely related to the exponential distribution of the W_i 's. For example, it is in general unknown which distribution $N(t)$ has and what the exact values of $EN(t)$ or $\text{var}(N(t))$ are. We will, however, see that the renewal processes and the homogeneous Poisson process have various *asymptotic* properties in common.

The first result of this kind is a strong law of large numbers for the renewal counting process.

Theorem 2.2.5 (Strong law of large numbers for the renewal process)

If the expectation $EW_1 = \lambda^{-1}$ of the inter-arrival times W_i is finite, N satisfies the strong law of large numbers:

$$\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \lambda \quad \text{a.s.}$$

Proof. We need a simple auxiliary result.

Lemma 2.2.6 Let (Z_n) be a sequence of random variables such that $Z_n \xrightarrow{\text{a.s.}} Z$ as $n \rightarrow \infty$ for some random variable Z , and let $(M(t))_{t \geq 0}$ be a stochastic process of integer-valued random variables such that $M(t) \xrightarrow{\text{a.s.}} \infty$ as $t \rightarrow \infty$. If M and (Z_n) are defined on the same probability space Ω , then

$$Z_{M(t)} \rightarrow Z \quad \text{a.s.} \quad \text{as } t \rightarrow \infty.$$

Proof. Write

$$\Omega_1 = \{\omega \in \Omega : M(t, \omega) \rightarrow \infty\} \quad \text{and} \quad \Omega_2 = \{\omega \in \Omega : Z_n(\omega) \rightarrow Z(\omega)\}.$$

By assumption, $P(\Omega_1) = P(\Omega_2) = 1$, hence $P(\Omega_1 \cap \Omega_2) = 1$ and therefore

$$P(\{\omega : Z_{M(t, \omega)}(\omega) \rightarrow Z(\omega)\}) \geq P(\Omega_1 \cap \Omega_2) = 1.$$

This proves the lemma. \square

Recall the following basic relation of a renewal process:

$$\{N(t) = n\} = \{T_n \leq t < T_{n+1}\}, \quad n \in \mathbb{N}_0.$$

Then it is immediate that the following sandwich inequalities hold:

$$\frac{T_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{T_{N(t)+1}}{N(t)+1} \frac{N(t)+1}{N(t)}. \quad (2.2.31)$$

By the strong law of large numbers for the iid sequence (W_n) we have

$$n^{-1} T_n \xrightarrow{\text{a.s.}} \lambda^{-1}.$$

In particular, $N(t) \rightarrow \infty$ a.s. as $t \rightarrow \infty$. Now apply Lemma 2.2.6 with $Z_n = T_n/n$ and $M = N$ to obtain

$$\frac{T_{N(t)}}{N(t)} \xrightarrow{\text{a.s.}} \lambda^{-1}. \quad (2.2.32)$$

The statement of the theorem follows by a combination of (2.2.31) and (2.2.32). \square

In the case of a homogeneous Poisson process we know the exact value of the expected renewal process: $EN(t) = \lambda t$. In the case of a general renewal

process N the strong law of large numbers $N(t)/t \xrightarrow{\text{a.s.}} \lambda = (EW_1)^{-1}$ suggests that the expectation $EN(t)$ of the renewal process is approximately of the order λt . A lower bound for $EN(t)/t$ is easily achieved. By an application of Fatou's lemma (see for example Williams [145])) and the strong law of large numbers for $N(t)$,

$$\lambda = E \liminf_{t \rightarrow \infty} \frac{N(t)}{t} \leq \liminf_{t \rightarrow \infty} \frac{EN(t)}{t}. \quad (2.2.33)$$

This lower bound can be complemented by the corresponding upper one which leads to the following standard result.

Theorem 2.2.7 (Elementary renewal theorem)

If the expectation $EW_1 = \lambda^{-1}$ of the inter-arrival times is finite, the following relation holds:

$$\lim_{t \rightarrow \infty} \frac{EN(t)}{t} = \lambda.$$

Proof. By virtue of (2.2.33) it remains to prove that

$$\limsup_{t \rightarrow \infty} \frac{EN(t)}{t} \leq \lambda. \quad (2.2.34)$$

We use a truncation argument which we borrow from Resnick [123], p. 191. Write for any $b > 0$,

$$W_i^{(b)} = \min(W_i, b), \quad T_i^{(b)} = W_1^{(b)} + \cdots + W_i^{(b)}, \quad i \geq 1.$$

Obviously, $(T_n^{(b)})$ is a renewal sequence and $T_n \geq T_n^{(b)}$ which implies $N_b(t) \geq N(t)$ for the corresponding renewal process

$$N_b(t) = \#\{i \geq 1 : T_i^{(b)} \leq t\}, \quad t \geq 0.$$

Hence

$$\limsup_{t \rightarrow \infty} \frac{EN(t)}{t} \leq \limsup_{t \rightarrow \infty} \frac{EN_b(t)}{t}. \quad (2.2.35)$$

We observe that, by definition of N_b ,

$$T_{N_b(t)}^{(b)} = W_1^{(b)} + \cdots + W_{N_b(t)}^{(b)} \leq t.$$

The following result is due to the fact that $N_b(t) + 1$ is a so-called *stopping time*²² with respect to the natural filtration generated by the sequence $(W_i^{(b)})$.

²² Let $\mathcal{F}_n = \sigma(W_i^{(b)}, i \leq n)$ be the σ -field generated by $W_1^{(b)}, \dots, W_n^{(b)}$. Then (\mathcal{F}_n) is the natural filtration generated by the sequence $(W_n^{(b)})$. An integer-valued random variable τ is a stopping time with respect to (\mathcal{F}_n) if $\{\tau = n\} \in \mathcal{F}_n$. If $E\tau < \infty$ Wald's identity yields $E(\sum_{i=1}^{\tau} W_i^{(b)}) = E\tau EW_1^{(b)}$. Notice that $\{N_b(t) = n\} = \{T_n^{(b)} \leq t < T_{n+1}^{(b)}\}$. Hence $N_b(t)$ is not a stopping time. However, the same argument shows that $N_b(t) + 1$ is a stopping time with respect to (\mathcal{F}_n) . The interested reader is referred to Williams's textbook [145] which gives a concise introduction to discrete-time martingales, filtrations and stopping times.

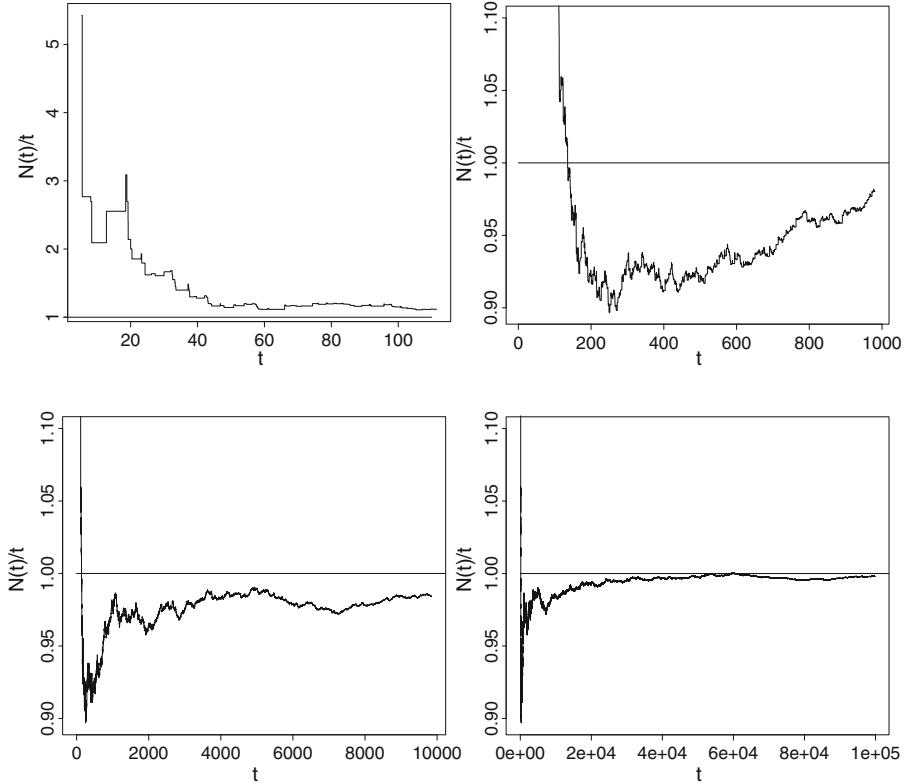


Figure 2.2.8 The ratio $N(t)/t$ for a renewal process with $n = 10^i$ jumps, $i = 2, 3, 4, 5$, and $\lambda = 1$. The strong law of large numbers forces $N(t)/t$ towards 1 for large t .

Then the relation

$$E(T_{N_b(t)+1}^{(b)}) = E(N_b(t) + 1) EW_1^{(b)} \quad (2.2.36)$$

holds by virtue of *Wald's identity*. Combining (2.2.35)-(2.2.36), we conclude that

$$\limsup_{t \rightarrow \infty} \frac{EN(t)}{t} \leq \limsup_{t \rightarrow \infty} \frac{E(T_{N_b(t)+1}^{(b)})}{t EW_1^{(b)}} \leq \limsup_{t \rightarrow \infty} \frac{t + b}{t EW_1^{(b)}} = (EW_1^{(b)})^{-1}.$$

Since by the monotone convergence theorem (see for example Williams [145]), letting $b \uparrow \infty$,

$$EW_1^{(b)} = E(\min(b, W_1)) \uparrow EW_1 = \lambda^{-1},$$

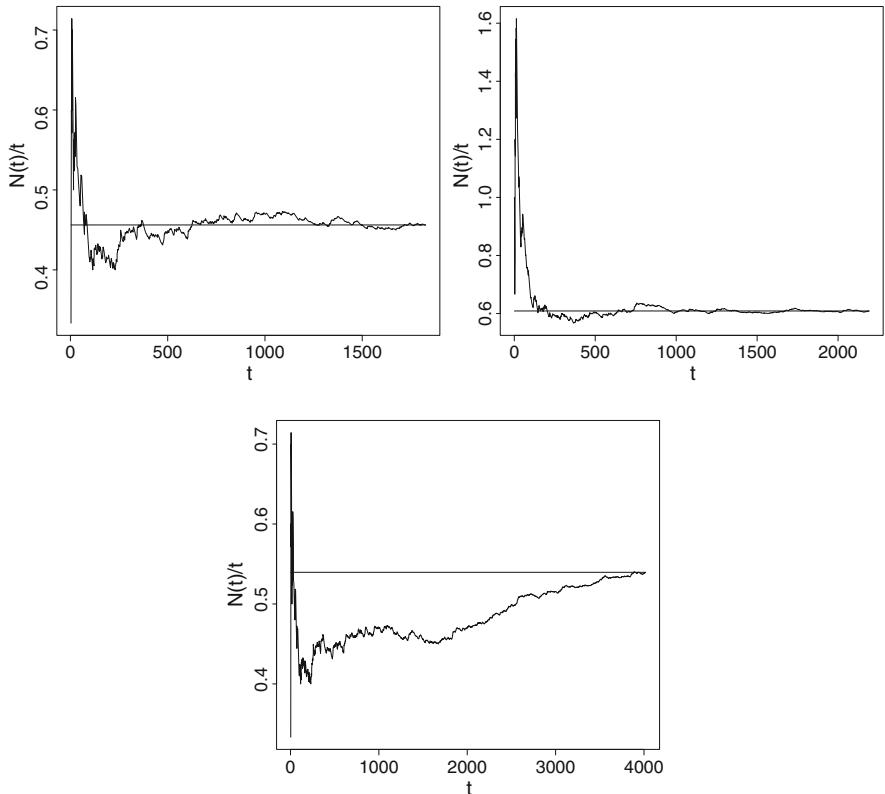


Figure 2.2.9 Visualization of the validity of the strong law of large numbers for the arrivals of the Danish fire insurance data 1980 – 1990; see Section 2.1.7 for a description of the data. Top left: The ratio $N(t)/t$ for 1980 – 1984, where $N(t)$ is the claim number at day t in this period. The values cluster around the value 0.46 which is indicated by the constant line. Top right: The ratio $N(t)/t$ for 1985 – 1990, where $N(t)$ is the claim number at day t in this period. The values cluster around the value 0.61 which is indicated by the constant line. Bottom: The ratio $N(t)/t$ for the whole period 1980 – 1990, where $N(t)$ is the claim number at day t in this period. The graph gives evidence about the fact that the strong law of large numbers does not apply to N for the whole period. This is caused by an increase of the annual intensity in 1985 – 1990 which can be observed in Figure 2.1.20. This fact makes the assumption of iid inter-arrival times over the whole period of 11 years questionable. We do, however, see in the top graphs that the strong law of large numbers works satisfactorily in the two distinct periods.

the desired relation (2.2.34) follows. This concludes the proof. \square

For further reference we include a result about the asymptotic behavior of $\text{var}(N(t))$. The proof can be found in Gut [65], Theorem 5.2.

Proposition 2.2.10 (The asymptotic behavior of the variance of the renewal process)

Assume $\text{var}(W_1) < \infty$. Then

$$\lim_{t \rightarrow \infty} \frac{\text{var}(N(t))}{t} = \frac{\text{var}(W_1)}{(EW_1)^3}.$$

Finally, we mention that $N(t)$ satisfies the central limit theorem; see Embrechts et al. [46], Theorem 2.5.13, for a proof.

Theorem 2.2.11 (The central limit theorem for the renewal process)

Assume that $\text{var}(W_1) < \infty$. Then the central limit theorem

$$(\text{var}(W_1)(EW_1)^{-3} t)^{-1/2} (N(t) - \lambda t) \xrightarrow{d} Y \sim N(0, 1). \quad (2.2.37)$$

holds as $t \rightarrow \infty$.

By virtue of Proposition 2.2.10, the normalizing constants $\sqrt{\text{var}(W_1)(EW_1)^{-3} t}$ in (2.2.37) can be replaced by the standard deviation $\sqrt{\text{var}(N(t))}$.

2.2.2 An Informal Discussion of Renewal Theory

Renewal processes model occurrences of events happening at random instants of time, where the inter-arrival times are approximately iid. In the context of non-life insurance these instants were interpreted as the arrival times of claims. Renewal processes play a major role in applied probability. Complex stochastic systems can often be described by one or several renewal processes as building blocks. For example, the Internet can be understood as the superposition of a huge number of ON/OFF processes. Each of these processes corresponds to one “source” (computer) which communicates with other sources. ON refers to an active period of the source, OFF to a period of silence. The ON/OFF periods of each source constitute two sequences of iid positive random variables, both defining renewal processes.²³ A renewal process is also defined by the sequence of renewals (times of replacement) of a technical device or tool, say the light bulbs in a lamp or the fuel in a nuclear power station. From these elementary applications the process gained its name.

Because of their theoretical importance renewal processes are among the best studied processes in applied probability theory. The object of main interest in renewal theory is the *renewal function*²⁴

²³ The approach to tele-traffic via superpositions of ON/OFF processes became popular in the 1990s; see Willinger et al. [146].

²⁴ The addition of one unit to the mean $EN(t)$ refers to the fact that $T_0 = 0$ is often considered as the first renewal time. This definition often leads to more elegant

$$m(t) = EN(t) + 1, \quad t \geq 0.$$

It describes the average behavior of the renewal counting process. In the insurance context, this is the expected number of claim arrivals in a portfolio. This number certainly plays an important role in the insurance business and its theoretical understanding is therefore essential. The iid assumption of the inter-arrival times is perhaps not the most realistic but is convenient for building up a theory.

The *elementary renewal theorem* (Theorem 2.2.7) is a simple but not very precise result about the average behavior of renewals: $m(t) = \lambda t (1 + o(1))$ as $t \rightarrow \infty$, provided $EW_1 = \lambda^{-1} < \infty$. Much more precise information is gained by *Blackwell's renewal theorem*. It says that for $h > 0$,

$$m(t, t+h] = EN(t, t+h] \rightarrow \lambda h, \quad t \rightarrow \infty.$$

(For Blackwell's renewal theorem and the further statements of this section we assume that the inter-arrival times W_i have a density.) Thus, for sufficiently large t , the expected number of renewals in the interval $(t, t+h]$ becomes independent of t and is proportional to the length of the interval. Since m is a non-decreasing function on $[0, \infty)$ it defines a measure m (we use the same symbol for convenience) on the Borel σ -field of $[0, \infty)$, the so-called *renewal measure*.

A special calculus has been developed for integrals with respect to the renewal measure. In this context, the crucial condition on the integrands is called *direct Riemann integrability*. Directly Riemann integrable functions on $[0, \infty)$ constitute quite a sophisticated class of integrands; it includes Riemann integrable functions on $[0, \infty)$ which have compact support (the function vanishes outside a certain finite interval) or which are non-increasing and non-negative. The *key renewal theorem* states that for a directly Riemann integrable function f ,

$$\int_0^t f(t-s) dm(s) \rightarrow \lambda \int_0^\infty f(s) ds. \quad (2.2.38)$$

Under general conditions, it is equivalent to Blackwell's renewal theorem which, in a sense, is a special case of (2.2.38) for indicator functions $f(x) = I_{(0,h]}(x)$ with $h > 0$ and for $t > h$:

$$\begin{aligned} \int_0^t f(t-s) dm(s) &= \int_{t-h}^t I_{(0,h]}(t-s) dm(s) = m(t-h, t] \\ &\rightarrow \lambda \int_0^\infty f(s) ds = \lambda h. \end{aligned}$$

theoretical formulations. Alternatively, we have learned on p. 57 that the process $N(t) + 1$ has the desirable theoretical property of a stopping time, which $N(t)$ does not have.

An important part of renewal theory is devoted to the *renewal equation*. It is a convolution equation of the form

$$U(t) = u(t) + \int_0^t U(t-y) dF_{T_1}(y), \quad (2.2.39)$$

where all functions are defined on $[0, \infty)$. The function U is *unknown*, u is a *known* function and F_{T_1} is the distribution function of the iid positive inter-arrival times $W_i = T_i - T_{i-1}$. The main goal is to find a solution U to (2.2.39). It is provided by the following general result which can be found in Resnick [123], p. 202.

Theorem 2.2.12 (W. Smith's key renewal theorem)

(1) *If u is bounded on every finite interval then*

$$U(t) = \int_0^t u(t-s) dm(s), \quad t \geq 0, \quad (2.2.40)$$

is the unique solution of the renewal equation (2.2.39) in the class of all functions on $(0, \infty)$ which are bounded on finite intervals. Here the right-hand integral has to be interpreted as $\int_{(-\infty, t]} u(t-s) dm(s)$ with the convention that $m(s) = u(s) = 0$ for $s < 0$.

(2) *If, in addition, u is directly Riemann integrable, then*

$$\lim_{t \rightarrow \infty} U(t) = \lambda \int_0^\infty u(s) ds.$$

Part (2) of the theorem is immediate from Blackwell's renewal theorem.

The renewal function itself satisfies the renewal equation with $u = I_{[0, \infty)}$. From this fact the general equation (2.2.39) gained its name.

Example 2.2.13 (The renewal function satisfies the renewal equation)
Observe that for $t \geq 0$,

$$\begin{aligned} m(t) &= EN(t) + 1 = 1 + E \left(\sum_{n=1}^{\infty} I_{[0,t]}(T_n) \right) = 1 + \sum_{n=1}^{\infty} P(T_n \leq t) \\ &= I_{[0,\infty)}(t) + \sum_{n=1}^{\infty} \int_0^t P(y + (T_n - T_1) \leq t) dF_{T_1}(y) \\ &= I_{[0,\infty)}(t) + \int_0^t \sum_{n=1}^{\infty} P(T_{n-1} \leq t-y) dF_{T_1}(y) \\ &= I_{[0,\infty)}(t) + \int_0^t m(t-y) dF_{T_1}(y). \end{aligned}$$

This is a renewal equation with $U(t) = m(t)$ and $u(t) = I_{[0,\infty)}(t)$. □

The usefulness of the renewal equation is illustrated in the following example.

Example 2.2.14 (Recurrence times of a renewal process)

In our presentation we closely follow Section 3.5 in Resnick [123]. Consider a renewal sequence (T_n) with $T_0 = 0$ and $W_n > 0$ a.s. Recall that

$$\{N(t) = n\} = \{T_n \leq t < T_{n+1}\}.$$

In particular, $T_{N(t)} \leq t < T_{N(t)+1}$. For $t \geq 0$, the quantities

$$F(t) = T_{N(t)+1} - t \quad \text{and} \quad B(t) = t - T_{N(t)}$$

are the *forward* and *backward recurrence times* of the renewal process, respectively. For obvious reasons, $F(t)$ is also called the *excess life* or *residual life*, i.e., it is the time until the next renewal, and $B(t)$ is called the *age process*. In an insurance context, $F(t)$ is the time until the next claim arrives, and $B(t)$ is the time which has evolved since the last claim arrived.

It is our aim to show that the function $P(B(t) \leq x)$ for fixed $0 \leq x < t$ satisfies a renewal equation. It suffices to consider the values $x < t$ since $B(t) \leq t$ a.s., hence $P(B(t) \leq x) = 1$ for $x \geq t$. We start with the identity

$$P(B(t) \leq x) = P(B(t) \leq x, T_1 \leq t) + P(B(t) \leq x, T_1 > t), \quad x > 0. \quad (2.2.41)$$

If $T_1 > t$, no jump has occurred by time t , hence $N(t) = 0$ and therefore $B(t) = t$. We conclude that

$$P(B(t) \leq x, T_1 > t) = (1 - F_{T_1}(t)) I_{[0,x]}(t). \quad (2.2.42)$$

For $T_1 \leq t$, we want to show the following result:

$$P(B(t) \leq x, T_1 \leq t) = \int_0^t P(B(t-y) \leq x) dF_{T_1}(y). \quad (2.2.43)$$

This means that, on the event $\{T_1 \leq t\}$, the process B “starts from scratch” at T_1 . We make this precise by exploiting a “typical renewal argument”. First observe that

$$\begin{aligned} P(B(t) \leq x, T_1 \leq t) &= P(t - T_{N(t)} \leq x, N(t) \geq 1) \\ &= \sum_{n=1}^{\infty} P(t - T_{N(t)} \leq x, N(t) = n) \\ &= \sum_{n=1}^{\infty} P(t - T_n \leq x, T_n \leq t < T_{n+1}). \end{aligned}$$

We study the summands individually by conditioning on $\{T_1 = y\}$ for $y \leq t$:

$$\begin{aligned}
& P(t - T_n \leq x, T_n \leq t < T_{n+1} \mid T_1 = y) \\
&= P\left(t - \left[y + \sum_{i=2}^n W_i\right] \leq x, y + \sum_{i=2}^n W_i \leq t < y + \sum_{i=2}^{n+1} W_i\right) \\
&= P(t - y - T_{n-1} \leq x, T_{n-1} \leq t - y \leq T_n) \\
&= P(t - y - T_{N(t-y)} \leq x, N(t-y) = n-1) .
\end{aligned}$$

Hence we have

$$\begin{aligned}
& P(B(t) \leq x, T_1 \leq t) \\
&= \sum_{n=0}^{\infty} \int_0^t P(t - y - T_{N(t-y)} \leq x, N(t-y) = n) dF_{T_1}(y) \\
&= \int_0^t P(B(t-y) \leq x) dF_{T_1}(y),
\end{aligned}$$

which is the desired relation (2.2.43). Combining (2.2.41)-(2.2.43), we arrive at

$$P(B(t) \leq x) = (1 - F_{T_1}(t)) I_{[0,x]}(t) + \int_0^t P(B(t-y) \leq x) dF_{T_1}(y). \quad (2.2.44)$$

This is a renewal equation of the form (2.2.39) with $u(t) = (1 - F_{T_1}(t)) I_{[0,x]}(t)$, and $U(t) = P(B(t) \leq x)$ is the unknown function.

A similar renewal equation can be given for $P(F(t) > x)$:

$$P(F(t) > x) = \int_0^t P(F(t-y) > x) dF_{T_1}(y) + (1 - F_{T_1}(t+x)). \quad (2.2.45)$$

We mentioned before, see (2.2.40), that the unique solution to the renewal equation (2.2.44) is given by

$$U(t) = P(B(t) \leq x) = \int_0^t (1 - F_{T_1}(t-y)) I_{[0,x]}(t-y) dm(y). \quad (2.2.46)$$

Now consider a homogeneous Poisson process with intensity λ . In this case, $m(t) = EN(t) + 1 = \lambda t + 1$, $1 - F_{T_1}(x) = \exp\{-\lambda x\}$. From (2.2.46) for $x < t$ and since $B(t) \leq t$ a.s. we obtain

$$P(B(t) \leq x) = P(t - T_{N(t)} \leq x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x < t, \\ 1 & \text{if } x \geq t. \end{cases}$$

A similar argument yields for $F(t)$,

$$P(F(t) \leq x) = P(T_{N(t)+1} - t \leq x) = 1 - e^{-\lambda x}, \quad x > 0.$$

The latter result is counterintuitive in a sense since, on the one hand, the inter-arrival times W_i are $\text{Exp}(\lambda)$ distributed and, on the other hand, the time $T_{N(t)+1} - t$ until the next renewal has the same distribution. This reflects the *forgetfulness property* of the exponential distribution of the inter-arrival times. We refer to Example 2.1.7 for further discussions and a derivation of the distributions of $B(t)$ and $F(t)$ for the homogeneous Poisson process by elementary means. \square

Comments

Renewal theory constitutes an important part of applied probability theory. Resnick [123] gives an entertaining introduction with various applications, among others, to problems of insurance mathematics. The advanced text on stochastic processes in insurance mathematics by Rolski et al. [127] makes extensive use of renewal techniques. Gut's book [65] is a collection of various useful limit results related to renewal theory and stopped random walks.

The notion of direct Riemann integrability has been discussed in various books; see Alsmeyer [2], p. 69, Asmussen [6], Feller [51], pp. 361-362, or Resnick [123], Section 3.10.1.

Smith's key renewal theorem will also be key to the asymptotic results on the ruin probability in the Cramér-Lundberg model in Section 4.2.2.

Exercises

- (1) Let (T_i) be a renewal sequence with $T_0 = 0$, $T_n = W_1 + \dots + W_n$, where (W_i) is an iid sequence of non-negative random variables.
 - (a) Which assumption is needed to ensure that the renewal process $N(t) = \#\{i \geq 1 : T_i \leq t\}$ has no jump sizes greater than 1 with positive probability?
 - (b) Can it happen that (T_i) has a limit point with positive probability? This would mean that $N(t) = \infty$ at some finite time t .
- (2) Let N be a homogeneous Poisson process on $[0, \infty)$ with intensity $\lambda > 0$.
 - (a) Show that $N(t)$ satisfies the central limit theorem as $t \rightarrow \infty$ i.e.,

$$\widehat{N}(t) = \frac{N(t) - \lambda t}{\sqrt{\lambda t}} \xrightarrow{d} Y \sim N(0, 1),$$

- (i) by using characteristic functions,
- (ii) by employing the known central limit theorem for the sequence $((N(n) - \lambda n)/\sqrt{\lambda n})_{n=1,2,\dots}$, and then by proving that

$$\max_{t \in (n, n+1]} (N(t) - N(n))/\sqrt{n} \xrightarrow{P} 0.$$

- (b) Show that N satisfies the multivariate central limit theorem for any $0 < s_1 < \dots < s_n$ as $t \rightarrow \infty$:

$$(\sqrt{\lambda t})^{-1} (N(s_1 t) - s_1 \lambda t, \dots, N(s_n t) - s_n \lambda t) \xrightarrow{d} \mathbf{Y} \sim N(\mathbf{0}, \Sigma),$$

where the right-hand distribution is multivariate normal with mean vector zero and covariance matrix Σ whose entries satisfy $\sigma_{i,j} = \min(s_i, s_j)$, $i, j = 1, \dots, n$.

- (3) Let $F(t) = T_{N(t)+1} - t$ be the forward recurrence time from Example 2.2.14.
- (a) Show that the probability $P(F(t) > x)$, considered as a function of t , for $x > 0$ fixed satisfies the renewal equation (2.2.45).
 - (b) Solve (2.2.45) in the case of iid $\text{Exp}(\lambda)$ inter-arrival times.

2.3 The Mixed Poisson Process

In Section 2.1.3 we learned that an inhomogeneous Poisson process N with mean value function μ can be derived from a standard homogeneous Poisson process \tilde{N} by a deterministic time change. Indeed, the process

$$\tilde{N}(\mu(t)), \quad t \geq 0,$$

has the same finite-dimensional distributions as N and is càdlàg, hence it is a possible representation of the process N . In what follows, we will use a similar construction by *randomizing the mean value function*.

Definition 2.3.1 (Mixed Poisson process)

Let \tilde{N} be a standard homogeneous Poisson process and μ be the mean value function of a Poisson process on $[0, \infty)$. Let $\theta > 0$ a.s. be a (non-degenerate) random variable independent of \tilde{N} . Then the process

$$N(t) = \tilde{N}(\theta \mu(t)), \quad t \geq 0,$$

is said to be a mixed Poisson process with mixing variable θ .

Example 2.3.2 (The negative binomial process as mixed Poisson process)
One of the important representatives of mixed Poisson processes is obtained by choosing $\mu(t) = t$ and θ gamma distributed. First recall that a $\Gamma(\gamma, \beta)$ distributed random variable θ has density

$$f_\theta(x) = \frac{\beta^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\beta x}, \quad x > 0. \quad (2.3.47)$$

Also recall that an integer-valued random variable Z is said to be negative binomially distributed with parameter (p, v) if it has individual probabilities

$$P(Z = k) = \binom{v+k-1}{k} p^v (1-p)^k, \quad k \in \mathbb{N}_0, \quad p \in (0, 1), \quad v > 0.$$

Verify that $N(t)$ is negative binomial with parameter $(p, v) = (\beta/(t+\beta), \gamma)$. \square

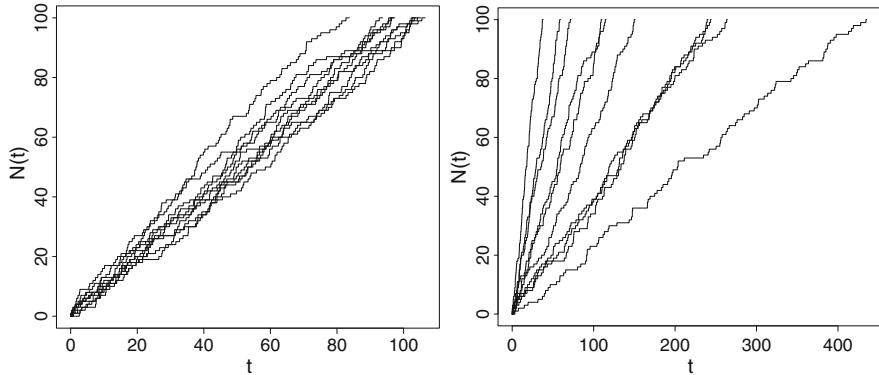


Figure 2.3.3 Left: Ten sample paths of a standard homogeneous Poisson process. Right: Ten sample paths of a mixed homogeneous Poisson process with $\mu(t) = t$. The mixing variable θ is standard exponentially distributed. The processes in the left and right graphs have the same mean value function $EN(t) = t$.

In an insurance context, a mixed Poisson process is introduced as a claim number process if one does not believe in one particular Poisson process as claim arrival generating process. As a matter of fact, if we observed only one sample path $\tilde{N}(\theta(\omega)\mu(t), \omega)$ of a mixed Poisson process, we would not be able to distinguish between this kind of process and a Poisson process with mean value function $\theta(\omega)\mu$. However, if we had several such sample paths we should see differences in the variation of the paths; see Figure 2.3.3 for an illustration of this phenomenon.

A mixed Poisson process is a special *Cox process* where the mean value function μ is a general random process with non-decreasing sample paths, independent of the underlying homogeneous Poisson process \tilde{N} . Such processes have proved useful, for example, in medical statistics where every sample path represents the medical history of a particular patient which has his/her “own” mean value function. We can think of such a function as “drawn” from a distribution of mean value functions. Similarly, we can think of θ representing different factors of influence on an insurance portfolio. For example, think of the claim number process of a portfolio of car insurance policies as a collection of individual sample paths corresponding to the different insured persons. The variable $\theta(\omega)$ then represents properties such as the driving skill, the age, the driving experience, the health state, etc., of the individual drivers.

In Figure 2.3.3 we see one striking difference between a mixed Poisson process and a homogeneous Poisson process: the shape and magnitude of the sample paths of the mixed Poisson process vary significantly. This property *cannot* be explained by the mean value function

$$EN(t) = E\tilde{N}(\theta \mu(t)) = E(E[\tilde{N}(\theta \mu(t)) | \theta]) = E[\theta \mu(t)] = E\theta \mu(t), \quad t \geq 0.$$

Thus, if $E\theta = 1$, as in Figure 2.3.3, the mean values of the random variables $\tilde{N}(\mu(t))$ and $N(t)$ are the same. The differences between a mixed Poisson and a Poisson process with the same mean value function can be seen in the variances. First observe that the Poisson property implies

$$E(N(t) | \theta) = \theta \mu(t) \quad \text{and} \quad \text{var}(N(t) | \theta) = \theta \mu(t). \quad (2.3.48)$$

Next we give an auxiliary result whose proof is left as an exercise.

Lemma 2.3.4 *Let A and B be random variables such that $\text{var}(A) < \infty$. Then*

$$\text{var}(A) = E[\text{var}(A | B)] + \text{var}(E[A | B]).$$

An application of this formula with $A = N(t) = \tilde{N}(\theta \mu(t))$ and $B = \theta$ together with (2.3.48) yields

$$\begin{aligned} \text{var}(N(t)) &= E[\text{var}(N(t) | \theta)] + \text{var}(E[N(t) | \theta]) \\ &= E[\theta \mu(t)] + \text{var}(\theta \mu(t)) \\ &= E\theta \mu(t) + \text{var}(\theta) (\mu(t))^2 \\ &= EN(t) \left(1 + \frac{\text{var}(\theta)}{E\theta} \mu(t) \right) \\ &> EN(t), \end{aligned}$$

where we assumed that $\text{var}(\theta) < \infty$ and $\mu(t) > 0$. The property

$$\text{var}(N(t)) > EN(t) \quad \text{for any } t > 0 \text{ with } \mu(t) > 0 \quad (2.3.49)$$

is called *over-dispersion*. It is one of the major differences between a mixed Poisson process and a Poisson process N , where $EN(t) = \text{var}(N(t))$.

We conclude by summarizing some of the important properties of the mixed Poisson process; some of the proofs are left as exercises.

The mixed Poisson process *inherits* the following properties of the Poisson process:

- It has the *Markov property*; see Section 2.1.2 for some explanation.
- It has the *order statistics property*: if the function μ has a continuous a.e. positive intensity function λ and N has arrival times $0 < T_1 < T_2 < \dots$, then for every $t > 0$,

$$(T_1, \dots, T_n | N(t) = n) \stackrel{d}{=} (X_{(1)}, \dots, X_{(n)}),$$

where the right-hand side is the ordered sample of the iid random variables X_1, \dots, X_n with common density $\lambda(x)/\mu(t)$, $0 \leq x \leq t$; cf. Theorem 2.1.11.

The order statistics property is remarkable insofar that it does not depend on the mixing variable θ . In particular, for a mixed homogeneous Poisson process the conditional distribution of $(T_1, \dots, T_{N(t)})$ given $\{N(t) = n\}$ is the distribution of the ordered sample of iid $U(0, t)$ distributed random variables.

The mixed Poisson process *loses* some of the properties of the Poisson process:

- It has *dependent increments*.
- In general, the distribution of $N(t)$ is *not Poisson*.
- It is *over-dispersed*; see (2.3.49).

Comments

For an extensive treatment of mixed Poisson processes and their properties we refer to the monograph by Grandell [61]. It can be shown that the mixed Poisson process and the Poisson process are the only *point processes* on $[0, \infty)$ which have the order statistics property; see Kallenberg [78]; cf. Grandell [61], Theorem 6.6.

Exercises

- (1) Consider the mixed Poisson process $(N(t))_{t \geq 0} = (\tilde{N}(\theta t))_{t \geq 0}$ with arrival times T_i , where \tilde{N} is a standard homogeneous Poisson process on $[0, \infty)$ and $\theta > 0$ is a non-degenerate mixing variable with $\text{var}(\theta) < \infty$, independent of \tilde{N} .
 - (a) Show that N does not have independent increments. (An easy way of doing this would be to calculate the covariance of $N(s, t]$ and $N(x, y]$ for disjoint intervals $(s, t]$ and $(x, y]$.)
 - (b) Show that N has the order statistics property, i.e., given $N(t) = n$, (T_1, \dots, T_n) has the same distribution as the ordered sample of the iid $U(0, t)$ distributed random variables U_1, \dots, U_n .
 - (c) Calculate $P(N(t) = n)$ for $n \in \mathbb{N}_0$. Show that $N(t)$ is not Poisson distributed.
 - (d) The negative binomial distribution on $\{0, 1, 2, \dots\}$ has the individual probabilities

$$p_k = \binom{v+k-1}{k} p^v (1-p)^k, \quad k \in \mathbb{N}_0, \quad p \in (0, 1), \quad v > 0.$$

Consider the mixed Poisson process N with gamma distributed mixing variable, i.e., θ has $\Gamma(\gamma, \beta)$ density

$$f_\theta(x) = \frac{\beta^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\beta x}, \quad x > 0.$$

Calculate the probabilities $P(N(t) = k)$ and give some reason why the process N is called *negative binomial process*.

- (2) Give an algorithm for simulating the sample paths of an arbitrary mixed Poisson process.

- (3) Prove Lemma 2.3.4.
- (4) Let $N(t) = \tilde{N}(\theta t)$, $t \geq 0$, be mixed Poisson, where \tilde{N} is a standard homogeneous Poisson process, independent of the mixing variable θ .
- (a) Show that N satisfies the strong law of large numbers with random limit θ :

$$\frac{N(t)}{t} \rightarrow \theta \quad \text{a.s.}$$

- (b) Show the following “central limit theorem”:

$$\frac{N(t) - \theta t}{\sqrt{\theta t}} \xrightarrow{d} Y \sim N(0, 1).$$

- (c) Show that the “naive” central limit theorem does not hold by showing that

$$\frac{N(t) - EN(t)}{\sqrt{\text{var}(N(t))}} \xrightarrow{\text{a.s.}} \frac{\theta - E\theta}{\sqrt{\text{var}(\theta)}}.$$

Here we assume that $\text{var}(\theta) < \infty$.

- (5) Let $N(t) = \tilde{N}(\theta t)$, $t \geq 0$, be mixed Poisson, where \tilde{N} is a standard homogeneous Poisson process, independent of the mixing variable $\theta > 0$. Write F_θ for the distribution function of θ and $\bar{F}_\theta = 1 - F_\theta$ for its right tail. Show that the following relations hold for integer $n \geq 1$,

$$P(N(t) > n) = t \int_0^\infty \frac{(tx)^n}{n!} e^{-tx} \bar{F}_\theta(x) dx,$$

$$P(\theta \leq x \mid N(t) = n) = \frac{\int_0^x y^n e^{-y t} dF_\theta(y)}{\int_0^\infty y^n e^{-y t} dF_\theta(y)},$$

$$E(\theta \mid N(t) = n) = \frac{\int_0^\infty y^{n+1} e^{-y t} dF_\theta(y)}{\int_0^\infty y^n e^{-y t} dF_\theta(y)}.$$

3

The Total Claim Amount

In Chapter 2 we learned about three of the most prominent claim number processes, N : the *Poisson process* in Section 2.1, the *renewal process* in Section 2.2, and the *mixed Poisson process* in Section 2.3. In this section we take a closer look at the *total claim amount process*, as introduced on p. 4:

$$S(t) = \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \tag{3.0.1}$$

where the *claim number process* N is independent of the iid *claim size sequence* (X_i). We also assume that $X_i > 0$ a.s. Depending on the choice of the process N , we get different models for the process S . In Example 2.1.3 we introduced the *Cramér-Lundberg model* as that particular case of model (3.0.1) when N is a homogeneous Poisson process. Another prominent model for S is called *renewal* or *Sparre-Anderson model*; it is model (3.0.1) when N is a renewal process.

In Section 3.1 we study the order of magnitude of the total claim amount $S(t)$ in the renewal model. This means we calculate the mean and the variance of $S(t)$ for large t , which give us a rough impression of the growth of $S(t)$ as $t \rightarrow \infty$. We also indicate that S satisfies the strong law of large numbers and the central limit theorem. The information about the asymptotic growth of the total claim amount enables one to give advise as to how much premium should be charged in a given time period in order to avoid bankruptcy or ruin in the portfolio. In Section 3.1.3 we collect some of the classical *premium calculation principles* which can be used as a rule of thumb for determining how big the premium income in a homogeneous portfolio should be.

We continue in Section 3.2 by considering some realistic claim size distributions and their properties. We consider exploratory statistical tools (QQ-plots, mean excess function) and apply them to real-life claim size data in order to get a preliminary understanding of which distributions fit real-life data. In this context, the issue of modeling large claims deserves particular attention. We discuss the notions of heavy- and light-tailed claim size distribu-

tions as appropriate for modeling large and small claims, respectively. Then, in Sections 3.2.5 and 3.2.6 we focus on the subexponential distributions and on distributions with regularly varying tails. The latter classes contain those distributions which are most appropriate for modeling large claims.

In Section 3.3 we study finally the distribution of the total claim amount $S(t)$ as a combination of claim number process and claim sizes. We start in Section 3.3.1 by investigating some theoretical properties of the total claim amount models. By applying characteristic function techniques, we learn about *mixture distributions* as useful tools in the context of compound Poisson and compound geometric processes. We show that the summation of independent compound Poisson processes yields a compound Poisson process and we investigate consequences of this result. In particular, we show in the framework of the Cramér-Lundberg model that the total claim amounts from disjoint layers for the claim sizes or over disjoint periods of time are independent compound Poisson variables. We continue in Section 3.3.3 with a numerical recursive procedure for determining the distribution of the total claim amount. In the insurance world, this technique is called *Panjer recursion*. In Sections 3.3.4 and 3.3.5 we consider alternative methods for determining *approximations* to the distribution of the total claim amount. These approximations are based on the central limit theorem or Monte Carlo techniques.

Finally, in Section 3.4 we apply the developed theory to the case of reinsurance treaties. The latter are agreements between a primary and a secondary insurer with the aim to protect the primary insurer against excessive losses which are caused by very large claim sizes or by a large number of small and moderate claim sizes. We discuss the most important forms of the treaties and indicate how previously developed theory can be applied to deal with their distributional properties.

3.1 The Order of Magnitude of the Total Claim Amount

Given a particular model for S , one of the important questions for an insurance company is to determine the order of magnitude of $S(t)$. This information is needed in order to determine a *premium* which covers the *losses* represented by $S(t)$.

Most desirably, one would like to know the distribution of $S(t)$. This, however, is in general a too complicated problem and therefore one often relies on numerical or simulation methods in order to approximate the distribution of $S(t)$. In this section we consider some simple means in order to get a rough impression of the size of the total claim amount. Those means include the expectation and variance of $S(t)$ (Section 3.1.1), the strong law of large numbers, and the central limit theorem for $S(t)$ as $t \rightarrow \infty$ (Section 3.1.2). In Section 3.1.3 we study the relationship of these results with premium calculation principles.

3.1.1 The Mean and the Variance in the Renewal Model

The expectation of a random variable tells one about its average size. For the total claim amount the expectation is easily calculated by exploiting the independence of (X_i) and $N(t)$, provided $EN(t)$ and EX_1 are finite:

$$ES(t) = E \left[E \left(\sum_{i=1}^{N(t)} X_i \mid N(t) \right) \right] = E(N(t) EX_1) = EN(t) EX_1.$$

Example 3.1.1 (Expectation of $S(t)$ in the Cramér-Lundberg and renewal models)

In the Cramér-Lundberg model, $EN(t) = \lambda t$, where λ is the intensity of the homogeneous Poisson process N . Hence

$$ES(t) = \lambda t EX_1.$$

Such a compact formula does not exist in the general renewal model. However, given $EW_1 = \lambda^{-1} < \infty$ we know from the elementary renewal Theorem 2.2.7 that $EN(t)/t \rightarrow \lambda$ a.s. as $t \rightarrow \infty$. Therefore

$$ES(t) = \lambda t EX_1 (1 + o(1)), \quad t \rightarrow \infty.$$

This is less precise information than in the Cramér-Lundberg model. However, this formula tells us that the expected total claim amount grows roughly linearly for large t . As in the Cramér-Lundberg case, the slope of the linear function is determined by the reciprocal of the expected inter-arrival time EW_1 and the expected claim size EX_1 . \square

The expectation does not tell one too much about the distribution of $S(t)$. We learn more about the order of magnitude of $S(t)$ if we combine the information about $ES(t)$ with the variance $\text{var}(S(t))$.

Assume that $\text{var}(N(t))$ and $\text{var}(X_1)$ are finite. Conditioning on $N(t)$ and exploiting the independence of $N(t)$ and (X_i) , we obtain

$$\begin{aligned} \text{var} \left[\sum_{i=1}^{N(t)} X_i \mid N(t) \right] &= \sum_{i=1}^{N(t)} \text{var}(X_i \mid N(t)) \\ &= N(t) \text{var}(X_1 \mid N(t)) = N(t) \text{var}(X_1), \end{aligned}$$

$$E \left[\sum_{i=1}^{N(t)} X_i \mid N(t) \right] = N(t) EX_1.$$

By virtue of Lemma 2.3.4 we conclude that

$$\begin{aligned} \text{var}(S(t)) &= E[N(t) \text{var}(X_1)] + \text{var}(N(t) EX_1) \\ &= EN(t) \text{var}(X_1) + \text{var}(N(t)) (EX_1)^2. \end{aligned}$$

Example 3.1.2 (Variance of $S(t)$ in the Cramér-Lundberg and renewal models)

In the Cramér-Lundberg model the Poisson distribution of $N(t)$ gives us $EN(t) = \text{var}(N(t)) = \lambda t$. Hence

$$\text{var}(S(t)) = \lambda t [\text{var}(X_1) + (EX_1)^2] = \lambda t E(X_1^2).$$

In the renewal model we again depend on some asymptotic formulae for $EN(t)$ and $\text{var}(N(t))$; see Theorem 2.2.7 and Proposition 2.2.10:

$$\begin{aligned} \text{var}(S(t)) &= [\lambda t \text{var}(X_1) + \text{var}(W_1) \lambda^3 t (EX_1)^2] (1 + o(1)) \\ &= \lambda t [\text{var}(X_1) + \text{var}(W_1) \lambda^2 (EX_1)^2] (1 + o(1)). \end{aligned}$$

□

We summarize our findings.

Proposition 3.1.3 (Expectation and variance of the total claim amount in the renewal model)

In the renewal model, if $EW_1 = \lambda^{-1}$ and EX_1 are finite,

$$\lim_{t \rightarrow \infty} \frac{ES(t)}{t} = \lambda EX_1,$$

and if $\text{var}(W_1)$ and $\text{var}(X_1)$ are finite,

$$\lim_{t \rightarrow \infty} \frac{\text{var}(S(t))}{t} = \lambda [\text{var}(X_1) + \text{var}(W_1) \lambda^2 (EX_1)^2].$$

In the Cramér-Lundberg model these limit relations degenerate to identities for every $t > 0$:

$$ES(t) = \lambda t EX_1 \quad \text{and} \quad \text{var}(S(t)) = \lambda t E(X_1^2).$$

The message of these results is that in the renewal model *both the expectation and the variance of the total claim amount grow roughly linearly as a function of t* . This is important information which can be used to give a rule of thumb about how much premium has to be charged for covering the losses $S(t)$: the premium should increase roughly linearly and with a slope larger than λEX_1 . In Section 3.1.3 we will consider some of the classical premium calculation principles and there we will see that this rule of thumb is indeed quite valuable.

3.1.2 The Asymptotic Behavior in the Renewal Model

In this section we are interested in the asymptotic behavior of the total claim amount process. Throughout we assume the renewal model (see p. 71) for the total claim amount process S . As a matter of fact, $S(t)$ satisfies quite a general strong law of large numbers and central limit theorem:

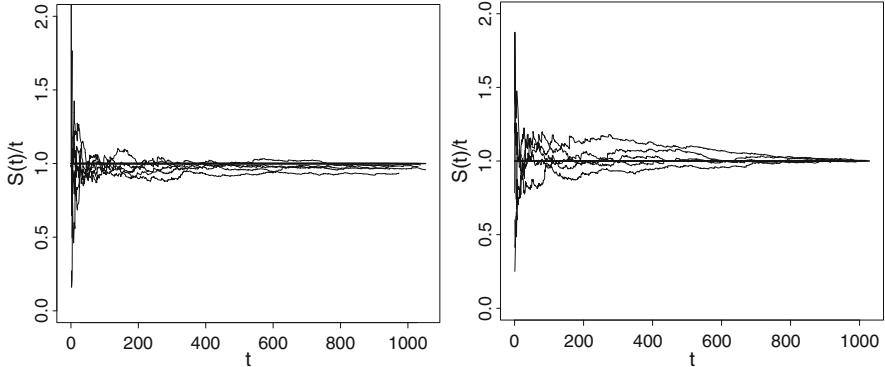


Figure 3.1.4 Visualization of the strong law of large numbers for the total claim amount S in the Cramér-Lundberg model with unit Poisson intensity. Five sample paths of the process $(S(t)/t)$ are drawn in the interval $[0, 1000]$. Left: Standard exponential claim sizes. Right: Pareto distributed claim sizes $X_i = 1 + (Y_i - EY_1)/\sqrt{\text{var}(Y_1)}$ for iid Y_i 's with distribution function $P(Y_i \leq x) = 1 - 2^4 x^{-4}$, $x \geq 2$. These random variables have mean and variance 1. The fluctuations of $S(t)/t$ around the mean 1 for small t are more pronounced than for exponential claim sizes. The right tail of the distribution of X_1 is much heavier than the right tail of the exponential distribution. Therefore much larger claim sizes may occur.

Theorem 3.1.5 (The strong law of large numbers and the central limit theorem in the renewal model)

Assume the renewal model for S .

- (1) If the inter-arrival times W_i and the claim sizes X_i have finite expectation, S satisfies the strong law of large numbers:

$$\lim_{t \rightarrow \infty} \frac{S(t)}{t} = \lambda E X_1 \quad \text{a.s.} \quad (3.1.2)$$

- (2) If the inter-arrival times W_i and the claim sizes X_i have finite variance, S satisfies the central limit theorem:

$$\sup_{x \in \mathbb{R}} \left| P \left(\frac{S(t) - E S(t)}{\sqrt{\text{var}(S(t))}} \leq x \right) - \Phi(x) \right| \rightarrow 0, \quad (3.1.3)$$

where Φ is the distribution function of the standard normal $N(0, 1)$ distribution.

Notice that the random sum process S satisfies essentially the same invariance principles, strong law of large numbers and central limit theorem, as the partial sum process

$$S_n = X_1 + \cdots + X_n, \quad n \geq 1.$$

Indeed, we know from a course in probability theory that (S_n) satisfies the strong law of large numbers

$$\lim_{n \rightarrow \infty} \frac{S_n}{n} = EX_1 \quad \text{a.s.}, \quad (3.1.4)$$

provided $EX_1 < \infty$, and the central limit theorem

$$P\left(\frac{S_n - ES_n}{\sqrt{\text{var}(S_n)}} \leq x\right) \rightarrow \Phi(x), \quad x \in \mathbb{R},$$

provided $\text{var}(X_1) < \infty$.

In both relations (3.1.2) and (3.1.3) we could use the asymptotic expressions for $ES(t)$ and $\text{var}(S(t))$ suggested in Proposition 3.1.3 for normalizing and centering purposes. Indeed, we have

$$\lim_{t \rightarrow \infty} \frac{S(t)}{ES(t)} = 1 \quad \text{a.s.}$$

and it can be shown by using some more sophisticated asymptotics for $ES(t)$ that as $t \rightarrow \infty$,

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{S(t) - \lambda EX_1 t}{\sqrt{\lambda t [\text{var}(X_1) + \text{var}(W_1) \lambda^2 (EX_1)^2]}} \leq x\right) - \Phi(x) \right| \rightarrow 0.$$

We also mention that the uniform version (3.1.3) of the central limit theorem is equivalent to the pointwise central limit theorem

$$P\left(\frac{S(t) - ES(t)}{\sqrt{\text{var}(S(t))}} \leq x\right) \rightarrow \Phi(x), \quad x \in \mathbb{R}.$$

This is a consequence of the well-known fact that convergence in distribution with continuous limit distribution function implies uniformity of this convergence; see Billingsley [18].

Proof. We only prove the first part of the theorem. For the second part, we refer to Embrechts et al. [46], Theorem 2.5.16. We have

$$\frac{S(t)}{t} = \frac{S(t)}{N(t)} \frac{N(t)}{t}. \quad (3.1.5)$$

Write

$$\Omega_1 = \{\omega : N(t)/t \rightarrow \lambda\} \quad \text{and} \quad \Omega_2 = \{\omega : S(t)/N(t) \rightarrow EX_1\}.$$

By virtue of (3.1.5) the result follows if we can show that $P(\Omega_1 \cap \Omega_2) = 1$. However, we know from the strong law of large numbers for N (Theorem 2.2.5)

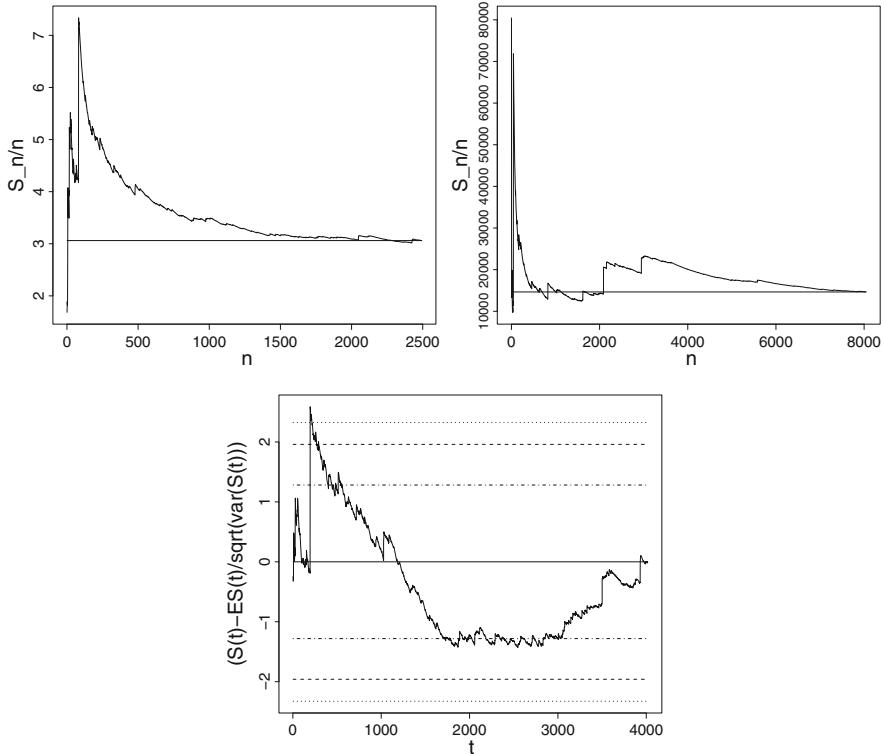


Figure 3.1.6 Top: Visualization of the strong law of large numbers for the Danish fire insurance data in million Danish Kroner (left) and the US industrial fire data (right). For a description of these data sets, see Example 3.2.11. The curves show the averaged sample sizes $S_n/n = (X_1 + \dots + X_n)/n$ as a function of n ; the solid straight line represents the overall sample mean. Both claim size samples contain very large values. This fact makes the ratio S_n/n converge to EX_1 very slowly. Bottom: The quantities $(S(t) - ES(t))/\sqrt{\text{var}(S(t))}$ for the Danish fire insurance data. The values of $ES(t)$ and $\text{var}(S(t))$ were evaluated from the asymptotic expressions suggested by Proposition 3.1.3. From bottom to top, the constant lines correspond to the 1%-
2.5%-
10%-
50%-
90%-
97.5%-
99%-quantiles of the standard normal distribution.

that $P(\Omega_1) = 1$. Moreover, since $N(t) \xrightarrow{\text{a.s.}} \infty$, an application of the strong law of large numbers (3.1.4) and Lemma 2.2.6 imply that $P(\Omega_2) = 1$. This concludes the proof. \square

The strong law of large numbers for the total claim amount process S is one of the important results which any insurance business has experienced since the foundation of insurance companies. As a matter of fact, the strong law of

large numbers can be observed in real-life data; see Figure 3.1.6. Its validity gives one confidence that large and small claims averaged over time converge to their theoretical mean value. The strong law of large numbers and the central limit theorem for S are backbone results when it comes to premium calculation. This is the content of the next section.

3.1.3 Classical Premium Calculation Principles

One of the basic questions of an insurance business is how one chooses a premium in order to cover the losses over time, described by the total claim amount process S . We think of the premium income $p(t)$ in the portfolio of those policies where the claims occur as a deterministic function.

A coarse, but useful approximation to the random quantity $S(t)$ is given by its expectation $ES(t)$. Based on the results of Sections 3.1.1 and 3.1.2 for the renewal model, we would expect that the insurance company loses on average if $p(t) < ES(t)$ for large t and gains if $p(t) > ES(t)$ for large t . Therefore it makes sense to choose a premium by “loading” the expected total claim amount by a certain positive number ρ .

For example, we know from Proposition 3.1.3 that in the renewal model

$$ES(t) = \lambda EX_1 t (1 + o(1)), \quad t \rightarrow \infty.$$

Therefore it is reasonable to choose $p(t)$ according to the equation

$$p(t) = (1 + \rho) ES(t) \quad \text{or} \quad p(t) = (1 + \rho) \lambda EX_1 t, \quad (3.1.6)$$

for some positive number ρ , called the *safety loading*. From the asymptotic results in Sections 3.1.1 and 3.1.2 it is evident that the insurance business is the more on the safe side the larger ρ . On the other hand, an overly large value ρ would make the insurance business less competitive: the number of contracts would decrease if the premium were too high compared to other premiums offered in the market. Since the success of the insurance business is based on the strong law of large numbers, one needs large numbers of policies in order to ensure the balance of premium income and total claim amount. Therefore, premium calculation principles more sophisticated than those suggested by (3.1.6) have also been considered in the literature. We briefly discuss some of them.

- The *net* or *equivalence principle*. This principle determines the premium $p(t)$ at time t as the expectation of the total claim amount $S(t)$:

$$p_{\text{Net}}(t) = ES(t).$$

In a sense, this is the “fair market premium” to be charged: the insurance portfolio does not lose or gain capital on average. However, the central limit theorem (Theorem 3.1.3) in the renewal model tells us that the deviation of $S(t)$ from its mean increases at an order comparable to its standard

deviation $\sqrt{\text{var}(S(t))}$ as $t \rightarrow \infty$. Moreover, these deviations can be both positive or negative with positive probability. Therefore it would be utterly unwise to charge a premium according to this calculation principle. It is of purely theoretical value, a “benchmark premium”. In Section 4.1 we will see that the net principle leads to “ruin” of the insurance business.

- The *expected value principle*.

$$p_{\text{EV}}(t) = (1 + \rho) ES(t),$$

for some positive *safety loading* ρ . The rationale of this principle is the strong law of large numbers of Theorem 3.1.5, as explained above.

- The *variance principle*.

$$p_{\text{Var}}(t) = ES(t) + \alpha \text{ var}(S(t)),$$

for some positive α . In the renewal model, this principle is equivalent in an asymptotic sense to the expected value principle with a positive loading. Indeed, using Proposition 3.1.3, it is not difficult to see that the ratio of the premiums charged by both principles converges to a positive constant as $t \rightarrow \infty$, and α plays the role of a positive safety loading.

- The *standard deviation principle*.

$$p_{\text{SD}}(t) = ES(t) + \alpha \sqrt{\text{var}(S(t))},$$

for some positive α . The rationale for this principle is the central limit theorem since in the renewal model (see Theorem 3.1.5),

$$P(S(t) - p_{\text{SD}}(t) \leq x) \rightarrow \Phi(\alpha), \quad x \in \mathbb{R},$$

where Φ is the standard normal distribution function. Convince yourself that this relation holds. In the renewal model, the standard deviation principle and the net principle are equivalent in the sense that the ratio of the two premiums converges to 1 as $t \rightarrow \infty$. This means that one charges a smaller premium by using this principle in comparison to the expected value and variance principles.

The interpretation of the premium calculation principles depends on the underlying model. In the renewal and Cramér-Lundberg models the interpretation follows by using the central limit theorem and the strong law of large numbers. If we assumed the mixed homogeneous Poisson process as the claim number process, the over-dispersion property, i.e., $\text{var}(N(t)) > EN(t)$, would lead to completely different statements. For example, for a mixed compound homogeneous Poisson process $p_{\text{Var}}(t)/p_{\text{EV}}(t) \rightarrow \infty$ as $t \rightarrow \infty$. Verify this!

Comments

Various other theoretical premium principles have been introduced in the literature; see for example Bühlmann [29], Kaas et al. [77] or Klugman et al.

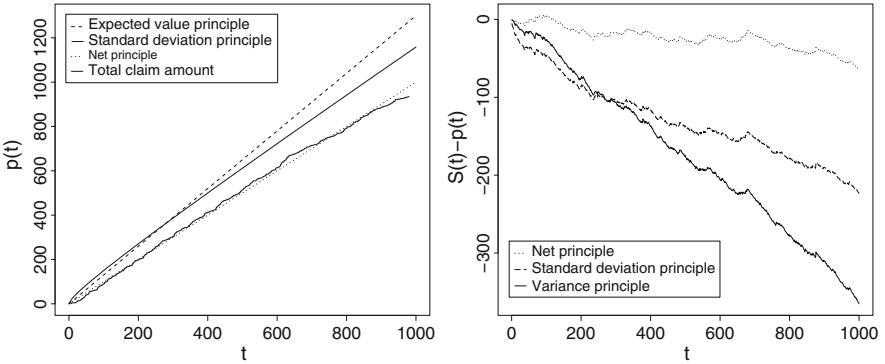


Figure 3.1.7 Visualization of the premium calculation principles in the Cramér-Lundberg model with Poisson intensity 1 and standard exponential claim sizes. Left: The premiums are: for the net principle $p_{Net}(t) = t$, for the standard deviation principle $p_{SD}(t) = t + 5\sqrt{2t}$ and for the expected value principle $p_{EV}(t) = 1.3t$ for $\rho = 0.3$. Equivalently, $p_{EV}(t)$ corresponds to the variance principle $p_{Var}(t) = 1.3t$ with $\alpha = 0.15$. One sample path of the total claim amount process S is also given. Notice that $S(t)$ can lie above or below $p_{Net}(t)$. Right: The differences $S(t) - p(t)$ are given. The upper curve corresponds to p_{Net} .

[86]. In Exercise 2 below one finds theoretical requirements taken from the actuarial literature that a “reasonable” premium calculation principle should satisfy. As a matter of fact, just one of these premium principles satisfies all requirements. It is the net premium principle which is not reasonable from an economic point of view since its application leads to ruin in the portfolio.

Exercises

- (1) Assume the renewal model for the total claim amount process S with $\text{var}(X_1) < \infty$ and $\text{var}(W_1) < \infty$.
 - (a) Show that the standard deviation principle is motivated by the central limit theorem, i.e., as $t \rightarrow \infty$,

$$P(S(t) - p_{SD}(t) \leq x) \rightarrow \Phi(\alpha), \quad x \in \mathbb{R},$$

where Φ is the standard normal distribution. This means that α is the $\Phi(\alpha)$ -quantile of the normal distribution.

- (b) Show that the net principle and the standard deviation principle are asymptotically equivalent in the sense that

$$\frac{p_{Net}(t)}{p_{SD}(t)} \rightarrow 1 \quad \text{as } t \rightarrow \infty.$$

- (c) Argue why the net premium principle and the standard deviation principle are “sufficient for a risk neutral insurer only”, i.e., these principles do not lead to a positive relative average profit in the long run: consider the relative gains $(p(t) - ES(t))/ES(t)$ for large t .
- (d) Show that for $h > 0$,

$$\lim_{t \rightarrow \infty} ES(t-h, t] = h \frac{EX_1}{EW_1}.$$

Hint: Appeal to Blackwell’s renewal theorem; see p. 61.

- (2) In the insurance literature one often finds theoretical requirements on the premium principles. Here are a few of them:
- *Non-negative loading* : $p(t) \geq ES(t)$.
 - *Consistency* : the premium for $S(t) + c$ is $p(t) + c$.
 - *Additivity* : for independent total claim amounts $S(t)$ and $S'(t)$ with corresponding premiums $p(t)$ and $p'(t)$, the premium for $S(t) + S'(t)$ should be $p(t) + p'(t)$.
 - *Homogeneity or proportionality* : for $c > 0$, the premium for $cS(t)$ should be $cp(t)$.

Which of the premium principles satisfies these conditions in the Cramér-Lundberg or renewal models?

- (3) Calculate the mean and the variance of the total claim amount $S(t)$ under the condition that N is mixed Poisson with $(N(t))_{t \geq 0} = (\tilde{N}(\theta t))_{t \geq 0}$, where \tilde{N} is a standard homogeneous Poisson process, $\theta > 0$ is a mixing variable with $\text{var}(\theta) < \infty$, and (X_i) is an iid claim size sequence with $\text{var}(X_1) < \infty$. Show that

$$p\text{Var}(t)/p\text{EV}(t) \rightarrow \infty, \quad t \rightarrow \infty.$$

Compare the latter limit relation with the case when N is a renewal process.

- (4) Assume the Cramér-Lundberg model with Poisson intensity $\lambda > 0$ and consider the corresponding risk process

$$U(t) = u + ct - S(t),$$

where $u > 0$ is the initial capital in the portfolio, $c > 0$ the premium rate and S the total claim amount process. The risk process and its meaning are discussed in detail in Chapter 4. In addition, assume that the moment generating function $m_{X_1}(h) = E \exp\{h X_1\}$ of the claim sizes X_i is finite in some neighborhood $(-h_0, h_0)$ of the origin.

- (a) Calculate the moment generating function of $S(t)$ and show that it exists in $(-h_0, h_0)$.
- (b) The premium rate c is determined according to the expected value principle: $c = (1 + \rho) \lambda EX_1$ for some positive safety loading ρ , where the value c (equivalently, the value ρ) can be chosen according to the *exponential premium principle*.¹ For its definition, write $v_\alpha(u) = e^{-\alpha u}$ for $u, \alpha > 0$. Then c is chosen as the solution to the equation

¹ This premium calculation principle is not intuitively motivated by the strong law of large numbers or the central limit theorem, but by so-called *utility theory*. The reader who wants to learn about the rationale of this principle is referred to Chapter 1 in Kaas et al. [77].

$$v_\alpha(u) = E[v_\alpha(U(t))] \quad \text{for all } t > 0. \quad (3.1.7)$$

Use (a) to show that a unique solution $c = c_\alpha > 0$ to (3.1.7) exists. Calculate the safety loading ρ_α corresponding to c_α and show that $\rho_\alpha \geq 0$.

- (c) Consider c_α as a function of $\alpha > 0$. Show that $\lim_{\alpha \downarrow 0} c_\alpha = \lambda EX_1$. This means that c_α converges to the value suggested by the net premium principle with safety loading $\rho = 0$.

3.2 Claim Size Distributions

In this section we are interested in the question:

What are realistic claim size distributions?

This question is about the goodness of fit of the claim size data to the chosen distribution. It is not our goal to give sophisticated statistical analyses, but we rather aim at introducing some classes of distributions used in insurance practice, which are sufficiently flexible and give a satisfactory fit to the data. In Section 3.2.1 we introduce QQ-plots and in Section 3.2.3 mean excess plots as two graphical methods for discriminating between different claim size distributions. Since realistic claim size distributions are very often heavy-tailed, we start in Section 3.2.2 with an informal discussion of the notions of heavy- and light-tailed distributions. In Section 3.2.4 we introduce some of the major claim size distributions and discuss their properties. In Sections 3.2.5 and 3.2.6 we continue to discuss natural heavy-tailed distributions for insurance: the classes of the distributions with regularly varying tails and the subexponential distributions. The latter class is by now considered as *the* class of distributions for modeling large claims.

3.2.1 An Exploratory Statistical Analysis: QQ-Plots

We consider some simple exploratory statistical tools and apply them to simulated and real-life claim size data in order to detect which distributions might give a reasonable fit to real-life insurance data. We start with a *quantile-quantile plot*, for short *QQ-plot*, and continue in Section 3.2.3 with a mean excess plot. Quantiles correspond to the “inverse” of a distribution function, which is not always well-defined (distribution functions are not necessarily strictly increasing). We focus on a left-continuous version.

Definition 3.2.1 (Quantile function)

The generalized inverse of the distribution function F , i.e.,

$$F^\leftarrow(t) = \inf\{x \in \mathbb{R} : F(x) \geq t\}, \quad 0 < t < 1,$$

is called the quantile function of the distribution function F . The quantity $x_t = F^\leftarrow(t)$ defines the t -quantile of F .

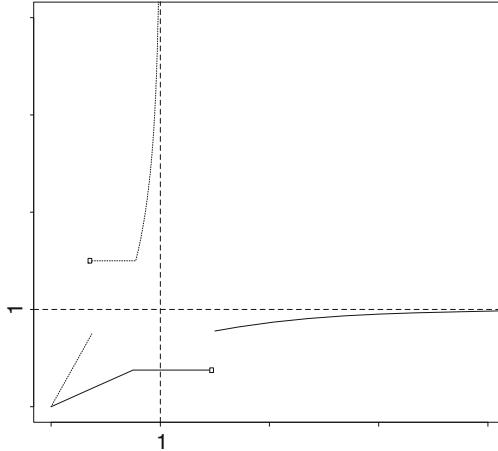


Figure 3.2.2 A distribution function F on $[0, \infty)$ and its quantile function F^\leftarrow . In a sense, F^\leftarrow is the mirror image of F with respect to the line $x = y$.

If F is monotone increasing (such as the distribution function Φ of the standard normal distribution), we see that $F^\leftarrow = F^{-1}$ on the image of F , i.e., the ordinary inverse of F . An illustration of the quantile function is given in Figure 3.2.2. Notice that intervals where F is constant turn into jumps of F^\leftarrow , and jumps of F turn into intervals of constancy for F^\leftarrow .

In this way we can define the generalized inverse of the *empirical distribution function* F_n of a sample X_1, \dots, X_n , i.e.,

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n I_{(-\infty, x]}(X_i), \quad x \in \mathbb{R}. \quad (3.2.8)$$

It is easy to verify that F_n has all properties of a distribution function:

- $\lim_{x \rightarrow -\infty} F_n(x) = 0$ and $\lim_{x \rightarrow \infty} F_n(x) = 1$.
- F_n is non-decreasing: $F_n(x) \leq F_n(y)$ for $x \leq y$.
- F_n is right-continuous: $\lim_{y \downarrow x} F_n(y) = F_n(x)$ for every $x \in \mathbb{R}$.

Let $X_{(1)} \leq \dots \leq X_{(n)}$ be the ordered sample of X_1, \dots, X_n . In what follows, we assume that the sample does not have ties, i.e., $X_{(1)} < \dots < X_{(n)}$ a.s. For example, if the X_i 's are iid with a density the sample does not have ties; see the proof of Lemma 2.1.9 for an argument.

Since the empirical distribution function of a sample is itself a distribution function, one can calculate its quantile function F_n^\leftarrow which we call the

empirical quantile function. If the sample has no ties then it is not difficult to see that

$$F_n(X_{(k)}) = k/n, \quad k = 1, \dots, n,$$

i.e., F_n jumps by $1/n$ at every value $X_{(k)}$ and is constant in $[X_{(k)}, X_{(k+1)})$ for $k < n$. This means that the empirical quantile function F_n^\leftarrow jumps at the values k/n by $X_{(k)} - X_{(k-1)}$ and remains constant in $((k-1)/n, k/n]$:

$$F_n^\leftarrow(t) = \begin{cases} X_{(k)} & t \in ((k-1)/n, k/n], \quad k = 1, \dots, n-1, \\ X_{(n)} & t \in ((n-1)/n, 1). \end{cases}$$

A fundamental result of probability theory, the *Glivenko-Cantelli lemma*, (see for example Billingsley [18], p. 275) tells us the following: if X_1, X_2, \dots is an iid sequence with distribution function F , then

$$\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \xrightarrow{\text{a.s.}} 0,$$

implying that $F_n(x) \approx F(x)$ uniformly for all x . One can show that the Glivenko-Cantelli lemma implies $F_n^\leftarrow(t) \rightarrow F^\leftarrow(t)$ a.s. as $n \rightarrow \infty$ for all continuity points t of F^\leftarrow ; see Resnick [122], p. 5. This observation is the basic idea for the *QQ-plot*: if X_1, \dots, X_n were a sample with known distribution function F , we would expect that $F_n^\leftarrow(t)$ is close to $F^\leftarrow(t)$ for all $t \in (0, 1)$, provided n is large. Thus, if we plot $F_n^\leftarrow(t)$ against $F^\leftarrow(t)$ for $t \in (0, 1)$ we should roughly see a straight line.

It is common to plot the graph

$$\left\{ \left(X_{(k)}, F^\leftarrow \left(\frac{k}{n+1} \right) \right), \quad k = 1, \dots, n \right\}$$

for a given distribution function F . Modifications of the plotting positions have been used as well. Chambers [32] gives the following properties of a QQ-plot:

- (a) Comparison of distributions. *If the data were generated from a random sample of the reference distribution, the plot should look roughly linear. This remains true if the data come from a linear transformation of the distribution.*
- (b) Outliers. *If one or a few of the data values are contaminated by gross error or for any reason are markedly different in value from the remaining values, the latter being more or less distributed like the reference distribution, the outlying points may be easily identified on the plot.*
- (c) Location and scale. *Because a change of one of the distributions by a linear transformation simply transforms the plot by the same transformation, one may estimate graphically (through the intercept and slope) location and scale parameters for a sample of data, on the assumption that the data come from the reference distribution.*

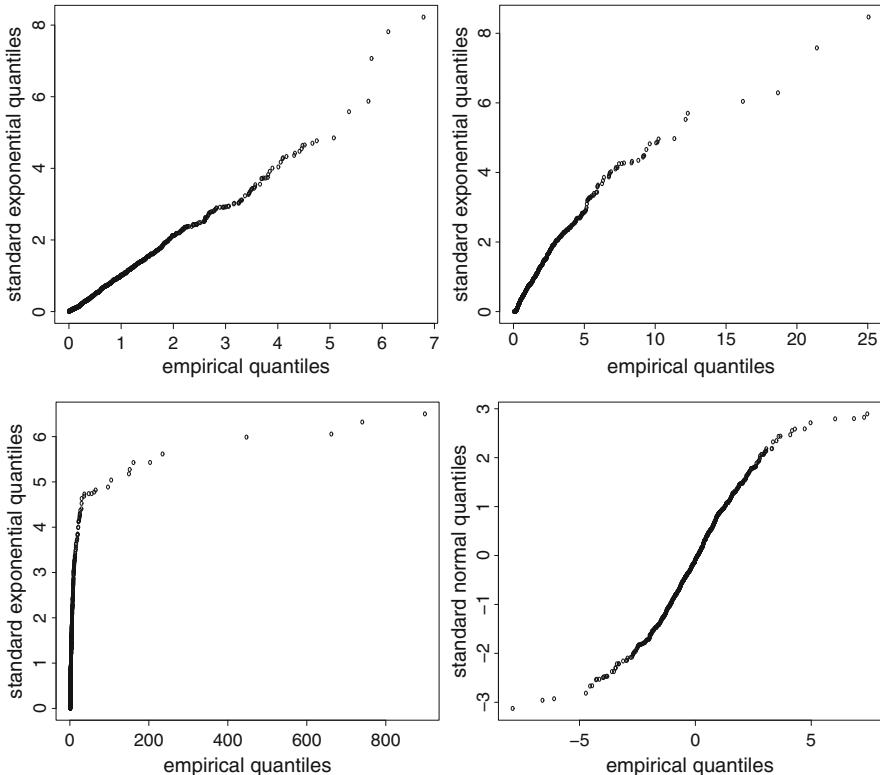


Figure 3.2.3 QQ-plots for samples of size 1 000. Standard exponential (top left), standard log-normal (top right) and Pareto distributed data with tail index 4 (bottom left) versus the standard exponential quantiles. Bottom right: t_4 -distributed data versus the quantiles of the standard normal distribution. The t_4 -distribution has tails $F(-x) = 1 - F(x) = cx^{-4}(1 + o(1))$ as $x \rightarrow \infty$, some $c > 0$, in contrast to the standard normal with tails $\Phi(-x) = 1 - \Phi(x) = (\sqrt{2\pi}x)^{-1} \exp\{-x^2/2\}(1 + o(1))$; see (3.2.9).

- (d) Shape. Some difference in distributional shape may be deduced from the plot. For example if the reference distribution has heavier tails (tends to have more large values) the plot will curve down at the left and/or up at the right.

For an illustration of (a) and (d), also for a two-sided distribution, see Figure 3.2.3. QQ-plots applied to real-life claim size data (Danish fire insurance, US industrial fire) are presented in Figures 3.2.5 and 3.2.15. QQ-plots

applied to the Danish fire insurance inter-arrival times are given in Figures 2.1.22 and 2.1.23.

3.2.2 A Preliminary Discussion of Heavy- and Light-Tailed Distributions

The Danish fire insurance data and the US industrial fire data presented in Figures 3.2.5 and 3.2.15, respectively, can be modeled by a very heavy-tailed distribution. Such claim size distributions typically occur in a reinsurance portfolio, where the largest claims are insured. In this context, the question arises:

What determines a heavy-tailed/light-tailed claim size distribution?

There is no clear-cut answer to this question. One common way to characterize the heaviness of the tails is by means of the exponential distribution as a benchmark. For example, if

$$\limsup_{x \rightarrow \infty} \frac{\bar{F}(x)}{e^{-\lambda x}} < \infty \quad \text{for some } \lambda > 0,$$

where

$$\bar{F}(x) = 1 - F(x), \quad x > 0,$$

denotes the right tail of the distribution function F , we could call F *light-tailed*, and if

$$\liminf_{x \rightarrow \infty} \frac{\bar{F}(x)}{e^{-\lambda x}} > 0 \quad \text{for all } \lambda > 0,$$

we could call F *heavy-tailed*.

Example 3.2.4 (Some well-known heavy- and light-tailed claim size distributions)

From the above definitions, the exponential $\text{Exp}(\lambda)$ distribution is light-tailed for every $\lambda > 0$.

A standard claim size distribution is the *truncated normal*. This means that the X_i 's have distribution function $F(x) = P(|Y| \leq x)$ for a normally distributed random variable Y . If we assume Y standard normal, $F(x) = 2(\Phi(x) - 0.5)$ for $x > 0$, where Φ is the standard normal distribution function with density

$$\varphi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \quad x \in \mathbb{R}.$$

An application of l'Hospital's rule shows that

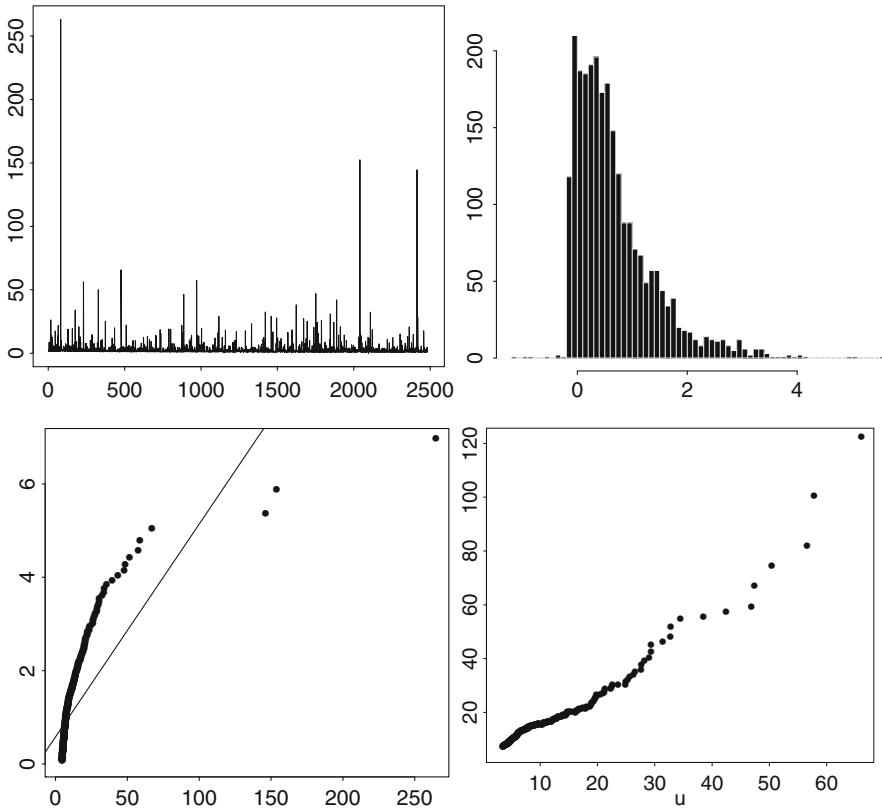


Figure 3.2.5 Top left: Danish fire insurance claim size data in millions of Danish Kroner (1985 prices). The data correspond to the period 1980 – 1992. There is a total of 2 493 observations. Top right: Histogram of the log-data. Bottom left: QQ-plot of the data against the standard exponential distribution. The graph is curved down at the right indicating that the right tail of the distribution of the data is significantly heavier than the exponential. Bottom right: Mean excess plot of the data. The graph increases in its whole domain. This is a strong indication of heavy tails of the underlying distribution. See Example 3.2.11 for some comments.

$$\lim_{x \rightarrow \infty} \frac{\bar{\Phi}(x)}{x^{-1}\varphi(x)} = 1. \quad (3.2.9)$$

The latter relation is often referred to as *Mill's ratio*. With Mill's ratio in mind, it is easy to verify that the truncated normal distribution is light-tailed. Using an analogous argument, it can be shown that the gamma distribution, for any choice of parameters, is light-tailed. Verify this.

A typical example of a heavy-tailed claim size distribution is the *Pareto distribution* with tail parameter $\alpha > 0$ and scale parameter $\kappa > 0$, given by

$$\bar{F}(x) = \frac{\kappa^\alpha}{(\kappa + x)^\alpha}, \quad x > 0.$$

Another prominent heavy-tailed distribution is the *Weibull distribution* with shape parameter $\tau < 1$ and scale parameter $c > 0$:

$$\bar{F}(x) = e^{-cx^\tau}, \quad x > 0.$$

However, for $\tau \geq 1$ the Weibull distribution is light-tailed. We refer to Tables 3.2.17 and 3.2.19 for more distributions used in insurance practice. \square

3.2.3 An Exploratory Statistical Analysis: Mean Excess Plots

The reader might be surprised about the rather arbitrary way in which we discriminated heavy-tailed distributions from light-tailed ones. There are, however, some very good theoretical reasons for the extraordinary role of the exponential distribution as a benchmark distribution, as will be explained in this section.

One tool in order to compare the thickness of the tails of distributions on $[0, \infty)$ is the *mean excess function*.

Definition 3.2.6 (Mean excess function)

Let Y be a non-negative random variable with finite mean, distribution F and $x_l = \inf\{x : F(x) > 0\}$ and $x_r = \sup\{x : F(x) < 1\}$. Then its mean excess or mean residual life function is given by

$$e_F(u) = E(Y - u \mid Y > u), \quad u \in (x_l, x_r).$$

For our purposes, we mostly consider distributions on $[0, \infty)$ which have support unbounded to the right. The quantity $e_F(u)$ is often referred to as the *mean excess over the threshold value u* . In an insurance context, $e_F(u)$ can be interpreted as the expected claim size in the unlimited layer, over priority u . Here $e_F(u)$ is also called the *mean excess loss function*. In a reliability or medical context, $e_F(u)$ is referred to as the *mean residual life function*. In a financial risk management context, switching from the right tail to the left tail, $e_F(u)$ is referred to as the *expected shortfall*.

The mean excess function of the distribution function F can be written in the form

$$e_F(u) = \frac{1}{\bar{F}(u)} \int_u^\infty \bar{F}(y) dy, \quad u \in [0, x_r). \quad (3.2.10)$$

This formula is often useful for calculations or for deriving theoretical properties of the mean excess function.

Another interesting relationship between e_F and the tail \bar{F} is given by

$$\bar{F}(x) = \frac{e_F(0)}{e_F(x)} \exp \left\{ - \int_0^x \frac{1}{e_F(y)} dy \right\}, \quad x > 0. \quad (3.2.11)$$

Here we assumed in addition that F is continuous and $F(x) > 0$ for all $x > 0$. Under these additional assumptions, F and e_F determine each other in a unique way. Therefore the tail \bar{F} of a non-negative distribution F and its mean excess function e_F are in a sense equivalent notions. The properties of \bar{F} can be translated into the language of the mean excess function e_F and vice versa.

Derive (3.2.10) and (3.2.11) yourself. Use the relation $EY = \int_0^\infty P(Y > y) dy$ which holds for any positive random variable Y .

Example 3.2.7 (Mean excess function of the exponential distribution)
Consider Y with exponential $\text{Exp}(\lambda)$ distribution for some $\lambda > 0$. It is an easy exercise to verify that

$$e_F(u) = \lambda^{-1}, \quad u > 0. \quad (3.2.12)$$

This property is another manifestation of the *forgetfulness property* of the exponential distribution; see p. 20. Indeed, the tail of the excess distribution function of Y satisfies

$$P(Y > u + x \mid Y > u) = P(Y > x), \quad x > 0.$$

This means that this distribution function corresponds to an $\text{Exp}(\lambda)$ random variable; it does not depend on the threshold u \square

Property (3.2.12) makes the exponential distribution unique: it offers another way of discriminating between heavy- and light-tailed distributions of random variables which are unbounded to the right. Indeed, if $e_F(u)$ converged to infinity for $u \rightarrow \infty$, we could call F *heavy-tailed*, if $e_F(u)$ converged to a finite constant as $u \rightarrow \infty$, we could call F *light-tailed*. In an insurance context this is quite a sensible definition since unlimited growth of $e_F(u)$ expresses the danger of the underlying distribution F in its right tail, where the large claims come from: given the claim size X_i exceeded the high threshold u , it is very likely that future claim sizes pierce an even higher threshold. On the other hand, for a light-tailed distribution F , the expectation of the excess $(X_i - u)_+$ (here $x_+ = \max(0, x)$) converges to zero (as for the truncated normal distribution) or to a positive constant (as in the exponential case), given $X_i > u$ and the threshold u increases to infinity. This means that claim sizes with light-tailed distributions are much less dangerous (costly) than heavy-tailed distributions.

In Table 3.2.9 we give the mean excess functions of some standard claim size distributions. In Figure 3.2.8 we illustrate the qualitative behavior of $e_F(u)$ for large u .

If one deals with claim size data with an unknown distribution function F , one does not know the mean excess function e_F . As it is often done in

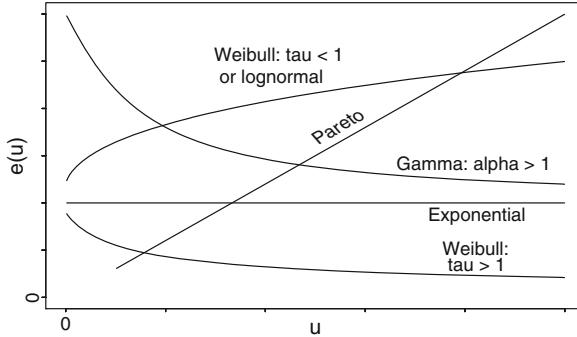


Figure 3.2.8 Graphs of the mean excess functions $e_F(u)$ for some standard distributions; see Table 3.2.9 for the corresponding parameterizations. Note that heavy-tailed distributions typically have $e_F(u)$ tending to infinity as $u \rightarrow \infty$.

Pareto	$\frac{\kappa + u}{\alpha - 1}, \quad \alpha > 1$
Burr	$\frac{u}{\alpha\tau - 1} (1 + o(1)), \quad \alpha\tau > 1$
Log-gamma	$\frac{u}{\alpha - 1} (1 + o(1)), \quad \alpha > 1$
Log-normal	$\frac{\sigma^2 u}{\log u - \mu} (1 + o(1))$
Benktander type I	$\frac{u}{\alpha + 2\beta \log u}$
Benktander type II	$\frac{u^{1-\beta}}{\alpha}$
Weibull	$\frac{u^{1-\tau}}{c\tau} (1 + o(1))$
Exponential	λ^{-1}
Gamma	$\beta^{-1} \left(1 + \frac{\alpha - 1}{\beta u} + o\left(\frac{1}{u}\right) \right)$
Truncated normal	$u^{-1} (1 + o(1))$

Table 3.2.9 Mean excess functions for some standard distributions. The parameterization is taken from Tables 3.2.17 and 3.2.19. The asymptotic relations are to be understood for $u \rightarrow \infty$.

statistics, we simply replace F in e_F by its sample version, the empirical distribution function F_n ; see (3.2.8). The resulting quantity e_{F_n} is called the *empirical mean excess function*. Since F_n has bounded support, we consider e_{F_n} only for $u \in [X_{(1)}, X_{(n)}]$:

$$\begin{aligned} e_{F_n}(u) &= E_{F_n}(Y - u \mid Y > u) = \frac{E_{F_n}(Y - u)_+}{\bar{F}_n(u)} \\ &= \frac{n^{-1} \sum_{i=1}^n (X_i - u)_+}{\bar{F}_n(u)}. \end{aligned} \quad (3.2.13)$$

An alternative expression for e_{F_n} is given by

$$e_{F_n}(u) = \frac{\sum_{i:i \leq n, X_i > u} (X_i - u)}{\#\{i \leq n : X_i > u\}}.$$

An application of the strong law of large numbers to (3.2.13) yields the following result.

Proposition 3.2.10 *Let X_i be iid non-negative random variables with distribution function F which are unbounded to the right. If $EX_1 < \infty$, then for every $u > 0$, $e_{F_n}(u) \xrightarrow{\text{a.s.}} e_F(u)$ as $n \rightarrow \infty$.*

A graphical test for tail behavior can now be based on e_{F_n} . A *mean excess plot (ME-plot)* consists of the graph

$$\{(X_{(k)}, e_{F_n}(X_{(k)})) : k = 1, \dots, n-1\}.$$

For our purposes, the ME-plot is used *only* as a graphical method, mainly for distinguishing between light- and heavy-tailed models; see Figure 3.2.12 for some simulated examples. Indeed caution is called for when interpreting such plots. Due to the sparseness of the data available for calculating $e_{F_n}(u)$ for large u -values, the resulting plots are very sensitive to changes in the data towards the end of the range; see Figure 3.2.13 for an illustration. For this reason, more robust versions like *median excess plots* and related procedures have been suggested; see for instance Beirlant et al. [14] or Rootzén and Tajvidi [128]. For a critical assessment concerning the use of mean excess functions in insurance, see Rytgaard [129].

Example 3.2.11 (Exploratory data analysis for some real-life data)

In Figures 3.2.5 and 3.2.15 we have graphically summarized some properties of two real-life data sets. The data underlying Figure 3.2.5 correspond to Danish fire insurance claims in millions of Danish Kroner (1985 prices). The data were communicated to us by Mette Havning and correspond to the period 1980-1992, inclusively. There is a total of $n = 2493$ observations.

The second insurance data, presented in Figure 3.2.15, correspond to a portfolio of US industrial fire data ($n = 8043$) reported over a two year period. This data set is definitely considered by the portfolio manager as

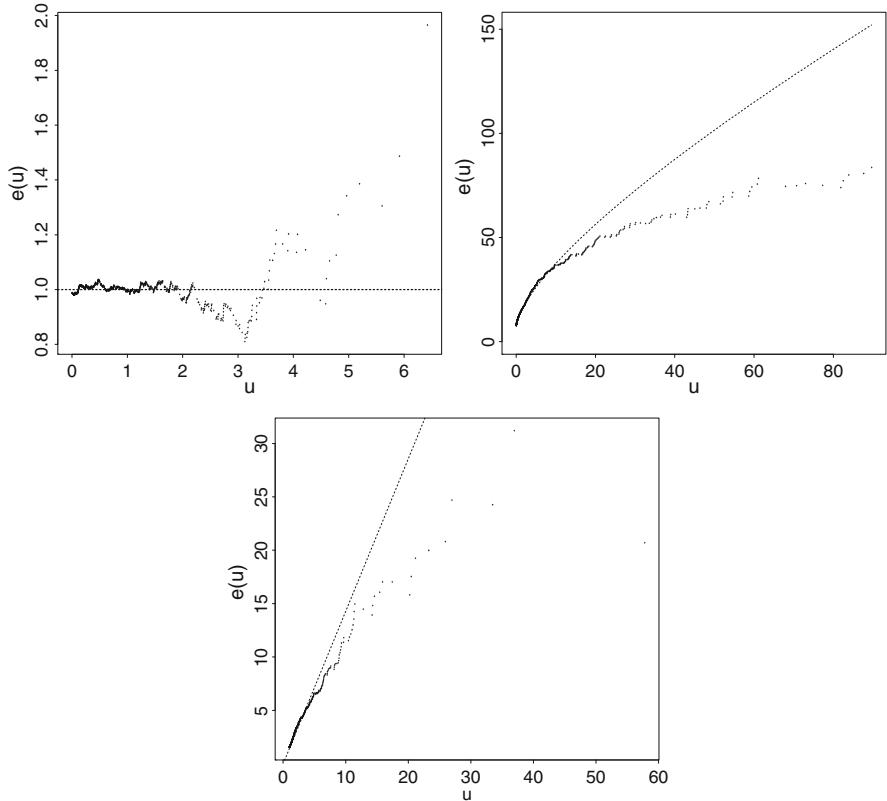


Figure 3.2.12 The mean excess function plot for 1 000 simulated data and the corresponding theoretical mean excess function e_F (solid line): standard exponential (top left), log-normal (top right) with $\log X \sim N(0, 4)$, Pareto (bottom) with tail index 1.7.

“dangerous”, i.e., large claim considerations do enter substantially in the final premium calculation.

A first glance at the figures and Table 3.2.14 for both data sets immediately reveals heavy-tailedness and skewedness to the right. The corresponding mean excess functions are close to a straight line which fact indicates that the underlying distributions may be modeled by Pareto-like distribution functions. The QQ-plots against the standard exponential quantiles also clearly show tails much heavier than exponential ones.

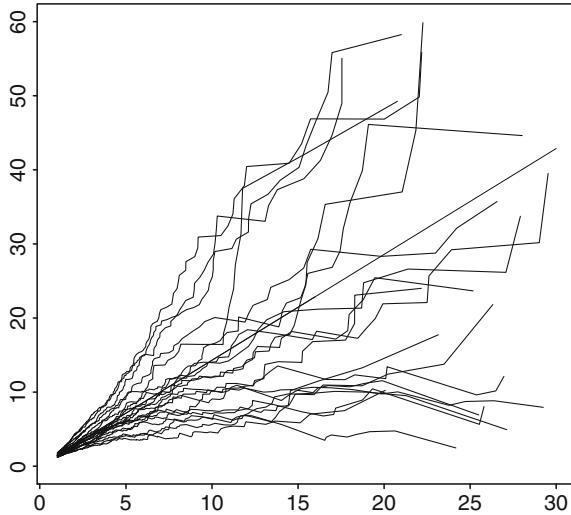


Figure 3.2.13 The mean excess function of the Pareto distribution $\bar{F}(x) = x^{-1.7}$, $x \geq 1$, (straight line), together with 20 simulated mean excess plots each based on simulated data ($n = 1000$) from the above distribution. Note the very unstable behavior, especially towards the higher values of u . This is typical and makes the precise interpretation of $e_{F_n}(u)$ difficult; see also Figure 3.2.12.

Data	Danish Industrial	
n	2 493	8 043
min	0.313	0.003
1st quartile	1.157	0.587
median	1.634	1.526
mean	3.063	14.65
3rd quartile	2.645	4.488
max	263.3	13 520
$\hat{x}_{0.99}$	24.61	184.0

Table 3.2.14 Basic statistics for the Danish and the industrial fire data; $\hat{x}_{0.99}$ stands for the empirical 99%-quantile.

□

Comments

The importance of the mean excess function (or plot) as a diagnostic tool for insurance data is nicely demonstrated in Hogg and Klugman [71]; see also Beirlant et al. [14] and the references therein.

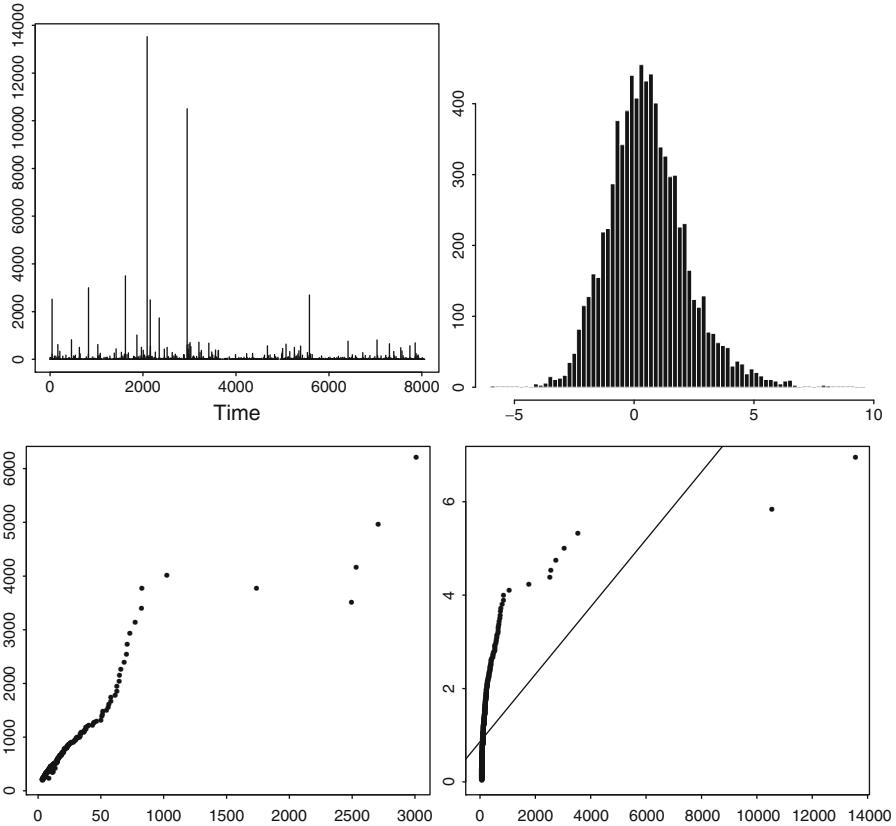


Figure 3.2.15 Exploratory data analysis of insurance claims caused by industrial fire: the data (top left), the histogram of the log-transformed data (top right), the ME-plot (bottom left) and a QQ-plot against standard exponential quantiles (bottom right). See Example 3.2.11 for some comments.

3.2.4 Standard Claim Size Distributions and Their Properties

Classical non-life insurance mathematics was most often concerned with claim size distributions with light tails in the sense which has been made precise in Section 3.2.3. We refer to Table 3.2.17 for a collection of such distributions. These distributions have mean excess functions $e_F(u)$ converging to some finite limit as $u \rightarrow \infty$, provided the support is infinite. For obvious reasons, we call them *small claim distributions*. One of the main reasons for the popularity of these distributions is that they are standard distributions in statistics. Classical statistics deals with the normal and the gamma distributions, among others, and in any introductory course on statistics we learn about these distributions because they have certain optimality conditions (closure

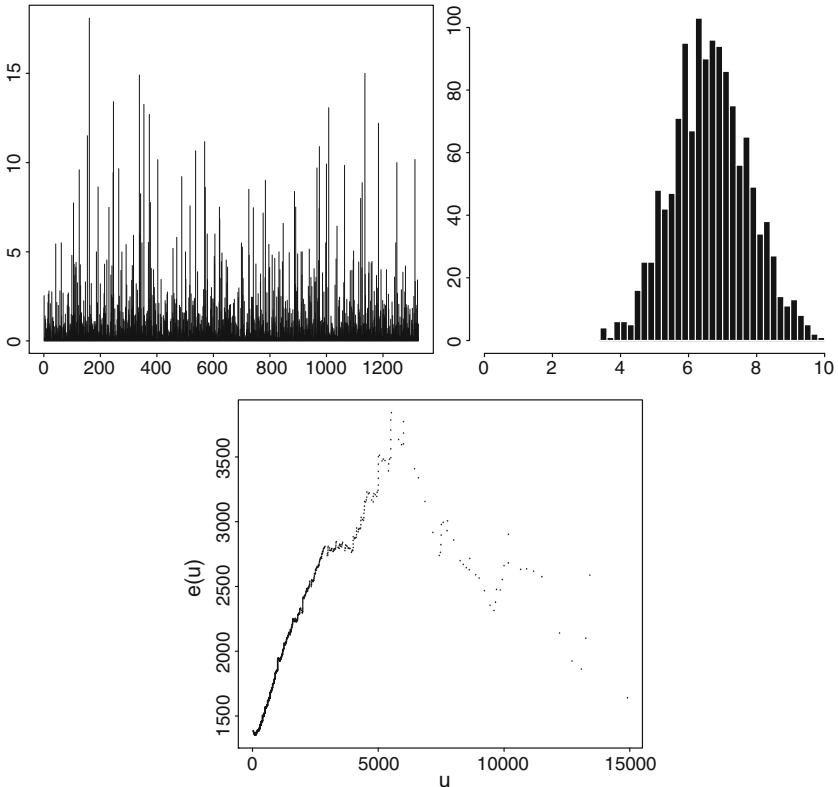


Figure 3.2.16 Exploratory data analysis of insurance claims caused by water: the data (top, left), the histogram of the log-transformed data (top, right), the ME-plot (bottom). Notice the kink in the ME-plot in the range (5 000, 6 000) reflecting the fact that the data seem to cluster towards some specific upper value.

of the normal and gamma distributions under convolutions, membership in exponential families, etc.) and therefore we can apply standard estimation techniques such as maximum likelihood.

In Figure 3.2.16 one can find a claim size sample which one could model by one of the distributions from Table 3.2.17. Indeed, notice that the mean excess plot of these data curves down at the right end, indicating that the right tail of the underlying distribution is not too dangerous. It is also common practice to fit distributions with bounded support to insurance claim data, for example by truncating any of the heavy-tailed distributions in Table 3.2.19 at a certain upper limit. This makes sense if the insurer has to cover claim sizes only up to this upper limit or for a certain layer. In this situation it is, however,

Name	Tail \bar{F} or density f	Parameters
Exponential	$\bar{F}(x) = e^{-\lambda x}$	$\lambda > 0$
Gamma	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$	$\alpha, \beta > 0$
Weibull	$\bar{F}(x) = e^{-cx^\tau}$	$c > 0, \tau \geq 1$
Truncated normal	$f(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2}$	—
Any distribution with bounded support		

Table 3.2.17 *Claim size distributions : “small claims”.*

reasonable to use the full data set (not just the truncated data) for estimating the parameters of the distribution.

Over the last few years the (re-)insurance industry has faced new challenges due to climate change, pollution, riots, earthquakes, terrorism, etc. We refer to Table 3.2.18 for a collection of the largest insured losses 1970-2007, taken from *Sigma* [136]. For this kind of data one would not use the distributions of Table 3.2.17, but rather those presented in Table 3.2.19. All distributions of this table are heavy-tailed in the sense that their mean excess functions $e_F(u)$ increase to infinity as $u \rightarrow \infty$; cf. Table 3.2.9. As a matter of fact, the distributions of Table 3.2.19 are not easily fitted since various of their characteristics (such as the tail index α of the Pareto distribution) can be estimated only by using the largest order statistics in the sample. In this case, *extreme value statistics* is called for. This means that, based on theoretical (semi-)parametric models from extreme value theory such as the extreme value distributions and the generalized Pareto distribution, one needs to fit those distributions from a relatively small number of upper order statistics or from the excesses of the underlying data over high thresholds. We refer to Embrechts et al. [46] for an introduction to the world of extremes.

We continue with some more specific comments on the distributions in Table 3.2.19. Perhaps with the exception of the log-normal distribution, these distributions are not most familiar from a standard course on statistics or probability theory.

The Pareto, Burr, log-gamma and truncated α -stable distributions have in common that their right tail is of the asymptotic form

$$\lim_{x \rightarrow \infty} \frac{\bar{F}(x)}{x^{-\alpha} (\log x)^\gamma} = c,$$

Losses	Date	Event	Country
68 515	08/25/05	Hurricane “Katrina”	US, Gulf of Mexico
23 654	08/23/92	Hurricane “Andrew”	US, Bahamas
21 999	09/11/01	Terrorist attack on WTC, Pentagon and other buildings	US
19 593	01/17/94	Northridge earthquake in California	US
14 115	09/02/04	Hurricane “Ivan”	US, Caribbean
13 339	10/19/05	Hurricane “Wilma”	US, Mexico
10 704	09/20/05	Hurricane “Rita”	US, Gulf of Mexico
8 840	08/11/04	Hurricane “Charley”	US, Cuba
8 599	09/27/91	Typhoon “Mireille”	Japan
7 650	09/15/89	Hurricane “Hugo”	US, P. Rico
7 413	01/25/90	Winter storm “Daria”	Europe
7 223	12/25/99	Winter storm “Lothar”	Europe
6 097	01/18/07	Winter storm “Kyrill”	Europe
5 659	10/15/87	Storm and floods	Europe
5 650	02/25/04	Hurricane “Frances”	US, Bahamas
5 066	02/25/90	Winter storm “Vivian”	Europe
5 031	09/22/99	Typhoon “Bart” hits the south of the country	Japan
4 492	09/20/98	Hurricane “Georges”	US, Caribbean
4 220	06/05/01	Tropical storm “Allison”; flooding	US
4 174	09/13/04	Hurricane “Jeanne”	US, Caribbean
3 937	09/06/04	Typhoon “Songda”	Japan, S. Korea
3 614	05/02/03	Thunderstorms, tornadoes, hail	US
3 515	09/10/99	Hurricane “Floyd”, heavy down-pours, flooding	US, Bahamas
3 508	07/06/88	Explosion on “Piper Alpha” offshore oil rig	UK
3 411	10/01/95	Hurricane “Opal”	US, Mexico
3 365	01/17/95	Great “Hanshin” earthquake in Kobe	Japan
2 989	12/27/99	Winter storm “Martin”	France, Spain, CH
2 818	03/10/93	Blizzard, tornados	US, Mexico, Canada
2 662	08/06/02	Severe floods	UK, Spain, Germany
2 589	10/20/91	Forest fire which spread to urban area	US
2 577	04/06/01	Hail, floods and tornadoes	US
2 488	06/25/07	Heavy rainfall, floods	UK
2 443	09/18/03	Hurricane “Isabel”	US, Canada
2 404	09/05/96	Hurricane “Fran”	US
2 372	12/03/99	Winter storm “Anatol”	N. Europe
2 365	09/11/92	Hurricane “Iniki”	US, North Pacific
2 282	08/29/79	Hurricane “Frederic”	US
2 255	08/19/05	Heavy rainfall, floods	Central Europe
2 217	10/23/89	Explosion at Philips Petroleum	US
2 196	12/26/04	Earthquake, tsunami	Indonesia, Thailand

Table 3.2.18 The 40 most costly insurance losses 1970–2007. Losses are in million \$US indexed to 2007 prices. The table is taken from Sigma [136].

Name	Tail \bar{F} or density f	Parameters
Log-normal	$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-(\log x - \mu)^2/(2\sigma^2)}$	$\mu \in \mathbb{R}, \sigma > 0$
Pareto	$\bar{F}(x) = \left(\frac{\kappa}{\kappa + x}\right)^\alpha$	$\alpha, \kappa > 0$
Burr	$\bar{F}(x) = \left(\frac{\kappa}{\kappa + x^\tau}\right)^\alpha$	$\alpha, \kappa, \tau > 0$
Benktander type I	$\bar{F}(x) = (1 + 2(\beta/\alpha) \log x) e^{-\beta(\log x)^2 - (\alpha+1) \log x}$	$\alpha, \beta > 0$
Benktander type II	$\bar{F}(x) = e^{\alpha/\beta} x^{-(1-\beta)} e^{-\alpha x^\beta/\beta}$	$\alpha > 0$ $0 < \beta < 1$
Weibull	$\bar{F}(x) = e^{-cx^\tau}$	$c > 0$ $0 < \tau < 1$
Log-gamma	$f(x) = \frac{\alpha^\beta}{\Gamma(\beta)} (\log x)^{\beta-1} x^{-\alpha-1}$	$\alpha, \beta > 0$
Truncated α -stable	$\bar{F}(x) = P(X > x)$ where X is an α -stable random variable	$1 < \alpha < 2$

Table 3.2.19 *Claim size distributions : “large claims”. All distributions have support $(0, \infty)$ except for the Benktander cases and the log-gamma with $(1, \infty)$. For the definition of an α -stable distribution, see p. 357.*

for some constants $\alpha, c > 0$ and $\gamma \in \mathbb{R}$. Tails of this kind are called *regularly varying*. We will come back to this notion in Section 3.2.5.

The log-gamma, Pareto and log-normal distributions are obtained by an exponential transformation of a random variable with gamma, exponential and normal distribution, respectively. For example, let Y be $N(\mu, \sigma^2)$ distributed. Then $\exp\{Y\}$ has the log-normal distribution with density given in Table 3.2.19. The goal of these exponential transformations of random variables with a standard *light-tailed* distribution is to create *heavy-tailed* distributions in a simple way. An advantage of this procedure is that by a logarithmic transformation of the data one returns to the standard light-tailed distributions. In particular, one can use standard theory for the estimation of the underlying parameters.

Some of the distributions in Table 3.2.19 were introduced as extensions of the Pareto, log-normal and Weibull ($\tau < 1$) distributions as classical heavy-tailed distributions. For example, the Burr distribution differs from the Pareto distribution only by the additional shape parameter τ . As a matter of fact,

practice in extreme value statistics (see for example Chapter 6 in Embrechts et al. [46], or convince yourself by a simulation study) shows that it is hard, if not impossible, to distinguish between the log-gamma, Pareto, Burr distributions based on parameter (for example maximum likelihood) estimation. It is indeed difficult to estimate the tail parameter α , the shape parameter τ or the scale parameter κ accurately in any of the cases. Similar remarks apply to the Benktander type I and the log-normal distributions, as well as the Benktander type II and the Weibull ($\tau < 1$) distributions. The Benktander distributions were introduced in the insurance world for one particular reason: one can explicitly calculate their mean excess functions; cf. Table 3.2.9.

3.2.5 Regularly Varying Claim Sizes and Their Aggregation

Although the distribution functions F in Table 3.2.19 look different, some of them are quite similar with regard to their asymptotic tail behavior. Those include the Pareto, Burr, stable and log-gamma distributions. In particular, their right tails can be written in the form

$$\bar{F}(x) = 1 - F(x) = \frac{L(x)}{x^\alpha}, \quad x > 0,$$

for some constant $\alpha > 0$ and a positive measurable function $L(x)$ on $(0, \infty)$ satisfying

$$\lim_{x \rightarrow \infty} \frac{L(cx)}{L(x)} = 1 \quad \text{for all } c > 0. \quad (3.2.14)$$

A function with this property is called *slowly varying* (at infinity). Examples of such functions are:

constants, logarithms, powers of logarithms, iterated logarithms.

Every slowly varying function has the representation

$$L(x) = c_0(x) \exp \left\{ \int_{x_0}^x \frac{\varepsilon(t)}{t} dt \right\}, \quad \text{for } x \geq x_0, \text{ some } x_0 > 0, \quad (3.2.15)$$

where $\varepsilon(t) \rightarrow 0$ as $t \rightarrow \infty$ and $c_0(t)$ is a positive function satisfying $c_0(t) \rightarrow c_0$ for some positive constant c_0 . Using representation (3.2.15), one can show that for every $\delta > 0$,

$$\lim_{x \rightarrow \infty} \frac{L(x)}{x^\delta} = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} x^\delta L(x) = \infty, \quad (3.2.16)$$

i.e., L is “small” compared to any power function, x^δ .

Definition 3.2.20 (Regularly varying function and regularly varying random variable)

Let L be a slowly varying function in the sense of (3.2.14).

(1) For any $\delta \in \mathbb{R}$, the function

$$f(x) = x^\delta L(x), \quad x > 0,$$

is said to be regularly varying with index δ .

(2) A positive random variable X and its distribution are said to be regularly varying² with (tail) index $\alpha \geq 0$ if the right tail of the distribution has the representation

$$P(X > x) = L(x) x^{-\alpha}, \quad x > 0.$$

An alternative way of defining regular variation with index δ is to require

$$\lim_{x \rightarrow \infty} \frac{f(cx)}{f(x)} = c^\delta \quad \text{for all } c > 0. \quad (3.2.17)$$

Regular variation is one possible way of describing “small” deviations from exact power law behavior. It is hard to believe that social or natural phenomena can be described by exact power law behavior. It is, however, known that various phenomena, such as Zipf’s law, fractal dimensions, the probability of exceedances of high thresholds by certain iid data, the world income distribution, etc., can be well described by functions which are “almost power” functions; see Schroeder [135] for an entertaining study of power functions and their application to different scaling phenomena. Regular variation is an appropriate concept in this context. It has been carefully studied for many years and arises in different areas, such as summation theory of independent or weakly dependent random variables, or in extreme value theory as a natural condition on the tails of the underlying distributions. We refer to Bingham et al. [19] for an encyclopedic treatment of regular variation.

Regularly varying distributions with positive index, such as the Pareto, Burr, α -stable, log-gamma distributions, are claim size distributions with some of the heaviest tails which have ever been fitted to claim size data. Although it is theoretically possible to construct distributions with tails which are heavier than any power law, statistical evidence shows that there is no need for such distributions. As a matter of fact, if X is regularly varying with index $\alpha > 0$, then

$$EX^\delta \begin{cases} = \infty & \text{for } \delta > \alpha, \\ < \infty & \text{for } \delta < \alpha, \end{cases}$$

² This definition differs from the standard usage of the literature which refers to X as a random variable with regularly varying tail and to its distribution as distribution with regularly varying tail.

i.e., moments below order α are finite, and moments above α are infinite.³ (Verify these moment relations by using representation (3.2.15).) The value α can be rather low for claim sizes occurring in the context of reinsurance. It is not atypical that α is below 2, sometimes even below 1, i.e., the variance or even the expectation of the distribution fitted to the data can be infinite. We refer to Example 3.2.11 for two data sets, where statistical estimation procedures provide evidence for values α close to or even below 2; see Chapter 6 in Embrechts et al. [46] for details.

As we have learned in the previous sections, one of the important quantities in insurance mathematics is the total claim amount $S(t) = \sum_{i=1}^{N(t)} X_i$. It is a random partial sum process with iid positive claim sizes X_i as summands, independent of the claim number process N . A complicated but important practical question is to get exact formulae or good approximations (by numerical or Monte Carlo methods) to the distribution of $S(t)$. Later in this course we will touch upon this problem; see Section 3.3.

In this section we focus on a simpler problem: the tail asymptotics of the distribution of the first n aggregated claim sizes

$$S_n = X_1 + \cdots + X_n, \quad n \geq 1.$$

We want to study how heavy tails of the claim size distribution function F influence the tails of the distribution function of S_n . From a reasonable notion of heavy-tailed distributions we would expect that the heavy tails do not disappear by aggregating independent claim sizes. This is exactly the content of the following result.

Lemma 3.2.21 *Assume that X_1 and X_2 are independent regularly varying random variables with the same index $\alpha > 0$, i.e.,*

$$\bar{F}_i(x) = P(X_i > x) = \frac{L_i(x)}{x^\alpha}, \quad x > 0.$$

for possibly different slowly varying functions L_i . Then $X_1 + X_2$ is regularly varying with the same index. More precisely, as $x \rightarrow \infty$,

$$\begin{aligned} P(X_1 + X_2 > x) &= [P(X_1 > x) + P(X_2 > x)] (1 + o(1)) \\ &= x^{-\alpha} [L_1(x) + L_2(x)] (1 + o(1)). \end{aligned}$$

Proof. Write $G(x) = P(X_1 + X_2 \leq x)$ for the distribution function of $X_1 + X_2$. Using $\{X_1 + X_2 > x\} \supset \{X_1 > x\} \cup \{X_2 > x\}$, one easily checks that

$$\bar{G}(x) \geq (\bar{F}_1(x) + \bar{F}_2(x)) (1 - o(1)).$$

³ These moment relations do not characterize a regularly varying distribution. A counterexample is the Peter-and-Paul distribution with distribution function $F(x) = \sum_{k \geq 1: 2^k \leq x} 2^{-k}$, $x \geq 0$. This distribution has finite moments of order $\delta < 1$ and infinite moments of order $\delta \geq 1$, but it is not regularly varying with index 1. See Exercise 7 on p. 108.

If $0 < \delta < 1/2$, then from

$$\{X_1 + X_2 > x\} \subset \{X_1 > (1 - \delta)x\} \cup \{X_2 > (1 - \delta)x\} \cup \{X_1 > \delta x, X_2 > \delta x\},$$

it follows that

$$\begin{aligned}\overline{G}(x) &\leq \overline{F}_1((1 - \delta)x) + \overline{F}_2((1 - \delta)x) + \overline{F}_1(\delta x)\overline{F}_2(\delta x) \\ &= \left(\overline{F}_1((1 - \delta)x) + \overline{F}_2((1 - \delta)x) \right) (1 + o(1)).\end{aligned}$$

Hence

$$1 \leq \liminf_{x \rightarrow \infty} \frac{\overline{G}(x)}{\overline{F}_1(x) + \overline{F}_2(x)} \leq \limsup_{x \rightarrow \infty} \frac{\overline{G}(x)}{\overline{F}_1(x) + \overline{F}_2(x)} \leq (1 - \delta)^{-\alpha},$$

and the result is established upon letting $\delta \downarrow 0$. \square

An important corollary, obtained via induction on n , is the following:

Corollary 3.2.22 *Assume that X_1, \dots, X_n are n iid regularly varying random variables with index $\alpha > 0$ and distribution function F . Then S_n is regularly varying with index α , and*

$$P(S_n > x) = n \overline{F}(x) (1 + o(1)), \quad x \rightarrow \infty.$$

Suppose now that X_1, \dots, X_n are iid with distribution function F , as in the above corollary. Denote the partial sum of X_1, \dots, X_n by $S_n = X_1 + \dots + X_n$ and their partial maximum by $M_n = \max(X_1, \dots, X_n)$. Then for $n \geq 2$ as $x \rightarrow \infty$,

$$P(M_n > x) = \overline{F^n}(x) = \overline{F}(x) \sum_{k=0}^{n-1} F^k(x) = n \overline{F}(x) (1 + o(1)).$$

Therefore, with the above notation, Corollary 3.2.22 can be reformulated as: if X_i is regularly varying with index $\alpha > 0$ then

$$\lim_{x \rightarrow \infty} \frac{P(S_n > x)}{P(M_n > x)} = 1, \quad \text{for } n \geq 2.$$

This implies that for distributions with regularly varying tails, the tail of the distribution of the sum S_n is essentially determined by the tail of the distribution of the maximum M_n . This is in fact one of the intuitive notions of heavy-tailed or large claim distributions. Hence, stated in a somewhat vague way: *under the assumption of regular variation, the tail of the distribution of the maximum claim size determines the tail of the distribution of the aggregated claim sizes*.

Comments

Surveys on regularly varying functions and distributions can be found in many standard textbooks on probability theory and extreme value theory; see for example Feller [51], Embrechts et al. [46] or Resnick [122]. The classical reference to regular variation is the book by Bingham et al. [19].

3.2.6 Subexponential Distributions

We learned in the previous section that for iid regularly varying random variables X_1, X_2, \dots with positive index α , the tail of the sum $S_n = X_1 + \dots + X_n$ is essentially determined by the tail of the maximum $M_n = \max_{i=1,\dots,n} X_i$. To be precise, we found that $P(S_n > x) = P(M_n > x)(1 + o(1))$ as $x \rightarrow \infty$ for every $n = 1, 2, \dots$. The latter relation can be taken as a natural definition for “heavy-tailedness” of a distribution:

Definition 3.2.23 (Subexponential distribution)

The positive random variable X with unbounded support and its distribution are said to be subexponential if for a sequence (X_i) of iid random variables with the same distribution as X the following relation holds:

$$\text{For all } n \geq 2: \quad P(S_n > x) = P(M_n > x)(1 + o(1)), \quad \text{as } x \rightarrow \infty. \quad (3.2.18)$$

The set of subexponential distributions is denoted by \mathcal{S} .

One can show that the defining property (3.2.18) holds for *all* $n \geq 2$ if it holds for *some* $n \geq 2$; see Section 1.3.2 in [46] for details.

As we have learned in Section 3.2.5, $P(M_n > x) = n\bar{F}(x)(1 + o(1))$ as $x \rightarrow \infty$, where F is the common distribution function of the X_i 's, and therefore the defining property (3.2.18) can also be formulated as

$$\text{For all } n \geq 2: \quad \lim_{x \rightarrow \infty} \frac{P(S_n > x)}{\bar{F}(x)} = n.$$

We consider some properties of subexponential distributions.

Lemma 3.2.24 (Basic properties of subexponential distributions)

(1) *If $F \in \mathcal{S}$, then for any $y > 0$,*

$$\lim_{x \rightarrow \infty} \frac{\bar{F}(x - y)}{\bar{F}(x)} = 1. \quad (3.2.19)$$

(2) *If (3.2.19) holds for every $y > 0$ then, for all $\varepsilon > 0$,*

$$e^{\varepsilon x} \bar{F}(x) \rightarrow \infty, \quad x \rightarrow \infty.$$

- (3) If $F \in \mathcal{S}$ then, given $\varepsilon > 0$, there exists a finite constant K so that for all $n \geq 2$,

$$\frac{P(S_n > x)}{\overline{F}(x)} \leq K(1 + \varepsilon)^n, \quad x \geq 0. \quad (3.2.20)$$

For the proof of (3), see Lemma 1.3.5 in [46].

Proof. (1) Write $G(x) = P(X_1 + X_2 \leq x)$ for the distribution function of $X_1 + X_2$. For $x \geq y > 0$,

$$\begin{aligned} \frac{\overline{G}(x)}{\overline{F}(x)} &= 1 + \int_0^y \frac{\overline{F}(x-t)}{\overline{F}(x)} dF(t) + \int_y^x \frac{\overline{F}(x-t)}{\overline{F}(x)} dF(t) \\ &\geq 1 + F(y) + \frac{\overline{F}(x-y)}{\overline{F}(x)} (F(x) - F(y)). \end{aligned}$$

Thus, if x is large enough so that $F(x) - F(y) \neq 0$,

$$1 \leq \frac{\overline{F}(x-y)}{\overline{F}(x)} \leq \left(\frac{\overline{G}(x)}{\overline{F}(x)} - 1 - F(y) \right) (F(x) - F(y))^{-1}.$$

In the latter estimate, the right-hand side tends to 1 as $x \rightarrow \infty$. This proves (3.2.19).

(2) By virtue of (1), the function $\overline{F}(\log y)$ is slowly varying. But then the conclusion that $y^\varepsilon \overline{F}(\log y) \rightarrow \infty$ as $y \rightarrow \infty$ follows immediately from the representation theorem for slowly varying functions; see (3.2.16). Now write $y = e^x$. \square

Lemma 3.2.24(2) justifies the name ‘‘subexponential’’ for $F \in \mathcal{S}$; indeed $\overline{F}(x)$ decays to 0 slower than any exponential function $e^{-\varepsilon x}$ for $\varepsilon > 0$. Furthermore, since for any $\varepsilon > 0$,

$$Ee^{\varepsilon X} \geq E(e^{\varepsilon X} I_{(y, \infty)}) \geq e^{\varepsilon y} \overline{F}(y), \quad y \geq 0,$$

it follows from Lemma 3.2.24(2) that for $F \in \mathcal{S}$, $Ee^{\varepsilon X} = \infty$ for all $\varepsilon > 0$.

Therefore the moment generating function of a subexponential distribution does not exist in any neighborhood of the origin.

Property (3.2.19) holds for larger classes of distributions than the subexponential distributions. It can be taken as another definition of heavy-tailed distributions. It means that the tails $P(X > x)$ and $P(X + y > x)$ are not significantly different, for any fixed y and large x . In particular, it says that for any $y > 0$ as $x \rightarrow \infty$,

$$\begin{aligned} \frac{P(X > x+y)}{P(X > x)} &= \frac{P(X > x+y, X > x)}{P(X > x)} \\ &= P(X > x+y \mid X > x) \rightarrow 1. \end{aligned} \quad (3.2.21)$$

Thus, once X has exceeded a high threshold, x , it is very likely to exceed an even higher threshold $x + y$. This situation changes completely when we look, for example, at an exponential or a truncated normal random variable. For these two distributions you can verify that the above limit exists, but its value is less than 1.

Property (3.2.19) helps one to exclude certain distributions from the class \mathcal{S} . However, it is in general difficult to determine whether a given distribution is subexponential.

Example 3.2.25 (Examples of subexponential distributions)

The large claim distributions in Table 3.2.19 are subexponential. The small claim distributions in Table 3.2.17 are not subexponential. However, the tail of a subexponential distribution can be very close to an exponential distribution. For example, the heavy-tailed Weibull distributions with tail

$$\bar{F}(x) = e^{-c x^\tau}, \quad x \geq 0, \quad \text{for some } \tau \in (0, 1),$$

and also the distributions with tail

$$\bar{F}(x) = e^{-x (\log x)^{-\beta}}, \quad x \geq x_0, \quad \text{for some } \beta, x_0 > 0,$$

are subexponential. We refer to Sections 1.4.1 and A3.2 in [46] for details. See also Exercise 11 on p. 108. \square

Comments

The subexponential distributions constitute a natural class of heavy-tailed claim size distributions from a theoretical but also from a practical point of view. In insurance mathematics subexponentiality is considered as a synonym for heavy-tailedness. The class \mathcal{S} is very flexible insofar that it contains distributions with very heavy tails such as the regularly varying subclass, but also distributions with moderately heavy tails such as the log-normal and Weibull ($\tau < 1$) distributions. In contrast to regularly varying random variables, log-normal and Weibull distributed random variables have finite power moments, but none of the subexponential distributions has a finite moment generating function in some neighborhood of the origin.

An extensive treatment of subexponential distributions, their properties and use in insurance mathematics can be found in Embrechts et al. [46]. A more recent survey on \mathcal{S} and related classes of distributions is given in Goldie and Klüppelberg [59].

We re-consider subexponential claim size distributions when we study ruin probabilities in Section 4.2.4. There subexponential distributions will turn out to be the most natural class of large claim distributions.

Exercises

Section 3.2.2

- (1) We say that a distribution is light-tailed (compared to the exponential distribution) if

$$\limsup_{x \rightarrow \infty} \frac{\bar{F}(x)}{e^{-\lambda x}} < \infty$$

for some $\lambda > 0$ and heavy-tailed if

$$\liminf_{x \rightarrow \infty} \frac{\bar{F}(x)}{e^{-\lambda x}} > 0$$

for all $\lambda > 0$.

- (a) Show that the gamma and the truncated normal distributions are light-tailed.
 (b) Consider a Pareto distribution given via its tail in the parameterization

$$\bar{F}(x) = \frac{\kappa^\alpha}{(\kappa + x)^\alpha}, \quad x > 0. \quad (3.2.22)$$

Show that F is heavy-tailed.

- (c) Show that the Weibull distribution with tail $\bar{F}(x) = e^{-cx^\tau}$, $x > 0$, for some $c, \tau > 0$, is heavy-tailed for $\tau < 1$ and light-tailed for $\tau \geq 1$.

Section 3.2.3

- (2) Let F be the distribution function of a positive random variable X with infinite right endpoint, finite expectation and $F(x) > 0$ for all $x > 0$.
 (a) Show that the mean excess function e_F satisfies the relation

$$e_F(x) = \frac{1}{\bar{F}(x)} \int_x^\infty \bar{F}(y) dy, \quad x > 0.$$

- (b) A typical heavy-tailed distribution is the Pareto distribution given via its tail in the parameterization

$$\bar{F}(x) = \gamma^\alpha x^{-\alpha}, \quad x > \gamma, \quad (3.2.23)$$

for positive γ and α . Calculate the mean excess function e_F for $\alpha > 1$ and verify that $e_F(x) \rightarrow \infty$ as $x \rightarrow \infty$. Why do we need the condition $\alpha > 1$?

- (c) Assume F is continuous and has support $(0, \infty)$. Show that

$$\bar{F}(x) = \frac{e_F(0)}{e_F(x)} \exp \left\{ - \int_0^x (e_F(y))^{-1} dy \right\}, \quad x > 0.$$

Hint: Interpret $-1/e_F(y)$ as logarithmic derivative.

- (3) The *generalized Pareto distribution* plays a major role in extreme value theory and extreme value statistics; see Embrechts et al. [46], Sections 3.4 and 6.5. It is given by its distribution function

$$G_{\xi,\beta}(x) = 1 - \left(1 + \xi \frac{x}{\beta}\right)^{-1/\xi}, \quad x \in D(\xi, \beta).$$

Here $\xi \in \mathbb{R}$ is a shape parameter and $\beta > 0$ a scale parameter. For $\xi = 0$, $G_{0,\beta}(x)$ is interpreted as the limiting distribution as $\xi \rightarrow 0$:

$$G_{0,\beta}(x) = 1 - e^{-x/\beta}.$$

The domain $D(\xi, \beta)$ is defined as follows:

$$D(\xi, \beta) = \begin{cases} [0, \infty) & \xi \geq 0, \\ [0, -1/\xi] & \xi < 0. \end{cases}$$

Show that $G_{\xi,\beta}$ has the mean excess function

$$e_G(u) = \frac{\beta + \xi u}{1 - \xi}, \quad \beta + u\xi > 0,$$

for u in the support of $G_{\xi,\beta}$ and $\xi < 1$.

Sections 3.2.4-3.2.5

- (4) Some properties of Pareto-like distributions.
- Verify for a random variable X with Pareto distribution function F given by (3.2.22) that $EX^\delta = \infty$ for $\delta \geq \alpha$ and $EX^\delta < \infty$ for $\delta < \alpha$.
 - Show that a Pareto distributed random variable X whose distribution has parameterization (3.2.23) is obtained by the transformation $X = \gamma \exp\{Y/\alpha\}$ for some standard exponential random variable Y and $\gamma, \alpha > 0$.
 - A Burr distributed random variable Y is obtained by the transformation $Y = X^{1/c}$ for some positive c from a Pareto distributed random variable X with tail (3.2.22). Determine the tail $\bar{F}_Y(x)$ for the Burr distribution and check for which $p > 0$ the moment EY^p is finite.
 - The log-gamma distribution has density

$$f(y) = \frac{\delta^\gamma \lambda^\delta}{\Gamma(\gamma)} \frac{(\log(y/\lambda))^{\gamma-1}}{y^{\delta+1}}, \quad y > \lambda.$$

for some $\lambda, \gamma, \delta > 0$. Check by some appropriate bounds for $\log x$ that the log-gamma distribution has finite moments of order less than δ and infinite moments of order greater than δ . Check that the tail \bar{F} satisfies

$$\lim_{x \rightarrow \infty} \frac{\bar{F}(x)}{(\delta^{\gamma-1} \lambda^\delta / \Gamma(\gamma)) (\log(x/\lambda))^{\gamma-1} x^{-\delta}} = 1.$$

- (e) Let X have a Pareto distribution with tail (3.2.23). Consider a positive random variable $Y > 0$ with $EY^\alpha < \infty$, independent of X . Show that

$$\lim_{x \rightarrow \infty} \frac{P(XY > x)}{P(X > x)} = EY^\alpha.$$

Hint: Use a conditioning argument.

- (5) Consider the Pareto distribution in the parameterization (3.2.23), where we assume the constant γ to be known. Determine the maximum likelihood estimator of α based on an iid sample X_1, \dots, X_n with distribution function F and the distribution of $1/\hat{\alpha}_{\text{MLE}}$. Why is this result not surprising? See (4,b).
- (6) Recall the representation (3.2.15) of a slowly varying function.
- Show that (3.2.15) defines a slowly varying function.
 - Use representation (3.2.15) to show that for any slowly varying function L and $\delta > 0$, the properties $\lim_{x \rightarrow \infty} x^\delta L(x) = \infty$ and $\lim_{x \rightarrow \infty} x^{-\delta} L(x) = 0$ hold.
- (7) Consider the Peter-and-Paul distribution function given by

$$F(x) = \sum_{k \geq 1: 2^k \leq x} 2^{-k}, \quad x \geq 0.$$

- Show that \bar{F} is not regularly varying.
- Show that for a random variable X with distribution function F , $EX^\delta = \infty$ for $\delta \geq 1$ and $EX^\delta < \infty$ for $\delta < 1$.

Section 3.2.6

- (8) Show by different means that the exponential distribution is not subexponential.
- Verify that the defining property (3.2.18) of a subexponential distribution does not hold.
 - Verify that condition (3.2.21) does not hold. The latter condition is necessary for subexponentiality.
 - Use an argument about the exponential moments of a subexponential distribution.
- (9) Show that the light-tailed Weibull distribution given by $\bar{F}(x) = e^{-c x^\tau}$, $x > 0$, for some $c > 0$ and $\tau \geq 1$ is not subexponential.
- (10) Show that a claim size distribution with finite support cannot be subexponential.
- (11) Pitman [118] gave a complete characterization of subexponential distribution functions F with a density f in terms of their *hazard rate function* $q(x) = f(x)/\bar{F}(x)$. In particular, he showed the following.

Assume that $q(x)$ is eventually decreasing to 0. Then

- $F \in \mathcal{S}$ if and only if

$$\lim_{x \rightarrow \infty} \int_0^x e^{y q(y)} f(y) dy = 1.$$

- If the function $g(x) = e^{x q(x)} f(x)$ is integrable on $[0, \infty)$, then $F \in \mathcal{S}$.

Apply these results in order to show that the distributions of Table 3.2.19 are subexponential.

- (12) Let (X_i) be an iid sequence of positive random variables with common distribution function F . Write $S_n = X_1 + \dots + X_n$, $n \geq 1$.
- Show that for every $n \geq 1$ the following relation holds:

$$\liminf_{x \rightarrow \infty} \frac{P(S_n > x)}{n \bar{F}(x)} \geq 1.$$

- (b) Show that the definition of a subexponential distribution function F is equivalent to the following relation

$$\limsup_{x \rightarrow \infty} \frac{P(S_n > x)}{P(X_i > x \text{ for some } i \leq n \text{ and } X_j \leq x \text{ for } 1 \leq j \neq i \leq n)} \leq 1,$$

for all $n \geq 2$.

- (c) Show that for a subexponential distribution function F and $1 \leq k \leq n$,

$$\lim_{x \rightarrow \infty} P(X_1 + \cdots + X_k > x \mid X_1 + \cdots + X_n > x) = \frac{k}{n}.$$

- (d) The relation (3.2.19) can be shown to hold uniformly on bounded y -intervals for subexponential F . Use this information to show that

$$\lim_{x \rightarrow \infty} P(X_1 \leq z \mid X_1 + X_2 > x) = 0.5 F(z), \quad z > 0.$$

3.3 The Distribution of the Total Claim Amount

In this section we study the distribution of the total claim amount

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

under the standard assumption that the claim number process N and the iid sequence (X_i) of positive claims are independent. We often consider the case of fixed t , i.e., we study the *random variable* $S(t)$, *not the stochastic process* $(S(t))_{t \geq 0}$. When t is fixed, we will often suppress the dependence of $N(t)$ and $S(t)$ on t and write $N = N(t)$, $S = S(t)$ and

$$S = \sum_{i=1}^N X_i,$$

thereby abusing our previous notation since we have used the symbols N for the claim number process and S for the total claim amount process before. It will, however, be clear from the context what S and N denote in the different sections.

In Section 3.3.1 we investigate the distribution of the total claim amount in terms of its characteristic function. We introduce the class of *mixture distributions* which turn out to be useful for characterizing the distribution of the total claim amount, in particular for compound Poisson processes. The most important results of this section say that sums of independent compound Poisson variables are again compound Poisson. Moreover, given a compound Poisson process (such as the total claim amount process in the Cramér-Lundberg model), it can be decomposed into independent compound Poisson processes by introducing a disjoint partition of time and claim size space. These results

are presented in Section 3.3.2. They are extremely useful, for example, if one is interested in the total claim amount over smaller periods of time or in the total claim amount of claim sizes assuming values in certain layers. We continue in Section 3.3.3 with a numerical procedure, the *Panjer recursion*, for determining the *exact distribution* of the total claim amount. This procedure works for integer-valued claim sizes and for a limited number of claim number distributions. In Sections 3.3.4 and 3.3.5 we consider alternative methods for determining *approximations* to the distribution of the total claim amount. They are based on the central limit theorem or Monte Carlo techniques.

3.3.1 Mixture Distributions

In this section we are interested in some theoretical properties of the distribution of $S = S(t)$ for fixed t . The distribution of S is determined by its *characteristic function*

$$\phi_S(s) = E e^{isS}, \quad s \in \mathbb{R},$$

and we focus here on techniques based on characteristic functions. Alternatively, we could use the moment generating function

$$m_S(h) = E e^{hS}, \quad h \in (-h_0, h_0),$$

provided the latter is finite for some positive $h_0 > 0$. Indeed, m_S also determines the distribution of S . However, $m_S(h)$ is finite in some neighborhood of the origin if and only if the tail $P(S > x)$ decays exponentially fast, i.e.,

$$P(S > x) \leq c e^{-\gamma x}, \quad x > 0,$$

for some positive c, γ . This assumption is not satisfied for S with the heavy-tailed claim size distributions introduced in Table 3.2.19, and therefore we prefer using characteristic functions,⁴ which are defined for any random variable S .

Exploiting the independence of N and (X_i) , a conditioning argument yields the following useful formula:

$$\begin{aligned} \phi_S(s) &= E \left(E \left[e^{is(X_1 + \dots + X_N)} \mid N \right] \right) \\ &= E \left([E e^{isX_1}]^N \right) = E([\phi_{X_1}(s)]^N) \\ &= E e^{N \log \phi_{X_1}(s)} = m_N(\log \phi_{X_1}(s)). \end{aligned} \tag{3.3.24}$$

⁴ As a second alternative to characteristic functions we could use the Laplace-Stieltjes transform $\hat{f}_S(s) = m_S(-s)$ for $s > 0$ which is well-defined for non-negative random variables S and determines the distribution of S . The reader who feels uncomfortable with the notion of characteristic functions could switch to moment generating functions or Laplace-Stieltjes transforms; most of the calculations can easily be adapted to either of the two transforms. We refer to p. 178 for a brief introduction to Laplace-Stieltjes transforms.

(The problems we have mentioned with the moment generating function do not apply in this situation, since we consider m_N at the complex argument $\log \phi_{X_1}(s)$. The quantities in (3.3.24) are all bounded in absolute value by 1, since we deal with a characteristic function.) We apply this formula to two important examples: the compound Poisson case, i.e., when N has a Poisson distribution, and the compound geometric case, i.e., when N has a geometric distribution.

Example 3.3.1 (Compound Poisson sum)

Assume that N is $\text{Pois}(\lambda)$ distributed for some $\lambda > 0$. Straightforward calculation yields

$$m_N(h) = e^{-\lambda(1-e^h)}, \quad h \in \mathbb{C}.$$

Then we conclude from (3.3.24) that

$$\phi_S(s) = e^{-\lambda(1-\phi_{X_1}(s))}, \quad s \in \mathbb{R}.$$

□

Example 3.3.2 (Compound geometric sum)

We assume that N has a geometric distribution with parameter $p \in (0, 1)$, i.e.,

$$P(N = n) = p q^n, \quad n = 0, 1, 2, \dots, \quad \text{where } q = 1 - p.$$

Moreover, let X_1 be exponentially $\text{Exp}(\lambda)$ distributed. It is not difficult to verify that

$$\phi_{X_1}(s) = \frac{\lambda}{\lambda - is}, \quad s \in \mathbb{R}.$$

We also have

$$m_N(h) = \sum_{n=0}^{\infty} e^{nh} P(N = n) = \sum_{n=0}^{\infty} e^{nh} p q^n = \frac{p}{1 - e^h q}$$

provided $|h| < -\log q$. Plugging ϕ_{X_1} and m_N in formula (3.3.24), we obtain

$$\phi_S(s) = \frac{p}{1 - \lambda(\lambda - is)^{-1}q} = p + q \frac{\lambda p}{\lambda p - is}, \quad s \in \mathbb{R}.$$

We want to interpret the right-hand side in a particular way. Let J be a random variable assuming two values with probabilities p and q , respectively. For example, choose $P(J = 1) = p$ and $P(J = 2) = q$. Consider the random variable

$$S' = I_{\{J=1\}} 0 + I_{\{J=2\}} Y,$$

where Y is $\text{Exp}(\lambda p)$ distributed and independent of J . This means that we choose either the random variable 0 or the random variable Y according as $J = 1$ or $J = 2$. Writing F_A for the distribution function of any random variable A , we see that S' has distribution function

$$F_{S'}(x) = p F_0(x) + q F_Y(x) = p I_{[0,\infty)}(x) + q F_Y(x), \quad x \in \mathbb{R}, \quad (3.3.25)$$

and characteristic function

$$E e^{is S'} = P(J=1) E e^{is 0} + P(J=2) E e^{is Y} = p + q \frac{\lambda p}{\lambda p - is}, \quad s \in \mathbb{R}.$$

In words, this is the characteristic function of S , and therefore $S \stackrel{d}{=} S'$:

$$S \stackrel{d}{=} I_{\{J=1\}} 0 + I_{\{J=2\}} Y.$$

A distribution function of the type (3.3.25) determines a *mixture distribution*. \square

We fix this notion in the following definition.

Definition 3.3.3 (Mixture distribution)

Let $(p_i)_{i=1,\dots,n}$ be a distribution on the integers $\{1, \dots, n\}$ and F_i , $i = 1, \dots, n$, be distribution functions of real-valued random variables. Then the distribution function

$$G(x) = p_1 F_1(x) + \dots + p_n F_n(x), \quad x \in \mathbb{R}, \quad (3.3.26)$$

defines a mixture distribution of F_1, \dots, F_n .

The above definition of mixture distribution can immediately be extended to distributions (p_i) on $\{1, 2, \dots\}$ and a sequence (F_i) of distribution functions by defining

$$G(x) = \sum_{i=1}^{\infty} p_i F_i(x), \quad x \in \mathbb{R}.$$

For our purposes, finite mixtures are sufficient.

As in Example 3.3.2 of a compound geometric sum, we can interpret the probabilities p_i as the distribution of a discrete random variable J assuming the values i : $P(J=i) = p_i$. Moreover, assume J is independent of the random variables Y_1, \dots, Y_n with distribution functions $F_{Y_i} = F_i$. Then a conditioning argument shows that the random variable

$$Z = I_{\{J=1\}} Y_1 + \dots + I_{\{J=n\}} Y_n$$

has the mixture distribution function

$$F_Z(x) = p_1 F_{Y_1}(x) + \cdots + p_n F_{Y_n}(x), \quad x \in \mathbb{R},$$

with the corresponding characteristic function

$$\phi_Z(s) = p_1 \phi_{Y_1}(s) + \cdots + p_n \phi_{Y_n}(s), \quad s \in \mathbb{R}. \quad (3.3.27)$$

It is interesting to observe that the dependence structure of the Y_i 's does not matter here.

An interesting result in the context of mixture distributions is the following.

Proposition 3.3.4 (Sums of independent compound Poisson variables are compound Poisson)

Consider the independent compound Poisson sums

$$S_i = \sum_{j=1}^{N_i} X_j^{(i)}, \quad i = 1, \dots, n,$$

where N_i is $\text{Pois}(\lambda_i)$ distributed for some $\lambda_i > 0$ and, for every fixed i , $(X_j^{(i)})_{j=1,2,\dots}$ is an iid sequence of claim sizes. Then the sum

$$\tilde{S} = S_1 + \cdots + S_n$$

is again compound Poisson with representation

$$\tilde{S} \stackrel{d}{=} \sum_{i=1}^{N_\lambda} Y_i, \quad N_\lambda \sim \text{Pois}(\lambda), \quad \lambda = \lambda_1 + \cdots + \lambda_n,$$

and (Y_i) is an iid sequence, independent of N_λ , with mixture distribution (3.3.26) given by

$$p_i = \lambda_i / \lambda \quad \text{and} \quad F_i = F_{X_1^{(i)}}. \quad (3.3.28)$$

Proof. Recall the characteristic function of a compound Poisson variable from Example 3.3.1:

$$\phi_{S_j}(s) = \exp \left\{ -\lambda_j \left(1 - \phi_{X_1^{(j)}}(s) \right) \right\}, \quad s \in \mathbb{R}.$$

By independence of the S_j 's and the definition (3.3.28) of the p_j 's,

$$\begin{aligned} \phi_{\tilde{S}}(s) &= \phi_{S_1}(s) \cdots \phi_{S_n}(s) \\ &= \exp \left\{ -\lambda \sum_{j=1}^n p_j \left(1 - \phi_{X_1^{(j)}}(s) \right) \right\} \\ &= \exp \left\{ -\lambda \left(1 - E \exp \left\{ is \sum_{j=1}^n I_{\{J=j\}} X_1^{(j)} \right\} \right) \right\}, \quad s \in \mathbb{R}, \end{aligned}$$

where J is independent of the $X_1^{(j)}$'s and has distribution $(P(J = i))_{i=1,\dots,n} = (p_i)_{i=1,\dots,n}$. This is the characteristic function of a compound Poisson sum with summands whose distribution is described in (3.3.27), where (p_i) and (F_i) are specified in (3.3.28). \square

The fact that sums of independent compound Poisson random variables are again compound Poisson is a nice closure property which has interesting applications in insurance. We illustrate this in the following example.

Example 3.3.5 (Applications of the compound Poisson property)

(1) Consider a Poisson process $N = (N(t))_{t \geq 0}$ with mean value function μ and assume that the claim sizes in the portfolio in year i constitute an iid sequence $(X_j^{(i)})$ and that all sequences $(X_j^{(i)})$ are mutually independent and independent of the claim number process N . The total claim amount in year i is given by

$$S_i = \sum_{j=N(i-1)+1}^{N(i)} X_j^{(i)}.$$

Since N has independent increments and the iid sequences $(X_j^{(i)})$ are mutually independent, we observe that

$$\left(\sum_{j=N(i-1)+1}^{N(i)} X_j^{(i)} \right)_{i=1,\dots,n} \stackrel{d}{=} \left(\sum_{j=1}^{N(i-1,i]} X_j^{(i)} \right)_{i=1,\dots,n}. \quad (3.3.29)$$

A formal proof of this identity is easily provided by identifying the joint characteristic functions of the vectors on both sides. This verification is left as an exercise. Since $(N(i-1, i])$ is a sequence of independent random variables, independent of the independent sequences $(X_j^{(i)})$, the annual total claim amounts S_i are mutually independent. Moreover, each of them is compound Poisson: let N_i be $\text{Pois}(\mu(i-1, i])$ distributed, independent of $(X_j^{(i)})$, $i = 1, \dots, n$. Then

$$S_i \stackrel{d}{=} \sum_{j=1}^{N_i} X_j^{(i)}.$$

We may conclude from Proposition 3.3.4 that the total claim amount $S(n)$ in the first n years is again compound Poisson, i.e.,

$$S(n) = S_1 + \cdots + S_n \stackrel{d}{=} \sum_{i=1}^{N_\lambda} Y_i,$$

where the random variable

$$N_\lambda \sim \text{Pois}(\lambda), \quad \lambda = \mu(0, 1] + \cdots + \mu(n-1, n] = \mu(n),$$

is independent of the iid sequence (Y_i) . Each of the Y_i 's has representation

$$Y_i \stackrel{d}{=} I_{\{J=1\}} X_1^{(1)} + \cdots + I_{\{J=n\}} X_1^{(n)}, \quad (3.3.30)$$

where J is independent of the $X_1^{(j)}$'s, with distribution $P(J = i) = \mu(i - 1, i]/\lambda$.

In other words, the total claim amount $S(n)$ in the first n years with possibly different claim size distributions in each year has representation as a compound Poisson sum with Poisson counting variable N_λ which has the same distribution as $N(n)$ and with iid claim sizes Y_i with the mixture distribution presented in (3.3.30).

(2) Consider n independent portfolios with total claim amounts in a fixed period of time given by the compound Poisson sums

$$S_i = \sum_{j=1}^{N_i} X_j^{(i)}, \quad N_i \sim \text{Pois}(\lambda_i).$$

The claim sizes $X_j^{(i)}$ in the i th portfolio are iid, but the distributions may differ from portfolio to portfolio. For example, think of each portfolio as a collection of policies corresponding to one particular type of car insurance or, even simpler, think of each portfolio as the claim history in one particular policy. Now, Proposition 3.3.4 ensures that the aggregation of the total claim amounts from the different portfolios, i.e.,

$$\tilde{S} = S_1 + \cdots + S_n,$$

is again compound Poisson with counting variable which has the same Poisson distribution as $N_1 + \cdots + N_n \sim \text{Pois}(\lambda)$, $\lambda = \lambda_1 + \cdots + \lambda_n$, with iid claim sizes Y_i . A sequence of the Y_i 's can be realized by independent repetitions of the following procedure:

- (a) Draw a number $i \in \{1, \dots, n\}$ with probability $p_i = \lambda_i/\lambda$.
- (b) Draw a realization from the claim size distribution of the i th portfolio.

□

3.3.2 Space-Time Decomposition of a Compound Poisson Process

In this section we prove a converse result to Proposition 3.3.4: we decompose a compound Poisson process into independent compound Poisson processes by partitioning time and (claim size) space. In this context, we consider a general compound Poisson process

$$S(t) = \sum_{i=1}^{N(t)} X_i, \quad t \geq 0,$$

where N is a Poisson process on $[0, \infty)$ with mean value function μ and arrival sequence (T_i) , independent of the iid sequence (X_i) of positive claim sizes with common distribution F . The mean value function μ generates a measure on the Borel σ -field of $[0, \infty)$, the mean measure of the Poisson process N , which we also denote by μ .

The points (T_i, X_i) assume values in the state space $E = [0, \infty)^2$ equipped with the Borel σ -field \mathcal{E} . We have learned in Section 2.1.8 that the counting measure

$$M(A) = \#\{i \geq 1 : (T_i, X_i) \in A\}, \quad A \in \mathcal{E},$$

is a Poisson random measure with mean measure $\nu = \mu \times F$. This means in particular that for any disjoint partition A_1, \dots, A_n of E , i.e.,

$$\bigcup_{i=1}^n A_i = E, \quad A_i \cap A_j = \emptyset, \quad 1 \leq i < j \leq n,$$

the random variables $M(A_1), \dots, M(A_n)$ are independent and $M(A_i) \sim \text{Pois}(\nu(A_i))$, $i = 1, \dots, n$, where we interpret $M(A_i) = \infty$ if $\nu(A_i) = \infty$. But even more is true, as the following theorem shows:

Theorem 3.3.6 (Space-time decomposition of a compound Poisson sum)
Assume that the mean value function μ of the Poisson process N on $[0, \infty)$ has an a.e. positive continuous intensity function λ . Let A_1, \dots, A_n be a disjoint partition of $E = [0, \infty)^2$. Then the following statements hold.

(1) *For every $t \geq 0$, the random variables*

$$S_j(t) = \sum_{i=1}^{N(t)} X_i I_{A_j}((T_i, X_i)), \quad j = 1, \dots, n,$$

are mutually independent.

(2) *For every $t \geq 0$, $S_j(t)$ has representation as a compound Poisson sum*

$$S_j(t) \stackrel{d}{=} \sum_{i=1}^{N(t)} X_i I_{A_j}((Y_i, X_i)), \quad (3.3.31)$$

where (Y_i) is an iid sequence of random variables with density $\lambda(x)/\mu(t)$ $0 \leq x \leq t$, independent of N and (X_i) .

Proof. Since μ has an a.e. positive continuous intensity function λ we know from the order statistics property of the one-dimensional Poisson process N (see Theorem 2.1.11) that

$$(T_1, \dots, T_k \mid N(t) = k) \stackrel{d}{=} (Y_{(1)}, \dots, Y_{(k)}),$$

where $Y_{(1)} \leq \dots \leq Y_{(k)}$ are the order statistics of an iid sample Y_1, \dots, Y_k with common density $\lambda(x)/\mu(t)$, $0 \leq x \leq t$. By a similar argument as in the proof of Proposition 2.1.16 we may conclude that

$$((S_j(t))_{j=1,\dots,n} | N(t) = k) \quad (3.3.32)$$

$$\stackrel{d}{=} \left(\sum_{i=1}^k X_i I_{A_j}((Y_{(i)}, X_i)) \right)_{j=1,\dots,n} \stackrel{d}{=} \left(\sum_{i=1}^k X_i I_{A_j}((Y_i, X_i)) \right)_{j=1,\dots,n},$$

where N , (Y_i) and (X_i) are independent. Observe that each of the sums on the right-hand side has iid summands. We consider the joint characteristic function of the $S_j(t)$'s. Exploiting relation (3.3.32), we obtain for any $s_i \in \mathbb{R}$, $i = 1, \dots, n$,

$$\begin{aligned} & \phi_{S_1(t), \dots, S_n(t)}(s_1, \dots, s_n) \\ &= E e^{i s_1 S_1(t) + \dots + i s_n S_n(t)} \\ &= \sum_{k=0}^{\infty} P(N(t) = k) E \left(e^{i s_1 S_1(t) + \dots + i s_n S_n(t)} \mid N(t) = k \right) \\ &= \sum_{k=0}^{\infty} P(N(t) = k) E \exp \left\{ i \sum_{l=1}^k \sum_{j=1}^n s_j X_l I_{A_j}((Y_l, X_l)) \right\} \\ &= E \exp \left\{ i \sum_{l=1}^{N(t)} \sum_{j=1}^n s_j X_l I_{A_j}((Y_l, X_l)) \right\}. \end{aligned}$$

Notice that the exponent in the last line is a compound Poisson sum. From the familiar form of its characteristic function and the disjointness of the A_j 's we may conclude that

$$\begin{aligned} & \log \phi_{S_1(t), \dots, S_n(t)}(s_1, \dots, s_n) \\ &= -\mu(t) \left(1 - E \exp \left\{ i \sum_{j=1}^n s_j X_1 I_{A_j}((Y_1, X_1)) \right\} \right) \\ &= -\mu(t) \left(1 - \left[\sum_{j=1}^n \left(E e^{i s_j X_1 I_{A_j}((Y_1, X_1))} - (1 - P((Y_1, X_1) \in A_j)) \right) \right] \right) \\ &= -\mu(t) \sum_{j=1}^n \left(1 - E e^{i s_j X_1 I_{A_j}((Y_1, X_1))} \right). \end{aligned} \quad (3.3.33)$$

The right-hand side in (3.3.33) is nothing but the sum of the logarithms of the characteristic functions $\phi_{S_j(t)}(s_j)$. Equivalently, the joint characteristic

function of the $S_j(t)$'s factorizes into the individual characteristic functions $\phi_{S_j(t)}(s_j)$. This means that the random variables $S_j(t)$ are mutually independent and each of them has compound Poisson structure as described in (3.3.31), where we again used the identity in law (3.3.32). This proves the theorem. \square

Theorem 3.3.6 has a number of interesting consequences.

Example 3.3.7 (Decomposition of time and claim size space in the Cramér-Lundberg model)

Consider the total claim amount process S in the Cramér-Lundberg model with Poisson intensity $\lambda > 0$ and claim size distribution function F .

(1) *Partitioning time.* Choose $0 = t_0 < t_1 < \dots < t_n = t$ and write

$$\Delta_1 = [0, t_1], \quad \Delta_i = (t_{i-1}, t_i], \quad i = 2, \dots, n, \quad \Delta_{n+1} = (t_n, \infty). \quad (3.3.34)$$

Then

$$A_i = \Delta_i \times [0, \infty), \quad i = 1, \dots, n + 1,$$

is a disjoint decomposition of the state space $E = [0, \infty)^2$. An application of Theorem 3.3.6 yields that the random variables

$$\sum_{i=1}^{N(t)} X_i I_{A_j}((T_i, X_i)) = \sum_{i=N(t_{j-1})+1}^{N(t_j)} X_i, \quad j = 1, \dots, n,$$

are independent. This is the well-known independent increment property of the compound Poisson process. It is also not difficult to see that the increments are stationary, i.e., $S(t) - S(s) \stackrel{d}{=} S(t-s)$ for $s < t$. Hence they are again compound Poisson sums.

(2) *Partitioning claim size space.* For fixed t , we partition the claim size space $[0, \infty)$ into the disjoint sets B_1, \dots, B_{n+1} . For example, one can think of disjoint *layers*

$$B_1 = [0, d_1], B_2 = (d_1, d_2], \dots, B_n = (d_{n-1}, d_n], B_{n+1} = (d_n, \infty),$$

where $0 < d_1 < \dots < d_n < \infty$ are finitely many limits which classify the order of magnitude of the claim sizes. Such layers are considered in a reinsurance context, where different insurance companies share the risk (and the premium) of a portfolio in its distinct layers. Then the sets

$$A_i = [0, t] \times B_i, \quad A'_i = (t, \infty) \times B_i, \quad i = 1, \dots, n + 1,$$

constitute a disjoint partition of the state space E . An application of Theorem 3.3.6 yields that the total claim amounts in the different parts of the partition

$$S_j(t) = \sum_{i=1}^{N(t)} X_i I_{A_j}((T_i, X_i)) = \sum_{i=1}^{N(t)} X_i I_{B_j}(X_i), \quad j = 1, \dots, n+1,$$

are mutually independent. Whereas the independent increment property of S is perhaps not totally unexpected because of the corresponding property of the Poisson process N , the independence of the quantities $S_j(t)$ is not obvious from their construction. Their compound Poisson structure is, however, immediate since the summands $X_i I_{B_j}(X_i)$ are iid and independent of $N(t)$.

(3) *General partitions.* So far we partitioned either time or the claim size space. But Theorem 3.3.6 allows one to consider *any* disjoint partition of the state space E . The message is always the same: the total claim amounts on the distinct parts of the partition are independent and have compound Poisson structure. This is an amazing and very useful result. \square

Theorem 3.3.6 has immediate consequences for the dependence structure of the compound Poisson processes of the decomposition of the total claim amount.

Corollary 3.3.8 *Under the conditions of Theorem 3.3.6, the processes $S_j = (S_j(t))_{t \geq 0}$, $j = 1, \dots, n$ are mutually independent and have independent increments.*

Proof. We start by showing the independent increment property for one process S_j . For $0 = t_0 < \dots < t_n$ and $n \geq 1$, define the Δ_i 's as in (3.3.34). The sets

$$A'_i = A_i \cap (\Delta_i \times [0, \infty)), \quad i = 1, \dots, n,$$

are disjoint. An application of Theorem 3.3.6 yields that the random variables

$$\sum_{i=1}^{N(t_n)} X_i I_{A'_j}((T_i, X_i)) = \sum_{i=N(t_{j-1})+1}^{N(t_j)} X_i I_{A_j}((T_i, X_i)) = S_j(t_{i-1}, t_i]$$

are mutually independent. This means that the process S_j has independent increments.

In order to show the independence of the processes S_j , $j = 1, \dots, n$, one has to show that the families of the random variables $(S_j(t_i^{(j)}))_{i=1, \dots, k_j}$, $j = 1, \dots, n$ for any choices of increasing $t_i^{(j)} \geq 0$ and integers $k_j \geq 1$ are mutually independent. Define the quantities $\Delta_i^{(j)}$ for $0 = t_0^{(j)} < \dots < t_{k_j}^{(j)} < \infty$, $j = 1, \dots, n$, in analogy to (3.3.34). Then

$$A_i^{(j)} = A_j \cap (\Delta_i^{(j)} \times [0, \infty)), \quad i = 1, \dots, k_j, \quad j = 1, \dots, n,$$

are disjoint subsets of E . By the same argument as above, the increments

$$S_j(t_{i-1}^{(j)}, t_i^{(j)})], \quad i = 1, \dots, k_j, \quad j = 1, \dots, n,$$

are independent. We conclude that the families of the random variables

$$\left(S_j(t_i^{(j)}) \right)_{i=1,\dots,k_j} = \left(\sum_{k=1}^i S_j(t_{k-1}^{(j)}, t_k^{(j)}) \right)_{i=1,\dots,k_j}, \quad j = 1, \dots, n,$$

are mutually independent: for each j , the $S_j(t_i^{(j)})$'s are constructed from increments which are mutually independent of the increments of S_k , $k \neq j$. \square

Comments

In Chapter 8 we reconsider the topic of this section. Based on the general point process theory of Chapter 7 we will give a much more elegant approach to the decomposition of the time and claim size space. There we will learn that the total claim amount corresponding to a set A in the time and claim size space can be written as a Poisson integral $\int_A f dN$ for some function f where N is a suitable Poisson random measure. For disjoint sets A_1, \dots, A_m the Poisson integrals $\int_{A_i} f dN$, $i = 1, \dots, m$, are mutually independent due to the mutual independence of the Poisson variables $N(A_1), \dots, N(A_m)$. In the context of the theory of Poisson random measures and Poisson integrals the results of this section will be straightforward and slightly tedious calculations (as in the present section) can be avoided. Point process techniques will give us more insight into the probabilistic structure of claim numbers and total claim amounts arising from different parts of the time and claim size space. Among others, we will be able to handle problems of delay in reporting and settlement of claims, including Incurred But Not Reported (IBNR) and Reported But Not Settled (RBNS) claims.

3.3.3 An Exact Numerical Procedure for Calculating the Total Claim Amount Distribution

In this section we consider one particular exact *numerical* technique which has become popular in insurance practice. As in Section 3.3.1, we consider $S(t)$ for fixed t , and therefore we suppress the dependence of $S(t)$ and $N(t)$ on t , i.e., we write

$$S = \sum_{i=1}^N X_i$$

for an integer-valued random variable N , independent of the iid claim size sequence (X_i) . We also write

$$S_0 = 0, \quad S_n = X_1 + \cdots + X_n, \quad n \geq 1,$$

for the partial sum process (random walk) generated by the claim sizes X_i .

The distribution function of S is given by

$$P(S \leq x) = E[P(S \leq x | N)] = \sum_{n=0}^{\infty} P(S_n \leq x) P(N = n).$$

From this formula we see that the total claim amount S has quite a complicated structure: even if we knew the probabilities $P(N = n)$ and the distribution of X_i , we would have to calculate the distribution functions of all partial sums S_n . This mission is impossible, in general. In general, we can say little about the exact distribution of S , and so one is forced to use Monte Carlo or numerical techniques for approximating the total claim amount distribution.

The numerical method we focus on yields the *exact* distribution of the total claim amount S . This procedure is often referred to as *Panjer recursion*, since its basic idea goes back to Harry Panjer [115]. The method is restricted to claim size distributions with support on a lattice (such as the integers) and to a limited class of claim number distributions. By now, high speed computers with a huge memory allow for efficient alternative Monte Carlo and numerical procedures in more general situations.

We start by giving the basic assumptions under which the method works.

- (1) The claim sizes X_i assume values in $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.
- (2) The claim number N has distribution of type

$$q_n = P(N = n) = \left(a + \frac{b}{n} \right) q_{n-1}, \quad n = 1, 2, \dots,$$

for some $a, b \in \mathbb{R}$.

Condition (1) is slightly more general than it seems. Alternatively, one could assume that X_i assumes values in the lattice $d\mathbb{N}_0$ for some $d > 0$. Indeed, we then have $S = d \sum_{i=1}^N (X_i/d)$, and the random variables X_i/d assume values in \mathbb{N}_0 .

Condition (1) rules out all continuous claim size distributions, in particular, those with a density. One might argue that this is not really a restriction since

- (a) every continuous claim size distribution on $[0, \infty)$ can be approximated by a lattice distribution arbitrarily closely (for example, in the sense of uniform or total variation distance) if one chooses the span of the lattice sufficiently small,
- (b) all real-life claim sizes are expressed in terms of prices which, necessarily, take values on a lattice.

Note, however, that fact (a) does not give any information about the goodness of the approximation to the distribution of S , if the continuous claim size distribution is approximated by a distribution on a lattice. As regards (b), observe that all claim size distributions which have been relevant in the history

of insurance mathematics (see Tables 3.2.17 and 3.2.19) have a density and would therefore fall outside the considerations of the present section.

Condition (2) is often referred to as *(a, b)-condition*. It is not difficult to verify that three standard claim number distributions satisfy this condition:

- (a) The Poisson $\text{Pois}(\lambda)$ distribution⁵ with $a = 0, b = \lambda \geq 0$. In this case one obtains the (a, b) -region

$$R_{\text{Pois}} = \{(a, b) : a = 0, b \geq 0\}.$$

- (b) The binomial $\text{Bin}(n, p)$ distribution⁶ with $a = -p/(1-p) < 0, b = -a(n+1), n \geq 0$. In this case one obtains the (a, b) -region

$$R_{\text{Bin}} = \{(a, b) : a < 0, b = -a(n+1) \text{ for some integer } n \geq 0\}.$$

- (c) The negative binomial distribution with parameters (p, v) , see Example 2.3.2, with $0 < a = 1 - p < 1, b = (1 - p)(v - 1)$ and $a + b > 0$. In this case one obtains the (a, b) -region

$$R_{\text{Negbin}} = \{(a, b) : 0 < a < 1, a + b > 0\}.$$

These three distributions are the only distributions on \mathbb{N}_0 satisfying the (a, b) -condition. In particular, only for the (a, b) -parameter regions indicated above the (a, b) -condition yields genuine distributions (q_n) on \mathbb{N}_0 . The verification of these statements is left as an exercise; see Exercise 7 on p. 140.

Now we formulate the *Panjer recursion scheme*.

Theorem 3.3.9 (Panjer recursion scheme)

Assume conditions (1) and (2) on the distributions of X_i and N . Then the probabilities $p_n = P(S = n)$ can be calculated recursively as follows:

$$p_0 = \begin{cases} q_0 & \text{if } P(X_1 = 0) = 0, \\ E([P(X_1 = 0)]^N) & \text{otherwise.} \end{cases}$$

$$p_n = \frac{1}{1 - a P(X_1 = 0)} \sum_{i=1}^n \left(a + \frac{bi}{n} \right) P(X_1 = i) p_{n-i}, \quad n \geq 1.$$

Since the parameter a is necessarily less than 1, all formulae for p_n are well-defined.

Proof. We start with

$$p_0 = P(N = 0) + P(S = 0, N > 0).$$

The latter relation equals q_0 if $P(X_1 = 0) = 0$. Otherwise,

⁵ The case $\lambda = 0$ corresponds to the distribution of $N = 0$.

⁶ The case $n = 0$ corresponds to the distribution of $N = 0$.

$$\begin{aligned}
p_0 &= q_0 + \sum_{i=1}^{\infty} P(X_1 = 0, \dots, X_i = 0) P(N = i) \\
&= q_0 + \sum_{i=1}^{\infty} [P(X_1 = 0)]^i P(N = i) \\
&= E([P(X_1 = 0)]^N).
\end{aligned}$$

Now we turn to the case p_n , $n \geq 1$. A conditioning argument and the (a, b) -condition yield

$$p_n = \sum_{i=1}^{\infty} P(S_i = n) q_i = \sum_{i=1}^{\infty} P(S_i = n) \left(a + \frac{b}{i} \right) q_{i-1}. \quad (3.3.35)$$

Notice that

$$\begin{aligned}
E \left(a + \frac{b X_1}{n} \mid S_i = n \right) &= E \left(a + \frac{b X_1}{X_1 + \dots + X_i} \mid S_i = n \right) \\
&= a + \frac{b}{i},
\end{aligned} \quad (3.3.36)$$

since by the iid property of the X_i 's

$$1 = E \left(\frac{S_i}{S_i} \mid S_i \right) = \sum_{k=1}^i E \left(\frac{X_k}{S_i} \mid S_i \right) = i E \left(\frac{X_1}{S_i} \mid S_i \right).$$

We also observe that

$$\begin{aligned}
E \left(a + \frac{b X_1}{n} \mid S_i = n \right) &= \sum_{k=0}^n \left(a + \frac{b k}{n} \right) P(X_1 = k \mid S_i = n) \\
&= \sum_{k=0}^n \left(a + \frac{b k}{n} \right) \frac{P(X_1 = k, S_i - X_1 = n - k)}{P(S_i = n)} \\
&= \sum_{k=0}^n \left(a + \frac{b k}{n} \right) \frac{P(X_1 = k) P(S_{i-1} = n - k)}{P(S_i = n)}. \quad (3.3.37)
\end{aligned}$$

Substitute (3.3.36) and (3.3.37) into (3.3.35) and interchange the order of summation:

$$\begin{aligned}
p_n &= \sum_{i=1}^{\infty} \sum_{k=0}^n \left(a + \frac{bk}{n} \right) P(X_1 = k) P(S_{i-1} = n - k) q_{i-1} \\
&= \sum_{k=0}^n \left(a + \frac{bk}{n} \right) P(X_1 = k) \left[\sum_{i=1}^{\infty} P(S_{i-1} = n - k) q_{i-1} \right] \\
&= \sum_{k=0}^n \left(a + \frac{bk}{n} \right) P(X_1 = k) P(S = n - k) \\
&= \sum_{k=0}^n \left(a + \frac{bk}{n} \right) P(X_1 = k) p_{n-k}.
\end{aligned}$$

Thus we finally obtain

$$p_n = a P(X_1 = 0) p_n + \sum_{k=1}^n \left(a + \frac{bk}{n} \right) P(X_1 = k) p_{n-k},$$

which gives the final result for p_n . \square

Example 3.3.10 (Stop-loss reinsurance contract)

We consider a so-called *stop-loss* reinsurance contract with retention level s ; see also Section 3.4. This means that the reinsurer covers the excess $(S - s)_+$ of the total claim amount S over the threshold s . Suppose the company is interested in its net premium, i.e., the expected loss:

$$p(s) = E(S - s)_+ = \int_s^{\infty} P(S > x) dx.$$

Now assume that S is integer-valued and $s \in \mathbb{N}_0$. Then

$$p(s) = \sum_{k=s}^{\infty} P(S > k) = p(s-1) - P(S > s-1).$$

This yields a recursive relation for $p(s)$:

$$p(s) = p(s-1) - [1 - P(S \leq s-1)].$$

The probability $P(S \leq s-1) = \sum_{i=0}^{s-1} p_i$ can be calculated by Panjer recursion from p_0, \dots, p_{s-1} . Now, starting with the initial value $p(0) = ES = EN EX_1$, we have a recursive scheme for calculating the net premium of a stop-loss contract. \square

Comments

Papers on extensions of Panjer's recursion have frequently appeared in the journal *ASTIN Bulletin*. The interested reader is referred, for example, to

Sundt [143] or Hess et al. [70]. The book by Kaas et al. [77] contains a variety of numerical methods for the approximation of the total claim amount distribution and examples illustrating them. See also the book by Willmot and Lin [147] on approximations to compound distributions. The monographs by Asmussen [5] and Rolski et al. [127] contain chapters about the approximation of the total claim amount distribution.

The following papers on the computation of compound sum distributions can be highly recommended: Grübel and Hermesmeier [62, 63] and Embrechts et al. [45]. These papers discuss the use of transform methods such as the Fast Fourier Transform (FFT) for computing the distribution of compound sums as well as the discretization error one encounters when a claim size distribution is replaced by a distribution on a lattice. Embrechts et al. [45] give some basic theoretical results. Grübel and Hermesmeier [62] discuss the so-called *aliasing error* which occurs in transform methods. In recursion and transform methods one has to truncate the calculation at a level n , say. This means that one calculates a finite number of probabilities p_0, p_1, \dots, p_n , where $p_k = P(S = k)$. With recursion methods one can calculate these probabilities *in principle without error*.⁷ In transform methods an additional aliasing error is introduced which is essentially a wraparound effect due to the replacement of the usual summation of the integers by summation modulo the truncation point n . However, it is shown in [62] that the complexity of the FFT method is of the order $n \log n$, i.e., one needs an operation count (number of multiplications) of this order. Recursion methods require an operation count of the order n^2 . With respect to this criterion, transform methods clearly outperform recursion methods. Grübel and Hermesmeier [63] also suggest an extrapolation method in order to reduce the discretization error when continuous distributions are replaced by distributions on a lattice, and they also give bounds for the discretization error.

3.3.4 Approximation to the Distribution of the Total Claim Amount Using the Central Limit Theorem

In this section we consider some approximation techniques for the total claim amount based on the central limit theorem. This is in contrast to Section 3.3.3, where one could determine the exact probabilities $P(S(t) = n)$ for integer-valued $S(t)$ and distributions of $N(t)$ which are in the (a, b) -class. The latter two restrictions are not needed in this section.

In our notation we switch back to the time dependent total claim amount process $S = (S(t))_{t \geq 0}$. Throughout we assume the renewal model

$$S(t) = \sum_{i=1}^{N(t)} X_i, \quad t \geq 0,$$

⁷ There is, of course, an error one encounters from floating point representations of the numbers by the computer.

where the iid sequence (X_i) of positive claim sizes is independent of the renewal process $N = (N(t))_{t \geq 0}$ with arrival times $0 < T_1 < T_2 < \dots$; see Section 2.2. Denoting the iid positive inter-arrival times as usual by $W_n = T_n - T_{n-1}$ and $T_0 = 0$, we learned in Theorem 3.1.5 about the central limit theorem for S : if $\text{var}(W_1) < \infty$ and $\text{var}(X_1) < \infty$, then

$$\sup_{x \in \mathbb{R}} \left| P \left(\frac{S(t) - ES(t)}{\sqrt{\text{var}(S(t))}} \leq x \right) - \Phi(x) \right| \quad (3.3.38)$$

$$= \sup_{y \in \mathbb{R}} \left| P(S(t) \leq y) - \Phi((y - ES(t))/\sqrt{\text{var}(S(t))}) \right| \rightarrow 0, \quad (3.3.39)$$

where Φ is the distribution function of the standard normal $N(0, 1)$ distribution. As in classical statistics, where one is interested in the construction of asymptotic confidence bands for estimators and in hypothesis testing, one could take this central limit theorem as justification for replacing the distribution of $S(t)$ by the normal distribution with mean $ES(t)$ and variance $\text{var}(S(t))$: for large t ,

$$P(S(t) \leq y) \approx \Phi((y - ES(t))/\sqrt{\text{var}(S(t))}). \quad (3.3.40)$$

Then, for example,

$$P \left(S(t) \in [ES(t) - 1.96 \sqrt{\text{var}(S(t))}, ES(t) + 1.96 \sqrt{\text{var}(S(t))}] \right) \approx 0.95.$$

Relation (3.3.39) is a uniform convergence result, but it does not tell us anything about the error we encounter in (3.3.40). Moreover, when we deal with heavy-tailed claim size distributions the probability $P(S(t) > y)$ can be non-negligible even for large values of y and fixed t ; see Example 3.3.12 below. The normal approximation to the tail probabilities $P(S(t) > y)$ and $P(S(t) \leq -y)$ for large y is not satisfactory (also not in the light-tailed case).

Improvements on the central limit theorem (3.3.39) have been considered starting in the 1950s. We refer to Petrov's classical monograph [116] which gives a very good overview for these kinds of results. It covers, among other things, *rates of convergence in the central limit theorem* for the partial sums

$$S_0 = 0, \quad S_n = X_1 + \dots + X_n, \quad n \geq 1,$$

and *asymptotic expansions* for the distribution function of S_n . In the latter case, one adds more terms to $\Phi(x)$ which depend on certain moments of X_i . This construction can be shown to improve upon the normal approximation (3.3.38) substantially. The monograph by Hall [67] deals with asymptotic expansions with applications to statistics. Jensen's [76] book gives very precise approximations to probabilities of rare events (such as $P(S(t) > y)$ for values y larger than $ES(t)$), extending asymptotic expansions to *saddlepoint approximations*. Asymptotic expansions have also been derived for the distribution of

the random sums $S(t)$; Chossy and Rapp [33] consider them with applications to insurance.

A rather precise tool for measuring the distance between Φ and the distribution of S_n is the so-called *Berry-Esséen inequality*. It says that

$$\sup_x (1 + |x|^3) \left| P\left(\frac{S_n - n EX_1}{\sqrt{n \text{var}(X_1)}} \leq x\right) - \Phi(x) \right| \leq \frac{c}{\sqrt{n}} \frac{E|X_1 - EX_1|^3}{(\sqrt{\text{var}(X_1)})^3}, \quad (3.3.41)$$

where $c = 0.7655 + 8(1 + e) = 30.51\dots$ is a universal constant. Here we assumed that $E|X_1|^3 < \infty$; see Petrov [116]. The constant c can be replaced by 0.7655 if one cancels $1 + |x|^3$ on the left-hand side of (3.3.41).

Relation (3.3.41) is rather precise for various discrete distributions. For example, one can show⁸ that one can derive a lower bound in (3.3.41) of the order $1/\sqrt{n}$ for iid Bernoulli random variables X_i with $P(X_i = \pm 1) = 0.5$. For distributions with a smooth density the estimate (3.3.41) is quite pessimistic, i.e., the right-hand side can often be replaced by better bounds. However, inequality (3.3.41) should be a warning to anyone who uses the central limit theorem without thinking about the error he/she encounters when the distribution of S_n is replaced by a normal distribution. It tells us that we need a sufficiently high sample size n to enable us to work with the normal distribution. But we also have to take into account the ratio $E|X_1 - EX_1|^3/(\sqrt{\text{var}(X_1)})^3$, which depends on the individual distribution of X_1 .

It is not possible to replace S_n by the total claim amount $S(t)$ without further work. However, we obtain a bound in the central limit theorem for $S(t)$, conditionally on $N(t) = n(t)$. Indeed, for a realization $n(t) = N(t, \omega)$ of the claim number process N we immediately have from (3.3.41) that for every $x \in \mathbb{R}$,

$$\begin{aligned} & \left| P\left(\frac{S(t) - n(t)EX_1}{\sqrt{n(t)\text{var}(X_1)}} \leq x \mid N(t) = n(t)\right) - \Phi(x) \right| \\ & \leq \frac{c}{\sqrt{n(t)}} \frac{1}{1 + |x|^3} \frac{E|X_1 - EX_1|^3}{(\sqrt{\text{var}(X_1)})^3}. \end{aligned} \quad (3.3.42)$$

Since $n(t) = N(t, \omega) \xrightarrow{\text{a.s.}} \infty$ in the renewal model, this error bound can give some justification for applying the central limit theorem to the distribution of $S(t)$, conditionally on $N(t)$, although it does not solve the original problem for the unconditional distribution of $S(t)$. In a portfolio with a large number $n(t)$ of claims, relation (3.3.42) tells us that the central limit theorem certainly gives a good approximation in the center of the distribution of $S(t)$ around $ES(t)$, but it shows how dangerous it is to use the central limit theorem when it comes to considering probabilities

⁸ Calculate the asymptotic order of the probability $P(S_{2n} = 0)$.

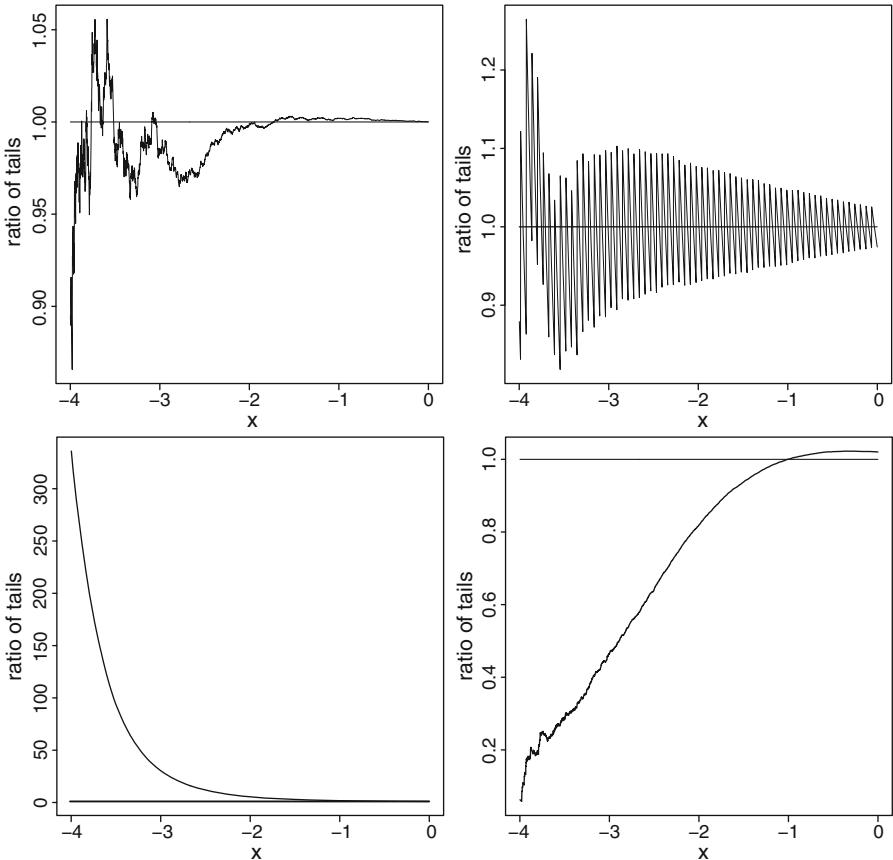


Figure 3.3.11 A plot of the tail ratio $r_n(x) = P((S_n - ES_n)/\sqrt{\text{var}(S_n)} \leq -x)/\Phi(-x)$, $x \geq 0$, for the partial sums $S_n = X_1 + \dots + X_n$ of iid random variables X_i . Here Φ stands for the standard normal distribution function. The order of magnitude of the deviation $r_n(x)$ from the constant 1 (indicated by the straight line) is a measure of the quality of the validity of the central limit theorem in the left tail of the distribution function of S_n . Top left: $X_1 \sim U(0, 1)$, $n = 100$. The central limit theorem gives a good approximation for $x \in [-2, 0]$, but is rather poor outside this area. Top right: $X_1 \sim \text{Bin}(5, 0.5)$, $n = 200$. The approximation by the central limit theorem is poor everywhere. Bottom left: X_1 has a student t_3 -distribution, $n = 2000$. This distribution has infinite 3rd moment and it is subexponential; cf. also Example 3.3.12. The approximation outside the area $x \in [-1, 0]$ is very poor due to very heavy tails of the t_3 -distribution. Bottom right: $X_1 \sim \text{Exp}(1)$, $n = 200$. Although the tail of this distribution is much lighter than for the t_3 -distribution the approximation below $x = -1$ is not satisfactory.

$$P(S(t) > y \mid N(t) = n(t)) = P\left(\frac{S(t) - n(t) EX_1}{\sqrt{n(t) \text{ var}(X_1)}} > \frac{y - n(t) EX_1}{\sqrt{n(t) \text{ var}(X_1)}}\right)$$

for large y . The normal approximation is poor if $x = (y - n(t) EX_1)/\sqrt{n(t) \text{ var}(X_1)}$ is too large. In particular, it can happen that the error bound on the right-hand side of (3.3.42) is larger than the approximated probability $1 - \Phi(x)$.

Example 3.3.12 (The tail of the distribution of $S(t)$ for subexponential claim sizes)

In this example we want to contrast the approximation of $P(S(t) > x)$ for $t \rightarrow \infty$ and fixed x , as provided by the central limit theorem, with an approximation for fixed t and large x . We assume the Cramér-Lundberg model and consider subexponential claim sizes. Therefore recall from p. 103 the definition of a subexponential distribution: writing $S_0 = 0$ and $S_n = X_1 + \dots + X_n$ for the partial sums and $M_n = \max(X_1, \dots, X_n)$ for the partial maxima of the iid claim size sequence (X_n) , the distribution of X_1 and its distribution function F_{X_1} are said to be subexponential if

For every $n \geq 2$: $P(S_n > x) = P(M_n > x)(1 + o(1)) = n \bar{F}_{X_1}(x)(1 + o(1))$,

as $x \rightarrow \infty$. We will show that a similar relation holds if the partial sums S_n are replaced by the random sums $S(t)$.

We have, by conditioning on $N(t)$,

$$\frac{P(S(t) > x)}{\bar{F}_{X_1}(x)} = \sum_{n=0}^{\infty} P(N(t) = n) \frac{P(S_n > x)}{\bar{F}_{X_1}(x)} = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \frac{P(S_n > x)}{\bar{F}_{X_1}(x)}.$$

If we interchange the limit as $x \rightarrow \infty$ and the infinite series on the right-hand side, the subexponential property of F_{X_1} yields

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{P(S(t) > x)}{\bar{F}_{X_1}(x)} &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \lim_{x \rightarrow \infty} \frac{P(S_n > x)}{\bar{F}_{X_1}(x)} \\ &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} n = EN(t) = \lambda t. \end{aligned}$$

This is the analog of the subexponential property for the random sum $S(t)$. It shows that the central limit theorem is not a good guide in the tail of the distribution of $S(t)$; in this part of the distribution the heavy right tail of the claim size distribution determines the decay which is much slower than for the tail $\bar{\Phi}$ of the standard normal distribution.

We still have to justify the interchange of the limit as $x \rightarrow \infty$ and the infinite series $\sum_{n=0}^{\infty}$. We apply a domination argument. Namely, if we can find a sequence (f_n) such that

$$\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} f_n < \infty \quad \text{and} \quad \frac{P(S_n > x)}{\bar{F}_{X_1}(x)} \leq f_n \quad \text{for all } x > 0, \quad (3.3.43)$$

then we are allowed to interchange these limits by virtue of the Lebesgue dominated convergence theorem; see Williams [145]. Recall from Lemma 3.2.24(3) that for any $\varepsilon > 0$ we can find a constant K such that

$$\frac{P(S_n > x)}{\bar{F}_{X_1}(x)} \leq K(1 + \varepsilon)^n, \quad \text{for all } n \geq 1.$$

With the choice $f_n = K(1 + \varepsilon)^n$ for any $\varepsilon > 0$, it is not difficult to see that (3.3.43) is satisfied. \square

Comments

The aim of this section was to show that an unsophisticated use of the normal approximation to the distribution of the total claim amount should be avoided, typically when one is interested in the probability of rare events, for example of $\{S(t) > x\}$ for x exceeding the expected claim amount $ES(t)$. In this case, other tools (asymptotic expansions for the distribution of $S(t)$, large deviation probabilities for the very large values x , saddlepoint approximations) can be used as alternatives. We refer to the literature mentioned in the text and to Embrechts et al. [46], Chapter 2, to get an impression of the complexity of the problem.

3.3.5 Approximation to the Distribution of the Total Claim Amount by Monte Carlo Techniques

One way out of the situation we encountered in Section 3.3.4 is to use the power and memory of modern computers to approximate the distribution of $S(t)$. For example, if we knew the distributions of the claim number $N(t)$ and of the claim sizes X_i , we could simulate an iid sample N_1, \dots, N_m from the distribution of $N(t)$. Then we could draw iid samples

$$X_1^{(1)}, \dots, X_{N_1}^{(1)}, \dots, X_1^{(m)}, \dots, X_{N_m}^{(m)}$$

from the distribution of X_1 and calculate iid copies of $S(t)$:

$$S_1 = \sum_{i=1}^{N_1} X_i^{(1)}, \dots, S_m = \sum_{i=1}^{N_m} X_i^{(m)}.$$

The probability $P(S(t) \in A)$ for some Borel set A could be approximated by virtue of the strong law of large numbers:

$$\hat{p}_m = \frac{1}{m} \sum_{i=1}^m I_A(S_i) \xrightarrow{\text{a.s.}} P(S(t) \in A) = p = 1 - q \quad \text{as } m \rightarrow \infty.$$

Notice that $m\hat{p}_m \sim \text{Bin}(m, p)$. The approximation of p by the relative frequencies \hat{p}_m of the event A is called (crude) *Monte Carlo simulation*.

The rate of approximation could be judged by applying the central limit theorem with Berry-Esséen specification, see (3.3.41):

$$\sup_x (1 + |x|^3) \left| P\left(\frac{\hat{p}_m - p}{\sqrt{pq/m}} \leq x\right) - \Phi(x) \right| \leq c \frac{p^3 q + (1-p)^3 p}{(\sqrt{pq})^3 \sqrt{m}} = c \frac{p^2 + q^2}{\sqrt{mpq}}. \quad (3.3.44)$$

We mentioned in the previous section that this bound is quite precise for a binomial distribution, i.e., for sums of Bernoulli random variables. This is encouraging, but for small probabilities p the Monte Carlo method is problematic. For example, suppose you want to approximate the probability $p = 10^{-k}$ for some $k \geq 1$. Then the rate on the right-hand side is of the order $10^{k/2}/\sqrt{m}$. This means you would need sample sizes m much larger than 10^k in order to make the right-hand side smaller than 1, and if one is interested in approximating small values of $\Phi(x)$ or $1 - \Phi(x)$, the sample sizes have to be chosen even larger. This is particularly unpleasant if one needs the whole distribution function of $S(t)$, i.e., if one has to calculate many probabilities of type $P(S(t) \leq y)$.

If one needs to approximate probabilities of very small order, say $p = 10^{-k}$ for some $k \geq 1$, then the crude Monte Carlo method does not work. This can be seen from the following argument based on the central limit theorem (3.3.44). The value p falls with 95% probability into the asymptotic confidence interval given by

$$[\hat{p}_m - 1.96 \sqrt{pq/m}; \hat{p}_m + 1.96 \sqrt{pq/m}].$$

For practical purposes one would have to replace p in the latter relation by its estimator \hat{p}_m . For small p this bound is inaccurate even if m is relatively large. One essentially has to compare the orders of magnitude of p and $1.96\sqrt{pq/m}$:

$$\frac{1.96\sqrt{pq/m}}{p} = \frac{1.96\sqrt{q}}{\sqrt{m}p} \approx 10^{k/2} \frac{1.96}{\sqrt{m}}.$$

This means we need sample sizes m much larger than 10^k in order to get a satisfactory approximation for p .

The crude Monte Carlo approximation can be significantly improved for small probabilities p and moderate sample sizes m . Over the last 30 years special techniques such as *importance sampling* have been developed and run under the name of *rare event simulation*; see Asmussen and Glynn [8] and Glasserman [56]. In an insurance context, rare events such as the WTC disaster or windstorm claims can have substantial impact on the insurance business; see Table 3.2.18. Therefore it is important to know that there are various techniques available which allow one to approximate such probabilities efficiently.

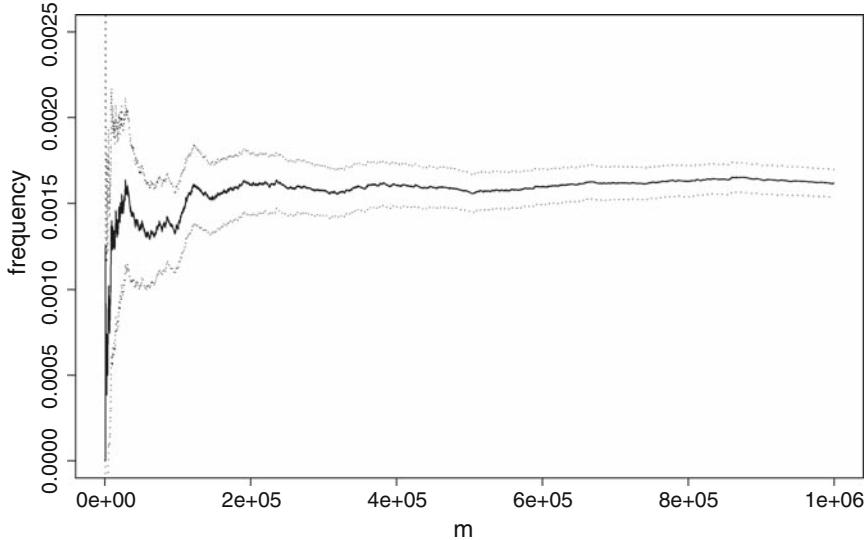


Figure 3.3.13 Crude Monte Carlo simulation for the probability $p = P(S(t) > ES(t) + 3.5\sqrt{\text{var}(S(t))})$, where $S(t)$ is the total claim amount in the Cramér-Lundberg model with Poisson intensity $\lambda = 0.5$ and Pareto distributed claim sizes with tail parameter $\alpha = 3$, scaled to variance 1. We have chosen $t = 360$ corresponding to one year. The intensity $\lambda = 0.5$ corresponds to expected inter-arrival times of 2 days. We plot \hat{p}_m for $m \leq 10^6$ and indicate 95% asymptotic confidence intervals prescribed by the central limit theorem. For $m = 10^6$ one has 1 618 values of $S(t)$ exceeding the threshold $ES(t) + 3.5\sqrt{\text{var}(S(t))}$, corresponding to $\hat{p}_m = 0.001618$. For $m \leq 20\ 000$ the estimates \hat{p}_m are extremely unreliable and the confidence bands are often wider than the approximated probability.

By virtue of Poisson's limit theorem, rare events are more naturally approximated by Poisson probabilities. Approximations to the binomial distribution with small success probability by the Poisson distribution have been studied for a long time and optimal rates of this approximation were derived; see for example Barbour et al. [11]. Alternatively, the Poisson approximation is an important tool for evaluating the probability of rare events in the context of catastrophic or extremal events; see Section 9.2 for a short introduction to the world of extremes.

In the rest of this section we consider a statistical simulation technique which has become quite popular among statisticians and users of statistics over the last 25 years: Efron's [43] *bootstrap*. In contrast to the approximation techniques considered so far it does a priori not require any information about the distribution of the X_i 's; all it uses is the information contained in the data available. In what follows, we focus on the case of an iid claim size sample

X_1, \dots, X_n with common distribution function F and empirical distribution function

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n I_{(-\infty, x]}(X_i), \quad x \in \mathbb{R}.$$

Then the Glivenko-Cantelli result (see Billingsley [18]) ensures that

$$\sup_x |F_n(x) - F(x)| \xrightarrow{\text{a.s.}} 0.$$

The latter relation has often been taken as a justification for replacing quantities depending on the *unknown* distribution function F by the same quantities depending on the *known* distribution function F_n . For example, in Section 3.2.3 we constructed the empirical mean excess function from the mean excess function in this way. The bootstrap extends this idea substantially: it suggests to *sample* from the empirical distribution function and to simulate pseudo-samples of iid random variables with distribution function F_n .

We explain the basic ideas of this approach. Let

$$x_1 = X_1(\omega), \dots, x_n = X_n(\omega)$$

be the values of an observed iid sample which we consider as fixed in the sequel, i.e., the empirical distribution function F_n is a given discrete distribution function with equal probability at the x_i 's. Suppose we want to approximate the distribution of a function $\theta_n = \theta_n(X_1, \dots, X_n)$ of the data, for example of the sample mean

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

The bootstrap is then given by the following algorithm.

- (a) Draw with replacement from the distribution function F_n the iid realizations

$$X_1^*(1), \dots, X_n^*(1), \dots, X_1^*(B), \dots, X_n^*(B)$$

for some large number B . *In principle*, using computer power we could make B *arbitrarily large*.

- (b) Calculate the iid sample

$$\theta_n^*(1) = \theta_n(X_1^*(1), \dots, X_n^*(1)), \dots, \theta_n^*(B) = \theta_n(X_1^*(B), \dots, X_n^*(B)).$$

In what follows, we write $X_i^* = X_i^*(1)$ and $\theta_n^* = \theta_n^*(1)$.

- (c) Approximate the distribution of θ_n^* and its characteristics such as moments, quantiles, etc., either by direct calculation or by using the strong law of large numbers.

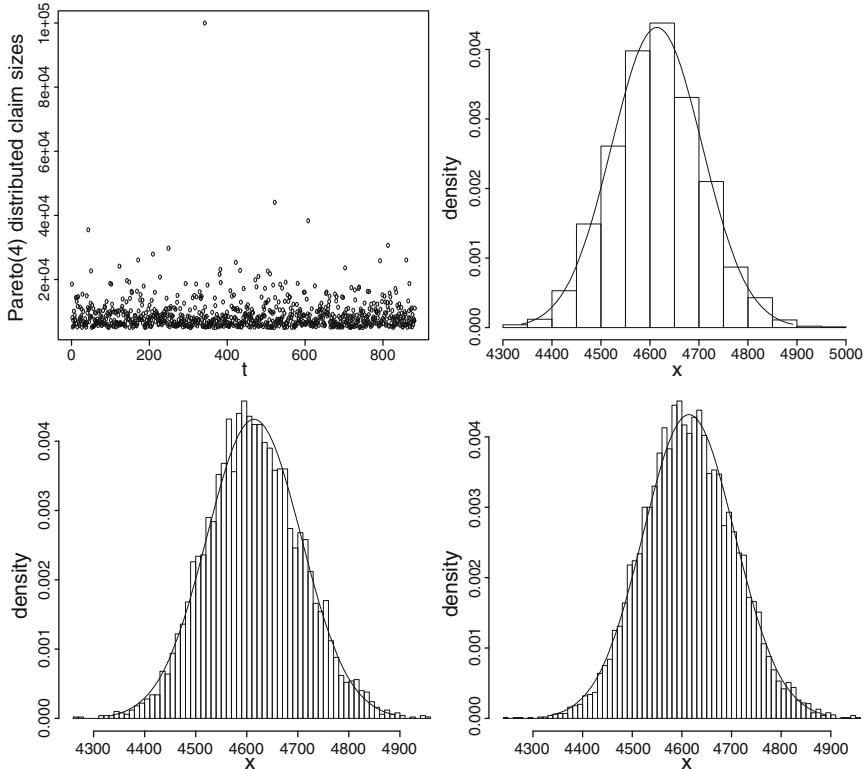


Figure 3.3.14 The bootstrap for the sample mean of 3 000 Pareto distributed claim sizes with tail index $\alpha = 4$; see Table 3.2.19. The largest value is 10 000 \$US. The claim sizes X_n which exceed the threshold of 5 000 \$US are shown in the top left graph. The top right, bottom left, bottom right graphs show histograms of the bootstrap sample mean with bootstrap sample size $B = 2\,000$, $B = 5\,000$ and $B = 10\,000$, respectively. For comparison we draw the normal density curve with the mean and variance of the data in the histograms.

We illustrate the meaning of this algorithm for the sample mean.

Example 3.3.15 (The bootstrap sample mean)

The sample mean $\theta_n = \bar{X}_n$ is an unbiased estimator of the expectation $\theta = EX_1$, provided the latter expectation exists and is finite. The *bootstrap sample mean* is the quantity

$$\bar{X}_n^* = \frac{1}{n} \sum_{i=1}^n X_i^*.$$

Since the (conditionally) iid X_i^* 's have the discrete distribution function F_n ,

$$\begin{aligned} E^*(X_1^*) &= E_{F_n}(X_1^*) = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}_n, \\ \text{var}^*(X_1^*) &= \text{var}_{F_n}(X_1^*) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x}_n)^2 = s_n^2. \end{aligned}$$

Now using the (conditional) independence of the X_i^* 's, we obtain

$$\begin{aligned} E^*(\bar{X}_n^*) &= \frac{1}{n} \sum_{i=1}^n E^*(X_i^*) = E^*(X_1^*) = \bar{x}_n, \\ \text{var}^*(\bar{X}_n^*) &= \frac{1}{n^2} \sum_{i=1}^n \text{var}^*(X_i^*) = n^{-1} \text{var}^*(X_1^*) = n^{-1} s_n^2. \end{aligned}$$

For more complicated functionals of the data it is in general not possible to get such simple expressions as for \bar{X}_n^* . For example, suppose you want to calculate the distribution function of \bar{X}_n^* at x :

$$P^*(\bar{X}_n^* \leq x) = E_{F_n} \left(I_{(-\infty, x]}(\bar{X}_n^*) \right) = \frac{1}{n^n} \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n I_{(-\infty, x]} \left(\frac{1}{n} \sum_{j=1}^n x_{i_j} \right).$$

This means that, in principle, one would have to evaluate n^n terms and sum them up. Even with modern computers and for small sample sizes such as $n = 10$ this would be a too difficult computational problem. On the other hand, the Glivenko-Cantelli result allows one to approximate $P^*(\bar{X}_n^* \leq x)$ arbitrarily closely by choosing a large bootstrap sample size B :

$$\sup_x \left| \frac{1}{B} \sum_{i=1}^B I_{(-\infty, x]}(\bar{X}_n^*(i)) - P^*(\bar{X}_n^* \leq x) \right| \rightarrow 0 \quad \text{as } B \rightarrow \infty,$$

with probability 1, where this probability refers to a probability measure which is constructed from F_n . In practical simulations one can make B very large. Therefore it is in general not considered a problem to approximate the distribution of functionals of X_1^*, \dots, X_n^* as accurately as one wishes. \square

The bootstrap is mostly used to approximate the distributional characteristics of functionals θ_n of the data such as the expectation, the variance and quantiles of θ_n in a rather unsophisticated way. In an insurance context, the method allows one to approximate the distribution of the aggregated claim sizes $n\bar{X}_n = X_1 + \dots + X_n$ by its bootstrap version $X_1^* + \dots + X_n^*$ or of the total claim amount $S(t)$ conditionally on the claim number $N(t)$ by approximation through the bootstrap version $X_1^* + \dots + X_{N(t)}^*$, and bootstrap

methods can be applied to calculate confidence bands for the parameters of the claim number and claim size distributions.

Thus it seems as if the bootstrap solves all statistical problems of this world without too much sophistication. This was certainly the purpose of its inventor Efron [43], see also the text by Efron and Tibshirani [44]. However, the replacement of the X_i 's with distribution function F with the corresponding bootstrap quantities X_i^* with distribution function F_n in a functional $\theta_n(X_1, \dots, X_n)$ has actually a continuity problem. This replacement does not always work even for rather simple functionals of the data; see Bickel and Freedman [16] for some counterexamples. Therefore one has to be careful; as for the crude Monte Carlo method considered above the *naive* bootstrap can one lead into the wrong direction, i.e., the bootstrap versions θ_n^* can have distributions which are far away from the distribution of θ_n . Moreover, in order to show that the bootstrap approximation “works”, i.e., it is close to the distribution of θ_n , one needs to apply asymptotic techniques for $n \rightarrow \infty$. This is slightly disappointing because the original idea of the bootstrap was to be applicable to small sample size.

As a warning we also mention that the naive bootstrap for the total claim amount does not work if one uses very heavy-tailed distributions. Then bootstrap sampling forces one to draw the largest values in the sample too often, which leads to deviations of the bootstrap distribution from the distribution of θ_n ; see Figure 3.3.16 for an illustration of this phenomenon. Moreover, the bootstrap does not solve the problem of calculating the probability of rare events such as $P(S(t) > x)$ for values x far beyond the mean $ES(t)$; see the previous discussions. Since the empirical distribution function stops increasing at the maximum of the data, the bootstrap does not extrapolate into the tails of the distribution of the X_i 's. For this purpose one has to depend on special parametric or semi-parametric methods such as those provided in extreme value theory; cf. Embrechts et al. [46], Chapter 6.

Comments

Monte Carlo simulations and the bootstrap are rather recent computer-based methods, which have an increasing appeal since the quality of the computers has enormously improved over the last 15-20 years. These methods provide an ad hoc approach to problems whose exact solution had been considered hopeless. Nevertheless, none of these methods is perfect. Pitfalls may occur even in rather simple cases. Therefore one should not use these methods without consulting the relevant literature. Often theoretical means such as the central limit theorem of Section 3.3.4 give the same or even better approximation results. Simulation should only be used if nothing else works.

The book by Efron and Tibshirani [44] is an accessible introduction to the bootstrap. Books such as Hall [67] or Mammen [104] show the limits of the method, but also require knowledge on mathematical statistics.

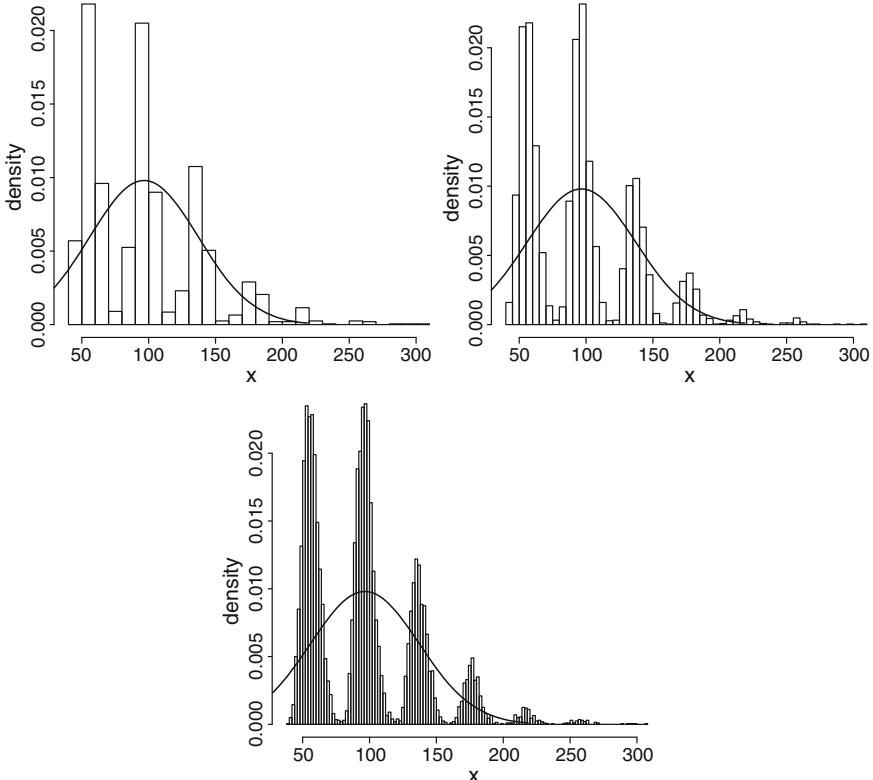


Figure 3.3.16 The bootstrap for the sample mean of 3 000 Pareto distributed claim sizes with tail index $\alpha = 1$. The graphs show histograms of the bootstrap sample mean with bootstrap sample size $B = 2\,000$ (top left), $B = 5\,000$ (top right) and $B = 10\,000$ (bottom). For comparison we draw the normal density curve with the sample mean and sample variance of the data in the histograms. It is known that the Pareto distribution with tail index $\alpha = 1$ does not satisfy the central limit theorem with normal limit distribution (e.g. [46], Chapter 2), but with a skewed Cauchy limit distribution. Therefore the misfit of the normal distribution is not surprising, but the distribution of the bootstrap sample mean is also far from the Cauchy distribution which has a unimodal density. In the case of infinite variance claim size distributions, the (naive) bootstrap does not work for the sample mean.

Asmussen and Glynn [8] and Glasserman [56] give introductions to the simulation of stochastic processes and distributions, see also Chapter X in Asmussen [5] which is devoted to simulation methodology, in particular for rare events. Glasserman's book [56] focuses on simulation techniques in the

framework of financial engineering. Survey papers about rare event simulation include Asmussen and Rubinstein [9] and Heidelberger [68]. Rare event simulation is particularly difficult when heavy-tailed distributions are involved. This is, for example, documented in Asmussen et al. [7].

Exercises

Section 3.3.1

- (1) Decomposition of the claim size space for discrete distribution.
 - (a) Let N_1, \dots, N_n be independent Poisson random variables with $N_i \sim \text{Pois}(\lambda_i)$ for some $\lambda_i > 0$, x_1, \dots, x_n be positive numbers. Show that $x_1 N_1 + \dots + x_n N_n$ has a compound Poisson distribution.
 - (b) Let $S = \sum_{k=1}^N X_k$ be compound Poisson where $N \sim \text{Pois}(\lambda)$, independent of the iid claim size sequence (X_k) and $P(X_1 = x_i) = p_i$, $i = 1, \dots, n$, for some distribution (p_i) . Show that $S \stackrel{d}{=} x_1 N_1 + \dots + x_n N_n$ for appropriate independent Poisson variables N_1, \dots, N_n .
 - (c) Assume that the iid claim sizes X_k in an insurance portfolio have distribution $P(X_k = x_i) = p_i$, $i = 1, \dots, n$. The sequence (X_k) is independent of the Poisson claim number N with parameter λ . Consider a disjoint partition A_1, \dots, A_m of the possible claim sizes $\{x_1, \dots, x_n\}$. Show that the total claim amount $S = \sum_{k=1}^N X_k$ has the same distribution as

$$\sum_{i=1}^m \sum_{k=1}^{N_i} X_k^{(i)},$$

where $N_i \sim \text{Pois}(\lambda_i)$, $\lambda_i = \lambda \sum_{k:x_k \in A_i} p_k$, are independent Poisson variables, independent of $(X_k^{(i)})$ and for each i , $X_k^{(i)}$, $k = 1, 2, \dots$, are iid with distribution $P(X_k^{(i)} = x_l) = p_l / \sum_{s:x_s \in A_i} p_s$. This means that one can split the claim sizes into distinct categories (for example one can introduce layers $A_i = (a_i, b_i]$ for the claim sizes or one can split the claims into small and large ones according as $x_i \leq u$ or $x_i > u$ for a threshold u) and consider the total claim amount from each category as a compound Poisson variable.

- (2) Consider the total claim amount $S(t) = \sum_{i=1}^{N(t)} X_i$ in the Cramér-Lundberg model for fixed t , where N is homogeneous Poisson and independent of the claim size sequence (X_i) .
 - (a) Show that

$$S(t) \stackrel{d}{=} \sum_{i=1}^{N_1(t)+N_2(t)} X_i \stackrel{d}{=} \sum_{i=1}^{N_1(t)} X_i + \sum_{i=1}^{N_2(t)} X'_i,$$

where N_1 and N_2 are independent homogeneous Poisson processes with intensities λ_1 and λ_2 , respectively, such that $\lambda_1 + \lambda_2 = \lambda$, (X'_i) is an independent copy of (X_i) , and N_1 , N_2 , (X_i) and (X'_i) are independent.

- (b) Show relation (3.3.29) by calculating the joint characteristic functions of the left- and right-hand expressions.

- (3) We consider the mixed Poisson processes $N_i(t) = \tilde{N}_i(\theta_i t)$, $t \geq 0$, $i = 1, \dots, n$. Here \tilde{N}_i are mutually independent standard homogeneous Poisson processes, θ_i are mutually independent positive mixing variables, and (\tilde{N}_i) and (θ_i) are independent. Consider the independent *compound mixed Poisson sums*

$$S_j = \sum_{i=1}^{N_j(1)} X_i^{(j)}, \quad j = 1, \dots, n,$$

where $(X_i^{(j)})$ are iid copies of a sequence (X_i) of iid positive claim sizes, independent of (N_j) . Show that $S = S_1 + \dots + S_n$ is again a compound mixed Poisson sum with representation

$$S \stackrel{d}{=} \sum_{i=1}^{\tilde{N}_1(\theta_1 + \dots + \theta_n)} X_i.$$

- (4) Let $S = \sum_{i=1}^N X_i$ be the total claim amount at a fixed time t , where the claim number N and the iid claim size sequence (X_i) are independent.
- (a) Show that the *Laplace-Stieltjes transform* of S , i.e., $\hat{f}_S(s) = m_S(-s) = Ee^{-sS}$ always exists for $s \geq 0$.
 - (b) Show that

$$P(S > x) \leq c e^{-hx} \quad \text{for all } x > 0, \text{ some } c > 0, \quad (3.3.45)$$

if $m_S(h) < \infty$ for some $h > 0$. Show that (3.3.45) implies that the moment generating function $m_S(s) = Ee^{sS}$ is finite in some neighborhood of the origin.

- (5) Recall the negative binomial distribution

$$p_k = \binom{v+k-1}{k} p^v (1-p)^k, \quad k = 0, 1, 2, \dots, \quad p \in (0, 1), \quad v > 0. \quad (3.3.46)$$

Recall from Example 2.3.2 that the negative binomial process $(N(t))_{t \geq 0}$ is a mixed standard homogeneous Poisson process with mixing variable θ with gamma $\Gamma(\gamma, \beta)$ density

$$f_\theta(x) = \frac{\beta^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\beta x}, \quad x > 0.$$

Choosing $v = \gamma$ and $p = \beta/(1+\beta)$, $N(1)$ then has distribution (3.3.46).

- (a) Use this fact to calculate the characteristic function of a negative binomial random variable with parameters p and ν .
- (b) Let $N \sim \text{Pois}(\lambda)$ be the number of accidents in a car insurance portfolio in a given period, X_i the claim size in the i th accident and assume that the claim sizes X_i are iid positive and integer-valued with distribution

$$P(X_i = k) = \frac{k^{-1} p^k}{-\log(1-p)}, \quad k = 1, 2, \dots$$

for some $p \in (0, 1)$. Verify that these probabilities define a distribution, the so-called *logarithmic distribution*. Calculate the characteristic function of

the compound Poisson variable $S = \sum_{i=1}^N X_i$. Verify that it has a negative binomial distribution with parameters $\tilde{v} = -\lambda/\log(1-p)$ and $\tilde{p} = 1-p$. Hence a random variable with a negative binomial distribution has representation as a compound Poisson sum with logarithmic claim size distribution.

- (6) A distribution F is said to be *infinitely divisible* if for every $n \geq 1$, its characteristic function ϕ can be written as a product of characteristic functions ϕ_n :

$$\phi(s) = (\phi_n(s))^n, \quad s \in \mathbb{R}.$$

In other words, for every $n \geq 1$, there exist iid random variables $Y_{n,1}, \dots, Y_{n,n}$ with common characteristic function ϕ_n such that for a random variable Y with distribution F the following identity in distribution holds:

$$Y \stackrel{d}{=} Y_{n,1} + \dots + Y_{n,n}.$$

Almost every familiar distribution with unbounded support which is used in statistics or probability theory has this property although it is often very difficult to prove this fact for concrete distributions. We refer to Lukacs [97] or Sato [132] for more information on this class of distributions. See also Chapter 10 for an introduction to Lévy processes and their relation with infinitely divisible distributions.

- (a) Show that the normal, Poisson and gamma distributions are infinitely divisible.
- (b) Show that the distribution of a compound Poisson variable is infinitely divisible.
- (c) Consider a compound Poisson process $S(t) = \sum_{i=1}^{N(t)} X_i$, $t \geq 0$, where N is a homogeneous Poisson process on $[0, \infty)$ with intensity $\lambda > 0$, independent of the iid claim sizes X_i . Show that the process S obeys the following infinite divisibility property: for every $n \geq 1$ there exist iid compound Poisson processes S_i such that $S \stackrel{d}{=} S_1 + \dots + S_n$, where $\stackrel{d}{=}$ refers to identity of the finite-dimensional distributions. Hint: Use the fact that S and S_i have independent and stationary increments.

Section 3.3.3

- (7) The (a, b) -class of distributions.
 (a) Verify the (a, b) -condition

$$q_n = P(N = n) = \left(a + \frac{b}{n} \right) q_{n-1} \quad (3.3.47)$$

for the Poisson, binomial and negative binomial claim number distributions (q_n) and appropriate choices of the parameters a, b . Determine the region R of possible (a, b) -values for these distributions.

- (b) Show that the (a, b) -condition (3.3.47) for values $(a, b) \notin R$ does not define a probability distribution (q_n) of a random variable N with values in \mathbb{N}_0 .
- (c) Show that the Poisson, binomial and negative binomial distributions are the only possible distributions on \mathbb{N}_0 satisfying an (a, b) -condition, i.e., (3.3.47) implies that (q_n) is necessarily Poisson, binomial or negative binomial, depending on the choice of $(a, b) \in R$.

Sections 3.3.4 and 3.3.5

- (8) Consider an iid sample X_1, \dots, X_n and the corresponding empirical distribution function:

$$F_n(x) = \frac{1}{n} \#\{i \leq n : X_i \leq x\}.$$

By X^* we denote any random variable with distribution function F_n , given X_1, \dots, X_n .

- (a) Calculate the expectation, the variance and the third absolute moment of X^* .
- (b) For (conditionally) iid random variables $X_i^*, i = 1, \dots, n$, with distribution function F_n calculate the mean and variance of the sample mean $\bar{X}_n^* = n^{-1} \sum_{i=1}^n X_i^*$.
- (c) Apply the strong law of large numbers to show that the limits of $E^*(\bar{X}_n^*)$ and $n \text{var}^*(\bar{X}_n^*)$ as $n \rightarrow \infty$ exist and coincide with their deterministic counterparts EX_1 and $\text{var}(X_1)$, provided the latter quantities are finite. Here E^* and var^* refer to expectation and variance with respect to the distribution function F_n of the (conditionally) iid random variables X_i^* 's.
- (d) Apply the Berry-Esséen inequality to

$$\begin{aligned} P^* \left(\frac{\sqrt{n}}{\sqrt{\text{var}^*(X_1^*)}} (\bar{X}_n^* - E^*(\bar{X}_n^*)) \leq x \right) - \Phi(x) \\ = P \left(\frac{\sqrt{n}}{\sqrt{\text{var}^*(X_1^*)}} (\bar{X}_n^* - E^*(\bar{X}_n^*)) \leq x \mid X_1, \dots, X_n \right) - \Phi(x), \end{aligned}$$

where Φ is the standard normal distribution function and show that the (conditional) central limit theorem applies⁹ to (X_i^*) if $E|X_1|^3 < \infty$, i.e., the above differences converge to 0 with probability 1.

Hint: It is convenient to use the elementary inequality

$$|x + y|^3 \leq (2 \max(|x|, |y|))^3 = 8 \max(|x|^3, |y|^3) \leq 8(|x|^3 + |y|^3), \quad x, y \in \mathbb{R}.$$

- (9) Let X_1, X_2, \dots be an iid sequence with finite variance (without loss of generality assume $\text{var}(X_1) = 1$) and mean zero. Then the central limit theorem and the continuous mapping theorem (see Billingsley [17]) yield

$$T_n = n(\bar{X}_n)^2 = \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \right)^2 \xrightarrow{d} Y^2,$$

where Y has a standard normal distribution. The naive bootstrap version of T_n is given by

$$T_n^* = n(\bar{X}_n^*)^2 = \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i^* \right)^2,$$

where (X_i^*) is an iid sequence with common empirical distribution function F_n based on the sample X_1, \dots, X_n , i.e., (X_i^*) are iid, conditionally on X_1, \dots, X_n .

⁹ As a matter of fact, the central limit theorem applies to (X_i^*) under the weaker assumption $\text{var}(X_1) < \infty$; see Bickel and Freedman [16].

- (a) Verify that the bootstrap does not work for T_n^* by showing that (T_n^*) has no limit distribution with probability 1. In particular, show that the following limit relation does not hold as $n \rightarrow \infty$:

$$\begin{aligned} P^*(T_n^* \leq x) &= P(T_n^* \leq x \mid X_1, \dots, X_n) \\ &\rightarrow P(Y^2 \leq x), \quad x \geq 0. \end{aligned} \quad (3.3.48)$$

Hints: (i) You may assume that we know that the central limit theorem

$$P^*(\sqrt{n}(\bar{X}_n^* - \bar{X}_n) \leq x) \rightarrow \Phi(x) \quad \text{a.s.}, \quad x \in \mathbb{R},$$

holds as $n \rightarrow \infty$; see Exercise 8 above.

(ii) Show that $(\sqrt{n}\bar{X}_n)$ does not converge with probability 1.

- (b) Choose an appropriate centering sequence for (T_n^*) and propose a modified bootstrap version of T_n^* which obeys the relation (3.3.48).
 (10) Let (X_i^*) be a (conditionally) iid bootstrap sequence corresponding to the iid sample X_1, \dots, X_n .
 (a) Show that the bootstrap sample mean \bar{X}_n^* has representation

$$\bar{X}_n^* \stackrel{d}{=} \frac{1}{n} \sum_{j=1}^n X_j \sum_{i=1}^n I_{((j-1)/n, j/n]}(U_i),$$

where (U_i) is an iid $U(0, 1)$ sequence, independent of (X_i) .

- (b) Write

$$M_{n,j} = \sum_{i=1}^n I_{((j-1)/n, j/n]}(U_i).$$

Show that the vector $(M_{n,1}, \dots, M_{n,n})$ has a multinomial distribution

$$\text{Mult}(n; n^{-1}, \dots, n^{-1}).$$

3.4 Reinsurance Treaties

In this section we introduce some reinsurance treaties which are standard in the literature. For the sake of illustration we assume the Cramér-Lundberg model with iid positive claim sizes X_i and Poisson intensity $\lambda > 0$.

Reinsurance treaties are mutual agreements between different insurance companies with the aim to reduce the risk in a particular insurance portfolio by sharing the risk of the occurring claims as well as the premium in this portfolio. In a sense, reinsurance is insurance for insurance companies. Reinsurance is a necessity for portfolios which are subject to catastrophic risks such as earthquakes, failure of nuclear power stations, major windstorms, industrial fire, tanker accidents, flooding, war, riots, etc. Often various insurance companies have mutual agreements about reinsuring certain parts of their portfolios. Major insurance companies such as Swiss and Munich Re or Lloyd's have specialized in reinsurance products and belong to the world's largest companies of their kind.

It is convenient to distinguish between two different types of reinsurance treaties:

- treaties of *random walk type*,
- treaties of *extreme value type*.

These names refer to the way how the treaties are constructed: *either* the total claim amount $S(t)$ (or a modified version of it) *or* some of the largest order statistics of the claim size sample are used for the construction of the treaty.

We start with *reinsurance treaties of random walk type*.

- (1) *Proportional reinsurance*. This is a common form of reinsurance for claims of “moderate” size. Here simply a fraction $p \in (0, 1)$ of each claim (hence the p th fraction of the whole portfolio) is covered by the reinsurer. Thus the reinsurer pays for the amount $R_{\text{Prop}}(t) = p S(t)$ whatever the size of the claims.
- (2) *Stop-loss reinsurance*. The reinsurer covers losses in the portfolio exceeding a well-defined limit K , the so-called *ceding company’s retention level*. This means that the reinsurer pays for $R_{\text{SL}}(t) = (S(t) - K)_+$, where $x_+ = \max(x, 0)$. This type of reinsurance is useful for protecting the company against insolvency due to excessive claims on the coverage.¹⁰
- (3) *Excess-of-loss reinsurance*. The reinsurance company pays for all individual losses in excess of some limit D , i.e., it covers $R_{\text{ExL}}(t) = \sum_{i=1}^{N(t)} (X_i - D)_+$. The limit D has various names in the different branches of insurance. In life insurance, it is called the *ceding company’s retention level*. In non-life insurance, where the size of loss is unknown in advance, D is called *deductible*. The reinsurer may in reality not insure the whole risk exceeding some limit D but rather buy a layer of reinsurance corresponding to coverage of claims in the interval $(D_1, D_2]$, $0 < D_1 < D_2 < \infty$. This can be done directly or by itself obtaining reinsurance from another reinsurer.

Notice that any of the quantities $R_i(t)$ defined above is closely related to the total claim amount $S(t)$; the same results and techniques which were developed in the previous sections can be used to evaluate the distribution and the distributional characteristics of $R_i(t)$. For example,

$$P(R_{\text{SL}}(t) \leq x) = P(S(t) \leq K) + P(K < S(t) \leq x + K), \quad x \geq 0,$$

and the processes R_{Prop} and R_{ExL} have total claim amount structure with claim sizes $p X_i$ and $(X_i - D)_+$, respectively.

Treaties of *extreme value type* aim at covering the largest claims in a portfolio. Consider the iid claim sizes $X_1, \dots, X_{N(t)}$ which occurred up to time t

¹⁰ The stop-loss treaty bears some resemblance with the terminal value of a so-called *European call option*. In this context, $S(t)$ is the price of a risky asset at time t such (as a share price, a foreign exchange rate or a stock index) and $(S(T) - K)_+$ is the value of the option with strike price K at time T of maturity. Mathematical finance deals with the pricing and hedging of such contracts; we refer to Björk [20] for a mathematical introduction to the field and to Mikosch [107] for an elementary approach.

and the corresponding ordered sample

$$X_{(1)} \leq \cdots \leq X_{(N(t))}.$$

- (4) *Largest claims reinsurance.* At the time when the contract is underwritten (i.e., at $t = 0$) the reinsurance company guarantees that the k largest claims in the time frame $[0, t]$ will be covered. For example, the company will cover the 10 largest annual claims in a portfolio over a period of 5 years, say.

This means that one has to study the quantity

$$R_{LC}(t) = \sum_{i=1}^k X_{(N(t)-i+1)}$$

either for a fixed k or for a k which grows sufficiently slowly with t .

- (5) *ECOMOR reinsurance (Excédent du coût moyen relatif).* This form of a treaty can be considered as an excess-of-loss reinsurance with a random deductible which is determined by the k th largest claim in the portfolio. This means that the reinsurer covers the claim amount

$$\begin{aligned} R_{ECOMOR}(t) &= \sum_{i=1}^{N(t)} (X_{(N(t)-i+1)} - X_{(N(t)-k+1)})_+ \\ &= \sum_{i=1}^{k-1} X_{(N(t)-i+1)} - (k-1)X_{(N(t)-k+1)} \end{aligned}$$

for a fixed number $k \geq 2$.

Treaties of random walk type can be studied by using tools for random walks such as the strong law of large numbers, the central limit theorem and ruin probabilities as considered in Chapter 4. In contrast to the latter, treaties of extreme value type need to be studied by extreme value theory techniques. We refer to Embrechts et al. [46] for an introduction to extreme value theory, in particular, to Section 8.7, where reinsurance treaties are considered. See also Section 9.2 below.

With the mathematical theory we have learned so far we can solve some problems which are related to reinsurance treaties:

- (1) How many claim sizes can occur in a layer $(D_1, D_2]$ or (D_2, ∞) , $0 < D_1 < D_2 < \infty$, up to time t ?
- (2) What can we say about the distribution of the largest claims?

It turns out that we can use similar techniques for answering these questions: we embed the pairs (T_i, X_i) in a Poisson process.

We start with the first question.

Example 3.4.1 (Distribution of the number of claim sizes in a layer)

We learned in Section 2.1.8 that (T_i, X_i) constitute the points of a Poisson process M with state space $[0, \infty)^2$ and mean measure $(\lambda \text{Leb}) \times F_{X_1}$, where Leb is Lebesgue measure on $[0, \infty)$. Concerning question (1), we are interested in the distribution of the quantity

$$M((0, t] \times A) = \#\{i \geq 1 : X_i \in A, T_i \leq t\} = \sum_{i=1}^{N(t)} I_A(X_i)$$

for some Borel set A and fixed $t > 0$. Since M is a Poisson process with mean measure $(\lambda \text{Leb}) \times F_{X_1}$, we immediately have the distribution of $M((0, t] \times A)$:

$$M((0, t] \times A) \sim \text{Pois}(F_{X_1}(A) \lambda t).$$

This solves problem (1) for limited layers $A_1 = (D_1, D_2]$ or unlimited layers $A_2 = (D_2, \infty)$. From the properties of the Poisson process M we also know that $M((0, t] \times A_1)$ and $M((0, t] \times A_2)$ are independent. Even more is true: we know from Section 3.3.2 that the corresponding total claim amounts $\sum_{i=1}^{N(t)} X_i I_{A_1}(X_i)$ and $\sum_{i=1}^{N(t)} X_i I_{A_2}(X_i)$ are independent. \square

As regards the second question, we can give exact formulae for the distribution of the largest claims:

Example 3.4.2 (Distribution of the largest claim sizes)

We proceed in a similar way as in Example 3.4.1 and use the same notation. Observe that

$$\{X_{(N(t)-k+1)} \leq x\} = \{M((0, t] \times (x, \infty)) < k\}.$$

Since $M((0, t] \times (x, \infty)) \sim \text{Pois}(\bar{F}_{X_1}(x) \lambda t)$,

$$P(X_{(N(t)-k+1)} \leq x) = \sum_{i=0}^{k-1} e^{-\bar{F}_{X_1}(x) \lambda t} \frac{(\bar{F}_{X_1}(x) \lambda t)^i}{i!}.$$

\square

As a matter of fact, it is much more complicated to deal with sums of order statistics as prescribed by the largest claims and the ECOMOR treaties. In general, it is impossible to give exact distributional characteristics of R_{LC} and R_{ECOMOR} . One of the few exceptions is the case of exponential claim sizes.

Example 3.4.3 (Treaties of extreme value type for exponential claim sizes)

Assume that the claim sizes X_i are iid $\text{Exp}(\gamma)$ distributed. From Exercise 13 on p. 49 we learn that the ordered sample $X_{(1)} < \dots < X_{(n)}$ of the X_1, \dots, X_n have the representation

$$(X_{(1)}, \dots, X_{(n)}) \stackrel{d}{=} \left(\frac{X_n}{n}, \frac{X_n}{n} + \frac{X_{n-1}}{n-1}, \dots, \frac{X_n}{n} + \frac{X_{n-1}}{n-1} + \dots + \frac{X_2}{2}, \right. \\ \left. \frac{X_n}{n} + \frac{X_{n-1}}{n-1} + \dots + \frac{X_1}{1} \right).$$

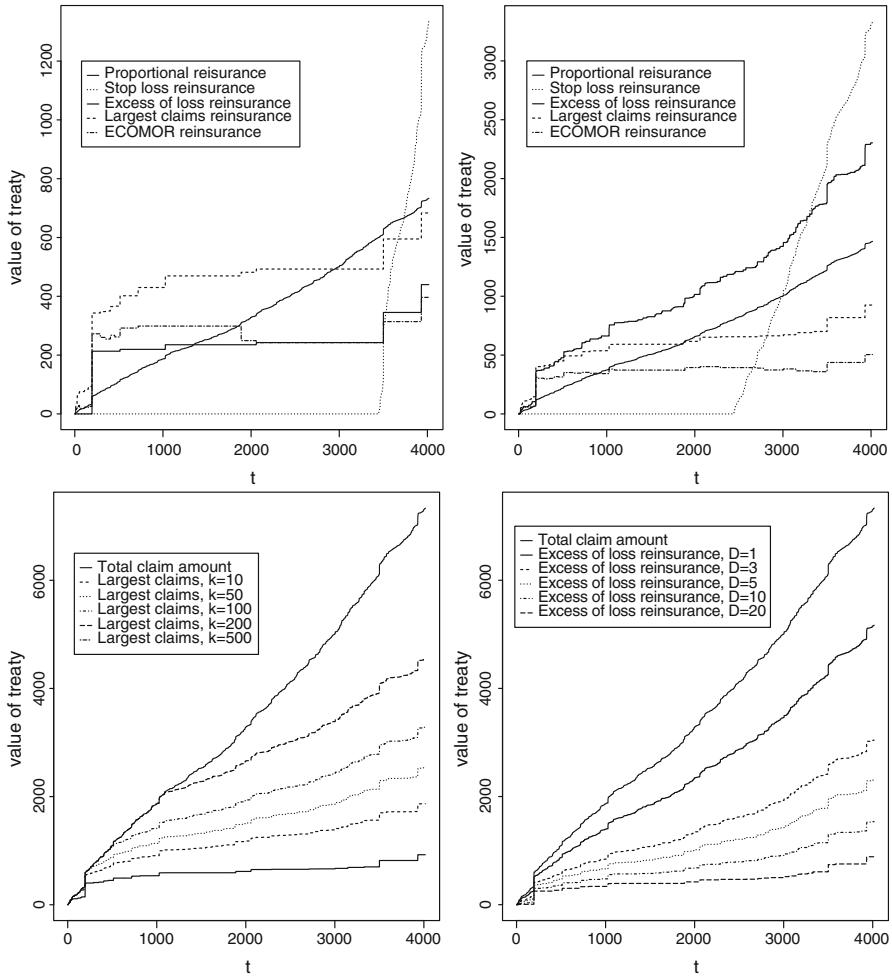


Figure 3.4.4 The values of the reinsurance treaties as a function of time for the Danish fire insurance data from January 1, 1980, until 31 December, 1990; see Section 2.1.7 for a description of the data. Prices on the y-axis are in thousands of Kroner. Top left: Proportional with $p = 0.1$, stop-loss with $K = 6$ millions, excess-of-loss with $D = 50\,000$, largest claims and ECOMOR with $k = 5$. Top right: Proportional with $p = 0.2$, stop-loss with $K = 4$ millions, excess-of-loss with $D = 5\,000$, largest claims and ECOMOR with $k = 10$. Notice the differences in scale on the y-axis. Bottom left: Largest claims reinsurance for different claim numbers k . Bottom right: Excess-of-loss reinsurance for different deductibles D .

This implies that

$$\sum_{i=1}^k X_{(n-i+1)} \stackrel{d}{=} \sum_{i=1}^k \left(\frac{X_i}{i} + \cdots + \frac{X_n}{n} \right) = \sum_{i=1}^k X_i + k \sum_{i=k+1}^n \frac{X_i}{i}$$

and

$$\sum_{i=1}^{k-1} X_{(n-i+1)} - (k-1) X_{(n-k+1)} \stackrel{d}{=} X_1 + \cdots + X_{k-1}.$$

Since the right-hand side in the latter relation does not depend on n the ECOMOR treaty has distribution

$$R_{\text{ECOMOR}}(t) \stackrel{d}{=} X_1 + \cdots + X_{k-1} \sim \Gamma(k-1, \gamma), \quad k \geq 2,$$

irrespective of t . The largest claims treaty has a less attractive distribution, but one can determine a limit distribution as $t \rightarrow \infty$. First observe that for every $t \geq 0$,

$$\begin{aligned} R_{\text{LC}}(t) &\stackrel{d}{=} \sum_{i=1}^k X_i + k \sum_{i=k+1}^{N(t)} \frac{X_i}{i} \\ &= \sum_{i=1}^k X_i + k EX_1 \sum_{i=k+1}^{N(t)} i^{-1} + k \sum_{i=k+1}^{N(t)} \frac{X_i - EX_1}{i}. \end{aligned}$$

The homogeneous Poisson process has the property $N(t) \xrightarrow{\text{a.s.}} \infty$ as $t \rightarrow \infty$ since it satisfies the strong law of large numbers $N(t)/t \xrightarrow{\text{a.s.}} \lambda$. Therefore,

$$\sum_{i=k+1}^{N(t)} \frac{X_i - EX_1}{i} \xrightarrow{\text{a.s.}} \sum_{i=k+1}^{\infty} \frac{X_i - EX_1}{i}.$$

The existence of the limit on the right-hand side is justified by Lemma 2.2.6 and the fact that the infinite series $\sum_{i=1}^{\infty} i^{-1}(X_i - EX_1)$ converges a.s. This statement can be verified by using the 3-series theorem or by observing that the infinite series has finite variance, cf. Billingsley [18], Theorems 22.6 and 22.8. It is well-known that the limit $\sum_{i=1}^n i^{-1} - \log n \rightarrow E$ exists as $n \rightarrow \infty$, where $E = 0.5772 \dots$ is Euler's constant. We conclude that as $t \rightarrow \infty$,

$$\begin{aligned} \sum_{i=k+1}^{N(t)} i^{-1} - \log(\lambda t) &= \left(\sum_{i=1}^{N(t)} i^{-1} - \log N(t) \right) - \sum_{i=1}^k i^{-1} + \log(N(t)/(\lambda t)) \\ &\xrightarrow{\text{a.s.}} E - \sum_{i=1}^k i^{-1} = C_k, \end{aligned}$$

where we also used the strong law of large numbers for $N(t)$. Collecting the above limit relations, we end up with

$$R_{LC}(t) - k \gamma^{-1} \log(\lambda t) \xrightarrow{d} \sum_{i=1}^k X_i + k \sum_{i=k+1}^{\infty} i^{-1} (X_i - \gamma^{-1}) + k \gamma^{-1} C_k. \quad (3.4.49)$$

The limit distribution can be evaluated by using Monte Carlo methods; see Figure 3.4.5. \square

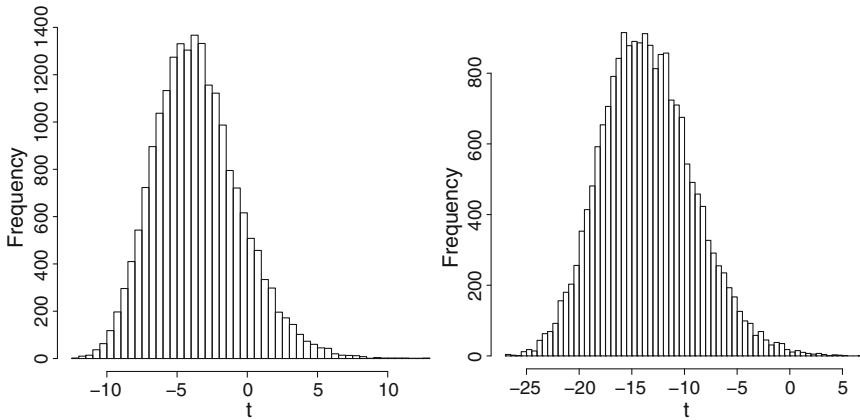


Figure 3.4.5 Histogram of 50 000 iid realizations of the limiting distribution in (3.4.49) with $k = 5$ (left), $k = 10$ (right), and $\lambda = \gamma = 1$.

In Examples 9.3.3 and 9.3.4 we return to the reinsurance treaties of extreme value type. There we will determine the limit distributions of $R_{ECOMOR}(t)$ and $R_{LC}(t)$ by using the asymptotic theory for a finite number of upper order statistics in a sample.

Comments

Over the last few years, traditional reinsurance has been complemented by financial products which are sold by insurance companies. Those include catastrophe insurance bonds or derivatives such as options and futures based on some catastrophe insurance index comparable to a composite stock index such as the S&P 500, the Dow Jones, DAX, etc. This means that reinsurance has attracted the interest of a far greater audience. The interested reader is referred to Section 8.7 in Embrechts et al. [46] and the references therein for an introduction to this topic. The websites of Munich Re WWW.MUNICHRE.COM,

Swiss Re www.swissre.com and Lloyd's www.lloyds.com give more recent information about the problems the reinsurance industry has to face.

The philosophy of classical non-life insurance is mainly based on the idea that large claims in a large portfolio have less influence and are “averaged out” by virtue of the strong law of large numbers and the central limit theorem. Over the last few years, extremely large claims have hit the reinsurance industry. Those include the claims which are summarized in Table 3.2.18. In order to deal with those claims, averaging techniques are insufficient; the expectation and the variance of a claim size sample tell one very little about the largest claims in the portfolio. Similar observations have been made in climatology, hydrology and meteorology: extreme events are not described by the normal distribution and its parameters.

In those areas special techniques have been developed to deal with extremes. They run under the name of extreme value theory and extreme value statistics. We refer to the monograph Embrechts et al. [46] and the references therein for a comprehensive treatment of these topics. In Section 9.2 we give a short introduction to the asymptotic theory of the maxima and upper order statistics for iid sequences of observations. In particular, we embed this asymptotic theory in the weak convergence of the so-called point processes of exceedances. We use these results in order to determine the limit distributions of the reinsurance treaties of extreme value type under suitable restrictions on the claim size distributions.

Exercises

- (1) An *extreme value distribution* F satisfies the following property: for every $n \geq 1$ there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that for iid random variables X_i with common distribution F ,

$$c_n^{-1} (\max(X_1, \dots, X_n) - d_n) \xrightarrow{d} X_1.$$

- (a) Verify that the Gumbel distribution with distribution function $\Lambda(x) = e^{-e^{-x}}$, $x \in \mathbb{R}$, the Fréchet distribution with distribution function $\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}$, $x > 0$, for some $\alpha > 0$, and the Weibull distribution with distribution function $\Psi_\alpha(x) = \exp\{-|x|^\alpha\}$, $x < 0$, for some $\alpha > 0$, are extreme value distributions. It can be shown that, up to changes of scale and location, these three distributions are the only extreme value distributions.
- (b) The extreme value distributions are known to be the only non-degenerate limit distributions for partial maxima $M_n = \max(X_1, \dots, X_n)$ of iid random variables X_i after suitable scaling and centering, i.e., there exist $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$c_n^{-1} (M_n - d_n) \xrightarrow{d} Y \sim H \in \{\Lambda, \Phi_\alpha, \Psi_\alpha\}. \quad (3.4.50)$$

Find suitable constants $c_n > 0$, $d_n \in \mathbb{R}$ and extreme value distributions H such that (3.4.50) holds for (i) Pareto, (ii) exponentially distributed, (iii) uniformly distributed claim sizes.

Ruin Theory

In Chapter 3 we studied the distribution and some distributional characteristics of the total claim amount $S(t)$ for fixed t as well as for $t \rightarrow \infty$. Although we sometimes used the structure of $S = (S(t))_{t \geq 0}$ as a stochastic process, for example of the renewal model, we did not really investigate the *finite-dimensional distributions* of the process S or any functional of S on a finite interval $[0, T]$ or on the interval $[0, \infty)$. Early on, with the path-breaking work of Crámer [36], the so-called *ruin probability* was introduced as a measure of risk which takes into account the temporal aspect of the insurance business over a finite or infinite time horizon. It is the aim of this section to report about Cramér's ruin bound and to look at some extensions. We start in Section 4.1 by introducing the basic notions related to ruin, including the net profit condition and the risk process. In Section 4.2 we collect some bounds on the probability of ruin. Those include the famous Lundberg inequality and Cramér's fundamental result in the case of small claim sizes. We also consider the large claim case. It turns out that the large and the small claim case lead to completely different bounds for ruin probabilities. In the small claim case ruin occurs as a collection of "atypical" claim sizes, whereas in the large claim case ruin happens as the result of one large claim size.

4.1 Risk Process, Ruin Probability and Net Profit Condition

Throughout this section we consider the total claim amount process

$$S(t) = \sum_{i=1}^{N(t)} X_i, \quad t \geq 0,$$

in the renewal model. This means that the iid sequence (X_i) of positive claim sizes with common distribution function F is independent of the claim arrival sequence (T_n) given by the renewal sequence

$$T_0 = 0, \quad T_n = W_1 + \cdots + W_n, \quad n \geq 1,$$

where the positive inter-arrival times W_n are assumed to be iid. Then the claim number process

$$N(t) = \#\{n \geq 1 : T_n \leq t\}, \quad t \geq 0,$$

is a renewal process which is independent of the claim size sequence (X_i) .

In what follows we assume a continuous *premium income* $p(t)$ in the homogeneous portfolio which is described by the renewal model. We also assume for simplicity that p is a deterministic function and even *linear*:

$$p(t) = ct.$$

We call $c > 0$ the *premium rate*. The *surplus* or *risk process* of the portfolio is then defined by

$$U(t) = u + p(t) - S(t), \quad t \geq 0.$$

The quantity $U(t)$ is nothing but the insurer's capital balance at a given time t , and the process $U = (U(t))_{t \geq 0}$ describes the cashflow in the portfolio over time. The function $p(t)$ describes the inflow of capital into the business by time t and $S(t)$ describes the outflow of capital due to payments for claims occurred in $[0, t]$. If $U(t)$ is positive, the company has gained capital, if $U(t)$ is negative it has lost capital. The constant value $U(0) = u > 0$ is called *initial capital*. It is not further specified, but usually supposed to be a "huge" value.¹ Later on, the large size of u will be indicated by taking limits as $u \rightarrow \infty$.

In the top graph of Figure 4.1.2 we see an idealized path of the process U . The process U starts at the initial capital u . Then the path increases linearly with slope c until time $T_1 = W_1$, when the first claim happens. The process decreases by the size X_1 of the first claim. In the interval $[T_1, T_2]$ the process again increases with slope c until a second claim occurs at time T_2 , when it jumps downward by the amount of X_2 , etc. In the figure we have also indicated that negative values are possible for $U(t)$ if there is a sufficiently large claim X_i which pulls the path of U below zero. The event that U ever falls below zero is called *ruin*.

Definition 4.1.1 (Ruin, ruin time, ruin probability)

The event that U ever falls below zero is called ruin:

$$\text{Ruin} = \{U(t) < 0 \quad \text{for some } t > 0\}.$$

¹ The assumption of a large initial capital is not just a mathematical assumption but also an economic necessity, which is reinforced by the supervisory authorities. In any civilized country it is not possible to start up an insurance business without a sufficiently large initial capital (reserve), which prevents the business from bankruptcy due to too many small or a few large claim sizes in the first period of its existence, before the premium income can balance the losses and the gains.

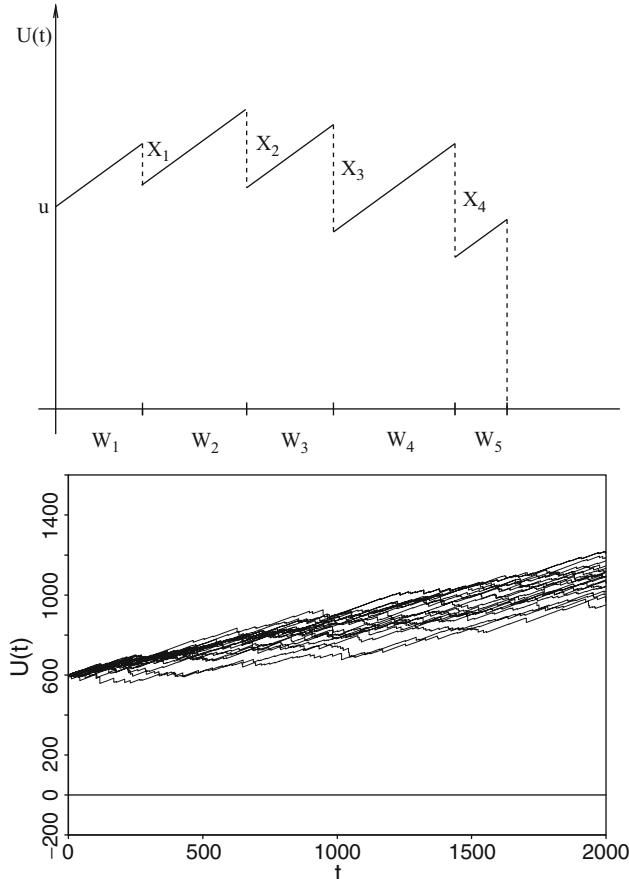


Figure 4.1.2 Top: An idealized realization of the risk process U . Bottom: Some realizations of the risk process U for exponential claim sizes and a homogeneous Poisson claim number process N . Ruin does not occur in this graph: all paths stay positive.

The time T when the process falls below zero for the first time is called ruin time:

$$T = \inf \{t > 0 : U(t) < 0\}.$$

The probability of ruin is then given by

$$\psi(u) = P(\text{Ruin} \mid U(0) = u) = P(T < \infty), \quad u > 0. \quad (4.1.1)$$

In the definition we made use of the fact that

$$\text{Ruin} = \bigcup_{t \geq 0} \{U(t) < 0\} = \left\{ \inf_{t \geq 0} U(t) < 0 \right\} = \{T < \infty\}.$$

The random variable T is not necessarily real-valued. Depending on the conditions on the renewal model, T may assume the value ∞ with positive probability. In other words, T is an *extended random variable*.

Both the event of ruin and the ruin time depend on the initial capital u , which we often suppress in the notation. The condition $U(0) = u$ in the ruin probability in (4.1.1) is artificial since $U(0)$ is a constant. This “conditional probability” is often used in the literature in order to indicate what the value of the initial capital is.

By construction of the risk process U , ruin can occur only at the times $t = T_n$ for some $n \geq 1$, since U linearly increases in the intervals $[T_n, T_{n+1})$. We call the sequence $(U(T_n))$ the *skeleton process* of the risk process U . Using the skeleton process, we can express ruin in terms of the inter-arrival times W_n , the claim sizes X_n and the premium rate c .

$$\begin{aligned} \text{Ruin} &= \left\{ \inf_{t>0} U(t) < 0 \right\} = \left\{ \inf_{n \geq 1} U(T_n) < 0 \right\} \\ &= \left\{ \inf_{n \geq 1} [u + p(T_n) - S(T_n)] < 0 \right\} \\ &= \left\{ \inf_{n \geq 1} \left[u + c T_n - \sum_{i=1}^n X_i \right] < 0 \right\}. \end{aligned}$$

In the latter step we used the fact that

$$N(T_n) = \#\{i \geq 1 : T_i \leq T_n\} = n \quad \text{a.s.}$$

since we assumed that $W_j > 0$ a.s. for all $j \geq 1$. Write

$$Z_n = X_n - cW_n, \quad S_n = Z_1 + \cdots + Z_n, \quad n \geq 1, \quad S_0 = 0.$$

Then we have the following alternative expression for the ruin probability $\psi(u)$ with initial capital u :

$$\psi(u) = P \left(\inf_{n \geq 1} (-S_n) < -u \right) = P \left(\sup_{n \geq 1} S_n > u \right). \quad (4.1.2)$$

Since each of the sequences (W_i) and (X_i) consists of iid random variables and the two sequences are mutually independent, the ruin probability $\psi(u)$ is nothing but the tail probability of the supremum functional of the random walk (S_n) . It is clear by its construction that this probability is not easily evaluated since one has to study a very complicated functional of a sophisticated random process. Nevertheless, the ruin probability has attracted enormous attention in the literature on applied probability theory. In particular, the asymptotic behavior of $\psi(u)$ as $u \rightarrow \infty$ has been of interest. The quantity $\psi(u)$ is a complex measure of the global behavior of an insurance portfolio as

time goes by. The main aim is to avoid ruin with probability 1, and the probability that the random walk (S_n) exceeds the high threshold u should be so small that the event of ruin can be excluded from any practical considerations if the initial capital u is sufficiently large.

Since we are dealing with a random walk (S_n) we expect that we can conclude, from certain asymptotic results for the sample paths of (S_n), some elementary properties of the ruin probability. In what follows, we assume that both EW_1 and EX_1 are finite. This is a weak regularity condition on the inter-arrival times and the claim sizes which is met in most cases of practical interest. But then we also know that $EZ_1 = EX_1 - cEW_1$ is well-defined and finite. The random walk (S_n) satisfies the strong law of large numbers:

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} EZ_1 \quad \text{as } n \rightarrow \infty,$$

which in particular implies that $S_n \xrightarrow{\text{a.s.}} +\infty$ or $-\infty$ a.s. according to whether EZ_1 is positive or negative. Hence if $EZ_1 > 0$, ruin is unavoidable whatever the initial capital u .

If $EZ_1 = 0$ it follows from some deep theory on random walks (e.g. Spitzer [138]) that for a.e. ω there exists a subsequence $(n_k(\omega))$ such that $S_{n_k}(\omega) \rightarrow \infty$ and another subsequence $(m_k(\omega))$ such that $S_{m_k}(\omega) \xrightarrow{\text{a.s.}} -\infty$. Hence $\psi(u) = 1$ in this case as well.²

In any case, we may conclude the following:

Proposition 4.1.3 (Ruin with probability 1)

If EW_1 and EX_1 are finite and the condition

$$EZ_1 = EX_1 - cEW_1 \geq 0 \tag{4.1.3}$$

holds then, for every fixed $u > 0$, ruin occurs with probability 1.

From Proposition 4.1.3 we learn that any insurance company should choose the premium $p(t) = ct$ in such a way that $EZ_1 < 0$. This is the only way to avoid ruin occurring with probability 1. If $EZ_1 < 0$ we may hope that $\psi(u)$ is different from 1.

Because of its importance we give a special name to the converse of condition (4.1.3).

Definition 4.1.4 (Net profit condition)

We say that the renewal model satisfies the net profit condition (NPC) if

$$EZ_1 = EX_1 - cEW_1 < 0. \tag{4.1.4}$$

² Under the stronger assumptions $EZ_1 = 0$ and $\text{var}(Z_1) < \infty$ one can show that the multivariate central limit theorem implies $\psi(u) = 1$ for every $u > 0$; see Exercise 1 on p. 156.

The interpretation of the NPC is rather intuitive. In a given unit of time the expected claim size EX_1 has to be smaller than the premium income in this unit of time, represented by the expected premium cEW_1 . In other words, the average cashflow in the portfolio is on the positive side: on average, more premium flows into the portfolio than claim sizes flow out. This does not mean that ruin is avoided since the expectation of a stochastic process says relatively little about the fluctuations of the process.

Example 4.1.5 (NPC and premium calculation principle)

The relation of the NPC with the premium calculation principles mentioned in Section 3.1.3 is straightforward. For simplicity, assume the Cramér-Lundberg model; see p. 12. We know that

$$ES(t) = EN(t) EX_1 = \lambda t EX_1 = \frac{EX_1}{EW_1} t.$$

If we choose the premium $p(t) = ct$ with $c = EX_1/EW_1$, we are in the net premium calculation principle. In this case, $EZ_1 = 0$, i.e., ruin is unavoidable with probability 1. This observation supports the intuitive argument against the net principle we gave in Section 3.1.3.

Now assume that we have the expected value or the variance premium principle. Then for some positive safety loading ρ ,

$$p(t) = (1 + \rho) ES(t) = (1 + \rho) \frac{EX_1}{EW_1} t.$$

This implies the premium rate

$$c = (1 + \rho) \frac{EX_1}{EW_1}. \quad (4.1.5)$$

In particular, $EZ_1 < 0$, i.e., the NPC is satisfied. \square

Exercises

- (1) We know that the ruin probability $\psi(u)$ in the renewal model has representation

$$\psi(u) = P\left(\sup_{n \geq 1} S_n > u\right), \quad (4.1.6)$$

where $S_n = Z_1 + \dots + Z_n$ is a random walk with iid step sizes $Z_i = X_i - cW_i$. Assume that the conditions $EZ_1 = 0$ and $\text{var}(Z_1) < \infty$ hold.

- (a) Apply the central limit theorem to show that

$$\lim_{u \rightarrow \infty} \psi(u) \geq 1 - \Phi(0) = 0.5,$$

where Φ is the standard normal distribution function. Hint: Notice that $\psi(u) \geq P(S_n > u)$ for every $n \geq 1$.

- (b) Let (Y_n) be an iid sequence of standard normal random variables. Show that for every $n \geq 1$,

$$\lim_{u \rightarrow \infty} \psi(u) \geq P(\max(Y_1, Y_1 + Y_2, \dots, Y_1 + \dots + Y_n) \geq 0).$$

Hint: Apply the multivariate central limit theorem and the continuous mapping theorem; see for example Billingsley [17].

- (c) Standard Brownian motion $(B_t)_{t \geq 0}$ is a stochastic process with independent stationary increments and continuous sample paths, starts at zero, i.e., $B_0 = 0$ a.s., and $B_t \sim N(0, t)$ for $t \geq 0$. Show that

$$\lim_{u \rightarrow \infty} \psi(u) \geq P\left(\max_{0 \leq s \leq 1} B_s \geq 0\right).$$

Hint: Use (b).

- (d) It is a well-known fact (see, for example, Resnick [123], Corollary 6.5.3 on p. 499) that Brownian motion introduced in (c) satisfies the reflection principle

$$P\left(\max_{0 \leq s \leq 1} B_s \geq x\right) = 2P(B_1 > x), \quad x \geq 0.$$

Use this result and (c) to show that $\lim_{u \rightarrow \infty} \psi(u) = 1$.

- (e) Conclude from (d) that $\psi(u) = 1$ for every $u > 0$. Hint: Notice that $\psi(u) \geq \psi(u')$ for $u \leq u'$.

- (2) Consider the total claim amount process

$$S(t) = \sum_{i=1}^{N(t)} X_i, \quad t \geq 0,$$

where (X_i) are iid positive claim sizes, independent of the Poisson process N with an a.e. positive and continuous intensity function λ . Choose the premium such that

$$p(t) = c \int_0^t \lambda(s) ds = c \mu(t),$$

for some premium rate $c > 0$ and consider the ruin probability

$$\psi(u) = P\left(\inf_{t \geq 0}(u + p(t) - S(t)) < 0\right),$$

for some positive initial capital u . Show that $\psi(u)$ coincides with the ruin probability in the Cramér-Lundberg model with Poisson intensity 1, initial capital u and premium rate c . Which condition is needed in order to avoid ruin with probability 1?

4.2 Bounds for the Ruin Probability

4.2.1 Lundberg's Inequality

In this section we derive an elementary upper bound for the ruin probability $\psi(u)$. We always assume the renewal model with the NPC (4.1.4). In addition,

we assume a *small claim condition*: the existence of the moment generating function of the claim size distribution in a neighborhood of the origin

$$m_{X_1}(h) = E e^{h X_1}, \quad h \in (-h_0, h_0) \quad \text{for some } h_0 > 0. \quad (4.2.7)$$

By Markov's inequality, for $h \in (0, h_0)$,

$$P(X_1 > x) \leq e^{-h x} m_{X_1}(h) \quad \text{for all } x > 0.$$

Therefore $P(X_1 > x)$ decays to zero exponentially fast. We have learned in Section 3.2 that this condition is perhaps not the most realistic condition for real-life claim sizes, which often tend to have heavier tails, in particular, their moment generating function is not finite in any neighborhood of the origin. However, we present this material here for small claims since the classical work by Lundberg and Cramér was done under this condition.

The following notion will be crucial.

Definition 4.2.1 (Adjustment or Lundberg coefficient)

Assume that the moment generating function of Z_1 exists in some neighborhood $(-h_0, h_0)$, $h_0 > 0$, of the origin. If a unique positive solution r to the equation

$$m_{Z_1}(h) = E e^{h(X_1 - c W_1)} = 1 \quad (4.2.8)$$

exists it is called the adjustment or Lundberg coefficient.

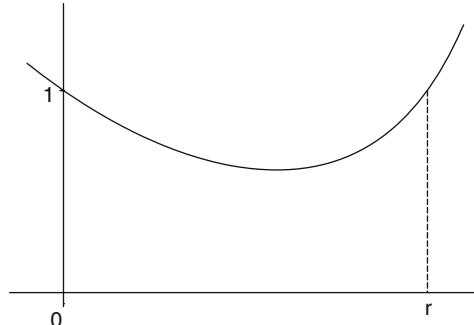


Figure 4.2.2 A typical example of the function $f(h) = m_{Z_1}(h)$ with the Lundberg coefficient r .

The existence of the moment generating function $m_{X_1}(h)$ for $h \in [0, h_0)$ implies the existence of $m_{Z_1}(h) = m_{X_1}(h)m_{cW_1}(-h)$ for $h \in [0, h_0)$ since $m_{cW_1}(-h) \leq 1$ for all $h \geq 0$. For $h \in (-h_0, 0)$ the same argument implies that $m_{Z_1}(h)$ exists if $m_{cW_1}(-h)$ is finite. Hence the moment generating function of

Z_1 exists in a neighborhood of zero if the moment generating functions of X_1 and cW_1 do. In the Cramér-Lundberg model with intensity λ for the claim number process N , $m_{cW_1}(h) = \lambda/(\lambda - ch)$ exists for $h < \lambda/c$.

In Definition 4.2.1 it was implicitly mentioned that r is unique, provided it exists as the solution to (4.2.8). The uniqueness can be seen as follows. The function $f(h) = m_{Z_1}(h)$ has derivatives of all orders in $(-h_0, h_0)$. This is a well-known property of moment generating functions. Moreover, $f'(0) = EZ_1 < 0$ by the NPC and $f''(h) = E(Z_1^2 \exp\{hZ_1\}) > 0$ since $Z_1 \neq 0$ a.s. The condition $f'(0) < 0$ and continuity of f imply that f decreases in some neighborhood of zero. On the other hand, $f''(h) > 0$ implies that f is convex. This implies that, if there exists some $h_c \in (0, h_0)$ such that $f'(h_c) = 0$, then f changes its monotonicity behavior from decrease to increase at h_c . For $h > h_c$, f increases; see Figure 4.2.2 for some illustration. Therefore the solution r of the equation $f(h) = 1$ is unique, provided the moment generating function exists in a sufficiently large neighborhood of the origin. A sufficient condition for this to happen is that there exists $0 < h_1 \leq \infty$ such that $f(h) < \infty$ for $h < h_1$ and $\lim_{h \uparrow h_1} f(h) = \infty$. This means that the moment generating function $f(h)$ increases continuously to infinity. In particular, it assumes the value 1 for sufficiently large h .

From this argument we also see that the existence of the adjustment coefficient as the solution to (4.2.8) is not automatic; the existence of the moment generating function of Z_1 in some neighborhood of the origin is not sufficient to ensure that there is some $r > 0$ with $f(r) = 1$.

Now we are ready to formulate one of the classical results in insurance mathematics.

Theorem 4.2.3 (The Lundberg inequality)

Assume the renewal model with NPC (4.1.4). Also assume that the adjustment coefficient r exists. Then the following inequality holds for all $u > 0$:

$$\psi(u) \leq e^{-r u}.$$

The exponential bound of the Lundberg inequality ensures that the probability of ruin is very small if one starts with a large initial capital u . Clearly, the bound also depends on the magnitude of the adjustment coefficient. The smaller r is, the more risky is the portfolio. In any case, the result tells us that, under a small claim condition and with a large initial capital, there is *in principle* no danger of ruin in the portfolio. We will see later in Section 4.2.4 that this statement is incorrect for portfolios with large claim sizes. We also mention that this result is much more informative than we ever could derive from the average behavior of the portfolio given by the strong law of large numbers for $S(t)$ supplemented by the central limit theorem for $S(t)$.

Proof. We will prove the Lundberg inequality by induction. Write

$$\psi_n(u) = P \left(\max_{1 \leq k \leq n} S_k > u \right) = P(S_k > u \text{ for some } k \in \{1, \dots, n\})$$

and notice that $\psi_n(u) \uparrow \psi(u)$ as $n \rightarrow \infty$ for every $u > 0$. Thus it suffices to prove that

$$\psi_n(u) \leq e^{-r u} \quad \text{for all } n \geq 1 \text{ and } u > 0. \quad (4.2.9)$$

We start with $n = 1$. By Markov's inequality and the definition of the adjustment coefficient,

$$\psi_1(u) \leq e^{-r u} m_{Z_1}(r) = e^{-r u}.$$

This proves (4.2.9) for $n = 1$. Now assume that (4.2.9) holds for $n = k \geq 1$. In the induction step we use a typical renewal argument. Write F_{Z_1} for the distribution function of Z_1 . Then

$$\begin{aligned} \psi_{k+1}(u) &= P\left(\max_{1 \leq n \leq k+1} S_n > u\right) \\ &= P(Z_1 > u) + P\left(\max_{2 \leq n \leq k+1} (Z_1 + (S_n - Z_1)) > u, Z_1 \leq u\right) \\ &= \int_{(u, \infty)} dF_{Z_1}(x) + \int_{(-\infty, u]} P\left(\max_{1 \leq n \leq k} [x + S_n] > u\right) dF_{Z_1}(x) \\ &= p_1 + p_2. \end{aligned}$$

We consider p_2 first. Using the induction assumption for $n = k$, we have

$$\begin{aligned} p_2 &= \int_{(-\infty, u]} P\left(\max_{1 \leq n \leq k} S_n > u - x\right) dF_{Z_1}(x) = \int_{(-\infty, u]} \psi_k(u - x) dF_{Z_1}(x) \\ &\leq \int_{(-\infty, u]} e^{r(x-u)} dF_{Z_1}(x). \end{aligned}$$

Similarly, by Markov's inequality,

$$p_1 \leq \int_{(u, \infty)} e^{r(x-u)} dF_{Z_1}(x).$$

Hence, by the definition of the adjustment coefficient r ,

$$p_1 + p_2 \leq e^{-r u} m_{Z_1}(r) = e^{-r u},$$

which proves (4.2.9) for $n = k + 1$ and concludes the proof. \square

Next we give a benchmark example for the Lundberg inequality.

Example 4.2.4 (Lundberg inequality for exponential claims)

Consider the Cramér-Lundberg model with iid exponential $\text{Exp}(\gamma)$ claim sizes and Poisson intensity λ . This means in particular that the W_i 's are iid exponential $\text{Exp}(\lambda)$ random variables. The moment generating function of an $\text{Exp}(a)$ distributed random variable A is given by

$$m_A(h) = \frac{a}{a-h}, \quad h < a.$$

Hence the moment generating function of $Z_1 = X_1 - cW_1$ takes the form

$$m_{Z_1}(h) = m_{X_1}(h)m_{cW_1}(-h) = \frac{\gamma}{\gamma-h} \frac{\lambda}{\lambda+ch}, \quad -\lambda/c < h < \gamma.$$

The adjustment coefficient is then the solution to the equation

$$1 + h \frac{c}{\lambda} = \frac{\gamma}{\gamma-h} = \frac{1}{1 - h EX_1}, \quad (4.2.10)$$

where $\gamma = (EX_1)^{-1}$. Now recall that the NPC holds:

$$\frac{EX_1}{EW_1} = \frac{\lambda}{\gamma} < c$$

Under this condition, straightforward calculation shows that equation (4.2.10) has a unique positive solution given by

$$r = \gamma - \frac{\lambda}{c} > 0.$$

In Example 4.1.5 we saw that we can interpret the premium rate c in terms of the expected value premium calculation principle:

$$c = \frac{EX_1}{EW_1} (1 + \rho) = \frac{\lambda}{\gamma} (1 + \rho).$$

Thus, in terms of the safety loading ρ ,

$$r = \gamma \frac{\rho}{1 + \rho}. \quad (4.2.11)$$

We summarize: In the Cramér-Lundberg model with iid $\text{Exp}(\gamma)$ distributed claim sizes and Poisson intensity λ , the Lundberg inequality for the ruin probability $\psi(u)$ is of the form

$$\psi(u) \leq \exp \left\{ -\gamma \frac{\rho}{1 + \rho} u \right\}, \quad u > 0. \quad (4.2.12)$$

From this inequality we get the intuitive meaning of the ruin probability $\psi(u)$ as a risk measure: ruin is very unlikely if u is large. However, the Lundberg bound is the smaller the larger we choose the safety loading ρ since $\rho/(1+\rho) \uparrow 1$ as $\rho \uparrow \infty$. The latter limit relation also tells us that the bound does not change significantly if ρ is sufficiently large. The right-hand side of (4.2.12) is also influenced by $\gamma = (EX_1)^{-1}$: the smaller the expected claim size, the smaller the ruin probability.

We will see in Example 4.2.13 that (4.2.12) is an almost precise estimate for $\psi(u)$ in the case of exponential claims: $\psi(u) = C \exp\{-u \gamma \rho/(1 + \rho)\}$ for some positive C . \square

Comments

It is in general difficult, if not impossible, to determine the adjustment coefficient r as a function of the distributions of the claim sizes and the inter-arrival times. A few well-known examples where one can determine r explicitly can be found in Asmussen [5] and Rolski et al. [127]. In general, one depends on numerical or Monte Carlo approximations to r .

4.2.2 Exact Asymptotics for the Ruin Probability: the Small Claim Case

In this section we consider the Cramér-Lundberg model, i.e., the renewal model with a homogeneous Poisson process with intensity λ as claim number process. It is our aim to get bounds on the ruin probability $\psi(u)$ from above and from below.

The following result is one of the most important results of risk theory, due to Cramér [36].

Theorem 4.2.5 (Cramér's ruin bound)

Consider the Cramér-Lundberg model with NPC (4.1.4). In addition, assume that the claim size distribution function F_{X_1} has a density, the moment generating function of X_1 exists in some neighborhood $(-h_0, h_0)$ of the origin, the adjustment coefficient (see (4.2.8)) exists and lies in $(0, h_0)$. Then there exists a constant $C > 0$ such that

$$\lim_{u \rightarrow \infty} e^{ru} \psi(u) = C.$$

The value of the constant C is given in (4.2.25). It involves the adjustment coefficient r , the expected claim size EX_1 and other characteristics of F_{X_1} as well as the safety loading ρ . We have chosen to express the NPC by means of ρ ; see (4.1.5):

$$\rho = c \frac{EW_1}{EX_1} - 1 > 0.$$

The proof of this result is rather technical. In what follows, we indicate some of the crucial steps in the proof. We introduce some additional notation. The non-ruin probability is given by

$$\varphi(u) = 1 - \psi(u).$$

As before, we write F_A for the distribution function of any random variable A and $\bar{F}_A = 1 - F_A$ for its tail.

The following auxiliary result is key to Theorem 4.2.5.

Lemma 4.2.6 (Fundamental integral equation for the non-ruin probability)
Consider the Cramér-Lundberg model with NPC and $EX_1 < \infty$. In addition,

assume that the claim size distribution function F_{X_1} has a density. Then the non-ruin probability $\varphi(u)$ satisfies the integral equation

$$\varphi(u) = \varphi(0) + \frac{1}{(1+\rho) EX_1} \int_0^u \bar{F}_{X_1}(y) \varphi(u-y) dy. \quad (4.2.13)$$

Remark 4.2.7 Write

$$F_{X_1,I}(y) = \frac{1}{EX_1} \int_0^y \bar{F}_{X_1}(z) dz, \quad y > 0.$$

for the *integrated tail distribution function* of X_1 . Notice that $F_{X_1,I}$ is indeed a distribution function since for any positive random variable A we have $EA = \int_0^\infty \bar{F}_A(y) dy$ and, therefore, $F_{X_1,I}(y) \uparrow 1$ as $y \uparrow \infty$. Now one can convince oneself that (4.2.13) takes the form

$$\varphi(u) = \varphi(0) + \frac{1}{1+\rho} \int_0^u \varphi(u-y) dF_{X_1,I}(y), \quad (4.2.14)$$

which reminds one of a renewal equation; see (2.2.39). Recall that in Section 2.2.2 we considered some renewal theory. It will be the key to the bound of Theorem 4.2.5. \square

Remark 4.2.8 The constant $\varphi(0)$ in (4.2.13) can be evaluated. Observe that $\varphi(u) \uparrow 1$ as $u \rightarrow \infty$. This is a consequence of the NPC and the fact that $S_n \rightarrow -\infty$ a.s., hence $\sup_{n \geq 1} S_n < \infty$ a.s. By virtue of (4.2.14) and the monotone convergence theorem,

$$\begin{aligned} 1 &= \lim_{u \uparrow \infty} \varphi(u) = \varphi(0) + \frac{1}{1+\rho} \lim_{u \uparrow \infty} \int_0^\infty I_{\{y \leq u\}} \varphi(u-y) dF_{X_1,I}(y) \\ &= \varphi(0) + \frac{1}{1+\rho} \int_0^\infty 1 dF_{X_1,I}(y) \\ &= \varphi(0) + \frac{1}{1+\rho}. \end{aligned}$$

Hence $\varphi(0) = (1+\rho)^{-1}$. \square

We continue with the proof of Lemma 4.2.6.

Proof. We again use a renewal argument. Recall from (4.1.2) that

$$\psi(u) = P \left(\sup_{n \geq 1} S_n > u \right) = 1 - \varphi(u),$$

where (S_n) is the random walk generated from the iid sequence (Z_n) with $Z_n = X_n - cW_n$. Then

$$\varphi(u) = P \left(\sup_{n \geq 1} S_n \leq u \right) = P(S_n \leq u \text{ for all } n \geq 1) \quad (4.2.15)$$

$$\begin{aligned}
&= P(Z_1 \leq u, S_n - Z_1 \leq u - Z_1 \text{ for all } n \geq 2) \\
&= E[I_{\{Z_1 \leq u\}} P(S_n - Z_1 \leq u - Z_1 \text{ for all } n \geq 2 | Z_1)] \\
&= \int_{w=0}^{\infty} \int_{x=0}^{u+cw} P(S_n - Z_1 \leq u - (x - cw) \text{ for all } n \geq 2) dF_{X_1}(x) dF_{W_1}(w) \\
&= \int_{w=0}^{\infty} \int_{x=0}^{u+cw} P(S_n \leq u - (x - cw) \text{ for all } n \geq 1) dF_{X_1}(x) \lambda e^{-\lambda w} dw.
\end{aligned} \tag{4.2.16}$$

Here we used the independence of $Z_1 = X_1 - cW_1$ and the sequence $(S_n - Z_1)_{n \geq 2}$. This sequence has the same distribution as $(S_n)_{n \geq 1}$, and the random variable W_1 has $\text{Exp}(\lambda)$ distribution. An appeal to (4.2.15) and (4.2.16) yields

$$\varphi(u) = \int_{w=0}^{\infty} \int_{x=0}^{u+cw} \varphi(u - x + cw) dF_{X_1}(x) \lambda e^{-\lambda w} dw.$$

With the substitution $z = u + cw$ we arrive at

$$\varphi(u) = \frac{\lambda}{c} e^{u\lambda/c} \int_{z=u}^{\infty} e^{-\lambda z/c} \int_{x=0}^z \varphi(z - x) dF_{X_1}(x) dz. \tag{4.2.17}$$

Since we assumed that F_{X_1} has a density, the function

$$g(z) = \int_0^z \varphi(z - x) dF_{X_1}(x)$$

is continuous. By virtue of (4.2.17),

$$\varphi(u) = \frac{\lambda}{c} e^{u\lambda/c} \int_{z=u}^{\infty} e^{-\lambda z/c} g(z) dz,$$

and, hence, φ is even differentiable. Differentiating (4.2.17), we obtain

$$\varphi'(u) = \frac{\lambda}{c} \varphi(u) - \frac{\lambda}{c} \int_0^u \varphi(u - x) dF_{X_1}(x).$$

Now integrate the latter identity and apply partial integration:

$$\begin{aligned}
&\varphi(t) - \varphi(0) - \frac{\lambda}{c} \int_0^t \varphi(u) du \\
&= -\frac{\lambda}{c} \int_0^t \int_0^u \varphi(u - x) dF_{X_1}(x) du \\
&= -\frac{\lambda}{c} \int_0^t \left[\varphi(u - x) F_{X_1}(x) \Big|_0^u + \int_0^u \varphi'(u - x) F_{X_1}(x) dx \right] du \\
&= -\frac{\lambda}{c} \int_0^t \left[\varphi(0) F_{X_1}(u) + \int_0^u \varphi'(u - x) F_{X_1}(x) dx \right] du.
\end{aligned}$$

In the last step we used $F_{X_1}(0) = 0$ since $X_1 > 0$ a.s. Now interchange the integrals:

$$\begin{aligned}
 & \varphi(t) - \varphi(0) \\
 &= \frac{\lambda}{c} \int_0^t \varphi(u) du - \frac{\lambda}{c} \varphi(0) \int_0^t F_{X_1}(u) du - \frac{\lambda}{c} \int_0^t F_{X_1}(x) [\varphi(t-x) - \varphi(0)] dx \\
 &= \frac{\lambda}{c} \int_0^t \varphi(t-u) du - \frac{\lambda}{c} \int_0^t F_{X_1}(x) \varphi(t-x) dx \\
 &= \frac{\lambda}{c} \int_0^t \bar{F}_{X_1}(x) \varphi(t-x) dx. \tag{4.2.18}
 \end{aligned}$$

Observe that

$$\frac{\lambda}{c} = \frac{1}{1+\rho} \frac{1}{EX_1},$$

see (4.1.5). The latter relation and (4.2.18) prove the lemma. \square

Lemma 4.2.6 together with Remarks 4.2.7 and 4.2.8 ensures that the non-ruin probability φ satisfies the equation

$$\varphi(u) = \frac{\rho}{1+\rho} + \frac{1}{1+\rho} \int_0^u \varphi(u-y) dF_{X_1,I}(y), \tag{4.2.19}$$

where

$$F_{X_1,I}(x) = \frac{1}{EX_1} \int_0^x \bar{F}_{X_1}(y) dy, \quad x > 0,$$

is the integrated tail distribution function of the claim sizes X_i . Writing

$$q = \frac{1}{1+\rho}$$

and switching in (4.2.19) from $\varphi = 1 - \psi$ to ψ , we obtain the equation

$$\psi(u) = q \bar{F}_{X_1,I}(u) + \int_0^u \psi(u-x) d(q F_{X_1,I}(x)). \tag{4.2.20}$$

This looks like a renewal equation, see (2.2.39):

$$R(t) = u(t) + \int_{[0,t]} R(t-y) dF(y), \tag{4.2.21}$$

where F is the distribution function of a positive random variable, u is a function on $[0, \infty)$ bounded on every finite interval and R is an unknown function. However, there is one crucial difference between (4.2.20) and (4.2.21): in the

former equation one integrates with respect to the measure $q F_{X_1,I}$ which is *not* a probability measure since $\lim_{x \rightarrow \infty} (q F_{X_1,I}(x)) = q < 1$. Therefore (4.2.20) is called a *defective renewal equation*. Before one can apply standard renewal theory, one has to transform (4.2.20) into the standard form (4.2.21) for some distribution function F .

Only at this point the notion of adjustment coefficient r comes into consideration. We define the distribution function $F^{(r)}$ for $x > 0$:

$$\begin{aligned} F^{(r)}(x) &= \int_0^x e^{ry} d(q F_{X_1,I}(y)) = q \int_0^x e^{ry} dF_{X_1,I}(y) \\ &= \frac{q}{EX_1} \int_0^x e^{ry} \bar{F}_{X_1}(y) dy. \end{aligned}$$

The distribution generated by $F^{(r)}$ is said to be the *Esscher transform* or the *exponentially tilted distribution* of F . This is indeed a distribution function since $F^{(r)}(x)$ is non-decreasing and has a limit as $x \rightarrow \infty$ given by

$$\frac{q}{EX_1} \int_0^\infty e^{ry} \bar{F}_{X_1}(y) dy = 1. \quad (4.2.22)$$

This identity can be shown by partial integration and the definition of the adjustment coefficient r . Verify (4.2.22); see also Exercise 3 on p. 178.

Multiplying both sides of (4.2.20) by e^{ru} , we obtain the equation

$$\begin{aligned} e^{ru} \psi(u) &= q e^{ru} \bar{F}_{X_1,I}(u) + \int_0^u e^{r(u-x)} \psi(u-x) e^{rx} d(q F_{X_1,I}(x)) \\ &= q e^{ru} \bar{F}_{X_1,I}(u) + \int_0^u e^{r(u-x)} \psi(u-x) dF^{(r)}(x), \end{aligned} \quad (4.2.23)$$

which is of renewal type (4.2.21) with $F = F^{(r)}$, $u(t) = q e^{rt} \bar{F}_{X_1,I}(t)$ and unknown function $R(t) = e^{rt} \psi(t)$. The latter function is bounded on finite intervals. Therefore we may apply Smith's key renewal Theorem 2.2.12(1) to conclude that the renewal equation (4.2.23) has solution

$$\begin{aligned} R(t) &= e^{rt} \psi(t) = \int_{[0,t]} u(t-y) dm^{(r)}(y) \\ &= q \int_{[0,t]} e^{r(t-y)} \bar{F}_{X_1,I}(t-y) dm^{(r)}(y), \end{aligned} \quad (4.2.24)$$

where $m^{(r)}$ is the renewal function corresponding to the renewal process whose inter-arrival times have common distribution function $F^{(r)}$. In general, we do not know the function $m^{(r)}$. However, Theorem 2.2.12(2) gives us the asymptotic order of the solution to (4.2.23) as $u \rightarrow \infty$:

$$C = \lim_{u \rightarrow \infty} e^{ru} \psi(u) = \lambda q \int_0^\infty e^{ry} \bar{F}_{X_1,I}(y) dy.$$

For the application of Theorem 2.2.12(2) we would have to verify whether $u(t) = q e^{r t} \bar{F}_{X_1, I}(t)$ is directly Riemann integrable. We refer to p. 31 in Embrechts et al. [46] for an argument. Calculation yields

$$C = \left[\frac{r}{\rho E X_1} \int_0^\infty x e^{r x} \bar{F}_{X_1}(x) dx \right]^{-1}. \quad (4.2.25)$$

This finishes the proof of the Cramér ruin bound of Theorem 4.2.5. \square

We mention in passing that the definition of the constant C in (4.2.25) requires more than the existence of the moment generating function $m_{X_1}(h)$ at $h = r$. This condition is satisfied since we assume that $m_{X_1}(h)$ exists in an open neighborhood of the origin, containing r .

Example 4.2.9 (The ruin probability in the Cramér-Lundberg model with exponential claim sizes)

As mentioned above, the solution (4.2.24) to the renewal equation for $e^{r u} \psi(u)$ is in general not explicitly given. However, if we assume that the iid claim sizes X_i are $\text{Exp}(\gamma)$ for some $\gamma > 0$, then this solution can be calculated. Indeed, the exponentially tilted distribution function $F^{(r)}$ is then $\text{Exp}(\gamma - r)$ distributed, where $\gamma - r = \gamma/(1 + \rho) = \gamma q$; see (4.2.11). Recall that the renewal function $m^{(r)}$ is given by $m^{(r)}(t) = EN^{(r)}(t) + 1$, where $N^{(r)}$ is the renewal process generated by the iid inter-arrival times $W_i^{(r)}$ with common distribution function $F^{(r)}$. Since $F^{(r)}$ is $\text{Exp}(\gamma q)$, the renewal process $N^{(r)}$ is homogeneous Poisson with intensity γq and therefore

$$m^{(r)}(t) = \gamma q t + 1, \quad t > 0.$$

According to Theorem 2.2.12(1), we have to interpret the integral in (4.2.24) such that $m^{(r)}(y) = 0$ for $y < 0$. Taking the jump of $m^{(r)}$ at zero into account, (4.2.24) reads as follows:

$$\begin{aligned} e^{r t} \psi(t) &= q e^{r t} e^{-\gamma t} + \gamma q^2 \int_0^t e^{r(t-y)} e^{-\gamma(t-y)} dy \\ &= q e^{-t(\gamma-r)} + \gamma q^2 \frac{1}{\gamma-r} \left(1 - e^{-t(\gamma-r)} \right) \\ &= q. \end{aligned}$$

This means that one gets the exact ruin probability $\psi(t) = q e^{-r t}$. \square

Example 4.2.10 (The tail of the distribution of the solution to a stochastic recurrence equation)

The following model has proved useful in various applied contexts:

$$Y_t = A_t Y_{t-1} + B_t, \quad t \in \mathbb{Z}, \quad (4.2.26)$$

where A_t and B_t are random variables, possibly dependent for each t , and the sequence of pairs (A_t, B_t) constitutes an iid sequence. Various popular

models for financial *log-returns*³ are closely related to the *stochastic recurrence equation* (4.2.26). For example, consider an *autoregressive conditionally heteroscedastic process of order 1* (ARCH(1))

$$X_t = \sigma_t Z_t, \quad t \in \mathbb{Z},$$

where (Z_t) is an iid sequence with unit variance and mean zero.⁴ The *squared volatility sequence* (σ_t^2) is given by the relation

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2, \quad t \in \mathbb{Z},$$

where α_0, α_1 are positive constants. Notice that $Y_t = X_t^2$ satisfies the stochastic recurrence equation (4.2.26) with $A_t = \alpha_1 Z_t^2$ and $B_t = \alpha_0 Z_t^2$:

$$X_t^2 = (\alpha_0 + \alpha_1 X_{t-1}^2) Z_t^2 = [\alpha_1 Z_t^2] X_{t-1}^2 + [\alpha_0 Z_t^2]. \quad (4.2.27)$$

An extension of the ARCH(1) model is the GARCH(1,1) model (*generalized ARCH model of order (1, 1)*) given by the equation

$$X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \quad t \in \mathbb{Z}.$$

Here (Z_t) is again an iid sequence with mean zero and unit variance, and α_0, α_1 and β_1 are positive constants. The squared log-return series (X_t^2) does not satisfy a stochastic recurrence equation of type (4.2.26). However, the squared volatility sequence (σ_t^2) satisfies such an equation with $A_t = \alpha_1 Z_{t-1}^2 + \beta_1$ and $B_t = \alpha_0$:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 Z_{t-1}^2 + \beta_1 \sigma_{t-1}^2 = \alpha_0 + [\alpha_1 Z_{t-1}^2 + \beta_1] \sigma_{t-1}^2.$$

In an insurance context, equation (4.2.26) has interpretation as present value of future accumulated payments which are subject to stochastic discounting. At the instants of time $t = 0, 1, 2, \dots$ a payment B_t is made. Previous payments Y_{t-1} are discounted by the stochastic discount factor A_t , i.e., A_t^{-1} is the interest paid for one price unit in the t th period, for example, in year t . Then $Y_t = A_t Y_{t-1} + B_t$ is the present value of the payments after t time steps.

In what follows, we assume that (A_t) is an iid sequence of positive random variables and, for the ease of presentation, we only consider the case $B_t \equiv 1$. It is convenient to consider all sequences with index set \mathbb{Z} . Iteration of equation (4.2.26) yields

³ For a price P_t of a risky asset (share price of stock, composite stock index, foreign exchange rate,...) which is reported at the times $t = 0, 1, 2, \dots$ the log-differences $R_t = \log P_t - \log P_{t-1}$ constitute the log-returns. In contrast to the prices P_t , it is believed that the sequence (R_t) can be modeled by a stationary process.

⁴ The sequence (Z_t) is often supposed to be iid standard normal.

$$\begin{aligned}
Y_t &= A_t A_{t-1} Y_{t-2} + A_t + 1 \\
&= A_t A_{t-1} A_{t-2} Y_{t-2} + A_t A_{t-1} + A_t + 1 \\
&\vdots \quad \vdots \quad \vdots \\
&= A_t \cdots A_1 Y_0 + \sum_{i=1}^{t-1} \prod_{j=i+1}^t A_j + 1.
\end{aligned}$$

The natural question arises as to whether “infinite iteration” yields anything useful, i.e., as to whether the sequence (Y_t) has series representation

$$Y_t = 1 + \sum_{i=-\infty}^{t-1} \prod_{j=i+1}^t A_j, \quad t \in \mathbb{Z}. \quad (4.2.28)$$

Since we deal with an infinite series we first have to study its convergence behavior; this means we have to consider the question of its existence. If $E \log A_1$ is well-defined, the strong law of large numbers yields

$$|t-i|^{-1} T_{i,t} = |t-i|^{-1} \sum_{j=i+1}^t \log A_j \xrightarrow{\text{a.s.}} E \log A_1 \quad \text{as } i \rightarrow -\infty.$$

Now assume that $-\infty \leq E \log A_1 < 0$ and choose $c \in (0, 1)$ such that $E \log A_1 < \log c < 0$. Then the strong law of large numbers implies

$$\prod_{j=i+1}^t A_j = \exp \{ |t-i| [|t-i|^{-1} T_{i,t}] \} \leq \exp \{ |t-i| \log c \} = c^{|t-i|}$$

for $i \leq i_0 = i_0(\omega)$, with probability 1. This means that $\prod_{j=i+1}^t A_j \xrightarrow{\text{a.s.}} 0$ exponentially fast as $i \rightarrow -\infty$ and, hence, the right-hand infinite series in (4.2.28) converges a.s. (Verify this fact.) Write

$$Y'_t = 1 + \sum_{i=-\infty}^{t-1} \prod_{j=i+1}^t A_j = f(A_t, A_{t-1}, \dots). \quad (4.2.29)$$

For every fixed $n \geq 1$, the distribution of the vectors

$$\mathbf{A}_{t,n} = ((A_s)_{s \leq t}, \dots, (A_s)_{s \leq t+n-1})$$

is independent of t , i.e., $\mathbf{A}_{t,n} \stackrel{d}{=} \mathbf{A}_{t+h,n}$ for every $t, h \in \mathbb{Z}$. Since f in (4.2.29) is a measurable function of $(A_s)_{s \leq t}$, one may conclude that

$$(Y'_t, \dots, Y'_{t+n-1}) \stackrel{d}{=} (Y'_{t+h}, \dots, Y'_{t+h+n-1}).$$

This means that (Y'_t) is a *strictly stationary* sequence.⁵ Obviously, (Y'_t) is a solution to the stochastic recurrence equation (4.2.26). If there exists another strictly stationary sequence (Y''_t) satisfying (4.2.26), then iteration of (4.2.26) yields for $i \geq 1$,

$$|Y'_t - Y''_t| = A_t \cdots A_{t-i+1} |Y'_{t-i} - Y''_{t-i}|. \quad (4.2.30)$$

By the same argument as above,

$$A_t \cdots A_{t-i+1} = \exp \{i [i^{-1} T_{t-i,t}] \} \xrightarrow{\text{a.s.}} 0$$

as $i \rightarrow \infty$, provided $E \log A_1 < 0$. Hence the right-hand side of (4.2.30) converges to zero in probability as $i \rightarrow \infty$ (verify this) and therefore $Y'_t = Y''_t$ a.s. Now we can identify the stationary sequence (Y'_t) as the a.s. unique solution (Y_t) to the stochastic recurrence equation (4.2.26).

Since, by stationarity, $Y_t \stackrel{d}{=} Y_0$, it is not difficult to see that

$$Y_t \stackrel{d}{=} 1 + \sum_{i=-\infty}^{-1} \prod_{j=i+1}^0 A_j \stackrel{d}{=} 1 + \sum_{i=1}^{\infty} \prod_{j=1}^i A_j.$$

Then we may conclude that for $x > 0$,

$$\begin{aligned} P(Y_0 > x) \\ \geq P \left(\sup_{n \geq 1} \prod_{j=1}^n A_j > x \right) = P \left(\sup_{n \geq 1} \sum_{j=1}^n \log A_j > \log x \right) \\ = \tilde{\psi}(\log x). \end{aligned}$$

The event on the right-hand side reminds one of the skeleton process representation of the ruin event; see (4.1.2). Indeed, since $E \log A_1 < 0$ the process $S'_n = \sum_{j=1}^n \log A_j$ constitutes a random walk with negative drift as in the case of the ruin probability for the renewal model with NPC; see Section 4.1. If we interpret the random walk (S'_n) as the skeleton process underlying a certain risk process, i.e., if we write $\log A_t = Z_t$, we can apply the bounds for the “ruin probability” $\tilde{\psi}(x)$. For example, the Lundberg inequality yields

$$\tilde{\psi}(\log x) \leq \exp \{-r \log x\} = x^{-r}, \quad x \geq 1,$$

provided that the equation $EA_1^h = Ee^{h \log A_1} = 1$ has a unique positive solution r . The proof of this fact is analogous to the proof of Theorem 4.2.3.

This upper bound for $\tilde{\psi}(\log x)$ does, however, not give one information about the decay of the tail $P(Y_0 > x)$. The Cramér bound of Theorem 4.2.5 is

⁵ We refer to Brockwell and Davis [24] or Billingsley [18] for more information about stationary sequences.

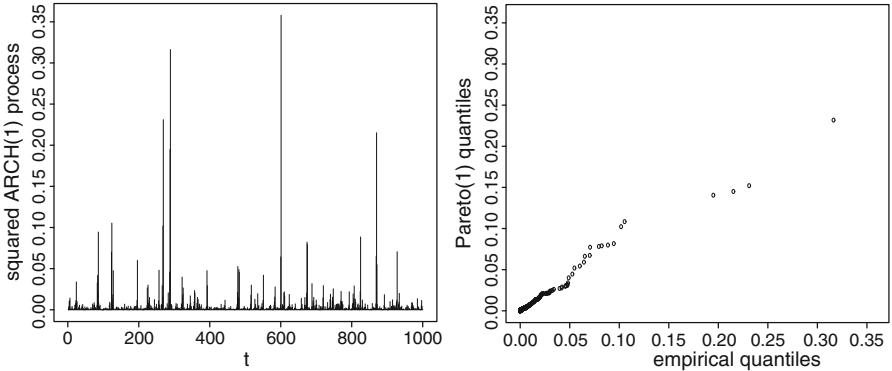


Figure 4.2.11 Left: Simulation of 1 000 values X_t^2 from the squared ARCH(1) stochastic recurrence equation (4.2.27) with parameters $\alpha_0 = 0.001$ and $\alpha_1 = 1$. Since $\text{var}(Z_1) = 1$ the equation $EA_1^h = E|Z_1|^{2h} = 1$ has the unique positive solution $r = 1$. Thus we may conclude that $P(X_t^2 > x) = Cx^{-1} (1 + o(1))$ for some positive constant $C > 0$ as $x \rightarrow \infty$. Right: QQ-plot of the sample of the squares X_t^2 against the Pareto distribution with tail parameter 1. The QQ-plot is in good agreement with the fact that the right tail of X_1^2 is Pareto like.

in general not applicable since we required the Cramér-Lundberg model, i.e., we assumed that the quantities Z_t have the special structure $Z_t = X_t - cW_t$, where (W_t) is an iid exponential sequence, independent of the iid sequence (X_i) . Nevertheless, it can be shown under additional conditions that the Cramér bound remains valid in this case, i.e., there exists a constant $C > 0$ such that

$$\tilde{\psi}(\log x) = (1 + o(1)) C e^{-r \log x} = (1 + o(1)) C x^{-r}, \quad x \rightarrow \infty.$$

This gives a lower asymptotic power law bound for the tail $P(Y_0 > x)$. It can even be shown that this bound is precise:

$$P(Y_0 > x) = (1 + o(1)) C x^{-r}, \quad x \rightarrow \infty,$$

provided that the “adjustment coefficient” $r > 0$ solves the equation $EA_1^h = 1$ and some further conditions on the distribution of A_1 are satisfied. We refer to Section 8.4 in Embrechts et al. [46] for an introduction to the subject of stochastic recurrence equations and related topics. The proofs in [46] are essentially based on work by Goldie [58]. Kesten [84] extended the results on power law tails for solutions to stochastic recurrence equations to the multivariate case. Power law tail behavior (regular variation) is a useful fact when one is interested in the analysis of extreme values in financial time series; see Mikosch [108] for a survey paper. \square

4.2.3 The Representation of the Ruin Probability as a Compound Geometric Probability

In this section we assume the Cramér-Lundberg model with NPC and use the notation of Section 4.2.2. Recall from Lemma 4.2.6 and (4.2.19) that the following equation for the non-ruin probability $\varphi = 1 - \psi$ was crucial for the derivation of Cramér's fundamental result:

$$\varphi(u) = \frac{\rho}{1 + \rho} + \frac{1}{1 + \rho} \int_0^u \varphi(u - y) dF_{X_1,I}(y). \quad (4.2.31)$$

According to the conditions in Lemma 4.2.6, for the validity of this equation one only needs to require that the claim sizes X_i have a density with finite expectation and that the NPC holds.

In this section we study equation (4.2.31) in some detail. First, we interpret the right-hand side of (4.2.31) as the distribution function of a compound geometric sum. Recall the latter notion from Example 3.3.2. Given a geometrically distributed random variable M ,

$$p_n = P(M = n) = p q^n, \quad n = 0, 1, 2, \dots, \quad \text{for some } p = 1 - q \in (0, 1),$$

the random sum

$$S_M = \sum_{i=1}^M X_i$$

has a compound geometric distribution, provided M and the iid sequence (X_i) are independent. Straightforward calculation yields the distribution function

$$\begin{aligned} P(S_M \leq x) &= p_0 + \sum_{n=1}^{\infty} p_n P(X_1 + \dots + X_n \leq x) \\ &= p + p \sum_{n=1}^{\infty} q^n P(X_1 + \dots + X_n \leq x). \end{aligned} \quad (4.2.32)$$

This result should be compared with the following one. In order to formulate it, we introduce a useful class of functions:

$$\begin{aligned} \mathcal{G} = \{G : \text{The function } G : \mathbb{R} \rightarrow [0, \infty) \text{ is non-decreasing, bounded,} \\ \text{right-continuous, and } G(x) = 0 \text{ for } x < 0\}. \end{aligned}$$

In words, $G \in \mathcal{G}$ if and only if $G(x) = 0$ for negative x and there exist $c \geq 0$ and a distribution function F of a non-negative random variable such that $G(x) = c F(x)$ for $x \geq 0$.

Proposition 4.2.12 (Representation of the non-ruin probability as compound geometric probability)
Assume the Cramér-Lundberg model with $EX_1 < \infty$ and NPC. In addition, assume the claim sizes X_i have a density. Let $(X_{I,n})$ be an iid sequence with common distribution function $F_{X_{1,I}}$. Then the function φ given by

$$\varphi(u) = \frac{\rho}{1+\rho} \left[1 + \sum_{n=1}^{\infty} (1+\rho)^{-n} P(X_{I,1} + \cdots + X_{I,n} \leq u) \right], \quad u > 0. \quad (4.2.33)$$

satisfies (4.2.31). Moreover, the function φ defined in (4.2.33) is the only solution to (4.2.31) in the class \mathcal{G} .

The identity (4.2.33) will turn out to be useful since one can evaluate the right-hand side in some special cases. Moreover, a glance at (4.2.32) shows that the non-ruin probability φ has interpretation as the distribution function of a compound geometric sum with iid summands $X_{I,i}$ and $q = (1+\rho)^{-1}$.

Proof. We start by showing that φ given by (4.2.33) satisfies (4.2.31). It will be convenient to write $q = (1+\rho)^{-1}$ and $p = 1 - q = \rho(1+\rho)^{-1}$. Then we have

$$\begin{aligned} \varphi(u) &= p + qp \left[F_{X_{1,I}}(u) + \right. \\ &\quad \left. \sum_{n=2}^{\infty} q^{n-1} \int_0^u P(y + X_{I,2} + \cdots + X_{I,n} \leq u) dF_{X_{1,I}}(y) \right] \\ &= p + q \int_0^u p \left[1 + \sum_{n=1}^{\infty} q^n P(X_{I,1} + \cdots + X_{I,n} \leq u-y) \right] dF_{X_{1,I}}(y) \\ &= p + q \int_0^u \varphi(u-y) dF_{X_{1,I}}(y). \end{aligned}$$

Hence φ satisfies (4.2.31).

It is not obvious that (4.2.33) is the only solution to (4.2.31) in the class \mathcal{G} . In order to show this it is convenient to use Laplace-Stieltjes transforms. The *Laplace-Stieltjes transform*⁶ of a function $G \in \mathcal{G}$ is given by

$$\widehat{g}(t) = \int_{[0,\infty)} e^{-tx} dG(x), \quad t \geq 0.$$

Notice that, for a distribution function G , $\widehat{g}(t) = Ee^{-tX}$, where X is a non-negative random variable with distribution function G . An important property

⁶ The reader who would like to learn more about Laplace-Stieltjes transforms is referred for example to the monographs Bingham et al. [19], Feller [51] or Resnick [123]. See also Exercise 5 on p. 178 for some properties of Laplace-Stieltjes transforms.

of Laplace-Stieltjes transforms is that for any $G_1, G_2 \in \mathcal{G}$ with Laplace-Stieltjes transforms $\widehat{g}_1, \widehat{g}_2$, respectively, $\widehat{g}_1 = \widehat{g}_2$ implies that $G_1 = G_2$. This property can be used to show that φ given in (4.2.33) is the only solution to (4.2.31) in the class \mathcal{G} . We leave this as an exercise; see Exercise 5 on p. 178 for a detailed explanation of this problem. \square

It is now an easy exercise to calculate $\psi(u)$ for exponential claim sizes by using Proposition 4.2.12.

Example 4.2.13 (The ruin probability in the Cramér-Lundberg model with exponential claim sizes)

For iid $\text{Exp}(\gamma)$ claim sizes X_i , Proposition 4.2.12 allows one to get an exact formula for $\psi(u)$. Indeed, formula (4.2.33) can be evaluated since the integrated tail distribution $F_{X_1,I}$ is again $\text{Exp}(\gamma)$ distributed and $X_{I,1} + \dots + X_{I,n}$ has a $\Gamma(n, \gamma)$ distribution whose density is well-known. Use this information to prove that

$$\psi(u) = \frac{1}{1+\rho} \exp\left\{-\gamma \frac{\rho}{1+\rho} u\right\}, \quad u > 0.$$

Compare with Lundberg's inequality (4.2.12) in the case of exponential claim sizes. The latter bound is almost exact up to the constant multiple $(1+\rho)^{-1}$. \square

4.2.4 Exact Asymptotics for the Ruin Probability: the Large Claim Case

In this section we again work under the hypothesis of the Cramér-Lundberg model with NPC.

The Cramér bound for the ruin probability $\psi(u)$

$$\psi(u) = Ce^{-r u} (1 + o(1)), \quad u \rightarrow \infty, \tag{4.2.34}$$

see Theorem 4.2.5, was obtained under a small claim condition: the existence of the moment generating function of X_1 in a neighborhood of the origin was a necessary assumption for the existence of the adjustment coefficient r given as the unique positive solution r to the equation $m_{Z_1}(h) = 1$.

It is the aim of this section to study what happens when the claim sizes are large. We learned in Section 3.2.6 that the subexponential distributions provide appropriate models of large claim sizes. The following result due to Embrechts and Veraverbeke [47] gives an answer to the ruin problem for large claims.

Theorem 4.2.14 (Ruin probability when the integrated claim size distribution is subexponential)

Assume the Cramér-Lundberg model with $EX_1 < \infty$ and NPC. In addition, assume that the claim sizes X_i have a density and that the integrated claim size distribution $F_{X_1,I}$ is subexponential. Then the ruin probability $\psi(u)$ satisfies the asymptotic relationship

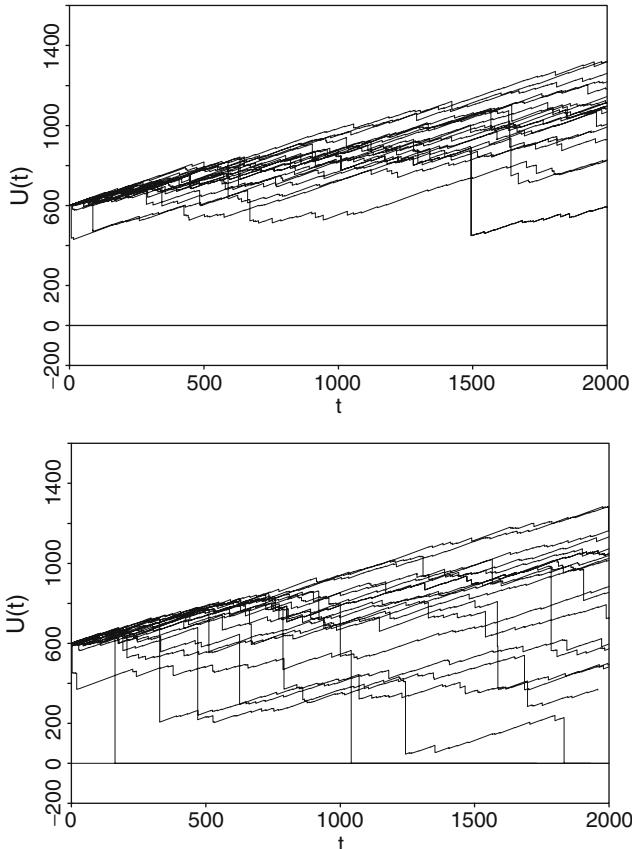


Figure 4.2.15 Some realizations of the risk process U for log-normal (top) and Pareto distributed claim sizes (bottom). In the bottom graph one can see that ruin occurs due to a single very large claim size. This is typical for subexponential claim sizes.

$$\lim_{u \rightarrow \infty} \frac{\psi(u)}{\bar{F}_{X_1,I}(u)} = \rho^{-1}. \quad (4.2.35)$$

Embrechts and Veraverbeke [47] even showed the much stronger result that (4.2.35) is equivalent to each of the conditions $F_{X_1,I} \in \mathcal{S}$ and $(1 - \psi) \in \mathcal{S}$.

Relations (4.2.35) and the Cramér bound (4.2.34) show the crucial difference between heavy- and light-tailed claim size distributions. Indeed, (4.2.35) indicates that the probability of ruin $\psi(u)$ is essentially of the same order as $\bar{F}_{X_1,I}(u)$, which is non-negligible even if the initial capital u is large. For example, if the claim sizes are Pareto distributed with index $\alpha > 1$ (only in this case $EX_1 < \infty$), $\bar{F}_{X_1,I}$ is regularly varying with index $\alpha - 1$, and therefore

$\psi(u)$ decays at a power rate instead of an exponential rate in the light-tailed case. This means that portfolios with heavy-tailed claim sizes are dangerous; the largest claims have a significant influence on the overall behavior of the portfolio in a long term horizon. In contrast to the light-tailed claim size case, ruin happens spontaneously in the heavy-tailed case and is caused by one very large claim size; see Embrechts et al. [46], Section 8.3, for a theoretical explanation of this phenomenon.

The assumption of $F_{X_1,I}$ instead of F_{X_1} being subexponential is not verified in a straightforward manner even in the case of simple distribution functions F_{X_1} such as the log-normal or the Weibull ($\tau < 1$) distributions. There exists one simple case where one can verify subexponentiality of $F_{X_1,I}$ directly: the case of regularly varying F_{X_1} with index $\alpha > 1$. Then $F_{X_1,I}$ is regularly varying with index $\alpha - 1$; see Exercise 11 on p. 181. Sufficient conditions for $F_{X_1,I}$ to be subexponential are given in Embrechts et al. [46], p. 55. In particular, all large claim distributions collected in Table 3.2.19 are subexponential and so are their integrated tail distributions.

We continue with the proof of Theorem 4.2.14.

Proof. The key is the representation of the non-ruin probability $\varphi = 1 - \psi$ as compound geometric distribution, see Proposition 4.2.12, which in terms of ψ reads as follows:

$$\frac{\psi(u)}{\bar{F}_{X_1,I}(u)} = \frac{\rho}{1 + \rho} \sum_{n=1}^{\infty} (1 + \rho)^{-n} \frac{P(X_{I,1} + \dots + X_{I,n} > u)}{\bar{F}_{X_1,I}(u)}.$$

By subexponentiality of $F_{X_1,I}$,

$$\lim_{u \rightarrow \infty} \frac{P(X_{I,1} + \dots + X_{I,n} > u)}{\bar{F}_{X_1,I}(u)} = n, \quad n \geq 1.$$

Therefore a formal interchange of the limit $u \rightarrow \infty$ and the infinite series $\sum_{n=1}^{\infty}$ yields the desired relation:

$$\lim_{u \rightarrow \infty} \frac{\psi(u)}{\bar{F}_{X_1,I}(u)} = \frac{\rho}{1 + \rho} \sum_{n=1}^{\infty} (1 + \rho)^{-n} n = \rho^{-1}.$$

The justification of the interchange of limit and infinite series follows along the lines of the proof in Example 3.3.12 by using Lebesgue dominated convergence and exploiting the properties of subexponential distributions. We leave this verification to the reader. \square

Comments

The literature about ruin probabilities is vast. We refer to the monographs by Asmussen [5], Embrechts et al. [46], Grandell [60], Rolski et al. [127] for some recent overviews and to the literature cited therein. The notion of ruin probability can be directly interpreted in terms of the tail of the distribution of the

stationary workload in a stable queue and therefore this notion also describes the average behavior of real-life queuing systems and stochastic networks.

The probability of ruin gives one a fine description of the long-run behavior in a homogeneous portfolio. In contrast to the results in Section 3.3, where the total claim amount $S(t)$ is treated as a *random variable* for fixed t or as $t \rightarrow \infty$, the ruin probability characterizes the total claim amount S as a *stochastic process*, i.e., as a random element assuming functions as values. The distribution of $S(t)$ for a fixed t is not sufficient for characterizing a complex quantity such as $\psi(u)$, which depends on the sample path behavior of S , i.e., on the whole distribution of the stochastic process.

The results of Cramér and Embrechts-Veraverbeke are of totally different nature; they nicely show the *phase transition* from heavy- to light-tailed distributions we have encountered earlier when we introduced the notion of subexponential distribution. The complete Embrechts-Veraverbeke result (Theorem 4.2.14 and its converse) shows that subexponential distributions constitute the most appropriate class of heavy-tailed distributions in the context of ruin. In fact, Theorem 4.2.14 can be dedicated to various authors; we refer to Asmussen [5], p. 260, for a historical account.

The ruin probability $\psi(u) = P(\inf_{t \geq 0} U(t) < 0)$ is perhaps not the most appropriate risk measure from a practical point of view. Indeed, ruin in an infinite horizon is not the primary issue which an insurance business will actually be concerned about. As a matter of fact, ruin in a finite time horizon has also been considered in the above mentioned references, but it leads to more technical problems and often to less attractive theoretical results.

With a few exceptions, the ruin probability $\psi(u)$ cannot be expressed as an explicit function of the ingredients of the risk process. This calls for numerical or Monte Carlo approximations to $\psi(u)$, which is an even more complicated task than the approximation to the total claim amount distribution at a fixed instant of time. In particular, the subexponential case is a rather subtle issue. We again refer to the above-mentioned literature, in particular Asmussen [5] and Rolski et al. [127], who give overviews of the techniques needed.

Exercises

Sections 4.2.1 and 4.2.2

- (1) Consider the Cramér-Lundberg model with Poisson intensity λ and $\Gamma(\gamma, \beta)$ distributed claim sizes X_i with density $f(x) = (\beta^\gamma / \Gamma(\gamma)) x^{\gamma-1} e^{-\beta x}$, $x > 0$.
 - (a) Calculate the moment generating function $m_{X_1}(h)$ of X_1 . For which $h \in \mathbb{R}$ is the function well-defined?
 - (b) Derive the NPC.
 - (c) Calculate the adjustment coefficient under the NPC.
 - (d) Assume the claim sizes are $\Gamma(n, \beta)$ distributed for some integer $n \geq 1$. Write $\psi^{(n)}(u)$ for the corresponding ruin probability with initial capital $u > 0$. Suppose that the same premium $p(t) = ct$ is charged for $\Gamma(n, \beta)$ and $\Gamma(n+1, \beta)$ distributed claim sizes. Show that $\psi^{(n)}(u) \leq \psi^{(n+1)}(u)$, $u > 0$.

- (2) Consider the risk process $U(t) = u + ct - S(t)$ in the Cramér-Lundberg model.
- Show that $S(s) = \sum_{i=1}^{N(s)} X_i$ is independent of $S(t) - S(s)$ for $s < t$. Hint: Use characteristic functions.
 - Use (a) to calculate

$$E \left(e^{-h U(t)} \mid S(s) \right) \quad (4.2.36)$$

for $s < t$ and some $h > 0$. Here we assume that $E e^{h S(t)}$ is finite. Under the assumption that the Lundberg coefficient r exists show the following relation:⁷

$$E \left(e^{-r U(t)} \mid S(s) \right) = e^{-r U(s)} \quad \text{a.s.} \quad (4.2.37)$$

- (c) Under the assumptions of (b) show that $E e^{-r U(t)}$ does not depend on t .
- (3) Consider the risk process with premium rate c in the Cramér-Lundberg model with Poisson intensity λ . Assume that the adjustment coefficient r exists as the unique solution to the equation $1 = E e^{r(X_1 - cW_1)}$. Write $m_A(t)$ for the moment generating function of any random variable A and $\rho = c/(\lambda E X_1) - 1 > 0$ for the safety loading. Show that r can be determined as the solution to each of the following equations.

$$\lambda + c r = \lambda m_{X_1}(r),$$

$$0 = \int_0^\infty [e^{rx} - (1 + \rho)] P(X_1 > x) dx,$$

$$e^{cr} = m_{S(1)}(r),$$

$$c = \frac{1}{r} \log m_{S(1)}(r).$$

- (4) Assume the Cramér-Lundberg model with the NPC. We also suppose that the moment generating function $m_{X_1}(h) = E \exp\{h X_1\}$ of the claim sizes X_i is finite for all $h > 0$. Show that there exists a unique solution $r > 0$ (Lundberg coefficient) to the equation $1 = E \exp\{h(X_1 - cW_1)\}$.

Section 4.2.3

- (5) Let \mathcal{G} be the class of non-decreasing, right-continuous, bounded functions $G : \mathbb{R} \rightarrow [0, \infty)$ such that $G(x) = 0$ for $x < 0$. Every such G can be written as $G = cF$ for some (probability) distribution function F of a non-negative random variable and some non-negative constant c . In particular, if $c = 1$, G is a distribution function. The *Laplace-Stieltjes transform* of $G \in \mathcal{G}$ is given by

⁷ The knowledgeable reader will recognize that (4.2.37) ensures that the process $M(t) = \exp\{-r U(t)\}$, $t \geq 0$, is a martingale with respect to the natural filtration generated by S , where one also uses the Markov property of S , i.e., $E(\exp\{-hU(t)\} \mid S(y), y \leq s) = E(\exp\{-hU(t)\} \mid S(s))$, $s < t$. Since the expectation of a martingale does not depend on t , we have $EM(t) = EM(0)$. This is the content of part (c) of this exercise.

$$\widehat{g}(t) = \int_{[0,\infty)} e^{-tx} dG(x), \quad t \geq 0.$$

It is not difficult to see that \widehat{g} is well-defined. Here are some of the important properties of Laplace-Stieltjes transforms.

(i) Different Laplace-Stieltjes transforms \widehat{g} correspond to different functions $G \in \mathcal{G}$. This means the following: if \widehat{g}_1 is the Laplace-Stieltjes transform of $G_1 \in \mathcal{G}$ and \widehat{g}_2 the Laplace-Stieltjes transform of $G_2 \in \mathcal{G}$, then $\widehat{g}_1 = \widehat{g}_2$ implies that $G_1 = G_2$. See Feller [51], Theorem XIII.1.

(ii) Let $G_1, G_2 \in \mathcal{G}$ and $\widehat{g}_1, \widehat{g}_2$ be the corresponding Laplace-Stieltjes transforms. Write

$$(G_1 * G_2)(x) = \int_0^x G_1(x-y) dG_2(y), \quad x \geq 0,$$

for the *convolution* of G_1 and G_2 . Then $G_1 * G_2$ has Laplace-Stieltjes transform $\widehat{g}_1 \widehat{g}_2$.

(iii) Let G^{n*} be the n -fold convolution of $G \in \mathcal{G}$, i.e., $G^{1*} = G$ and $G^{n*} = G^{(n-1)*} * G$. Then G^{n*} has Laplace-Stieltjes transform \widehat{g}^n .

(iv) The function $G = I_{[0,\infty)}$ has Laplace-Stieltjes transform $\widehat{g}(t) = 1$, $t \geq 0$.

(v) If $c \geq 0$ and $G \in \mathcal{G}$, cG has Laplace-Stieltjes transform $c\widehat{g}$.

(a) Show property (ii). Hint: Use the fact that for independent random variables A_1, A_2 with distribution functions G_1, G_2 , respectively, the relation $(G_1 * G_2)(x) = P(A_1 + A_2 \leq x)$, $x \geq 0$, holds.

(b) Show properties (iii)-(v).

(c) Let H be a distribution function with support on $[0, \infty)$ and $q \in (0, 1)$. Show that the function

$$G(u) = (1-q) \sum_{n=0}^{\infty} q^n H^{n*}(u), \quad u \geq 0, \quad (4.2.38)$$

is a distribution function on $[0, \infty)$. We interpret $H^{0*} = I_{[0,\infty)}$.

(d) Let H be a distribution function with support on $[0, \infty)$ and with density h . Let $q \in (0, 1)$. Show that the equation

$$G(u) = (1-q) + q \int_0^u G(u-x) h(x) dx, \quad u \geq 0. \quad (4.2.39)$$

has a solution G which is a distribution function with support on $[0, \infty)$.

Hint: Look at the proof of Proposition 4.2.12.

(e) Show that (4.2.38) and (4.2.39) define the same distribution function G . Hint: Show that (4.2.38) and (4.2.39) have the same Laplace-Stieltjes transforms.

(f) Determine the distribution function G for $H \sim \text{Exp}(\gamma)$ by direct calculation from (4.2.38). Hint: H^{n*} is a $\Gamma(n, \gamma)$ distribution function.

(6) Consider the Cramér-Lundberg model with NPC, safety loading $\rho > 0$ and iid $\text{Exp}(\gamma)$ claim sizes.

(a) Show that the ruin probability is given by

$$\psi(u) = \frac{1}{1+\rho} e^{-\gamma u \rho/(1+\rho)}, \quad u > 0. \quad (4.2.40)$$

Hint: Use Exercise 5(f) and Proposition 4.2.12.

- (b) Compare (4.2.40) with the Lundberg inequality.
- (7) Consider the risk process $U(t) = u + ct - S(t)$ with total claim amount $S(t) = \sum_{i=1}^{N(t)} X_i$, where the iid sequence (X_i) of $\text{Exp}(\gamma)$ distributed claim sizes is independent of the mixed homogeneous Poisson process N . In particular, we assume

$$(N(t))_{t \geq 0} = (\tilde{N}(\theta t))_{t \geq 0},$$

where \tilde{N} is a standard homogeneous Poisson process, independent of the positive mixing variable θ .

- (a) Conditionally on θ , determine the NPC and the probability of ruin for this model, i.e.,

$$P\left(\inf_{t \geq 0} U(t) < 0 \mid \theta\right).$$

- (b) Apply the results of part (a) to determine the ruin probability

$$\psi(u) = P\left(\inf_{t \geq 0} U(t) < 0\right).$$

- (c) Use part (b) to give conditions under which $\psi(u)$ decays exponentially fast to zero as $u \rightarrow \infty$.
- (d) What changes in the above calculations if you choose the premium $p(t) = (1+\rho)(\theta/\gamma)t$ for some $\rho > 0$? This means that you consider the risk process $U(t) = u + p(t) - S(t)$ with random premium adjusted to θ .
- (8) Consider a reinsurance company with risk process $U(t) = u + ct - S(t)$, where the total claim amount $S(t) = \sum_{i=1}^{N(t)} (X_i - x)_+$ corresponds to an excess-of-loss treaty, see p. 143. Moreover, N is homogeneous Poisson with intensity λ , independent of the iid sequence (X_i) of $\text{Exp}(\gamma)$ random variables. We choose the premium rate according to the expected value principle:

$$c = (1 + \rho) \lambda E[(X_1 - x)_+]$$

for some positive safety loading ρ .

- (a) Show that $c = (1 + \rho) \lambda e^{-\gamma x} \gamma$.
- (b) Show that

$$\phi_{(X_1-x)_+}(t) = E e^{it(X_1-x)_+} = 1 + \frac{it}{\gamma - it} e^{-x\gamma}, \quad t \in \mathbb{R}.$$

- (c) Show that $S(t)$ has the same distribution as $\tilde{S}(t) = \sum_{i=1}^{\tilde{N}(t)} X_i$, where \tilde{N} is a homogeneous Poisson process with intensity $\tilde{\lambda} = \lambda e^{-\gamma x}$, independent of (X_i) .
- (d) Show that the processes S and \tilde{S} have the same finite-dimensional distributions. Hint: The compound Poisson processes S and \tilde{S} have independent stationary increments. See Corollary 3.3.8. Use (c).
- (e) Define the risk process $\tilde{U}(t) = u + ct - \tilde{S}(t)$, $t \geq 0$. Show that

$$\psi(u) = P\left(\inf_{t \geq 0} U(t) < 0\right) = P\left(\inf_{t \geq 0} \tilde{U}(t) < 0\right)$$

and calculate $\psi(u)$. Hint: Use (d).

Section 4.2.4

- (9) Give a detailed proof of Theorem 4.2.14.
- (10) Verify that the integrated tail distribution corresponding to a Pareto distribution is subexponential.
- (11) Let $f(x) = x^\delta L(x)$ be a regularly varying function, where L is slowly varying and δ is a real number; see Definition 3.2.20. A well-known result which runs under the name *Karamata's theorem* (see Feller [51]) says that, for any $y_0 > 0$,

$$\lim_{y \rightarrow \infty} \frac{\int_y^\infty f(x) dx}{y f(y)} = -(1 + \delta)^{-1} \text{ if } \delta < -1$$

and

$$\lim_{y \rightarrow \infty} \frac{\int_{y_0}^y f(x) dx}{y f(y)} = (1 + \delta)^{-1} \text{ if } \delta > -1.$$

Use this result to show that the integrated tail distribution of any regularly varying distribution with index $\alpha > 1$ is subexponential.

Part II

Experience Rating

In Part I we focused on the overall or average behavior of a homogeneous insurance portfolio, where the claim number process occurred independently of the iid claim size sequence. As a matter of fact, this model disregards the policies, where the claims come from. For example, in a portfolio of car insurance policies the driving skill and experience, the age of the driver, the gender, the profession, etc., are factors which are not of interest. The policyholders generate iid claims which are aggregated in the total claim amount. The goal of collective risk theory is to determine the order of magnitude of the total claim amount in order to judge the risk represented by the claims in the portfolio as time goes by.

Everybody will agree that it is to some extent unfair and perhaps even unwise if every policyholder had to pay the same premium. A driver with poor driving skills would have to pay the same premium as a policyholder who drives carefully and has never caused any accident in his/her life. Therefore it seems reasonable to build an *individual model* for every policyholder which takes his or her claim history into account for determining a premium, as well as the overall behavior of the portfolio. This is the basic idea of *credibility theory*, which was popularized and propagated by Hans Bühlmann in his monograph [29] and in the articles [27, 28]. The monograph [29] was one of the first rigorous treatments of non-life insurance which used modern probability theory. It is one of the classics in the field and has served generations of actuaries as a guide for insurance mathematics.

In Chapter 5 we sketch the theory on *Bayes estimation* of the premium for an individual policy based on the data available in the policy. Instead of the expected total claim amount, which was the crucial quantity for the premium calculation principles in a *portfolio* (see Section 3.1.3), premium calculation in a *policy* is based on the expected claim size/claim number, conditionally on the experience in the policy. This so-called *Bayes estimator* of the individual premium minimizes the mean square deviation from the conditional expectation in the class of all finite variance measurable functions of the data. Despite the elegance of the theory, the generality of the class of approximating functions leads to problems when it comes to determining the Bayes estimator for concrete examples.

For this reason, the class of *linear Bayes* or *credibility estimators* is introduced in Chapter 6. Here the mean square error is minimized over a subclass of all measurable functions of the data having finite variance: the class of *linear* functions of the data. This minimization procedure leads to mathematically tractable expressions. The coefficients of the resulting *linear Bayes estimator* are determined as the solution to a system of linear equations. It turns out that the linear Bayes estimator can be understood as the convex combination of the overall portfolio mean and of the sample mean in the individual policy. Depending on the experience in the policy, more or less weight is given to the individual experience or to the portfolio experience. This means that the data of the policy become *more credible* if a lot of experience about the policy is available. This is the fundamental idea of *credibility theory*. We consider the

basics on linear Bayes estimation in Section 6.1. In Sections 6.2-6.4 we apply the theory to two of the best known models in this context: the Bühlmann and the Bühlmann-Straub models.

Bayes Estimation

In this chapter we consider the basics of experience rating in a policy. The heterogeneity model is fundamental. It combines the experience about the claims in an individual policy with the experience of the claims in the whole portfolio; see Section 5.1. In this model, a random parameter is attached to every policy. According to the outcome of this parameter in a particular policy, the distribution of the claims in the policy is chosen. This random *heterogeneity parameter* determines essential properties of the policy. Conditionally on this parameter, the expected claim size (or claim number) serves as a means for determining the premium in the policy. Since the heterogeneity parameter of a policy is not known *a priori*, one uses the data of the policy to *estimate* the conditional expectation in the policy. In this chapter, an estimator is obtained by minimizing the mean square deviation of the estimator (which can be any finite variance measurable function of the data) from the conditional expectation in the policy. The details of this so-called *Bayes estimation* procedure and the estimation error are discussed in Section 5.2. There we also give some intuition on the name *Bayes estimator*.

5.1 The Heterogeneity Model

In this section we introduce an *individual model* which describes one particular policy and its inter-relationship with the portfolio. We assume that the claim history of the i th policy in the portfolio is given by a time series of non-negative observations

$$x_{i,1}, \dots, x_{i,n_i}.$$

The latter sequence of numbers is interpreted as a realization of the sequence of non-negative random variables

$$X_{i,1}, \dots, X_{i,n_i}.$$

Here $X_{i,t}$ is interpreted as the claim size or the claim number occurring in the i th policy in the t th period. Periods can be measured in months, half-years, years, etc. The number n_i is then the sample size in the i th policy.

A natural question to ask is

How can one determine a premium for the i th policy by taking the claim history into account?

A simple means to determine the premium would be to calculate the expectation of the $X_{i,t}$'s. For example, if $(X_{i,t})_{t \geq 1}$ constituted an iid sequence and n_i were large we could use the strong law of large numbers to get an approximation of $EX_{i,t}$:

$$\bar{X}_i = \frac{1}{n_i} \sum_{t=1}^{n_i} X_{i,t} \approx EX_{i,1} \quad \text{a.s.}$$

There are, however, some arguments against this approach. If n_i is not large enough, the variation of \bar{X}_i around the mean $EX_{i,1}$ can be quite large which can be seen by a large variance $\text{var}(\bar{X}_i)$, provided the latter quantity is finite. Moreover, if a new policy started, no experience about the policyholder would be available: $n_i = 0$. One can also argue that the claims caused in one policy are not really independent. For example, in car insurance the individual driver is certainly a factor which has significant influence on the size and the frequency of the claims.

Here an additional modeling idea is needed: to every policy we assign a random parameter θ which contains essential information about the policy. For example, it tells one how much driving skill or experience the policyholder has. Since one usually does not know these properties before the policy is purchased, one assumes that the sequence of θ_i 's, where θ_i corresponds to the i th policy, constitutes an iid random sequence. This means that all policies behave on average in the same way; what matters is the random realization $\theta_i(\omega)$ which determines the individual properties of the i th policy, and the totality of the values θ_i determines the heterogeneity in the portfolio.

Definition 5.1.1 (The heterogeneity model)

- (1) *The i th policy is described by the pair $(\theta_i, (X_{i,t})_{t \geq 1})$, where the random parameter θ_i is the heterogeneity parameter and $(X_{i,t})_{t \geq 1}$ is the sequence of claim sizes or claim numbers in the policy.*
- (2) *The sequence of pairs $(\theta_i, (X_{i,t})_{t \geq 1})$, $i = 1, 2, \dots$, is iid.*
- (3) *Given θ_i , the sequence $(X_{i,t})_{t \geq 1}$ is iid with distribution function $F(\cdot | \theta_i)$.*

The conditions of this model imply that the claim history of the i th policy, given by the sequence of claim sizes or claim numbers, is mutually independent of the other policies. This is a natural condition which says that the different policies do not interfere with each other. Dependence is only possible between the claim sizes/claim numbers $X_{i,t}$, $t = 1, 2, \dots$, within the i th portfolio. The assumption that these random variables are iid conditionally on θ_i is certainly

an idealization which has been made for mathematical convenience. Later, in Chapter 6, we will replace this assumption by a weaker condition.

The $X_{i,t}$'s are identically distributed with distribution function

$$\begin{aligned} P(X_{i,t} \leq x) &= E[P(X_{i,t} \leq x | \theta_i)] = E[P(X_{i,1} \leq x | \theta_i)] \\ &= E[F(x | \theta_i)] = E[F(x | \theta_1)]. \end{aligned}$$

Now we come back to the question how we could determine a premium in the i th policy by taking into account the individual claim history. Since expectations $EX_{i,t}$ are not sensible risk measures in this context, a natural surrogate quantity is given by

$$\mu(\theta_i) = E(X_{i,1} | \theta_i) = \int_{\mathbb{R}} x dF(x | \theta_i),$$

where we assume the latter quantity is well-defined, the condition $EX_{1,1} < \infty$ being sufficient. Notice that $\mu(\theta_i)$ is a measurable function of the random variable θ_i . Since the sequence (θ_i) is iid, so is $(\mu(\theta_i))$.

In a sense, $\mu(\theta_i)$ can be interpreted as a net premium (see Section 3.1.3) in the i th policy which gives one an idea how much premium one should charge.

Under the conditions of the heterogeneity model, the strong law of large numbers implies that $\bar{X}_i \xrightarrow{\text{a.s.}} \mu(\theta_i)$ as $n_i \rightarrow \infty$. (Verify this relation! Hint: first apply the strong law of large numbers conditionally on θ_i .) Therefore \bar{X}_i can be considered as one possible approximation to $\mu(\theta_i)$. It is the aim of the next section to show how one can find *best approximations* (in the mean square sense) to $\mu(\theta_i)$ from the available data. These so-called *Bayes estimators* or not necessarily linear functions of the data.

5.2 Bayes Estimation in the Heterogeneity Model

In this section we assume the heterogeneity model; see Definition 5.1.1. It is our aim to find a reasonable approximation to the quantity $\mu(\theta_i) = E(X_{i,1} | \theta_i)$ by using all available data $X_{i,t}$.

Write

$$\mathbf{X}_i = (X_{i,1}, \dots, X_{i,n_i})', \quad i = 1, \dots, r,$$

for the samples of data available in the r independent policies. Since the samples are mutually independent, it seems unlikely that \mathbf{X}_j , $j \neq i$, will contain any useful information about $\mu(\theta_i)$. This conjecture will be confirmed soon.

In what follows, we assume that $\text{var}(\mu(\theta_i))$ is finite. Then it makes sense to consider the quantity

$$\rho(\hat{\mu}) = E[(\mu(\theta_i) - \hat{\mu})^2],$$

where $\hat{\mu}$ is any measurable real-valued function of the data $\mathbf{X}_1, \dots, \mathbf{X}_r$ with finite variance. The notation $\rho(\hat{\mu})$ is slightly misleading since ρ is not a function of the random variable $\hat{\mu}$ but of the joint distribution of $(\hat{\mu}, \mu(\theta_i))$. We will nevertheless use this symbol since it is intuitively appealing.

We call the quantity $\rho(\hat{\mu})$ the (quadratic) *risk* or the *mean square error* of $\hat{\mu}$ (with respect to $\mu(\theta_i)$). The choice of the quadratic risk is mainly motivated by mathematical tractability.¹ We obtain an approximation (estimator) $\hat{\mu}_B$ to $\mu(\theta_i)$ by minimizing $\rho(\hat{\mu})$ over a suitable class of distributions of $(\hat{\mu}, \mu(\theta_i))$.

Theorem 5.2.1 (Minimum risk estimation of $\mu(\theta_i)$)

The minimizer of the risk $\rho(\hat{\mu})$ in the class of all measurable functions $\hat{\mu}$ of $\mathbf{X}_1, \dots, \mathbf{X}_r$ with $\text{var}(\hat{\mu}) < \infty$ exists and is unique with probability 1. It is attained for

$$\hat{\mu}_B = E(\mu(\theta_i) | \mathbf{X}_i)$$

with corresponding risk

$$\rho(\hat{\mu}_B) = E[\text{var}(\mu(\theta_i) | \mathbf{X}_i)].$$

The index B indicates that $\hat{\mu}_B$ is a so-called *Bayes estimator*. We will give an argument for the choice of this name in Example 5.2.4 below.

Proof of Theorem 5.2.1. The result is a special case of a well-known fact on conditional expectations which we recall and prove here for convenience.

Lemma 5.2.2 *Let X be a random variable defined on the probability space (Ω, \mathcal{G}, P) and \mathcal{F} be a sub- σ -field of \mathcal{G} . Assume $\text{var}(X) < \infty$. Denote the set of random variables on (Ω, \mathcal{F}, P) with finite variance by $L^2(\Omega, \mathcal{F}, P)$. Then the minimizer of $E[(X - Y)^2]$ in the class of all random variables $Y \in L^2(\Omega, \mathcal{F}, P)$ exists and is a.s. unique. It is attained at $Y = E(X | \mathcal{F})$ with probability 1.²*

Proof. Since both X and Y have finite variance and live on the same probability space, we can define $E[(X - Y)^2]$ and $E(X | \mathcal{F})$. Then

$$E[(X - Y)^2] = E \left[([X - E(X | \mathcal{F})] + [E(X | \mathcal{F}) - Y])^2 \right]. \quad (5.2.1)$$

Notice that $X - E(X | \mathcal{F})$ and $E(X | \mathcal{F}) - Y$ are uncorrelated. Indeed, $X - E(X | \mathcal{F})$ has mean zero, and exploiting the fact that both Y and $E(X | \mathcal{F})$ are \mathcal{F} -measurable,

¹ The theory in Chapters 5 and 6 is based on Hilbert space theory; the resulting estimators can be interpreted as projections from the space of all square integrable random variables into smaller Hilbert sub-spaces.

² If one wants to be mathematically correct, one has to consider $L^2(\Omega, \mathcal{F}, P)$ as the collection of equivalence classes of random variables modulo P whose representatives have finite variance and are \mathcal{F} -measurable.

$$\begin{aligned}
& E\left(\left[X - E(X \mid \mathcal{F})\right] \left[E(X \mid \mathcal{F}) - Y\right]\right) \\
&= E\left(E\left[\left[X - E(X \mid \mathcal{F})\right] \left[E(X \mid \mathcal{F}) - Y\right] \mid \mathcal{F}\right]\right) \\
&= E\left(\left[E(X \mid \mathcal{F}) - Y\right] E[X - E(X \mid \mathcal{F}) \mid \mathcal{F}]\right) \\
&= E\left(\left[E(X \mid \mathcal{F}) - Y\right] \left[E(X \mid \mathcal{F}) - E(X \mid \mathcal{F})\right]\right) \\
&= 0.
\end{aligned}$$

Hence relation (5.2.1) becomes

$$\begin{aligned}
E[(X - Y)^2] &= E\left(\left[X - E(X \mid \mathcal{F})\right]^2\right) + E\left(\left[E(X \mid \mathcal{F}) - Y\right]^2\right) \\
&\geq E\left(\left[X - E(X \mid \mathcal{F})\right]^2\right).
\end{aligned}$$

Obviously, in the latter inequality one achieves equality if and only if $Y = E(X \mid \mathcal{F})$ a.s. This means that minimization in the class $L^2(\Omega, \mathcal{F}, P)$ of all \mathcal{F} -measurable random variables Y with finite variance yields $E(X \mid \mathcal{F})$ as the only candidate, with probability 1. \square

Now turn to the proof of the theorem. We assume that all random vectors considered are defined on the measurable space (Ω, \mathcal{G}) . We denote by $\mathcal{F} = \sigma(\mathbf{X}_1, \dots, \mathbf{X}_r)$ the sub- σ -field of \mathcal{G} generated by the data $\mathbf{X}_1, \dots, \mathbf{X}_r$. Then the theorem aims at minimizing

$$\rho(\hat{\mu}) = E[(\mu(\theta_i) - \hat{\mu})^2]$$

in the class $L^2(\Omega, \mathcal{F}, P)$ of finite variance measurable functions $\hat{\mu}$ of the data $\mathbf{X}_1, \dots, \mathbf{X}_r$. This is the same as saying that $\hat{\mu}$ is \mathcal{F} -measurable and $\text{var}(\hat{\mu}) < \infty$. Then Lemma 5.2.2 tells us that the minimizer of $\rho(\hat{\mu})$ exists, is a.s. unique and given by

$$\hat{\mu}_B = E(\mu(\theta_i) \mid \mathcal{F}) = E(\mu(\theta_i) \mid \mathbf{X}_1, \dots, \mathbf{X}_r) = E(\mu(\theta_i) \mid \mathbf{X}_i).$$

In the last step we used the fact that θ_i and \mathbf{X}_j , $j \neq i$, are mutually independent.

It remains to calculate the risk:

$$\begin{aligned}
\rho(\hat{\mu}_B) &= E\left[\left(\mu(\theta_i) - E(\mu(\theta_i) \mid \mathbf{X}_i)\right)^2\right] \\
&= E\left(E\left[\left(\mu(\theta_i) - E(\mu(\theta_i) \mid \mathbf{X}_i)\right)^2 \mid \mathbf{X}_i\right]\right) \\
&= E[\text{var}(\mu(\theta_i) \mid \mathbf{X}_i)].
\end{aligned}$$

This proves the theorem. \square

From Theorem 5.2.1 it is immediate that the minimum risk estimator $\hat{\mu}_B$ only

depends on the data in the i th portfolio. Therefore we suppress the index i in the notation wherever we focus on one particular policy. We write θ for θ_i and X_1, X_2, \dots for $X_{i,1}, X_{i,2}, \dots$, but also \mathbf{X} instead of \mathbf{X}_i and n instead of n_i .

The calculation of the Bayes estimator $E(\mu(\theta) | \mathbf{X})$ very much depends on the knowledge of the conditional distribution of $\theta | \mathbf{X}$. The following lemma contains some useful rules how one can calculate the conditional density $\theta | \mathbf{X}$ provided the latter exists.

Lemma 5.2.3 (Calculation of the conditional density of θ given the data)
Assume the heterogeneity model, that θ has density f_θ and the conditional density $f_\theta(y | \mathbf{X} = \mathbf{x})$, $y \in \mathbb{R}$, of the one-dimensional parameter θ given \mathbf{X} exists for \mathbf{x} in the support of \mathbf{X} .

(1) *If X_1 has a discrete distribution then $\theta | \mathbf{X}$ has density*

$$\begin{aligned} f_\theta(y | \mathbf{X} = \mathbf{x}) & \quad (5.2.2) \\ &= \frac{f_\theta(y) P(X_1 = x_1 | \theta = y) \cdots P(X_1 = x_n | \theta = y)}{P(\mathbf{X} = \mathbf{x})}, \quad y \in \mathbb{R}, \end{aligned}$$

on the support of \mathbf{X} .

(2) *If (\mathbf{X}, θ) have the joint density $f_{\mathbf{X}, \theta}$, then $\theta | \mathbf{X}$ has density*

$$f_\theta(y | \mathbf{X} = \mathbf{x}) = \frac{f_\theta(y) f_{X_1}(x_1 | \theta = y) \cdots f_{X_1}(x_n | \theta = y)}{f_{\mathbf{X}}(\mathbf{x})}, \quad y \in \mathbb{R},$$

on the support of \mathbf{X} .

Proof. (1) Since the conditional density of $\theta | \mathbf{X}$ is assumed to exist we have

$$P(\theta \leq x | \mathbf{X} = \mathbf{x}) = \int_{-\infty}^x f_\theta(y | \mathbf{X} = \mathbf{x}) dy, \quad x \in \mathbb{R}. \quad (5.2.3)$$

Since the X_i 's are iid conditionally on θ , for $x \in \mathbb{R}$,

$$\begin{aligned} P(\theta \leq x | \mathbf{X} = \mathbf{x}) & \\ &= [P(\mathbf{X} = \mathbf{x})]^{-1} E[P(\theta \leq x, \mathbf{X} = \mathbf{x} | \theta)] \\ &= [P(\mathbf{X} = \mathbf{x})]^{-1} E[I_{(-\infty, x]}(\theta) P(\mathbf{X} = \mathbf{x} | \theta)] \\ &= [P(\mathbf{X} = \mathbf{x})]^{-1} \int_{-\infty}^x P(\mathbf{X} = \mathbf{x} | \theta = y) f_\theta(y) dy \\ &= \int_{-\infty}^x [P(\mathbf{X} = \mathbf{x})]^{-1} P(X_1 = x_1 | \theta = y) \cdots P(X_1 = x_n | \theta = y) f_\theta(y) dy. \end{aligned} \quad (5.2.4)$$

By the Radon-Nikodym theorem, the integrands in (5.2.3) and (5.2.4) coincide a.e. This gives (5.2.2).

(2) The conditional density of $\mathbf{X} | \theta$ satisfies

$$f_{\mathbf{X}}(\mathbf{x} | \theta = y) = f_{\mathbf{X}, \theta}(\mathbf{x}, y) / f_{\theta}(y),$$

on the support of θ , see for example Williams [145], Section 15.6. On the other hand, in the heterogeneity model the \mathbf{X}_i 's are iid given θ . Hence

$$f_{\mathbf{X}}(\mathbf{x} | \theta) = f_{X_1}(x_1 | \theta) \cdots f_{X_n}(x_n | \theta).$$

We conclude that

$$f_{\theta}(y | \mathbf{X} = \mathbf{x}) = \frac{f_{\theta, \mathbf{X}}(y, \mathbf{x})}{f_{\mathbf{X}}(\mathbf{x})} = \frac{f_{\theta}(y) f_{X_1}(x_1 | \theta = y) \cdots f_{X_n}(x_n | \theta = y)}{f_{\mathbf{X}}(\mathbf{x})}.$$

This concludes the proof of (2). \square

Example 5.2.4 (Poisson distributed claim numbers and gamma distributed heterogeneity parameters)

Assume the claim numbers X_t , $t = 1, 2, \dots$, are iid with $\text{Pois}(\theta)$ distribution, given θ , and $\theta \sim \Gamma(\gamma, \beta)$ for some positive γ and β , i.e.,

$$f_{\theta}(x) = \frac{\beta^{\gamma}}{\Gamma(\gamma)} x^{\gamma-1} e^{-\beta x}, \quad x > 0.$$

It was mentioned in Example 2.3.2 that X_t is then negative binomially distributed with parameter $(\beta/(1+\beta), \gamma)$. Also recall that

$$E\theta = \frac{\gamma}{\beta} \quad \text{and} \quad \text{var}(\theta) = \frac{\gamma}{\beta^2}. \quad (5.2.5)$$

Since $X_1 | \theta$ is $\text{Pois}(\theta)$ distributed,

$$\mu(\theta) = E(X_1 | \theta) = \theta.$$

We intend to calculate the Bayes estimator $\hat{\mu}_B = E(\theta | \mathbf{X})$ of θ . We start by calculating the distribution of θ given \mathbf{X} . We apply formula (5.2.2):

$$\begin{aligned} f_{\theta}(x | \mathbf{X} = \mathbf{x}) &= P(X_1 = x_1 | \theta = x) \cdots P(X_n = x_n | \theta = x) f_{\theta}(x) [P(\mathbf{X} = \mathbf{x})]^{-1} \\ &= D_1(\mathbf{x}) x^{\gamma-1} e^{-\beta x} \prod_{t=1}^n \left(\frac{x^{x_t}}{x_t!} e^{-x} \right) \\ &= D_2(\mathbf{x}) x^{\gamma+x.-1} e^{-x(\beta+n)}, \end{aligned} \quad (5.2.6)$$

where $D_1(\mathbf{x})$ and $D_2(\mathbf{x})$ are certain multipliers which do not depend on x , and $x. = \sum_{t=1}^n x_t$. Since (5.2.6) represents a density, we may conclude from

its particular form that it is the density of the $\Gamma(\gamma + x., \beta + n)$ distribution, i.e., $\theta | \mathbf{X} = \mathbf{x}$ has this particular gamma distribution.

From (5.2.5) we can deduce the expectation and variance of $\theta | \mathbf{X}$:

$$E(\theta | \mathbf{X}) = \frac{\gamma + X.}{\beta + n} \quad \text{and} \quad \text{var}(\theta | \mathbf{X}) = \frac{\gamma + X.}{(\beta + n)^2},$$

where $X. = \sum_{t=1}^n X_t$. Hence the Bayes estimator $\hat{\mu}_B$ of $\mu(\theta) = \theta$ is

$$\hat{\mu}_B = \frac{\gamma + X.}{\beta + n}$$

and the corresponding risk is given by

$$\rho(\hat{\mu}_B) = E(\text{var}(\theta | \mathbf{X})) = E\left(\frac{\gamma + X.}{(\beta + n)^2}\right) = \frac{\gamma + n E X_1}{(\beta + n)^2} = \frac{\gamma}{\beta} \frac{1}{\beta + n},$$

where we used the fact that $E X_1 = E[E(X_1 | \theta)] = E\theta = \gamma/\beta$.

The Bayes estimator $\hat{\mu}_B$ of θ has representation

$$\hat{\mu}_B = (1 - w) E\theta + w \bar{X},$$

where $\bar{X} = n^{-1} X.$ is the sample mean in the policy and

$$w = \frac{n}{\beta + n}$$

is a positive weight. Thus the Bayes estimator of θ given the data \mathbf{X} is a weighted mean of the expected heterogeneity parameter $E\theta$ and the sample mean in the individual policy. Notice that $w \rightarrow 1$ if the sample size $n \rightarrow \infty$. This means that the Bayes estimator $\hat{\mu}_B$ gets closer to \bar{X} the larger the sample size. For small n , the variation of \bar{X} is too large in order to be representative of the policy. Therefore the weight w given to the policy average \bar{X} is small, whereas the weight $1 - w$ assigned to the expected value $E\theta$ of the portfolio heterogeneity is close to one. This means that the net premium represented by $\mu(\theta) = E(X_1 | \theta) = \theta$ is strongly influenced by the information available in the policy. In particular, if no such information is available, i.e., $n = 0$, premium calculation is solely based on the overall portfolio expectation. Also notice that the risk satisfies

$$\rho(\hat{\mu}_B) = (1 - w) \text{var}(\theta) = (1 - w) \frac{\gamma}{\beta^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Finally, we comment on the name *Bayes estimator*. It stems from Bayesian statistics, which forms a major part of modern statistics. Bayesian statistics has gained a lot of popularity over the years, in particular, since Bayesian techniques have taken advantage of modern computer power. One of the fundamental ideas of this theory is that the parameter of a distribution is not deterministic but has distribution in the parameter space considered. In the

context of our example, we *assumed* that the parameter θ has a gamma distribution with given parameters γ and β . This distribution has to be known (conjectured) in advance and is therefore referred to as the *prior distribution*. Taking into account the information which is represented by the sample \mathbf{X} , we then updated the distribution of θ , i.e., we were able to calculate the distribution of $\theta | \mathbf{X}$ and obtained the gamma distribution with parameters $\gamma + X_i$ and $\beta + n_i$. We see from this example that the data change the prior distribution in a particular way. The resulting gamma distribution is referred to as the *posterior distribution*. This reasoning might explain the notion of Bayes estimator. \square

Comments

The minimization of the risk $\rho(\hat{\mu})$ in the class of all finite variance measurable functions of the data leads in general to a situation where one cannot calculate the Bayes estimator $\hat{\mu}_B = E(\mu(\theta) | \mathbf{X})$ explicitly. In the next section we will therefore minimize the risk over the smaller class of linear functions of the data and we will see that this estimator can be calculated explicitly.

The idea of minimizing over the class of all measurable functions is basic to various concepts in probability theory and statistics. In this section we have already seen that the conditional expectation of a random variable with respect to a σ -field is such a concept. Similar concepts occur in the context of predicting future values of a time series based on the information contained in the past, in regression analysis, Kalman filtering or extrapolation in spatial processes. As a matter of fact, we have calculated an approximation to the “best prediction” $\mu(\theta_i) = E(X_{i,n_i+1} | \theta_i)$ of the next claim size/number X_{i,n_i+1} in the i th policy by minimizing the quadratic risk $E[(E(X_{i,n_i+1} | \theta_i) - \hat{\mu})^2]$ in the class of all measurable functions of the data $X_{i,1}, \dots, X_{i,n_i}$. Therefore the idea underlying the Bayes estimator considered in this section has been exploited in other areas as well and the theory in these other fields is often directly interpretable in terms of Bayes estimation. We refer for example to Brockwell and Davis [24] for prediction of time series and Kalman filtering, and to Cressie’s book [37] on spatial statistics.

Parts of standard textbooks on statistics are devoted to Bayesian statistics. We refer to the classical textbook of Lehmann [93] for an introduction to the theory. Bühlmann’s monograph [27] propagated the use of Bayesian methods for premium calculation in a policy. Since then, major parts of textbooks on non-life insurance mathematics have been devoted to the Bayes methodology; see for example Kaas et al. [77], Klugman et al. [86], Sundt [143], Straub [141].

Exercises

- (1) Assume the heterogeneity model.
 - (a) Give a necessary and sufficient condition for the independence of $X_{i,t}$, $t = 1, \dots, n_i$, in the i th policy.

- (b) Assume that $EX_{1,1} < \infty$. Show that $E(X_{i,1} | \theta_i)$ is well-defined and finite. Prove the following strong laws of large numbers as $n \rightarrow \infty$:

$$\frac{1}{n} \sum_{t=1}^n X_{i,t} \xrightarrow{\text{a.s.}} \mu(\theta_i) = E(X_{i,1} | \theta_i) \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^n X_{i,t} \xrightarrow{\text{a.s.}} EX_{1,1}.$$

- (2) Assume the heterogeneity model and consider the i th policy. We suppress the dependence on i in the notation. Given $\theta > 0$, let the claim sizes X_1, \dots, X_n in the policy be iid Pareto distributed with parameters (λ, θ) , i.e.,

$$\bar{F}(x | \theta) = P(X_i > x | \theta) = (\lambda/x)^\theta, \quad x > \lambda.$$

Assume that θ is $\Gamma(\gamma, \beta)$ distributed with density

$$f_{\gamma, \beta}(x) = \frac{\beta^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\beta x}, \quad x > 0.$$

- (a) Show that $\theta | \mathbf{X}$ with $\mathbf{X} = (X_1, \dots, X_n)'$ has density

$$f_{\gamma+n, \beta+\sum_{i=1}^n \log(X_i/\lambda)}(x).$$

- (b) A reinsurance company takes into account only the values X_i exceeding a known high threshold K . They “observe” the counting variables $Y_i = I_{(K, \infty)}(X_i)$ for a known threshold $K > \lambda$. The company is interested in estimating $P(X_1 > K | \theta)$.

- (i) Give a naive estimator of $P(X_1 > K | \theta)$ based on the empirical distribution function of X_1, \dots, X_n .

- (ii) Determine the a.s. limit of this estimator as $n \rightarrow \infty$. Does it coincide with $P(X_1 > K | \theta)$?

- (c) Show that Y_i , given θ , is $\text{Bin}(1, p(\theta))$ distributed, where $p(\theta) = E(Y_1 | \theta)$. Compare $p(\theta)$ with the limit in (b,ii).

- (d) Show that the Bayes estimator of $p(\theta) = E(Y_1 | \theta)$ based on the data X_1, \dots, X_n is given by

$$\frac{(\beta + \sum_{i=1}^n \log(X_i/\lambda))^{\gamma+n}}{(\beta + \sum_{i=1}^n \log(X_i/\lambda) + \log(K/\lambda))^{\gamma+n}}.$$

- (3) Assume the heterogeneity model and consider a policy with one observed claim number X and corresponding heterogeneity parameter θ . We assume that $X | \theta$ is $\text{Pois}(\theta)$ distributed, where θ has a continuous density f_θ on $(0, \infty)$. Notice that $E(X | \theta) = \theta$.

- (a) Determine the conditional density $f_\theta(y | X = k)$, $k = 0, 1, \dots$, of $\theta | X$ and use this information to calculate the Bayes estimator $m_k = E(\theta | X = k)$, $k = 0, 1, 2, \dots$

- (b) Show that

$$m_k = (k+1) \frac{P(X = k+1)}{P(X = k)}, \quad k = 0, 1, \dots$$

- (c) Show that

$$E(\theta^l | X = k) = \prod_{i=0}^{l-1} m_{k+i}, \quad k \geq 0, l \geq 1.$$

- (4) Consider the i th policy in a heterogeneity model. We suppress the dependence on i in the notation. We assume the heterogeneity parameter θ to be $\beta(a, b)$ -distributed with density

$$f_\theta(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1} (1-y)^{b-1}, \quad 0 < y < 1, \quad a, b > 0.$$

Given θ , the claim numbers X_1, \dots, X_n are iid $\text{Bin}(k, \theta)$ distributed.

- (a) Calculate the conditional density $f_\theta(y | \mathbf{X} = \mathbf{x})$ of θ given

$$\mathbf{X} = (X_1, \dots, X_n)' = \mathbf{x} = (x_1, \dots, x_n)'.$$

- (b) Calculate the Bayes estimator $\hat{\mu}_B$ of $\mu(\theta) = E(X_1 | \theta)$ and the corresponding risk. Hint: A $\beta(a, b)$ -distributed random variable θ satisfies the relations $E\theta = a/(a+b)$ and $\text{var}(\theta) = ab/[(a+b+1)(a+b)^2]$.
- (5) Consider the i th policy in a heterogeneity model. We suppress the dependence on i in the notation. We assume the heterogeneity parameter θ to be $N(\mu, \sigma^2)$ -distributed. Given θ , the claim sizes X_1, \dots, X_n are iid log-normal (θ, τ) -distributed. This means that $\log X_t$ has representation $\log X_t = \theta + \tau Z_t$ for an iid $N(0, 1)$ sequence (Z_t) independent of θ and some positive constant τ .

- (a) Calculate the conditional density $f_\theta(y | \mathbf{X} = \mathbf{x})$ of θ given

$$\mathbf{X} = (X_1, \dots, X_n)' = \mathbf{x} = (x_1, \dots, x_n)'.$$

- (b) Calculate the Bayes estimator $\hat{\mu}_B$ of $\mu(\theta) = E(X_1 | \theta)$ and the corresponding risk. It is useful to remember that

$$E e^{a+bZ_1} = e^{a+b^2/2} \text{ and } \text{var}\left(e^{a+bZ_1}\right) = e^{2a+b^2} \left(e^{b^2} - 1\right), \quad a \in \mathbb{R}, b > 0.$$

Linear Bayes Estimation

As mentioned at the end of Chapter 5, it is generally difficult, if not impossible, to calculate the Bayes estimator $\hat{\mu}_B = E(\mu(\theta_i) \mid \mathbf{X}_i)$ of the net premium $\mu(\theta_i) = E(X_{i,t} \mid \theta_i)$ in the i th policy based on the data $\mathbf{X}_i = (X_{i,1}, \dots, X_{i,n_i})'$. As before, we write $X_{i,t}$ for the claim size/claim number in the i th policy in the t th period. One way out of this situation is to minimize the risk,

$$\rho(\hat{\mu}) = E[(\mu(\theta_i) - \hat{\mu})^2],$$

not over the whole class of finite variance measurable functions $\hat{\mu}$ of the data $\mathbf{X}_1, \dots, \mathbf{X}_r$, but over a smaller class. In this section we focus on the class of linear functions

$$\mathcal{L} = \left\{ \hat{\mu} : \hat{\mu} = a_0 + \sum_{i=1}^r \sum_{t=1}^{n_i} a_{i,t} X_{i,t}, \quad a_0, a_{i,t} \in \mathbb{R} \right\}. \quad (6.0.1)$$

If a minimizer of the risk $\rho(\hat{\mu})$ in the class \mathcal{L} exists, we call it a *linear Bayes estimator* for $\mu(\theta_i)$, and we denote it by $\hat{\mu}_{LB}$.

We start in Section 6.1 by solving the above minimization problem in a wider context: we consider the best approximation (with respect to quadratic risk) of a finite variance random variable by linear functions of a given vector of finite variance random variables. The coefficients of the resulting linear function and the corresponding risk can be expressed as the solution to a system of linear equations, the so-called *normal equations*. This is an advantage compared to the Bayes estimator, where, in general, we could not give an explicit solution to the minimization problem. In Section 6.2 we apply the minimization result to the original question about estimation of the conditional policy mean $\mu(\theta_i)$ by linear functions of the data $\mathbf{X}_1, \dots, \mathbf{X}_n$. It turns out that the requirements of the heterogeneity model (Definition 5.1.1) can be relaxed. Indeed, the heterogeneity model is tailored for Bayes estimation, which requires one to specify the complete dependence structure inside and across the policies. Since linear Bayes estimation is concerned with the minimization of second moments, it is plausible in this context that one only needs

to assume suitable conditions about the first and second moments inside and across the policies. These attempts result in the so-called *Bühlmann model* of Section 6.2 and, in a more general context, in the *Bühlmann-Straub model* of Section 6.4. In Sections 6.3 and 6.4 we also derive the corresponding linear Bayes estimators and their risks.

6.1 An Excursion to Minimum Linear Risk Estimation

In this section we consider the more general problem of approximating a finite variance random variable X by linear functions of finite variance random variables Y_1, \dots, Y_m which are defined on the same probability space. Write $\mathbf{Y} = (Y_1, \dots, Y_m)'$. Then our task is to approximate X by any element of the class of linear functions

$$\mathcal{L}' = \{Y : Y = a_0 + \mathbf{a}' \mathbf{Y}, \quad a_0 \in \mathbb{R}, \mathbf{a} \in \mathbb{R}^m\}, \quad (6.1.2)$$

where $\mathbf{a} = (a_1, \dots, a_m)' \in \mathbb{R}^m$ is any column vector. In Section 6.3 we will return to the problem of estimating $X = \mu(\theta_i)$ by linear functions of the data $\mathbf{X}_1, \dots, \mathbf{X}_r$. There we will apply the theory developed in this section.

We introduce the *expectation vector* of the vector \mathbf{Y} :

$$E\mathbf{Y} = (EY_1, \dots, EY_m)',$$

the *covariance vector* of X and \mathbf{Y} :

$$\Sigma_{X,\mathbf{Y}} = (\text{cov}(X, Y_1), \dots, \text{cov}(X, Y_m))'$$

and the *covariance matrix* of \mathbf{Y} :

$$\Sigma_{\mathbf{Y}} = (\text{cov}(Y_i, Y_j))_{i,j=1,\dots,m},$$

where we assume that all quantities are well-defined and finite.

The following auxiliary result gives a complete answer to the approximation problem of X in the class \mathcal{L}' of linear functions Y of the random variables Y_i with respect to quadratic risk $E[(X - Y)^2]$.

Proposition 6.1.1 (Minimum risk estimation by linear functions)

Assume that $\text{var}(X) < \infty$ and $\text{var}(Y_i) < \infty$, $i = 1, \dots, m$. Then the following statements hold.

(1) Let (a_0, \mathbf{a}) be any solution of the system of linear equations

$$a_0 = EX - \mathbf{a}' E\mathbf{Y}, \quad \Sigma'_{X,\mathbf{Y}} = \mathbf{a}' \Sigma_{\mathbf{Y}}, \quad (6.1.3)$$

and $\widehat{Y} = a_0 + \mathbf{a}' \mathbf{Y}$. Then for any $Y \in \mathcal{L}'$ the risk $E[(X - Y)^2]$ is bounded from below by

$$E[(X - Y)^2] \geq E[(X - \hat{Y})^2] = \text{var}(X) - \mathbf{a}' \Sigma_{\mathbf{Y}} \mathbf{a}, \quad (6.1.4)$$

and the right-hand side does not depend on the particular choice of the solution (a_0, \mathbf{a}) to (6.1.3). This means that any $\hat{Y} \in \mathcal{L}'$ with (a_0, \mathbf{a}) satisfying (6.1.3) is a minimizer of the risk $E[(X - Y)^2]$. Conversely, (6.1.3) is a necessary condition for \hat{Y} to be a minimizer of the risk.

(2) The estimator \hat{Y} of X introduced in (1) satisfies the equations

$$EX = E\hat{Y}, \quad \text{cov}(X, Y_i) = \text{cov}(\hat{Y}, Y_i), \quad i = 1, \dots, m. \quad (6.1.5)$$

(3) If $\Sigma_{\mathbf{Y}}$ has inverse, then there exists a unique minimizer \hat{Y} of the risk $E[(X - Y)^2]$ in the class \mathcal{L}' given by

$$\hat{Y} = EX + \Sigma'_{X, \mathbf{Y}} \Sigma_{\mathbf{Y}}^{-1} (\mathbf{Y} - E\mathbf{Y}). \quad (6.1.6)$$

with risk given by

$$E[(X - \hat{Y})^2] = \text{var}(X) - \Sigma'_{X, \mathbf{Y}} \Sigma_{\mathbf{Y}}^{-1} \Sigma_{X, \mathbf{Y}} \quad (6.1.7)$$

$$= \text{var}(X) - \text{var}(\hat{Y}). \quad (6.1.8)$$

It is not difficult to see that (6.1.3) always has a solution (a_0, \mathbf{a}) (we have $m + 1$ linear equations for the $m + 1$ variables a_i), but it is not necessarily unique. However, any $\hat{Y} = a_0 + \mathbf{a}' \mathbf{Y}$ with (a_0, \mathbf{a}) satisfying (6.1.3) has the same (minimal) risk.

Relations (6.1.7)-(6.1.8) imply that

$$\text{var}(\hat{Y}) = \Sigma'_{X, \mathbf{Y}} \Sigma_{\mathbf{Y}}^{-1} \Sigma_{X, \mathbf{Y}}.$$

and that \hat{Y} and $X - \hat{Y}$ are uncorrelated.

Proof. (1) We start by verifying necessary conditions for the existence of a minimizer \hat{Y} of the risk in the class \mathcal{L}' . In particular, we will show that (6.1.3) is a necessary condition for $\hat{Y} = a_0 + \mathbf{a}' \mathbf{Y}$ to minimize the risk. Since the smallest risk $E[(X - Y)^2]$ for any $Y = a_0 + \mathbf{a}' \mathbf{Y} \in \mathcal{L}'$ can be written in the form

$$\inf_{\mathbf{a}, a_0} E \left[(X - (a_0 + \mathbf{a}' \mathbf{Y}))^2 \right] = \inf_{\mathbf{a}} \inf_{a_0} E \left[(X - (a_0 + \mathbf{a}' \mathbf{Y}))^2 \right],$$

one can use a two-step minimization procedure:

- (a) Fix \mathbf{a} and minimize the risk $E[(X - Y)^2]$ with respect to a_0 .
- (b) Plug the a_0 from (a) into the risk $E[(X - Y)^2]$ and minimize with respect to \mathbf{a} .

For fixed \mathbf{a} and any $Y \in \mathcal{L}'$, $E[(X - Y)^2] \geq \text{var}(X - Y)$ since $E(Z + c)^2 \geq \text{var}(Z)$ for any random variable Z and any constant $c \in \mathbb{R}$. Therefore the first of the equations in (6.1.3) determines a_0 . It ensures that $EX = E\hat{Y}$. Since we

fixed \mathbf{a} , the minimizer a_0 is a function of \mathbf{a} . Now plug this particular \mathbf{a} into the risk. Then straightforward calculation yields:

$$\begin{aligned}
 E[(X - Y)^2] &= \text{var}(X - Y) \\
 &= E \left[\left((X - EX) - \sum_{t=1}^m a_t (Y_t - EY_t) \right)^2 \right] \\
 &= \text{var}(X) + \text{var} \left(\sum_{t=1}^m a_t Y_t \right) - 2 \text{cov} \left(X, \sum_{t=1}^m a_t Y_t \right) \\
 &= \text{var}(X) + \sum_{t=1}^m \sum_{s=1}^m a_t a_s \text{cov}(Y_t, Y_s) - 2 \sum_{t=1}^m a_t \text{cov}(X, Y_t). \quad (6.1.9)
 \end{aligned}$$

Differentiating the latter relation with respect to a_k and setting the derivatives equal to zero, one obtains the system of linear equations

$$0 = \sum_{t=1}^m a_t \text{cov}(Y_k, Y_t) - \text{cov}(X, Y_k), \quad k = 1, \dots, m.$$

Using the notation introduced at the beginning of this section, we see that the latter equation says nothing but

$$\Sigma'_{X, \mathbf{Y}} = \mathbf{a}' \Sigma_{\mathbf{Y}}, \quad (6.1.10)$$

which is the desired second equation in (6.1.3).

So far we have proved that the coefficients (a_0, \mathbf{a}) of any minimizer $\hat{Y} = a_0 + \mathbf{a}' \mathbf{Y}$ of the risk $E[(X - Y)^2]$ in the class \mathcal{L}' necessarily satisfy relation (6.1.3). To complete the proof it remains to show that any solution to (6.1.3) minimizes the risk $E[(X - Y)^2]$ in \mathcal{L}' . One way to show this is by considering the matrix of second partial derivatives of (6.1.9) as a function of \mathbf{a} . Direct calculation shows that this matrix is $\Sigma_{\mathbf{Y}}$. Any covariance matrix is non-negative definite which condition is sufficient for the existence of a minimum of the function (6.1.9) at \mathbf{a} satisfying the necessary condition (6.1.3). A unique minimizer exists if the matrix of second partial derivatives is positive definite. This condition is satisfied if and only if $\Sigma_{\mathbf{Y}}$ is invertible.

An alternative way to verify that any \hat{Y} with (a_0, \mathbf{a}) satisfying (6.1.3) minimizes the risk goes as follows. Pick any $Y \in \mathcal{L}'$ with representation $Y = b_0 + \mathbf{b}' \mathbf{Y}$. Then

$$\begin{aligned}
 E[(X - Y)^2] &\geq \text{var}(X - Y) \\
 &= E \left[\left([(X - EX) - \mathbf{a}' (\mathbf{Y} - E\mathbf{Y})] + (\mathbf{a} - \mathbf{b})' (\mathbf{Y} - E\mathbf{Y}) \right)^2 \right].
 \end{aligned} \quad (6.1.11)$$

Since the coefficients a_t satisfy relation (6.1.10) it is not difficult to verify that the random variables $X - \mathbf{a}' \mathbf{Y}$ and $(\mathbf{a} - \mathbf{b})' \mathbf{Y}$ are uncorrelated. Hence we conclude from (6.1.11) and (6.1.10) that

$$\begin{aligned}
E[(X - Y)^2] &\geq \text{var}(X - \mathbf{a}' \mathbf{Y}) + \text{var}((\mathbf{a} - \mathbf{b})' \mathbf{Y}) \\
&\geq \text{var}(X - \mathbf{a}' \mathbf{Y}) \\
&= \text{var}(X) + \text{var}(\mathbf{a}' \mathbf{Y}) - 2 \text{cov}(X, \mathbf{a}' \mathbf{Y}) \\
&= \text{var}(X) + \mathbf{a}' \Sigma_{\mathbf{Y}} \mathbf{a} - 2 \mathbf{a}' \Sigma_{X, \mathbf{Y}} \\
&= \text{var}(X) - \mathbf{a}' \Sigma_{\mathbf{Y}} \mathbf{a}.
\end{aligned}$$

This relation implies that for any $Y \in \mathcal{L}'$ the risk $E[(X - Y)^2]$ is bounded from below by the risk $E[(X - (a_0 + \mathbf{a}' \mathbf{Y}))^2]$ for any (a_0, \mathbf{a}) satisfying (6.1.3). It remains to show that the risk does not depend on the particular choice of (a_0, \mathbf{a}) . Suppose both $\hat{Y}, \tilde{Y} \in \mathcal{L}'$ have coefficients satisfying (6.1.3). But then $E[(X - \hat{Y})^2] \geq E[(X - \tilde{Y})^2] \geq E[(X - \hat{Y})^2]$. Hence they have the same risk.

(2) We have to show the equivalence of (6.1.3) and (6.1.5). If (6.1.3) holds,

$$\hat{Y} = a_0 + \mathbf{a}' \mathbf{Y} = EX + \mathbf{a}' (\mathbf{Y} - E\mathbf{Y}),$$

and hence the identity $E\hat{Y} = EX$ is obvious. If (6.1.5) holds, take expectations in $\hat{Y} = a_0 + \mathbf{a}' \mathbf{Y}$ to conclude that $a_0 = EX - \mathbf{a}' E\mathbf{Y}$.

It is straightforward to see that

$$\text{cov}(\hat{Y}, Y_i) = \text{cov}(\mathbf{a}' \mathbf{Y}, Y_i) = \mathbf{a}' \Sigma_{Y_i, \mathbf{Y}}, \quad i = 1, \dots, m. \quad (6.1.12)$$

Assuming (6.1.3), the latter relations translate into

$$\Sigma'_{\hat{Y}, \mathbf{Y}} = \mathbf{a}' \Sigma_{\mathbf{Y}} = \Sigma'_{X, \mathbf{Y}}.$$

This proves the equality of the covariances in (6.1.5). Conversely, assuming (6.1.5) and again exploiting (6.1.12), it is straightforward to see that

$$\text{cov}(X, Y_i) = \mathbf{a}' \Sigma_{Y_i, \mathbf{Y}}, \quad i = 1, \dots, m,$$

implying the second relation in (6.1.3).

(3) From the first equation of (6.1.3) we know that any minimizer \hat{Y} of the risk in \mathcal{L}' can be written in the form

$$\hat{Y} = a_0 + \sum_{t=1}^m a_t Y_t = [EX - \mathbf{a}' E\mathbf{Y}] + \mathbf{a}' \mathbf{Y} = EX + \mathbf{a}' (\mathbf{Y} - E\mathbf{Y}).$$

Moreover, the system of linear equations $\Sigma'_{X, \mathbf{Y}} = \mathbf{a}' \Sigma_{\mathbf{Y}}$ in (6.1.3) has a unique solution if and only if $\Sigma_{\mathbf{Y}}^{-1}$ exists, and then

$$\Sigma'_{X, \mathbf{Y}} \Sigma_{\mathbf{Y}}^{-1} = \mathbf{a}'.$$

Plugging the latter relation into \hat{Y} , we obtain

$$\widehat{Y} = EX + \Sigma'_{X,Y} \Sigma_Y^{-1} (Y - EY).$$

This is the desired relation (6.1.6) for \widehat{Y} . The risk is derived in a similar way by taking into account the right-hand side of relation (6.1.4). This proves (6.1.7). Relation (6.1.8) follows by observing that $\text{var}(\widehat{Y}) = \text{var}(a'Y) = a'\Sigma_Y a$. \square

Both relations (6.1.3) and (6.1.5) determine the minimum risk estimator \widehat{Y} of X in the class \mathcal{L}' of linear functions of the Y_t 's. Because of their importance they get a special name.

Definition 6.1.2 (Normal equations, linear Bayes estimator)

Each of the equivalent relations (6.1.3) and (6.1.5) is called the normal equations. The minimum risk estimator $\widehat{Y} = a_0 + a'Y$ in the class \mathcal{L}' of linear functions of the Y_t 's, which is determined by the normal equations, is the linear Bayes estimator of X .

The name “linear Bayes estimator” is perhaps not most intuitive in this general context. We choose it because linear Bayes estimation will be applied to $X = \mu(\theta_i)$ in the next sections, where we want to compare it with the more complex Bayes estimator of $\mu(\theta_i)$ introduced in Chapter 5.

6.2 The Bühlmann Model

Now we return to our original problem of determining the minimum risk estimator of $\mu(\theta_i)$ in the class \mathcal{L} , see (6.0.1). An analysis of the proof of Proposition 6.1.1 shows that only expectations, variances and covariances were needed to determine the linear Bayes estimator. For this particular reason we introduce a model which is less restrictive than the general heterogeneity model; see Definition 5.1.1. The following model fixes the conditions for linear Bayes estimation.

Definition 6.2.1 (The Bühlmann model)

- (1) *The i th policy is described by the pair $(\theta_i, (X_{i,t})_{t \geq 1})$, where the random parameter θ_i is the heterogeneity parameter and $(X_{i,t})_{t \geq 1}$ is the sequence of claim sizes or claim numbers in the policy.*
- (2) *The pairs $(\theta_i, (X_{i,t})_{t \geq 1})$ are mutually independent.*
- (3) *The sequence (θ_i) is iid.*
- (4) *Conditionally on θ_i , the $X_{i,t}$'s are independent and their expectation and variance are given functions of θ_i :*

$$\mu(\theta_i) = E(X_{i,t} | \theta_i) \quad \text{and} \quad v(\theta_i) = \text{var}(X_{i,t} | \theta_i).$$

Since the functions $\mu(\theta_i)$ and $v(\theta_i)$ only depend on θ_i , it follows that $(\mu(\theta_i))$ and $(v(\theta_i))$ are iid sequences. It will be convenient to use the following notation:

$$\mu = E\mu(\theta_i), \quad \lambda = \text{var}(\mu(\theta_i)) \quad \text{and} \quad \varphi = Ev(\theta_i).$$

The Bühlmann model differs from the heterogeneity model in the following aspects:

- The sequence $((X_{i,t})_{t \geq 1})_{i \geq 1}$ consists of independent components $(X_{i,t})_{t \geq 1}$ which are not necessarily identically distributed.
- In particular, the $X_{i,t}$'s inside and across the policies can have different distributions.
- Only the conditional expectation $\mu(\theta_i)$ and the conditional variance $v(\theta_i)$ are the same for $X_{i,t}$, $t = 1, 2, \dots$. The remaining distributional characteristics of the $X_{i,t}$'s are not fixed.

The heterogeneity model is a special case of the Bühlmann model insofar that in the former case the random variables $X_{i,t}$, $t = 1, 2, \dots$, are iid given θ_i and that the $X_{i,t}$'s are identically distributed for all i, t .

We mention that the first two moments of the $X_{i,t}$'s are the same for all i and t , and so are the covariances. Since we will make use of these facts quite often, we collect here some of the relations needed.

Lemma 6.2.2 *Assume the conditions of the Bühlmann model and that the variances $\text{var}(X_{i,t})$ are finite for all i and t . Then the following relations are satisfied for $i \geq 1$ and $t \neq s$:*

$$\begin{aligned} EX_{i,t} &= E[E(X_{i,t} | \theta_i)] = E\mu(\theta_i) = \mu, \\ E(X_{i,t}^2) &= E[E(X_{i,t}^2 | \theta_i)] = E[\text{var}(X_{i,t} | \theta_i)] + E[(E(X_{i,t} | \theta_i))^2] \\ &= \varphi + E[(\mu(\theta_i))^2] = \varphi + \lambda + \mu^2, \\ \text{var}(X_{i,t}) &= \varphi + \lambda, \\ \text{cov}(X_{i,t}, X_{i,s}) &= E[E(X_{i,t} - EX_{i,1} | \theta_i) E(X_{i,s} - EX_{i,1} | \theta_i)] \\ &= \text{var}(\mu(\theta_i)) = \lambda, \\ \text{cov}(\mu(\theta_i), X_{i,t}) &= E[(\mu(\theta_i) - EX_{i,1}) E[X_{i,t} - EX_{i,1} | \theta_i]] = \text{var}(\mu(\theta_i)) = \lambda. \end{aligned}$$

Remark 6.2.3 By virtue of Lemma 6.2.2, the covariance matrix $\Sigma_{\mathbf{X}_i}$ is rather simple:

$$\text{cov}(X_{i,t}, X_{i,s}) = \begin{cases} \lambda + \varphi & \text{if } t = s, \\ \lambda & \text{if } t \neq s. \end{cases}$$

Therefore the inverse of $\Sigma_{\mathbf{X}_i}$ exists if and only if $\varphi > 0$, i.e., $\text{var}(X_{i,t} | \theta_i)$ is not equal to zero a.s. This is a very natural condition. Indeed, if $\varphi = 0$ one has $X_{i,t} = \mu(\theta_i)$ a.s., i.e., there is no variation inside the policies.

6.3 Linear Bayes Estimation in the Bühlmann Model

Writing

$$\mathbf{Y} = \text{vec}(\mathbf{X}_1, \dots, \mathbf{X}_r) = (X_{1,1}, \dots, X_{1,n_1}, \dots, X_{r,1}, \dots, X_{r,n_r})',$$

$$\mathbf{a} = \text{vec}(\mathbf{a}_1, \dots, \mathbf{a}_r) = (a_{1,1}, \dots, a_{1,n_1}, \dots, a_{r,1}, \dots, a_{r,n_r})',$$

we can identify \mathcal{L} in (6.0.1) and \mathcal{L}' in (6.1.2). Then Proposition 6.1.1 applies.

Theorem 6.3.1 (Linear Bayes estimator in the Bühlmann model)

Consider the Bühlmann model. Assume $\text{var}(X_{i,t}) < \infty$ for all i, t and $\varphi > 0$. Then the linear Bayes estimator $\hat{\mu}_{\text{LB}} = a_0 + \mathbf{a}' \mathbf{Y}$ of $\mu(\theta_i) = E(X_{i,t} | \theta_i)$ in the class \mathcal{L} of the linear functions of the data $\mathbf{X}_1, \dots, \mathbf{X}_r$ exists, is unique and given by

$$\hat{\mu}_{\text{LB}} = (1 - w)\mu + w\bar{X}_i, \quad (6.3.13)$$

where

$$w = \frac{n_i \lambda}{\varphi + n_i \lambda}. \quad (6.3.14)$$

The risk of $\hat{\mu}_{\text{LB}}$ is given by

$$\rho(\hat{\mu}_{\text{LB}}) = (1 - w)\lambda.$$

Similarly to the Bayes estimator $\hat{\mu}_{\text{B}}$ we observe that $\hat{\mu}_{\text{LB}}$ only depends on the data \mathbf{X}_i of the i th policy. This is not surprising in view of the independence of the policies.

It is worthwhile comparing the linear Bayes estimator (6.3.13) with the Bayes estimator in the special case of Example 5.2.4. Both are weighted means of $E X_{i,t} = \mu$ and \bar{X}_i . In general, the Bayes estimator does not have such a linear representation; see for example Exercise 2 on p. 196.

Proof. We have to verify the normal equations (6.1.3) for $X = \mu(\theta_i)$ and \mathbf{Y} as above. Since the policies are independent, $X_{i,t}$ and $X_{j,s}$, $i \neq j$, are independent. Hence

$$\text{cov}(X_{i,t}, X_{j,s}) = 0 \quad \text{for } i \neq j \text{ and any } s, t.$$

Therefore the second equation in (6.1.3) turns into

$$\mathbf{0} = \mathbf{a}'_j \Sigma_{\mathbf{X}_j}, \quad j \neq i, \quad \Sigma'_{\mu(\theta_i), \mathbf{X}_i} = \mathbf{a}'_i \Sigma_{\mathbf{X}_i}.$$

For $j \neq i$, $\mathbf{a}_j = \mathbf{0}$ is the only possible solution since $\Sigma_{\mathbf{X}_j}^{-1}$ exists; see Remark 6.2.3. Therefore the second equation in (6.1.3) turns into

$$\Sigma'_{\mu(\theta_i), \mathbf{X}_i} = \mathbf{a}'_i \Sigma_{\mathbf{X}_i}, \quad \mathbf{a}_j = \mathbf{0}, \quad j \neq i. \quad (6.3.15)$$

Since $EX_{i,t} = \mu$ and also $E\mu(\theta_i) = \mu$, see Lemma 6.2.2, the first equation in (6.1.3) yields

$$a_0 = \mu(1 - a_{i,\cdot}), \quad (6.3.16)$$

where $a_{i,\cdot} = \sum_{t=1}^{n^i} a_{i,t}$. Relations (6.3.15) and (6.3.16) imply that the linear Bayes estimator of $\mu(\theta_i)$ only depends on the data \mathbf{X}_i of the i th policy. For this reason, we suppress the index i in the notation for the rest of the proof.

An appeal to (6.3.15) and Lemma 6.2.2 yields

$$\begin{aligned} \lambda &= a_t \operatorname{var}(X_1) + (a_\cdot - a_t) \operatorname{var}(\mu(\theta)) = a_t (\lambda + \varphi) + (a_\cdot - a_t) \lambda \\ &= a_t \varphi + a_\cdot \lambda, \quad t = 1, \dots, n. \end{aligned} \quad (6.3.17)$$

This means that $a_t = a_1$, $t = 1, \dots, n$, with

$$a_1 = \frac{\lambda}{\varphi + n \lambda}.$$

Then, by (6.3.16),

$$a_0 = \mu(1 - n a_1) = \mu \frac{\varphi}{\varphi + n \lambda}.$$

Finally, write $w = n a_1$. Then

$$\hat{\mu}_{\text{LB}} = a_0 + \mathbf{a}' \mathbf{Y} = (1 - w) \mu + a_1 X_\cdot = (1 - w) \mu + w \bar{X}.$$

Now we are left to derive the risk of $\hat{\mu}_{\text{LB}}$. From (6.1.8) and Lemma 6.2.2 we know that

$$\rho(\hat{\mu}_{\text{LB}}) = \operatorname{var}(\mu(\theta)) - \operatorname{var}(\hat{\mu}_{\text{LB}}) = \lambda - \operatorname{var}(\hat{\mu}_{\text{LB}}).$$

Moreover,

$$\begin{aligned} \operatorname{var}(\hat{\mu}_{\text{LB}}) &= \operatorname{var}(w \bar{X}) \\ &= w^2 [E[\operatorname{var}(\bar{X} \mid \theta)] + \operatorname{var}(E(\bar{X} \mid \theta))] \\ &= w^2 [n^{-1} E[\operatorname{var}(X_1 \mid \theta)] + \operatorname{var}(\mu(\theta))] \\ &= w^2 [n^{-1} \varphi + \lambda] \\ &= \lambda \frac{n \lambda}{\varphi + n \lambda}. \end{aligned}$$

Now the risk is given by

$$\rho(\hat{\mu}_{\text{LB}}) = \lambda - \lambda \frac{n \lambda}{\varphi + n \lambda} = (1 - w) \lambda$$

This concludes the proof. \square

In what follows, we suppress the dependence on the policy index i in the notation.

Example 6.3.2 (The linear Bayes estimator for Poisson distributed claim numbers and a gamma distributed heterogeneity parameter)

We assume the conditions of Example 5.2.4 and use the same notation. We want to calculate the linear Bayes estimator $\hat{\mu}_{LB}$ for $\mu(\theta) = E(X_1|\theta) = \theta$. With $EX_1 = E\theta = \gamma/\beta$ and $\text{var}(\theta) = \gamma/\beta^2$ we have

$$\varphi = E[\text{var}(X_1 | \theta)] = E\theta = \gamma/\beta,$$

$$\lambda = \text{var}(\theta) = \gamma/\beta^2.$$

Hence the weight w in (6.3.14) turns into

$$w = \frac{n \lambda}{\varphi + n \lambda} = \frac{n \gamma/\beta^2}{\gamma/\beta + n \gamma/\beta^2} = \frac{n}{\beta + n}.$$

From Example 5.2.4 we conclude that the linear Bayes and the Bayes estimator coincide and have the same risk. In general we do not know the form of the Bayes estimator $\hat{\mu}_B$ of $\mu(\theta)$ and therefore we cannot compare it with the linear Bayes estimator $\hat{\mu}_{LB}$. \square

Bühlmann [29] coined the name (linear) *credibility estimator* for the linear Bayes estimator

$$\hat{\mu}_{LB} = (1 - w)\mu + w\bar{X}, \quad w = \frac{n \lambda}{\varphi + n \lambda} = \frac{n}{\varphi/\lambda + n},$$

w being the *credibility weight*. The larger w the more credible is the information contained in the data of the i th policy and the less important is the overall information about the portfolio represented by the expectation $\mu = E\mu(\theta)$. Since $w \rightarrow 1$ as $n \rightarrow \infty$ the credibility of the information in the policy increases with the sample size. But the size of w is also influenced by the ratio

$$\frac{\varphi}{\lambda} = \frac{E[\text{var}(X_t | \theta)]}{\text{var}(\mu(\theta))} = \frac{E[(X_t - \mu(\theta))^2]}{E[(\mu(\theta) - \mu)^2]}.$$

If φ/λ is small, w is close to 1. This phenomenon occurs if the variation of the claim sizes/claim numbers X_t in the individual policy is small compared to the variation in the whole portfolio. This can happen if there is a lot of heterogeneity in the portfolio, i.e., there is a lot of variation across the policies. This means that the expected claim size/claim number of the overall portfolio is quite meaningless when one has to determine the premium in a policy.

Any claim in the policy can be decomposed as follows

$$X_t = [X_t - \mu(\theta)] + [\mu(\theta) - \mu] + \mu. \quad (6.3.18)$$

The random variables $X_t - \mu(\theta)$ and $\mu(\theta) - \mu$ are uncorrelated. The quantity μ represents the expected claim number/claim size X_t in the portfolio. The difference $\mu(\theta) - \mu$ describes the deviation of the average claim number/claim

size in the individual policy from the overall mean, whereas $X_t - \mu(\theta)$ is the (annual, say) fluctuation of the claim sizes/claim numbers X_t around the policy average. The credibility estimator $\hat{\mu}_{LB}$ is based on the decomposition (6.3.18). The resulting formula for $\hat{\mu}_{LB}$ as a weighted average of the policy and portfolio experience is essentially a consequence of (6.3.18).

Comments

Linear Bayes estimation seems to be quite restrictive since the random variable $\mu(\theta_i) = E(X_{i,t} | \theta_i)$ is approximated only by linear functions of the data $X_{i,t}$ in the i th policy. However, the general linear Bayes estimation procedure of Section 6.1 also allows one to calculate the minimum risk estimator of $\mu(\theta_i)$ in the class of all linear functions of any functions of the $X_{i,t}$'s which have finite variance. For example, the space \mathcal{L}' introduced in (6.1.2) can be interpreted as the set of all linear functions of the powers $X_{i,t}^k$, $k \leq p$, for some integer $p \geq 1$. Then minimum linear risk estimation amounts to the best approximation of $\mu(\theta_i)$ by all polynomials of the $X_{i,t}$'s of order p . We refer to Exercise 1 on p. 211 for an example with quadratic polynomials.

6.4 The Bühlmann-Straub Model

The Bühlmann model was further refined by Hans Bühlmann and Erwin Straub [31]. Their basic idea was to allow for heterogeneity inside each policy: each claim number/claim size $X_{i,t}$ is subject to an individual risk exposure expressed by an additional parameter $p_{i,t}$. These weights express our knowledge about the *volume* of $X_{i,t}$. For example, you may want to think of $p_{i,t}$ as the size of a particular house which is insured against fire damage or of the type of a particular car. In this sense, $p_{i,t}$ can be interpreted as *risk unit per time unit*, for example, per year.

In his monograph [141], Straub illustrated the meaning of *volume* by giving the different positions of the Swiss Motor Liability Tariff. The main positions are private cars, automobiles for goods transport, motor cycles, buses, special risks and short term risks. Each of these risks is again subdivided into distinct subclasses. He also refers to the positions of the German Fire Tariff which includes warehouses, mines and foundries, stone and earth, iron and metal works, chemicals, textiles, leather, paper and printing, wood, nutritionals, drinks and tobacco, and other risks. The variety of risks in these portfolios is rather high, and the notion of volume aims at assigning a quantitative measure for them.

Definition 6.4.1 (The Bühlmann-Straub model)

The model is defined by the requirements (1)-(3) in Definition 6.2.1, and Condition (4) is replaced by

- (4') Conditionally on θ_i , the $X_{i,t}$'s are independent and their expectation and variance are given functions of θ_i :

$$\mu(\theta_i) = E(X_{i,t} | \theta_i) \quad \text{and} \quad \text{var}(X_{i,t} | \theta_i) = v(\theta_i)/p_{i,t}.$$

The weights $p_{i,t}$ are pre-specified deterministic positive risk units.

Since the heterogeneity parameters θ_i are iid, the sequences $(\mu(\theta_i))$ and $(v(\theta_i))$ are iid.

We use the same notation as in the Bühlmann model

$$\mu = E\mu(\theta_i), \quad \lambda = \text{var}(\mu(\theta_i)) \quad \text{and} \quad \varphi = Ev(\theta_i).$$

The following result is the analog of Theorem 6.3.1 for the linear Bayes estimator in the Bühlmann-Straub model.

Theorem 6.4.2 (Linear Bayes estimation in the Bühlmann-Straub model)
Assume $\text{var}(X_{i,t}) < \infty$ for $i, t \geq 1$ and $\Sigma_{\mathbf{X}_i}$ is invertible for every i . Then the linear Bayes estimator $\hat{\mu}_{\text{LB}}$ of $\mu(\theta_i)$ in the class \mathcal{L} of linear functions of the data $\mathbf{X}_1, \dots, \mathbf{X}_r$ exists, is unique and given by

$$\hat{\mu}_{\text{LB}} = (1 - w)\mu + w\bar{X}_{i,\cdot},$$

where

$$w = \frac{\lambda p_{i,\cdot}}{\varphi + \lambda p_{i,\cdot}} \quad \text{and} \quad \bar{X}_{i,\cdot} = \frac{1}{p_{i,\cdot}} \sum_{t=1}^{n_i} p_{i,t} X_{i,t}$$

The risk of $\hat{\mu}_{\text{LB}}$ is given by

$$\rho(\hat{\mu}_{\text{LB}}) = (1 - w)\lambda.$$

The proof of this result is completely analogous to the Bühlmann model (Theorem 6.3.1) and left as an exercise. We only mention that the normal equations in the i th portfolio, see Proposition 6.1.1, and the corresponding relations (6.3.16) and (6.3.17) in the proof of Theorem 6.3.1 boil down to the equations

$$\begin{aligned} a_0 &= \mu(1 - a_{i,\cdot}), \\ \lambda a_{i,\cdot} + \varphi \frac{a_{i,t}}{p_{i,t}}, \quad t &= 1, \dots, n. \end{aligned}$$

Comments

In the Bühlmann and Bühlmann-Straub models the global parameters μ , φ , λ of the portfolio have to be estimated from the data contained in all policies. In the exercises below we hint at some possible estimators of these quantities; see also the references below.

The classical work on credibility theory and experience rating is summarized in Bühlmann's classic text [29]. A more recent textbook treatment aimed at actuarial students is Kaas et al. [77]. Textbook treatments of credibility theory and related statistical questions can be found in the textbooks by Klugman et al. [86], Sundt [143], Straub [141]. A recent treatment of credibility theory and its statistics is Bühlmann and Gisler [30].

Exercises

- (1) We consider the i th policy in the heterogeneity model and suppress the dependence on i in the notation. Assume we have one claim number X in the policy which is $\text{Pois}(\theta)$ distributed, given some positive random variable θ . Assume that the moments $m_k = E(\theta^k) < \infty$, $k = 1, 2, 3, 4$, are known.
 - (a) Determine the linear Bayes estimator $\hat{\theta}$ for $\mu(\theta) = E(X | \theta) = \theta$ based on X only in terms of X , m_1 , m_2 . Express the minimal linear Bayes risk $\rho(\hat{\theta})$ as a function of m_1 and m_2 .
 - (b) Now we want to find the best estimator $\tilde{\theta}_{\text{LB}}$ of θ with respect to the quadratic risk $\rho(\tilde{\theta}) = E[(\theta - \tilde{\theta})^2]$ in the class of linear functions of X and $X(X - 1)$:

$$\tilde{\theta} = a_0 + a_1 X + a_2 X(X - 1), \quad a_0, a_1, a_2 \in \mathbb{R}.$$

This means that $\tilde{\theta}$ is the linear Bayes estimator of θ based on the data $\mathbf{X} = (X, X(X - 1))'$. Apply the normal equations to determine a_0, a_1, a_2 . Express the relevant quantities by the moments m_k .

Hint: Use the well-known identity $EY^{(k)} = \lambda^k$ for the factorial moments $EY^{(k)} = E[Y(Y - 1) \cdots (Y - k + 1)]$, $k \geq 1$, of a random variable $Y \sim \text{Pois}(\lambda)$.

- (2) For Exercise 2 on p. 196 calculate the linear Bayes estimate of $p(\theta) = E(Y_1 | \theta)$ based on the data X_1, \dots, X_n and the corresponding linear Bayes risk. Compare the Bayes and the linear Bayes estimators and their risks.
- (3) For Exercise 4 on p. 197 calculate the linear Bayes estimator of $E(X_1 | \theta)$ and the corresponding linear Bayes risk. Compare the Bayes and the linear Bayes estimators and their risks.
- (4) For Exercise 5 on p. 197 calculate the linear Bayes estimator of $E(X_1 | \theta)$ and the corresponding linear Bayes risk. Compare the Bayes and the linear Bayes estimators and their risks.
- (5) Consider a portfolio with n independent policies.
 - (a) Assume that the claim numbers $X_{i,t}$, $t = 1, 2, \dots$, in the i th policy are independent and $\text{Pois}(p_{i,t}\theta_i)$ distributed, given θ_i . Assume that $p_{i,t} \neq p_{i,s}$ for some $s \neq t$. Are the conditions of the Bühlmann-Straub model satisfied?
 - (b) Assume that the claim sizes $X_{i,t}$, $t = 1, 2, \dots$, in the i th policy are independent and $\Gamma(\gamma_{i,t}, \beta_{i,t})$ distributed, given θ_i . Give conditions on $\gamma_{i,t}, \beta_{i,t}$ under which the Bühlmann-Straub model is applicable. Identify the parameters μ, φ, λ and $p_{i,t}$.
- (6) Consider the Bühlmann-Straub model with r policies, where the claim sizes/claim numbers $X_{i,t}$, $t = 1, 2, \dots$, in policy i are independent, given θ_i . Let w_i be positive weights satisfying $\sum_{i=1}^n w_i = 1$ and $\bar{X}_{i,\cdot} = p_i^{-1} \sum_{t=1}^{n_i} p_{i,t} X_{i,t}$ be the (weighted) sample mean in the i th policy.

- (a) Show that

$$\hat{\mu} = \sum_{i=1}^r w_i \bar{X}_{i \cdot} \quad (6.4.19)$$

- is an unbiased estimator of $\mu = E\mu(\theta_i) = E[E(X_{i,t} | \theta_i)]$.
- (b) Calculate the variance of $\hat{\mu}$ in (6.4.19).
- (c) Choose the weights w_i in such a way that $\text{var}(\hat{\mu})$ is minimized and calculate the minimal value $\text{var}(\hat{\mu})$.
- (7) Consider the Bühlmann-Straub model.

- (a) Guess what is estimated by the statistics

$$s_1 = \sum_{i=1}^n \sum_{t=1}^n p_{i,t} (X_{i,t} - \bar{X}_{i \cdot})^2 \quad \text{and} \quad s_2 = \sum_{i=1}^n w_i (\bar{X}_{i \cdot} - \hat{\mu})^2,$$

where w_i are the optimal weights derived in Exercise 6 above and $\hat{\mu}$ is defined in (6.4.19).

- (b) Calculate the expectations of s_1 and s_2 . Are your guesses from (a) confirmed by these calculations?
- (c) Calculate Es_2 with the weights $w_i = p_{i \cdot} / p_{..}$, where $p_{..} = \sum_{i=1}^n p_{i \cdot}$. Modify s_1 and s_2 such they become unbiased estimators of the quantities which are suggested by (b).

Part III

A Point Process Approach to Collective Risk Theory

The General Poisson Process

In this part of the book we return to the collective risk model which we studied in detail in Part I. The key will be the powerful notion of the *general Poisson process* or *Poisson random measure*, which was briefly touched on in Section 2.1.8. For example, we will interpret the Cramér-Lundberg model as a special Poisson process. This general point of view will allow us to see the results of Part I in a different light: various results, such as the order statistics property of the Poisson process or the independent increment property of the compound Poisson process will be simple consequences of the general theory. Of course, a general theory requires more effort before its applications can be considered. The reward will be elegance and transparency of the results.

Poisson random measures are particular *point processes* for which a rich theory exists. We start by giving a brief introduction to point processes and their distributions in Sections 7.1 and consider several examples. We will learn that the Cramér-Lundberg and renewal models can be interpreted in the point process context, and we will introduce the point process of exceedances, which plays a major role in extreme value theory; see the discussion in Section 9.2. In Section 7.2 we consider basic properties and examples of Poisson random measures and introduce *Poisson integrals*, i.e., integrals with respect to a Poisson random measure. Such integrals will be interpreted as claim numbers or total claim amounts in an abstract time-claim size space; see Chapter 8. In Section 7.3 we introduce various principles for constructing new Poisson random measures from a given one. A combination of the techniques from the present chapter will be the basis for the point process analysis of the collective risk model provided in Chapter 8.

7.1 The Notion of a Point Process

7.1.1 Definition and First Examples

We are concerned with the question:

What is a point process, how can we describe its distribution, and what are some simple examples?

Consider a sequence (X_n) of random vectors in the state space E and define, for $A \subset E$,

$$N(A) = \#\{i \geq 1 : X_i \in A\},$$

i.e., $N(A)$ counts the number of X_i 's falling into A . Naturally, $N(A) = N(A, \omega)$ is random for a given set A and, for fixed ω , $N(\cdot, \omega)$ defines an ordinary counting measure with atoms $X_i(\omega)$ on a suitable σ -field \mathcal{E} of subsets of E . This is the intuitive meaning of the point process N ; see also Figure 7.1.2 for an illustration.

The state space E , where the points live, is a Borel subset of a finite-dimensional Euclidean space and E is equipped with the σ -field $\mathcal{E} = \mathcal{B}(E)$ of the Borel sets generated by the open sets of E . It is convenient to write a point process using the Dirac measure ε_x at $x \in E$:

$$\varepsilon_x(A) = I_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases} \quad A \in \mathcal{E}.$$

For a given sequence $(x_i)_{i \geq 1}$ in E ,

$$m(A) = \sum_{i=1}^{\infty} \varepsilon_{x_i}(A) = \#\{i \geq 1 : x_i \in A\}, \quad A \in \mathcal{E},$$

defines a *counting measure* on \mathcal{E} which is called a *point measure* if $m(K) < \infty$ for all compact sets $K \subset E$.¹ This means that any compact set K must not contain infinitely many points x_i .

Let $M_p(E)$ be the space of all point measures on E equipped with the smallest σ -field $\mathcal{M}_p(E)$ which contains all sets of the form

$$\{m \in M_p(E) : m(A) \in B\} \tag{7.1.1}$$

for any $A \in \mathcal{E}$ and any Borel set $B \subset [0, \infty]$, i.e., it is the smallest σ -field making the maps $m \rightarrow m(A)$ measurable for all $A \in \mathcal{E}$.

Definition 7.1.1 (Definition of a point process)

A point process N with state space E is a measurable map from the underlying outcome space Ω equipped with a σ -field \mathcal{F} to $(M_p(E), \mathcal{M}_p(E))$.

In other words, a point process N is a random element or a random function which assumes point measures as values: for every $\omega \in \Omega$ the value $m(\cdot) = N(\cdot, \omega)$ is a point measure. In particular, $N(K) < \infty$ for compact sets $K \subset E$. In the context of counting measures it is rather natural that $N(A)$ may also assume the value ∞ for non-compact sets A .

The next result gives a justification of the fact that we may interpret a point process N as a collection $(N(A))_{A \in \mathcal{E}}$ of the random variables $N(A)$ with values in $\{0, 1, \dots, \infty\}$.

¹ Recall that a compact set $K \subset E$ is bounded and closed relative to the set E .

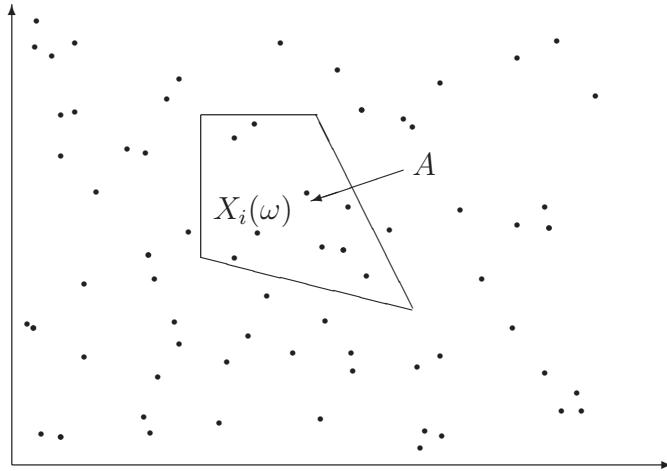


Figure 7.1.2 A configuration of random points X_i in $(0, \infty)^2$. The number of points that fall into the set A constitute the counting variable $N(A)$; in this case $N(A, \omega) = 9$.

Lemma 7.1.3 (A point process as a collection of random counting variables) *The mapping N from (Ω, \mathcal{F}) to $(M_p(E), \mathcal{M}_p(E))$ is a point process on E if and only if, for every $A \in \mathcal{E}$, $N(A)$ is a random variable with values in $\{0, 1, \dots, \infty\}$ such that $N(A) < \infty$ for compact $A \subset E$.*

Proof. Assume that N is a point process on E . By definition of a point process, the mapping $\omega \rightarrow N(\cdot, \omega)$ from (Ω, \mathcal{F}) to $(M_p(E), \mathcal{M}_p(E))$ is measurable. On the other hand, for a given Borel set $A \in \mathcal{E}$ the mapping $f_A : m \rightarrow m(A)$ from $(M_p(E), \mathcal{M}_p(E))$ to $([0, \infty], \mathcal{B}([0, \infty]))$ is measurable. This follows from the fact that the σ -field $\mathcal{M}_p(E)$ is generated from the sets (7.1.1). The composition map $N(A, \omega) = f_A(N(\cdot, \omega))$ is then also measurable, hence $N(A)$ is a random variable and, by definition of a point process, $N(A)$ is finite for compact A .

A proof of the converse, i.e., if $(N(A))_{A \in \mathcal{E}}$ is a collection of random variables with values in $\{0, 1, \dots, \infty\}$ with $N(A) < \infty$ for compact A , then N is a point process, can be found in Resnick [122], Proposition 3.1. \square

The point processes we are interested in can often be written in the form $N = \sum_{i=1}^{\infty} \varepsilon_{X_i}$ for a sequence (X_i) of d -dimensional random vectors such that, with probability 1, any bounded Borel set $B \in \mathcal{E}$ contains only finitely many points X_i . Then, for almost every $\omega \in \Omega$,

$$N(A, \omega) = \sum_{i=1}^{\infty} \varepsilon_{X_i(\omega)}(A), \quad A \in \mathcal{E},$$

defines a point measure on \mathcal{E} .

Assume that $m = \sum_{i=1}^{\infty} \varepsilon_{x_i}$ is a point measure on E . Let (y_i) be a subsequence of (x_i) containing all mutually distinct values x_i (with no repeats). Define the *multiplicity* of y_i as $n_i = \#\{j \geq 1 : y_i = x_j\}$. Then we may write $m = \sum_{i=1}^{\infty} n_i \varepsilon_{y_i}$. If $n_i = 1$ for all i , then m is called a *simple point measure*. Analogously, if the realizations of the point process N are simple point measures with probability 1, then N is a *simple point process*. Alternatively, a point process N with representation $N = \sum_{i=1}^{\infty} \varepsilon_{X_i}$ for a sequence of E -valued random vectors X_i is simple if all points X_i , $i = 1, 2, \dots$, are distinct with probability 1.

Example 7.1.4 (A renewal process defines a simple point process)

Consider a renewal sequence (T_i) , i.e., the T_i 's are the points of a random walk with iid positive step sizes Y_i :

$$T_0 = 0, \quad T_i = Y_1 + \cdots + Y_i, \quad i \geq 1. \quad (7.1.2)$$

By the strong law of large numbers, $T_i \uparrow \infty$ a.s. as $i \rightarrow \infty$ and therefore the realizations $(T_i(\omega))$ of (T_i) do not have finite limit points, with probability 1. Hence the random variable

$$N(A) = \sum_{i=1}^{\infty} \varepsilon_{T_i}(A) = \#\{i \geq 1 : T_i \in A\}$$

is finite a.s. for any bounded Borel set A of the state space $E = (0, \infty)$. Since $Y_i = T_i - T_{i-1} > 0$ a.s., all points T_i occur separated through time with probability 1, and therefore N is a simple point process. \square

Example 7.1.5 (The renewal model of non-life insurance mathematics as a simple point process)

Recall the renewal model of non-life insurance mathematics; see p. 71. The arrivals T_i of the claims are given by a renewal sequence; see (7.1.2). At time T_i a claim of size X_i occurs. We assume that (X_i) is an iid sequence of positive random variables, independent of (T_i) . The following random variables define a point process with state space $E = (0, \infty)^2$:

$$N(A) = \sum_{i=1}^{\infty} \varepsilon_{(T_i, X_i)}(A) = \#\{i \geq 1 : (T_i, X_i) \in A\}.$$

By the same argument as in Example 7.1.4, with probability 1, any bounded set $A \subset E$ contains only finitely many points (T_i, X_i) whose time components T_i occur separated through time. Hence, with probability 1, multiple points are excluded, and N is a simple point process. \square

Example 7.1.6 (Point process of exceedances)

Let (X_n) be a sequence of random variables and (u_n) a sequence of real

numbers. The *point process of exceedances* corresponding to the threshold u_n is given by

$$N_n(\cdot) = \sum_{i=1}^n \varepsilon_{n^{-1}i}(\cdot) I_{\{X_i > u_n\}}.$$

See Figure 7.1.7 for an illustration.

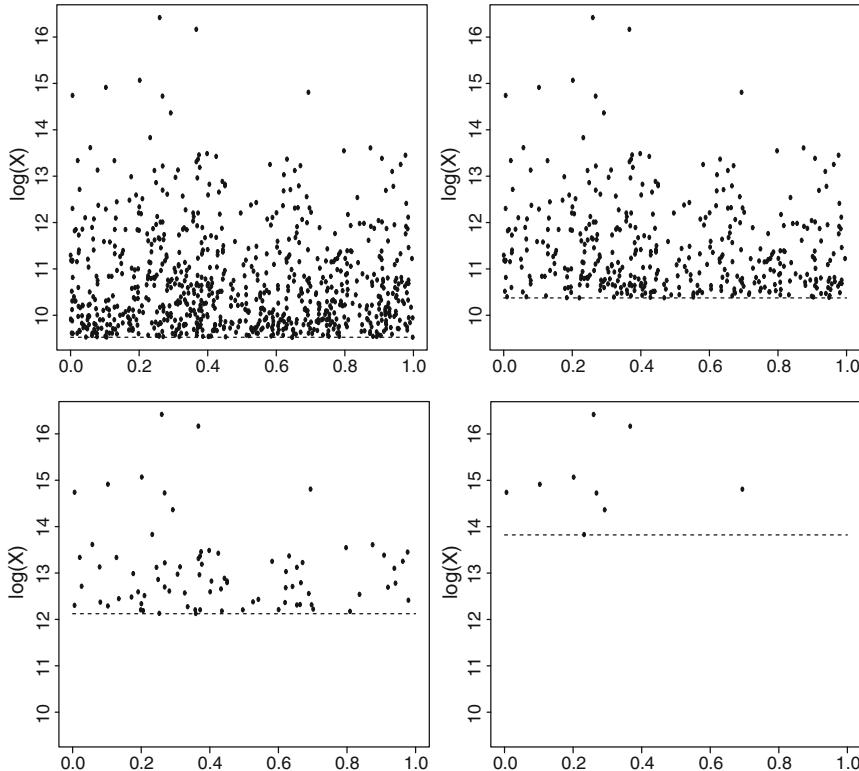


Figure 7.1.7 The point process of the exceedances of the logarithmic US industrial fire data with different thresholds $u_n = u(\alpha)$. The data are described in Example 3.2.11. The threshold $u(\alpha)$ is indicated as a dashed line. It represents the α -quantile of the data. Top left $\alpha = 0.90$, right $\alpha = 0.95$. Bottom left $\alpha = 0.99$, right $\alpha = 0.999$.

We choose the state space $E = (0, 1]$ which is common for all point processes N_n , $n = 1, 2, \dots$. Notice that N_n counts the number of exceedances of the threshold u_n by the sequence X_1, \dots, X_n . For example, take the whole interval $(0, 1]$. Then

$$\begin{aligned} N_n(0, 1] &= \#\{i : 0 < n^{-1}i \leq 1 \text{ and } X_i > u_n\} \\ &= \#\{i \leq n : X_i > u_n\}. \end{aligned}$$

Since there are only finitely many points $n^{-1}i$, $i = 1, \dots, n$, which are separated from each other, N_n is a simple point process.

We immediately see the close link with *extreme value theory*. For example, let $X_{(n-k+1)}$ denote the k th largest order statistic of the sample X_1, \dots, X_n . Then

$$\begin{aligned} \{N_n(0, 1] = 0\} &= \{\#\{i \leq n : X_i > u_n\} = 0\} \\ &= \{\text{None of the } X_i\text{'s, } i \leq n, \text{ exceeds } u_n\} \\ &= \{\max(X_1, \dots, X_n) \leq u_n\}, \\ \{N_n(0, 1] < k\} &= \{\#\{i \leq n : X_i > u_n\} < k\} \\ &= \{\text{Fewer than } k \text{ among the } X_i\text{'s, } i \leq n, \text{ exceed } u_n\} \\ &= \{\text{The order statistic } X_{(n-k+1)} \text{ does not exceed } u_n\} \\ &= \{X_{(n-k+1)} \leq u_n\}. \end{aligned}$$

We return to the point process of exceedances in the context of the convergence of affinely transformed maxima and upper order statistics of iid samples; see Section 9.2. \square

Example 7.1.8 (A non-simple point process)

Consider a continuous distribution function F on \mathbb{R} , and let $(Y_i)_{i \geq 0}$ be a sequence of iid random variables with distribution function \sqrt{F} . Then the sequence $X_i = \max(Y_{i-1}, Y_i)$, $i = 1, 2, \dots$, constitutes a strictly stationary² sequence which is *1-dependent*, i.e., whereas the neighbors X_i and X_{i+1} are dependent, for $k \geq 2$ the random variables X_i and X_{i+k} are independent. Moreover, X_i has distribution function

$$P(X_i \leq x) = P(Y_{i-1} \leq x, Y_i \leq x) = (\sqrt{F(x)})^2 = F(x), \quad x \in \mathbb{R}.$$

For $n \geq 2$ consider the point process

² Recall that (X_i) is strictly stationary if $(X_i) \stackrel{d}{=} (X_{i+h})$ for all positive integers h , where $\stackrel{d}{=}$ refers to equality of the finite-dimensional distributions.

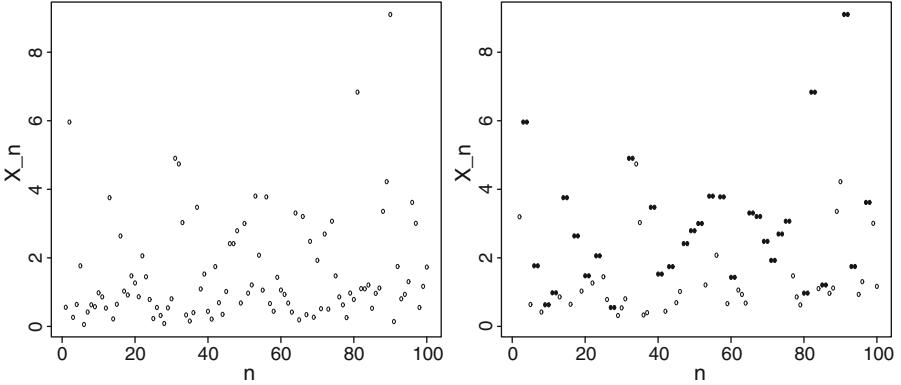


Figure 7.1.9 Left: Plot of 100 iid log-normal points Y_i . Right: The points $X_i = \max(Y_{i-1}, Y_i)$ of the non-simple point process N_{100} described in Example 7.1.8. One often sees pairs (X_i, X_{i+1}) with identical components (solid dots). This is in agreement with the calculations in Example 7.1.8: about one third of the Y_i 's satisfy $Y_{i-1} \leq Y_i$ and $Y_{i+1} < Y_i$ resulting in the ties $X_i = X_{i+1}$.

$$\begin{aligned} N_n &= \sum_{i=1}^n \varepsilon_{X_i} = \sum_{i=1}^n \varepsilon_{Y_i} I_{\{Y_i \geq Y_{i-1}\}} + \sum_{i=1}^n \varepsilon_{Y_{i-1}} I_{\{Y_i < Y_{i-1}\}} \\ &= \varepsilon_{Y_0} I_{\{Y_1 < Y_0\}} + \sum_{i=1}^{n-1} \varepsilon_{Y_i} [I_{\{Y_i \geq Y_{i-1}\}} + I_{\{Y_{i+1} < Y_i\}}] + \varepsilon_{Y_n} I_{\{Y_n \geq Y_{n-1}\}} \end{aligned}$$

with state space $E = \mathbb{R}$. This is a genuine non-simple point process. Indeed, for $i = 1, \dots, n-1$, and $n \geq 2$,

$$\begin{aligned} P(I_{\{Y_i \geq Y_{i-1}\}} + I_{\{Y_{i+1} < Y_i\}} = 2) &= P(Y_i \geq Y_{i-1}, Y_{i+1} < Y_i) \\ &= E[P(Y_1 \geq Y_0 | Y_1) P(Y_2 < Y_1 | Y_1)] \\ &= E[(P(Y_1 \geq Y_0 | Y_1))^2]. \end{aligned}$$

In the last step we have used the continuity of F . The probability on the right-hand side can be written as $E[F(Y_1)] = 1/3$ by observing that $\sqrt{F(Y_1)}$ has a uniform distribution on $(0, 1)$. Therefore it is rather likely that one observes two points X_i and X_{i-1} of the process N_n which coincide. Also notice that

$$E(I_{\{Y_i \geq Y_{i-1}\}} + I_{\{Y_{i+1} < Y_i\}}) = 2 P(Y_1 > Y_0) = 2 E\sqrt{F(Y_1)} = 1.$$

For an illustration of the points of the process N_n , see Figure 7.1.9. We also refer to Exercise 5(b) on p. 224 for an extension of this example to the case of a process with points which, with positive probability, have multiplicity $k \geq 2$. \square

7.1.2 Distribution and Laplace Functional

The realizations of a point process N are point measures. Therefore the *distribution* of N is defined on suitable subsets of point measures:

$$P_N(A) = P(N \in A), \quad A \in \mathcal{M}_p(E).$$

This distribution is not easy to imagine. Fortunately, the distribution of N is uniquely determined by the family of the finite-dimensional distributions of the random vectors

$$(N(A_1), \dots, N(A_m)) \tag{7.1.3}$$

for any choice of bounded Borel sets $A_1, \dots, A_m \in \mathcal{E}$ and $m \geq 1$; see Daley and Vere-Jones [38], Proposition 6.2.III. The collection of all these distributions is called the *finite-dimensional distributions of the point process*.

We can imagine the finite-dimensional distributions much more easily than the distribution P_N itself. Indeed, (7.1.3) is a random vector of integer-valued random variables which is completely given by the probabilities

$$P(N(A_1) = k_1, \dots, N(A_m) = k_m), \quad k_i \in \{0, 1, \dots, \infty\}, \quad i = 1, \dots, m.$$

From a course on probability theory we know that it is often convenient to describe the distribution of a random variable or of a random vector by some analytical means. For example, one uses a whole class of transforms: characteristic functions, Laplace-Stieltjes transforms, generating functions, etc. Each of them characterizes the distribution of the random object of interest; see for example Billingsley [18]. A similar tool exists for point processes:

Definition 7.1.10 (Laplace functional)

The Laplace functional of the point process N is given by

$$\Psi_N(g) = E e^{-\int_E g dN} = \int_{M_p(E)} e^{-\int_E g dm} P_N(dm), \tag{7.1.4}$$

where g is any non-negative bounded measurable function g on the state space E .

The expectation in (7.1.4) is always finite, since the integrand does not exceed 1. For $g \geq 0$ the integrals $\int_E g dN$ and $\int_E g dm$ appearing in (7.1.4) are well-defined as Lebesgue-Stieltjes integrals. To illustrate this for $\int_E g dN$, assume the representation $N = \sum_{i=1}^{\infty} \varepsilon_{X_i}$ for random vectors X_i with values in E and recall that, with probability 1, any realization of N is a point measure on the σ -field $\mathcal{E} = \mathcal{B}(E)$. Then

$$\int_E g dN = \sum_{i=1}^{\infty} g(X_i).$$

In particular, $\int_A dN = \int_E I_A dN = N(A)$.

Lemma 7.1.11 *The Laplace functional Ψ_N of a point process N uniquely determines the distribution P_N .*

Proof. Consider the following bounded non-negative functions on the state space E :

$$g_{\mathbf{z}} = z_1 I_{A_1} + \cdots + z_m I_{A_m}, \quad (7.1.5)$$

where $z_i \geq 0$ and $A_i \in \mathcal{E}$, $i = 1, \dots, m$. Then

$$\begin{aligned} \Psi_N(g_{\mathbf{z}}) &= E e^{-\int_E g_{\mathbf{z}} dN} \\ &= E e^{-(z_1 N(A_1) + \cdots + z_m N(A_m))}, \quad z_1 \geq 0, \dots, z_m \geq 0, \end{aligned}$$

is the Laplace-Stieltjes transform of the vector $\mathbf{N}_m = (N(A_1), \dots, N(A_m))'$. This transform uniquely determines the distribution of the vector \mathbf{N}_m ; see Billingsley [18]. Hence the collection of the quantities $\Psi_N(g_{\mathbf{z}})$, $z_i \geq 0$, $i = 1, \dots, m$, determines the finite-dimensional distributions of N . In view of the discussion on p. 222 the finite-dimensional distributions of N determine the distribution P_N of the point process N . \square

Laplace functionals have the property that they determine the distribution of the underlying point process. Moreover, their pointwise convergence is equivalent to the weak convergence of a sequence of point processes; see Section 9.1. In this context it can be useful to restrict the class of functions g . In Chapter 9 we will choose the class $\mathbb{C}_K^+(E)$, which consists of the non-negative continuous functions on E with compact support. It follows, for example, from Exercise 3.4.3 and Lemma 3.11 in Resnick [122] that the restricted Laplace functional $\Psi_N(g)$, $g \in \mathbb{C}_K^+(E)$, determines the distribution of N .

In the course of the proof of Lemma 7.1.11, we learned about another class of non-negative bounded functions g whose Laplace functional $\Psi_N(g)$ determines the distribution of the point process N : the simple functions $g = g_{\mathbf{z}}$ in (7.1.5). This class and $\mathbb{C}_K^+(E)$ are disjoint.

Comments

Point processes are special random measures and can therefore be treated in a much more general context; see for example Kallenberg [79, 80]. Classical monographs on point processes and random measures are Matthes et al. [105], Kallenberg [79] and Daley and Vere-Jones [38, 39, 40]. Point processes have also been treated in various books on general stochastic processes; see for example Jacod and Shiryaev [74] or Resnick [122, 123, 124].

Although the notion of a point process is intuitively appealing, it is an infinite-dimensional structure and therefore it is a rather complex object. In our presentation we have tried to avoid too much sophistication. The interested reader will find the necessary details in the references mentioned above.

Point processes have found a multitude of applications, for example, in spatial statistics (see Cressie [37], Ripley [125]), extreme value theory (see Leadbetter et al. [92], Resnick [122, 124], Embrechts et al. [46], and Section 9.2 below), queuing theory (see Bacelli and Bremaud [10], Bremaud [23]), Lévy process theory (see Samorodnitsky and Taqqu [131], Bertoin [15], Sato [132], Kyprianou [90], and Chapter 10 below) and stochastic geometry (see Stoyan et al. [140]).

In 1903, Lundberg [99] introduced the Poisson process in a non-life insurance context as a simple claim counting model. This was the first use of the Poisson process. In an insurance context, point process methods have been used for many decades but often without being explicitly mentioned. In the context of non-life insurance, Norberg has propagated the use of point process methodology; see for example [114]. Point processes are also the theoretical basis of Markov counting process techniques which are frequently used in life insurance for estimating the mortality of a population.

Exercises

Section 7.1.1

- (1) Let $x_i = i^{-1}$, $i = 1, 2, \dots$. Show that $m = \sum_{i=1}^{\infty} \varepsilon_{x_i}$ is not a point measure with state space $E = [0, 1]$ but it is a point measure with state space $E = (0, 1]$.
- (2) Let (X_i) be a sequence of random vectors with values in \mathbb{R}^d .
 - (a) Show that $N = \sum_{i=1}^{\infty} \varepsilon_{X_i}$ defines a point process with values in $E \subset \mathbb{R}^d$ if and only if (X_i) does not have limit points in E with probability 1.
 - (b) Let X_i be non-zero with probability 1 for any $i \geq 1$ and assume that $X_i \rightarrow \mathbf{0}$ a.s. Show that $N = \sum_{i=1}^{\infty} \varepsilon_{X_i}$ is a point process with state space $E = \mathbb{R}^d \setminus \{\mathbf{0}\}$.
 - (c) Let (X_i) be an iid sequence of real-valued random variables with $E|X_1| < \infty$. Consider the sequence of partial sums $S_n = X_1 + \dots + X_n$. Show that the random measure $N = \sum_{n=1}^{\infty} \varepsilon_{S_n}$ is not a point process with state space $E = \mathbb{R}^d$. Choose an alternative state space such that N becomes a point process.
- (3) Let $N = \sum_{i=1}^{\infty} \varepsilon_{X_i}$ for a sequence of random vectors X_i with values in the state space $E \subset \mathbb{R}^d$ equipped with its Borel σ -field \mathcal{E} . Define the *mean measure* μ of N by $\mu(A) = EN(A)$, $A \in \mathcal{E}$. Give a sufficient condition on μ which ensures that N is a point process.
- (4) Consider the point process of exceedances N_n defined in Example 7.1.6. Show that this point process has representation $N_n = \sum_{i=1}^n \varepsilon_{Y_i}$ for an appropriately chosen sequence (Y_i) of independent random vectors.
- (5) Let F be a continuous distribution function on \mathbb{R} .
 - (a) Consider an iid sequence $(Y_i)_{i \geq 1}$ with distribution function $F^{1/3}$. Construct the points $X_i = \max(Y_{i-2}, Y_{i-1}, Y_i)$ and show that $(X_i)_{i \geq 3}$ constitutes a strictly stationary process with marginal distribution F .
 - (b) In Figure 7.1.12 we see that it is likely that three successive values X_{i-2} , X_{i-1} , X_i of the process introduced in (a) coincide. This means that the point process $N_n = \sum_{i=3}^n \varepsilon_{X_i}$, $n \geq 3$, is a non-simple point process. Give

a theoretical explanation for this phenomenon by borrowing the argument from Example 7.1.8.

- (c) Suggest how one can modify Example 7.1.8 or part (b) of this exercise to get a non-simple point process whose points have multiplicity $k \geq 2$ with positive probability.

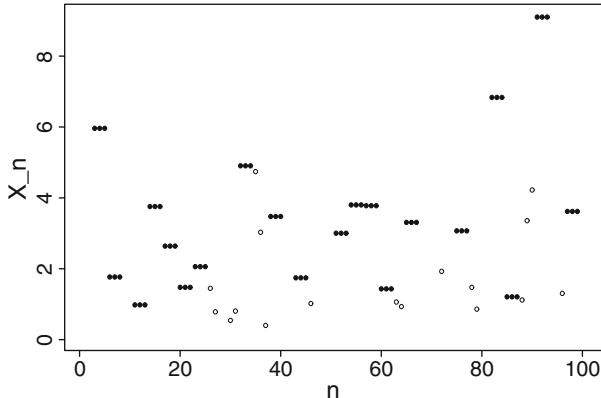


Figure 7.1.12 The points $X_i = \max(Y_{i-2}, Y_{i-1}, Y_i)$ of the point process N_{100} with iid log-normal Y_i 's considered in Exercise 5(b). The Y_i 's come from the same realization as in Figure 7.1.9. Observe that the components of various triples (X_{i-2}, X_{i-1}, X_i) coincide (solid dots).

- (6) Consider the counting process

$$N = \sum_{i=1}^{\tau} \varepsilon_{X_i}, \quad (7.1.6)$$

where τ is a non-negative integer-valued random variable, independent of the iid sequence of random vectors X_i with values in a Borel set $E \subset \mathbb{R}^d$ and common distribution F .

- (a) Show that N defines a point process on E .
 (b) Show that N has representation $N = \sum_{i=1}^{\infty} \varepsilon_{Y_i}$ for a sequence of appropriately chosen random vectors Y_i .

Section 7.1.2

- (7) Consider the point processes of exceedances from Example 7.1.6, i.e.,

$$N_n(\cdot) = \sum_{i=1}^n \varepsilon_{n^{-1}i}(\cdot) I_{\{X_i > u_n\}}, \quad n = 1, 2, \dots,$$

on the state space $E = (0, 1]$, where (X_i) is an iid sequence with common distribution F .

- (a) Calculate the Laplace functional Ψ_{N_n} of N_n .
 (b) Prove for any continuous non-negative function g on $[0, 1]$ that

$$\Psi_{N_n}(g) \rightarrow \exp \left\{ -\tau \int_0^1 (1 - e^{-g(t)}) dt \right\}, \quad n \rightarrow \infty,$$

provided $n \bar{F}(u_n) \rightarrow \tau$ for some $\tau \in (0, \infty)$.

The right-hand side is the Laplace functional $\Psi_N(g)$ of a Poisson random measure PRM(τ Leb) on $(0, 1]$, denoted by N ; see Lemma 7.2.7(1) below. The convergence $\Psi_{N_n}(g) \rightarrow \Psi_N(g)$ for all non-negative continuous functions g on $[0, 1]$ can be shown to be equivalent to the convergence in distribution $N_n \xrightarrow{d} N$; see Section 9.1 below.

- (8) Consider the point process N defined in (7.1.6) with a $\text{Pois}(\lambda)$ distributed random variable τ for some $\lambda > 0$, independent of the iid sequence (X_i) with common distribution F on $E \subset \mathbb{R}^d$. Calculate the Laplace functional of N . Compare with the Laplace functional of a Poisson random measure with mean measure λF ; see Lemma 7.2.7(1) below.

7.2 Poisson Random Measures

Poisson processes or, synonymously, *Poisson random measures* constitute one of the most important classes of point processes. They appear in a natural way as limits of “binomial point processes”, i.e., point processes with independent points. This is similar to Poisson’s limit theorem, where a Poisson random variable is the distributional limit of a sequence of binomial random variables. Poisson processes have appealing dependence and distributional properties. From the points of a Poisson process, one can construct richer structures, such as Lévy, infinitely divisible and max-stable processes. The new processes inherit some of the nice properties of Poisson processes.

We start in Section 7.2.1 with the definition of a Poisson random measure (PRM) and with some simple examples of PRMs. These include not only the homogeneous Poisson process with points in \mathbb{R}^d , but also the record sequence of an iid sequence. In Section 7.2.2 we derive the Laplace functional of a PRM. This functional allows one to identify a point process as a PRM and to determine its *mean measure*, which in turn characterizes the distribution of a PRM. Since Laplace functionals are defined via the integrals $\int_E g dN$, we study these for a PRM N . Such *Poisson integrals* often have compound Poisson structure. Moreover, the Poisson integrals $\int_E f_i dN$, $i = 1, 2, \dots$, have the remarkable property that they are mutually independent if and only if the functions f_i , $i = 1, 2, \dots$, have disjoint support.

In Section 7.3 we continue the investigation of PRMs. There we will construct new PRMs from operations acting on the points of given PRMs.

7.2.1 Definition and First Examples

As before, we assume that the state space $E \subset \mathbb{R}^d$ is equipped with its Borel σ -field \mathcal{E} . Recall that a measure μ on E is a *Radon measure* if it is finite on compact sets $A \in \mathcal{E}$.

Definition 7.2.1 (Poisson random measure (PRM))

Let μ be a Radon measure on E . A point process N is called a *Poisson process* or a *Poisson random measure* with mean measure μ (we write $\text{PRM}(\mu)$) if the following two conditions are satisfied:

- (1) For $A \in \mathcal{E}$, $N(A)$ is $\text{Pois}(\mu(A))$ distributed.
- (2) For any disjoint sets $A_1, \dots, A_m \in \mathcal{E}$ and $m \geq 1$, the random variables $N(A_1), \dots, N(A_m)$ are mutually independent.

For sets A with $\mu(A) = s \in \{0, \infty\}$, we will use the convention that the Poisson variable $N(A) = s$ a.s. This is in agreement with the relations $Y_\lambda \xrightarrow{P} s$ as $\lambda \rightarrow s$ for $\text{Pois}(\lambda)$ distributed Y_λ and $s \in \{0, \infty\}$.

The name *mean measure* of a $\text{PRM}(\mu)$, N , is justified by the fact that $EN(A) = \mu(A)$ for all $A \in \mathcal{E}$. It follows from the definition of a PRM that the mean measure μ determines the distribution of N . This is similar to the fact that a Poisson distribution is given by its mean value.

The *Radon property* of the mean measure μ ensures that, for any compact set $K \in \mathcal{E}$, $EN(K) = \mu(K) < \infty$, hence $N(K) < \infty$ a.s., as required for the definition of a point process; see p. 216.

Any PRM N with a finite mean measure μ , i.e., $\mu(E) < \infty$, has the representation

$$N(A) = \sum_{i=1}^{\tau} I_A(X_i), \quad A \in \mathcal{E}, \tag{7.2.7}$$

where (X_i) is an iid sequence of E -valued random vectors, independent of the $\text{Pois}(\mu(E))$ distributed random variable τ , and the X_i 's have the common distribution

$$P(X_1 \in A) = \frac{\mu(A)}{\mu(E)}, \quad A \in \mathcal{E}.$$

The proof of this result is left as Exercise 2 on p. 242, where it is also indicated how the construction (7.2.7) can be extended to a PRM with infinite mean measure μ .

Example 7.2.2 (Homogeneous PRM)

Consider a $\text{PRM}(\lambda \text{Leb})$, denoted by N , on the state space $E = [0, \infty)$ for some $\lambda > 0$, where Leb denotes Lebesgue measure on E . Define the stochastic process³ $N(t) = N[0, t]$, $t \geq 0$. This process has stationary increments since,

³ As in Part I of this book we write $N(a, b] = N((a, b]) = N(b) - N(a)$, $a < b$, for the increment of the process N on the interval $(a, b]$, and correspondingly $N[a, b] = N([a, b])$, $N[a, b) = N([a, b))$, etc.

for any $0 < a < b < \infty$ and $h > 0$, $N(a+h, b+h] \sim \text{Pois}(\lambda(b-a))$. Moreover, for $0 = t_0 < t_1 < \dots < t_m < \infty$ the sets $(t_{i-1}, t_i]$, $i = 1, \dots, m$, are disjoint, hence the increments $N(t_{i-1}, t_i]$, $i = 1, \dots, m$, are mutually independent. Since $EN(0) = EN(\{0\}) = \lambda |\{0\}| = 0$, where $|A| = \text{Leb}(A)$ for any Borel set A , we have $N(0) = 0$ a.s. Then, with the exception of the càdlàg property of the sample paths of N , we have rediscovered the properties of a homogeneous Poisson process $(N(t))_{t \geq 0}$ with intensity $\lambda > 0$; see Section 2.1.1. For this reason, we will call a PRM on $[0, \infty)$ with mean measure λLeb a *homogeneous Poisson process* or *homogeneous PRM* on the state space $[0, \infty)$.

The converse result, i.e., the fact that a homogeneous Poisson process (in the sense of Definition 2.1.1), $(N(t))_{t \geq 0}$, with intensity $\lambda > 0$ determines a homogeneous PRM (λLeb) is not trivial; see Proposition 6.2.III in Daley and Vere-Jones [38]. For every fixed ω , $N(t, \omega)$ can be interpreted as a counting measure of the interval $[0, t]$. Then the extension theorem for measures allows one to define an ordinary counting measure $N(\cdot, \omega)$ on \mathcal{E} . This construction works *for every fixed* ω , but one has to ensure that $N(\cdot, \omega)$ is a point measure, for almost all $\omega \in \Omega$, and this step requires more work. \square

Motivated by Example 7.2.2, we can define a *homogeneous Poisson process* or *homogeneous PRM* on any Borel state space $E \subset \mathbb{R}$. For example, $\text{PRM}(\lambda \text{Leb}(\cdot \cap (a, b]))$ defines a homogeneous PRM on the interval $E = (a, b]$. Since Lebesgue measure is also defined in the Euclidean space \mathbb{R}^d , one can define a homogeneous PRM N with intensity $\lambda > 0$ on the state space $E \subset \mathbb{R}^d$ simply by specifying the mean measure as $\lambda \text{Leb}(\cdot \cap E)$. (In this notation, we suppress the dependence of Lebesgue measure on the dimension d .) It is desirable that E have positive Lebesgue measure in order to avoid the trivial case of a point process vanishing on E .

More generally, if the mean measure μ of a PRM is absolutely continuous with respect to Lebesgue measure, i.e., there exists a non-negative function $\lambda(\cdot)$ such that

$$\mu(A) = \int_A \lambda(x) dx, \quad A \in \mathcal{E},$$

then $\lambda(\cdot)$ is the *intensity* or *rate function of the PRM*.

Example 7.2.3 (The restriction of a PRM to a smaller state space is a PRM) Consider a PRM N on the state space E with mean measure μ , and let $E' \in \mathcal{E}$ be a measurable subset of E . Define the point process

$$N'(A) = N(A), \quad A \in \mathcal{E}' = \mathcal{B}(E').$$

Then, by definition of a PRM on E , for $A \in \mathcal{E}'$, $N'(A)$ is $\text{Pois}(\mu(A))$ distributed and for any disjoint sets A_1, \dots, A_m in \mathcal{E}' , $N'(A_1), \dots, N'(A_m)$ are independent. This means that the restriction N' of N to the Borel set $E' \subset E$ is again a PRM with mean measure μ' which is the restriction of μ to \mathcal{E}' . \square

Example 7.2.4 (The records of an iid sequence constitute a Poisson process)⁴

Consider an iid sequence (X_i) with continuous common distribution function F and construct the corresponding sequence of partial maxima

$$M_1 = X_1, \quad M_n = \max_{i=1,\dots,n} X_i, \quad n \geq 2.$$

By definition, $M_1 = X_1$ is the *first record* in the sequence (X_n) . In general, a *new record* occurs if $X_n > M_{n-1}$, i.e., X_n exceeds all previous maxima. Clearly, $M_n = X_n$ is then the *new record value*. The *record times*, i.e., the times when the records occur, constitute an increasing sequence $1 = R_1 < R_2 < R_3 < \dots$, and the *record sequence* has the representation $(X_{R_n})_{n \geq 1} = (M_{R_n})_{n \geq 1}$.

The process (M_n) constitutes a Markov chain. This fact will be indicated by the following argument. Let $x_1 < \dots < x_n < \infty$ be values in the support of F . Then for $n \geq 2$ and $k \geq 0$, by independence of the X_i 's,

⁴ In this example we closely follow Resnick [122], Section 4.1.

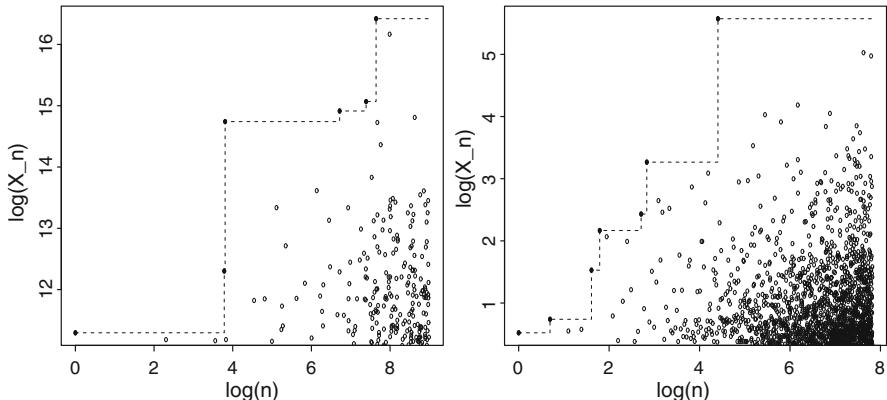


Figure 7.2.5 The records (solid dots on the dashed staircase) of the US industrial fire data (left) and the Danish fire insurance data (right); see Example 3.2.11 for a description of the data. Both the data (y-axis) and the index n (x-axis) of the time series are on log-scale. There are 6 records in the US fire data and 7 records in the Danish fire insurance data.

$$\begin{aligned}
& P(M_{n+k} \leq x_n \mid M_{n-1} \leq x_{n-1}, \dots, M_1 \leq x_1) \\
&= \frac{P(X_1 \leq x_1, \dots, X_{n-1} \leq x_{n-1}, \max_{i=n, \dots, n+k} X_i \leq x_n)}{P(X_1 \leq x_1, \dots, X_{n-1} \leq x_{n-1})} \\
&= \frac{F(x_1) \cdots F(x_{n-1}) F^{k+1}(x_n)}{F(x_1) \cdots F(x_{n-1})} \\
&= F^{k+1}(x_n).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
P(M_{n+k} \leq x_n \mid M_{n-1} \leq x_{n-1}) &= \frac{P(M_{n-1} \leq x_{n-1}, \max_{i=n, \dots, n+k} X_i \leq x_n)}{P(M_{n-1} \leq x_{n-1})} \\
&= \frac{F^{n-1}(x_{n-1}) F^{k+1}(x_n)}{F^{n-1}(x_{n-1})} \\
&= F^{k+1}(x_n).
\end{aligned}$$

A rigorous proof of the Markov property of (M_n) requires more effort; see Resnick [122], Section 4.1. The transition probabilities of this chain do not depend on n . Hence (M_n) is a homogeneous Markov chain.

A homogeneous Markov chain is determined by its initial distribution and the one-step transition probabilities; see Breiman [22], Chapter 15, or Meyn and Tweedie [106]. The record sequence $(X_{R_n}) = (M_{R_n})$ constitutes the *embedded jump chain* of (M_n) which is again a homogeneous Markov chain; see the argument in Resnick [122], Section 4.1, or Breiman [22], Section 5.5. Since $R_1 = 1$, the initial distribution of this Markov chain is F . For $x < y$ and x in the support of F , the one-step transition probabilities are given by

$$\begin{aligned}
\pi(x, (y, \infty)) &= P(X_{R_2} > y \mid X_{R_1} = x) \\
&= P(X_{R_2} > y, \max_{j=2, \dots, R_2-1} X_j \leq x \mid X_1 = x) \\
&= \sum_{n=2}^{\infty} P(X_n > y, \max_{j=2, \dots, n-1} X_j \leq x \mid X_1 = x) \\
&= \sum_{n=2}^{\infty} \bar{F}(y) F^{n-2}(x) \\
&= \bar{F}(y)/\bar{F}(x).
\end{aligned}$$

Now assume that $\bar{F}(x) = e^{-x}$, i.e., X_1 is a standard exponential random variable. Then the record sequence (X_{R_n}) is a homogeneous Markov chain with initial distribution F and one-step transition probabilities $\pi(x, (y, \infty)) = e^{-(y-x)}$.

On the other hand, consider the renewal sequence $T_n = Y_1 + \dots + Y_n$, $n \geq 1$, for an iid standard exponential sequence (Y_n) . Then (T_n) constitutes a homogeneous Markov chain with initial distribution F and one-step transition probabilities

$$P(T_2 > y \mid T_1 = x) = P(Y_2 + x > y) = e^{-(y-x)}.$$

Thus $(T_n) \stackrel{d}{=} (X_{R_n})$ and therefore the record sequence of an iid standard exponential sequence is nothing but the renewal sequence of a standard homogeneous Poisson process on $(0, \infty)$.

For general continuous F , calculation shows that

$$(X_n)_{n \geq 1} \stackrel{d}{=} (G^\leftarrow(Y_n))_{n \geq 1}. \quad (7.2.8)$$

Here (Y_n) is an iid standard exponential sequence, $G(x) = -\log \bar{F}(x)$ and

$$G^\leftarrow(y) = \inf\{z \in \mathbb{R} : G(z) \geq y\}$$

is the generalized inverse of G . The function G is well-defined and continuous on (x_l^F, x_r^F) , where x_l^F and x_r^F denote the left and right endpoints of the distribution F , respectively. Since G is continuous, G^\leftarrow is monotone increasing. Therefore and by virtue of (7.2.8), the record time sequences (R_n) of (X_n) and (\tilde{R}_n) of (Y_n) have the same distribution. Moreover, the record sequences (X_{R_n}) and $(G^\leftarrow(Y_{\tilde{R}_n}))$ have the same distribution. Since the points $Y_{\tilde{R}_n}$ constitute a standard homogeneous PRM on $(0, \infty)$, the points $G^\leftarrow(Y_{\tilde{R}_n})$ constitute a PRM N_0 with state space (x_l^F, x_r^F) and mean measure μ_0 given by

$$\begin{aligned} \mu_0(a, b] &= EN_0(a, b] \\ &= E(\#\{i \geq 1 : a < G^\leftarrow(Y_{\tilde{R}_n}) \leq b\}) \\ &= E(\#\{i \geq 1 : G(a) < Y_{\tilde{R}_n} \leq G(b)\}) \\ &= |(G(a), G(b)]| = G(b) - G(a), \quad (a, b] \subset (x_l^F, x_r^F). \end{aligned}$$

The above theory on record sequences is a nice application of point process theory. In particular, we have used a suitable transformation of the points of the record sequence of an iid exponential sequence. This is a particular example of the general transformation theory for the points of PRMs which will be given in Section 7.3.1.

The number of records in a sample of size n is given by

$$L_1 = 1, \quad L_n = 1 + \sum_{k=2}^n I_{\{X_k > M_{k-1}\}} = \#\{1 \leq i \leq n : R_i \leq n\}, \quad n \geq 2. \quad (7.2.9)$$

The following result gives some information about the size of L_n :

$$EL_n = \sum_{k=1}^n \frac{1}{k} \quad \text{and} \quad \text{var}(L_n) = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k^2} \right). \quad (7.2.10)$$

The proof is left as Exercise 4(a) on p. 243.

Notice that EL_n and $\text{var}(L_n)$ are both of the order $\log n$ as $n \rightarrow \infty$. More precisely, $EL_n - \log n \rightarrow \gamma$, where $\gamma = 0.5772\dots$ denotes Euler's constant. As a consequence: *the number of records of iid data grows very slowly*. Before reading further, guess the answer to the following question:

How many records would you expect in 100, 1 000 or 10 000 iid observations?

Table 7.2.6 contains the somewhat surprising answer.

$n = 10^k, k =$	EL_n	$\log n$	$\log n + \gamma$	s_n
1	2.9	2.3	2.9	1.2
2	5.2	4.6	5.2	1.9
3	7.5	6.9	7.5	2.4
4	9.8	9.2	9.8	2.8
5	12.1	11.5	12.1	3.2
6	14.4	13.8	14.4	3.6
7	16.7	16.1	16.7	3.9
8	19.0	18.4	19.0	4.2
9	21.3	20.7	21.3	4.4

Table 7.2.6 Expected number of records EL_n in an iid sequence (X_i) with a continuous distribution function F , together with the asymptotic approximations $\log n$, $\log n + \gamma$, and standard deviation $s_n = \sqrt{\text{var}(L_n)}$, based on (7.2.10). A comparison of the columns shows that the approximation $\log n + \gamma$ for EL_n works well.

We conclude that records are not very relevant for statistical purposes: their number is small even if the sample size n is large; see Table 7.2.6. For an illustration in the case of the US and Danish fire data, see Figure 7.2.5. \square

7.2.2 Laplace Functional and Non-Negative Poisson Integrals

We know from Lemma 7.1.11 that the distribution of a point process is uniquely determined by its Laplace functional. The Laplace functional of a PRM has a very characteristic form. It will be used in the sequel to identify point processes as PRMs and to determine their mean measures.

Recall that the Laplace functional of any point process with representation $N = \sum_{i=1}^{\infty} \varepsilon_{X_i}$ on the state space E is given by

$$\Psi_N(g) = Ee^{-\int_E g dN} = Ee^{-\sum_{i=1}^{\infty} g(X_i)},$$

for any bounded measurable function $g \geq 0$ on E . We call the Lebesgue-Stieltjes integral $\int_E g dN$ a *Poisson integral*. Since g is non-negative this integral is well-defined even if its value is infinite, and then $\exp\{-\int_E g dN\}$ is

evaluated as zero. In Lemma 7.2.7, we give a necessary and sufficient condition for $\int_E g dN$ to be finite a.s.

Lemma 7.2.7 (Laplace functional of PRM and finiteness of non-negative Poisson integrals)

(1) *The Laplace functional of $\text{PRM}(\mu)$ on the state space $E \subset \mathbb{R}^d$ is given by*

$$\Psi_N(g) = \exp \left\{ - \int_E \left(1 - e^{-g(x)} \right) \mu(dx) \right\}, \quad (7.2.11)$$

for any (not necessarily bounded) measurable function $g \geq 0$.

(2) *Let g be a non-negative measurable function on E . Then the integral $\int_E g dN$ is finite a.s. if and only if*

$$\int_E \min(g(x), 1) \mu(dx) < \infty. \quad (7.2.12)$$

In order to understand the relation between (7.2.11) and (7.2.12), consider the decomposition

$$\begin{aligned} \int_E \left(1 - e^{-g(x)} \right) \mu(dx) &= \left(\int_{\{x:g(x)>1\}} + \int_{\{x:g(x)\leq 1\}} \right) \left(1 - e^{-g(x)} \right) \mu(dx) \\ &= I_1 + I_2. \end{aligned}$$

The integrand in I_1 assumes values in $(1 - e^{-1}, 1]$. Hence I_1 is finite if and only if $\mu(\{x : g(x) > 1\}) = \int_{\{x:g(x)>1\}} \mu(dx) < \infty$. A Taylor expansion of the integrand $1 - \exp\{-g(x)\}$ shows that I_2 is finite if and only if $\int_{\{x:g(x)\leq 1\}} g(x) \mu(dx) < \infty$. Thus, condition (7.2.12) is equivalent to the fact that the integral in the Laplace functional (7.2.11) is finite.

Proof. In the proof we borrow arguments from the proofs of Lemma 10.2 and Theorem 10.15 in Kallenberg [80].

(1) We start with a simple function

$$g = \sum_{i=1}^m a_i I_{A_i} \quad (7.2.13)$$

for disjoint $A_1, \dots, A_m \in \mathcal{E}$ and non-negative a_i . Then

$$\int_E g dN = \sum_{i=1}^m a_i \int_E I_{A_i}(x) N(dx) = \sum_{i=1}^m a_i N(A_i).$$

By the PRM property, the random variables $N(A_i)$, $i = 1, \dots, m$, are mutually independent with corresponding mean values $\mu(A_i)$, $i = 1, \dots, m$, possibly infinite. If $\mu(A_i) = \infty$ and $a_i > 0$ for some i , then $N(A_i) = \infty$ a.s. by

definition of a Poisson variable with infinite mean, and then $\int_E g dN = \infty$ a.s. and $\Psi_N(g) = 0$. Now assume that all values $\mu(A_i)$ are finite. Then direct calculation yields

$$\Psi_N(g) = \prod_{i=1}^m E e^{-a_i N(A_i)},$$

where we have used the independence of the $N(A_i)$'s. For any Poisson variable M with mean $\lambda > 0$ and any real number z ,

$$E e^{zM} = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} e^{zk} = e^{-\lambda(1-e^z)}.$$

We conclude that

$$\begin{aligned} \Psi_N(g) &= \prod_{i=1}^m e^{-\mu(A_i)(1-e^{-a_i})} \\ &= \exp \left\{ - \sum_{i=1}^m \int_E \left(1 - e^{-a_i I_{A_i}(x)} \right) \mu(dx) \right\} \\ &= \exp \left\{ - \int_E \left(1 - e^{-g(x)} \right) \mu(dx) \right\}. \end{aligned}$$

Any non-negative measurable function g on E is the monotone limit of simple functions $g_n \geq 0$ of type (7.2.13), i.e., $g_n \uparrow g$. Then

$$\begin{aligned} \int_E g_n dN &\uparrow \int_E g dN \quad \text{a.s.}, \\ \int_E \left(1 - e^{-g_n(x)} \right) \mu(dx) &\uparrow \int_E \left(1 - e^{-g(x)} \right) \mu(dx), \end{aligned}$$

by the monotone convergence theorem. Finally, dominated convergence yields $\Psi_N(g_n) \rightarrow \Psi_N(g)$ as $n \rightarrow \infty$. Using the previously proved statement for the simple functions g_n , we have now shown that

$$\begin{aligned} \Psi_N(g_n) &= \exp \left\{ - \int_E \left(1 - e^{-g_n(x)} \right) \mu(dx) \right\} \\ &\rightarrow \exp \left\{ - \int_E \left(1 - e^{-g(x)} \right) \mu(dx) \right\} \\ &= \Psi_N(g), \quad n \rightarrow \infty. \end{aligned}$$

This establishes (7.2.11).

(2) First assume that $\int_E \min(g(x), 1) \mu(dx) < \infty$ for some measurable $g \geq 0$.

The argument given after Lemma 7.2.7 shows that the integral in the Laplace functional (7.2.11) of $\text{PRM}(\mu)$ is finite. Dominated convergence as $z \downarrow 0$ for the Laplace-Stieltjes transform of $\int_E g dN$ yields

$$\begin{aligned} P\left(\int_E g dN < \infty\right) &= \lim_{z \downarrow 0} E\left(e^{-z \int_E g dN} I_{\{\int_E g dN < \infty\}}\right) \\ &= \lim_{z \downarrow 0} E\left(e^{-z \int_E g dN}\right) \\ &= \exp\left\{-\lim_{z \downarrow 0} \int_E \left(1 - e^{-z g(x)}\right) \mu(dx)\right\} = e^0 = 1. \end{aligned}$$

This proves that $\int_E g dN < \infty$ a.s.

Now assume that the latter Poisson integral is finite a.s. It follows from the first part of the proof that $\int_E (1 - \exp\{-g(x)\}) \mu(dx)$ is finite. Equivalently, (7.2.12) holds. This proves the lemma. \square

An immediate consequence of Lemma 7.2.7 is the following representation of Poisson integrals as a compound Poisson sum.

Corollary 7.2.8 (Compound Poisson representation of Poisson integrals)
Let g be a non-negative measurable function on E such that (7.2.12) holds. If $0 < \mu(E) < \infty$, then the integral $\int_E g dN$ has representation as a compound Poisson sum, i.e.,

$$\int_E g dN \stackrel{d}{=} \sum_{i=1}^M Z_i, \quad (7.2.14)$$

where M is $\text{Pois}(\mu(E))$ distributed, independent of the iid sequence (Z_i) of non-negative random variables with common distribution

$$F_Z(B) = [G(g^{-1})](B) = G(\{x \in E : g(x) \in B\}), \quad B \in \mathcal{B}([0, \infty)),$$

where $G(dx) = \mu(dx)/\mu(E)$ is a probability measure on (E, \mathcal{E}) . In particular,

$$\begin{aligned} E\left(\int_E g dN\right) &= \int_E g(x) \mu(dx), \\ \text{var}\left(\int_E g dN\right) &= \int_E [g(x)]^2 \mu(dx). \end{aligned}$$

In what follows, we will very often refer to the compound Poisson representation (7.2.14) of a Poisson integral. For ease of notation, we will write

$$\sum_{i=1}^M Z_i \sim \text{CP}(EM, F_Z),$$

where EM denotes the expectation of the Poisson variable M and F_Z is the common distribution of the iid Z_i 's.

Proof. Since (7.2.12) holds, the integral $\int_E g dN$ is finite a.s. by virtue of Lemma 7.2.7(2). Since $0 < \mu(E) < \infty$ we have that $G(A) = \mu(A)/\mu(E)$, $A \in \mathcal{E}$, defines a probability distribution on (E, \mathcal{E}) . From Lemma 7.2.7(1) we know that the Laplace-Stieltjes transform of $\int_E g dN$ can be written in the form

$$\begin{aligned} E e^{-z \int_E g dN} &= \exp \left\{ -\mu(E) \int_E (1 - e^{-z g(x)}) G(dx) \right\} \\ &= \exp \left\{ -\mu(E) \int_{g(E)} (1 - e^{-z y}) [G(g^{-1})(dy)] \right\} \\ &= \exp \{ -\mu(E) (1 - E e^{-z Z}) \}, \quad z \geq 0, \end{aligned} \quad (7.2.15)$$

where $g(E) = \{g(x) : x \in E\} \subset [0, \infty)$ and the non-negative random variable Z has distribution $G(g^{-1})$ on $([0, \infty), \mathcal{B}([0, \infty)))$. It is not difficult to see that the Laplace-Stieltjes transform of the compound Poisson sum on the right-hand side of (7.2.14) coincides with formula (7.2.15). Therefore the distributions on the left- and right-hand sides of (7.2.14) coincide.

For the moments of the compound Poisson sum $\sum_{i=1}^M Z_i$ we use standard results from Section 3.1.1:⁵

$$\begin{aligned} E \left(\sum_{i=1}^M Z_i \right) &= EM EZ = \int_E g(x) \mu(dx), \\ \text{var} \left(\sum_{i=1}^M Z_i \right) &= EM E(Z^2) = \int_E [g(x)]^2 \mu(dx). \end{aligned}$$

□

7.2.3 Properties of General Poisson Integrals

Throughout this section, N is PRM(μ) on some state space $E \subset \mathbb{R}^d$. We study the Poisson integrals $\int_E f dN$ for general measurable functions f on E . In particular, we are interested in the dependence and moment structures of these integrals.

⁵ The random variables Z_i and M are in general defined on a probability space which is distinct from the probability space (Ω, \mathcal{F}, P) where the PRM N is defined. Indeed, from the proof we conclude that Z is a random variable defined on the probability space $([0, \infty), \mathcal{B}([0, \infty)), G(g^{-1}))$. However, for the ease of notation, we use the same symbols for the new probability measures, expectations, variances, etc., related to M and Z_i .

In Section 7.2.2 we started investigating Poisson integrals with non-negative integrands. The integral $\int_E f dN$ is understood in the Lebesgue-Stieltjes sense, and therefore it is finite a.s. if and only if $\int_E |f| dN < \infty$ a.s. Therefore we may conclude from Lemma 7.2.7(2) the following result:

Lemma 7.2.9 *Let N be PRM(μ) on $E \subset \mathbb{R}^d$ and f be a real-valued measurable function on E . The integral $\int_E f dN$ exists and is finite a.s. if and only if the condition*

$$\int_E \min(|f(x)|, 1) \mu(dx) < \infty \quad (7.2.16)$$

holds.

In what follows, we say that a function f on E has the measurable support $A \subset E$ if it vanishes on $A^c = E \setminus A$. This notion does not define the support of f in a unique way.

Poisson integrals $\int_E f_i dN$, $i = 1, 2, \dots$, have a very appealing property: they are independent if the functions f_i have disjoint support.

Lemma 7.2.10 (Independence of Poisson integrals with disjoint support)
Let N be PRM(μ) on the state space $E \subset \mathbb{R}^d$ and f_i , $i = 1, \dots, k$, be measurable real-valued functions on E with disjoint support. Assume that the integrals $\int_E f_i dN$, $i = 1, \dots, k$, exist and are finite a.s. Then the random variables $\int_E f_i dN$, $i = 1, \dots, k$, are mutually independent.

Recall from Lemma 7.2.9 that $\int_E f_i dN$ exists and is finite if and only if $\int_E \min(|f_i(x)|, 1) \mu(dx) < \infty$.

Proof. Denote the support of f_i by A_i . We have

$$\int_E f_i dN = \int_{A_i} f_i dN.$$

First assume that all f_i are non-negative simple measurable functions, i.e., there exist non-negative $a_j^{(i)}$ and disjoint sets $A_j^{(i)} \subset A_i$, $A_j^{(i)} \in \mathcal{E}$, such that

$$f_i = \sum_{j=1}^m a_j^{(i)} I_{A_j^{(i)}}. \quad (7.2.17)$$

Here we assume, without loss of generality, that the number m of the sets $A_j^{(i)}$ partitioning A_i is independent of i . By virtue of the disjoint supports A_i and the disjointness of the sets $A_j^{(i)}$, the PRM property immediately implies that

$$E \exp \left\{ - \sum_{i=1}^k z_i \int_{A_i} f_i dN \right\} = \prod_{i=1}^k E \exp \left\{ - z_i \int_{A_i} f_i dN \right\} \quad (7.2.18)$$

for $z_i \geq 0$, $i = 1, \dots, k$. But this factorization property of the joint Laplace-Stieltjes transform of the random variables $\int_E f_i dN$ means that they are mutually independent; see Billingsley [18].

Now let f_i be general non-negative measurable functions with corresponding supports A_i , $i = 1, \dots, m$, which we assumed disjoint. Each function f_i is the pointwise limit of a non-decreasing sequence of simple functions of type (7.2.17) with support A_i . Then multiple application of the monotone convergence theorem implies that (7.2.18) remains valid.

Finally, consider Poisson integrals $\int_E f_i dN$ with general integrands⁶

$$f_i = (f_i)_+ - (f_i)_-, \quad i = 1, \dots, k,$$

and disjoint supports. The functions $(f_i)_\pm$, $i = 1, \dots, k$, are non-negative and have disjoint supports. Then the above Laplace-Stieltjes transform argument yields that the integrals $\int_E (f_i)_\pm dN$, $i = 1, \dots, k$, are mutually independent, and so are the integrals $\int_E f_i dN = \int_E [(f_i)_+ - (f_i)_-] dN$, $i = 1, \dots, k$, whose existence and finiteness was assumed. This concludes the proof. \square

Lemma 7.2.10 is one of the important tools in Chapter 8, where we will consider the total claim amounts of a portfolio in disjoint parts of the time-claim size space. An immediate consequence will then be that claim numbers and total claim amounts arising from distinct parts of the space are mutually independent.

Example 7.2.11 Let N be a PRM(μ) on the state space $E \subset \mathbb{R}^d$ with points X_i . Consider measurable functions f_j , $j = 1, \dots, k$, on E with mutually disjoint supports and such that the integrals $\int_E f_j dN$ are well-defined and finite a.s. Then Lemma 7.2.10 tells us that the integrals

$$\int_E f_j(x) N(dx) = \sum_{i=1}^{\infty} f_j(X_i), \quad j = 1, \dots, k,$$

are mutually independent. In particular, let f be such that $\int_E f dN$ exists and is finite a.s. Define $f_j = f I_{A_j}$ for disjoint Borel sets $A_j \subset E$, $j = 1, \dots, k$. Then the random variables $\int_{A_j} f(x) N(dx) = \sum_{i:X_i \in A_j} f(X_i)$, $j = 1, \dots, k$, are mutually independent. \square

In Corollary 7.2.8 we calculated the expectation and the variance of the Poisson integral $\int_E g dN$ for some non-negative measurable function g and for a state space E whose mean measure satisfied $0 < \mu(E) < \infty$. In what follows, we will extend these results for general measurable functions f on a state space E which does not necessarily have finite mean measure.

⁶ Here and in what follows, for any real x , $x_\pm = \max(\pm x, 0)$ denote the positive and the negative parts of x , respectively. For real-valued functions f , we define f_\pm pointwise as $f_\pm(x) = (f(x))_\pm$.

Lemma 7.2.12 (Expectation, variance and covariance of general Poisson integrals)

Let N be PRM(μ) on $E \subset \mathbb{R}^d$ and f, g be real-valued measurable functions on E .

(1) Assume that $\int_E |f(x)| \mu(dx) < \infty$. Then

$$E \left(\int_E f dN \right) = \int_E f(x) \mu(dx). \quad (7.2.19)$$

(2) Assume that

$$\int_E \max([f(x)]^2, |f(x)|) \mu(dx) < \infty. \quad (7.2.20)$$

Then

$$\text{var} \left(\int_E f dN \right) = \int_E [f(x)]^2 \mu(dx), \quad (7.2.21)$$

and the right-hand side is finite.

(3) Assume that f satisfies (7.2.20) and g satisfies the corresponding condition. Then

$$c(f, g) = \text{cov} \left(\int_E f dN, \int_E g dN \right) = \int_E f(x) g(x) \mu(dx). \quad (7.2.22)$$

Proof. (1) Under the assumptions, the Poisson integrals $\int_E f^+ dN$ and $\int_E f^- dN$ are finite a.s. Moreover, they have disjoint support. Therefore they are independent by Lemma 7.2.10. Since the difference of two independent random variables has finite expectation if and only if the two variables have finite expectation (see Exercise 6 on p. 243), we conclude that $E(\int_E f^\pm dN) < \infty$ is necessary and sufficient for the expectation in (7.2.19) to be finite. Therefore assume without loss of generality that $f \geq 0$. Since μ is a Radon measure, there exist Borel sets $E_n \uparrow E$ such that $\mu(E_n) < \infty$ for all n . For every n , $E(\int_{E_n} f dN) = \int_{E_n} f(x) \mu(dx)$ by Corollary 7.2.8. Moreover, $f I_{E_n} \uparrow f$ and then the monotone convergence theorem implies that $\int_{E_n} f N(dx) \uparrow \int_E f N(dx)$ and $E(\int_{E_n} f N(dx)) \uparrow \int_E f(x) \mu(dx)$. This concludes the proof of part (1).

(2) The variance of the random variable $\int_E f dN$ is finite if and only if its second moment is finite. Since $\int_E f^+ dN$ and $\int_E f^- dN$ are independent and finite a.s. under the assumptions, $E[(\int_E f dN)^2] < \infty$ if and only if $E[(\int_E f^\pm dN)^2] < \infty$. Therefore assume without loss of generality that $f \geq 0$. By the same monotone convergence argument as above,

$$E \left[\left(\int_{E_n} f dN \right)^2 \right] \uparrow E \left[\left(\int_E f dN \right)^2 \right].$$

On the other hand, from Corollary 7.2.8 we conclude that

$$\begin{aligned} E \left[\left(\int_{E_n} f dN \right)^2 \right] &= \text{var} \left(\int_{E_n} f dN \right) + \left(E \left[\int_{E_n} f dN \right] \right)^2 \\ &= \int_{E_n} [f(x)]^2 \mu(dx) + \left(\int_{E_n} f(x) \mu(dx) \right)^2 \\ &\uparrow \int_E [f(x)]^2 \mu(dx) + \left(\int_E f(x) \mu(dx) \right)^2. \end{aligned}$$

The integrals on the right-hand side are finite under assumption (7.2.20). Together with part (1) we conclude that (7.2.21) holds.

(3) We start by observing that for any random variables Y_1 and Y_2 with finite variance the following elementary relation holds:

$$\text{cov}(Y_1, Y_2) = \frac{1}{4} [\text{var}(Y_1 + Y_2) - \text{var}(Y_1 - Y_2)]. \quad (7.2.23)$$

Now define $Y_1 = \int_E f dN$ and $Y_2 = \int_E g dN$. Then condition (7.2.20) and part (2) of the proof imply that Y_1 and Y_2 have finite variance. Moreover, we can apply equations (7.2.21) and (7.2.23) to conclude that

$$\begin{aligned} c(f, g) &= \frac{1}{4} \int_E [f(x) + g(x)]^2 \mu(dx) - \frac{1}{4} \int_E [f(x) - g(x)]^2 \mu(dx) \\ &= \int_E f(x) g(x) \mu(dx). \end{aligned}$$

□

An interesting consequence of Lemma 7.2.12(3) is the fact that zero covariance between Poisson integrals whose integrands do not change sign implies their independence. This is a rather unusual property. Indeed, zero covariance between two random variables does not, in general, imply their independence. Jointly Gaussian random variables constitute another important example where zero covariance implies their independence.

Corollary 7.2.13 (Uncorrelated Poisson integrals are independent)

Consider Poisson integrals $\int_E f_i dN$, $i = 1, 2, \dots$, for measurable functions f_i satisfying condition (7.2.20) (with f replaced by f_i) and such that either $f_i \geq 0$ or $f_i \leq 0$ for every i .

- (1) The integrals $\int_E f_i dN$, $i = 1, 2, \dots$, are mutually independent if and only if $c(f_i, f_j) = 0$ for all $i \neq j$.
- (2) The integrals $\int_E f_i dN$, $i = 1, 2, \dots$, are uncorrelated if and only if the f_i 's have disjoint supports $\text{supp}(f_i)$ in the sense that

$$\mu(\text{supp } f_i \cap \text{supp } f_j) = 0 \quad \text{for any } i \neq j. \quad (7.2.24)$$

The property that the integrands do not change sign on E is crucial. For general integrands f_i , zero correlation and independence of the Poisson integrals $\int_E f_i dN$ are different properties. The reader is encouraged to construct an example of two Poisson integrals, where $c(f_1, f_2) = 0$ but $\int_E f_1 dN$ and $\int_E f_2 dN$ are dependent.

Proof. We give the proof for two Poisson integrals with non-negative integrands and leave the remaining argument as Exercise 7 on p. 243.

First assume that $c(f_1, f_2) = 0$. Then we have by (7.2.22),

$$c(f_1, f_2) = \int_E f_1(x) f_2(x) \mu(dx) = 0. \quad (7.2.25)$$

Since we assumed $f_i \geq 0$ for $i = 1, 2$, we conclude from (7.2.25) that $f_1 f_2 = 0$ μ -a.e., which is only possible if f_1 and f_2 have disjoint support in the sense of (7.2.24). However, the latter condition implies that

$$\begin{aligned} \int_E f_i dN &= \int_{\text{supp } f_i} f_i dN \\ &= \int_{\text{supp } f_i \setminus (\text{supp } f_1 \cap \text{supp } f_2)} f_i dN \quad \text{a.s.}, \quad i = 1, 2. \end{aligned} \quad (7.2.26)$$

Therefore we may assume without loss of generality that f_1 and f_2 have disjoint support. Now an application of Lemma 7.2.10 implies that $\int_E f_1 dN$ and $\int_E f_2 dN$ are independent.

Conversely, if f_1 and f_2 have disjoint support in the sense of (7.2.24) we conclude from (7.2.26) that we may assume without loss of generality that $\text{supp } f_1 \cap \text{supp } f_2 = \emptyset$. Another application of Lemma 7.2.10 yields that $\int_E f_1 dN$ and $\int_E f_2 dN$ are independent, hence uncorrelated. \square

Comments

Since 1903, when Lundberg [99] introduced the Poisson process in a non-life insurance context, the Poisson process has become one of the most important stochastic processes in applications. In the general theory of point processes, which has evolved through the last couple of decades, Poisson random measures have been recognized as an instrumental class of stochastic models. For example, Poisson processes have been used as building blocks for other classes of stochastic processes, such as Lévy and infinitely divisible processes. We refer to Samorodnitsky and Taqqu [131], Bertoin [15], Sato [132], Kyprianou [90] for recent textbook treatments of Lévy processes; see also Chapter 10 for an introduction. In extreme value theory, the class of max-stable processes plays a similar role as Lévy processes in summation theory. Max-stable processes have representation as functionals of Poisson processes; see Resnick [122, 124].

Poisson random measures are treated in various textbooks on general point process theory; see, for example, Kallenberg [79], Daley and Vere-Jones [38, 39, 40]. Resnick [123] yields a lively introduction to applied stochastic processes. In particular, the section on PRMs is very accessible. Applications of PRMs to extremes and other heavy-tailed phenomena can be found in Resnick [122, 124]. Various other books have been devoted to topics which are closely related to Poisson processes, for example Kingman [85] and Barbour et al. [11].

Exercises

Section 7.2.1

- (1) (a) Let N be $\text{PRM}(\mu)$ on $E = [0, 1]$ with $0 < \mu(E) < \infty$ and $\mu(\{0\}) = 0$. Show that N is simple (see p. 218 for the definition) if and only if the function $f(x) = \mu[0, x]$, $x \in [0, 1]$, is continuous.

Hint: Use the renewal representation of a homogeneous Poisson \tilde{N} and the fact that the points of N can be obtained by a monotone increasing transformation of the points of \tilde{N} . It is advantageous to use the generalized inverse of f :

$$f^\leftarrow(y) = \inf\{x \in [0, 1] : f(x) \geq y\}, \quad y \in (0, f(1)).$$

- (b) Extend the result from (a) to a $\text{PRM}(\mu)$ with Radon mean measure μ on any finite interval $(a, b]$ with $a < b$ including the case that $\mu(a, b] = \infty$.
- (2) Let (E, \mathcal{E}, μ) with $E \subset \mathbb{R}^d$ and $\mathcal{E} = \mathcal{B}(E)$ be a measure space such that $0 < \mu(E) < \infty$ and τ be $\text{Pois}(\mu(E))$ distributed. Assume that τ is independent of the iid E -valued sequence (X_i) with common distribution given by

$$P(X_1 \in A) = \frac{\mu(A)}{\mu(E)}, \quad A \in \mathcal{E}.$$

- (a) Show that the counting process

$$N(A) = \sum_{i=1}^{\tau} I_A(X_i), \quad A \in \mathcal{E}, \tag{7.2.27}$$

is $\text{PRM}(\mu)$ on E , for example by calculating the joint characteristic function or Laplace-Stieltjes transform of the finite-dimensional distributions of N . Alternatively, one can identify N as a point process on E and calculate its Laplace functional.

- (b) Specify the construction of (a) in the case when $E = [0, 1]$ is equipped with the Borel σ -field and μ has an a.e. positive Lebesgue density λ . What is the relation with the order statistics property of the Poisson process N ? For this reason, recall the order statistics property of a Poisson process from Theorem 2.1.11.
- (c) Specify the construction of (a) in the case when $E = [0, 1]^d$ is equipped with the Borel σ -field and $\mu = \lambda \text{Leb}$ for some $\lambda > 0$. Propose how one could define an “order statistics property” for this (homogeneous) Poisson process with points in E .

- (d) Suggest how one could simulate the points of $\text{PRM}(\mu)$ on E .
 (e) Suggest how one could extend the construction of (7.2.27) to $\text{PRM}(\mu)$ with a general Radon measure μ , cf. Resnick [122], Section 3.3.1.
 Hint: Recall that any Radon measure μ on E can be represented in the form $\mu(\cdot) = \sum_{i=1}^{\infty} \mu(\cdot \cap E_i)$ where $E_i \in \mathcal{E}$, $i = 1, 2, \dots$, constitute a disjoint partition of E such that $\mu(E_i) < \infty$ for all i .
- (3) Let τ be a $\text{Pois}(1)$ random variable independent of the iid sequence (X_i) with common distribution function F and a positive Lebesgue density on $(0, \infty)$. Show that

$$N(t) = \sum_{i=1}^{\tau} I_{(0,t]}(X_i), \quad t \geq 0,$$

- defines a Poisson process on $(0, \infty)$ in the sense of Definition 2.1.1 and determine the mean measure of the corresponding PRM. Argue why this process cannot be homogeneous Poisson on the state space $E = (0, \infty)$.
- (4) Consider the number of records L_n in a sample of size n from an iid sequence (X_i) with a continuous common distribution function F ; see (7.2.9) for the definition of L_n .
- (a) Prove that EL_n and $\text{var}(L_n)$ are given by formula (7.2.10).
 Hint: It is convenient to use the fact from the beginning of Example 7.2.4 that the record time sequences of (X_i) and (U_i) for iid uniform $\text{U}(0, 1)$ random variables U_i have the same distribution.
- (b) Prove that the record times R_i of an iid sequence (X_i) with continuous common distribution function F do not constitute the points of a PRM.
- (c) However, the point process of the record times sequence (R_i) is “close” to a PRM in an asymptotic sense. Indeed, the sequence of the point processes $N_n = \sum_{i=1}^{\infty} \varepsilon_{n^{-1} R_i}$ converges in distribution on the state space $(0, \infty)$ to $\text{PRM}(\mu)$, denoted by N , with mean measure given by $\mu(a, b] = \log(b/a)$, $a < b$. See Resnick [122], Section 4.1, or Embrechts et al. [46], Section 5.4.3. For the definition of the convergence $N_n \xrightarrow{d} N$, see Section 9.1.
 Fix any $c > 0$. Show that the point process N restricted to (c, ∞) has representation $\sum_{i=1}^{\infty} \varepsilon_{c e^{T_i}}$, where $\sum_{i=1}^{\infty} \varepsilon_{T_i}$ is a standard homogeneous PRM(Leb) on $(0, \infty)$.
- (5) Consider an iid sequence (X_i) with continuous common distribution function F and write $M_n = \max_{i=1, \dots, n} X_i$, $n = 1, 2, \dots$. Define the records of (X_i) as follows. The value $X_1 = M_1$ is the first record. The value X_n is a record if $X_n \geq M_{n-1}$. Show that, with probability 1, this definition yields the same record sequence as the definition on p. 229.

Section 7.2.3

- (6) In the proof of Lemma 7.2.12 we used the fact that for two independent random variables X_1, X_2 and any $p > 0$, one has $E|X_1 - X_2|^p < \infty$ if and only if $E|X_i|^p < \infty$, $i = 1, 2$. Prove this.
- (7) Assume that the conditions of Corollary 7.2.13 hold in parts (a) and (b).
- (a) Extend the proof for two Poisson integrals to a sequence of Poisson integrals $\int_E f_i dN$, $i = 1, 2, \dots$, with integrands f_i which do not change sign on E . This means that either $f_i \geq 0$ or $f_i \leq 0$ for every $i \geq 1$.

- (b) Show that mutual and pairwise independence for a sequence of Poisson integrals satisfying the conditions in (a) are equivalent notions.
- (c) Find an example of two uncorrelated Poisson integrals $\int_E f_i dN$, $i = 1, 2$, which are *not* independent.

7.3 Construction of New Poisson Random Measures from Given Poisson Random Measures

In this section we construct new Poisson processes from a given Poisson random measure. The construction principles are rather elementary ones:

- *Measurable transformations* of the points of a PRM; see Section 7.3.1.
- *Independent marking* of the points of a PRM; see Section 7.3.2.
- *Aggregation* of independent PRMs; see Section 7.3.4.

The combination of the three principles will give us insight into the structure of the claim number and total claim amount processes. With these constructions we will be able to deal with phenomena such as delay in reporting, settlement of claims and merging of independent portfolios; see Chapter 8.

7.3.1 Transformation of the Points of a Poisson Random Measure

In this section we deal with the first basic construction principle of new PRMs: transformations of the points of a given PRM.

Proposition 7.3.1 (Transformed PRMs are PRMs)

Suppose the point process $N = \sum_{i=1}^{\infty} \varepsilon_{X_i}$ is PRM(μ) with state space $E \subset \mathbb{R}^d$ equipped with the Borel σ -field $\mathcal{E} = \mathcal{B}(E)$. Assume that the points X_i of N are transformed by a measurable map $\psi : E \rightarrow E'$, where $E' \subset \mathbb{R}^m$ is equipped with the Borel σ -field $\mathcal{E}' = \mathcal{B}(E')$. Further assume that one of the following conditions holds:

- (1) The inverse image $\psi^{-1}(B) \subset E$ is bounded, for any bounded Borel set $B \subset E'$.
- (2) The measure μ is finite on E .

Then the transformed point process

$$N_{\psi} = \sum_{i=1}^{\infty} \varepsilon_{\psi(X_i)}$$

is PRM($\mu(\psi^{-1})$) on E' .

This means that N_{ψ} is a PRM on E' with mean measure given by

$$\mu_{\psi}(A) = \mu(\psi^{-1}(A)) = \mu(\{x \in E : \psi(x) \in A\}), \quad A \in \mathcal{E}'.$$

Since μ_ψ is the mean measure of a PRM it must be Radon. Under condition (2) this is automatic since

$$\mu_\psi(E') = \mu(\psi^{-1}(E')) \leq \mu(E) < \infty.$$

If μ is an infinite measure, condition (1) ensures that μ_ψ is Radon. To see this, observe that any compact set $K \subset E'$ is bounded, hence by condition (1), $\psi^{-1}(K)$ is bounded in E . Then the closure of the set $\psi^{-1}(K)$ in E is compact, hence it has finite μ -measure by the Radon property of μ . This proves that $\mu_\psi(K) < \infty$.

In applications, μ is often a finite measure. Consequently, we will not have to worry about condition (1).

Proof. We will evaluate the Laplace functional of N_ψ by using the Laplace functional of N .

Let g be a bounded non-negative measurable function on E' and observe that $g(\psi)$ is then a bounded non-negative measurable function on E . Notice that

$$\int_{E'} g dN_\psi = \sum_{i=1}^{\infty} g(\psi(X_i)) = \int_E g(\psi) dN.$$

Hence, from the form of the Laplace functional of the PRM N ,

$$\begin{aligned} \Psi_{N_\psi}(g) &= E \exp \left\{ - \int_E g(\psi) dN \right\} \\ &= \exp \left\{ - \int_E \left(1 - e^{-g(\psi(x))} \right) \mu(dx) \right\} \\ &= \exp \left\{ - \int_{E'} \left(1 - e^{-g(y)} \right) \mu_\psi(dy) \right\}. \end{aligned}$$

In the last step we have used the change of variable formula for integrals. Since μ_ψ is Radon, as discussed above, the right-hand expression is the Laplace functional of $\text{PRM}(\mu_\psi)$ on E' . It determines the distribution of N_ψ . This concludes the proof of the proposition. \square

Example 7.3.2 (Inhomogeneous Poisson process on $(0, \infty)$)

Let T_i be the points of a standard homogeneous Poisson process N on $(0, \infty)$. Consider a non-decreasing càdlàg transformation $\nu : (0, \infty) \rightarrow [0, \infty)$ with $\nu(0+) = 0$, and define its *generalized inverse* by

$$\nu^\leftarrow(y) = \inf\{x > 0 : \nu(x) \geq y\}$$

for $y \in (0, \sup_{x>0} \nu(x))$. Then ν defines a measure (we use the same symbol ν for the transformation and the measure) on the Borel σ -field \mathcal{E} of $(0, \infty)$ given by its values on intervals:

$$\nu(a, b] = \nu(b) - \nu(a) \quad \text{for any } 0 < a < b < \infty.$$

Moreover, the points $\nu^\leftarrow(T_i)$ constitute a PRM(ν) with state space

$$E' = \nu^\leftarrow([0, \infty)) = \{\nu^\leftarrow(y) : y \in (0, \infty)\}$$

and mean measure ν . Indeed, by the properties of generalized inverse functions (see Resnick [122], Section 0.2) for $0 < a < b$,

$$\#\{i \geq 1 : a < \nu^\leftarrow(T_i) \leq b\} = \#\{i \geq 1 : \nu(a) < T_i \leq \nu(b)\} = N(\nu(a), \nu(b)],$$

and (recall that $|A|$ denotes the Lebesgue measure of any Borel set A)

$$EN(\nu(a), \nu(b)] = |(\nu(a), \nu(b))| = \nu(b) - \nu(a) = \nu(a, b].$$

If ν is non-linear (equivalently, if the measure ν is not of the form λLeb for some positive λ) it is common to call PRM(ν) an *inhomogeneous Poisson process* (or PRM) with *mean value function* ν . This convention also applies to Poisson processes which are not defined on $(0, \infty)$. This notion of inhomogeneous Poisson process is in agreement with the definition in Section 2.1.3.

In Example 7.2.4 we dealt with the point process of the records of an iid sequence. A study of this example reveals that we constructed an inhomogeneous PRM by transforming the points of a homogeneous Poisson process (the record sequence of an iid exponential sequence) with an appropriate measurable function. \square

7.3.2 Marked Poisson Random Measures

PRMs have another remarkable property. One can adhere an independent coordinate to the points of a PRM — for obvious reasons the additional coordinate is called a *mark* and the corresponding procedure *marking*. Then under some restrictions on the distribution of the mark sequence the new process will again be a PRM on a larger state space. This is the content of the next result.

Proposition 7.3.3 (Independent marking of a PRM)

Assume $N_X = \sum_{i=1}^{\infty} \varepsilon_{X_i}$ is PRM(μ) with state space $E_1 \subset \mathbb{R}^d$. Let (Y_n) be an iid sequence of random vectors with values in $E_2 \subset \mathbb{R}^m$ and common distribution F . If (X_n) and (Y_n) are independent, then the point process $N = \sum_{i=1}^{\infty} \varepsilon_{(X_i, Y_i)}$ is PRM($\mu \times F$) on the state space $E = E_1 \times E_2$.

We call the resulting process $\sum_{i=1}^{\infty} \varepsilon_{(X_i, Y_i)}$ of the marking procedure a *marked PRM*, the sequence (Y_i) is the *mark sequence* and the common distribution of the Y_i 's is the *mark distribution*. We also mention that any PRM N on $E = E_1 \times E_2$ with mean measure $\mu \times F$ on E , where μ is a Radon measure on E_1 and F a probability distribution on E_2 , has the interpretation of a marked PRM with mark distribution F ; see Exercise 4 on p. 257.

Proof. First notice that

$$\int_E g \, dN = \sum_{i=1}^{\infty} g(X_i, Y_i).$$

We consider the Laplace functional $\Psi_N(g)$ of N for any bounded measurable function $g \geq 0$ on E :

$$\Psi_N(g) = E \left[E \left(\exp \left\{ - \sum_{i=1}^{\infty} g(X_i, Y_i) \right\} \middle| (X_i) \right) \right].$$

Since (X_i) and (Y_i) are independent and (Y_i) is an iid sequence with common distribution F ,

$$\begin{aligned} \Psi_N(g) &= E \left(\prod_{i=1}^{\infty} \int_{E_2} e^{-g(X_i, y)} F(dy) \right) \\ &= E \left(\exp \left\{ \sum_{i=1}^{\infty} \log \left(\int_{E_2} e^{-g(X_i, y)} F(dy) \right) \right\} \right) \\ &= E \left(\exp \left\{ - \int_{E_1} f(x) N_X(dx) \right\} \right), \end{aligned}$$

where N_X is the PRM(μ) with points X_i and the function f is given by

$$f(x) = -\log \left(\int_{E_2} e^{-g(x, y)} F(dy) \right).$$

Since $\int_{E_2} e^{-g(x, y)} F(dy) \leq 1$, we have $f \geq 0$. Therefore we may apply Lemma 7.2.7(1):

$$\begin{aligned} \Psi_N(g) &= \exp \left\{ - \int_{E_1} \left(1 - e^{-f(x)} \right) \mu(dx) \right\} \\ &= \exp \left\{ - \int_{E_1} \left(1 - \int_{E_2} e^{-g(x, y)} F(dy) \right) \mu(dx) \right\} \\ &= \exp \left\{ - \int_{E_1 \times E_2} \left(1 - e^{-g(x, y)} \right) (\mu \times F)(dx, dy) \right\}. \end{aligned}$$

This quantity is the Laplace functional of the desired PRM($\mu \times F$) on $E = E_1 \times E_2$. \square

Example 7.3.4 (Independent thinning)

Let X_i , $i = 1, 2, \dots$, be the points of PRM(μ), denoted by N_X , on the state space $E \subset \mathbb{R}^d$. Mark this PRM by an iid sequence (Y_i) of Bernoulli random

variables with distribution F_Y given by $P(Y_i = 1) = p$ and $P(Y_i = -1) = q$ with $p + q = 1$ and $p \in (0, 1)$. The marked point process,

$$N_{X,Y} = \sum_{i=1}^{\infty} \varepsilon_{(X_i, Y_i)},$$

is $\text{PRM}(\mu \times F_Y)$ on $E \times \{+1, -1\}$. Then, in particular, the point processes

$$\begin{aligned} N_X^+ &= \sum_{i=1}^{\infty} \varepsilon_{X_i} I_{\{Y_i=+1\}} = N_{X,Y}(\cdot \cap E \times \{+1\}), \\ N_X^- &= \sum_{i=1}^{\infty} \varepsilon_{X_i} I_{\{Y_i=-1\}} = N_{X,Y}(\cdot \cap E \times \{-1\}), \end{aligned} \tag{7.3.28}$$

are independent, because they are defined on the disjoint sets $E \times \{+1\}$ and $E \times \{-1\}$. Moreover, N_X^+ and N_X^- are $\text{PRM}(p\mu)$ and $\text{PRM}(q\mu)$, respectively, and both have state space E . We leave the verification of these facts as Exercise 2 on p. 256.

The point processes N_X^+ and N_X^- are simple examples of an *independent thinning* procedure of the process N_X . Indeed, each point X_i of N_X is inspected and, depending on whether $Y_i = +1$ or $Y_i = -1$, a (random) decision

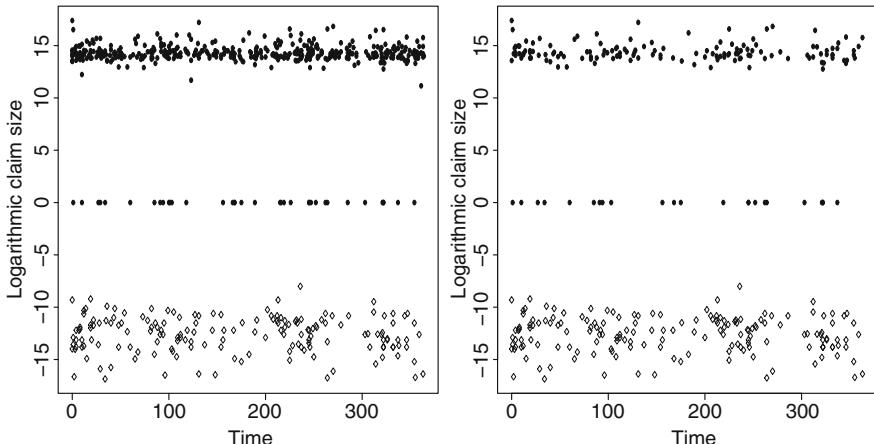


Figure 7.3.5 Left: *Danish fire insurance data reported in 2002. Each arrival T_i (day in 2002) is marked (solid dots) with the corresponding logarithmic claim size $\log(1 + X_i)$, $i = 1, \dots, 447$, where X_i stands for the insured damage to the building.* Right: *The thinned process (solid dots) consists of the retained points $(T_i, \log(1 + X_i))$ for which a positive loss of profit P_i was reported. After the thinning, 168 claims remain, corresponding to 37% of the annual claim number. The (negative) values $-\log(1 + P_i)$ are also shown in both graphs.*

is taken to retain the point X_i in the process or to exclude it from the process. This procedure is referred to as *thinning* of the original process.

We illustrate the thinning procedure in Figure 7.3.5. There we consider the claims of the Danish fire insurance data reported in 2002. The original point process consists of the pairs (T_i, X_i) (arrival and claim size corresponding to damage of building). The thinned process contains only the retained points for which a loss of profit was reported. This practice is typical for shops or office buildings which cannot be used after a fire. It may, of course, be doubted that the proposed thinning procedure is such that the decision of whether X_i is retained or deleted is really independent of X_i . \square

7.3.3 The Cramér-Lundberg and Related Models as Marked Poisson Random Measures

We start by reconsidering the Cramér-Lundberg model in the context of point process theory.

Example 7.3.6 (The Cramér-Lundberg model as a marked PRM)

Recall the point process from Example 7.1.4, which is generated from the points (T_i, X_i) , where (T_i) is a renewal process independent of the iid sequence (X_i) of random variables with values in $E_2 = (0, \infty)$ and distribution F . One possible interpretation is that X_i is the size of the claim arriving at time T_i . If we specify the T_i 's to be the points of a homogeneous Poisson process on $E_1 = (0, \infty)$ with intensity $\lambda > 0$, then we conclude from Proposition 7.3.3 that the points (T_i, X_i) constitute a marked PRM N on the state space $E = E_1 \times E_2$ with mean measure $\lambda \text{Leb} \times F$.

An immediate consequence of this fact is that the random variables $N(A_1), \dots, N(A_m)$ are mutually independent and Poisson distributed whenever A_1, \dots, A_m are disjoint.

A special case occurs if one decomposes time into disjoint intervals $(a_i, b_i]$, $i = 1, \dots, m$, and considers sets of the form $A_i = (a_i, b_i] \times B_i$ for any Borel sets B_i in $(0, \infty)$. Then the numbers of the claims that arrive in the time frame $(a_i, b_i]$ with corresponding claim sizes falling into the set B_i are mutually independent. For example, $(a_i, b_i]$ may denote disjoint years, months, days, etc.

Another important special case occurs if we decompose the claim size space into disjoint sets, for example into the disjoint layers $(c_i, d_i]$, $i = 1, \dots, m$. Then for any Borel sets C_i in $(0, \infty)$, $i = 1, \dots, m$, the claim numbers $N(C_i \times (c_i, d_i])$ with mean $\lambda|C_i|(F(d_i) - F(c_i))$ are mutually independent Poisson random variables. This means that any decomposition of the claim size space into disjoint layers yields independent claim numbers, whatever the time frame C_i .

But the embedding of the Cramér-Lundberg model in the point process context yields even more. Assume that A_1, \dots, A_m are disjoint Borel sets in E . Then, by Lemma 7.2.10, the integrals

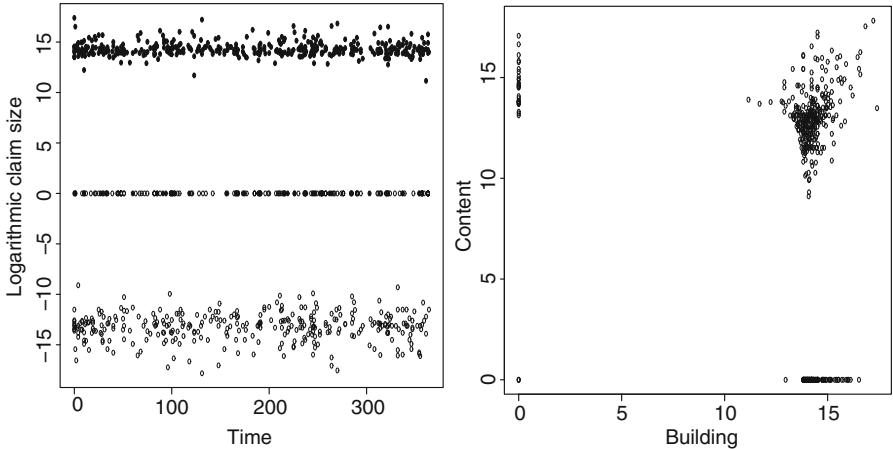


Figure 7.3.7 Left: Visualization of the marked point process of Danish fire insurance data in 2002. Each arrival T_i (day in 2002), $i = 1, \dots, 447$, on the x -axis is marked by a pair $(\log(1 + X_i), -\log(1 + Y_i))$ of logarithmic claim sizes, where X_i (solid dots) stands for the insured damage to the building and Y_i for the damage to the content of the building. The points that lie on the x -axis correspond to claims with zero damage to the building or the content. Right: Plot of the points $(\log(1 + X_i), \log(1 + Y_i))$. Most of the points are concentrated close to the line $x = y$ indicating that X_i and Y_i are dependent.

$$\int_{A_k} f_k dN = \sum_{i:(T_i, X_i) \in A_k} f_k(T_i, X_i), \quad k = 1, \dots, m,$$

are independent random variables provided these integrals are well-defined for the functions f_k . For example, take $f_k(t, x) = x$ and $A_k = (a_k, b_k] \times B_k$ for disjoint intervals $(a_k, b_k]$. Then we may conclude that the total claim amounts

$$\int_{A_k} f_k dN = \sum_{i:a_k < T_i \leq b_k, X_i \in B_k} X_i, \quad k = 1, \dots, m, \quad (7.3.29)$$

constitute independent random variables. This means that the total claim amounts over disjoint time periods (years, months, days, etc.) are independent. By a similar argument, the total claim amounts from disjoint layers $(c_i, d_i]$ are also independent, whatever the time period C_i . This means that the random variables

$$\sum_{i:T_i \in C_k, c_k < X_i \leq d_k} X_i, \quad k = 1, \dots, m, \quad (7.3.30)$$

are mutually independent.

Everything which has been said so far remains valid if the claim arrivals T_i come from an inhomogeneous Poisson. What changes is the mean measure μ of the Poisson process on the time axis. For example, with the same notation and under the same assumptions as above, the claim numbers $N(C_i \times (c_i, d_i])$ are mutually independent and $\text{Pois}(\mu(C_i)(F(d_i) - F(c_i)))$ distributed, and the claim amounts in (7.3.30) are independent. \square

Example 7.3.8 (Multivariate claim sizes)

We notice another interesting fact from Proposition 7.3.3: the marks of the PRM can also be multivariate. For example, T_i are the arrivals of a homogeneous Poisson process on $E_1 = (0, \infty)$ with intensity $\lambda > 0$, and the marks \mathbf{X}_i are iid d -dimensional random vectors with distribution F on the Borel σ -field of $E_2 = (0, \infty)^d$. Then (T_i, \mathbf{X}_i) are the points of a $\text{PRM}(\lambda \text{Leb} \times F)$ on $E = E_1 \times E_2 = (0, \infty)^{d+1}$. The vector \mathbf{X}_i can be interpreted as multiple claim size caused by an event that was incurred at time T_i . This model can be interpreted as a *multivariate Cramér-Lundberg model*.

For example, assume that T_i is the “arrival time” of a fire at an insured building in a portfolio, $X_i^{(1)}$ stands for the damage to the building, $X_i^{(2)}$ is the damage to the content of the building (furniture, valuables, machines, etc.) and $X_i^{(3)}$ is the claim size which corresponds to damages of neighboring buildings due to the fire. In this setting, $\mathbf{X}_i = (X_i^{(1)}, X_i^{(2)}, X_i^{(3)})$ is a multiple claim size. Although it is reasonable to assume that the random vectors \mathbf{X}_i are independent, it is less appealing to assume that the components $X_i^{(k)}$, $k = 1, 2, 3$, are independent. In the scenario described above, there will typically arise inter-dependencies between the components of \mathbf{X}_i , for every fixed i .

In the left graph of Figure 7.3.7 we illustrate multiple claim sizes of Danish fire insurance data which was incurred in 2002. In total, 447 fires were reported whose values exceeded 1 million Danish Kroner. Each fire claim generates a three-dimensional realization of a claim size: damage to the building, damage to the content and loss of profit. The latter category applies, for example, to office buildings which cannot be used after the fire. The graph shows the three-dimensional marked point process of the arrivals and claim sizes of the buildings and contents on a logarithmic scale. Some of the two-dimensional claim sizes have zero components, corresponding to the case that there has been no damage reported to the building or the content.

A rough evaluation of the data shows that there is dependence between the components of the two-dimensional claim sizes; see the right graph in Figure 7.3.7. For example, the estimated correlation between the components equals 0.43. \square

Example 7.3.9 (IBNR claims – delay in reporting)

Another interesting application of PRMs is in the context of IBNR (*Incurred But Not Reported*) claims. This means that the i th claim is not reported to the insurer at the claim time T_i , when it actually occurs, but at $T_i + D_i$ with some random time delay D_i . There is strong empirical evidence that

the majority of real-life claims are reported with a substantial time delay. Reasons for this phenomenon are manifold. For example, a claim might occur on a holiday or in the week between Christmas and New Year's Eve. Most offices (at least in Europe) are then closed and it might be difficult to get hold of any authorities. Alternatively, a driver could be involved in an accident and have to stay in hospital for a longer period without a chance of reporting the damage to his/her insurance company. In various cases, the insured might not be aware of the claim or he/she may be too busy to report it because of a catastrophic event. A situation like this one is often observed when a strong wind storm happens; it can take months before all claims are reported. In such a situation, the total claim amount due to one major event can be huge and therefore it is desirable to get an idea of the claim number and the total claim amount, perhaps based on similar claim histories.

A simplistic model in this context is the following. The claims arrive at the times T_i of a homogeneous Poisson process with intensity $\lambda > 0$. They are reported with delay D_i at times $T_i + D_i$. Now assume that (D_i) is an iid sequence of positive delay times with distribution F_D , independent of the arrival sequence (T_i) and of the iid claim size sequence (X_i) . Then the sequence of the pairs (X_i, D_i) is iid as well and independent of (T_i) . By virtue of Proposition 7.3.3, the points (T_i, X_i, D_i) constitute a PRM($\lambda \text{Leb} \times F \times F_D$) on the state space $E = (0, \infty)^3$.

In particular, from the transformation result of Proposition 7.3.1, we conclude that the points $T_i + D_i$ constitute a PRM N_{T+D} on $(0, \infty)$ with mean measure of $(0, t]$ given by

$$\begin{aligned}\nu(0, t] &= EN_{T+D}(0, t] = (\lambda \text{Leb} \times F_D)(\{(s, d) : 0 < s + d \leq t\}) \\ &= \lambda \int_{s=0}^t \int_{r=0}^{t-s} F_D(dr) ds \\ &= \lambda \int_0^t F_D(x) dx = \lambda t - \lambda \int_0^t P(D > x) dx.\end{aligned}\tag{7.3.31}$$

In particular, if $ED = \int_0^\infty P(D > x) dx < \infty$ we have

$$t^{-1}EN_{T+D}(0, t] = \lambda - O(t^{-1}), \quad t \rightarrow \infty.$$

This means that the average claim number is not much different from the Cramér-Lundberg case if t is large. For the expected claim number in a time interval of fixed length h (one year, say) we observe that

$$EN_{T+D}(t, t+h] = \lambda \int_t^{t+h} F_D(x) dx.$$

Then for large t and a continuous distribution function F_D ,

$$EN_{T+D}(t, t+h] \approx h \lambda F_D(t).$$

If the distribution F_D is heavy-tailed (e.g. subexponential; see Section 3.2.6) with finite mean and t is not large (i.e., the insurance business has not been running for a sufficiently long period of time), then the deviation of $\lambda h F_D(t)$ from the corresponding value $EN(t, t+h] = \lambda h$ in the Cramér-Lundberg case can be significant, showing the difference between the delayed and non-delayed cases.

If we mark the PRM N_{T+D} generated from the points $T_i + D_i$ with the iid claim size sequence (X_i) , then the points $(T_i + D_i, X_i)$ constitute a PRM($\nu \times F$) on $(0, \infty)^2$, where ν is given in (7.3.31) and F is the claim size distribution. As in Example 7.3.6, we can now decompose the time axis and/or the claim size space into disjoint sets resulting in independent claim numbers and total claim amounts taken over the different time periods and/or layers. \square

Example 7.3.10 (Compound Poisson representation of the aggregate claim amount on some subspace, continuation of Example 7.3.6)

We investigate the total claim amount in the Cramér-Lundberg model considered as a marked homogeneous Poisson process; see Example 7.3.6. Then T_i are the points of a homogeneous Poisson process on $(0, \infty)$ with intensity $\lambda > 0$, independent of the iid positive claim size sequence (X_i) with common distribution F . We conclude from Corollary 7.2.8 that the total claim amount on the Borel set $A \in \mathcal{E} = \mathcal{B}((0, \infty)^2)$ with $(\lambda \text{Leb} \times F)(A) < \infty$ has compound Poisson representation

$$\int_A x N(dt, dx) = \sum_{i:(T_i, X_i) \in A} X_i \stackrel{d}{=} \sum_{i=1}^M Z_i,$$

where $M \sim \text{Pois}((\lambda \text{Leb} \times F)(A))$ is independent of the iid non-negative random variables Z_i with common distribution function for $y > 0$,

$$F_Z(y) = G(\{(t, x) \in A : x \leq y\}) = \frac{(\text{Leb} \times F)(A \cap ((0, \infty) \times [0, y]))}{(\text{Leb} \times F)(A)}.$$

In particular, assume $A = B \times C$ for Borel sets $B, C \subset (0, \infty)$ with $F(C) > 0$ and $|B| > 0$. Then $M \sim \text{Pois}(\lambda |B| F(C))$ and

$$F_Z(y) = \frac{F(C \cap [0, y])}{F(C)} = P(X_1 \leq y \mid X_1 \in C), \quad y \geq 0, \quad (7.3.32)$$

i.e., the distribution of Z_i is nothing but the distribution of X_i conditioned to stay in the set C . It is left as Exercise 6(a) on p. 257 to show the following relations for $f(t, x) = x$:

$$E \left(\int_A f dN \right) = \lambda \int_A x dt F(dx) \quad \text{and} \quad \text{var} \left(\int_A f dN \right) = \lambda \int_A x^2 dt F(dx). \quad (7.3.33)$$

For the special case $A = B \times C$ we have simpler formulae whose proofs are left as an exercise:

$$E\left(\int_A f dN\right) = \lambda |B| \int_C x F(dx) = (\lambda \text{Leb} \times F)(A) E(X_1 \mid X_1 \in C), \quad (7.3.34)$$

$$\text{var}\left(\int_A f dN\right) = \lambda |B| \int_C x^2 F(dx) = (\lambda \text{Leb} \times F)(A) E(X_1^2 \mid X_1 \in C). \quad (7.3.35)$$

For the IBNR modification of the Cramér-Lundberg model (see Example 7.3.9) one can derive analogous formulae; see Exercise 6(b) on p. 257. \square

7.3.4 Aggregating Poisson Random Measures

In this section we learn about a third method to create new PRMs from given ones: the superposition or *aggregation* of PRMs. The basis for this method is the following result. It is the analog of the well-known fact that the sum $Y_1 + \dots + Y_m$ of independent $\text{Pois}(\lambda_i)$ distributed random variables Y_i , $i = 1, \dots, m$, is Poisson distributed with parameter $\lambda = \lambda_1 + \dots + \lambda_m$.

Proposition 7.3.11 *Assume N_1, \dots, N_m are mutually independent point processes on the same state space $E \subset \mathbb{R}^d$. If N_i is PRM(μ_i) with mean measure μ_i , $i = 1, \dots, m$, then the aggregated point process $N = N_1 + \dots + N_m$ is a PRM on E with mean measure $\mu = \mu_1 + \dots + \mu_m$.*

The proof of the proposition is straightforward either by directly verifying the defining properties of a PRM or by using a Laplace functional argument. The verification is left as Exercise 7(a) on p. 257.

The PRM property of the aggregated process N is convenient since we can apply the whole body of theory provided so far. For example, the integrals $\int_E f_i dN$ are mutually independent if the f_i 's have disjoint support. In particular, the total claim amounts $\int_{A_i} x dN$ over disjoint sets A_i , $i = 1, \dots, k$, of the state space E are independent; see Lemma 7.2.10.

We consider some consequences in the case when the N_i 's are mutually independent marked Poisson processes. Then the aggregated process N is not necessarily a marked Poisson process, i.e., a Poisson process with mean measure which is a product.

Example 7.3.12 (Aggregation of independent marked Poisson processes)
We assume that N_i , $i = 1, \dots, m$, are mutually independent marked PRMs on the state space $E = E_1 \times E_2$, where N_i has mean measure $\mu_i \times F_i$, μ_i is Radon on E_1 and F_i is a probability distribution on E_2 .

In the insurance context the aggregation of independent marked PRMs has a rather concrete meaning. Suppose we consider m independent non-life portfolios, each of which corresponds to a different business line, such

as fire insurance, motor insurance, house insurance. In the i th portfolio, let $T_j^{(i)}$, $j = 1, 2, \dots$, denote the claim arrivals which constitute a $\text{PRM}(\mu_i)$ on $(0, \infty)$ and are independent of the iid sequence $X_j^{(i)}$, $j = 1, 2, \dots$, of the claim sizes with common distribution F_i and values in E_2 , i.e., the pair $(T_j^{(i)}, X_j^{(i)})$ characterizes the j th claim in the i th portfolio. The point sequence $(T_j^{(i)}, X_j^{(i)})$, $j = 1, 2, \dots$, describes the evolution of the claims in the i th portfolio.

Now assume that the m independent portfolios are merged into one portfolio. This means that we no longer distinguish which portfolio the individual claims come from. Then we obtain only one claim sequence (T_k, X_k) , $k = 1, 2, \dots$, where any given T_k has the representation $T_k = T_j^{(i)}$, for some j and i , and the corresponding claim size $X_k = X_j^{(i)}$ then has distribution F_i .

A rather astonishing consequence of Proposition 7.3.11 is the fact that the point process $N = N_1 + \dots + N_m$ of the aggregated portfolios is again a $\text{PRM}(\mu)$ on $E_1 \times E_2$ with mean measure $\mu = \sum_{i=1}^m (\mu_i \times F_i)$. In particular, for any Borel sets $A \subset E_1$ and $B \subset E_2$, we thus obtain:

$$\mu(A \times B) = \sum_{i=1}^m \mu_i(A) F_i(B).$$

The latter relation cannot, in general, be written in the form $\mu(A \times B) = \nu(A) F(B)$ for an appropriate mean measure ν on E_1 , a distribution F on E_2 and any Borel sets $A \subset E_1$, $B \subset E_2$. Hence it is, in general, incorrect that μ has a representation as a product measure: $\mu = \nu \times F$. On the other hand, any marked Poisson process M on $E_1 \times E_2$ has mean measure $\nu \times F$ for a Radon measure ν on E_1 and a distribution F on E_2 , and this mean measure determines the distribution of M .

There are some exceptional cases when μ again has the product representation $\mu = \nu \times F$. We discuss one important case below and refer to Exercise 7(c) on p. 257 for another example.

Assume that $\mu_i = \lambda_i \gamma$, for positive numbers λ_i , $i = 1, \dots, m$, and a Radon measure γ on E_1 . Then, with

$$\lambda = \lambda_1 + \dots + \lambda_m, \quad F = \sum_{i=1}^m p_i F_i \quad \text{and} \quad p_i = \lambda_i / \lambda, \quad i = 1, \dots, m,$$

F is a distribution on E_2 , and we obtain, for Borel sets $A \subset E_1$ and $B \subset E_2$,

$$\mu(A \times B) = \lambda \gamma(A) \sum_{i=1}^m p_i F_i(B) = \lambda \gamma(A) F(B).$$

Then μ is again a product measure on $E_1 \times E_2$. Therefore the aggregated process N has interpretation as a marked PRM on $E_1 \times E_2$ with mean measure $\lambda \gamma \times F$. The aggregated process N has points (T_i, X_i) , where the T_i 's constitute a Poisson process on E_1 with mean measure $\lambda \gamma$, independent of

the iid sequence (X_i) with common distribution F . In particular, if $\gamma = \text{Leb}$, N is a marked homogeneous PRM with intensity λ . Obviously, F is a mixture distribution with weights p_i , $i = 1, \dots, m$, i.e., $p_i \geq 0$ and $\sum_{i=1}^m p_i = 1$. This was discussed in Section 3.3.1. In particular, X_i has representation in distribution

$$X_i \stackrel{d}{=} I_{\{J=1\}} X_1^{(1)} + \cdots + I_{\{J=m\}} X_1^{(m)},$$

where J has distribution $p_i = P(J = i)$, $i = 1, \dots, m$, and is independent of the mutually independent random variables $X_1^{(i)}$, $i = 1, \dots, m$.

If we interpret T_i as claim arrivals in the aggregated portfolio the corresponding iid claim sizes X_i can be understood as drawn by independent sampling with distribution (p_i) from the m individual claim size distributions F_i . \square

Comments

In this section we have studied some of the basic operations acting on point processes. The corresponding mathematical theory can be found in all texts on point processes, for example in Kallenberg [79], Daley and Vere-Jones [38, 39, 40], Resnick [122, 123].

The marking procedure can be extended to general point processes. Moreover, the mark sequence can be chosen in such a way that it depends on the unmarked point process. As a consequence, the thinning procedure, which was briefly mentioned in Example 7.3.4, can also be made dependent on the underlying point process. For example, if we interpret the Cramér-Lundberg model as a marked homogeneous Poisson process, a possible decision as to whether a claim should be retained in the portfolio could be based on the claim size: if it exceeds a given threshold, the claim could be removed from the portfolio and be covered by a reinsurance treaty. Thus, buying reinsurance can be considered as a thinning procedure.

Exercises

Section 7.3.1

- (1) Prove Proposition 7.3.1 by direct verification of the defining properties of $\text{PRM}(\mu(\psi^{-1}))$. Hint: use the representation $N = \sum_{i=1}^{\infty} \varepsilon_{\psi(X_i)}$.

Section 7.3.2

- (2) Consider the independent thinning procedure described in Example 7.3.4.
 (a) Show that the point processes N_X^+ and N_X^- are independent PRMs on E . Determine their mean measures.

- (b) Extend the independent thinning procedure as follows. Instead of a decision with two alternatives (retain/delete the point X_i) assume that the decision about retaining the point X_i in the point process is based on a random variable Y_i with distribution $p_k = P(Y_i = k)$, $k = 1, \dots, m$, where (Y_i) is iid, and the sequences (Y_i) and (X_i) are independent. Consider the m resulting thinned PRMs $N_k = \sum_{i=1}^{\infty} \varepsilon_{X_i} I_{\{Y_i=k\}}$, $k = 1, \dots, m$. Show that they are independent and determine their mean measures.
- (3) Let N be PRM($\mu_1 \times \mu_2$) on $E = E_1 \times E_2$, where μ_i is a Radon measure on E_i , $i = 1, 2$.
- Assume that $0 < \mu_2(E_2) < \infty$. Show that N has representation as a marked PRM($\mu(E_2)\mu_1 \times F$), where F is a probability distribution on E_2 given by $F(A) = \mu_2(A)/\mu(E_2)$ for any Borel sets $A \subset E_2$.
 - Let $\mu_1 = \lambda \text{Leb}$ on $E_1 = (0, \infty)$ and $B \subset E_2$ be a fixed Borel set such that $0 < \mu_2(B) < \infty$. Show that $N_B(C \times D) = N(C \times (D \cap B))$ for any Borel sets $C \subset E_1$, $D \subset E_2$ defines a marked homogeneous Poisson process on E with intensity $\lambda \mu_2(B)$ and mark distribution $\mu_2(A \cap B)/\mu_2(B)$ for any Borel set $A \subset E_2$.
 - Consider the process N_B defined in (b) with $E_2 = \mathbb{R}$. Show that $S(t) = \int_{(0,t] \times \mathbb{R}} x N_B(ds, dx)$, $t > 0$, defines a compound Poisson process on $[0, \infty)$, i.e., for every t , $S(t)$ has compound Poisson representation, and S has independent and stationary increments.
- (4) Let N be PRM($\mu_1 \times \mu_2$) on $E = E_1 \times E_2$, where μ_1 is a Radon measure on $E_1 \subset \mathbb{R}^d$ and μ_2 is a finite measure on $E_2 \subset \mathbb{R}^m$. Show that N has interpretation as a marked PRM on E . Determine the mark distribution.

Section 7.3.3

- (5) Assume the Cramér-Lundberg model as a marked PRM; see Example 7.3.6. Let $A, B \subset (0, \infty)$ be disjoint Borel sets. Decide whether the following claim numbers and total claim amounts are independent:
- $\#\{i \geq 1 : T_i \in A\}$ and $\sum_{i:T_i \in A} X_i$.
 - $\#\{i \geq 1 : T_i \in A\}$ and $\sum_{i:T_i \in B} X_i$.
 - $\#\{i \geq 1 : T_i \in A\}$ and $\sum_{i:T_i \in A} X_i I_B(X_i)$.
 - $\#\{i \geq 1 : T_i \in A, X_i \in B\}$ and $\#\{i \geq 1 : T_i \in B, X_i \in A\}$.
 - $\#\{i \geq 1 : T_i \in A, X_i \in B\}$ and $\sum_{i:T_i \in A} X_i I_A(X_i)$.
- (6) (a) Show the formulae (7.3.33)–(7.3.35) for the mean and variance of the total claim amount on a Borel set $A \subset (0, \infty)^2$ in the Cramér-Lundberg model.
(b) Modify the results of Example 7.3.10 for the IBNR Cramér-Lundberg model introduced in Example 7.3.9.

Section 7.3.4

- (7) (a) Prove Proposition 7.3.11 (i) by using Laplace functionals and (ii) by direct verification of the properties of a PRM for the aggregated point process N .
(b) Give an example of independent marked Poisson processes N_1 and N_2 on the same state space $E = E_1 \times E_2$ with mark space E_2 whose aggregated process $N = N_1 + N_2$ is not a marked Poisson process on E .
(c) Consider the case of independent marked PRMs on the same state space E with identical mark distributions. Show that the resulting aggregated process is again a marked Poisson process on E . Determine its mean measure.

- (8) Prove *Raikov's theorem* for point processes: Assume that N_1 and N_2 are independent point processes on the state space $E \subset \mathbb{R}^d$ and that $N = N_1 + N_2$ constitutes PRM(μ) for some Radon measure μ on E . Then N_1 and N_2 are PRMs with corresponding mean measures μ_1 and μ_2 and $\mu = \mu_1 + \mu_2$.

Hint: Use Raikov's theorem for Poisson random variables; see Exercise 4 on p. 47.

Poisson Random Measures in Collective Risk Theory

In Chapter 7 we collected the basic notions of point process theory. We have focused on Poisson random measures (PRMs) and their properties. In the present chapter we would like to apply the theory developed there, to models from collective risk theory. In particular, we will make considerable use of the marking and transformation techniques of PRMs introduced in Section 7.3, and we will intensively exploit the independence of Poisson claim numbers and Poisson integrals on disjoint parts of the time-claim size space. In Section 8.1, we consider different decompositions of the time-claim size space, such as decomposition by claim size, year of occurrence, year of reporting, etc. In Section 8.2, we study a major generalization of the Cramér-Lundberg model, called the *basic model*, which accounts for delays in reporting, claim settlements, as well as the payment process in the settlement period of the claim. We also decompose the time-claim size space into its basic ingredients, resulting in settled, incurred but not reported, and reported but not settled claims. We study the distributions of the corresponding claim numbers and total claim amounts.

This chapter was inspired by the ideas in Norberg's [114] article on point process techniques for non-life insurance.

8.1 Decomposition of the Time-Claim Size Space

The aim of this section is to decompose the time-claim size space in various ways into disjoint subsets. The resulting claim numbers and total claim amounts on the subspaces will be independent due to the underlying PRM structure. We will determine the distributions of these independent quantities.

8.1.1 Decomposition by Claim Size

Assume that claims arrive at times T_i according to $\text{PRM}(\mu)$ on the state space $(0, \infty)$, independently of the iid one-dimensional claim sizes X_i with common

distribution F on $(0, \infty)$. We know from Section 7.3.2 that the points (T_i, X_i) constitute PRM($\mu \times F$), denoted by N , on the state space $E = (0, \infty)^2$.

For $0 = c_0 < c_1 < \dots < c_m < \infty$ and $m \geq 1$, the claim size layers

$$A_i = (c_{i-1}, c_i], \quad i = 1, \dots, m, \quad \text{and} \quad A_{m+1} = (c_m, \infty),$$

are disjoint. For a given period of time $(a, b]$ for some $0 \leq a < b < \infty$, such as a year, a quarter, a month, etc., the claim numbers

$$N_i = N((a, b] \times A_i), \quad i = 1, \dots, m+1,$$

are mutually independent Poisson random variables. In particular, N_i has a Pois($\mu(a, b] F(A_i)$) distribution. This follows from the defining properties of a PRM.

By Lemma 7.2.10, the corresponding total claim amounts

$$\begin{aligned} S_i &= \int_{(a,b] \times A_i} x N(dt, dx) \\ &= \int_E x I_{(a,b] \times A_i}((t, x)) N(dt, dx) \\ &= \sum_{j: T_j \leq b, X_j \in A_i} X_j, \quad i = 1, \dots, m+1, \end{aligned}$$

are mutually independent, since the integrands $f_i(t, x) = x I_{(a,b] \times A_i}((t, x))$ have disjoint support. If we assume that $\mu(a, b] < \infty$, then

$$(\mu \times F)((a, b] \times A_i) = \mu(a, b] F(A_i) < \infty,$$

and therefore every S_i has compound Poisson representation

$$\text{CP}(\mu(a, b] F(A_i), P(X_1 \leq \cdot | X_1 \in A_i));$$

see Corollary 7.2.8 or Example 7.3.10.

An important special case corresponds to $m = 1$. Then the claim size space is divided into two layers $A_1 = (0, c]$ and $A_2 = (c, \infty)$, i.e., the portfolio splits into small and large claims. The quantity c can be interpreted as the *deductible by minimum franchise* or *first risk* in direct insurance, or as *retention level* in the context of excess-of-loss reinsurance; see Section 3.4 for terminology on reinsurance treaties. In this case, the primary insurer covers the amount

$$\begin{aligned} \sum_{j: T_j \in (a, b]} \min(X_j, c) &= \int_{(a,b] \times (0,c]} x N(dt, dx) + c \int_{(a,b] \times (c,\infty)} N(dt, dx) \\ &= S_1 + S_2, \end{aligned}$$

and the reinsurer covers

$$\sum_{j:T_j \in (a,b]} (X_j - c)_+ = \int_{(a,b] \times (c,\infty)} (x - c) N(dt, dx) = S_3.$$

The claim amounts $S_1 + S_2$ and S_3 are *not* independent, but the amounts S_1 and S_2 constituting the shares of the primary insurer are independent, and so are the claim amounts S_1 (to be paid by the primary insurer) and S_3 (to be paid by the reinsurer).

The situation with *proportional reinsurance* is different. Then the primary insurer covers the amount $p \int_{(a,b] \times (0,\infty)} x N(dt, dx)$ and the reinsurer $q \int_{(a,b] \times (0,\infty)} x N(dt, dx)$, where $p, q \in (0, 1)$ and $p + q = 1$. In this case, the two total claim amounts are strongly dependent. Indeed, they are linearly dependent and therefore their correlation is 1.

8.1.2 Decomposition by Year of Occurrence

As in the previous section, we assume that the points (T_i, X_i) in time-claim size space constitute a marked PRM($\mu \times F$), denoted by N , on $(0, \infty)^2$. We also assume that the accounting of the total claim amounts is provided on an annual basis. This means that we decompose time into the mutually disjoint sets (years)

$$A_i = (i - 1, i], \quad i = 1, 2, \dots.$$

Then it is immediate from the PRM property that the claim numbers $N(A_i \times (0, \infty))$ through the different years A_i are mutually independent and Pois($\mu(A_i)$) distributed. In particular, if the points T_j constitute a homogeneous Poisson process, then the distribution of $N(A_i \times (0, \infty))$ does not depend on the year. Similarly, the annual total claim amounts $\int_{A_i \times (0, \infty)} x N(dt, dx)$ are independent and have compound Poisson representation CP($\mu(A_i)$, F). In particular, for a homogeneous Poisson arrival process, the total claim amounts through the years constitute an iid sequence. These are properties we have already derived in Section 3.3.2. In contrast to that part of the book, the results in this section are simple byproducts of the theory of general Poisson processes.

The top graphs in Figure 8.1.1 show both the annual claim numbers and total claim amounts of the Danish fire insurance data 1980–2002. Claim sizes are evaluated in prices of 2002¹ by using the Danish Consumer Price Index (CPI) which is available from the website of Danmarks Statistik:

www.dst.dk/Statistik/seneste/Indkomst/Priser/FPI_inflation.aspx

The increase of the claim numbers through time can be explained for different reasons. First, not all companies might have reported their claims to the

¹ In Part I of this book we used the Danish fire insurance data 1980–1990 expressed in prices of 1985.

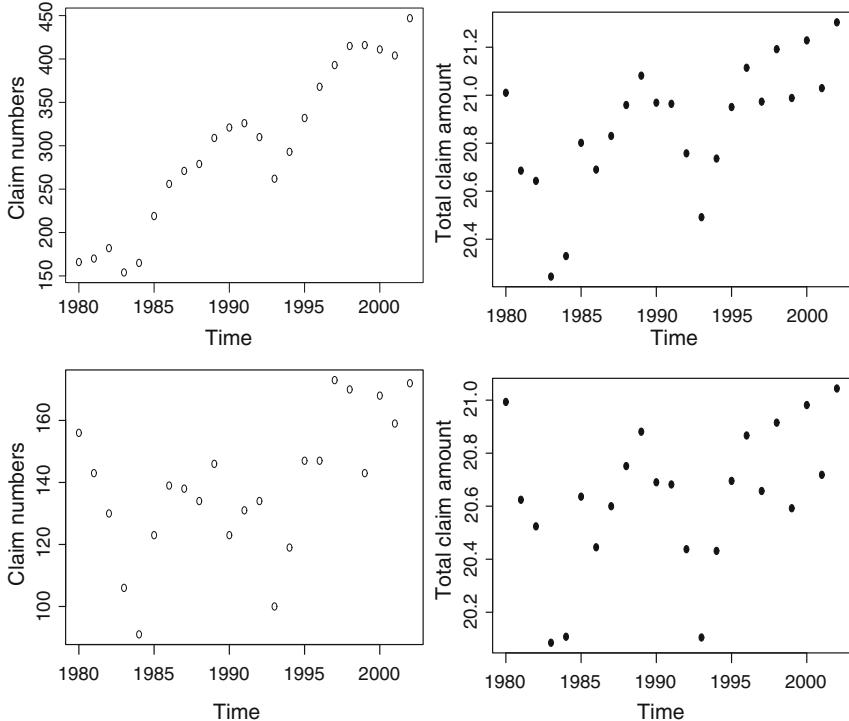


Figure 8.1.1 Top: *The Danish fire insurance data 1980–2002 in prices of 2002.* Bottom: *The data whose claim size exceeds 2.244 million Kroner in 2002 prices (corresponding to 1 million Kroner in 1980 prices).* Left column: *The annual claim numbers.* Right column: *The corresponding logarithmic annual total claim amounts.* Notice that the bottom graphs are more in agreement with the hypothesis of iid annual claim numbers and total claim amounts than the top graphs.

authorities in earlier years. Second, only those claims were reported which exceeded the value of 1 million Danish Kroner in the year of reporting. In prices of 2002, this threshold corresponds to 2.244 million Kroner in 1980. This means that many claims were not reported in 1980–2001 due to the use of different thresholds. For example, if the 1980 threshold of 2.244 million Kroner were applied in 2002, 172 out of the 447 reported claims (or 38%) would not be taken into account. Third, fire insurance and prices for buildings

are rather closely linked. Therefore the CPI might not be the best indicator for evaluating fire insurance.

In order to show the influence of inflation, in the bottom graphs of Figure 8.1.1 we plot the annual claim numbers and total claim amounts of those claims exceeding 2.244 million Kroner in 2002 prices (1 million Kroner in 1980 prices). The new graphs give the impression that the distributions of the annual claim numbers and total claim amounts do not significantly change through the years, although a slight increase in both categories is plausible. The bottom graphs are more in agreement with the PRM assumption on the claim arrivals and claim sizes than the top graphs, resulting in iid annual claim numbers and total claim amounts.

8.1.3 Decomposition by Year of Reporting

In this section we assume that the i th claim occurs at the time point T_i of a homogeneous Poisson process on $(0, \infty)$ with intensity $\lambda > 0$. The corresponding claim size X_i is reported with delay D_i . In the language of point processes, every arrival T_i is marked with the pair (D_i, X_i) with values in $(0, \infty)^2$ and joint distribution $F_{D,X}$, possibly with dependent components. The sequence of marks (D_i, X_i) , $i = 1, 2, \dots$, constitutes an iid sequence, independent of (T_i) . We write F_D for the distribution of D_i and F for the distribution of X_i .

In the remainder of this section we assume independence between D_i and X_i . We know from Example 7.3.9 that the points $(T_i + D_i, X_i)$ constitute $\text{PRM}(\nu \times F)$, denoted by $N_{T+D,X}$, on $(0, \infty)^2$, where

$$\nu(0, t] = \lambda \int_0^t F_D(y) dy, \quad t \geq 0.$$

We split time $(0, \infty)$ into disjoint periods (years say) $A_i = (i - 1, i]$, $i = 1, 2, \dots$. The time component of $N_{T+D,X}$ counts the claims *reported* in A_i ; they might have been *incurred* some periods ago. The corresponding pairs of claim numbers and total claim amounts

$$\left(N_{T+D,X}(A_i \times (0, \infty)), \int_{A_i \times (0, \infty)} x N_{T+D,X}(dt, dx) \right), \quad i = 1, 2, \dots,$$

are mutually independent. The claim number in the period A_i is $\text{Pois}(\nu(A_i))$ distributed, the corresponding claim amount has a $\text{CP}(\nu(A_i), F)$ distribution; see Example 7.3.10.

Write $N_{T,D}$ for the $\text{PRM}(\lambda \text{Leb} \times F_D)$ generated by the points (T_i, D_i) . The number of claims that were incurred in the i th period but were reported d periods later is given by the quantity

$$\begin{aligned} N_{i,d} &= \#\{j \geq 1 : i - 1 < T_j \leq i, i + d - 1 < T_j + D_j \leq i + d\} \\ &= N_{T,D}(\{(t, y) : t \in A_i, t + y \in A_{i+d}\}), \\ &\quad i = 1, 2, \dots, \quad d = 0, 1, \dots. \end{aligned}$$

A straightforward calculation yields that $N_{i,d}$ is Poisson distributed with mean

$$\begin{aligned} EN_{i,d} &= (\lambda \text{Leb} \times F_D)(\{(t, y) : t \in A_i, t + y \in A_{i+d}\}) \\ &= \lambda \int_{t \in (i-1, i]} \int_{t+y \in (i+d-1, i+d]} F_D(dy) dt \\ &= \lambda \int_d^{d+1} [F_D(z) - F_D(z-1)] dz. \end{aligned}$$

The distribution of $N_{i,d}$ is independent of i due to the homogeneity of the underlying Poisson process with points T_j . For different i , the quantities $N_{i,d}$ arise from disjoint subsets of the state space, hence $N_{i,d}$, $i = 1, 2, \dots$, are iid. The Poisson property also ensures that the corresponding total claim amounts

$$\sum_{j:i-1 < T_j \leq i, i+d-1 < T_j + D_j \leq i+d} X_j, \quad i = 1, 2, \dots, \quad (8.1.1)$$

are iid compound Poisson sums. It is left as Exercise 3(a) on p. 267 to calculate the parameters of their common distribution.

8.1.4 Effects of Dependence Between Delay in Reporting Time and Claim Size

We assume the conditions of Section 8.1.3, but we allow for possible dependence between the components D_i and X_i of the mutually independent pairs (D_i, X_i) . Then the reporting time $T_i + D_i$ of the i th claim depends on the claim size X_i . This assumption can be realistic. For example, a large claim size is more likely to be reported as early as possible than a small claim size. For an illustration of this phenomenon, see Example 8.1.2 below.

The points (T_i, D_i, X_i) constitute a $\text{PRM}(\lambda \text{Leb} \times F_{D,X})$, denoted by N , where $F_{D,X}$ denotes the joint distribution of (D_i, X_i) on $(0, \infty)^2$. This property implies, in particular, that for disjoint Borel sets $B_i \subset (0, \infty)^3$, the pairs $(N(B_i), \int_{B_i} x N(dt, dy, dx))$ of claim numbers and total claim amounts are mutually independent.

For any bounded Borel set $A \subset (0, \infty)^3$ the corresponding total claim amount $\int_A x N(dt, dy, dx)$ has $\text{CP}((\lambda \text{Leb} \times F_{D,X})(A), F_Z)$ distribution given by the distribution function

$$\begin{aligned} F_Z(y) &= \frac{(\text{Leb} \times F_{D,X})(A \cap \{(t, d, x) : x \leq y\})}{(\text{Leb} \times F_{D,X})(A)} \\ &= \frac{(\text{Leb} \times F_{D,X})(A \cap ((0, \infty)^2 \times [0, y]))}{(\text{Leb} \times F_{D,X})(A)}, \quad y > 0. \end{aligned}$$

Now specify the set A as follows:

$$A = (t_1, t_2] \times (d_1, d_2] \times (x_1, x_2], \quad 0 < t_1 < t_2, \quad 0 < d_1 < d_2, \quad 0 < x_1 < x_2.$$

Then $\int_A x N(dt, dy, dx)$ has distribution

$$\text{CP}(\lambda(t_2 - t_1) F_{D,X}((d_1, d_2] \times (x_1, x_2]), F_Z)$$

with corresponding distribution function

$$\begin{aligned} F_Z(y) &= \frac{F_{D,X}((d_1, d_2] \times (x_1, \min(x_2, y)))}{F_{D,X}((d_1, d_2] \times (x_1, x_2])} \\ &= P(X_1 \in (x_1, \min(x_2, y)] \mid D_1 \in (d_1, d_2], X_1 \in (x_1, x_2]), \quad y > 0. \end{aligned}$$

Example 8.1.2 (Large claims tend to be reported earlier than small ones)
If one has more information about the dependence between D_i and X_i one can specify the distribution $F_{D,X}$ in a meaningful way. Norberg [114], p. 112, assumed that the conditional distribution of D_1 given $X_1 = x$, $x > 0$, is $\text{Exp}(x\gamma)$ distributed for some positive $\gamma > 0$, and that X_1 has a $\Gamma(\alpha, \beta)$ distribution for some $\alpha, \beta > 0$.

The joint density $f_{D,X}$ of (D_i, X_i) can be calculated from the conditional density $f_D(y \mid X_1 = x)$ and the density f_X of X_1 :

$$\begin{aligned} f_{D,X}(y, x) &= \frac{f_{D,X}(y, x)}{f_X(x)} f_X(x) = f_D(y \mid X_1 = x) f_X(x) \\ &= \left((x\gamma) e^{-(x\gamma)y} \right) \left(\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \right) \\ &= \gamma \frac{\beta^\alpha}{\Gamma(\alpha)} x^\alpha e^{-x(\gamma y + \beta)}, \quad x, y \geq 0. \end{aligned} \tag{8.1.2}$$

The rationale for the choice of the density $f_{D,X}$ in (8.1.2) is that a large claim size $X_1 = x$ will increase the parameter of the exponential distribution $P(D_1 \leq y \mid X_1 = x)$, hence large claims will tend to be reported faster than small claims. This fact is also immediate from the following comparison of the tails: for $0 < x_1 < x_2$,

$$P(D_1 > y \mid X_1 = x_1) = e^{-(x_1\gamma)y} > e^{-(x_2\gamma)y} = P(D_1 > y \mid X_1 = x_2).$$

Integration with respect to x yields the density f_D of D_1 :

$$\begin{aligned} f_D(y) &= \int_0^\infty f_{D,X}(y, x) dx \\ &= \gamma \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(\gamma y + \beta)^{\alpha+1}} \int_0^\infty \frac{(\gamma y + \beta)^{\alpha+1}}{\Gamma(\alpha+1)} x^\alpha e^{-x(\gamma y + \beta)} dx \\ &= \frac{\gamma \alpha \beta^\alpha}{(\gamma y + \beta)^{\alpha+1}}, \quad y \geq 0. \end{aligned}$$

This means that the distribution of D_1 is in the location-scale family of a Pareto distribution with tail parameter $\alpha > 0$. This result is surprising: from the forms of the conditional density $f_D(y | X_1 = x)$ and the density f_X , it is difficult to guess that a Pareto distributed delay time D_1 appears. Since F_D is heavy-tailed (see Section 3.2.5), it is not unlikely that some claims will be reported with a very long delay. \square

8.1.5 Effects of Inflation and Interest

We again assume the conditions of Section 8.1.3. Then the reporting times $T_i + D_i$ of the claims constitute PRM(ν), denoted by N_{T+D} , with mean measure given by $\nu(0, t] = \lambda \int_0^t F_D(y) dy$, $t > 0$. We assume independence between (X_i) and (D_i) . Therefore the points $(T_i + D_i, X_i)$ constitute PRM($\nu \times F$), denoted by $N_{T+D, X}$, on the state space $(0, \infty)^2$.

Let $f(y, x)$ be a non-negative measurable function on $\mathbb{R} \times (0, \infty)$ such that $f(y, x) = 0$ for $y < 0$. We consider the stochastic process

$$\begin{aligned} S(t) &= \int_E f(t - y, x) N_{T+D, X}(dy, dx) \\ &= \int_{(0, t] \times (0, \infty)} f(t - y, x) N_{T+D, X}(dy, dx) \quad (8.1.3) \\ &= \sum_{i=1}^{\infty} f(t - (T_i + D_i), X_i) \\ &= \sum_{i=1}^{N_{T+D}(0, t]} f(t - (T_i + D_i), X_i), \quad t \geq 0. \end{aligned}$$

If we further specify

$$f(y, x) = e^{-r y} I_{(0, \infty)}(y) x$$

for some $r \in \mathbb{R}$, we obtain

$$S(t) = \sum_{i=1}^{N_{T+D}(0, t]} e^{-r(t-(T_i+D_i))} X_i, \quad t \geq 0. \quad (8.1.4)$$

If r is positive, then we can interpret r as the *inflation rate*. Assume that $t > 0$ is present time. Then the value of the claim size X_i which was reported at time $T_i + D_i$ in the past has the discounted value $e^{-r(t-(T_i+D_i))} X_i$ in terms of present prices. If r is negative we can interpret r as *interest rate*. Then the present value of a payment X_i made at time $T_i + D_i$ in the past is given by the increased amount $e^{-r(t-(T_i+D_i))} X_i$ due to compounded interest.

The stochastic process S considered in (8.1.4) is a modification of the total claim amount process in the Cramér-Lundberg model; the latter process is obtained by choosing $r = 0$ and $D_i = 0$ a.s. In contrast to the compound Poisson process in the original Cramér-Lundberg model, the process (8.1.4) has, in general, neither independent nor stationary increments even if one assumes no delay in reporting, i.e., $D_i = 0$ a.s. However, $S(t)$ has representation as a Poisson integral (8.1.3) and therefore, by Corollary 7.2.8, it has representation as a compound Poisson sum. We leave the verification of the details as Exercise 4 on p. 267.

Exercises

Sections 8.1.2

- (1) Consider the situation in Section 8.1.2 from the point of view of a reinsurer who covers the amount $g_i(X_j)$ of any claim size X_j occurring in year i . Here g_i , $i = 1, 2, \dots$, are non-negative measurable functions on $(0, \infty)$ with the property $0 \leq g_i(x) \leq x$.
 - (a) Show that the reinsurer's annual total claim amounts $R_i = \sum_{j:T_j \in A_i} g_i(X_j)$, $i = 1, 2, \dots$, are mutually independent.
 - (b) Determine the distribution of R_i defined in (a).
 - (c) Show that the amounts R_i covered by the reinsurer and $P_i = \sum_{j:T_j \in A_i} (X_j - g_i(X_j))$ covered by the primary insurer in year i are in general dependent. In which circumstances are R_i and P_i independent?

Section 8.1.3

- (2) Consider the situation of Section 8.1.3 but assume that the arrival sequence (T_i) is PRM(μ) on $(0, \infty)$ with a positive intensity function $\lambda(t)$, $t \geq 0$. Derive the distributions of the claim number and total claim amount corresponding to the claims reported in the i th period $A_i = (i-1, i]$.
- (3) Assume the conditions of Section 8.1.3 and that D_i and X_i are independent for every i .
 - (a) Determine the parameters of the compound Poisson representation of the total claim amounts (8.1.1).
 - (b) Determine the joint distribution of the claim numbers

$$\#\{j \geq 1 : 0 < T_j \leq 1, \quad d-1 < T_j + D_j \leq d\}, \quad d = 1, 2, \dots,$$

i.e., of those claims which occurred in the first period but were reported $d-1$ years later. Determine the joint distribution of the corresponding total claim amounts.

Section 8.1.5

- (4) Consider the process S in (8.1.4) with $r > 0$ and without delay in reporting, i.e., $D_i = 0$ a.s. Write N_T for the homogeneous Poisson process of the arrivals T_i with intensity $\lambda > 0$ and $N_T(t) = N_T(0, t]$, $t > 0$.

- (a) Assume in addition that X_i has finite variance. Show that S neither has independent nor stationary increments on $(0, \infty)$.
 (b) For fixed $t > 0$ show that $S(t)$ has compound Poisson representation

$$S(t) \stackrel{d}{=} \sum_{i=1}^{N_T(t)} e^{-r t U_i} X_i.$$

Here (U_i) is an iid uniform $U(0, 1)$ sequence and $N_T(t)$, (X_i) , (U_i) are mutually independent.

- (c) Show that for every $t > 0$,

$$e^{-rt} \sum_{i=1}^{N_T(t)} e^{r T_i} X_i \stackrel{d}{=} \sum_{i=1}^{N_T(t)} e^{-r T_i} X_i. \quad (8.1.5)$$

This identity in distribution has an interesting interpretation. We suppose that $r > 0$ is the inflation rate in $[0, t]$. First assume that all claim sizes X_i are known at time 0. Then the quantity $e^{r T_i} X_i$ stands for the value of X_i at time T_i . If we interpret T_i as the time when a payment to the insured is executed, $e^{r T_i} X_i$ is the amount to be paid at time T_i . The quantity $\sum_{i=1}^{N_T(t)} e^{r T_i} X_i$ is the total amount of all payments in $[0, t]$ in terms of inflated prices. The amount $e^{-rt} \sum_{i=1}^{N_T(t)} e^{r T_i} X_i$ is the corresponding deflated amount in terms of prices at time zero. The right-hand side of (8.1.5) has a different meaning. Here we assume that the claim size X_i occurs at time $T_i \leq t$ and $e^{-r T_i} X_i$ is its value in terms of prices at time 0. The quantity $\sum_{i=1}^{N_T(t)} e^{-r T_i} X_i$ is then the total amount of those claims that were incurred in $[0, t]$ in terms of prices at time 0.

8.2 A General Model with Delay in Reporting and Settlement of Claim Payments

8.2.1 The Basic Model and the Basic Decomposition of Time-Claim Size Space

In this section we study an extension of the basic model used in Part I of the book. We again call it *the basic model*. It is given by the following conditions.

The Basic Model

- The i th claim is associated with the quadruple (T_i, D_i, S_i, X_i) . The incident causing the claim arrives at time T_i with size X_i and is reported at time $T_i + D_i$. In the period $[T_i + D_i, T_i + D_i + S_i]$ the claim is settled, i.e., the amount X_i is paid to the insured.
- The *claim arrival sequence* (T_i) constitutes a homogeneous Poisson process on $(0, \infty)$ with intensity $\lambda > 0$.
- The *claim size sequence* (X_i) is iid with common distribution F on $(0, \infty)$.

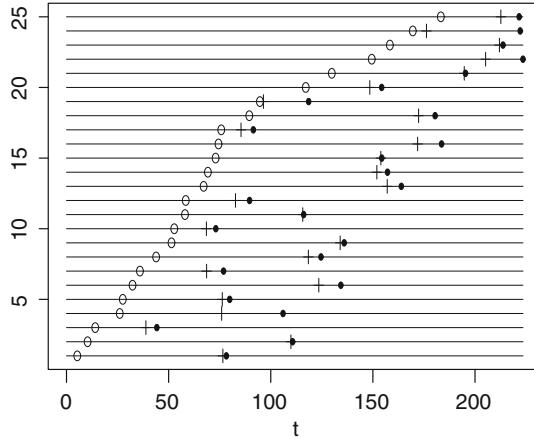


Figure 8.2.1 Visualization of the time components in the basic model. Each line corresponds to one claim. On the i th line, the claim arrival T_i (left dot), the reporting time $T_i + D_i$ (small vertical line) and the time of settlement $T_i + D_i + S_i$ (right bullet) are shown.

- The *delay sequence* (D_i) is iid with common distribution F_D on $(0, \infty)$.
- The *duration of settlement sequence* (S_i) is iid with common distribution F_S on $(0, \infty)$.
- The sequences (T_i) , (X_i) , (D_i) and (S_i) are *mutually independent*.

This is a simple model which takes into account some of the major ingredients of an insurance business. Of course, various of these assumptions deserve some criticism, for example, the homogeneity of the Poisson process, but also the independence of D_i , S_i and X_i . One also needs to specify in which way a claim is settled: one has to define a payment function on the settlement interval $[T_i + D_i, T_i + D_i + S_i]$ which yields the amount X_i at time $T_i + D_i + S_i$. In Section 8.2.3 we will discuss a simplistic payment function, and we will continue in Section 11.3 discussing a more realistic approach.

“More realistic” assumptions lead to a higher theoretical complexity. It is our aim to illustrate the problem of determining the distribution of the total claim amount of a portfolio under the “simple” but “still realistic” assumptions described in the basic model. This “simple model” will already turn out to be sufficiently complex.

Our first observation is that the points (T_i, D_i, S_i, X_i) constitute a marked PRM($\lambda \text{Leb} \times F_D \times F_S \times F$) on the state space $(0, \infty)^4$.

Indeed, the sequence (T_i) constitutes $\text{PRM}(\lambda \text{Leb})$, independent of the iid points (D_i, S_i, X_i) with common distribution $F_D \times F_S \times F$. Then the statement follows from Proposition 7.3.3.

Throughout the section, N denotes the basic process generated by the points (T_i, D_i, S_i, X_i) . We will also work with other PRMs derived from N by transformations of its points; see Section 7.3.1 for the theoretical background. We have already introduced the $\text{PRM}(\nu \times F)$ of the points $(T_i + D_i, X_i)$ on $(0, \infty)^2$, denoted by $N_{T+D,X}$, with $\nu(0, t] = \lambda \int_0^t F_D(y) dy$, $t > 0$; see Example 7.3.9. We will also work with the PRM generated from the points

$$(T_i, T_i + D_i, T_i + D_i + S_i, X_i).$$

In particular, we will use the fact that the points $(T_i + D_i + S_i, X_i) \in (0, \infty)^2$ constitute $\text{PRM}(\gamma \times F)$, denoted by $N_{T+D+S,X}$, where

$$\gamma(0, t] = \lambda \int_0^t F_{D+S}(y) dy = \lambda E(t - D_1 - S_1)_+, \quad t > 0, \quad (8.2.6)$$

defines the mean measure of the PRM which consists of the points $T_i + D_i + S_i$. Here F_{D+S} is the distribution function of $D_i + S_i$ given by

$$F_{D+S}(y) = \int_0^y F_D(y-s) F_S(ds), \quad y \geq 0.$$

We leave the verification of (8.2.6) as Exercise 1 on p. 286.

In the context of the basic model, we understand the state space $E = (0, \infty)^4$ of the point process N as the corresponding *time-claim size space*. At a given time $T > 0$ which we interpret as the present time we decompose this space into four disjoint subsets:

$$E = E_{\text{Settled}} \cup E_{\text{RBNS}} \cup E_{\text{IBNR}} \cup E_{\text{Not incurred}}.$$

They are characterized as follows.

The Basic Decomposition of Time-Claim Size Space

- The set

$$E_{\text{Settled}} = \{(t, d, s, x) : t + d + s \leq T\}$$

describes the claims which are settled by time T , i.e., the insurance company has paid the amount X_i to the insured by time T .

- The set

$$E_{\text{RBNS}} = \{(t, d, s, x) : t + d \leq T < t + d + s\}$$

describes the claims that have been incurred and are reported by time T , but they are not completely settled, i.e., the payment process for these claims is not finished yet. It is standard to call these claims **Reported But Not Settled** or simply *RBNS claims*.

- The set

$$E_{IBNR} = \{(t, d, s, x) : t \leq T < t + d\}$$

describes the claims that have been incurred but have not yet been reported at time T . It is standard to call these claims **Incurred But Not Reported** or simply *IBNR claims*.

- The set

$$E_{\text{Not incurred}} = \{(t, d, s, x) : T < t\}.$$

describes the claims that will be incurred after time T .

We notice that $E_{\text{Not incurred}}$ contains infinitely many points of the point process N with probability 1. In order to avoid this situation one can consider the insurance business over a finite time horizon, $T \leq T_0$, for some $T_0 < \infty$.

Although the sets of the basic decomposition depend on the time T we will often suppress this dependence in the notation.

8.2.2 The Basic Decomposition of the Claim Number Process

As a consequence of the disjointness of the sets in the basic decomposition of the time-claim space the claim number of concern for the insurance business at time T can be decomposed into three mutually independent Poisson numbers

$$\begin{aligned} N(E_{\text{Settled}}) + N(E_{\text{RBNS}}) + N(E_{IBNR}) &= N((0, T] \times (0, \infty)^3) \\ &= \#\{i \geq 1 : T_i \leq T\}. \end{aligned}$$

The points $T_i + D_i + S_i$ constitute $\text{PRM}(\gamma)$ on $(0, \infty)$ with mean measure γ given in (8.2.6). Then the process

$$N(E_{\text{Settled by time } T}) = \#\{i \geq 1 : T_i + D_i + S_i \leq T\}, \quad T > 0, \quad (8.2.7)$$

is inhomogeneous Poisson on $(0, \infty)$ with mean value function $\gamma(T) = \gamma(0, T]$, $T > 0$. In particular, $(N(E_{\text{Settled by time } T}))_{T>0}$ has independent increments.

The processes $(N(E_{\text{RBNS}} \text{ at time } T))_{T>0}$ and $(N(E_{IBNR} \text{ at time } T))_{T>0}$ do not have the property of a Poisson process on $(0, \infty)$. For example, they do not have independent increments; see Exercise 2(a) on p. 286. However, at any fixed time $T > 0$, the random variables $N(E_{\text{RBNS}})$ and $N(E_{IBNR})$ are Poisson distributed.

Next we collect some characteristic properties of the claim numbers corresponding to the basic decomposition.

Lemma 8.2.2 (Characterization of the Poisson claim numbers of the basic decomposition)

For every $T > 0$ the claim numbers $N(E_{\text{Settled}})$, $N(E_{\text{RBNS}})$ and $N(E_{IBNR})$ are independent Poisson random variables whose distribution has the following properties.

- (1) The process $(N(E_{\text{Settled at time } T}))_{T>0}$ is inhomogeneous Poisson on $(0, \infty)$ with mean value function

$$\begin{aligned}\gamma(T) &= \lambda \int_0^T F_{D+S}(y) dy \\ &= \lambda E(T - D_1 - S_1)_+, \quad T > 0.\end{aligned}\tag{8.2.8}$$

- (2) For every $T > 0$, the Poisson random variable $N(E_{\text{RBNS}})$ has mean value

$$\begin{aligned}&\lambda E[(T - D_1)_+ - (T - D_1 - S_1)_+] \\ &= \lambda E[S_1 I_{\{D_1 + S_1 \leq T\}}] + \lambda E[(T - D_1) I_{\{D_1 \leq T < D_1 + S_1\}}].\end{aligned}$$

- (3) For every $T > 0$, the Poisson random variable $N(E_{\text{IBNR}})$ has mean value

$$\lambda E[T - (T - D_1)_+] = \lambda E[D_1 I_{\{D_1 \leq T\}}] + \lambda T P(D_1 > T).$$

Proofs. (1) The Poisson process property is immediate from the representation (8.2.7). The mean value function γ was given in (8.2.6).

(2) Since the claim arrival process is homogeneous Poisson with intensity $\lambda > 0$, we observe that for $t > 0$,

$$\begin{aligned}\lambda t &= E\#\{i \geq 1 : T_i \leq t\} = E\left(\sum_{i=1}^{\infty} I_{(0,t]}(T_i)\right) \\ &= \sum_{i=1}^{\infty} P(T_i \leq t).\end{aligned}\tag{8.2.9}$$

A conditioning argument and an application of Fubini's theorem yield the following series of equations:

$$\begin{aligned}EN(E_{\text{RBNS}}) &= E\left(\sum_{i=1}^{\infty} I_{\{T_i + D_i \leq T < T_i + D_i + S_i\}}\right) \\ &= \sum_{i=1}^{\infty} P(T_i + D_1 \leq T < T_i + D_1 + S_1) \\ &= \sum_{i=1}^{\infty} E\left[\int_{((T-D_1-S_1)_+, (T-D_1)_+)} dP(T_i \leq t)\right] \\ &= E\left(\int_{((T-D_1-S_1)_+, (T-D_1)_+)} d(\lambda t)\right).\end{aligned}$$

In the last step we used Fubini's theorem and relation (8.2.9). Thus we arrive at the desired relation

$$\begin{aligned} EN(E_{\text{RBNS}}) &= \lambda [E(T - D_1)_+ - E(T - D_1 - S_1)_+] \\ &= \lambda E[S_1 I_{\{D_1 + S_1 \leq T\}}] + \lambda E[(T - D_1)I_{\{D_1 \leq T < D_1 + S_1\}}]. \end{aligned}$$

(3) The calculations are similar to part (2). They are left as Exercise 2(b) on p. 286. \square

8.2.3 The Basic Decomposition of the Total Claim Amount

In this section we study the total claim amounts at time T corresponding to the different parts in the basic decomposition of the time-claim size space. The total claim amount of the portfolio at time T is given by

$$\begin{aligned} S(T) &= \left(\int_{E_{\text{Settled}}} + \int_{E_{\text{RBNS}}} + \int_{E_{\text{IBNR}}} \right) x N(dt, dr, ds, dx) \\ &= S_{\text{Settled}} + S_{\text{RBNS}} + S_{\text{IBNR}} \\ &= \sum_{i:T_i \leq T} X_i. \end{aligned}$$

As for the claim numbers, we will often suppress the dependence on T in the notation. Since the three Poisson integrals S_{Settled} , S_{RBNS} and S_{IBNR} are defined on disjoint sets of the state space, they are mutually independent. Each of them can be represented as a compound Poisson sum.

The Settled and IBNR Total Claim Amounts

The amount $S_{\text{Settled at time } T}$ is that part of the total claim amount which corresponds to the claims arising from the set $E_{\text{Settled at time } T}$. For the points of the latter set, the amounts X_i have been paid to the insured by time T . Hence the corresponding total claim amount process is given by

$$S_{\text{Settled at time } T} = \sum_{i=1}^{N(E_{\text{Settled at time } T})} X_i, \quad T > 0. \quad (8.2.10)$$

We know from Lemma 8.2.2(1) that the counting process $(N(E_{\text{Settled at time } T}))$ constitutes an inhomogeneous Poisson process on $(0, \infty)$ with mean value function $\gamma(T) = \lambda E(T - D_1 - S_1)_+$, $T > 0$. The process (8.2.10) has independent but, in general, non-stationary increments. For every fixed $T > 0$, S_{Settled} has a $\text{CP}(\gamma(T), F)$ representation. We leave the verification of the details as an exercise.

The IBNR part of the total claim amount by time T is dealt with in a similar way. Since the reporting times occur after time T , the following amount is outstanding:

$$S_{\text{IBNR}} = \int_{E_{\text{IBNR}}} x N(dt, dr, ds, dx) = \sum_{i: (T_i, D_i, S_i, X_i) \in E_{\text{IBNR}}} X_i.$$

The condition defining E_{IBNR} only restricts the points (T_i, D_i) . Therefore the claim number $N(E_{\text{IBNR}})$ with a $\text{Pois}(\lambda E(T - (T - D_1)_+))$ distribution is independent of (X_i) , and S_{IBNR} has compound Poisson representation $\text{CP}(\lambda E(T - (T - D_1)_+), F)$. The process $(S_{\text{IBNR}} \text{ at time } t)_{t>0}$ does not have independent increments since the corresponding counting process does not constitute a Poisson process on $(0, \infty)$; see the discussion before Lemma 8.2.2.

The RBNS Total Claim Amount

For the RBNS part of the liability one has to make some assumptions about the cash flow from the insurer to the insured in the settlement period $[T_i + D_i, T_i + D_i + S_i]$. We assume that at each reporting time $T_i + D_i$ a stochastic (preferably non-decreasing càdlàg) *cash flow* or *payment process* starts which finishes at time $T_i + D_i + S_i$ with the settlement value X_i , i.e., with the actual claim size.

Although a stochastic payment process might be more realistic, we will restrict ourselves to a simplistic cash flow process which equals zero at $T_i + D_i$, is X_i at $T_i + D_i + S_i$ and increases linearly between these two instants of time. Then the settled part of the RBNS claims at time T amounts to

$$\begin{aligned} S_{\text{Settled RBNS}} &= \int_{E_{\text{RBNS}}} x \frac{T - t - r}{s} N(dt, dr, ds, dx) \\ &= \sum_{i: (T_i, D_i, S_i, X_i) \in E_{\text{RBNS}}} X_i \frac{T - T_i - D_i}{S_i}. \end{aligned} \quad (8.2.11)$$

Since $S_{\text{Settled RBNS}}$ is a Poisson integral, it has compound Poisson representation $\text{CP}(EN(E_{\text{RBNS}}), F_Z)$ according to Corollary 7.2.8. We know from Lemma 8.2.2(2) that

$$EN(E_{\text{RBNS}}) = \lambda E[(T - D_1)_+ - (T - D_1 - S_1)_+].$$

The integrand $f(t, r, s, x) = x s^{-1} (T - t - r)$ in the Poisson integral above is rather complex and therefore it seems difficult to evaluate F_Z . Writing

$$\pi = \pi(t, r, s) = s^{-1} (T - t - r),$$

we obtain the following formula for $F_Z(y)$ from Corollary 7.2.8 for $y > 0$:

$$F_Z(y) = \frac{(\lambda \text{Leb} \times F_D \times F_S \times F)(\{(t, d, s, x) : \pi \in [0, 1], x \pi \leq y\})}{EN(E_{\text{RBNS}})}.$$

Here we have made use of the fact that $T_i + D_i \leq T < T_i + D_i + S_i$ is equivalent to the fact that

$$\pi_i = S_i^{-1}(T - T_i - D_i) \in [0, 1].$$

The evaluation of $F_Z(y)$ simplifies by observing that the points π_i constitute a PRM on the state space $[0, 1]$. Indeed, since $\pi_i I_{[0,1]}(\pi)$ is a measurable function of the points (t, r, s) , we conclude from the results for the transformed points of a PRM that $\pi_i I_{[0,1]}(\pi_i)$ constitute the points of PRM(α) on $[0, 1]$. The mean measure α is given by

$$\begin{aligned} \alpha[0, z] &= (\lambda \text{Leb} \times F_D \times F_S) (\{(t, d, s) : 0 \leq s^{-1}(T - t - d) \leq z\}) \\ &= (\lambda \text{Leb} \times F_D \times F_S) (\{(t, d, s) : t + d \leq T \leq t + d + z s\}) \\ &= E(\#\{i \geq 1 : T_i + D_i \leq T \leq T_i + D_i + z S_i\}), \quad z \in [0, 1]. \end{aligned}$$

The same calculations as in the proof of Lemma 8.2.2(2) yield that

$$\alpha[0, z] = \lambda E[(T - D_1)_+ - (T - D_1 - z S_1)_+], \quad z \in [0, 1]. \quad (8.2.12)$$

Also notice that $\alpha[0, 1] = EN(E_{\text{RBNS}})$, and

$$\tilde{\alpha}(z) = \frac{\alpha[0, z]}{\alpha[0, 1]}, \quad z \in [0, 1], \quad (8.2.13)$$

defines a distribution function on $[0, 1]$.

Now we are in the position to rewrite $F_Z(y)$ in a much more accessible form:

$$\begin{aligned} F_Z(y) &= (\tilde{\alpha} \times F)(\{(\pi, x) : \pi \in [0, 1], x \pi \leq y\}) \\ &= \int_{\pi \in [0, 1]} \int_{x \pi \leq y} F(dx) \tilde{\alpha}(d\pi) \\ &= \int_{\pi \in (0, 1)} F(y/\pi) \tilde{\alpha}(d\pi). \end{aligned} \quad (8.2.14)$$

This means that Z in the compound Poisson representation of the Poisson integral has representation as a product

$$Z \stackrel{d}{=} X \Pi,$$

where $X \stackrel{d}{=} X_1$, Π has distribution $\tilde{\alpha}$, and X and Π are independent.

We summarize our findings.

Lemma 8.2.3 (Settled part of the RBNS claims by time T)

Assume the basic model of Section 8.2.1 and assume a linear cash flow function such that the insurer starts paying to the insured at time $T_i + D_i$ and finishes the payment X_i at time $T_i + D_i + S_i$. Then the amount $S_{\text{Settled RBNS}}$ of the RBNS claims which is settled by time T has compound Poisson representation $\text{CP}(\alpha[0, 1], F_Z)$, where α is given by (8.2.12) and F_Z by (8.2.14). In particular,

$$S_{\text{Settled RBNS}} \stackrel{d}{=} \sum_{i=1}^M X_i \Pi_i, \quad (8.2.15)$$

where M is $\text{Pois}(\alpha[0, 1])$ distributed, independent of the mutually independent iid sequences (X_i) with common distribution F and (Π_i) with common distribution $\tilde{\alpha}$ given in (8.2.13).

Notice the differences between the representations (8.2.11) and (8.2.15). In (8.2.11) the counting variable depends on the points $\pi_i = S_i^{-1}(T - T_i - D_i)$, whereas M in (8.2.15) is independent of (Π_i) . Also notice that π_i and Π_i have different distributions. Whereas (Π_i) is an iid sequence, the points π_i are not independent and have different distributions.

We can deal with the *non-settled* or *outstanding part* of the RBNS claims in a similar way. This means we are interested in the remaining total claim amount

$$\begin{aligned} S_{\text{Outstanding RBNS}} &= \int_{E_{\text{RBNS}}} x \left[1 - \frac{T - t - r}{s} \right] N(dt, dr, ds, dx) \\ &= \sum_{i:(T_i, D_i, S_i, X_i) \in E_{\text{RBNS}}} X_i (1 - \pi_i). \end{aligned}$$

The latter sum is meaningful because $EN(E_{\text{RBNS}})$ is finite and hence, with probability 1, there are only finitely many points (T_i, D_i, S_i, X_i) in E_{RBNS} . Also notice that $1 - \pi_i \in (0, 1]$ for any RBNS point (T_i, D_i, S_i, X_i) .

Proceeding in the same way as above, one sees that the points $1 - \pi_i$ constitute $\text{PRM}(\beta)$ on $(0, 1]$ with mean measure β given by

$$\begin{aligned} \beta[0, z] &= (\lambda \text{Leb} \times F_D \times F_S) (\{(t, d, s) : 0 \leq 1 - s^{-1}(T - t - d) \leq z\}) \\ &= (\lambda \text{Leb} \times F_D \times F_S) (\{(t, d, s) : 1 - z \leq s^{-1}(T - t - d) \leq 1\}) \\ &= \alpha[1 - z, 1], \quad z \in [0, 1]. \end{aligned}$$

Notice that $\beta[0, 1] = \alpha[0, 1]$.

Now calculations similar to those which led to Lemma 8.2.3 yield an analogous result for the outstanding part of the RBNS claims.

Lemma 8.2.4 (Outstanding part of the RBNS claims at time T)

We assume the conditions of Lemma 8.2.3. Then the amount $S_{\text{Outstanding RBNS}}$ of the RBNS claims which is outstanding at time T has compound Poisson representation $\text{CP}(\alpha[0, 1], F_{Z'})$, where α is given by (8.2.12), $Z' \stackrel{d}{=} X(1 - \Pi)$, $X \stackrel{d}{=} X_1$ and Π are independent and Π has distribution $\tilde{\alpha}$. In particular,

$$S_{\text{Outstanding RBNS}} \stackrel{d}{=} \sum_{i=1}^M X_i (1 - \Pi_i),$$

where M is $\text{Pois}(\alpha[0, 1])$ distributed, independent of the mutually independent iid sequences (X_i) with common distribution F and (Π_i) with common distribution $\tilde{\alpha}$.

Finally, the distribution of

$$\begin{aligned} S_{\text{RBNS}} &= S_{\text{Settled RBNS}} + S_{\text{Outstanding RBNS}} \\ &= \sum_{i:(T_i, D_i, S_i, X_i) \in E_{\text{RBNS}}} X_i, \end{aligned}$$

has a $\text{CP}(\alpha[0, 1], F)$ distribution. Considered as a function of T , the process S_{RBNS} does not have independent increments.

Writing $N_{\pi, X}$ for the marked $\text{PRM}(\alpha \times F)$ of the points (π_i, X_i) , we have the following Poisson integral representations:

$$S_{\text{Settled RBNS}} = \int_{\pi \in [0, 1]} x \pi N_{\pi, X}(d\pi, dx), \quad (8.2.16)$$

$$S_{\text{Outstanding RBNS}} = \int_{\pi \in [0, 1]} x (1 - \pi) N_{\pi, X}(d\pi, dx). \quad (8.2.17)$$

The integrands in these two integrals are non-negative and do not have disjoint support. Therefore the resulting Poisson integrals are dependent; see Corollary 7.2.13.

Using similar arguments, the theory above can be derived for any non-decreasing payment function f from $[0, 1]$ to $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1$. The corresponding settled and outstanding amounts are then given by the Poisson integrals

$$S_{\text{Settled RBNS}} = \int_{\pi \in [0, 1]} x f(\pi) N_{\pi, X}(d\pi, dx), \quad (8.2.18)$$

$$S_{\text{Outstanding RBNS}} = \int_{\pi \in [0, 1]} x (1 - f(\pi)) N_{\pi, X}(d\pi, dx). \quad (8.2.19)$$

They have compound Poisson representations

$$S_{\text{Settled RBNS}} \stackrel{d}{=} \sum_{i=1}^M X_i f(\Pi_i) \quad \text{and} \quad S_{\text{Outstanding RBNS}} \stackrel{d}{=} \sum_{i=1}^M X_i (1 - f(\Pi_i)),$$

where M , (X_i) , (Π_i) have the same distributions and dependence structure as in Lemmas 8.2.3 and 8.2.4. We encourage the conscientious and non-passive reader to verify these formulae.

Even more generality is achieved if one chooses integrand functions $g(\pi, x)$ with $0 \leq g(\pi, x) \leq x$ for $\pi \in [0, 1]$, non-decreasing in the π -component and such that $g(1, x) = x$. Then, for example,

$$S_{\text{Settled RBNS}} = \int_{\pi \in [0,1]} g(\pi, x) N_{\pi, X}(d\pi, dx) \stackrel{d}{=} \sum_{i=1}^M g(\Pi_i, X_i),$$

and $S_{\text{Outstanding RBNS}}$ is defined correspondingly. The choice of the function g allows one to determine the speed at which the insurer pays the insured in the settlement period of a claim. A simplistic (but not totally unrealistic) example is provided by the function

$$g(\pi, x) = \begin{cases} 0, & \pi \in [0, 1), \\ x, & \pi = 1. \end{cases}$$

Here the insurer pays nothing until the very end of the settlement period.

In real-life applications, the form of the payment function will depend on the circumstances surrounding each individual claim. In order to model such *random* phenomena, one would have to assume stochastic payment functions, i.e., at each reporting time $T_i + D_i$, a stochastic process starts which describes the individual settlement history of the claim. In Section 11.3 an attempt is made to take into account the stochastic nature of payments.

8.2.4 An Excursion to Teletraffic and Long Memory: The Stationary IBNR Claim Number Process

In this section we make an excursion into the active research area of large data or teletraffic networks such as the Internet or the Local Area Computer Network of a large company or a university. Such networks are highly complex and therefore the need for simple models arises which nevertheless describe some of the essential features of real-life networks. One of the properties of modern data networks is the presence of *long memory* or *long range dependence* in the counting process of packets registered by a sniffer or counter per time unit. We will give an explanation for this phenomenon in the model considered.

In what follows, we consider the counting process of the IBNR claims as a possible generating model for the activities in a large data network. Indeed, a modification of this process has been used for a long time as one of the standard models in the literature. Of course, the arrivals T_i and the delays D_i have a completely different meaning in this context. We will think of T_i as the *arrival of an activity* to the network. For example, this can be the arrival of a packet in your computer. The packet is processed, i.e., queued and routed to its destination. This activity creates an amount of work. The interval $[T_i, T_i + D_i]$ describes the period when the packet is processed. Assuming that the work is always processed at the same rate, the length D_i of this interval multiplied by the rate is then considered as a measure of the work initiated by the packet.

With these different meanings of T_i and D_i in mind, we modify the basic model of Section 8.2.1 insofar that we assume that the arrivals T_i come from

a homogeneous Poisson process with intensity λ on the whole real line.² We consider an increasing enumeration of the points T_i such that

$$\cdots < T_{-2} < T_{-1} < 0 \leq T_1 < T_2 < \cdots \quad \text{a.s.}$$

We mark each point T_i by a positive random variable D_i , which now stands for the amount of work brought into the system. The iid sequence $(D_i)_{i \in \mathbb{Z}_0}$ of positive random variables D_i is again independent of (T_i) . Here we write $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$ for the set of the non-zero integers. Hence the points (T_i, D_i) constitute $\text{PRM}(\lambda \text{Leb} \times F_D)$, denoted by $N_{T,D}$, on the state space $\mathbb{R} \times (0, \infty)$.

We consider the following analog of the IBNR claim number process which in this context represents the number of active sources at time T :

$$\begin{aligned} M(T) &= N_{T,D}(\{(t, d) : t \leq T < t + d\}) = \#\{i \in \mathbb{Z}_0 : T_i \leq T < T_i + D_i\} \\ &= N_{T,D}(\{(t, d) : t < 0, T < t + d\}) + N_{T,D}(\{(t, d) : 0 \leq t \leq T < t + d\}) \\ &= \#\{i \leq -1 : T < T_i + D_i, T_i < 0\} + \#\{i \geq 1 : 0 \leq T_i \leq T < T_i + D_i\} \\ &= M_-(T) + M_+(T), \quad T \geq 0. \end{aligned} \tag{8.2.20}$$

Notice that $M_-(T)$ counts the number of those arrivals which occurred before time 0 and whose activity period reaches into the future after time T . The claim number $M_+(T)$ coincides with the claim number $N(E_{\text{IBNR at time } T})$ considered in the previous sections. Since $M_+(T)$ and $M_-(T)$ arise from Poisson points (T_i, D_i) in disjoint subsets of the state space $\mathbb{R} \times (0, \infty)$ of $N_{T,D}$, they are independent and Poisson distributed.

In contrast to the quantities $\#\{i \in \mathbb{Z}_0 : T_i \leq T\}$, which are infinite a.s. at any time T (see Exercise 6(a) on p. 287), the random variables $M(T)$ are finite a.s. for every $T \geq 0$, provided D_1 has finite expectation. This is easily seen since the mean values $EM_{\pm}(T)$ are finite for any $T \geq 0$. Indeed, from Lemma 8.2.2(3) we know that

$$EM_+(T) = EN(E_{\text{IBNR at time } T}) = \lambda E(T - (T - D_1)_+).$$

For $M_-(T)$ we proceed in the same way as in the proof of Lemma 8.2.2(2). First observe that

$$(-T_i)_{i \leq -1} \stackrel{d}{=} (T_i)_{i \geq 1}.$$

Then we have

² In an insurance context, this would mean that the business does not start at time 0 but it has always been running.

$$\begin{aligned}
EM_-(T) &= E \left(\sum_{i=-\infty}^{-1} I_{\{T < T_i + D_i\}} \right) \\
&= \sum_{i=-\infty}^{-1} P(0 \leq -T_i < (T - D_1)_-) \\
&= \sum_{i=1}^{\infty} P(0 \leq T_i < (T - D_1)_-).
\end{aligned}$$

A Fubini argument and relation (8.2.9) yield:

$$\begin{aligned}
EM_-(T) &= E \left(\int_{[0, (T - D_1)_-)} d \left[\sum_{i=1}^{\infty} P(T_i \leq t) \right] \right) \\
&= \lambda E \left(\int_{[0, (T - D_1)_-)} dt \right) = \lambda E(T - D_1)_-.
\end{aligned}$$

Thus we have proved that

$$EM(T) = EM_+(T) + EM_-(T) = \lambda ED_1, \quad T \geq 0,$$

and this quantity is finite for $ED_1 < \infty$.

Since the expectation $EM(T)$ does not depend on T this is a first indication of the fact that $(M(T))_{T \geq 0}$ constitutes a *strictly stationary process*, i.e.,

$$(M(T))_{T \geq 0} \stackrel{d}{=} (M(T + h))_{T \geq 0} \quad \text{for } h \geq 0,$$

where $\stackrel{d}{=}$ refers to equality of the finite-dimensional distributions. A proof of strict stationarity of the process M is left as Exercise 6(c) on p. 287.

We restrict ourselves to the problem of showing *second order stationarity*, in the sense that the covariance function $C_M(T, T + h)$ of the process M does not depend on T :

$$C_M(T, T + h) = \text{cov}(M(T), M(T + h)), \quad h \geq 0, \quad T \geq 0.$$

This property is easily verified since both $M(T)$ and $M(T + h)$ can be represented as Poisson integrals:

$$M(s) = \int_{\mathbb{R} \times (0, \infty)} I_{\{t \leq s < t+r\}}((t, r)) N_{T, D}(dt, dr), \quad s \in \mathbb{R}.$$

Then we conclude from relation (7.2.22) that

$$\begin{aligned}
C_M(T, T+h) &= \lambda \int_{\mathbb{R} \times (0, \infty)} I_{\{t \leq T < t+r\}}((t, r)) I_{\{t \leq T+h < t+r\}}((t, r)) F_D(dr) dt \\
&= \lambda \int_{t=-\infty}^T \int_{r=T-t+h}^{\infty} F_D(dr) dt \\
&= \lambda \int_{-\infty}^T \bar{F}_D(T-t+h) dt \\
&= \lambda \int_h^{\infty} \bar{F}_D(s) ds.
\end{aligned}$$

We summarize these results.

Lemma 8.2.5 *The counting process $(M(T))_{T \geq 0}$ defined in (8.2.20) based on the points (T_i, D_i) of $\text{PRM}(\lambda \text{Leb} \times F_D)$ with $ED_1 < \infty$ is a strictly stationary process whose one-dimensional marginal distributions are Poisson. In particular,*

$$\begin{aligned}
EM(T) &= \lambda ED_1, \\
\text{cov}(M(T), M(T+h)) &= \lambda \int_h^{\infty} \bar{F}_D(s) ds \\
&= \gamma_M(h), \quad T \geq 0, h \geq 0.
\end{aligned} \tag{8.2.21}$$

Since we have assumed $ED_1 < \infty$, the covariance function $\gamma_M(h)$ is finite for any h and satisfies $\gamma_M(h) \downarrow 0$ as $h \uparrow \infty$. The decay rate of $\gamma_M(h)$ to zero as $h \rightarrow \infty$ is often interpreted as *range of memory* or *range of dependence* in the stationary process M . It is clear that the lighter the tail \bar{F}_D , the faster $\gamma_M(h)$ in (8.2.21) tends to zero. It is in general not possible to calculate γ_M more explicitly than (8.2.21). But then it often suffices to have results about the asymptotic behavior of $\gamma_M(h)$ as $h \rightarrow \infty$. One such case is described in the following example.

Example 8.2.6 (Regularly varying D_1)

Recall from Definition 3.2.20 on p. 99 that a positive random variable D_1 is said to be *regularly varying with index $\alpha \geq 0$* if the right tail of F_D has form

$$\bar{F}_D(x) = L(x) x^{-\alpha}, \quad x > 0,$$

for some $\alpha \geq 0$ and some slowly varying function L , i.e., a non-negative measurable function on $(0, \infty)$ satisfying $L(cx)/L(x) \rightarrow 1$ as $x \rightarrow \infty$ for every $c > 0$. Regularly varying functions satisfy some asymptotic integration rule which runs under the name *Karamata's theorem*; see p. 181. An application of Karamata's theorem to (8.2.21) yields, for $\alpha > 1$,

$$\gamma_M(h) \sim (\alpha - 1)^{-1} h \bar{F}_D(h), \quad \text{as } h \rightarrow \infty. \tag{8.2.22}$$

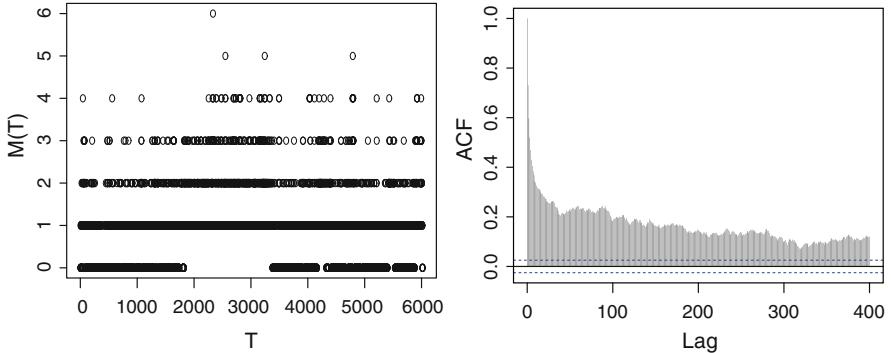


Figure 8.2.7 Left: A realization of the strictly stationary process $(M(n))_{n=1,\dots,6000}$ with Poisson intensity $\lambda = 3$, D_1 is regularly varying with index 1.5. Right: The sample autocorrelation function $\hat{\rho}_M$ at the first 400 lags. It decays very slowly. The dashed lines indicate lag-wise 95% asymptotic confidence intervals for $\hat{\rho}_M(h)$ under the hypothesis of iid Gaussian noise for the underlying sample.

This means that the covariance function $\gamma_M(h)$ decays very slowly to zero — like a power law with exponent $1 - \alpha < 0$. Another application of Karamata's theorem yields for $\alpha \in (1, 2)$ that

$$\int_0^\infty \gamma_M(h) dh = \infty.$$

In Figure 8.2.7 we visualize the process M at the discrete instants of time $k = 1, 2, \dots, 6000$. The D_i 's are regularly varying with index $\alpha = 1.5$. Then the arguments above apply and the autocovariance function $\gamma_M(h)$ decays to zero as described in (8.2.22). Since the function γ_M is in general not explicitly known it is common practice in time series analysis to estimate its standardized version $\rho_M(h) = \gamma_M(h)/\gamma_M(0)$, the *autocorrelation function* of the stationary time series $(M(k))$, from the sample $(M(k))_{k=1,\dots,n}$. The corresponding *sample autocovariances* $\hat{\gamma}_M(h)$ and *sample autocorrelations* $\hat{\rho}_M(h)$ are then given by

$$\begin{aligned} \hat{\gamma}_M(h) &= n^{-1} \sum_{k=1}^{n-h} (M(k) - \bar{M}_n)(M(k+h) - \bar{M}_n), \\ \hat{\rho}_M(h) &= \frac{\hat{\gamma}_M(h)}{\hat{\gamma}_M(0)}, \quad h \geq 0, \end{aligned}$$

where \bar{M}_n denotes the sample mean. The sample autocovariances and sample autocorrelations are consistent estimators of their deterministic counterparts

if the underlying process ($M(n)$) is strictly stationary and ergodic. This follows by an application of the ergodic theorem; see Krengel [89]. The process ($M(n)$) inherits ergodicity from the ergodicity of the underlying homogeneous Poisson process; see Daley and Vere-Jones [38].

Thus slow decay of the sample autocorrelation function $\hat{\rho}_M(h)$ as a function of the lag h is an indication of slow decay of $\rho_M(h)$. \square

For any stationary process with covariance function γ the property

$$\int_0^\infty |\gamma(h)| = \infty$$

is often referred to as *long range dependence* or *long memory*. It describes extremely slow decay of the covariance function $\gamma(h)$ to zero as $h \rightarrow \infty$. This definition seems arbitrary and, indeed, there exist various other ones based on different arguments; see for example Samorodnitsky and Taqqu [131], Chapter 7, Doukhan et al. [41], and Samorodnitsky [130].

An alternative way of defining long memory is to require that $\gamma(h)$ is a regularly varying function with index in $(-1, 0)$, i.e., $\gamma(h) = L(h)h^{2(H-1)}$ for a slowly varying function L and a parameter $H \in (0.5, 1)$, called the *Hurst coefficient*. This assumption is satisfied for the process M provided D_1 is regularly varying with index $\alpha \in (1, 2)$. Then H assumes the value $(3 - \alpha)/2$; see (8.2.22). Such a definition is more reasonable from a statistical point of view since it allows one to estimate the Hurst parameter H , which characterizes the range of dependence in the stationary process.

The property of power law decay of the covariance function is also observed for certain fractional Gaussian noises, i.e., the increment process of fractional Brownian motion (see Exercise 9 on p. 289), and for certain fractional ARIMA processes. We refer to Samorodnitsky and Taqqu [131], Chapter 7, for fractional Brownian motion and Gaussian noise, and Brockwell and Davis [24] for fractional ARIMA processes.

The notion of long memory or long range dependence has attracted a lot of attention over the last 40 years. It is a phenomenon which is empirically observed in areas as diverse as physics, telecommunications, hydrology, climatology, and finance. We refer to Doukhan et al. [41] and Samorodnitsky [130] for recent surveys on the theory and applications of long memory processes.

The interest in the notion of long memory is explained by the fact that long memory stationary processes, in contrast to short memory processes, often exhibit asymptotic behavior different from standard central limit theory. For example, the workload process $\int_0^T M(t) dt$ of the strictly stationary teletraffic process M defined in (8.2.20) does not, in general, satisfy standard central limit theory in the sense of functional distributional convergence with limiting Brownian motion; cf. Billingsley [17]. On the contrary, the workload process with regularly varying D_1 with index $\alpha \in (1, 2)$ has a less familiar limiting

process,³ and the normalization in this limit result significantly differs from the common \sqrt{T} -scaling constants.

As a matter of fact, the process M defined in (8.2.20) with regularly varying D_1 with index $\alpha \in (1, 2)$ has attracted a lot of attention in the teletraffic community; see for example Mikosch et al. [110], Faÿ et al. [50], Mikosch and Samorodnitsky [111], cf. Resnick [124] and the references given therein. The PRM($\lambda \text{Leb} \times F_D$) model generating the process M is often referred to as $M/G/\infty$ queue model or as *infinite source Poisson model* in the probability literature on queuing and telecommunications. It is a simple model for real-life teletraffic, in particular for the Internet. The model is simplistic but allows for the description of phenomena which are also observed in teletraffic data: long memory, heavy-tailed components (such as file sizes or transmission lengths) and self-similarity⁴ of the limiting process of the workload process $\int_0^T M(t) dt$. In addition, the process M is easily simulated. In the teletraffic context, the quantities D_i are most relevant. Their size determines typical behavior of the whole system. The memory in the activity process $M(T)$ at a given time T is then determined by the range of the activities described by the length of the D_i 's.

8.2.5 A Critique of the Basic Model

In the previous sections, we decomposed the time-claim size space E at the present time $T > 0$ into the disjoint sets E_{Settled} , E_{RBNS} , E_{IBNR} and $E_{\text{Not incurred}}$. The corresponding pairs of claim number and total claim amount

$$(N(E_{\text{Settled}}), S_{\text{Settled}}), \quad (N(E_{\text{RBNS}}), S_{\text{RBNS}}), \quad (N(E_{\text{IBNR}}), S_{\text{IBNR}})$$

are mutually independent. These quantities are functions of the PRM N with points (T_i, D_i, S_i, X_i) . In our presentation we have assumed that the claim arrivals T_i come from a homogeneous Poisson process and that the three-dimensional iid marks (D_i, S_i, X_i) have independent components. These conditions can be weakened. For example, the arrivals may arise from an inhomogeneous Poisson process or the components D_i , S_i and X_i may be dependent; see Section 8.1.4 for an example of dependence between D_i and X_i . In this more general context, one can often follow the arguments given above without major difficulties. Of course, one has to pay a price for more generality: calculations become more tedious and the resulting formulae are more complex.

³ The limiting process is spectrally positive α -stable Lévy motion; see Example 10.5.2 for its definition and properties. Alternatively, fractional Brownian motion may occur as limit if the intensity $\lambda = \lambda(T)$ of the underlying homogeneous Poisson process grows sufficiently fast with time T ; see Mikosch et al. [110], cf. Resnick [124].

⁴ See Exercise 9(d) on p. 289 for the definition and examples of some self-similar processes, including fractional Brownian motion.

The basic model is statistically tractable. Based on historical data one can estimate the underlying Poisson intensity and the distribution of the iid observations (D_i, S_i, X_i) . If the components D_i , S_i and X_i are independent, one can use one-dimensional statistical methods to fit the distributions of F_D , F_S and F separately. The statistics become much more complicated if one aims at fitting a three-dimensional distribution with dependencies between D_i , S_i and X_i .

Based on historical information and on the fitted distributions one knows *in principle* the distributions of the settled and outstanding claim numbers and total claim amounts of an insurance business. In most cases of interest the distributions will not be tractable without Monte-Carlo or numerical methods. For example, we learned in Section 3.3.3 about *Panjer recursion* as a numerical technique for evaluating compound Poisson distributions.

An advantage of the presented theory is the consequent use of Poisson processes. The Poisson ideology allows one to decompose the total claim amount into its essential parts (IBNR, RBNS, settled and outstanding, say). These are independent due to the Poisson nature of the underlying counting process. One loses the elegance of the theory if one gives up the Poisson assumption. Nevertheless, even in the case of a non-Poissonian marked point process, several of the calculations given above can be provided by using general point process techniques: most of the moment and covariance calculations are still possible; see Daley and Vere-Jones [38, 39, 40].

As mentioned above, the basic model can be extended and generalized in different directions. A way of introducing a “more realistic” model is to assume a genuine stochastic process model Y_i which describes the payment of the i th claim size X_i in the period $[T_i + D_i, T_i + D_i + S_i]$. In Section 8.2.3 we have assumed a simple linear model $Y_i(T) = X_i S_i^{-1}(T - T_i - D_i)$ for $T \in [T_i + D_i, T_i + D_i + S_i]$. Unfortunately, every claim has its own characteristics and therefore it would be rather optimistic to believe that a linear function is in agreement with real-life data.

It is possible to assume very general pay-off functions Y_i and to develop some theory for the resulting total claim amounts; for some asymptotic results see Klüppelberg et al. [81, 82, 83] who worked with Poisson shot noise models. The latter class of models is closely related to the Poisson models considered above. In the language of marked Poisson processes, the claim arrivals T_i are then marked with an iid sequence of quadruples (D_i, S_i, X_i, Y_i) , where the meaning of (D_i, S_i, X_i) is as above and Y_i is a stochastic process whose sample paths describe the payment process for the i th claim. In this context it is reasonable to let Y_i have non-decreasing sample paths on the interval of interest $[T_i + D_i, T_i + D_i + S_i]$: choose Y_i such that $Y_i(t) = 0$ a.s. for $t < 0$, the process $Y_i(T - T_i - D_i)$ gets activated at the reporting time $T = T_i + D_i$ (possibly with a positive initial payment), it does not decrease until time $T_i + D_i + S_i$, where it achieves its largest value X_i and becomes deactivated at times $T > T_i + D_i + S_i$, i.e., $Y_i(T - T_i - D_i) = 0$ a.s.

One faces a major problem: the choice of a reasonable model for the payment process Y_i . A practical solution would be to work with historical sample paths from a sufficiently large portfolio over a sufficiently long period of time. Then the distribution of the total claim amount could be approximated by Monte-Carlo simulations from the empirical distribution of the historical sample paths. This approach is close in spirit to the bootstrap; see Efron and Tibshirani [44] for an elementary introduction, cf. Section 3.3.5. However, this approach is ad hoc and requires a theoretical justification.

Motivated by Bayesian ideas, Norberg [114] suggested modeling the payment processes Y_i by suitable gamma and Dirichlet processes. He demonstrated that one can predict outstanding claims by calculating their expectation conditionally on information about past payments for the claim. While the required assumptions seem ad hoc, they are as realistic (or unrealistic) as assuming a non-decreasing smooth payment function, as we did on pp. 274–278.

In Chapter 11 we shall look at some models which we will call *cluster point processes*. There we will describe the payment processes for individual claims by stochastic processes. It will again be convenient to assume a simplifying Poisson structure of the points of these processes. In Section 11.3, this structure will allow us to get explicit expressions for predicted claim numbers and total claim amounts based on historical information.

Exercises

Section 8.2.1

- (1) Let N be the $\text{PRM}(\lambda \text{Leb} \times F_D \times F_S \times F)$ generated from the points (T_i, D_i, S_i, X_i) in the basic model; see p. 268.
 - (a) Show that the point process $N_{T, T+D, T+D+S, X}$ of the points $(T_i, T_i + D_i, T_i + D_i + S_i, X_i)$ is PRM.
 - (b) Determine the mean measure of $N_{T, T+D, T+D+S, X}$.
 - (c) Show that the point process $N_{T+D+S, X}$ of the points $(T_i + D_i + S_i, X_i)$ is $\text{PRM}(\gamma \times F)$ on $(0, \infty)^2$, where γ is defined by (8.2.6).

Section 8.2.2

- (2) Consider the basic decomposition of the time-claim size space; see p. 270.
 - (a) Show that the processes $(N(E_{\text{RBNS at time } T}))_{T>0}$, $(N(E_{\text{IBNR at time } T}))_{T>0}$ do not have independent increments.
Hint: It is advantageous to calculate the covariance of increments on disjoint intervals.
 - (b) Prove Lemma 8.2.2(3).
 - (c) Assume the conditions of the basic model (see p. 268) with one exception: the T_i 's constitute a renewal sequence. This means that $T_n = Y_1 + \dots + Y_n$, $n \geq 1$, for iid positive random variables Y_i with finite mean value. Also assume that $ED_1 < \infty$.

Recall the notion of renewal function

$$m(t) = 1 + E\#\{i \geq 1 : T_i \leq t\}, \quad t \geq 0;$$

see Section 2.2.2.

Show that the following relations hold:

$$EN(E_{RBNS}) = E[m((T - D_1)_+) - m((T - D_1 - S_1)_+)] ,$$

$$EN(E_{IBNR}) = E[m(T) - m((T - D_1)_+)] .$$

Section 8.2.3

- (3) Consider the process $(S_{\text{Settled at time } T})_{T>0}$ given in (8.2.10).
 - (a) Prove that the process has independent increments.
 - (b) Prove that S_{Settled} for fixed $T > 0$ has $\text{CP}(\gamma(T), F)$ representation, where the mean value function γ is given in (8.2.8). In particular, conclude that $(S_{\text{Settled at time } T})_{T>0}$ does in general not have stationary increments.
- (4) Assume that $\text{var}(X_1) < \infty$. Calculate the covariance between $S_{\text{Settled RBNS}}$ and $S_{\text{Outstanding RBNS}}$.
Hint: It is advantageous to use the Poisson integral representations (8.2.16) and (8.2.17).
- (5) (a) Modify the calculations leading to Lemma 8.2.3 for $S_{\text{Settled RBNS}}$ such that you prove Lemma 8.2.4 for $S_{\text{Outstanding RBNS}}$.
(b) Repeat the calculations in (a) for the claim amount

$$S_{\text{Settled RBNS}} = \int_{E_{\text{RBNS}}} g(s^{-1}(T - t - r), x) N(dt, dr, ds, dx) ,$$

where $g(\pi, x) \in [0, x]$ is continuous and non-decreasing in the π -component such that $g(1, x) = x$.

Section 8.2.4

- (6) Consider a homogeneous Poisson process with points T_i , $i \in \mathbb{Z}_0$, on \mathbb{R} such that $\dots < T_{-2} < T_{-1} < 0 \leq T_1 < T_2 < \dots$, i.e., $T_i \geq 0$ for $i \geq 1$ and $T_i < 0$ for $i \leq -1$. We mark the points T_i with positive random variables D_i such that the iid sequence (D_i) is independent of (T_i) .
 - (a) Show that for any $T \geq 0$, $\#\{i \in \mathbb{Z}_0 : T_i \leq T\} = \infty$ a.s.
 - (b) Show that

$$(-T_i)_{i \leq -1} \stackrel{d}{=} (T_i)_{i \geq 1} .$$

- (c) Show strict stationarity of the process

$$M(T) = \#\{i \in \mathbb{Z}_0 : T_i \leq T < T_i + D_i\} , \quad T \geq 0 ,$$

in the sense that the finite-dimensional distributions of $(M(T+h))_{T \geq 0}$ do not depend on $h \geq 0$.

Hint: Show that the PRMs with points (T_i, D_i) and $(T_i + h, D_i)$ have the same distribution for any $h \in \mathbb{R}$.

- (d) Calculate the covariance function γ_M of the strictly stationary process M given in (8.2.21) for (i) D_1 with an exponential $\text{Exp}(a)$, $a > 0$, distribution and (ii) a Pareto distribution with parameterization $\bar{F}_D(x) = x^{-\alpha}$, $x \geq 1$, for some $\alpha > 1$. Explain why the assumption $\alpha > 1$ is relevant.

- (7) Let N be a homogeneous Poisson process on \mathbb{R} with intensity $\lambda > 0$. For any $s > 0$, define the process

$$O(T) = \int_{-\infty}^T e^{-s(T-t)} N(dt), \quad T \in \mathbb{R}.$$

This process is an analog of the classical *Ornstein-Uhlenbeck process* where N is replaced by Brownian motion. The Ornstein-Uhlenbeck process is one of the most popular Gaussian processes. It has a multitude of applications in areas as diverse as physics and finance. For example, in finance a ramification of the Ornstein-Uhlenbeck process, known as the Vasicek model, is used as a model for interest rates; see Björk [20].

We refer to the process O as the *Ornstein-Uhlenbeck Poisson process*.

- (a) Show that the Poisson integral $O(T)$ exists and is finite a.s. for every T .
 - (b) Calculate the mean value function $E[O(T)]$, $T \in \mathbb{R}$, and the covariance function $C_O(T, T + h) = \text{cov}(O(T), O(T + h))$, $T, h \in \mathbb{R}$.
 - (c) Show that O is a strictly stationary process on \mathbb{R} .
- Hint: It is convenient to use a Laplace-Stieltjes transform argument for the finite-dimensional distributions of the processes $(O(T))_{T \in \mathbb{R}}$ and $(O(T + h))_{T \in \mathbb{R}}$.
- (d) Consider a discrete-time version of O given by $(O(n))_{n \in \mathbb{Z}}$. Conclude that this is a strictly stationary process, i.e., $(O(n))_{n \in \mathbb{Z}} \stackrel{d}{=} (O(n+k))_{n \in \mathbb{Z}}$ for any integer k . Calculate the mean value and covariance functions of this process.
 - (e) Prove that the discrete-time process $(O(n))_{n \in \mathbb{Z}}$ considered in (d) satisfies the difference equation

$$O(n) = e^{-s} O(n-1) + Z_n, \quad n \in \mathbb{Z}, \quad (8.2.23)$$

where (Z_n) is an iid sequence. Determine the distribution of Z_n .

A discrete-time real-valued process with index set \mathbb{Z} is often called a time series. For a general iid sequence (Z_n) , a solution to the difference equation (8.2.23) defines an *autoregressive process of order 1* or AR(1) process. The AR(1) processes are natural discrete-time analogs of an Ornstein-Uhlenbeck process. The AR(1) process is a prominent member of the class of stationary ARMA processes. The latter class consists of those time series models which are used most often in applications. We refer to the books by Brockwell and Davis [24, 25] for introductions to time series analysis and ARMA models.

Ornstein-Uhlenbeck processes can also be defined for classes of driving processes N much wider than Brownian motion or the Poisson process. For example, it can be defined for certain classes of Lévy processes, i.e., processes with independent stationary increments (see Section 10 for their definition and properties), or even for fractional Brownian motion (see Exercise 9 below), i.e., Gaussian Markov processes with stationary increments; see for example Mikosch and Norvaiša [109] for the definition and properties of such Ornstein-Uhlenbeck processes and Samorodnitsky and Taqqu [131], Chapter 7, for further reading on fractional Brownian motion.

- (8) Let N be a homogeneous Poisson process on \mathbb{R} with intensity $\lambda > 0$. Define the stochastic process

$$\eta(t) = \int_{-\infty}^t f(t-s) N(ds), \quad t \in \mathbb{R},$$

for a non-negative measurable function f on \mathbb{R} .

- (a) Give a condition on f guaranteeing that $\eta(t) < \infty$ a.s. for every $t \in \mathbb{R}$.
- (b) Assume that $\eta(t) < \infty$ a.s. for every $t \in \mathbb{R}$; see (a). Show that η is a strictly stationary process.
- (c) Show that there exist processes $\eta^{(i)}(t) = \int_{-\infty}^t f_i(t-s) N(ds)$, $i = 1, 2$, with $f_1 \neq f_2$ a.e. with respect to Lebesgue measure but such that the autocovariance functions

$$\gamma_{\eta^{(i)}}(h) = \text{cov}(\eta^{(i)}(s), \eta^{(i)}(t)), \quad s, t \in \mathbb{R}, \quad i = 1, 2,$$

of the processes $\eta^{(i)}$, $i = 1, 2$, coincide.

- (9) *Fractional Brownian motion* $(B_t^{(H)})_{t \geq 0}$ is a mean zero Gaussian process given by its covariance function

$$C_B(t, s) = \text{cov}(B_t^{(H)}, B_s^{(H)}) = \frac{\sigma^2}{2} \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right), \quad s, t \geq 0,$$

for $\sigma > 0$ and $H \in (0, 1]$. The increment process

$$\xi^{(H)}(n) = B_n^{(H)} - B_{n-1}^{(H)}, \quad n = 1, 2, \dots,$$

is called *fractional Gaussian noise*.

- (a) Show that $B^{(H)}$ has stationary increments and that $B_0^{(H)} = 0$ a.s.
- (b) Show that $(\xi^{(H)}(n))_{n=1,2,\dots}$ constitutes a strictly stationary process.
- (c) Calculate the autocovariance function

$$\gamma_{\xi^{(H)}}(h) = \text{cov}(\xi^{(H)}(1), \xi^{(H)}(1+h)), \quad h \geq 0,$$

and show that fractional Gaussian noise for $H \in (0, 0.5) \cup (0.5, 1)$ satisfies the relation

$$\gamma_{\xi^{(H)}}(h) \sim c h^{2(H-1)} \quad \text{as } h \rightarrow \infty \text{ for some constant } c > 0.$$

Conclude that $(\xi^{(H)}(n))$ for $H \in (0.5, 1)$ exhibits long range dependence in the sense that

$$\sum_{h=1}^{\infty} |\gamma_{\xi^{(H)}}(h)| = \infty.$$

- (d) Verify that $B^{(H)}$, $0 < H < 1$ is a *self-similar process* in the sense that for any $c > 0$ (here $\stackrel{d}{=}$ refers to equality of the finite-dimensional distributions),

$$c^H (B_t^{(H)})_{t \geq 0} \stackrel{d}{=} (B_{ct}^{(H)})_{t \geq 0}.$$

Another self-similar process — symmetric α -stable Lévy motion — is discussed in Example 10.5.2 on p. 358.

- (e) Show that the case $H = 0.5$ corresponds to Brownian motion, i.e., $B^{(0.5)}$ is mean zero Gaussian with independent stationary increments.
- (f) Show that the distribution of any continuous-time mean zero Gaussian process $(\eta_t)_{t \geq 0}$ with stationary increments is determined by its variance function $\sigma_{\eta}^2(t) = \text{var}(\eta_t)$, $t \geq 0$.

Weak Convergence of Point Processes

One of the fundamental results of probability theory is Poisson's limit theorem. It tells us that a sequence of binomial $\text{Bin}(p_n, n)$ distributed random variables B_n , $n = 1, 2, \dots$, converges in distribution to a Poisson random variable Y with parameter $\lambda > 0$ if and only if $n p_n = EB_n \rightarrow EY = \lambda$. Here one deals with a rare event approximation because p_n , the success probability of the binomial random variable B_n , necessarily converges to zero. It is a remarkable result insofar that the distributional convergence relation $B_n \xrightarrow{d} Y$ is equivalent to the convergence of the expectations $EB_n \rightarrow EY$. We will see later that this property remains valid in a sense for the convergence in distribution of "binomial point processes" towards a Poisson random measure (PRM): convergence of the underlying mean measures of the "binomial point processes" implies their convergence in distribution towards a PRM with the limiting mean measure.

Any binomial $\text{Bin}(n, p)$ random variable B_n can be interpreted as the counting number of the successes in n independent trials with success probability $p \in (0, 1)$: if (X_i) is an iid sequence which describes an experiment with the success event A with probability p , then the number of successes

$$B_n = \sum_{i=1}^n I_A(X_i), \quad n \geq 1, \tag{9.0.1}$$

is binomially distributed with parameter (n, p) . This representation bears some resemblance to a point process. Indeed, we can introduce point processes

$$N_n = \sum_{i=1}^n \varepsilon_{X_i}, \quad n = 1, 2, \dots,$$

generated from the points X_1, X_2, \dots , on a suitable state space E . Then $B_n = N_n(A)$, and we can vary the event A over some σ -field \mathcal{E} . Since we know the notion of a PRM as a special point process it is reasonable to think

about distributional convergence of the “binomial processes” N_n to a PRM. Of course, since $EN_n(A) = n P(X_1 \in A) \rightarrow \infty$ or $= 0$ according as $P(X_1 \in A) > 0$ or $= 0$ we need to ensure that A depends on n and then $P(X_1 \in A_n) \rightarrow 0$ at a certain rate; we will often achieve this goal by considering the point process of the normalized and centered points $c_n^{-1}(X_i - d_n)$ for suitable constants $c_n > 0$ and $d_n \in \mathbb{R}$.

In what follows, we will deal with two major problems. *First*, point processes are random measures: each such random measure can be understood as a collection of random variables indexed by the sets of an appropriate σ -field which typically contains infinitely many elements. We have to clarify the meaning of convergence in distribution for these infinite-dimensional objects; see Section 9.1.1. *Second*, we need to explain the meaning of convergence of (possibly infinite) mean measures. Weak convergence of probability measures will be a guide. It will be one of the topics in Section 9.1.2, where we introduce the concept of *vague convergence* of measures. There we will also see that the convergence in distribution of point processes is equivalent to the pointwise convergence of the underlying Laplace functionals.

In Section 9.2 we apply the results of Section 9.1 in the context of extreme value theory. Extremes are important for applications in non-life insurance, in particular, in reinsurance. We study the point processes of exceedances and their weak convergence to a PRM. This convergence is equivalent to the convergence of maxima and upper order statistics for iid sequences. In Section 9.2.2 we characterize the possible limit distributions for maxima of iid random variables. The celebrated Fisher-Tippett Theorem (Theorem 9.2.7) summarizes the asymptotic theory for maxima. In Section 9.2.3 we consider maximum domains of attraction, i.e., we find conditions on a distribution F such that the normalized and centered partial maxima of an iid sequence with distribution F converge to a non-degenerate limit. In Section 9.2.4 we modify the point process of exceedances insofar that we assume that we consider a random sample indexed by a renewal process. The weak convergence of these processes is applied to reinsurance treaties of extreme value type as introduced in Section 3.4. In Section 9.3 we return to these treaties and study their limiting behavior.

9.1 Definition and Basic Examples

9.1.1 Convergence of the Finite-Dimensional Distributions

First of all we have to clarify:

What is the meaning of weak convergence of point processes?

This question cannot be answered at a completely elementary level. Consider point processes N, N_1, N_2, \dots on the same state space $E \subset \mathbb{R}^d$. We know from Section 7.1.2 that the distribution of these point processes in $M_p(E)$, the space

of all point measures on E , is determined by their finite-dimensional distributions. Thus a natural requirement for weak convergence¹ of (N_n) towards N would be that, for any choice of “good” Borel sets $A_1, \dots, A_m \in \mathcal{E} = \mathcal{B}(E)$ and for any integer $m \geq 1$,

$$(N_n(A_1), \dots, N_n(A_m)) \xrightarrow{d} (N(A_1), \dots, N(A_m)). \quad (9.1.2)$$

Every point process N_n can be considered as a stochastic process $(N_n(A))_{A \in \mathcal{E}}$ indexed by the sets $A \in \mathcal{E}$. From the theory of weak convergence of probability measures (see for example Billingsley [17]) we know that the convergence of the finite-dimensional distributions of stochastic processes is in general not sufficient for the weak convergence of their distributions: one needs an extra *tightness condition* meaning that the mass of the converging probability distributions should not disappear from compact sets.

Perhaps unexpectedly, point processes are user-friendly in the sense that tightness follows from the convergence of their finite-dimensional distributions; see for instance Daley and Vere-Jones [38], Theorem 9.1.VI. Hence we obtain quite an intuitive notion of weak convergence which is equivalent to the usual definition of weak convergence.

Definition 9.1.1 (Weak convergence of point processes)

Let N, N_1, N_2, \dots , be point processes on the state space $E \subset \mathbb{R}^d$ equipped with the σ -field \mathcal{E} of its Borel sets. We say that the sequence of point processes (N_n) converges weakly to the point process N in $M_p(E)$ (we write $N_n \xrightarrow{d} N$) if (9.1.2) is satisfied for all possible choices of bounded sets $A_i \in \mathcal{E}$ satisfying $P(N(\partial A_i) = 0) = 1$, $i = 1, \dots, m$, $m \geq 1$, where ∂A denotes the boundary of A in E .

Assume for the moment that the state space E is an interval $(a, b] \subset \mathbb{R}$. Convergence of the finite-dimensional distributions can sometimes be checked by surprisingly simple means as the following result shows. Recall the notion of a *simple point process* from p. 218, i.e., the values of this process are simple point measures.

Theorem 9.1.2 (Kallenberg’s Theorem for weak convergence to a simple point process on an interval; see Kallenberg [79]; cf. Resnick [122], Proposition 3.22) Let N, N_1, N_2, \dots , be point processes on $E = (a, b] \subset \mathbb{R}$ and assume that N is simple. Suppose the following two conditions hold:

$$EN_n(c, d] \rightarrow EN(c, d], \quad a < c < d \leq b, \quad (9.1.3)$$

$$P(N_n(B) = 0) \rightarrow P(N(B) = 0), \quad (9.1.4)$$

¹ Instead of *weak convergence of point processes* it would be appropriate to refer to *weak convergence of the distributions of the point processes* or to *convergence in distribution of the point processes*; see Billingsley [17]. However, our usage is not uncommon in the literature; see for example Daley and Vere-Jones [38] and Resnick [122].

for any union $B = \cup_{i=1}^k (c_i, d_i]$ of disjoint intervals $(c_i, d_i] \subset (a, b]$ with $\min_{i=1, \dots, k} c_i > a$, $k \geq 1$. Then $N_n \xrightarrow{d} N$ in $M_p(E)$.

We apply Kallenberg's result to the weak convergence of Poisson processes.

Example 9.1.3 (Poisson processes converging to a simple Poisson process)
 We consider a sequence of Poisson processes N_n on $E = (a, b]$ with corresponding mean measures μ_n such that $N_n \xrightarrow{d} N$ in $M_p(E)$ for a simple Poisson process N with mean measure μ . Recall from Exercise 1 on p. 242 that a Poisson process N on $(a, b]$ is simple if and only if

$$\mu(\{x\}) = 0 \quad \text{for all } x \in (a, b]. \quad (9.1.5)$$

In particular, this condition is satisfied if μ has an intensity function $\lambda(x)$ on $(a, b]$.

Condition (9.1.3) of Kallenberg's Theorem then turns into

$$\mu_n(c, d] \rightarrow \mu(c, d] \quad \text{for } a < c < d \leq b. \quad (9.1.6)$$

Since the increments $N_n(c_i, d_i]$ for disjoint sets $(c_i, d_i]$ are independent condition (9.1.4) reads as

$$\begin{aligned} P(N_n(c, d] = 0) &= e^{-\mu_n(c, d]} \\ \rightarrow P(N(c, d] = 0) &= e^{-\mu(c, d]} \quad \text{for } a < c < d \leq b, \end{aligned} \quad (9.1.7)$$

which is equivalent to (9.1.6). Thus condition (9.1.6) is sufficient for $N_n \xrightarrow{d} N$.

Now assume that $N_n \xrightarrow{d} N$ in $M_p(E)$ for a simple PRM N . By Definition 9.1.1 we conclude that $N_n(c, d] \xrightarrow{d} N(c, d]$ for all $a < c < d \leq b$ such that $\mu(\partial(c, d]) = \mu(\{c, d\}) = 0$. The latter condition is satisfied in view of (9.1.5). Therefore (9.1.7) holds, implying (9.1.6).

We conclude that for PRMs N, N_1, N_2, \dots on $(a, b]$ with mean measure μ of the limit process N satisfying (9.1.5), the relations $N_n \xrightarrow{d} N$ and (9.1.6) are equivalent. For homogeneous Poisson processes N_n and N with corresponding intensities λ_n and λ relation (9.1.6) turns into $\lambda_n \rightarrow \lambda$. \square

9.1.2 Convergence of Laplace Functionals

The Laplace functional (see p. 222) plays a similar role for point processes as characteristic functions for random variables. In particular, the continuity theorem of characteristic functions ensures that the distributional convergence of a sequence of random variables is equivalent to the pointwise convergence of the corresponding characteristic functions. The weak convergence of a sequence of point processes is equivalent to the convergence of their Laplace functionals indexed by a suitable family of bounded non-negative functions.

Such a set of functions is

$$\mathbb{C}_K^+(E) = \{g : g \text{ is a continuous, non-negative function on } E \text{ with compact support}\}.$$

Recall that a real-valued function g has compact support if there exists a compact set $K \subset E$ such that $g(x) = 0$ on K^c , the complement of K relative to E . We have already stressed on p. 223 that the distribution of a point process N on E is determined by its Laplace functional Ψ_N restricted to the functions $g \in \mathbb{C}_K^+(E)$. We can also test for weak convergence by checking Laplace functionals converge along $\mathbb{C}_K^+(E)$.

Theorem 9.1.4 (Criterion for weak convergence of point processes via convergence of Laplace functionals; see Daley and Vere-Jones [38], Proposition 9.1.VII, Resnick [122], Proposition 3.19.)

The sequence of point processes (N_n) converges weakly to the point process N in $M_p(E)$ if and only if the corresponding Laplace functionals converge for every $g \in \mathbb{C}_K^+(E)$:

$$\begin{aligned} \Psi_{N_n}(g) &= E \exp \left\{ - \int_E g dN_n \right\} \\ \rightarrow \quad \Psi_N(g) &= E \exp \left\{ - \int_E g dN \right\}, \quad n \rightarrow \infty. \end{aligned} \quad (9.1.8)$$

Observe that $g \in \mathbb{C}_K^+(E)$ if and only if $zg \in \mathbb{C}_K^+(E)$ for every $z > 0$. Hence (9.1.8) for any $g \in \mathbb{C}_K^+(E)$ is equivalent to the convergence of the Laplace-Stieltjes transforms

$$E \exp \left\{ -z \int_E g dN_n \right\} \rightarrow E \exp \left\{ -z \int_E g dN \right\}, \quad z \geq 0.$$

In turn, this relation is equivalent to $\int_E g dN_n \xrightarrow{d} \int_E g dN$; see Feller [51], XIII 1, Theorem 2a.

Since we know the Laplace functional of a PRM it is straightforward to apply Theorem 9.1.4 to the weak convergence of PRMs towards a PRM. Also notice that we studied a special case of this convergence in Example 9.1.3, where we required the limiting process to be simple. This restriction is not needed below.

Example 9.1.5 (Convergence of PRMs towards a PRM)

From (7.2.11) recall the Laplace functional of a PRM(μ), denoted by N , on $E \subset \mathbb{R}^d$:

$$\Psi_N(g) = \exp \left\{ - \int_E \left(1 - e^{-g(x)} \right) \mu(dx) \right\}, \quad g \in \mathbb{C}_K^+(E).$$

We consider a sequence of PRMs N_n on E with corresponding Radon mean measures μ_n . According to Theorem 9.1.4, the relation $N_n \xrightarrow{d} N$ is equivalent to (9.1.8) for every $g \in \mathbb{C}_K^+(E)$:

$$\begin{aligned}\Psi_{N_n}(g) &= \exp \left\{ - \int_E \left(1 - e^{-g(x)} \right) \mu_n(dx) \right\} \\ \rightarrow \quad \Psi_N(g) &= \exp \left\{ - \int_E \left(1 - e^{-g(x)} \right) \mu(dx) \right\}.\end{aligned}\quad (9.1.9)$$

In order to understand the latter relation consider a compact set $K \subset E$ with $\mu(\partial K) = 0$. Then a *formal* application of (9.1.9) to the indicator function $g = I_K$ yields

$$\begin{aligned}\mu_n(K) &= -(1 - e^{-1})^{-1} \log(\Psi_{N_n}(g)) \\ \rightarrow \quad \mu(K) &= -(1 - e^{-1})^{-1} \log(\Psi_N(g)).\end{aligned}$$

Unfortunately, the function I_K is not continuous, hence not in $\mathbb{C}_K^+(E)$. However, the condition $\mu(\partial K) = 0$ ensures that I_K is continuous with the exception of a set of μ -measure zero, namely the boundary of K . In what follows, we show that the convergence of the Laplace functionals (9.1.9) for $g \in \mathbb{C}_K^+(E)$ implies the relation

$$\mu_n(K) \rightarrow \mu(K) \quad \text{for any compact set } K \subset E \text{ satisfying } \mu(\partial K) = 0. \quad (9.1.10)$$

Before we continue with the proof we mention that the relation $N_n \xrightarrow{d} N$ immediately yields (9.1.10). Indeed, by the definition of $N_n \xrightarrow{d} N$ we have $N_n(A) \xrightarrow{d} N(A)$ for any bounded Borel set $A \subset E$ with the property

$$P(N(\partial A) = 0) = e^{-\mu(\partial A)} = 1, \quad \text{i.e.,} \quad \mu(\partial A) = 0,$$

and then in particular,

$$P(N_n(A) = 0) = e^{-\mu_n(A)} \rightarrow P(N(A) = 0) = e^{-\mu(A)}.$$

This implies (9.1.10).

A standard argument shows that one can approximate $g = I_K$ arbitrarily closely and uniformly by suitable functions $g_\delta \in \mathbb{C}_K^+(E)$ with support $K_\delta \downarrow K$ as $\delta \downarrow 0$ and such that $g \leq g_\delta \leq I_{K_\delta} \downarrow g$ and $g_\delta = g$ on K . Possible choices are

$$g_\delta(x) = 1 - \min(1, \rho(x, K)/\delta) \quad \text{and} \quad K_\delta = \{x \in E : \rho(x, K) \leq \delta\}, \quad (9.1.11)$$

where for any set A , $\rho(x, A) = \inf\{y \in A : |x - y|\}$. By construction of g_δ ,

$$g_\delta(x) = \begin{cases} 0 & \text{for } x \notin K_\delta, \\ g(x) & \text{for } x \in K. \end{cases}$$

Using this property, we obtain

$$\begin{aligned}
& |\log(\Psi_{N_n}(g)) - \log(\Psi_N(g))| \\
& \leq |\log(\Psi_{N_n}(g)) - \log(\Psi_{N_n}(g_\delta))| + |\log(\Psi_{N_n}(g_\delta)) - \log(\Psi_N(g_\delta))| \\
& \quad + |\log(\Psi_N(g_\delta)) - \log(\Psi_N(g))| \\
& = \int_{K_\delta \setminus K} \left(1 - e^{-g_\delta(x)}\right) \mu_n(dx) + |\log(\Psi_{N_n}(g_\delta)) - \log(\Psi_N(g_\delta))| \\
& \quad + \int_{K_\delta \setminus K} \left(1 - e^{-g_\delta(x)}\right) \mu(dx) \\
& \leq \int_{K_\delta \setminus K} \left(1 - e^{-g_\delta(x)}\right) \mu_n(dx) + |\log(\Psi_{N_n}(g_\delta)) - \log(\Psi_N(g_\delta))| + \mu(K_\delta \setminus K).
\end{aligned} \tag{9.1.12}$$

Continuity of the measure μ implies that $\mu(K_\delta \setminus K) \downarrow \mu(\emptyset) = 0$ as $\delta \downarrow 0$. Hence the third expression on the right-hand side can be made arbitrarily small. Theorem 9.1.4 yields $\Psi_{N_n}(g_\delta) \rightarrow \Psi_N(g_\delta)$ as $n \rightarrow \infty$ because $g_\delta \in \mathbb{C}_K^+(E)$. This makes the second expression vanish as $n \rightarrow \infty$. The first expression in (9.1.12) is bounded by $\mu_n(\overline{K_\delta \setminus K})$, where for any set $A \subset E$, \overline{A} denotes its closure relative to E . For any $\epsilon > 0$ define $f_\epsilon(x) = 1 - \min(1, \rho(x, \overline{K_\delta \setminus K})/\epsilon)$. This function is in $\mathbb{C}_K^+(E)$ and has compact support $(\overline{K_\delta \setminus K})_\epsilon$. Similar arguments as above show that

$$\begin{aligned}
(1 - e^{-1}) \limsup_{n \rightarrow \infty} \mu_n(\overline{K_\delta \setminus K}) & \leq \lim_{n \rightarrow \infty} \int_E \left(1 - e^{-f_\epsilon(x)}\right) \mu_n(dx) \\
& = \int_E \left(1 - e^{-f_\epsilon(x)}\right) \mu(dx) \\
& \leq \mu((\overline{K_\delta \setminus K})_\epsilon).
\end{aligned}$$

Since $(\overline{K_\delta \setminus K})_\epsilon \downarrow \overline{K_\delta \setminus K}$ as $\epsilon \downarrow 0$ and $\overline{K_\delta \setminus K} \downarrow \partial K$ as $\delta \downarrow 0$ we conclude that

$$(1 - e^{-1}) \limsup_{n \rightarrow \infty} \mu_n(\overline{K_\delta \setminus K}) \leq \mu(\partial K) = 0.$$

Now collecting the arguments above, we conclude that (9.1.10) holds.

On the other hand, any $g \in \mathbb{C}_K^+(E)$ can be uniformly approximated by linear combinations of indicator functions I_K for suitable compact sets $K \subset E$ with $\mu(\partial K) = 0$. Using this uniform approximation and the relation (9.1.10) (which is equivalent to $\Psi_{N_n}(I_K) \rightarrow \Psi_N(I_K)$ for the sets K considered) one can show that (9.1.10) holds for any $g \in \mathbb{C}_K^+(E)$.

We conclude that for PRMs N, N_1, N_2, \dots the weak convergence result $N_n \xrightarrow{d} N$ is equivalent to the convergence of the mean measures in the sense of (9.1.10). \square

In the previous example the weak convergence of PRMs towards a PRM is completely described by the convergence of their mean measures. We formalize this kind of convergence.

Definition 9.1.6 (Vague convergence of measures)

Let μ and μ_n , $n = 1, 2, \dots$, be Radon measures on (E, \mathcal{E}) . We say that the sequence (μ_n) converges vaguely to μ if $\mu_n(A) \rightarrow \mu(A)$ for every relatively compact set $A \subset E$ such that $\mu(\partial A) = 0$. We write $\mu_n \xrightarrow{v} \mu$.

Vague convergence is similar to weak convergence of probability measures. Actually, weak and vague convergence coincide if μ_n and μ are probability measures. Another property which carries over from weak to vague convergence is the fact that it suffices to verify $\mu_n(A) \rightarrow \mu(A)$ for sets A which generate this kind of convergence. In particular, if the state space E is an interval of \mathbb{R} it suffices to verify the condition $\mu_n(c, d] \rightarrow \mu(c, d]$ for all $(c, d] \subset E$ with $\mu(\{c, d\}) = 0$. Thus relation (9.1.6), which turned up in Example 9.1.3 as a necessary and sufficient condition for the weak convergence of Poisson processes towards a simple Poisson process, is equivalent to the vague convergence of the underlying mean measures.

Example 9.1.7 (Continuation of Example 9.1.5)

Assume that N_n is a $\text{PRM}(\mu_n)$ for a Radon measure μ_n on E for every $n = 1, 2, \dots$. It is an immediate consequence of the definition of the weak convergence relation $N_n \xrightarrow{d} N$ and the PRM property of the N_n 's that the random variables $N(A_1), \dots, N(A_m)$ are mutually independent and Poisson distributed for any choice of disjoint bounded sets $A_i \in \mathcal{E}$ satisfying $P(N(\partial A_i) = 0) = 1$, $i = 1, \dots, m$; see Exercise 1 on p. 299. Moreover, the corresponding Poisson parameters $\mu(A_i)$ of $N(A_i)$, $i = 1, \dots, m$, satisfy the additivity property:

$$\mu \left(\bigcup_{i=1}^m A_i \right) = \sum_{i=1}^m \mu(A_i).$$

Therefore a natural question arises:

Do the relation $N_n \xrightarrow{d} N$ and the PRM property of the point processes N_n imply that N is $\text{PRM}(\mu)$ for some mean measure μ ?

In other words, one needs to show that there exists a Radon measure μ on E that coincides with the additive measure μ on bounded sets A with the property $P(N(\partial A) = 0) = 1$. This fact is indeed true as a more advanced argument shows. Since $\mu_n(A) \rightarrow \mu(A)$ for all relatively compact sets $A \subset E$ with $\mu(\partial A) = 0$, Proposition 3.16 in Resnick [122] implies that the set of measures $\{\mu_n, n = 1, 2, \dots\}$ is vaguely relatively compact. This means that for any subsequence (n_k) of the positive integers there exists a subsequence (m_ℓ) (say) such that $\mu_{m_\ell} \xrightarrow{v} \tilde{\mu}$ for a Radon measure $\tilde{\mu}$ on E , possibly depending on the subsequence. However, for all such measures $\tilde{\mu}$, $\tilde{\mu}(K) = \mu(K)$ for any compact set K with $\mu(\partial K) = 0$ and therefore $\tilde{\mu} = \mu$. \square

Comments

The distribution of a point process is defined on the σ -field $\mathcal{M}_p(E)$ of subsets of $M_p(E)$. Therefore convergence in distribution of point processes is naturally defined as weak convergence of the distributions of the point processes on $\mathcal{M}_p(E)$. In order to avoid the infinite-dimensional setting and to keep the understanding of weak convergence at an intuitive level, we have chosen to define weak convergence via convergence of the finite-dimensional distributions. For our choice of state spaces, this definition is equivalent to the general notion of weak convergence of point processes; see Daley and Vere-Jones [38], Theorem 9.1.VI.

The weak convergence of point processes is treated in standard texts such as Kallenberg [79], Resnick [122, 124], Daley and Vere-Jones [38]. It plays a major role in the asymptotic theory of extremes for independent and strictly stationary sequences; see Leadbetter et al. [92], Resnick [122, 124], Embrechts et al. [46]. Moreover, the weak convergence of “binomial point processes” with iid points towards a PRM gives some justification for the use of general PRMs.

The importance of the concept of vague convergence for point measures was stressed in Kallenberg [79] and Resnick [121, 122] rather early on. It is one of the major tools for dealing with the weak convergence of point processes.

Exercises

Section 9.1.1

- (1) Let (N_n) be Poisson random measures on the state space $E \subset \mathbb{R}^d$ such that N_n has mean measure μ_n . Assume that $N_n \xrightarrow{d} N$ in the sense of Definition 9.1.1 and that N is a non-degenerate point process.
 - (a) Let $A \subset E$ be any bounded Borel set such that $P(N(\partial A) = 0) = 1$. Prove that there exists a number $\mu(A)$ such that $\mu_n(A) \rightarrow \mu(A)$ and that $\mu(A) = EN(A)$.
 - (b) Let $A_1, \dots, A_m \in \mathcal{E}$ be disjoint bounded Borel sets such that $P(N(\partial A_i) = 0) = 1$, $i = 1, \dots, m$. Show that the random vector $(N(A_1), \dots, N(A_m))$ has independent components and that $N(A_i)$ is $\text{Pois}(\mu(A_i))$ distributed, $i = 1, \dots, m$. Moreover, show that $\mu(\cup_{i=1}^m A_i) = \sum_{i=1}^m \mu(A_i)$.

Section 9.1.2

- (2) Consider Example 9.1.5.
 - (a) Show that for any compact set $K \subset E$, $I_K \leq g_\delta \leq I_{K_\delta} \downarrow I_K$ as $\delta \downarrow 0$ for g_δ as defined in (9.1.11).
 - (b) Show that any function $g \in \mathbb{C}_K^+(E)$ can be uniformly approximated by linear combinations of indicator functions I_K for suitable compact sets $K \subset E$ with $\mu(\partial K) = 0$.
- (3) Let (μ_n) be a sequence of probability measures on (E, \mathcal{E}) .
 - (a) Let μ be a probability measure on (E, \mathcal{E}) . Show that weak and vague convergence of (μ_n) towards μ are equivalent notions.

- (b) Give an example of a measure μ such that $\mu_n \xrightarrow{v} \mu$ but (μ_n) does not converge weakly.
- (4) Let X be a positive regularly varying random variable with distributional tail $\bar{F}(x) = x^{-\alpha} L(x)$, $x > 0$, where L is slowly varying and $\alpha > 0$. Choose any sequence $c_n \rightarrow \infty$ such that $n \bar{F}(c_n) \rightarrow 1$. Define the measures μ_n on $E = (0, \infty)$ by

$$\mu_n(x, \infty) = n \bar{F}(x c_n), \quad x > 0.$$

- Prove that $\mu_n \xrightarrow{v} \mu$ on E , where μ is given by $\mu(x, \infty) = x^{-\alpha}$, $x > 0$.
- (5) Let μ_n be the discrete uniform distribution on the set $\{n^{-1}, 2n^{-1}, \dots, 1\} \subset (0, 1]$. Show that $\mu_n \xrightarrow{v} \mu$ on E , where μ is the uniform $U(0, 1)$ distribution.

9.2 Point Processes of Exceedances and Extremes

9.2.1 Convergence of the Point Processes of Exceedances

In this section we consider one of the basic point processes which has major applications in extreme value theory: *the point process of exceedances*. We recall its definition from Example 7.1.6: for a sequence (X_i) of iid random variables with common distribution F and a sequence of thresholds $u_n \in \mathbb{R}$ the corresponding sequence of point processes of exceedances is given by

$$N_n = \sum_{i=1}^n \varepsilon_{n^{-1}i} I_{\{X_i > u_n\}}, \quad n = 1, 2, \dots. \quad (9.2.13)$$

See Figure 9.2.2 for an illustration. Even though $E_n = \{n^{-1}, 2n^{-1}, \dots, 1\}$ is the support set of the measure N_n , it is more convenient, especially for asymptotics, to consider all N_n 's as measures on the same state space $E = (0, 1]$. As we learned in Example 7.1.6, the choice of these point processes is motivated by their relation with the maxima and order statistics of the samples X_1, \dots, X_n , $n = 1, 2, \dots$.

Our objective is to find conditions which ensure the weak convergence of the sequence of the point processes of exceedances (N_n) towards a homogeneous Poisson process N on $(0, 1]$. Of course, these conditions must be in terms of the threshold $u = u_n$ combined with the distribution F of the X_i 's.

Proposition 9.2.1 (Weak convergence of the point processes of exceedances) *Suppose that (X_n) is a sequence of iid random variables with common distribution F and, for a given sequence (u_n) of real numbers, (N_n) is the corresponding sequence of point processes given in (9.2.13). For any $\tau \in (0, \infty)$, the following statements are equivalent:*

- (1) *The limit $n \bar{F}(u_n) \rightarrow \tau$ exists.*
- (2) *The relation $N_n \xrightarrow{d} N$ holds in $M_p(E)$, where N is a homogeneous Poisson process on $E = (0, 1]$ with intensity τ .*

(3) For the sequence of partial maxima

$$M_n = \max_{i=1,\dots,n} X_i, \quad n = 1, 2, \dots,$$

the limit relation $P(M_n \leq u_n) \rightarrow e^{-\tau}$ holds.

(4) Consider the ordered sample²

$$X_{(1)} \leq \dots \leq X_{(n)} \quad (9.2.14)$$

of the iid sample X_1, \dots, X_n . For the sequence $(X_{(n-k+1)})$ of the k th largest order statistics and any fixed integer $k \geq 1$ the following limit relation holds:

$$\lim_{n \rightarrow \infty} P(X_{(n-k+1)} \leq u_n) = e^{-\tau} \sum_{i=0}^{k-1} \frac{\tau^i}{i!}. \quad (9.2.15)$$

Proof. (2) \Leftrightarrow (1) First assume $N_n \xrightarrow{d} N$. In particular,

$$B_n = N_n(0, 1] \xrightarrow{d} N(0, 1] = Y.$$

This relation means that the sequence of the binomial $\text{Bin}(n, \bar{F}(u_n))$ random variables B_n converges in distribution to the $\text{Pois}(\tau)$ distributed random variable Y . By Poisson's limit theorem (or by a simple check of the converging characteristic functions or Laplace-Stieltjes transforms) the condition $n \bar{F}(u_n) \rightarrow \tau$ follows.

Now assume that $n \bar{F}(u_n) \rightarrow \tau$ holds. Since the limiting process N is assumed to be a homogeneous Poisson process, in particular simple, we may apply Kallenberg's Theorem 9.1.2.

For $(a, b] \subset (0, 1]$ the random variable

$$N_n(a, b] = \sum_{i=1}^n \varepsilon_{n-1} i(a, b] I_{\{X_i > u_n\}} = \sum_{i:a < n^{-1} i \leq b} I_{\{X_i > u_n\}}$$

is $\text{Bin}(n(a, b], \bar{F}(u_n))$ distributed, where

$$n(a, b] = \#\{1 \leq i \leq n : na < i \leq nb\}.$$

Then, by the condition $n \bar{F}(u_n) \rightarrow \tau$,

$$EN_n(a, b] = n(a, b] \bar{F}(u_n) \sim (n(b-a)) \bar{F}(u_n) \sim (b-a)\tau = EN(a, b],$$

² The notation (9.2.14) for the ordered sample of X_1, \dots, X_n is slightly imprecise. It would be more consequent to indicate the dependence of the order statistics on the sample size n , for example by using double indices. For the ease of presentation we avoid this approach, assuming implicitly that the order statistics have to be recalculated for every n .

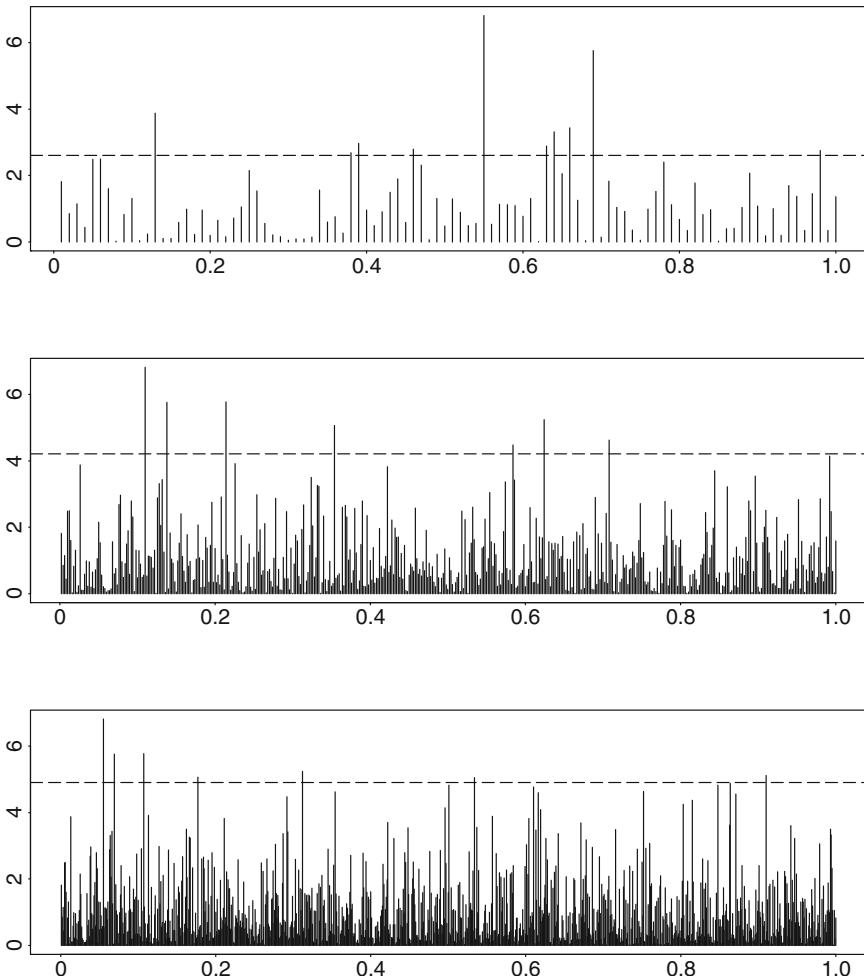


Figure 9.2.2 Visualization of the Poisson approximation for the exceedances of iid standard exponential random variables. The threshold increases with the sample size $n = 100$ (top), $n = 500$ (middle) and $n = 1000$ (bottom). Notice that the first sample also appears at the beginning of the second and the second at the beginning of the third.

which proves condition (9.1.3) of Kallenberg's Theorem. Here and in what follows, $f(x) \sim g(x)$ as $x \rightarrow x_0$ for positive f and g means that $f(x)/g(x) \rightarrow 1$ as $x \rightarrow x_0$.

Thus it remains to show the second condition (9.1.4). We start by observing that for any $0 < a < b \leq 1$,

$$\begin{aligned}
P(N_n(a, b] = 0) &= P\left(\max_{na < i \leq nb} X_i \leq u_n\right) \\
&= (F(u_n))^{n(a, b]} \sim (F(u_n))^{n(b-a)} \\
&= e^{(n(b-a)) \log(1 - \bar{F}(u_n))} \\
&\rightarrow e^{-\tau(b-a)}. \tag{9.2.16}
\end{aligned}$$

In the last step we used the Taylor expansion $\log(1 + x) = x(1 + o(1))$ as $x \rightarrow 0$ and the fact that $n\bar{F}(u_n) \rightarrow \tau$ implies $\bar{F}(u_n) \rightarrow 0$.

Now recall the definition of the set B in (9.1.4). The independence of the random variables X_j and the disjointness of the sets $(c_i, d_i] \subset (0, 1]$, $c_i > 0$, which constitute the union B , imply that the counting variables $N_n(c_i, d_i]$ are independent. Therefore

$$\begin{aligned}
P(N_n(B) = 0) &= P(N_n(c_i, d_i] = 0, \quad i = 1, \dots, k) \\
&= \prod_{i=1}^k P(N_n(c_i, d_i] = 0) \\
&\rightarrow \prod_{i=1}^k e^{-\tau(d_i - c_i)}.
\end{aligned}$$

In the last step we made multiple use of (9.2.16). On the other hand, by the Poisson property of N ,

$$P(N(B) = 0) = e^{-\tau|B|} = e^{-\tau \sum_{i=1}^k (d_i - c_i)}.$$

This relation proves $N_n \xrightarrow{d} N$ by virtue of Kallenberg's Theorem 9.1.2.

(2) \Rightarrow (3) Since $N_n \xrightarrow{d} N$ we have $N_n(0, 1] \xrightarrow{d} N(0, 1]$, and $N(0, 1]$ has a $\text{Pois}(\tau)$ distribution. Then it is immediate that

$$P(M_n \leq u_n) = P(N_n(0, 1] = 0) \rightarrow P(N(0, 1] = 0) = e^{-\tau}.$$

(3) \Rightarrow (1) We have

$$P(M_n \leq u_n) = (F(u_n))^n = e^{-n \log(1 - \bar{F}(u_n))} = e^{-n \bar{F}(u_n)(1+o(1))} \rightarrow e^{-\tau},$$

where we again used the properties $\bar{F}(u_n) \rightarrow 0$ and $\log(1 + x) = x(1 + o(1))$ as $x \rightarrow 0$. Now (1) is immediate.

In order to show the equivalence of (4) and one of the conditions (1)–(3) it suffices to consider the case that (4) holds for any fixed $k > 1$. Indeed, the case $k = 1$ coincides with (3).

(4) \Rightarrow (1) First we observe that for any $k \geq 1$,

$$\{X_{(n-k+1)} \leq u_n\} = \{N_n(0, 1] < k\} \quad (9.2.17)$$

and that $N_n(0, 1]$ has a $\text{Bin}(n, \bar{F}(u_n))$ distribution. Therefore for any $k \geq 1$,

$$P(X_{(n-k+1)} \leq u_n) = \sum_{i=0}^{k-1} \binom{n}{i} [\bar{F}(u_n)]^i [F(u_n)]^{n-i}.$$

Since (9.2.15) is assumed for any fixed $k > 1$ the probability $F(u_n)$ necessarily converges to 1 as $n \rightarrow \infty$. Therefore as $n \rightarrow \infty$,

$$\begin{aligned} P(X_{(n-k+1)} \leq u_n) &\sim [F(u_n)]^n \sum_{i=0}^{k-1} \frac{[n \bar{F}(u_n)]^i}{i!} \\ &\sim e^{-n \bar{F}(u_n)} \sum_{i=0}^{k-1} \frac{[n \bar{F}(u_n)]^i}{i!}. \end{aligned} \quad (9.2.18)$$

Here we again made use of a Taylor expansion argument: $n \log(1 - \bar{F}(u_n)) \sim -n \bar{F}(u_n)$. The function

$$f_k(x) = e^{-x} \sum_{i=0}^{k-1} \frac{x^i}{i!}, \quad x > 0,$$

is continuous and decreasing, hence its inverse f_k^{-1} exists and is continuous. By virtue of (9.2.15) and (9.2.18) we have

$$f_k(n \bar{F}(u_n)) \rightarrow f_k(\tau).$$

By invertibility and continuity of f_k this implies that $n \bar{F}(u_n) \rightarrow \tau$, i.e., (1) holds.

(1) \Rightarrow (4) The key relations are again (9.2.17) and (9.2.18). Since $n \bar{F}(u_n) \rightarrow \tau$ the limit relation (9.2.15) is immediate for any fixed $k \geq 1$.

This proves the proposition. \square

A short proof of the equivalence of (1) and (2) in Proposition 9.2.1 can be based on the equivalence of $N_n \xrightarrow{d} N$ and the convergence of the corresponding Laplace functionals $\Psi_{N_n}(f) \rightarrow \Psi_N(f)$. Indeed, for $f \in \mathbb{C}_K^+(0, 1])$ direct calculation shows that

$$\begin{aligned} \Psi_{N_n}(f) &= \prod_{i=1}^n \left(1 - \bar{F}(u_n) (1 - e^{-f(i/n)})\right) \\ &\rightarrow \Psi_N(f) = \exp \left\{ -\tau \int_{(0,1]} \left(1 - e^{-f(x)}\right) dx \right\}, \end{aligned} \quad (9.2.19)$$

if and only if $n \bar{F}(u_n) \rightarrow \tau$ for some $\tau \in (0, \infty)$. The reader is encouraged to verify this relation; see Exercise 2 on p. 322.

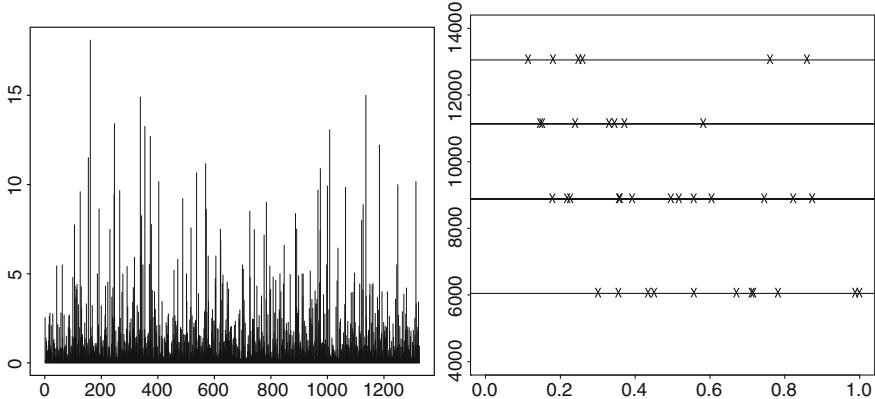


Figure 9.2.3 Left: Insurance claim sizes (in 1 000\$ units) caused by water damage, $n = 1\,762$ observations are available. Right: Visualization of the point processes of exceedances for this data set. For the sample size $[n/4]$ we choose the threshold $u_{[n/4]} = 6\,046$, correspondingly $u_{n/2} = 8\,880$, $u_{[3n/4]} = 11\,131$ and $u_n = 13\,051$. The exceedances times (renormalized to the interval $(0, 1]$) appear uniformly distributed.

The Poisson approximation to the point processes of exceedances is visualized in Figure 9.2.3. We assume that these claim size data are iid. In view of the condition $n \bar{F}(u_n) \rightarrow \tau \in (0, \infty)$ the threshold u_n is raised in such a way that we roughly see the same number of exceedances for each threshold. According to Proposition 9.2.1 we may assume that the point process at the (renormalized) exceedance times i/n of the high threshold u_n is approximated by a homogeneous Poisson process. In agreement with the order statistics property of the homogeneous Poisson process we expect that the exceedance times occur uniformly on $(0, 1)$. This hypothesis is supported by the graph.

9.2.2 Convergence in Distribution of Maxima and Order Statistics Under Affine Transformations

In this section we study the links between the weak convergence of the point processes of the exceedances and the convergence in distribution of centered and normalized maxima and order statistics of iid samples when the sample size increases. Throughout we assume that (X_n) is a sequence of iid random variables with common distribution F . For a given sample X_1, \dots, X_n consider the ordered sample

$$X_{(1)} \leq \dots \leq X_{(n)} = M_n .$$

First we investigate the convergence in distribution of the sequence of maxima (M_n) under affine transformations:

Which conditions on F ensure that there exist constants $c_n > 0$, $d_n \in \mathbb{R}$, $n = 1, 2, \dots$, and a non-constant limit random variable Y such that

$$c_n^{-1}(M_n - d_n) \xrightarrow{d} Y ? \quad (9.2.20)$$

If a non-degenerate limit distribution H in (9.2.20) exists it is called an *extreme value distribution*.

Example 9.2.4 (Three classes of extreme value distributions)

We consider the following three distributions:

Fréchet distribution: $\Phi_\alpha(x) = e^{-x^{-\alpha}}$ for some $\alpha > 0$, $x > 0$,

Weibull distribution: $\Psi_\alpha(x) = e^{-|x|^\alpha}$ for some $\alpha > 0$, $x < 0$,

Gumbel distribution: $\Lambda(x) = e^{-e^{-x}}$ for $x \in \mathbb{R}$.

Of course, $\Psi_\alpha(x) = 1$ for $x \geq 0$ and $\Phi_\alpha(x) = 0$ for $x \leq 0$ in order to ensure that Ψ_α and Φ_α are proper distributions. We also notice that Ψ_α is *not* the Weibull claim size distribution; see pp. 96 and 98. The latter distribution has support $(0, \infty)$. See Figure 9.2.5 for a visualization of the densities of these distributions.

Now assume that (X_i) is an iid sequence with common distribution H . Straightforward calculation yields for any $n \geq 1$,

$$\begin{aligned} n^{-1/\alpha} M_n &\stackrel{d}{=} X_1 & \text{for } H = \Phi_\alpha, \\ n^{1/\alpha} M_n &\stackrel{d}{=} X_1 & \text{for } H = \Psi_\alpha, \\ M_n - \log n &\stackrel{d}{=} X_1 & \text{for } H = \Lambda. \end{aligned}$$

Hence there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$c_n^{-1}(M_n - d_n) \stackrel{d}{=} X_1 \quad \text{for any } n \geq 1. \quad (9.2.21)$$

In particular, relation (9.2.20) holds trivially and therefore any of these distributions H is an extreme value distribution. Relation (9.2.21) remains valid for any distribution in the location-scale family $(H(ax+b))_{x \in \mathbb{R}}$, $a > 0$, $b \in \mathbb{R}$, for suitable choices of $c_n > 0$ and d_n ; see Exercise 6 on p. 322. We refer to this location-scale family as *the type of the distribution H* . Hence a distribution in the type of any of the distributions Φ_α , Ψ_α or Λ is an extreme value distribution as well. As a matter of fact, these types cover all extreme value distributions, as the celebrated Fisher-Tippett Theorem shows; see Theorem 9.2.7 below. Because of the validity of relation (9.2.21) for suitable constants $c_n > 0$ and $d_n \in \mathbb{R}$ for any $n \geq 1$, the extreme value distributions are also called *maxstable distributions*. \square

All extreme value distributions H are continuous. Therefore (9.2.20) can be written in the form

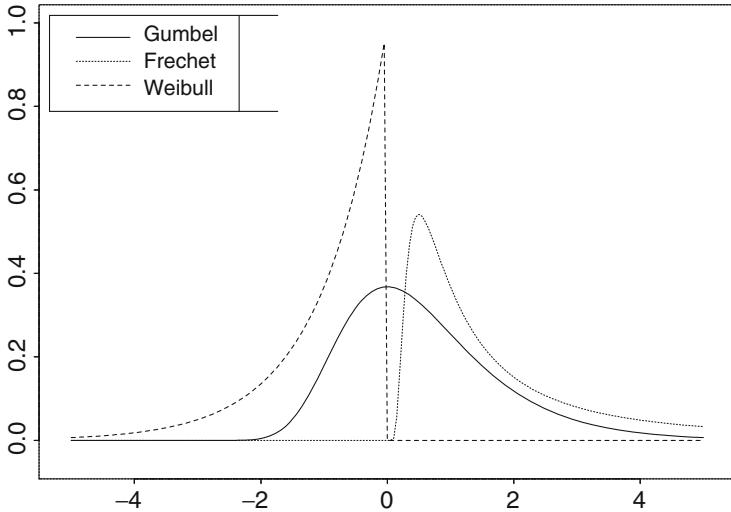


Figure 9.2.5 Densities of some extreme value distributions. We choose $\alpha = 1$ for the Fréchet and Weibull distributions.

$$\lim_{n \rightarrow \infty} P(M_n \leq c_n x + d_n) = H(x), \quad x \in \mathbb{R}.$$

It clearly suffices to consider x in the support of H such that $H(x) \in (0, 1)$. For these x 's, we can rewrite the latter relation as

$$\lim_{n \rightarrow \infty} P(M_n \leq u_n(x)) = e^{-\tau(x)},$$

where $u_n(x) = c_n x + d_n$ and $\tau(x) = -\log H(x)$. Now we are in the framework of Proposition 9.2.1.

Corollary 9.2.6 (Criterion for convergence in distribution of affinely transformed maxima and order statistics)

Let (X_i) be an iid sequence with common distribution F and H an extreme value distribution. Moreover, let $c_n > 0$ and $d_n \in \mathbb{R}$, $n \geq 1$, be given sequences of constants. Then the following statements are equivalent:

- (1) For any x in the support of H the following relation holds:

$$n \bar{F}(c_n x + d_n) \rightarrow -\log H(x).$$

- (2) For any x in the support of H the corresponding point processes of exceedances

$$N_{n,x} = \sum_{i=1}^n \varepsilon_{n-1,i} I_{\{X_i > c_n x + d_n\}}, \quad n = 1, 2, \dots,$$

converge in distribution to a homogeneous Poisson process on $(0, 1]$ with intensity $-\log H(x)$.

- (3) The sequence $(c_n^{-1}(M_n - d_n))$ converges in distribution to a random variable with distribution H .
- (4) For any x in the support of H and any fixed integer $k \geq 1$ the limit

$$\lim_{n \rightarrow \infty} P(c_n^{-1}(X_{(n-k+1)} - d_n) \leq x) = H(x) \sum_{i=0}^{k-1} \frac{(-\log H(x))^i}{i!}$$

exists and defines a proper distribution.

We have defined the extreme value distributions as the set of all non-degenerate limit distributions of centered and normalized maxima M_n of iid random variables X_i . In Example 9.2.4 we found that the Fréchet, Weibull and Gumbel distributions and their types are extreme value distributions. The following fundamental result shows that these are the only possible extreme value distributions.

Theorem 9.2.7 (Fisher-Tippett Theorem; cf. Embrechts et al. [46], Theorem 3.2.3, Resnick [122], Proposition 0.3)

Let (X_i) be a sequence of iid random variables. The relation $c_n^{-1}(M_n - d_n) \xrightarrow{d} Y$ holds for some constants $c_n > 0$, $d_n \in \mathbb{R}$ and a random variable Y with non-degenerate distribution H if and only if H is in the type of one of the distributions Φ_α , Ψ_α for some $\alpha > 0$, or Λ defined in Example 9.2.4.

The class of extreme value distributions can be written as a three-parameter family

$$H_{\xi;\mu,\sigma}(x) = \exp \left\{ -(1 + \xi \frac{x - \mu}{\sigma})^{-1/\xi} \right\}. \quad (9.2.22)$$

Here $\xi \in \mathbb{R}$ is a shape parameter, $\mu \in \mathbb{R}$ and $\sigma > 0$ are location and scale parameters, respectively. The arguments x are chosen such that

$$1 + \xi \frac{x - \mu}{\sigma} > 0$$

in order to ensure that $H_{\xi;\mu,\sigma}$ defines a proper distribution. The case $\xi = 0$ is interpreted in the limiting sense as $\xi \rightarrow 0$ for fixed x :

$$H_{0;\mu,\sigma}(x) = \exp \left\{ -e^{-(x - \mu)/\sigma} \right\}, \quad x \in \mathbb{R}.$$

Obviously, the case $\xi = 0$ corresponds to the type of the Gumbel distribution Λ . The type of the Fréchet distribution Φ_α corresponds to $\xi = 1/\alpha > 0$ and the type of the Weibull distribution Ψ_α to $\xi = -1/\alpha < 0$. We leave it as Exercise 9 on p. 323 to verify that the types of Φ_α , Ψ_α and Λ are covered by the representation (9.2.22).

The joint three-parameter representation (9.2.22) of the three different types of the extreme value distributions is referred to as the *generalized extreme value distribution* which is often abbreviated as GEV. It is also referred to as the *Jenkinson-von Mises representation* of the extreme value distributions; see Section 3.4 in Embrechts et al. [46]. This representation was introduced for statistical applications. It allows one to estimate the parameters ξ, μ, σ by standard techniques such as the maximum likelihood method or the method of probability weighted moments in a unified way; see Embrechts et al. [46], Chapter 6.

9.2.3 Maximum Domains of Attraction

Consider a sequence (X_i) of iid random variables with common distribution F and assume that the relation

$$c_n^{-1}(M_n - d_n) \xrightarrow{d} Y \quad (9.2.23)$$

holds for suitable constants $c_n > 0$, $d_n \in \mathbb{R}$ and a random variable Y with an extreme value distribution H . Then the distribution F is said to be in the *maximum domain of attraction* of H ($F \in \text{MDA}(H)$). According to Corollary 9.2.6 the limit relation (9.2.23) is equivalent to the condition

$$n \bar{F}(c_n x + d_n) \rightarrow -\log H(x) \quad (9.2.24)$$

for x in the support of H . In various cases of interest it is possible to guess what the distribution H and the constants $c_n > 0$ and d_n are. However, a complete characterization of membership in $\text{MDA}(H)$ for a given extreme value distribution H is not trivial. We also mention that there exist distributions which do not belong to $\text{MDA}(H)$ for any extreme value distribution H . These include various well-known discrete distributions such as the Poisson, binomial, geometric, and negative binomial distributions; see Exercise 5(d) on p. 322.

It is clear from (9.2.24) that the constants c_n and d_n need to be expressed in terms of the right tail \bar{F} . This is not a very difficult task for F in the MDAs of the Fréchet and Weibull distributions, but it is rather sophisticated for the MDA of the Gumbel distribution. For complete descriptions of the MDAs, see for example Embrechts et al. [46], Chapter 3, or Resnick [122], Chapter 1. In these references one finds the following result and its proof. For its formulation recall the *quantile function* of the distribution F given by

$$F^\leftarrow(p) = \inf\{x \in \mathbb{R} : F(x) \geq p\}, \quad p \in (0, 1),$$

as well as the *right endpoint* of F :

$$x_F = \sup\{x \in \mathbb{R} : F(x) < 1\}.$$

We also recall that a measurable positive function L on $(0, \infty)$ is slowly varying if it satisfies $L(cx)/L(x) \rightarrow 1$ as $x \rightarrow \infty$ for any $c > 0$.

Theorem 9.2.8 (Characterization of MDAs)

Let (X_i) be an iid sequence with common distribution F .

- (1) The distribution $F \in \text{MDA}(\Phi_\alpha)$ for some $\alpha > 0$ if and only if

$$\bar{F}(x) = x^{-\alpha} L(x), \quad x > 0, \quad L \text{ slowly varying.}$$

The constants in (9.2.23) can be chosen such that

$$c_n = F^{\leftarrow}(1 - n^{-1}) \quad \text{and} \quad d_n = 0.$$

- (2) The distribution $F \in \text{MDA}(\Psi_\alpha)$ for some $\alpha > 0$ if and only if $x_F < \infty$ and

$$\bar{F}(x_F - x^{-1}) = x^{-\alpha} L(x), \quad x > 0, \quad L \text{ slowly varying.}$$

The constants in (9.2.23) can be chosen such that

$$c_n = x_F - F^{\leftarrow}(1 - n^{-1}) \quad \text{and} \quad d_n = x_F.$$

- (3) The distribution $F \in \text{MDA}(\Lambda)$ if and only if $F(\{x_F\}) = 0$ for $x_F < \infty$, and for $x_F \leq \infty$ there exist $x_0 < x_F$ and a positive absolutely continuous function $a(x)$, $x \in (x_0, x_F)$, such that the Lebesgue density $a'(x) \rightarrow 0$ as $x \uparrow x_F$ and

$$\bar{F}(x) \sim c \exp \left\{ - \int_{x_0}^x [a(t)]^{-1} dt \right\}, \quad x \uparrow x_F, \quad \text{for some constant } c > 0.$$

The constants in (9.2.23) can be chosen such that

$$d_n = F^{\leftarrow}(1 - n^{-1}) \quad \text{and} \quad c_n = a(d_n).$$

We refer to Tables 9.2.9, 9.2.10 and 9.2.14, where the MDA conditions and constants are summarized and the constants c_n, d_n are calculated for various distributions.

By its definition, a slowly varying function is positive for all $x > 0$. Therefore $F \in \text{MDA}(\Phi_\alpha)$ requires $x_F = \infty$. The MDA of the Fréchet distribution contains heavy-tailed distributions (those with a regularly varying tail), whereas the MDA of the Gumbel distribution ranges over a wide spectrum of distributional tails. It includes subexponential distributions (such as the log-normal distribution), but also light-tailed distributions with infinite right endpoint (such as the normal and the exponential distributions), and it also contains distributions with $x_F < \infty$. A distribution $F \in \text{MDA}(\Psi_\alpha)$ always satisfies $x_F < \infty$.

In what follows, we look at the MDAs in more detail and consider some prominent members in these classes of distributions.

The MDA of the Fréchet Distribution

This class of distributions contains various large claim distributions which we characterized as *heavy-tailed*; see for example Section 3.2.5. Recall that a (positive) regularly varying random variable X with tail index $\alpha > 0$ does not have finite moments of order greater than α . The MDA of Φ_α contains prominent members such as the Pareto, log-gamma, student, Cauchy, Burr and infinite variance stable distributions; see Table 9.2.9 for an overview.

The normalizing constants c_n in $\text{MDA}(\Phi_\alpha)$ can be derived in a rather simple way. Possible choices³ for c_n are the quantiles $F^\leftarrow(1 - n^{-1})$ or any sequence (c'_n) satisfying the asymptotic relation $n \bar{F}(c'_n) \rightarrow 1$. Of course, by virtue of the relation

$$n \bar{F}(c_n x + d_n) \rightarrow -\log \Phi_\alpha(x) = x^{-\alpha}, \quad x > 0, \quad (9.2.25)$$

setting $d_n = 0$ and $x = 1$, the quantities $c_n = F^\leftarrow(1 - n^{-1})$ satisfy the relation $n \bar{F}(c_n) \rightarrow 1$. We leave it as Exercise 11 on p. 323 to show that $c_n/c'_n \rightarrow 1$.

For a tail of the form $\bar{F}(x) = x^{-\alpha} L(x)$ for some $\alpha > 0$ and a slowly varying function L the MDA condition (9.2.25) is easily verified. Indeed, choose any sequence (c'_n) such that $n \bar{F}(c'_n) \rightarrow 1$. Then by regular variation of $\bar{F}(x)$ and since $c'_n \rightarrow \infty$,

$$n \bar{F}(c'_n x) \sim \frac{\bar{F}(c'_n x)}{\bar{F}(c'_n)} \sim x^{-\alpha} \frac{L(c'_n x)}{L(c'_n)} \rightarrow x^{-\alpha}, \quad x > 0.$$

If $\bar{F}(x) \sim K x^{-\alpha}$ for some positive constants $\alpha, K > 0$ it is not difficult to see that $(Kn)^{-1/\alpha} M_n \xrightarrow{d} Y$ where Y has distribution Φ_α . Such tail behavior is observed for the Pareto, Burr, infinite variance stable, Cauchy and student distributions. If the slowly varying function L in the tail $\bar{F}(x) = x^{-\alpha} L(x)$ is not asymptotically constant (as in the previously mentioned examples) the calculation of c_n is more complicated. However, it follows from the theory of regular variation (see for example Bingham et al. [19]) that $c_n = n^{1/\alpha} \ell(n)$ for a slowly varying function ℓ . This means that the shape of the sequence (c_n) is essentially determined by the powers $n^{1/\alpha}$. The slowly varying $\ell(n)$ is negligible in comparison to $n^{1/\alpha}$ since $\ell(n)/n^\delta \rightarrow 0$ and $\ell(n) n^\delta \rightarrow \infty$ for any $\delta > 0$; see Section 3.2.5.

³ In any limit relation $a_n^{-1}(A_n - b_n) \xrightarrow{d} Y$ the constants $a_n > 0$ and b_n are only determined up to asymptotic equivalence. Indeed, if $a'_n/a_n \rightarrow 1$ and $(b'_n - b_n)/a_n \rightarrow 0$ then we also have $(a'_n)^{-1}(A_n - b'_n) \xrightarrow{d} Y$. See also the Convergence to Types Theorem formulated on p. 323.

Fréchet distribution	$\Phi_\alpha(x) = e^{-x^{-\alpha}}, \quad x > 0, \quad \alpha > 0$
MDA (Φ_α)	$x_F = \infty, \quad \bar{F}(x) = x^{-\alpha} L(x), \quad L$ slowly varying
Constants	$c_n = F^{(-)}(1 - n^{-1}) = n^{1/\alpha} \ell(n), \quad \ell$ slowly varying, $d_n = 0$
Examples	
Cauchy	$f(x) = (\pi(1 + x^2))^{-1}, \quad x \in \mathbb{R}$ $c_n = n/\pi$
Pareto, Burr, stable ($\alpha < 2$)	$\bar{F}(x) \sim K x^{-\alpha}, \quad x \rightarrow \infty, \quad K, \alpha > 0$ $c_n = (Kn)^{1/\alpha}$
Log-gamma	$f(x) = \frac{\alpha^\beta}{\Gamma(\beta)} (\log x)^{\beta-1} x^{-\alpha-1}, \quad x > 1, \quad \alpha, \beta > 0$ $c_n = [(\Gamma(\beta))^{-1} (\log n)^{\beta-1}]^{1/\alpha}$

Table 9.2.9 Maximum domain of attraction of the Fréchet distribution. The symbol f stands for the Lebesgue density of the distribution.

The MDA of the Weibull Distribution

Distributions $F \in \text{MDA}(\Psi_\alpha)$ have finite right endpoint. Therefore they describe risks which potentially cannot get arbitrarily large. Such risks are bounded by a threshold such as claims covered by an excess-of-loss reinsurance treaty. There is also statistical evidence that the life span distribution of human beings is in the MDA of the Weibull distribution. In particular, extreme value methods yield rather convincing approximations to the empirical distribution of life span data at very old age; see Aarsen and de Haan [1] for a study of the life span of the Dutch population.

Prominent members of the MDA of the Weibull distribution are the uniform and beta distributions. Both exhibit power law behavior at the right endpoint of the distribution, i.e., $\bar{F}(x_F - x^{-1}) \sim K x^{-\alpha}$, and therefore the normalizing constants c_n can be determined from relation (9.2.24) which reads as

$$n \bar{F}(c_n x + x_F) \rightarrow -\log \Psi_\alpha(x) = |x|^\alpha, \quad x < 0.$$

Setting $x = -1$, we immediately see that we can choose $c_n = (K n)^{-1/\alpha}$. If the slowly varying function in the tail $\bar{F}(x_F - x^{-1}) = x^{-\alpha} L(x)$ is not asymptotically constant, calculation of the constants c_n becomes more involved. See also Table 9.2.10 for an overview.

The MDAs of the Weibull and the Fréchet distributions are closely related because they are described by regular variation conditions on the tail of the

distribution at its right endpoint x_F . This analogy allows one to translate the limit theory in $\text{MDA}(\Phi_\alpha)$ to the corresponding limit theory in $\text{MDA}(\Psi_\alpha)$, and vice versa. For example, consider a sequence (X_i) of iid random variables with common distribution $F \in \text{MDA}(\Psi_\alpha)$ for some $\alpha > 0$. Then the transformed variables $Y_i = (x_F - X_i)^{-1}$ have tail $P(Y_1 > x) = L(x)x^{-\alpha}$, $x > 0$, for some slowly varying L , hence they are in the MDA of the Fréchet distribution Φ_α and then the limit theory in $\text{MDA}(\Phi_\alpha)$ can be applied to $\max_{i=1,\dots,n} Y_i$. The reader is encouraged to work out the details in Exercise 15 on p. 323.

Weibull distribution	$\Psi_\alpha(x) = e^{- x ^\alpha}$, $x < 0$, $\alpha > 0$
$\text{MDA } (\Psi_\alpha)$	$x_F < \infty$, $\bar{F}(x_F - x^{-1}) = x^{-\alpha} L(x)$, L slowly varying
Constants	$c_n = x_F - F^{\leftarrow}(1 - n^{-1}) = n^{-1/\alpha} \ell(n)$, ℓ slowly varying, $d_n = x_F$
Examples	
Uniform	$f(x) = 1$, $x \in (0, 1)$ $c_n = n^{-1}$, $d_n = 1$
Power law at x_F	$\bar{F}(x) \sim K(x_F - x)^\alpha$, $x \uparrow x_F$ $K, \alpha > 0$ $c_n = (K n)^{-1/\alpha}$, $d_n = x_F$
Beta	$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}$, $0 < x < 1$, $a, b > 0$ $c_n = (n [\Gamma(a+b)/(\Gamma(a)\Gamma(b+1))])^{-1/b}$, $d_n = 1$

Table 9.2.10 Maximum domain of attraction of the Weibull distribution. The symbol f stands for the Lebesgue density of the distribution.

The MDA of the Gumbel Distribution

The MDA of the Gumbel distribution A contains a wide range of tail behaviors; see Table 9.2.14. It contains distributions with moderately heavy tails such as the log-normal distribution, the Weibull claim size distribution with tail $\bar{F}(x) = e^{-cx^\tau}$, $x > 0$, for constants $c > 0$, $\tau < 1$, and the Benktander distributions. These distributions are subexponential; see Section 3.2.6. In contrast to the claim size distributions in $\text{MDA}(\Phi_\alpha)$ the last mentioned distributions have finite moments of any order. However, their moment generating function does not exist as a consequence of their subexponentiality. But $\text{MDA}(A)$ also contains light-tailed distributions such as the gamma,

exponential, normal distributions and the Weibull claim size distribution with shape parameter $\tau \geq 1$ as well as certain distributions with finite right endpoint.

Because of its wide range of tail behaviors the characterization of $\text{MDA}(\Lambda)$ is rather complicated. It contains all distributions F with $x_F \leq \infty$ whose tail has representation

$$\bar{F}(x) = c(x) \exp \left\{ - \int_{x_0}^x [a(t)]^{-1} dt \right\} \quad \text{for } x \in (x_0, x_F), \quad (9.2.26)$$

with $F(\{x_F\}) = 0$ for $x_F < \infty$. The functions $c(x)$ and $a(x)$ are positive such that $c(x) \rightarrow c$ for some constant $c \in (0, \infty)$, $a(x)$ has a Lebesgue density $a'(x)$ for $x \in (x_0, x_F)$ with the property that $a'(x) \rightarrow 0$ as $x \uparrow x_F$.

The function $a(x)$ is called an *auxiliary function*. It is not uniquely determined; there is a trade-off between the functions $c(x)$ and $a(x)$ possible. A possible choice of the auxiliary function is given by the *mean excess function*⁴

$$a(x) = E_F(X - x \mid X > x) = \frac{1}{\bar{F}(x)} \int_x^{x_F} \bar{F}(y) dy, \quad x < x_F.$$

An important subclass of $\text{MDA}(\Lambda)$ constitute the *von Mises functions*. They are distributions F where the function $c(x)$ in (9.2.26) can be chosen as a positive constant c . Most distributions in the MDA of the Gumbel distribution which are of interest in statistics or probability theory are von Mises functions. These include the Weibull claim size distribution, the normal, exponential, gamma distributions; see Table 9.2.12 for a list of von Mises functions and Table 9.2.14 for an overview of the MDA. Notice that von Mises distribution functions are smooth in the sense that they are absolutely continuous and monotone increasing on (x_0, x_F) . In contrast to von Mises functions a general distribution $F \in \text{MDA}(\Lambda)$ may have discontinuities or regions where it is constant. The following lemma shows that it actually suffices to focus the asymptotic extreme value theory on von Mises functions.

Lemma 9.2.11 (Tail equivalence of distributions in $\text{MDA}(\Lambda)$ and von Mises functions)

Assume that $F \in \text{MDA}(\Lambda)$ has representation (9.2.26) with $c(x) \rightarrow c > 0$ and G is the corresponding von Mises function with $c(x)$ replaced by c . Let (X_i) and (Y_i) be two sequences of iid random variables with common distributions F and G , respectively, and denote the corresponding partial maxima by $M_n(X)$ and $M_n(Y)$, $n = 1, 2, \dots$. Write Z for a Gumbel distributed random variable. Then there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that $c_n^{-1}(M_n(X) - d_n) \xrightarrow{d} Z$ if and only if $c_n^{-1}(M_n(Y) - d_n) \xrightarrow{d} Z$.

Proof. Both distributions F and G have representation (9.2.26). Hence they are in $\text{MDA}(\Lambda)$. By Corollary 9.2.6, there exist constants $c_n > 0$ and d_n such that

⁴ See Section 3.2.3 for properties and interpretation of the mean excess function.

Distribution	Tail and auxiliary function
Exponential	$\bar{F}(x) = e^{-\lambda x}, \quad x \geq 0, \quad \lambda > 0$ $a(x) = \lambda^{-1}$
Weibull claim size distribution	$\bar{F}(x) = e^{-c x^\tau}, \quad x \geq 0, \quad c, \tau > 0$ $a(x) = c^{-1} \tau^{-1} x^{1-\tau}, \quad x > 0$
Erlang i.e., $\Gamma(n, \lambda)$	$\bar{F}(x) = e^{-\lambda x} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!}, \quad x \geq 0, \quad \lambda > 0, \quad n \geq 1$ $a(x) = \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-k-1)!} \lambda^{-(k+1)} x^{-k}, \quad x > 0$
Exponential at finite x_F	$\bar{F}(x) = K e^{-\alpha/(x_F - x)}, \quad x < x_F < \infty, \quad \alpha, K > 0$ $a(x) = (x_F - x)^2 / \alpha, \quad x < x_F$
Benktander type I	$\bar{F}(x) = (1 + 2(\beta/\alpha) \log x) e^{-(\beta(\log x)^2 + (\alpha+1) \log x)}, \quad x \geq 1, \quad \alpha, \beta > 0$ $a(x) = x / (\alpha + 2\beta \log x), \quad x \geq 1$
Benktander type II	$\bar{F}(x) = x^{-(1-\beta)} e^{-(\alpha/\beta)(x^\beta - 1)}, \quad x \geq 1, \quad \alpha > 0, 0 < \beta < 1$ $a(x) = \alpha^{-1} x^{1-\beta}, \quad x \geq 1$

Table 9.2.12 Von Mises functions F and their auxiliary functions $a(x)$.

$$n \bar{F}(c_n x + d_n) \rightarrow -\log \Lambda(x) = e^{-x}, \quad x \in \mathbb{R}. \quad (9.2.27)$$

In particular, $\bar{F}(c_n x + d_n) \rightarrow 0$ and therefore $c_n x + d_n \rightarrow x_F$. By assumption, $\bar{F}(y)/\bar{G}(y) \rightarrow 1$ as $y \uparrow x_F$ and therefore (9.2.27) holds if and only if

$$n \bar{G}(c_n x + d_n) \rightarrow -\log \Lambda(x) = e^{-x}, \quad x \in \mathbb{R}.$$

Equivalently, $c_n^{-1}(M_n(Y) - d_n) \xrightarrow{d} Z$. This concludes the proof. \square

For any distribution F with a positive density f we can write for suitable choices of $x_0 < x_F$ and constant $C > 0$,

$$\begin{aligned} \bar{F}(x) &= e^{\log \bar{F}(x)} \\ &= C e^{\int_{x_0}^x (\log \bar{F}(y))' dy} \\ &= C e^{-\int_{x_0}^x f(y)/\bar{F}(y) dy}, \quad x \in (x_0, x_F). \end{aligned}$$

By its definition, any von Mises function F has an a.e. positive Lebesgue density f on (x_0, x_F) , say. Then a possible choice of the auxiliary function $a(x)$ is given by⁵

$$a(x) = \bar{F}(x)/f(x), \quad x \in (x_0, x_F). \quad (9.2.28)$$

⁵ Note that $f(x)/\bar{F}(x)$ is the *hazard rate* of the distribution F ; see p. 108.

This equation follows from the a.e. uniqueness of the Radon-Nikodym derivative of a function (see Billingsley [18]). Moreover, if the function $a(x)$ defined in (9.2.28) has a Lebesgue density $a'(x)$ such that $a'(x) \rightarrow 0$ then necessarily $F \in \text{MDA}(A)$. Using representation (9.2.28), it is not difficult to identify the auxiliary functions of numerous von Mises functions.

The distributions $F \in \text{MDA}(A)$ have much lighter right tail than in the MDA of the Fréchet distribution. For $x_F < \infty$ this is straightforward. For $x_F = \infty$ the following property holds:

$$\lim_{x \rightarrow \infty} \frac{\bar{F}(xt)}{\bar{F}(x)} = \begin{cases} 0, & \text{if } t > 1, \\ \infty, & \text{if } 0 < t < 1. \end{cases} \quad (9.2.29)$$

It is called *rapid variation* of the tail \bar{F} in contrast to regular variation.

We summarize some of the properties of von Mises functions:

Lemma 9.2.13 (Properties of von Mises functions)

Every von Mises function F is absolutely continuous on (x_0, x_F) , for some $x_0 < x_F$, with positive Lebesgue density f . The auxiliary function can be chosen as $a(x) = \bar{F}(x)/f(x)$. Moreover, the following properties hold.

- (1) *If $x_F = \infty$ then \bar{F} is rapidly varying in the sense of (9.2.29) and*

$$\lim_{x \rightarrow \infty} \frac{xf(x)}{\bar{F}(x)} = \infty.$$

- (2) *If $x_F < \infty$ then (9.2.29) holds with $\bar{F}(x)$ replaced by $\bar{G}(x) = \bar{F}(x_F - x^{-1})$, $x > 0$, and*

$$\lim_{x \uparrow x_F} \frac{(x_F - x)f(x)}{\bar{F}(x)} = \infty.$$

We leave the proof as Exercise 17 on p. 324.

9.2.4 The Point Process of Exceedances at the Times of a Renewal Process

In a non-life insurance context one is interested in the magnitude of the largest claim sizes in a given period of time such as a year. The number of claims that arrive in this period will usually be considered as a random variable, and then we need to study the extremes of a random sample of claim sizes. The extreme value theory developed so far is not directly applicable to such objects. The purpose of this section is to introduce a modified point process of exceedances which takes into account random claim numbers and to study the corresponding random maxima and order statistics. The results will be applied to *reinsurance treaties of extreme value type* in Section 9.3. These treaties can be considered as functionals of the largest claim sizes in a random sample.

Gumbel	$\Lambda(x) = e^{-e^{-x}}, \quad x \in \mathbb{R}$
MDA(Λ)	$x_F \leq \infty, \quad \bar{F}(x) = c(x) \exp \left\{ - \int_{x_0}^x [a(t)]^{-1} dt \right\}, \quad x_0 < x < x_F,$ where $c(x) \rightarrow c > 0, \quad a(x) > 0, \quad a'(x) \rightarrow 0 \quad \text{as } x \uparrow x_F$
Constants	$d_n = F^{*-}(1 - n^{-1}), \quad c_n = a(d_n)$
Examples	
Exponential-like	$\bar{F}(x) \sim K e^{-\lambda x}, \quad x \rightarrow \infty, \quad K, \lambda > 0$ $c_n = \lambda^{-1}, \quad d_n = \lambda^{-1} \log(Kn)$
Weibull-like	$\bar{F}(x) \sim K x^\alpha e^{-cx^\tau}, \quad x \rightarrow \infty, \quad K, c, \tau > 0, \quad \alpha \in \mathbb{R}$ $c_n = (c\tau)^{-1} (c^{-1} \log n)^{1/\tau-1},$ $d_n = (c^{-1} \log n)^{1/\tau} + c_n [\alpha \tau^{-1} \log(c^{-1} \log n) + \log K]$
Gamma	$f(x) = [\beta^\alpha / \Gamma(\alpha)] x^{\alpha-1} e^{-\beta x}, \quad x > 0, \quad \alpha, \beta > 0$ $c_n = \beta^{-1}, \quad d_n = \beta^{-1} [\log n + (\alpha - 1) \log \log n - \log \Gamma(\alpha)]$
Normal	$f(x) = (2\pi)^{-1/2} e^{-x^2/2}, \quad x \in \mathbb{R}$ $a_n = (2 \log n)^{1/2}, \quad c_n = a_n^{-1},$ $d_n = a_n - [\log(4\pi) + \log \log n] / (2 a_n)$
Log-normal	$f(x) = (2\pi)^{-1/2} (\sigma x)^{-1} e^{-(\log x - \mu)^2 / (2\sigma^2)}, \quad x > 0, \quad \mu \in \mathbb{R}, \sigma > 0$ $a_n = (2 \log n)^{1/2}, \quad c_n = \sigma a_n^{-1} d_n,$ $d_n = \exp \{ \mu + \sigma (a_n - [\log(4\pi) + \log \log n] / (2 a_n)) \}$
Exponential at $x_F < \infty$	$\bar{F}(x) = K e^{-\alpha/(x_F-x)}, \quad x < x_F, \quad \alpha, K > 0$ $c_n = \alpha [\log(Kn)]^{-2}, \quad d_n = x_F - \alpha [\log(Kn)]^{-1}$
Benktander type I	$\bar{F}(x) = (1 + 2(\beta/\alpha) \log x) e^{-(\beta(\log x)^2 + (\alpha+1) \log x)}, \quad x > 1, \quad \alpha, \beta > 0$ $a_n = e^{-(\alpha+1)/(2\beta) + (\log n/\beta)^{1/2}}, \quad c_n = a_n / (2(\beta \log n)^{1/2}),$ $d_n = a_n \exp \{ [\log \log n + \log(4\beta/\alpha^2) + (\alpha+1)^2 / (2\beta)] / (4(\beta \log n)^{1/2}) \}$
Benktander type II	$\bar{F}(x) = x^{-(1-\beta)} \exp \{ -(\alpha/\beta)(x^\beta - 1) \}, \quad x > 1, \quad \alpha > 0, 0 < \beta < 1$ $a_n = \beta \alpha^{-1} \log n, \quad c_n = \alpha^{-1} a_n^{1/\beta-1},$ $d_n = a_n^{1/\beta} + (\alpha\beta)^{-1} a_n^{1/\beta-1} \{ \alpha - (1-\beta) [\log \log n + \log(\beta/\alpha)] \}$

Table 9.2.14 Maximum domain of attraction of the Gumbel distribution. The symbol f stands for the Lebesgue density of the distribution.

The theory developed below parallels the results in the previous sections: we again exploit Kallenberg's Theorem. The advanced reader who feels comfortable with weak convergence of point processes may skip this section and immediately go to Section 9.3.

We describe the claim arrivals by the points of a renewal sequence

$$T_n = Y_1 + \cdots + Y_n, \quad n \geq 1,$$

with iid positive inter-arrival times Y_1, Y_2, \dots , and define the corresponding renewal process by

$$N_T(t) = \#\{i \geq 1 : T_i \leq t\}, \quad t \geq 0.$$

We mark every arrival T_n by a positive claim size X_n and assume that the sequences (T_n) and (X_n) are independent and (X_n) is iid.

A possible modification of the *point process of exceedances* for a given threshold u_n is then given by

$$\tilde{N}_n = \sum_{i=1}^{N_T(n)} \varepsilon_{n^{-1}T_i} I_{\{X_i > u_n\}}. \quad (9.2.30)$$

Here we again rescale time by n and count the exceedances of the increasing thresholds u_n by the random sample $X_1, \dots, X_{N_T(n)}$. As state space we choose the interval $E = (0, 1]$.

Notice that for any $0 < a < b \leq 1$,

$$\begin{aligned} \tilde{N}_n(a, b] &= \#\{i \geq 1 : na < T_i \leq nb, X_i > u_n\} \\ &= \#\{N_T(na) < i \leq N_T(nb) : X_i > u_n\}, \end{aligned}$$

i.e., $\tilde{N}_n(a, b]$ counts the number of exceedances of u_n by the claim sizes X_i that arrive in the interval $(na, nb]$.

The following result is analogous to Proposition 9.2.1 in the case of the standard point process of exceedances N_T .

Proposition 9.2.15 (Weak convergence of the point processes of exceedances at the times of a renewal process)

Suppose that (X_n) is a sequence of iid random variables with common distribution F and \tilde{N}_n is the point process of the exceedances of the threshold u_n given in (9.2.30). We also assume that

$$\lambda^{-1} = EY_1 \in (0, \infty) \quad (9.2.31)$$

holds and that $n\bar{F}(u_n) \rightarrow \tau$ for some real number $\tau \in (0, \infty)$. Then the following statements are valid:

- (1) The relation $\tilde{N}_n \xrightarrow{d} N$ holds in $M_p(E)$, where N is a homogeneous Poisson process on $E = (0, 1]$ with intensity $\lambda\tau$.

(2) Consider the ordered sample

$$X_{(1)} \leq \cdots \leq X_{(N_T(n))}$$

of the random sample $X_1, \dots, X_{N_T(n)}$. For the sequence $(X_{(N_T(n)-k+1)})$ of the k th largest order statistics in a sample of size $N_T(n)$ and for any fixed integer $k \geq 1$ the following limit relation holds:

$$\lim_{n \rightarrow \infty} P(X_{(N_T(n)-k+1)} \leq u_n) = e^{-\lambda \tau} \sum_{i=0}^{k-1} \frac{(\lambda \tau)^i}{i!}.$$

Proof. We start by proving (1). Since the limit process is a homogeneous Poisson process it is simple. Therefore we may apply Kallenberg's Theorem 9.1.2. Then it suffices to show that

$$E\tilde{N}_n(a, b] \rightarrow EN(a, b] = \lambda \tau (b - a), \quad 0 < a < b \leq 1, \quad (9.2.32)$$

$$P(\tilde{N}_n(B) = 0) \rightarrow P(N(B) = 0), \quad (9.2.33)$$

for all sets $B = \cup_{i=1}^k (c_i, d_i]$ with disjoint intervals $(c_i, d_i] \subset (0, 1]$ such that $\min_{i=1, \dots, k} c_i > 0$, $k \geq 1$. Using the independence between N_T and (X_i) , we have

$$\begin{aligned} E\tilde{N}_n(a, b] &= E \left(\sum_{i=N_T(na)+1}^{N_T(nb)} I_{\{X_i > u_n\}} \right) \\ &= EN_T(na, nb] \overline{F}(u_n) \\ &= (n^{-1} EN_T(na, nb]) [n \overline{F}(u_n)] \\ &\rightarrow \lambda(b - a) \tau. \end{aligned}$$

In the last step we employed the assumption $n\overline{F}(u_n) \rightarrow \tau$ and the elementary renewal theorem $n^{-1}EN_T(nx) \rightarrow \lambda x$, $x > 0$; see Theorem 2.2.7. This proves (9.2.32).

Now we turn to the proof of (9.2.33). For simplicity we restrict ourselves to the set $B = (c_1, d_1] \cup (c_2, d_2]$, where $(c_i, d_i]$, $i = 1, 2$, are disjoint. The case of $k > 2$ disjoint sets $(c_i, d_i] \subset (0, 1]$ can be treated similarly. Conditioning on (T_i) and using the independence of (T_i) and (X_i) , we obtain

$$\begin{aligned} P(\tilde{N}_n(B) = 0) &= P \left(\max_{N_T(nc_1) < i \leq N_T(nd_1)} X_i \leq u_n, \quad \max_{N_T(nc_2) < i \leq N_T(nd_2)} X_i \leq u_n \right) \\ &= E \left([F(u_n)]^{N_T(nc_1, nd_1) + N_T(nc_2, nd_2)} \right) \\ &= E \exp \left\{ n^{-1} (N_T(nc_1, nd_1) + N_T(nc_2, nd_2)) [n \log(1 - \overline{F}(u_n))] \right\} \end{aligned}$$

$$\begin{aligned} &\rightarrow \exp \{-\lambda \tau ((d_1 - c_1) + (d_2 - c_2))\} \\ &= P(N(B) = 0). \end{aligned}$$

In this limit relation we applied the strong law of large numbers for renewal processes (Theorem 2.2.5) and relation $n \bar{F}(u_n) \rightarrow \tau$ in combination with Lebesgue dominated convergence. This proves (9.2.33) and, by Kallenberg's Theorem, the first statement.

Part (2) is an immediate consequence of (1). Indeed, for any $k \geq 1$,

$$\begin{aligned} P(X_{(N_T(n)-k+1)} \leq u_n) &= P(\tilde{N}_n(0, 1] < k) \\ &\rightarrow P(N(0, 1] < k). \end{aligned}$$

The random variable $N(0, 1]$ is $\text{Pois}(\lambda \tau)$ distributed which fact yields the form of the limit probability $P(N(0, 1] < k)$. This proves the proposition. \square

Proposition 9.2.15 enables one to derive the following analog of Corollary 9.2.6.

Corollary 9.2.16 *Let (X_i) be an iid sequence with common distribution F and $c_n > 0$, $d_n \in \mathbb{R}$, $n \geq 1$, be constants. For every $x \in \mathbb{R}$, consider the point process of the exceedances of the threshold $c_n x + d_n$:*

$$\tilde{N}_{n,x} = \sum_{i=1}^{N_T(n)} \varepsilon_{n^{-1}T_i} I_{\{X_i > c_n x + d_n\}}.$$

Assume that the relation $n \bar{F}(c_n x + d_n) \rightarrow -\log H(x)$ holds for any x in the support of an extreme value distribution H . Moreover, assume that (9.2.31) is satisfied. Then the following statements are valid:

- (1) *For any x in the support of H the sequence $(\tilde{N}_{n,x})$ of the point processes of exceedances converges in distribution to a homogeneous Poisson process on $(0, 1]$ with intensity $-\log([H(x)]^\lambda)$.*
- (2) *The sequence $(c_n^{-1}(M_{N_T(n)} - d_n))$ converges in distribution to a random variable with distribution H^λ .*
- (3) *For any x in the support of H and any integer $k \geq 1$ the limit*

$$\lim_{n \rightarrow \infty} P(c_n^{-1}(X_{(N_T(n)-k+1)} - d_n) \leq x) = [H(x)]^\lambda \sum_{i=0}^{k-1} \frac{(-\log([H(x)]^\lambda))^i}{i!}$$

exists and defines a proper distribution.

It is mentioned in Exercise 7 on p. 322 that for any extreme value distribution H , the power H^λ , $\lambda > 0$, is again an extreme value distribution. In particular, H and H^λ are of the same type as one of the standard extreme value distributions Φ_α , Ψ_α or Λ .

Comments

Weak convergence of point processes and random measures has been treated in all standard texts on the topic. We refer to Daley and Vere-Jones [38, 39, 40], Kallenberg [79], Matthes et al. [105], and Resnick [122, 123].

As mentioned above, a rigorous treatment of weak convergence of point processes requires one to consider them as random elements in an appropriate metric space. The general theory on weak convergence in metric spaces can be found for example in Billingsley [17]. One way to metrize weak convergence of point processes is via vague convergence of measures. Rigorous treatments are given in Kallenberg [79], Daley and Vere-Jones [38], Chapter 9, or Resnick [122], Chapter 3. Weak convergence of point processes and vague convergence are closely related to regular variation in \mathbb{R}^d ; see Resnick [121, 122].

The weak convergence of the point processes of exceedances has been exploited in extreme value theory for a long time; see the texts Leadbetter et al. [92], Resnick [122], Falk et al. [48], Embrechts et al. [46]. In the case of an iid sequence the limiting point processes are homogeneous Poisson processes. The situation typically changes for the point processes of exceedances for strictly stationary sequences. Then the limiting processes often have multiple points located at the points of a homogeneous Poisson process. Such a process is also called a compound Poisson process. The fact that the limiting point process is not simple is caused by clusters of exceedances of high thresholds; due to the dependence in a stationary sequence exceedances of these thresholds occur not separated through time, but there is a tendency for large values to occur roughly at the same time. The extreme value theory for stationary sequences is still under development; the texts mentioned above treat the asymptotic theory for maxima and order statistics for various classes of strictly stationary sequences. Most of the recent theory is only available in journals.

Extreme value theory is a classical part of probability theory and statistics. In addition to the texts mentioned above, textbook treatments which address different levels of the theory and applications of extremes are the following ones: Gumbel [64], Galambos [52], Kotz and Nadarajah [88], Coles [34], Beirlant et al. [14, 13], de Haan and Ferreira [66], Reiss and Thomas [120], Resnick [124]. Treatments of multivariate extremes are given in Galambos [52], Resnick [121, 124], Kotz and Nadarajah [88], Coles [34], Falk et al. [48], de Haan and Ferreira [66].

Exercises

Section 9.2.1

- (1) Prove the following version of *Poisson's limit theorem*. Let (X_n) be an iid sequence of random variables with common distribution F , (u_n) be a sequence of real thresholds, $\tau \in (0, \infty)$. Then the relation $n \bar{F}(u_n) \rightarrow \tau$ holds if and only if the binomial $\text{Bin}(n, \bar{F}(u_n))$ random variable $B_n = \sum_{i=1}^n I_{\{X_i > u_n\}}$ converges in distribution to a $\text{Pois}(\tau)$ distributed random variable Y .

- (2) Prove by using Laplace functionals that the sequence of the point processes of exceedances N_n with thresholds u_n weakly converges to PRM(τ Leb), denoted by N , on $E = (0, 1]$ if and only if $n \bar{F}(u_n) \rightarrow \tau$ for some $\tau \in (0, \infty)$; see (9.2.19).
- (3) Assume that there exist constants u_n such that $n \bar{F}(u_n) \rightarrow \tau$ for some $\tau \in (0, \infty)$. Show that for any $\gamma \in (0, \infty)$ there exists a sequence $(u_n(\gamma))$ of constants such that $n \bar{F}(u_n(\gamma)) \rightarrow \gamma$.

Section 9.2.2

- (4) Let (X_i) be an iid sequence of random variables with common distribution F , (u_n) be a sequence of real numbers, $\tau \in [0, \infty]$ and $M_n = \max(X_1, \dots, X_n)$. Give a proof of the equivalence of the relations $P(M_n \leq u_n) \rightarrow e^{-\tau}$ and $n \bar{F}(u_n) \rightarrow \tau$ without point process arguments.
- (5) Let (X_i) be an iid sequence of random variables with common distribution F such that the right endpoint

$$x_F = \inf\{x : F(x) < 1\}$$

of the distribution F is finite. Assume that F has an atom at x_F , i.e., $\bar{F}(x_F) = F(x_F) - F(x_F-) > 0$.

- (a) Assume that $P(M_n \leq u_n) \rightarrow \rho$ for some $\rho \in [0, 1]$ and a sequence (u_n) . Show that necessarily $\rho = 0$ or $\rho = 1$.
- (b) Prove that there do not exist sequences of constants $c_n > 0$, $d_n \in \mathbb{R}$ and a non-degenerate limit distribution H such that $c_n^{-1}(M_n - d_n) \xrightarrow{d} Y$, where Y has distribution H .
- (c) The result of part (a) can be extended to general distributions F . Assume that $x_F \leq \infty$ and let $\tau \in (0, \infty)$. There exists a sequence (u_n) satisfying $n \bar{F}(u_n) \rightarrow \tau$ if and only if

$$\lim_{x \uparrow x_F} \frac{\bar{F}(x)}{\bar{F}(x-)} = 1, \quad (9.2.34)$$

and $F(x_F-) = 1$ in the case $x_F < \infty$. See Leadbetter et al. [92], Theorem 1.7.13, for a proof.

Now assume that $x_F = \infty$ and F has support on the integers. Show that condition (9.2.34) is equivalent to the relation

$$\lim_{n \rightarrow \infty} \frac{\bar{F}(n)}{\bar{F}(n-)} = 1.$$

- (d) Let (X_i) be an iid sequence with a common distribution F which is either binomial, Poisson, geometric or negative binomial. Show that there do not exist sequences of constants $c_n > 0$, $d_n \in \mathbb{R}$ and a non-degenerate distribution H such that $c_n^{-1}(M_n - d_n) \xrightarrow{d} Y$ for some random variable Y with distribution H .
- (6) Let H be any of the extreme value distributions Φ_α , Ψ_α or A . Show that the distribution $(H(ax + b))_{x \in \mathbb{R}}$ for any $a > 0$, $b \in \mathbb{R}$, is again an extreme value distribution; see Example 9.2.4.
- (7) Let H be an extreme value distribution and $\lambda > 0$. Show that H^λ is an extreme value distribution. In particular, show that H and H^λ are of the same type as one of the standard extreme value distributions Φ_α , Ψ_α or A .

- (8) Let X be a positive random variable with a Fréchet distribution Φ_α for some $\alpha > 0$. Show that $\log(X^\alpha)$ has distribution Λ , whereas $-X^{-1}$ has distribution Ψ_α .
- (9) Show that the Jenkinson-von Mises representation (9.2.22) covers the types of the extreme value distributions Φ_α , Ψ_α and Λ .
- (10) Let (X_i) be an iid sequence with common distribution F . Assume there exist constants $c_n > 0$ and d_n such that $c_n^{-1}(M_n - d_n) \xrightarrow{d} Y$ for some random variable Y with non-degenerate distribution H . Show that H is a *max-stable distribution*, i.e., for any $n \geq 1$ there exist $a_n > 0$ and b_n such that $a_n^{-1}(M_n(Y) - b_n) \xrightarrow{d} Y$, where Y, Y_1, Y_2, \dots , constitute an iid sequence and $M_n(Y) = \max_{i=1,\dots,n} Y_i$. We refer to Theorem 3.2.2 in Embrechts et al. [46].
Hints: Show that the sequences $(c_{nk}^{-1}(M_{nk} - d_{nk}))$ and $(c_n^{-1}(M_{nk} - d_n))$ converge in distribution to some non-constant limits for any integer $k \geq 1$ and identify the limit distributions as a function of H . Then apply the *Convergence to Types Theorem* (see Bingham et al. [19], Lemma 8.0.1, Resnick [122], Embrechts et al. [46], Theorem A1.5):

Let A, B, A_1, A_2, \dots , be random variables and $r_n, r'_n > 0$, s_n, s'_n be constants. Assume that $r_n^{-1}(A_n - s_n) \xrightarrow{d} A$. Then the relation

$$(r'_n)^{-1}(A_n - s'_n) \xrightarrow{d} B \quad (9.2.35)$$

holds if and only if the limits $\lim_{n \rightarrow \infty} r_n/r'_n = r \in [0, \infty)$ and $\lim_{n \rightarrow \infty} (s_n - s'_n)/r'_n = s \in \mathbb{R}$ exist. If (9.2.35) holds then $B \xrightarrow{d} rA + s$, and r, s are the unique constants for which this equality holds.

Section 9.2.3

- (11) Consider $F \in \text{MDA}(\Phi_\alpha)$ for some $\alpha > 0$. Show that any sequence (c'_n) satisfying the relation $n\bar{F}(c'_n) \rightarrow 1$ is asymptotically equivalent to $c_n = F^{\leftarrow}(1 - n^{-1})$, i.e., $c_n/c'_n \rightarrow 1$.
- (12) Let H be an extreme value distribution. Show that $H \in \text{MDA}(H)$.
- (13) The log-gamma distribution has density

$$f(x) = \frac{\alpha^\beta}{\Gamma(\beta)} (\log x)^{\beta-1} x^{-\alpha-1}, \quad x > 1, \quad \alpha, \beta > 0.$$

- (a) Determine the asymptotic behavior of $\bar{F}(x)$ as $x \rightarrow \infty$. Show that the log-gamma distribution is in the domain of attraction of the Fréchet distribution Φ_α . Hint: Use Karamata's theorem.
- (b) Determine the corresponding normalizing constants c_n .
- (14) The beta distribution has density

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}, \quad 0 < x < 1, \quad a, b > 0.$$

- (a) Determine the asymptotic behavior of $\bar{F}(x)$ as $x \uparrow 1$. Show that the beta distribution is in the domain of attraction of the Weibull distribution Ψ_b .
- (b) Determine the corresponding constants c_n and d_n .
- (15) Let (X_i) be an iid sequence with common distribution $F \in \text{MDA}(\Psi_\alpha)$ for some positive α .

- (a) Show that the distribution of the iid random variables $Y_i = (x_F - X_i)^{-1}$ has right tail $\bar{F}_Y(x) = P(Y_1 > x) = x^{-\alpha} L(x)$, $x > 0$, for a slowly varying function L .
- (b) Use the fact that $F_Y \in \text{MDA}(\Phi_\alpha)$ to prove that $c_n^{-1}(M_n - x_F) \xrightarrow{d} Z$, where Z has distribution Ψ_α and $c_n = x_F - F^{-1}(1 - n^{-1})$.
- (16) Let $a(x)$ be an auxiliary function in the representation (9.2.26) of $F \in \text{MDA}(A)$. Show that $\lim_{x \rightarrow \infty} x^{-1}a(x) = 0$ if $x_F = \infty$ and $a(x) = o(x_F - x) = o(1)$ as $x \uparrow x_F$ if $x_F < \infty$.
- (17) Prove Lemma 9.2.13.
- (18) Let (X_i) be a sequence of iid random variables with common distribution F which is a von Mises function. Show that there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that $c_n^{-1}(M_n - d_n) \xrightarrow{d} Y$, where Y has distribution A . The constants are given by $c_n = F^{-1}(1 - n^{-1})$ and $c_n = a(d_n)$, where F^{-1} is the inverse of F and $a(x)$ is the auxiliary function in representation (9.2.26).

Hints: First show that

$$\lim_{x \uparrow x_F} \frac{\bar{F}(x + a(x)t)}{\bar{F}(x)} = e^{-t}, \quad t \in \mathbb{R},$$

by using the form of the von Mises functions given in (9.2.26). Then choose c_n such that $\bar{F}(c_n) = n^{-1}$ and apply Corollary 9.2.6.

Section 9.2.4

- (19) Consider an iid sequence (Y_i) of positive random variables with finite mean $\lambda^{-1} = EY_1$ and the corresponding renewal sequence $T_n = Y_1 + \dots + Y_n$, $n \geq 1$. Prove the uniform strong law of large numbers

$$\sup_{x \in [0, 1]} \left| \frac{T_{[nx]}}{n} - \frac{x}{\lambda} \right| \xrightarrow{\text{a.s.}} 0.$$

9.3 Asymptotic Theory for the Reinsurance Treaties of Extreme Value Type

In this section we consider the same setting as in Section 9.2.4, i.e., we consider an iid claim size sequence (X_n) of positive random variables with common distribution F and a renewal claim arrival sequence (T_n) which is independent of (X_n) . We again write

$$N_T(t) = \#\{i \geq 1 : T_i \leq t\}, \quad t \geq 0,$$

for the corresponding renewal counting process. For ease of presentation, we restrict ourselves to a homogeneous Poisson process N_T with intensity λ . However, the asymptotic theory developed below can be extended to the case of general renewal processes N_T with $\lambda^{-1} = EY_1 \in (0, \infty)$; see Embrechts et al. [46], Section 8.7.

Reinsurance treaties of extreme value type (see Section 3.4), in contrast to *treaties of random walk type*, only involve the largest claims in the random sample $X_1, \dots, X_{N_T(t)}$ which have occurred up to time t .

- (1) *Largest claims reinsurance.* In this treaty the reinsurer covers the claim amount

$$R_{LC}(t) = \sum_{i=1}^k X_{(N_T(t)-i+1)}$$

in the period $[0, t]$, $t \geq 0$, for a fixed integer $k \geq 1$.

- (2) *ECOMOR reinsurance (Excédent du coût moyen relatif).* This treaty has the form of an excess-of-loss reinsurance treaty with a random deductible which is determined by the k th largest claim in the portfolio. The reinsurer covers the claim amount

$$\begin{aligned} R_{ECOMOR}(t) &= \sum_{i=1}^{N_T(t)} (X_{(N_T(t)-i+1)} - X_{(N_T(t)-k+1)})_+ \\ &= \sum_{i=1}^{k-1} X_{(N_T(t)-i+1)} - (k-1)X_{(N_T(t)-k+1)} \end{aligned} \quad (9.3.36)$$

in the period $[0, t]$, $t \geq 0$, for a fixed integer $k \geq 2$.

Example 9.3.1 (The joint distribution of the k largest claim sizes)

The distribution of $X_{(N_T(t)-k+1)}$ is easily calculated if N_T is a homogeneous Poisson process with intensity $\lambda > 0$. Indeed, writing N for the marked PRM of the points (T_i, X_i) with mean measure $\lambda \text{Leb} \times F$ and state space $E = (0, \infty)^2$, the Poisson property immediately yields for the k th largest claim size in the period $(0, t]$,

$$\begin{aligned} P(X_{(N_T(t)-k+1)} \leq x) &= P(N((0, t] \times (x, \infty)) < k) \\ &= \sum_{i=0}^{k-1} e^{-\lambda t \bar{F}(x)} \frac{(\lambda t \bar{F}(x))^i}{i!}. \end{aligned}$$

We have already derived this formula in Example 3.4.2 by some elementary considerations. At this point we can use advanced point process theory and gain much more information about the joint distribution of the largest claim sizes. For example, by a geometric argument it is also possible to determine the joint distribution of the vector $(X_{(N_T(t)-k+1)}, \dots, X_{(N_T(t))})$. Indeed, for $x_k < \dots < x_1$, $k \geq 1$, the sets

$$\begin{aligned} A_k &= (0, t] \times (x_k, x_{k-1}], \dots, A_2 = (0, t] \times (x_2, x_1], \\ A_1 &= (0, t] \times (x_1, \infty), \end{aligned}$$

are disjoint and therefore the PRM property implies that the counting variables $K_i = N(A_i)$, $i = 1, \dots, k$, are mutually independent. Hence for any integers $\ell_i \geq 0$, $i = 1, \dots, k$,

$$P(K_1 = \ell_1, \dots, K_k = \ell_k) = P(K_1 = \ell_1) \cdots P(K_k = \ell_k),$$

and then

$$\begin{aligned}
& P(X_{(N_T(t)-k+1)} \leq x_k, \dots, X_{(N_T(t))} \leq x_1) \\
&= P(N((0, t] \times (x_k, \infty)) < k, \dots, N((0, t] \times (x_1, \infty)) < 1) \\
&= P(K_k + \dots + K_1 < k, K_{k-1} + \dots + K_1 < k-1, \dots, K_1 < 1) \\
&= P(K_k + \dots + K_2 \leq k-1, \dots, K_3 + K_2 \leq 2, K_2 \leq 1, K_1 = 0).
\end{aligned}$$

The latter relation can be evaluated by some elementary calculations. For example, for $x_2 < x_1$,

$$\begin{aligned}
& P(X_{(N_T(t)-1)} \leq x_2, X_{(N_T(t))} \leq x_1) \\
&= P(K_1 = 0, K_2 \leq 1) \\
&= P(K_1 = 0) [P(K_2 = 0) + P(K_2 = 1)] \\
&= e^{-\lambda t \bar{F}(x_1)} \left[e^{-\lambda t F(x_2, x_1)} (1 + \lambda t F(x_2, x_1)) \right] \\
&= e^{-\lambda t \bar{F}(x_2)} (1 + \lambda t F(x_2, x_1)),
\end{aligned}$$

and for $x_3 < x_2 < x_1$,

$$\begin{aligned}
& P(X_{(N_T(t)-2)} \leq x_3, X_{(N_T(t)-1)} \leq x_2, X_{(N_T(t))} \leq x_1) \\
&= P(K_1 = 0, K_2 \leq 1, K_2 + K_3 \leq 2) \\
&= P(K_1 = 0) \sum_{\ell_2=0}^1 P(K_2 = \ell_2) \sum_{\ell_3=0}^{2-\ell_2} P(K_3 = \ell_3) \\
&= e^{-\lambda t \bar{F}(x_3)} \left(1 + \lambda t F(x_3, x_1) + (\lambda t)^2 F(x_3, x_2) \left(\frac{1}{2} F(x_3, x_1) + F(x_2, x_1) \right) \right).
\end{aligned}$$

Obviously, these calculations become even more tedious the larger k :

$$\begin{aligned}
& P(X_{(N_T(t)-k+1)} \leq x_k, \dots, X_{(N_T(t))} \leq x_1) \\
&= P(K_1 = 0, K_2 \leq 1, K_2 + K_3 \leq 2, \dots, K_2 + \dots + K_k \leq k-1) \\
&= \sum_{\ell_2=0}^1 \sum_{\ell_3=0}^{2-\ell_2} \sum_{\ell_4=0}^{3-\ell_3-\ell_2} \dots \sum_{\ell_k=0}^{k-1-\ell_{k-1}-\dots-\ell_2} \prod_{l=0}^k P(K_l = \ell_l) \\
&= e^{-\lambda t \bar{F}(x_k)} \sum_{\ell_2=0}^1 \sum_{\ell_3=0}^{2-\ell_2} \sum_{\ell_4=0}^{3-\ell_3-\ell_2} \dots \sum_{\ell_k=0}^{k-1-\ell_{k-1}-\dots-\ell_2} \prod_{l=0}^k \frac{(\lambda t F(x_l, x_{l-1}))^{\ell_l}}{\ell_l!}.
\end{aligned} \tag{9.3.37}$$

□

Because we explicitly know the joint distribution of the vector

$$(X_{(N_T(t)-k+1)}, \dots, X_{(N_T(t))})$$

of the largest claim sizes in a portfolio we can *in principle* calculate the distribution of any measurable function of this vector, in particular, the distributions of $R_{LC}(t)$ and $R_{ECOMOR}(t)$. However, despite the “simplicity” of the structure of the functions $R_{LC}(t)$ and $R_{ECOMOR}(t)$, they are still too complicated in order to be tractable for general distributions F . In order to better understand the distribution of these reinsurance treaties asymptotic extreme value techniques as considered in the previous sections can be employed. We illustrate this aspect in the remainder of this section.

In what follows, we assume that the claim size distribution F belongs to the maximum domain of attraction (MDA) of some extreme value distribution H . We conclude from Corollary 9.2.16 that for each of the order statistics constituting the quantities $R_{LC}(n)$ and $R_{ECOMOR}(n)$ we have an asymptotic extreme value theory at our disposal. However, in order to evaluate the asymptotic behavior of these treaties we need to know the joint limit distribution of the vectors $c_n^{-1}(X_{(N_T(n)-k+1)} - d_n, \dots, X_{(N_T(n))} - d_n)$ for suitable normalizing and centering constants $c_n > 0$ and $d_n \in \mathbb{R}$, $n \geq 1$. Recall from Corollary 9.2.16 that the condition

$$\bar{F}(u_n(x)) \rightarrow -\log H(x), \quad u_n(x) = c_n x + d_n, \quad (9.3.38)$$

for any x in the support of H is sufficient for the convergence result

$$c_n^{-1}(X_{(N_T(n))} - d_n) \xrightarrow{d} Y,$$

where Y has distribution H^λ . An application of condition (9.3.38) to formula (9.3.37) yields a more sophisticated limit relation for any $x_k < \dots < x_1$ in the support of the extreme value distribution H :

$$\begin{aligned} P(c_n^{-1}(X_{(N_T(n))-i+1}) - d_n) \leq x_i, i = 1, \dots, k) \\ = e^{-\lambda n \bar{F}(u_n(x_k))} \times \\ \times \sum_{\ell_2=0}^1 \sum_{\ell_3=0}^{2-\ell_2} \sum_{\ell_4=0}^{3-\ell_3-\ell_2} \dots \sum_{\ell_k=0}^{k-1-\ell_{k-1}-\dots-\ell_2} \prod_{l=0}^k \frac{(\lambda n F(u_n(x_l), u_n(x_{l-1}))])^{\ell_l}}{\ell_l!} \\ \rightarrow H^\lambda(x_k) \times \\ \times \sum_{\ell_2=0}^1 \sum_{\ell_3=0}^{2-\ell_2} \sum_{\ell_4=0}^{3-\ell_3-\ell_2} \dots \sum_{\ell_k=0}^{k-1-\ell_{k-1}-\dots-\ell_2} \prod_{l=0}^k \frac{[\log(H^\lambda(x_{l-1})/H^\lambda(x_l))]^{\ell_l}}{\ell_l!}. \end{aligned} \quad (9.3.39)$$

Every extreme value distribution H has a Lebesgue density h . Therefore partial differentiation with respect to all x_i yields the density of the limiting distribution in (9.3.39). The interested reader may derive the form of the density.

We omit further details and simply read off the result from Theorems 4.2.8 and 4.3.4 in Embrechts et al. [46]:

Proposition 9.3.2 (Joint limit distribution of k upper order statistics)

Assume that (9.3.38) holds for any x in the support of an extreme value distribution H and suitable constants $c_n > 0$, $d_n \in \mathbb{R}$, $n \geq 1$. Then for every $k \geq 1$,

$$(c_n^{-1} (X_{(N_T(n)-i+1)} - d_n))_{i=1,\dots,k} \xrightarrow{d} (Y_\lambda^{(i)})_{i=1,\dots,k}, \quad (9.3.40)$$

where the limit vector is the k -dimensional extremal variate corresponding to the extreme value distribution H^λ . It has density

$$h_\lambda(x_1, \dots, x_k) = H^\lambda(x_k) \prod_{i=1}^k \frac{h_\lambda(x_i)}{H^\lambda(x_i)}, \quad x_k < \dots < x_1,$$

where the arguments x_i are chosen from the support of H , $h_\lambda = \lambda h H^{\lambda-1}$ is the Lebesgue density of H^λ and h is the density of H .

For $\lambda = 1$, the extremal variate has density for $x_k < \dots < x_1$ in the support of H given by

$$\begin{aligned} H = \Phi_\alpha: \quad \varphi_\alpha(x_1, \dots, x_k) &= \alpha^k \exp \left\{ -x_k^{-\alpha} - (\alpha + 1) \sum_{j=1}^k \log x_j \right\}, \\ H = \Psi_\alpha: \quad \psi_\alpha(x_1, \dots, x_k) &= \alpha^k \exp \left\{ -(-x_k)^\alpha + (\alpha - 1) \sum_{j=1}^k \log(-x_j) \right\}, \\ H = \Lambda: \quad \lambda(x_1, \dots, x_k) &= \exp \left\{ -e^{-x_k} - \sum_{j=1}^k x_j \right\}. \end{aligned}$$

Notice that the cases $\lambda \neq 1$ and $\lambda = 1$ are closely related. Indeed, since $h_\lambda = \lambda h H^{\lambda-1}$ we have

$$\begin{aligned} h_\lambda(x_1, \dots, x_k) &= \lambda^k H^\lambda(x_k) \prod_{i=1}^k \frac{h(x_i)}{H(x_i)}, \\ &= \lambda^k H^{\lambda-1}(x_k) h_1(x_1, \dots, x_k), \quad x_k < \dots < x_1. \end{aligned} \quad (9.3.41)$$

Example 9.3.3 (The limit distribution of the value of an ECOMOR reinsurance treaty with claim size distribution in the MDA of the Gumbel distribution)

Recall the value of an ECOMOR reinsurance treaty from (9.3.36): for any fixed $k \geq 2$,

$$\begin{aligned}
R_{\text{ECOMOR}}(n) &= \sum_{i=1}^{k-1} X_{(N_T(n)-i+1)} - (k-1)X_{(N_T(n)-k+1)} \\
&= \sum_{i=1}^{k-1} (X_{(N_T(n)-i+1)} - X_{(N_T(n)-k+1)}) \\
&= \sum_{i=1}^{k-1} \sum_{j=i}^{k-1} (X_{(N_T(n)-j+1)} - X_{(N_T(n)-j)}) \\
&= \sum_{j=1}^{k-1} j (X_{(N_T(n)-j+1)} - X_{(N_T(n)-j)}).
\end{aligned}$$

Now assume that $F \in \text{MDA}(\Lambda)$ for suitable normalizing and centering constants $c_n > 0$ and $d_n \in \mathbb{R}$. An application of (9.3.40) jointly with the continuous mapping theorem (see Billingsley [17]) yields

$$\begin{aligned}
c_n^{-1} R_{\text{ECOMOR}}(n) &= \sum_{i=1}^{k-1} i [(c_n^{-1}(X_{(N_T(n)-i+1)} - d_n) - c_n^{-1}(X_{(N_T(n)-i)} - d_n))] \\
&\xrightarrow{d} \sum_{i=1}^{k-1} i [Y_\lambda^{(i)} - Y_\lambda^{(i+1)}]. \tag{9.3.42}
\end{aligned}$$

Here $(Y_\lambda^{(i)})_{i=1,\dots,k}$ is the extremal variate corresponding to the extreme value distribution Λ^λ . We have

$$(Y_\lambda^{(1)}, \dots, Y_\lambda^{(k)}) \stackrel{d}{=} (Y_1^{(1)} + \log \lambda, \dots, Y_1^{(k)} + \log \lambda). \tag{9.3.43}$$

This equality can be checked by comparison of the densities on both sides, using relation (9.3.41). We conclude that the limit distribution in (9.3.42) does not depend on λ . Therefore we assume without loss of generality that $\lambda = 1$.

One way to get more information about the limit distribution in (9.3.42) is to choose some special distribution $F \in \text{MDA}(\Lambda)$ for which one can calculate the distribution of $c_n^{-1} R_{\text{ECOMOR}}(n)$ explicitly for every $n \geq 1$. Such an F is the standard exponential distribution. Direct calculation shows that

$$n \bar{F}(x + \log n) \rightarrow e^{-x}, \quad x \in \mathbb{R}.$$

Hence we can choose $c_n = 1$ and $d_n = \log n$. Relation (9.3.42) then yields for $\lambda = 1$

$$R_{\text{ECOMOR}}(n) \xrightarrow{d} \sum_{i=1}^{k-1} i [Y_1^{(i)} - Y_1^{(i+1)}].$$

On the other hand, we know from Example 3.4.3 that for an iid standard exponential claim size sequence $R_{\text{ECOMOR}}(n)$ has a $\Gamma(k-1, 1)$ distribution. This is the limit distribution in (9.3.42).

The asymptotic theory for an ECOMOR treaty with claim size distribution $F \in \text{MDA}(\Lambda)$ is rather attractive. The limit distribution is well known. Moreover, the limit relation (9.3.42) does not require knowledge of the Poisson intensity λ and the centering constants d_n . If we know the distribution F we can read off the normalizing constants c_n for some of the standard distributions in the MDA of the Gumbel distribution; see for example Table 9.2.14. \square

Example 9.3.4 (The limit distribution of the largest claims reinsurance treaty with claim size distribution in the MDA of the Gumbel distribution) It follows from the arguments given in Example 9.3.3 that for $F \in \text{MDA}(\Lambda)$ one can find constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$\begin{aligned} c_n^{-1}(R_{LC}(n) - k d_n) &= \sum_{i=1}^k [c_n^{-1}(X_{(N_T(n)-i+1)} - d_n)] \\ &\xrightarrow{d} \sum_{i=1}^k Y_\lambda^{(i)} \stackrel{d}{=} \sum_{i=1}^k Y_1^{(i)} + k \log \lambda, \end{aligned} \quad (9.3.44)$$

where $(Y_\lambda^{(1)}, \dots, Y_\lambda^{(k)})$ is the extremal variate corresponding to the extreme value distribution Λ^λ . We assume without loss of generality that $\lambda = 1$.

In order to identify the limit distribution in (9.3.44) we consider the special case of an iid standard exponential claim size sequence. We know from Example 9.3.3 that we can choose $c_n = 1$ and $d_n = \log n$. We recall from (3.4.49) that

$$R_{LC}(n) - k \log n \xrightarrow{d} \sum_{i=1}^k E_i + k \sum_{i=k+1}^{\infty} i^{-1}(E_i - 1) + k C_k = R, \quad (9.3.45)$$

where (E_i) is an iid standard exponential sequence, $C_k = \gamma - \sum_{i=1}^k i^{-1}$, and $\gamma = 0.5772\dots$ is Euler's constant.

The limit distribution R in (9.3.45) is rather unfamiliar. Writing

$$R_1 = \sum_{i=1}^k E_i \quad \text{and} \quad R_2 = k \sum_{i=k+1}^{\infty} i^{-1}(E_i - 1),$$

we see that the distribution of R is determined by the constant $k C_k$ and the independent summands R_1 and R_2 . Of course, R_1 has a $\Gamma(k, 1)$ distribution. The characteristic function of R can be easily evaluated and from it one can derive the moments. An elementary calculation yields

$$ER = k + k C_k \sim -k \log k, \quad k \rightarrow \infty,$$

$$\text{var}(R) = k + k \sum_{i=k+1}^{\infty} i^{-2} \sim k, \quad k \rightarrow \infty.$$

Thus the expectation and variance of R for large k are essentially determined by $kC_k \sim -k \log k$ and $\text{var}(R_1) = k$, respectively. For more detailed quantitative analyses of the distribution of R it is useful to employ simulations. Figure 3.4.5 gives one an idea about the density of the distribution of R . \square

Comments

Extreme value theory has natural applications in non-life insurance, in particular in reinsurance. This aspect is stressed in Embrechts et al. [46]. In the presence of large claims, extreme value theory advises one *not* to use the classical approach to collective risk theory by applying mean value and variance analyses of a portfolio or asymptotic analyses based on the strong law of large numbers and the central limit theorem. The study of the asymptotic behavior of the reinsurance treaties of extreme value type above shows that rather unfamiliar distributional limits may occur. In Embrechts et al. [46], Section 8.7, an asymptotic theory is also developed for these treaties when the underlying claim size distribution is in the maximum domain of attraction of the Fréchet distribution. However, the resulting limits are even more complicated than those in $\text{MDA}(\Lambda)$.

Exercises

- (1) Let (E_i) be an iid standard exponential sequence. Show that the exponential spacings $E_{(n-i+1)} - E_{(n-i)}$ of the ordered sample $E_{(1)} < \dots < E_{(n)}$ a.s. are mutually independent for any $n \geq 1$ and that the random variables $i(E_{(n-i+1)} - E_{(n-i)})$ are iid with common standard exponential distribution.

Hint: Start by solving Exercise 13 on p. 49 which gives a representation of the ordered sample. Then apply a linear transformation to this vector and the density transformation theorem.

- (2) Prove relation (9.3.43) by using (9.3.41) for $H = \Lambda$.

Part IV

Special Topics

An Excursion to Lévy Processes

Over the last 10–15 years the class of *Lévy processes* has attracted a lot of attention. This is due to the fact that this is a rather flexible class for modeling jumps and crashes in real-life processes and therefore it has in particular found a multitude of applications in mathematical finance and physics. In insurance mathematics, the Poisson and compound Poisson Lévy processes have been used for more than 100 years as basic models for the claim number and the total claim amount in a homogeneous non-life portfolio.

The compound Poisson process is a basic Lévy process from which all pure jump Lévy processes can be generated. The aim of this chapter is to show how any pure jump Lévy process can be constructed as the limit of Poisson integrals with respect to suitable marked PRMs. We will see that our knowledge of PRMs is sufficient to understand the structure of Lévy processes which is one of the major classes of stochastic processes. In particular, we will discover that the models and techniques used in non-life insurance mathematics are very close to those in Lévy process theory.

10.1 Definition and First Examples of Lévy Processes

We start with a formal definition of a Lévy process.

Definition 10.1.1 (Lévy process)

The real-valued process $S = (S(t))_{t \geq 0}$ is a Lévy process on $[0, \infty)$ if the following properties are satisfied:

- (1) *It starts at zero: $S(0) = 0$ a.s.*
- (2) *It has independent increments: for any $0 = t_0 < t_1 < \dots < t_n$ and $n \geq 1$, the increments $S(t_{i-1}, t_i]$, $i = 1, \dots, n$, are mutually independent.*
- (3) *It has stationary increments: for any $0 \leq s < t$ and $h \geq 0$, $S(s, t] \stackrel{d}{=} S(s + h, t + h]$.*
- (4) *It is stochastically continuous: for any $\varepsilon > 0$,*

$$\lim_{h \downarrow 0} P(|S(h)| > \varepsilon) = 0. \quad (10.1.1)$$

The restriction $(S(t))_{t \in [a,b]}$ of a Lévy process S to the interval $[a,b]$ is called a Lévy process on $[a,b]$.

The stochastic continuity property allows one to define a *version*¹ of the process S with càdlàg sample paths; see Sato [132], Chapter 2. Alternatively, the requirement (10.1.1) can be replaced by the assumption that S has càdlàg sample paths, i.e., these paths are right-continuous and have limits from the left. In view of the stationary increments of S , relation (10.1.1) can be written in the form

$$\lim_{t \rightarrow s} P(|S(t) - S(s)| > \varepsilon) = 0, \quad \varepsilon > 0,$$

for any $s > 0$ which illustrates the idea of “continuity” in the sense of convergence in probability. Of course, (10.1.1) can be written in the form $S_h \xrightarrow{P} 0$ as $h \downarrow 0$.

The most simple example of a Lévy process is the deterministic function $f(t) = ct$, $t \geq 0$, for some constant c . From a course on stochastic processes we know a prime example of a Lévy process: *Brownian motion* B . In addition to the defining properties of a Lévy process, $B(t)$ is normally $N(0, t\sigma^2)$ distributed for every $t \geq 0$ and some constant $\sigma > 0$. Then B is a Gaussian process, i.e., its finite-dimensional distributions are multivariate Gaussian distributions. The Gaussianity makes it rather special: it is the only non-trivial Lévy process whose sample paths can be chosen continuous with probability 1; see Sato [132], Theorem 21.1. See Figure 10.1.3 for the visualization of a Brownian sample path. In our treatment we are less interested in Brownian motion; we focus on pure jump Lévy processes, i.e., Lévy processes without a Gaussian component. They are functions of a PRM.

Two prime examples of Lévy jump processes are the homogeneous Poisson and the compound Poisson processes.

Example 10.1.2 (The Poisson and compound Poisson Lévy processes)
Consider the marked PRM $(\lambda \text{Leb} \times F)$, $\lambda > 0$, on $[0, \infty) \times \mathbb{R}$ with points (T_i, X_i) corresponding to the Cramér-Lundberg model. We denote this PRM by N and the homogeneous Poisson process of the points T_i on $[0, \infty)$ by N_T . We write $N_T(t) = N_T[0, t]$ for $t \geq 0$.

It follows from the defining properties of the PRM N_T that $(N_T(t))_{t \geq 0}$ has stationary and independent increments and that $N(0) = 0$ a.s. The stochastic continuity property (10.1.1) is immediate by Markov’s inequality: for $h > 0$,

$$P(N_T(h) > \varepsilon) \leq \varepsilon^{-1} E N_T(h) = \varepsilon^{-1} \lambda h \downarrow 0.$$

Moreover, $N_T(t)$ is $\text{Pois}(\lambda t)$ distributed for every $t \geq 0$. This property characterizes the Poisson Lévy process N_T .

¹ This means that there exists a process $\eta = (\eta(t))_{t \geq 0}$ such that $S \stackrel{d}{=} \eta$, where $\stackrel{d}{=}$ refers to the equality of the finite-dimensional distributions.

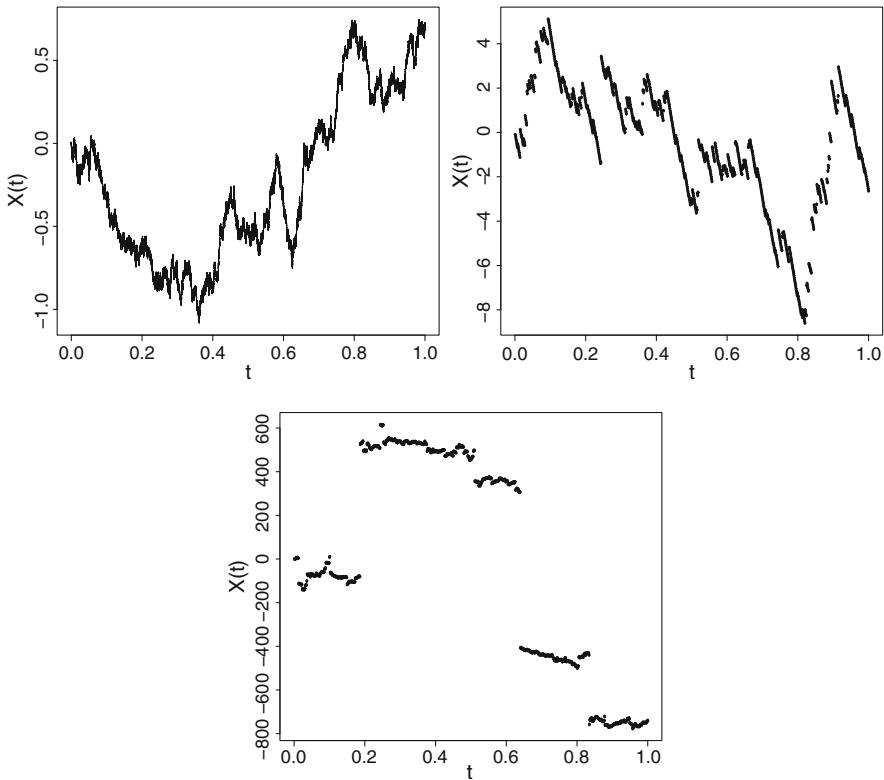


Figure 10.1.3 Visualization of Lévy processes on $[0, 1]$. Top left: Standard Brownian motion. Top right: Centered gamma process; see Example 10.5.1. This process has only upward jumps. Bottom: Cauchy process; see Example 10.5.2. This process has both upward and downward jumps.

The corresponding compound Poisson process

$$\begin{aligned} S(t) &= \int_{[0,t] \times \mathbb{R}} x N(ds, dx) \\ &= \sum_{i \geq 1: 0 \leq T_i \leq t} X_i = \sum_{i=1}^{N_T(t)} X_i, \quad t \geq 0, \end{aligned} \tag{10.1.2}$$

has stationary and independent increments, as a check of the properties of Poisson integrals shows. Moreover, $N_T(0) = 0$ a.s. and therefore $S(0) = 0$ a.s. The stochastic continuity follows for example from the convergence of the characteristic functions: for any $t \in \mathbb{R}$,

$$Ee^{itS(h)} = e^{-\lambda h(1-Ee^{itX_1})} \rightarrow e^0 = 1, \quad h \downarrow 0.$$

By the continuity theorem for characteristic functions, $S(h) \xrightarrow{P} 0$ as $h \downarrow 0$. \square

Exercises

- (1) Prove that the deterministic function $f(t) = ct$, $t \geq 0$, for constant c has the properties of a Lévy process on $[0, \infty)$.

10.2 Some Basic Properties of Lévy Processes

In what follows, we list some of the basic properties of Lévy processes. We start with some operations on Lévy processes.

Lemma 10.2.1 (Linear combinations and weak limits)

- (1) *A linear combination of independent Lévy processes is a Lévy process.*
 (2) *Let (X_n) be a sequence of Lévy processes on $[0, \infty)$ and X be a stochastic process on $[0, \infty)$ such that the finite-dimensional distributions of X_n converge to those of X : for any choice of $0 \leq t_1 < \dots < t_m$ and $m \geq 1$,*

$$(X_n(t_1), \dots, X_n(t_m)) \xrightarrow{d} (X(t_1), \dots, X(t_m)).$$

Then X has stationary independent increments and $X(0) = 0$ a.s.

The proofs are left as Exercise 1 on p. 340.

Next we calculate the first two moments.

Lemma 10.2.2 (Expectation and variance)

Let X be a Lévy process on $[0, \infty)$.

- (1) *If $E|X(t)| < \infty$ for some $t > 0$ then $E|X(1)| < \infty$ and*

$$EX(s) = sEX(1), \quad s \geq 0.$$

- (2) *If $\text{var}(X(t)) < \infty$ for some $t > 0$ then $\text{var}(X(1)) < \infty$ and*

$$\text{var}(X(s)) = s\text{var}(X(1)), \quad s \geq 0.$$

Proof. (1) Assume that $E|X(t)| < \infty$. Since $X(t) = X(s) + X(s, t]$, and $X(s, t]$ and $X(s)$ are independent for $s \leq t$, both $EX(s)$ and $EX(s, t] = EX(t - s)$ exist and are finite; see Exercise 6 on p. 243. Hence $EX(s)$ exists and is finite for $0 \leq s \leq t$. Since $X((k-1)t, kt) \xrightarrow{d} X(t)$ for all integers $k \geq 1$, it follows for all integers $n \geq 1$ that $X(nt) = \sum_{k=1}^n X((k-1)t, kt]$ has finite first moment and therefore $X(s)$, $0 \leq s \leq nt$, has finite first moment for all $n \geq 1$. Hence $EX(s)$ is finite for all $s \geq 0$.

Write $m(t) = EX(t)$, $t \geq 0$. It follows from the linearity of the expectation and the stationary increment property that

$$m(t) + m(s) = m(t+s), \quad t, s \geq 0. \quad (10.2.3)$$

By (10.2.3), for all positive integers $n, k \geq 1$,

$$m(n/k) = n m(1/k) = (n/k) [k m(1/k)] = (n/k) m(1).$$

Since $X(0) = 0$ a.s. we also have $m(0) = 0$. In other words, $m(t) = t m(1)$ for rational $t \geq 0$. We will show that $m(t) = t m(1)$ for all $t \geq 0$. In particular, $m(t) = t E X(1)$ which yields the desired result.

Thus assume that $t > 0$ is an irrational number and that (t_n) is a sequence of rational numbers such that $t_n \downarrow t$. Since by the stationary increment property

$$m(t) = [m(t) - m(t_n)] + m(t_n) = -m(t_n - t) + t_n m(1),$$

it suffices to show that $m(s) \rightarrow 0$ as $s \downarrow 0$. We exploit the stochastic continuity property of the Lévy process to show this.

Let $M > 1$ by any positive number and $x_{\pm} = \max(0, \pm x)$ for any $x \in \mathbb{R}$. Then for $s > 0$,

$$\begin{aligned} m(s) &= E[(X(s))_+] - E[(X(s))_-] \\ &= \int_0^\infty [P(X(s) > x) - P(X(s) < -x)] dx \\ &= \left(\int_{[0, M^{-1}]} + \int_{(M^{-1}, M]} + \int_{(M, \infty)} \right) [P(X(s) > x) - P(X(s) < -x)] dx \\ &= I_1(M, s) + I_2(M, s) + I_3(M, s). \end{aligned}$$

Obviously,

$$\limsup_{M \rightarrow \infty} |I_1(M, s)| \leq \lim_{M \rightarrow \infty} M^{-1} = 0,$$

and by the stochastic continuity property of X ,

$$\limsup_{s \downarrow 0} |I_2(M, s)| \leq (M - M^{-1}) \lim_{s \downarrow 0} P(|X(s)| > M^{-1}) = 0.$$

Furthermore, for $s \leq s_0$ and s_0 to be chosen later,

$$|I_3(M, s)| \leq \int_{(M, \infty)} P \left(\sup_{0 \leq r \leq s_0} |X(r)| > x \right) dx. \quad (10.2.4)$$

It follows by a Lévy-Ottaviani inequality (Gihman and Skorohod [54], Lemma 2, p. 420) that

$$P \left(\sup_{0 \leq r \leq s_0} |X(r)| > x \right) \leq \frac{P(|X(s_0)| > x/2)}{1 - \sup_{0 \leq r \leq s_0} P(|X(r)| > x/2)}, \quad (10.2.5)$$

where we may assume that the denominator on the right-hand side is positive for small s_0 in view of $X(r) \xrightarrow{P} 0$ as $r \downarrow 0$. Hence the right-hand side in (10.2.4) is dominated by

$$\frac{\int_{(M,\infty)} P(|X(s_0)| > x/2) dx}{1 - \sup_{0 \leq r \leq s_0} P(|X(r)| > M/2)} \leq \frac{2 E|X(s_0)|}{1 - \sup_{0 \leq r \leq s_0} P(|X(r)| > M/2)} < \infty.$$

An application of the Lebesgue dominated convergence theorem and the stochastic continuity property of X yield

$$\limsup_{s \downarrow 0} |I_3(M, s)| \leq \int_{(M,\infty)} \lim_{s \downarrow 0} P(|X(s)| > x) dx = 0.$$

(2) The proof is analogous to (1) by using the stationary independent increments of X . The proof is left as Exercise 2(a) on p. 340. \square

A significantly simpler proof of Lemma 10.2.2 can be based on relation (10.2.6) below, which links the moments of a random variable and the derivatives of its characteristic function, and by exploiting the particular form of the characteristic function of $X(t)$ in (10.3.9) below. For details, see Exercise 2(b).

Comments

Equation (10.2.3) is the *Cauchy functional equation*. If m is known to be measurable or bounded on some interval the only solutions are given by $m(t) = ct$, $t \geq 0$, for constant c ; see Bingham et al. [19], Section 1.1.3. If these conditions on m are not satisfied there exist non-measurable solutions of the Cauchy equation which are not of the form $m(t) = ct$.

There exists a manifold of inequalities relating the tail of the supremum of a process η on the interval $[0, T]$, $T > 0$, to the tail of η_T , i.e., of the process at its right endpoint. The Lévy-Ottaviani inequality (10.2.5) is just one of them. Such inequalities have been proved for processes with independent increments and random walks (see Petrov [116, 117]) and further developed, for example for Markov processes; see Gihman and Skorohod [54], Lemma 2, p. 420.

Exercises

- (1) Prove Lemma 10.2.1.
- (2) (a) Prove Lemma 10.2.2(2) by the same methods as used in the proof of the first part of the lemma.
 (b) Let X be a random variable with n th finite moment for some integer $n \geq 1$ and $\phi_X(s) = E e^{isX}$, $s \in \mathbb{R}$, be its characteristic function. The moment $E(X^n)$ is then given by the relation

$$i^n E(X^n) = \phi_X^{(n)}(0) = \left. \frac{d^n \phi_X(s)}{ds^n} \right|_{s=0}. \quad (10.2.6)$$

Use this relation and the characteristic function of $X(t)$ given in (10.3.9) to determine $EX(t)$ and $\text{var}(X(t))$.

10.3 Infinite Divisibility: The Lévy-Khintchine Formula

A random variable X and its distribution are called *infinitely divisible* if for every $n \geq 1$ there exist iid random variables X_{ni} , $i = 1, \dots, n$, such that

$$X \stackrel{d}{=} X_{n1} + \dots + X_{nn}.$$

There is a very close relationship between infinitely divisible distributions and Lévy processes.

Lemma 10.3.1 (Infinite divisibility of marginal distributions)

- (1) *Let X be a Lévy process on $[0, \infty)$. For any $t \geq 0$, the distribution of $X(t)$ is infinitely divisible.*
- (2) *For every infinitely divisible distribution F on \mathbb{R} one can construct a Lévy process X on $[0, \infty)$ such that $X(1)$ has distribution F .*

The proof of part 1 is elementary and left as Exercise 1 on p. 347. The proof of the converse part 2 follows by an application of Kolmogorov's consistency theorem which allows one to construct a stochastic process with a given dependence structure determined by the finite-dimensional distributions of the process; see Sato [132], Theorem 7.10.

Example 10.3.2 (Some well-known infinitely divisible distributions)

Most well-known distributions with infinite support are infinitely divisible. It is in general hard to verify this property; see for example Lukacs [97]. In the latter reference, one finds sophisticated methods for detection of infinite divisibility of a distribution from the form of the corresponding characteristic function.

Infinite divisibility is easily verified for the symmetric stable distributions which have characteristic function

$$E e^{isX} = e^{-\sigma^\alpha |s|^\alpha}, \quad s \in \mathbb{R}, \tag{10.3.7}$$

for positive constants σ and $\alpha \in (0, 2]$. The case $\alpha = 2$ corresponds to the normal $N(0, 2\sigma^2)$ distribution. In particular, we have for iid copies X_i of X and any $n \geq 1$,

$$X \stackrel{d}{=} n^{-1/\alpha} (X_1 + \dots + X_n).$$

This equation follows by calculating the characteristic function of the sum on the right-hand side. The case $\alpha = 1$ yields the Cauchy distribution. It has the remarkable property that the sample mean of iid copies X_i of X has the same distribution as X :

$$X \stackrel{d}{=} \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i. \tag{10.3.8}$$

Condition $E|X| < \infty$ is necessary and sufficient for the strong law of large numbers $\bar{X}_n \xrightarrow{\text{a.s.}} EX$ which is violated in view of (10.3.8), hence $E|X| = \infty$. In general, α -stable random variables with $\alpha < 2$ have infinite α th moment, but finite moments of order $\delta < \alpha$; see Feller [51], Zolotarev [148].

The Poisson distribution is infinitely divisible as well: for a $\text{Pois}(\lambda)$ distributed random variable X ,

$$X \stackrel{d}{=} X_{n1} + \cdots + X_{nn},$$

where for fixed n , X_{ni} , $i = 1, \dots, n$, are iid $\text{Pois}(\lambda/n)$ distributed. This relation again follows by a characteristic function argument. Notice the difference between the Poisson and the stable cases: in the stable case, the distribution of X is obtained by an affine transformation of a sum of iid copies of X . Such a relation is not possible for the Poisson distribution: affine transformations of sums of iid Poisson random variables are in general not Poisson distributed.

A compound Poisson sum $\sum_{i=1}^M Z_i$ for a Poisson distributed random variable M independent of the iid sequence (Z_i) inherits infinite divisibility from the Poisson distribution. This statement is easily checked by a characteristic function argument.

Another infinitely divisible distribution is the gamma distribution. This property follows from the well-known convolution closure of the gamma family, i.e., if Y_i , $i = 1, 2$, are independent with corresponding $\Gamma(\alpha_i, \beta)$ distributions for $\alpha_i, \beta > 0$, $i = 1, 2$, then $Y_1 + Y_2$ is $\Gamma(\alpha_1 + \alpha_2, \beta)$ distributed. Infinite divisibility of the gamma $\Gamma(\alpha, \beta)$ distributed random variable X also follows from the form of its characteristic function

$$\phi_X(s) = (1 - i s / \beta)^{-\alpha}, \quad s \in \mathbb{R}.$$

We encourage the reader to verify the infinite divisibility for the mentioned classes of distributions; see Exercise 2(a) on p. 347. \square

For other well-known distributions the property of infinite divisibility has been established, but it is not easily verified. Among those infinitely divisible distributions are the Pareto, log-normal and student distributions; see the references in Sato [132], Remark 8.12.

One property is useful for concluding a distribution is *not* infinitely divisible: distributions with bounded support are not infinitely divisible; see Exercise 3 on p. 347. This fact immediately excludes distributions such as the uniform and the beta distributions.

The Characteristic Function of a Lévy Process

Due to the stationary independent increment property the finite-dimensional distributions of a Lévy process X are completely determined by the one-dimensional marginal distributions, i.e., the distributions of $X(t)$, $t \geq 0$. In turn, the marginal distributions are determined by their characteristic functions.

Lemma 10.3.3 (Characteristic function of marginal distribution)

The characteristic function of a Lévy process X at $t \geq 0$ has representation

$$Ee^{isX(t)} = (Ee^{isX(1)})^t, \quad s \in \mathbb{R}. \quad (10.3.9)$$

This result shows that the distribution of $X(t)$ for any $t > 0$ is completely determined by the distribution of $X(1)$. In this sense, one can call (10.3.9) the characteristic function of the Lévy process X .

Proof. Since for any $t \geq 0$ and integers $k \geq 1$,

$$X(kt) = X(t) + X(t, 2t] + \cdots + X((k-1)t, kt],$$

the stationary independent increment property implies

$$Ee^{isX(kt)} = (Ee^{isX(t)})^k, \quad s \in \mathbb{R}.$$

Therefore for any positive integers $n, m \geq 1$,

$$Ee^{isX(n/m)} = (Ee^{isX(1/m)})^n = ([Ee^{isX(1/m)}]^m)^{n/m} = (Ee^{isX(1)})^{n/m}.$$

This proves (10.3.9) for positive rational t .

For irrational $t > 0$, we can find a sequence (t_n) of rational numbers such that $t_n \downarrow t$. By the stationary increments and the stochastic continuity,

$$X(t_n) - X(t) \stackrel{d}{=} X(t_n - t) \xrightarrow{P} 0, \quad n \rightarrow \infty.$$

Then the independent increments yield for $s \in \mathbb{R}$,

$$\begin{aligned} (Ee^{isX(1)})^{t_n} &= Ee^{isX(t_n)} \\ &= Ee^{is(X(t_n) - X(t))} Ee^{isX(t)} \\ &\rightarrow Ee^{isX(t)}. \end{aligned}$$

On the other hand,

$$(Ee^{isX(1)})^{t_n} \rightarrow (Ee^{isX(1)})^t.$$

This concludes the proof of (10.3.9). \square

The class of infinitely divisible distributions has been studied for decades; see for example Gnedenko and Kolmogorov [57], Lukacs [97], Feller [51] or Sato [132]. One of the important classical results is the *Lévy-Khintchine formula* which can be found in any of the references mentioned. This formula yields a unique representation of the characteristic function of an infinitely divisible distribution.

Theorem 10.3.4 (Lévy-Khintchine formula)

Let X be a random variable with an infinitely divisible distribution. Then it has characteristic function

$$Ee^{isX} = \exp \left\{ i s c - \sigma^2 \frac{s^2}{2} - \int_{\mathbb{R}} (1 - e^{isx} + i s x I_{[0,1]}(|x|)) \mu(dx) \right\}, \quad (10.3.10)$$

where $c \in \mathbb{R}$, $\sigma \geq 0$ are constants and μ is a measure on \mathbb{R} satisfying the condition

$$\mu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min(x^2, 1) \mu(dx) < \infty. \quad (10.3.11)$$

The values c , σ and the measure μ determine the distribution of X in a unique way.

It is easily seen that condition (10.3.11) ensures that the integral in (10.3.10) is well-defined and finite. Moreover, (10.3.11) implies that μ is finite on the sets $\{x : |x| > \delta\}$ for any $\delta > 0$, whereas it is not excluded that

$$\mu(\{x : |x| \leq \delta\}) = \infty.$$

A measure μ on \mathbb{R} satisfying the additional condition (10.3.11) is called a *Lévy measure*. It will play a prominent role in the Lévy-Itô representation of a Lévy process given in Section 10.4. There it will have interpretation as mean measure of a PRM describing the magnitude of the jump sizes of a Lévy process per unit of time.

Combining Lemmas 10.3.1 and 10.3.3 with Theorem 10.3.4, we conclude the following result.

Corollary 10.3.5 (Lévy-Khintchine formula for Lévy process)

Let X be a Lévy process on $[0, \infty)$. Then there exist unique values $c \in \mathbb{R}$, $\sigma \geq 0$, and a unique Lévy measure μ on \mathbb{R} such that for every $t \geq 0$

$$\begin{aligned} Ee^{isX(t)} \\ = \exp \left\{ t \left[i s c - \sigma^2 \frac{s^2}{2} - \int_{\mathbb{R}} (1 - e^{isx} + i s x I_{[0,1]}(|x|)) \mu(dx) \right] \right\}, \quad s \in \mathbb{R}. \end{aligned}$$

Moreover, the triplet (c, σ, μ) characterizes the distribution of the Lévy process X in a unique way.

For obvious reasons, the triplet (c, σ, μ) is often called the *characteristic triplet* of the Lévy process X or of the infinitely divisible distribution of $X(1)$.

The Lévy-Khintchine representation has a straightforward interpretation. Let N be a standard normal random variable independent of an infinitely divisible random variable Y with triplet $(0, 0, \mu)$. Then X has decomposition

$$X \stackrel{d}{=} c + \sigma N + Y.$$

Indeed, it is easily verified that the characteristic function of the right-hand side coincides with (10.3.10). In Section 10.4 we will show that this decomposition has an important extension to Lévy processes: c corresponds to a linear drift term, σN to a Brownian motion independent of a pure jump Lévy process whose jumps are characterized by the Lévy measure.

Example 10.3.6 (Lévy-Khintchine formula for some infinitely divisible distributions)

In Example 10.3.2 we mentioned some well-known infinitely divisible distributions. Here we give their characteristic triplets; see also Table 10.3.7. The verification of these formulae is left as Exercise 2(b) on p. 347.

From the well-known characteristic function

$$\phi_X(s) = e^{ics - \sigma^2(s^2/2)}, \quad s \in \mathbb{R},$$

of a normal $N(c, \sigma^2)$ distribution we conclude that it has triplet (c, σ, o) , where o denotes the null measure on \mathbb{R} .

Recall the characteristic function of a symmetric stable distribution with parameters $\sigma > 0$ and $\alpha \in (0, 2)$ from (10.3.7). Direct calculation (which is tricky; see Resnick [124], p. 155) yields that its Lévy measure is given by

$$\mu_\alpha(x, \infty) = \mu_\alpha(-\infty, -x) = \frac{1}{2} \frac{\sigma^\alpha}{C_\alpha} x^{-\alpha}, \quad x > 0, \quad (10.3.12)$$

where

$$C_\alpha = \alpha \int_0^\infty (1 - \cos x) x^{-\alpha-1} dx. \quad (10.3.13)$$

Alternatively, μ_α has Lebesgue density

$$f_{\mu_\alpha}(x) = \frac{\alpha}{2} \frac{\sigma^\alpha}{C_\alpha} |x|^{-\alpha-1}, \quad x \neq 0.$$

The characteristic triplet is given by $(0, 0, \mu_\alpha)$.

A $\text{Pois}(\lambda)$, $\lambda > 0$, distributed random variable X has characteristic function

$$\phi_X(s) = e^{-\lambda(1-e^{is})}, \quad s \in \mathbb{R}.$$

Its Lévy measure $\mu_\lambda = \lambda \varepsilon_1$ is degenerate at $x = 1$. It has characteristic triplet $(\lambda, 0, \mu_\lambda)$.

A compound Poisson sum $S = \sum_{i=1}^M Z_i$ has the well-known characteristic function

$$\begin{aligned} \phi_S(s) &= \exp \{-\lambda (1 - E e^{is Z_1})\} \\ &= \exp \left\{ -\lambda \int_{\mathbb{R}} (1 - e^{is x}) F(dx) \right\}, \quad s \in \mathbb{R}. \end{aligned}$$

Here M is $\text{Pois}(\lambda)$ distributed for some $\lambda > 0$, independent of the iid summands Z_i with common distribution F on \mathbb{R} . As before, we write $\text{CP}(\lambda, F)$ for the distribution of S . The corresponding characteristic triplet is given by $(\lambda \int_{\{x:|x|\leq 1\}} x F(dx), 0, \lambda F)$.

The gamma $\Gamma(\alpha, \beta)$ distribution has characteristic function (a derivation is given in Sato [132], Example 8.10)

$$\phi_X(s) = (1 - i s / \beta)^{-\alpha} = \exp \left\{ -\alpha \int_0^\infty (1 - e^{i s x}) x^{-1} e^{-\beta x} dx \right\}, \quad s \in \mathbb{R},$$

with corresponding triplet $((\alpha/\beta)(1 - e^{-\beta}), 0, \mu_\gamma)$, and the Lévy measure μ_γ has Lebesgue density

$$f_{\mu_\gamma}(x) = \alpha x^{-1} e^{-\beta x}, \quad x > 0. \quad (10.3.14)$$

□

Distribution of $X(1)$	Parameters	Triplet
Degenerate at c	$c \in \mathbb{R}$	$(c, 0, o)$
Normal $N(c, \sigma^2)$	$c \in \mathbb{R}, \sigma > 0$	(c, σ, o)
Symmetric stable, see (10.3.7)	$\alpha \in (0, 2),$ $\sigma > 0$	$(0, 0, \mu_\alpha),$ see (10.3.12)
Poisson $\text{Pois}(\lambda)$	$\lambda > 0$	$(\lambda, 0, \lambda \varepsilon_1)$
Compound Poisson $\text{CP}(\lambda, F)$,	$\lambda > 0,$ F distribution on \mathbb{R}	$\lambda \left(\int_{\{x: x \leq 1\}} x F(dx), 0, F \right)$
Gamma $\Gamma(\alpha, \beta)$	$\alpha > 0,$ $\beta > 0$	$((\alpha/\beta)(1 - e^{-\beta}), 0, \mu_\gamma),$ see (10.3.14)

Table 10.3.7 Characteristic triplets of some Lévy processes, cf. Example 10.3.6.

Comments

In Example 10.3.2 we introduced the α -stable distributions with $\alpha < 2$ via their characteristic functions. These distributions have densities which in

general can only be written as slowly converging series and there are explicit formulae only in a small number of cases. The Cauchy distribution is a real exception: it is the student distribution with one degree of freedom with density $f(x) = \pi^{-1}(1 + x^2)^{-1}$, $x \in \mathbb{R}$.

Infinite divisibility of specific classes of distributions has been studied in the monographs Linnik [94], Lukacs [97, 98], Linnik and Ostrovskii [95], Feller [51], Bondesson [21], Steutel and van Harn [139].

Exercises

- (1) Let X be a Lévy process on $[0, \infty)$. Show that for any $t \geq 0$, the distribution of $X(t)$ is infinitely divisible.
- (2) (a) Verify that the symmetric stable (including the normal case), the gamma and the Poisson distributions are infinitely divisible, cf. Example 10.3.2. Show that a compound Poisson sum is infinitely divisible.
 (b) Verify the form of the characteristic triplets of the infinitely divisible distributions from (a) as indicated in Example 10.3.6 and Table 10.3.7.
- (3) Show that a non-degenerate infinitely divisible distribution has infinite support; cf. Sato [132], Section 5.24.
- (4) Consider a Lévy process X on $[0, \infty)$. Show that the finite-dimensional distributions of X are determined by the one-dimensional marginal distributions:
 (a) Determine the joint characteristic function of a vector $(X(t_1), \dots, X(t_n))$ for any $0 \leq t_1 < \dots < t_n$ and $n \geq 1$. Show that this characteristic function only depends on the characteristic function of $X(1)$; see Lemma 10.3.3.
 (b) Show that the distribution of a Lévy process X is completely determined by the distribution of $X(t)$ for an arbitrary value $t > 0$.
- (5) Consider a Lévy process X on $[0, 1]$. A discrete-time approximation of the sample paths of this process is given by

$$X^{(n)}(t) = \sum_{i=1}^{[nt]} X((i-1)/n, i/n), \quad t \in [0, 1], \quad n \geq 1.$$

This construction yields a method for simulating sample paths of X given one knows the distribution of the iid increments $X((i-1)/n, i/n)$, $i = 1, \dots, n$. In general, the distribution of these increments is unknown.

- (a) Consider the Lévy processes in Table 10.3.7 and decide whether one can determine the distribution of their iid increments $X((i-1)/n, i/n)$.
- (b) Simulate sample paths $X^{(n)}$ for the Lévy processes in Table 10.3.7 for which one knows the distribution of their increments $X((i-1)/n, i/n)$.
- (c) Argue that $X^{(n)}$ is not a Lévy process on $[0, 1]$.
- (d) Show that the finite-dimensional distributions of the approximation $X^{(n)}$ converge to those of X as $n \rightarrow \infty$, i.e., for any $0 < t_1 < \dots < t_m \leq 1$ and $m \geq 1$,

$$(X^{(n)}(t_i))_{i=1,\dots,m} \xrightarrow{d} (X(t_i))_{i=1,\dots,m}.$$

10.4 The Lévy-Itô Representation of a Lévy Process

A general Lévy jump process can be constructed as an a.s. limit of Poisson integrals which have representation as compound Poisson processes as in (10.1.2). To make this statement precise, we proceed in several construction steps.

In what follows, N is PRM($\text{Leb} \times \mu$) on the state space $[0, \infty) \times \mathbb{R}$, where μ is a measure on \mathbb{R} such that $\text{Leb} \times \mu$ is Radon. Later we will require that μ is a Lévy measure, but for the moment we keep μ as general as possible.

Lemma 10.4.1 *Assume that N is PRM($\text{Leb} \times \mu$) on the state space $[0, \infty) \times \mathbb{R}$ for some measure μ on \mathbb{R} . Let $B \subset \mathbb{R}$ be a Borel set such that $0 < \mu(B) < \infty$. Then the process*

$$S_B(t) = \int_{[0,t] \times B} x N(ds, dx), \quad t \geq 0, \quad (10.4.15)$$

defines a compound Poisson process on $[0, \infty)$. In particular, it is a Lévy process.

Proof. In view of Lemma 7.2.9 the Poisson integral $S_B(t)$ is finite with probability 1 if and only if

$$\begin{aligned} & \int_{[0,t] \times B} \min(|x|, 1) ds \mu(dx) \\ &= t \mu(\{x : |x| > 1\} \cap B) + t \int_{\{x : |x| \leq 1\} \cap B} |x| \mu(dx) < \infty. \end{aligned}$$

This condition is trivially satisfied since the right-hand side is bounded by $t \mu(B) < \infty$. Therefore the process S_B is well-defined on $[0, \infty)$.

Now introduce the point process

$$N_B(\cdot) = N(\cdot \cap ([0, \infty) \times B)).$$

According to Example 7.2.3, N_B is PRM($\text{Leb} \times \mu_B$), where μ_B is the restriction of μ to B . The measure

$$F_B(\cdot) = \frac{\mu(\cdot \cap B)}{\mu(B)} = \frac{\mu_B(\cdot)}{\mu(B)}$$

is a probability distribution on B . Moreover, the mean measure of N_B can be written in the form

$$\text{Leb} \times \mu_B = \mu(B) \text{Leb} \times F_B. \quad (10.4.16)$$

This is the mean measure of a marked PRM and therefore N_B has representation as a marked PRM. The process S_B can be written in the form

$$S_B(t) = \int_{[0,t] \times \mathbb{R}} x N_B(ds, dx), \quad t \geq 0.$$

In view of Example 10.1.2, S_B is a compound Poisson process. \square

Corollary 10.4.2 *Let μ be a measure on \mathbb{R} such that*

$$\mu(\{x : |x| > \delta\}) < \infty \quad \text{for some } \delta \in (0, 1). \quad (10.4.17)$$

Then the processes

$$\begin{aligned} S_{\{x:|x|>1\}}(t) &= \int_{[0,t] \times \{x:|x|>1\}} x N(ds, dx), \quad t \geq 0, \\ S_{\{x:\delta<|x|\leq 1\}}(t) &= \int_{[0,t] \times \{x:\delta<|x|\leq 1\}} x N(ds, dx), \quad t \geq 0, \end{aligned}$$

are independent compound Poisson processes on $[0, \infty)$.

The independence of the processes $S_{\{x:|x|>1\}}$ and $S_{\{x:\delta<|x|\leq 1\}}$ follows from the fact that they have Poisson integral representations defined on disjoint subsets of the state space underlying the point process N .

For any constant c , $f(t) = ct$, $t \geq 0$, is a Lévy process. In particular, we can choose the constant

$$c_\delta = E(S_{\{x:\delta<|x|\leq 1\}}(1)) = \int_{\{x:\delta<|x|\leq 1\}} x \mu(dx).$$

The last equality follows from (7.2.19). By the same argument,

$$E(S_{\{x:\delta<|x|\leq 1\}}(t)) = c_\delta t, \quad t \geq 0.$$

We conclude from Lemma 10.2.1(1) that the sum of the two independent Lévy processes

$$\begin{aligned} S_{\{x:\delta<|x|\}}(t) - c_\delta t \\ = \left[S_{\{x:\delta<|x|\leq 1\}}(t) - t \int_{\{x:\delta<|x|\leq 1\}} x \mu(dx) \right] + S_{\{x:|x|>1\}}(t), \quad t \geq 0, \end{aligned}$$

constitutes a Lévy process, provided condition (10.4.17) holds.

In a final step we want to let $\delta \downarrow 0$. This limit relation requires some additional conditions on the measure μ , namely that μ is a Lévy measure.

Theorem 10.4.3 (Lévy-Itô representation of a pure jump Lévy process² on $[0, \infty)$)

² See p. 354 for some insight into the notion *pure jump Lévy process*.

Assume that N is PRM($\text{Leb} \times \mu$) on the state space $[0, \infty) \times \mathbb{R}$ with a Lévy measure μ on \mathbb{R} , i.e., it satisfies

$$\mu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min(x^2, 1) \mu(dx) < \infty. \quad (10.4.18)$$

(1) *The limit process*

$$\begin{aligned} S(t) &= \lim_{\delta \downarrow 0} [S_{\{x: \delta < |x|\}}(t) - c_{\delta} t] \\ &= \lim_{\delta \downarrow 0} \left[\int_{[0,t] \times \{x: \delta < |x| \leq 1\}} x N(ds, dx) - t \int_{\{x: \delta < |x| \leq 1\}} x \mu(dx) \right] \\ &\quad + \int_{[0,t] \times \{x: |x| > 1\}} x N(ds, dx), \quad t \geq 0, \end{aligned} \quad (10.4.19)$$

exists for every $t \geq 0$ with probability 1.

(2) *The process S defined in (10.4.19) is a Lévy process with characteristic function at $t \geq 0$ given by*

$$E e^{isS(t)} = \exp \left\{ -t \int_{\mathbb{R}} (1 - e^{isx} + isx I_{[0,1]}(|x|)) \mu(dx) \right\}, \quad s \in \mathbb{R}. \quad (10.4.20)$$

Condition (10.4.18) is equivalent to the conditions $\mu(\{0\}) = 0$,

$$\int_{\{x: 0 < |x| \leq 1\}} x^2 \mu(dx) < \infty \quad \text{and} \quad \mu(\{x: |x| > 1\}) < \infty. \quad (10.4.21)$$

This condition implies in particular that

$$\mu(\{x: |x| > \delta\}) < \infty \quad \text{for every } \delta > 0.$$

It is however not excluded that $\mu([-\delta, \delta]) = \infty$ for any $\delta > 0$. If the latter condition holds $\text{Leb} \times \mu$ is not Radon on $[0, \infty) \times \mathbb{R}$ and therefore one has to make precise the meaning of a PRM with this mean measure. This goal is achieved by modifying the state space. Since μ has no mass at zero we simply exclude this point from the state space and consider $\text{Leb} \times \mu$ on $[0, \infty) \times \mathbb{R} \setminus \{0\}$. Finally, one needs to change the topology of $\mathbb{R} \setminus \{0\}$ in such a way that the roles of 0 and $\pm\infty$ are interchanged. Then bounded sets are those which are bounded away from zero, i.e., they are contained in a set $\{x: |x| > \delta\}$ for some $\delta > 0$. These sets have finite μ -measure. With these modifications, N is a PRM with Radon mean measure $\text{Leb} \times \mu$. We refer to Resnick [124] for details of this construction. In what follows, we understand N as a PRM in the sense explained above.

Proof. (1) Let $(\delta_k)_{k \geq 0}$ be any monotone decreasing sequence of positive numbers such that $\delta_0 = 1$ and $\delta_k \downarrow 0$. Consider the mutually independent centered Poisson integrals (the integrands have disjoint support)

$$Z_k = \int_{[0,t] \times \{x: \delta_k < |x| \leq \delta_{k-1}\}} x N(ds, dx) - t \int_{\{x: \delta_k < |x| \leq \delta_{k-1}\}} x \mu(dx).$$

The limit (10.4.19) exists a.s. if the infinite series

$$\begin{aligned} \sum_{k=1}^{\infty} Z_k &= \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n Z_k \right] \\ &= \lim_{n \rightarrow \infty} \left[\int_{[0,t] \times \{x: \delta_n < |x| \leq 1\}} x N(ds, dx) - t \int_{\{x: \delta_n < |x| \leq 1\}} x \mu(dx) \right] \end{aligned}$$

converges a.s. By the three-series theorem (see for example Feller [51], Petrov [116] or Billingsley [18]), it suffices to verify that

$$\begin{aligned} \sum_{k=1}^{\infty} \text{var}(Z_k) &= \sum_{k=1}^{\infty} t \int_{\{x: \delta_k < |x| \leq \delta_{k-1}\}} x^2 \mu(dx) \\ &= t \int_{\{x: 0 < |x| \leq 1\}} x^2 \mu(dx) < \infty, \end{aligned}$$

where the formula for $\text{var}(Z_k)$ follows from (7.2.21). The right-hand side is finite by virtue of condition (10.4.18); cf. (10.4.21).

(2) It follows from Lemma 10.2.1 and the fact that a.s. convergence for every t implies convergence in distribution in the sense of the finite-dimensional distributions that S has stationary and independent increments and $S(0) = 0$ with probability 1.

Stochastic continuity of S follows from the stationary independent increments of S and the characteristic function (10.4.20) (a derivation is given next) which implies that for $s \in \mathbb{R}$,

$$E e^{isS(t)} = (E e^{isS(1)})^t \rightarrow 1, \quad t \downarrow 0,$$

hence $S(t) \xrightarrow{P} 0$ as $t \downarrow 0$.

It remains to show (10.4.20). Since the integrands of the Poisson integrals $S_{\{x: \delta < |x| \leq 1\}}(t)$ and $S_{\{x: |x| > 1\}}(t)$ have disjoint support they are independent for every $\delta \in (0, 1)$. Since these Poisson integrals have compound Poisson representation it follows that

$$\begin{aligned}
& E \exp \left\{ i s \left(S_{\{x:\delta<|x|\leq 1\}}(t) - c_\delta t + S_{\{x:|x|>1\}}(t) \right) \right\} \\
&= E \exp \left\{ i s \left(S_{\{x:\delta<|x|\leq 1\}}(t) - c_\delta t \right) \right\} E \exp \left\{ i s S_{\{x:|x|>1\}}(t) \right\} \\
&= E \exp \left\{ -t \int_{\{x:\delta<|x|\leq 1\}} (1 - e^{isx} + isx) \mu(dx) \right. \\
&\quad \left. -t \int_{\{x:|x|>1\}} (1 - e^{isx}) \mu(dx) \right\}.
\end{aligned}$$

Letting $\delta \downarrow 0$, we obtain the desired limiting characteristic function of $S(t)$ given in (10.4.20). \square

We proved that the process S in (10.4.19) exists in a pointwise sense for every fixed $t > 0$. It is possible (and also more rigorous from a mathematical point of view) to show that $S(t)$ converges uniformly with probability 1 on compact t -intervals; see Sato [132], Sections 4.19 and 4.20. Among other things, uniform convergence ensures that the càdlàg property of the limit S follows from càdlàguity of the converging processes.

The construction of the process S is rather arbitrary as regards the partition of the support of the measure μ into the disjoint subsets $\{x : |x| > 1\}$ and $\{x : 0 < |x| \leq 1\}$. In the steps of the proof it has been indicated that there exist alternative representations of S . Start from a Borel set $B \subset \mathbb{R}$ such that $0 < \mu(B) < \infty$ and define S_B as in (10.4.15). Then consider an increasing sequence of subsets $B_n \subset \mathbb{R} \setminus B$ such that $B_n \uparrow \mathbb{R} \setminus B$, $\mu(B_n) < \infty$ and $\int_{B_n} |x| \mu(dx) < \infty$ for all n . Under the condition (10.4.18), the pointwise limits

$$\tilde{S}(t) = \lim_{n \rightarrow \infty} [S_{B_n}(t) - ES_{B_n}(t)] + S_B(t), \quad (10.4.22)$$

exist and are finite with probability 1. In particular, one can choose

$$B = \{x : \delta < |x|\} \quad \text{for any } \delta > 0.$$

One can follow the lines of the proofs above. The reader is encouraged to give a detailed proof; see Exercise 1 on p. 355.

The meaning of the equivalent conditions (10.4.18) and (10.4.21) is illuminated in the proof of Theorem 10.4.3. Finiteness of μ on the sets $\{x : |x| > \delta\}$ ensures that the Poisson integral process $S_{\{x:|x|>\delta\}}$ is finite a.s. for every $\delta > 0$; see Lemma 7.2.9. Finiteness of the integral $\int_{\{x:0<|x|\leq 1\}} x^2 \mu(dx)$ ensures the a.s. convergence of the centered Poisson integrals

$$\int_{[0,t] \times \{x:\delta<|x|\leq 1\}} x N(ds, dx) - t \int_{\{x:\delta<|x|\leq 1\}} x \mu(dx) \quad (10.4.23)$$

to a finite limit as $\delta \downarrow 0$. Without centering these Poisson integrals will in general not converge a.s.; see the three-series argument on p. 351.

However, if the condition

$$\mu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min(|x|, 1) \mu(dx) < \infty \quad (10.4.24)$$

holds then the centering constants in (10.4.23) converge to the finite constant $t \int_{\{x:0<|x|\leq 1\}} x \mu(dx)$ as $\delta \downarrow 0$. Therefore centering can be avoided under the assumption (10.4.24) which is stronger than condition (10.4.18). The reader is encouraged to modify the proof of Theorem 10.4.3 and to prove the following result.

Proposition 10.4.4 (Lévy-Itô representation of a pure jump Lévy process without centering)

Assume that N is PRM(Leb \times μ) on $[0, \infty) \times \mathbb{R}$ with a measure μ satisfying (10.4.24). Then the following statements hold.

(1) *The Poisson integral process*

$$S(t) = \int_{[0,t] \times \mathbb{R}} x N(ds, dx) \quad (10.4.25)$$

is finite for every $t \geq 0$ with probability 1.

(2) *The process S defined in (10.4.25) is a Lévy process with characteristic function at $t \geq 0$ given by*

$$E e^{i s S(t)} = \exp \left\{ -t \int_{\mathbb{R}} (1 - e^{i s x}) \mu(dx) \right\}, \quad s \in \mathbb{R}. \quad (10.4.26)$$

We are now in the position to give the Lévy-Itô representation of a general Lévy process.

Theorem 10.4.5 (Lévy-Itô representation of a general Lévy process)

(1) *Any Lévy process X on $[0, \infty)$ has representation*

$$X(t) = ct + \sigma B(t) + S(t), \quad t \geq 0, \quad (10.4.27)$$

where B is standard Brownian motion independent of the process S and $c \in \mathbb{R}$, $\sigma \geq 0$ are constants. The process S has integral representation (10.4.19) with respect to PRM(Leb \times μ), where μ is a Lévy measure on \mathbb{R} , i.e., it satisfies (10.4.18).

(2) *The decomposition (10.4.27) is uniquely determined by the characteristic triplet (c, σ, μ) of the infinitely divisible distribution of $X(1)$ arising in the Lévy-Khintchine formula (10.3.10).*

Proof. We know from Corollary 10.3.5 that every Lévy process X is uniquely determined by its Lévy-Khintchine formula and the corresponding characteristic triplet (c, σ, μ) . The form of the characteristic function of $ct + \sigma B(t)$, the independence of B and S , and Theorem 10.4.3 yield that the process X

in (10.4.27) has Lévy-Khintchine formula with triplet (c, σ, μ) . \square

The Brownian component in the decomposition (10.4.27) has continuous sample paths with probability 1, whereas the process S has representation as a.s. limit of compound Poisson processes. The resulting process S has discontinuous sample paths with probability 1. This fact justifies the name *pure jump process* for S .

The Lévy measure μ is sometimes referred to as *jump measure*. This name is motivated by the Lévy-Itô representation of a pure jump Lévy process given in (10.4.19). Indeed, assuming the representation

$$N = \sum_{i=1}^{\infty} \varepsilon_{(T_i, \Delta_i)}$$

of the PRM N , the Lévy-Itô representation of the pure jump Lévy process S turns into

$$\begin{aligned} S(t) &= \lim_{\delta \downarrow 0} \left[\sum_{i:T_i \leq t, \delta < |\Delta_i| \leq 1} \Delta_i - t \int_{\{x: \delta < |x| \leq 1\}} x \mu(dx) \right] \\ &\quad + \sum_{i:T_i \leq t, |\Delta_i| > 1} \Delta_i, \quad t \geq 0. \end{aligned} \tag{10.4.28}$$

Each of the sums in this representation has finitely many summands since they have compound Poisson representation. The value Δ_i can be interpreted as the jump size of the process S at the jump time T_i . Therefore the Lévy-Itô representation of S divides the jumps of the process into large ones (with size $|\Delta_i| > 1$) and small ones (with size $|\Delta_i| \leq 1$). The PRM property of N ensures that the two resulting sums of jump sizes are independent.

A Lévy process with $\mu(\mathbb{R}) = \infty$ has infinitely many jumps in any interval $[0, t]$. This follows for example from the argument in the proof of Lemma 10.4.1. An inspection of the mean measure (10.4.16) shows that

$$\int_{[0,t] \times \{x: \delta < |x| \leq 1\}} x N(ds, dx) = \sum_{i:T_i \leq t, \delta < |\Delta_i| \leq 1} \Delta_i \tag{10.4.29}$$

has compound Poisson CP($t \lambda_\delta, F_\delta$) representation with parameters

$$\lambda_\delta = \mu(\{x : \delta < |x| \leq 1\}) \quad \text{and} \quad F_\delta(\cdot) = \frac{\mu(\cdot \cap \{x : \delta < |x| \leq 1\})}{\mu(\{x : \delta < |x| \leq 1\})}.$$

For fixed $\delta > 0$, (10.4.29) defines a compound Poisson process whose intensity λ_δ increases to infinity when $\delta \downarrow 0$.

The heuristic argument above can be made precise. In addition, one can show that the infinitely many jumps of a Lévy process with $\mu(\mathbb{R}) = \infty$ are countable and dense in $[0, \infty)$; see Sato [132], Theorem 21.3.

Exercises

- (1) Assume condition (10.4.18). Give a proof of the representation of the pure jump Lévy process \tilde{S} given in (10.4.22), where the sequence of Borel sets (B_n) satisfies the conditions described before (10.4.22). Derive the characteristic function of $\tilde{S}(t)$.
- (2) Prove Proposition 10.4.4 by following the hints in the discussion before the formulation of the proposition.
- (3) Prove the Lévy-Itô representation for a d -dimensional Lévy process X , i.e., $X(t)$ assumes values in \mathbb{R}^d .
- (4) Determine the characteristic triplet from the characteristic function (10.4.26).

10.5 Some Special Lévy Processes

In this section we consider some special Lévy processes and consequences of the Lévy-Itô decomposition. We start with processes which have positive jumps, the so-called spectrally positive Lévy processes.

Example 10.5.1 (Subordinators and spectrally positive Lévy processes)
The total claim amount process in the Cramér-Lundberg model has representation

$$S(t) = \int_{[0,t] \times (0,\infty)} x N(ds, dx), \quad t \geq 0,$$

where N is PRM(Leb $\times F$) on $(0, \infty)^2$ and F is the claim size distribution on $(0, \infty)$. This process jumps upward at the Poisson claim arrivals and the jump sizes are the corresponding positive claim sizes. Such a process is called a *subordinator*. Its sample paths are non-decreasing with probability 1. Consider the process $S(t) - ct$ with some positive premium rate c . This process has positive jumps at the Poisson arrival times, but it is linear with a negative slope between two arrivals. Such a process is called *spectrally positive*.

We have learned on p. 354 that the Lévy measure has interpretation as jump measure of the underlying Lévy jump process. If $\mu(-\infty, 0) = 0$ the Lévy process does not have upward jumps. It is called a *spectrally positive Lévy process*. Similarly, if $\mu(0, \infty) = 0$, the Lévy process is *spectrally negative*, i.e., it does not have positive jumps. A spectrally positive process is called a *subordinator* if it has non-decreasing sample paths with probability 1.

A glance at the two independent parts of the Lévy-Itô decomposition (10.4.28) shows that a pure jump Lévy process has non-decreasing sample paths with probability 1 if and only if the Poisson integral of the small jumps does not need centering and the jump sizes Δ_i are non-negative. Hence, in addition to the condition $\mu(-\infty, 0) = 0$ on the Lévy measure μ one needs to ensure that centering is not necessary. According to the argument leading to Proposition 10.4.4 we have to require the additional condition

$$\int_{(0,\infty)} \min(x, 1) \mu(dx) < \infty. \quad (10.5.30)$$

Under these conditions a subordinator is given by the a.s. converging Poisson integrals

$$\int_{[0,t] \times (0,\infty)} x N(ds, dx) = \sum_{i \geq 1: T_i \leq t} \Delta_i.$$

Here we used the notation introduced on p. 354, where the pair (T_i, Δ_i) stands for the jump time T_i and the corresponding jump size $\Delta_i > 0$.

Simple examples of subordinators are the homogeneous Poisson process and the compound Poisson process with claim size distribution supported on $(0, \infty)$. Their Lévy measures are finite on \mathbb{R} . Examples with infinite Lévy measures are the α -stable subordinators for $\alpha < 1$ (see Example 10.5.2) and the gamma process. This process has Lévy measure μ_γ derived from the gamma distribution; see Example 10.3.6 and Table 10.3.7. Direct calculation shows that the condition (10.5.30) on the Lévy measure μ_γ is satisfied. We refer to Figure 10.5.3 for the visualization of a gamma subordinator and Figure 10.1.3 for a centered gamma process which represents a spectrally positive Lévy process. \square

Example 10.5.2 (Stable Lévy processes)

Recall the notion of a symmetric α -stable distribution for $\alpha \in (0, 2]$ from

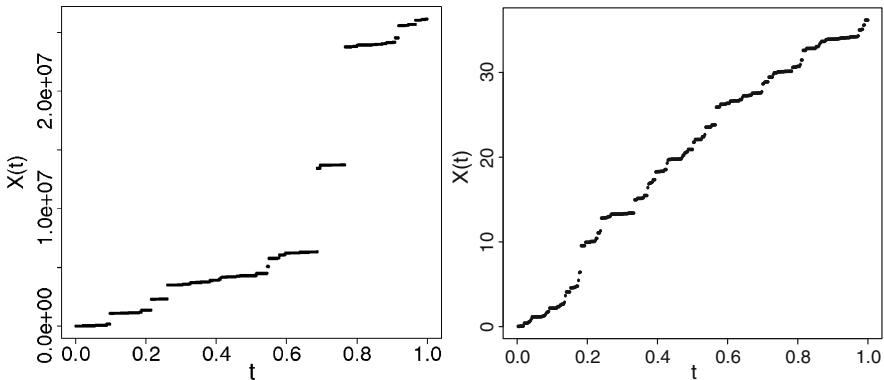


Figure 10.5.3 Visualization of some subordinators. Left: 0.5-stable subordinator; see Example 10.5.2. Right: Gamma subordinator; see p. 356. In contrast to the gamma process whose moments $E(|X(t)|^r)$ are finite for any $t > 0$ and $r > 0$, the 0.5-stable subordinator has a tail of the form $P(X(t) > x) \sim c(t) x^{-0.5}$, $x \rightarrow \infty$, for some constant $c(t) > 0$. This tail behavior is responsible for the extremely large jumps of X in the left graph.

(10.3.7), given via its characteristic function. The corresponding Lévy measure μ_α for $\alpha \in (0, 2)$ is provided in (10.3.12); the case $\alpha = 2$ corresponds to Brownian motion. The resulting Lévy process X is called a *symmetric α -stable Lévy process* or *symmetric α -stable Lévy motion*. The case $\alpha = 1$ is referred to as *Cauchy process* because $X(t)$ is Cauchy distributed; see Example 10.3.2. In Figure 10.1.3 a sample path of the Cauchy process is shown.

Due to the symmetry of the distribution of $X(t)$, $t \geq 0$, and the independent increments it is easily seen that $X \stackrel{d}{=} -X$ in the sense of the finite-dimensional distributions. In what follows, we exclude the case $\alpha = 2$ (Brownian motion) which has the null Lévy measure and a version with continuous sample paths.

In the case $\alpha < 2$, the condition of symmetry of the marginal distributions can be relaxed. Starting with the Lévy measure

$$\mu_\alpha(-\infty, -x] = q C x^{-\alpha} \quad \text{and} \quad \mu_\alpha(x, \infty) = p C x^{-\alpha}, \quad x > 0, \quad (10.5.31)$$

with constants $p, q \geq 0$, $p + q = 1$, $C > 0$, and plugging μ_α into the Lévy-Khintchine formula (10.3.10), one can identify the distribution of $X(1)$ via its characteristic function. In turn, the distribution of $X(1)$ determines the finite-dimensional distributions of the *α -stable Lévy motion* X with Lévy measure μ_α given in (10.5.31), cf. Lemma 10.3.3.

Calculation yields the following characteristic function which is often referred to as the *spectral representation of an α -stable distribution*:

$$E e^{i s X(1)} = \exp \{i c s - \sigma^\alpha |s|^\alpha (1 - i \beta \operatorname{sign}(s) z(s, \alpha))\}, \quad s \in \mathbb{R}, \quad (10.5.32)$$

where $\sigma > 0$, $\alpha \in (0, 2)$, $\beta \in [-1, 1]$, $c \in \mathbb{R}$, are constants and

$$z(s, \alpha) = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right) & \text{if } \alpha \neq 1, \\ -\frac{2}{\pi} \log |s| & \text{if } \alpha = 1. \end{cases}$$

The parameter σ is a scaling constant, β describes the skewness, c the location, and α is a shape parameter which characterizes the distribution up to skewness and scale. The case $\beta = 0$ yields a real-valued characteristic function, hence $X(1)$ is symmetric. This case corresponds to $p = q = 0.5$ in (10.5.31).

In general, the parameters of the Lévy measure μ_α and the characteristic function (10.5.32) are related as follows:

$$C = \frac{1}{2} \frac{\sigma^\alpha}{C_\alpha}, \quad p = \frac{1 + \beta}{2} \quad \text{and} \quad q = \frac{1 - \beta}{2}, \quad (10.5.33)$$

where C_α is defined in (10.3.13). The reader who is interested in verifying the equivalence of the Lévy-Khintchine formula with Lévy measure (10.5.31) and

(10.5.32) is referred to Sato [132], Section 14, Gnedenko and Kolmogorov [57], Feller [51] or Samorodnitsky and Taqqu [131].

From (10.5.33) and the form of the Lévy measure we see that $\mu_\alpha(-\infty, 0) = 0$ whenever $q = 0$ (or $\beta = 1$). In this case, we learned in Example 10.5.1 that X is spectrally positive, i.e., the process X jumps only upward. For X to be a subordinator we further needed the condition

$$\int_{(0,\infty)} \min(x, 1) \mu_\alpha(dx) = \alpha C \int_{(0,1]} x^{-\alpha} dx + \mu_\alpha(1, \infty) < \infty.$$

This means that only in the case $\alpha < 1$ can one define an α -stable subordinator, i.e., an α -stable Lévy process with non-decreasing sample paths. See Figure 10.5.3 for a realization of a 0.5-stable subordinator. In comparison to the gamma subordinator in the same figure the jump sizes in the stable case are much larger. This observation is due to the fact that $X(1)$ does not have finite moments of order $r \geq 0.5$ in contrast to the gamma case where $X(1)$ has moments of any order.

The name α -stability refers to the following *stability property*. Let X, X_1, X_2, \dots , be iid random variables with characteristic function (10.5.32). Then it is not difficult to see that for every $n \geq 1$ there exist real constants d_n such that

$$X \stackrel{d}{=} n^{-1/\alpha} (X_1 + \cdots + X_n - d_n). \quad (10.5.34)$$

The constants d_n can be chosen to be zero for every $n \geq 1$ in the following situations:

- $\alpha < 1$ and $c = 0$ in (10.5.32),
- $\alpha = 1$, $c = 0$ and $\beta = 0$, i.e., X is symmetric,
- $\alpha \in (1, 2)$ and $EX = 0$.

In particular, $d_n = 0$ for symmetric X for any $\alpha \in (0, 2)$. An α -stable distribution satisfying (10.5.34) with $d_n = 0$ for every $n \geq 1$, is said to be *strictly stable*; see Sato [132], Section 14, for a complete characterization of the strictly stable distributions.

Symmetric α -stable Lévy motion enjoys the *self-similarity property*:

$$c^{1/\alpha} X(\cdot) \stackrel{d}{=} X(c \cdot) \quad \text{for any } c > 0, \quad (10.5.35)$$

where $\stackrel{d}{=}$ denotes equality of the finite-dimensional distributions. The process X is said to be $1/\alpha$ -self-similar. Notice that self-similarity is shared with Brownian motion which is 0.5-self-similar. Self-similarity is particularly useful for simulations: it suffices to simulate a self-similar process on the interval $[0, 1]$, say. Then a sample path of the same process on $[0, T]$ for any $T > 0$ is obtained by scaling the values of the process with $T^{1/\alpha}$ and rescaling time by the factor T . We leave the verification of (10.5.35) as Exercise 2(a) on p. 361. \square

Example 10.5.4 (Representation of a Lévy process as difference of two spectrally positive Lévy processes)

Recall the Lévy-Itô representation of a pure jump Lévy process S from (10.4.19) which we rewrite in a slightly different equivalent form:

$$S(t) = S_+(t) - S_-(t), \quad t \geq 0, \quad (10.5.36)$$

where

$$\begin{aligned} S_+(t) &= \lim_{\delta \downarrow 0} \left[\int_{[0,t] \times \{x: \delta < x \leq 1\}} x N(ds, dx) - t \int_{\{x: \delta < x \leq 1\}} x \mu(dx) \right] \\ &\quad + \int_{[0,t] \times \{x: x > 1\}} x N(ds, dx), \\ -S_-(t) &= \lim_{\delta \downarrow 0} \left[\int_{[0,t] \times \{x: -1 \leq x < -\delta\}} x N(ds, dx) - t \int_{\{x: -1 \leq x < -\delta\}} x \mu(dx) \right] \\ &\quad + \int_{[0,t] \times \{x: x < -1\}} x N(ds, dx). \end{aligned}$$

The justification of this decomposition is left as Exercise 3 on p. 361. From their definition, the processes S_{\pm} are defined on disjoint parts of the state space underlying the PRM N , hence the two processes are independent. According to the Lévy-Itô representation, both S_+ and S_- represent Lévy processes with Lévy measures supported on $(0, \infty)$. Hence S_{\pm} are spectrally positive Lévy processes; see Example 10.5.1. This means that any Lévy process S can be written as the difference of two independent spectrally positive Lévy processes S_{\pm} .

If the Lévy measure μ satisfies the additional condition (10.4.24) we learned in Proposition 10.4.4(1) that centering in the Lévy-Itô representation can be avoided, and the same arguments as above lead to the representation

$$\begin{aligned} S(t) &= \int_{[0,t] \times (0,\infty)} x N(ds, dx) - \left(- \int_{[0,t] \times (-\infty,0)} x N(ds, dx) \right) \\ &= \tilde{S}_+(t) - \tilde{S}_-(t), \quad t \geq 0, \end{aligned}$$

where both \tilde{S}_+ and \tilde{S}_- are subordinators. This means in particular that the sample paths of S , with probability 1, have bounded variation on compact t -intervals. Any α -stable Lévy motion with $\alpha < 1$ can be written in this form, where \tilde{S}_+ and \tilde{S}_- are α -stable subordinators; see Example 10.5.2. \square

Comments

Lévy processes constitute a major class of stochastic processes. In view of their stationary and independent increments they can be understood as continuous-time analogs of random walks. In contrast to a random walk, the marginal

distributions are restricted to the class of infinitely divisible distributions; see Lemma 10.3.1.

The beginning of the studies of infinitely divisible distributions and Lévy processes is closely related with some of the major names in probability theory of the first half of the 20th century, including Khintchine, Kolmogorov, Lévy, Feller. The development of the theory of infinitely divisible distributions is summarized in monographs such as Gnedenko and Kolmogorov [57], Linnik [94], Feller [51], Lukacs [97, 98], Linnik and Ostrovskii [95], Petrov [116, 117], Steutel and van Harn [139]. An early monograph dedicated to Lévy processes is Skorokhod [137]. Texts such as Gihman and Skorohod [54, 55], Jacod and Shiryaev [74], Liptser and Shiryaev [96] treat Lévy processes as major building blocks of martingales, local martingales, semimartingales and diffusions.

More recently, over the last 15 years it has been realized that the general classes of semimartingales and diffusion processes are not necessarily best suited for applications and that it can be useful to consider particular classes of processes. Since the end of the 1990s, monographs such as Bertoin [15] and Sato [132] summarized the classical theory and introduced modern concepts of Lévy processes in an accessible way. Among other things, they worked out the links between Lévy jump processes and point process theory. Since then various books on Lévy processes and related areas have reached the market at different levels of mathematical sophistication and have been written for different audiences; see among others Barndorff-Nielsen et al. [12], Applebaum [4], Kyprianou [90], Cont and Tankov [35].

Lévy process theory has found fruitful applications in financial mathematics (see Eberlein [42], Schoutens [134], Kyprianou et al. [91], Cont and Tankov [35]) and physics (see Janicki and Weron [75]). Various subclasses of Lévy processes and processes with infinitely divisible distributions have attracted special attention. Apart from Brownian motion, the class of *stable processes* has received interest. Their theory is most advanced, probably due to the relatively simple form of the Lévy measure which has power law tails and the resulting nice scaling properties of such processes. The theory of stable processes has initiated additional research on Lévy and infinitely divisible processes, where the stable processes often serve as a toy example. Samorodnitsky and Taqqu [131] give both an introduction and advanced treatment of stable processes.

In insurance mathematics the compound Poisson processes have played an important role since Lundberg's [99] original work. It has a straightforward interpretation: jump times are claim arrivals and jump sizes are claim sizes. A general Lévy process with infinitely many and arbitrarily small jump sizes on any finite time interval is more difficult to interpret. Nevertheless, general Lévy processes have been incorporated into insurance models. For example, Veraverbeke [144] considered the following model (which was originally proposed by H. Gerber) as a risk process:

$$R(t) = u + c t - S(t) + B(t), \quad t \geq 0,$$

where u , c and S are defined in the framework of the Cramér-Lundberg model as initial capital, premium rate and total claim amount process, respectively. The process B is Brownian motion independent of the compound Poisson process S . Of course, $R - u$ is a Lévy process. While S is the physically observable total claim amount process, B describes deviations from the Cramér-Lundberg model which are for example due to delays or errors in reporting and accounting. Veraverbeke [144] derived an estimate similar to the ruin bound in the Cramér-Lundberg model for subexponential claim size distributions.

Similar interpretations are possible if the process $S - B$ is replaced by a general Lévy process. A corresponding theory, in particular ruin bounds in the light- and heavy-tailed cases, can be found in Kyprianou [90] and the references given therein. He summarizes the known recent results and extends some of them significantly.

The theory provided in this chapter can be extended without great difficulty to d -dimensional Lévy processes X when $X(t)$, $t \geq 0$, assumes values in \mathbb{R}^d . The reader is encouraged to go through the calculations, following the lines of the proofs above.

Exercises

- (1) Let X be α -stable Lévy motion for some $\alpha \in (0, 2]$.
 - (a) Show that a symmetric α -stable Lévy motion X as defined in Example 10.5.2 is a symmetric process in the sense that $X \stackrel{d}{=} -X$ for the finite-dimensional distributions.
 - (b) Show that the measure μ_α defined in (10.5.31) is a Lévy measure.
- (2) Consider symmetric α -stable Lévy motion X on $[0, \infty)$ for some $\alpha \in (0, 2]$; see Example 10.5.2.
 - (a) Show that X is $1/\alpha$ -self-similar, i.e., for any $0 = t_0 \leq t_1 < \dots < t_n$, $n \geq 1$ and $c > 0$ the following relation holds:

$$(X(t_1), \dots, X(t_n)) \stackrel{d}{=} c^{-1/\alpha} (X(ct_1), \dots, X(ct_n)).$$

- (b) Show for the t_i 's as in part (a) that

$$(X(t_1), \dots, X(t_n)) \stackrel{d}{=} \left(\sum_{i=1}^k (t_i - t_{i-1})^{1/\alpha} X_i \right)_{k=1, \dots, n},$$

where X_i , $i = 1, \dots, n$, are iid copies of $X(1)$.

- (3) Prove the decomposition (10.5.36).

Hints: (i) Use the fact that $N([0, t] \times \{x : |x| > 1\}) < \infty$ a.s. for $t \geq 0$ in order to deal with the large jumps of S .
(ii) Show that

$$\begin{aligned}
& \lim_{\delta \downarrow 0} \left[\int_{[0,t] \times \{x: \delta < x \leq 1\}} x N(ds, dx) - t \int_{\{x: \delta < x \leq 1\}} x \mu(dx) \right] + \\
& \lim_{\delta \downarrow 0} \left[\int_{[0,t] \times \{x: -1 \leq x < -\delta\}} x N(ds, dx) - t \int_{\{x: -1 \leq x < -\delta\}} x \mu(dx) \right] \\
& = \lim_{\delta \downarrow 0} \left[\int_{[0,t] \times \{x: \delta < |x| \leq 1\}} x N(ds, dx) - t \int_{\{x: \delta < |x| \leq 1\}} x \mu(dx) \right].
\end{aligned}$$

In particular, show that the two a.s. limits on the left-hand side exist. One may follow the lines of the proof of Theorem 10.4.3(1).

- (4) Consider the Lévy-Itô representation of an α -stable Lévy motion X for $\alpha \in (1, 2)$.
(a) Show that $E|X(1)| < \infty$ and that

$$EX(t) = t \int_{\{x: |x| > 1\}} x \mu_\alpha(dx), \quad (10.5.37)$$

where the Lévy measure μ_α is given in (10.5.31).

Hint: Use a Lebesgue dominated convergence argument to show that the a.s. limit of the centered sums of the small jumps in the Lévy-Itô representation (10.4.19) has mean zero.

- (b) Calculate the value of $EX(t)$ by using (10.5.37).
(5) Let S be a Lévy process with characteristic triplet $(0, 0, \mu)$ given by its Lévy-Itô representation. Show that

$$ES(t) = t \int_{\{x: |x| > 1\}} x \mu(dx),$$

provided $\int_{\{x: |x| > 1\}} |x| \mu(dx) < \infty$.

- (6) Let (Γ_i) be the points of a standard homogeneous Poisson process on $(0, \infty)$ and (Γ'_i) a copy of (Γ_i) . Moreover, let (γ_i) be an iid sequence of Bernoulli random variables with $P(\gamma_1 = 1) = 1 - P(\gamma_1 = -1) = 0.5$. Moreover, assume that (Γ_i) , (Γ'_i) and (γ_i) are mutually independent.

Prove that the infinite series

$$\sum_{i \geq 1: \Gamma'_i \leq t} \gamma_i \Gamma_i^{-1/\alpha}, \quad t \geq 0,$$

converges a.s. for $\alpha < 2$ and represents symmetric α -stable Lévy motion.

Hint: Use the fact that $N = \sum_{i=1}^{\infty} \varepsilon_{(\Gamma'_i, \Gamma_i, \gamma_i)}$ are the points of $\text{PRM}(\text{Leb} \times \text{Leb} \times F_\gamma)$, on $(0, \infty)^2 \times \{\pm 1\}$.

- (7) Consider the gamma Lévy process X with Lévy measure μ_γ defined in (10.3.14). Show that X is a subordinator.

Cluster Point Processes

In this chapter we consider a *cluster point process model*. This means that at every point of a point process on $(0, \infty)$ a cluster of activities starts. We interpret this point as the arrival time of a claim which triggers a random stream of payments from the insurer to the insured. This model includes the chain ladder which is used by many practicing actuaries for forecasting the claim numbers and total claim amounts in future periods, given the total claim amounts and claim numbers from previous years. After the definition of the general cluster model, in Section 11.2 we immediately turn to the chain ladder model. The treatment of chain ladder techniques does not really require genuine point process techniques. It is however convenient to use the language of point processes as a general framework, in order to contrast and compare with the theoretical models considered in Section 11.3. In this section we study processes where the clusters start at the points of a homogeneous Poisson process. A Poisson point represents the arrival time of a claim, whereas the newly started process describes the times and amounts of the payments from the insurer to the insured for this particular claim. Since the resulting claim number and total claim amount processes have Poisson integral structure we can derive their first and second order moments. We also discuss the problem of predicting future claim numbers and total claim amounts given the past number of payments in the special case when the payment process is Poisson.

11.1 The General Cluster Process

We commence by introducing the model.

Definition 11.1.1 (The general cluster model)

The general cluster model satisfies the following two conditions.

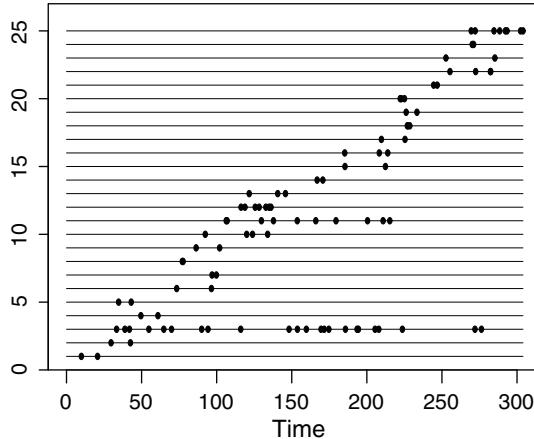


Figure 11.1.2 Visualization of the times of payments. Each line corresponds to one claim. The first dot at the left indicates the claim arrival T_i , the following dots indicate the instants T_{ij} when payments to the insured are executed.

- (1) Claims arrive at the random instants of time $0 < T_1 < T_2 < \dots, i = 1, 2, \dots^1$
- (2) The i th claim causes a random stream or cluster of payments from the insurer to the insured. The j th payment for the i th claim is a positive random variable X_{ij} executed at time

$$T_{ij} = T_i + \sum_{k=1}^j Y_{ik}, \quad 1 \leq j \leq K_i,$$

where $(Y_{ik})_{k \geq 1}$ is a sequence of positive random variables and K_i is a positive integer-valued random variable.

Thus the payment process for the i th claim is described by the pairs (T_{ij}, X_{ij}) , $j = 1, \dots, K_i$. The i th claim is settled at time $T_{iK_i} = T_i + \sum_{k=1}^{K_i} Y_{ik}$ with corresponding claim size $\sum_{k=1}^{K_i} X_{ik}$. See Figure 11.1.2 for an illustration.

We mark each claim arrival T_i by the random element

$$A_i = ((Y_{ik})_{k \geq 1}, (X_{ik})_{k \geq 1}, K_i), \quad i = 1, 2, \dots.$$

The points A_i assume values in

¹ Alternatively, one can interpret T_i as the reporting time of the i th claim. In this section we do not take into account delays in reporting.

$$E_A = (0, \infty)^\infty \times (0, \infty)^\infty \times \mathbb{N},$$

where $(0, \infty)^\infty$ denotes the space of sequences with positive components:

$$(0, \infty)^\infty = \{(x_k)_{k \geq 1} : x_k \in (0, \infty)\}.$$

In Section 11.3 we will interpret

$$N = \sum_{i=1}^{\infty} \varepsilon_{(T_i, A_i)} \quad (11.1.1)$$

as a marked point process with state space $E = (0, \infty) \times E_A$.

11.2 The Chain Ladder Method

11.2.1 The Chain Ladder Model

The chain ladder is a model which is frequently used in insurance practice for estimating reserves. For the formulation of the model it is convenient to introduce the periods of time (years say)

$$C_i = (i-1, i], \quad i = 1, 2, \dots$$

but also the aggregated periods

$$C_{i,j} = C_i \cup \dots \cup C_{i+j} = (i-1, i+j], \quad j = 0, 1, \dots$$

By $N_{i,i+j}$ and $S_{i,i+j}$ we denote the respective number and total amount of those claims that occurred in the year C_i and for which a payment was made in the period $C_{i,j}$, $j = 0, 1, \dots$. Using the point process notation of the general cluster model with N introduced in (11.1.1), the claim numbers can be written in the form

$$\begin{aligned} N_{i,i+j} &= \int_E \sum_{l=1}^k I_{\{t \in C_i, t + (y_1 + \dots + y_l) \in C_{i,i+j}\}} N(dt, d(y_r), d(x_r), dk) \\ &= \sum_{n=1}^{\infty} \sum_{l=1}^{K_n} I_{\{T_n \in C_i, T_n + (Y_{n1} + \dots + Y_{nl}) \in C_{i,i+j}\}} \\ &= \sum_{n=1}^{\infty} \sum_{l=1}^{K_n} I_{\{T_n \in C_i, T_{nl} \in C_{i,i+j}\}}, \quad j = 0, 1, \dots, \end{aligned}$$

and correspondingly the claim amounts

$$\begin{aligned}
S_{i,i+j} &= \int_E \sum_{l=1}^k x_l I_{\{t \in C_i, t+(y_1+\dots+y_l) \in C_{i,i+j}\}} N(dt, d(y_r), d(x_r), dk) \\
&= \sum_{n=1}^{\infty} \sum_{l=1}^{K_n} X_{nl} I_{\{T_n \in C_i, T_n + (Y_{n1} + \dots + Y_{nl}) \in C_{i,i+j}\}} \\
&= \sum_{n=1}^{\infty} \sum_{l=1}^{K_n} X_{nl} I_{\{T_n \in C_i, T_{nl} \in C_{i,i+j}\}}, \quad j = 0, 1, \dots.
\end{aligned}$$

The *chain ladder* is the triangular array of the pairs $(N_{i,i+j}, S_{i,i+j})$, $i = 1, \dots, m$, $1 \leq i+j \leq m$:

$$\begin{array}{cccccc}
(N_{11}, S_{11}) & (N_{12}, S_{12}) & (N_{13}, S_{13}) & \cdots & (N_{1m}, S_{1m}) \\
(N_{22}, S_{22}) & (N_{23}, S_{23}) & \cdots & (N_{2m}, S_{2m}) \\
(N_{33}, S_{33}) & \cdots & (N_{3m}, S_{3m}) \\
& \ddots & & \vdots & \\
& & (N_{mm}, S_{mm}).
\end{array}$$

If we interpret C_m as the present year the chain ladder contains the complete aggregated annual information about all claim payments in the previous and present years. This information consists of the number of payments $N_{i,i+j}$ and the corresponding total amount $S_{i,i+j}$ for claims that occurred in the year C_i for some $i \leq m$ and for which payments were executed in the period $C_{i,i+j}$ for $i+j \leq m$.

11.2.2 Mack's Model

A practical problem is the following:

How can one use the information represented by the past and present observations $(N_{i,i+j}, S_{i,i+j})$, $i = 1, \dots, m$, $1 \leq i+j \leq m$, in order to predict the future total claim amounts $S_{i,m+1}$ and numbers of payments $N_{i,m+1}$, $i = 1, \dots, m$?

With this question in mind, Mack [100] introduced some additional conditions in the chain ladder model.

Definition 11.2.1 (Mack's model)

For the chain ladder model we require the following additional assumptions:

- (3) *The numbers and total amounts of payments for claims originating in different years are iid, i.e., the processes $(N_{i,i+j}, S_{i,i+j})_{j=0,1,\dots}$, $i = 1, 2, \dots$, are iid.²*

² Mack [100] actually assumed the condition that $(N_{i,i+j}, S_{i,i+j})_{j=0,1,\dots}$, $i = 1, 2, \dots$, be independent which is weaker than condition (3). For most of the results below the independence condition suffices. The iid assumption is however convenient for the asymptotic theory in Section 11.2.3.

- (4) The random variables $N_{i,i+j}, S_{i,i+j}$, $i = 1, 2, \dots$, $j = 0, 1, \dots$, are a.s. positive and have finite expectation. There exist non-negative real numbers f_j and g_j such that for $i = 1, 2, \dots$, $j = 0, 1, \dots$,

$$E(N_{i,i+j+1} | N_{ii}, \dots, N_{i,i+j}) = f_j N_{i,i+j}, \quad (11.2.2)$$

$$E(S_{i,i+j+1} | S_{ii}, \dots, S_{i,i+j}) = g_j S_{i,i+j}. \quad (11.2.3)$$

For ease of presentation, we introduce the generic sequences

$$(N_j)_{j \geq 0} = (N_{1,1+j})_{j \geq 0} \quad \text{and} \quad (S_j)_{j \geq 0} = (S_{1,1+j})_{j \geq 0}.$$

Then conditions (11.2.2) and (11.2.3) read as follows:

$$E(N_{j+1} | N_0, \dots, N_j) = f_j N_j, \quad (11.2.4)$$

$$E(S_{j+1} | S_0, \dots, S_j) = g_j S_j. \quad (11.2.5)$$

The requirement of finiteness of the moments EN_j and ES_j in condition (4) is needed in order to ensure that the conditional expectations in (11.2.4) and (11.2.5) are finite with probability 1. Condition (4) is often modified in the literature. For example, it is not uncommon to assume the less restrictive conditions

$$E(N_{j+1} | N_j) = f_j N_j \quad \text{and} \quad E(S_{j+1} | S_j) = g_j S_j, \quad j \geq 0.$$

Most of the arguments below can easily be adapted to this situation.

The assumption that N_j and S_j be a.s. positive is a serious restriction since it excludes various standard distributions. For example, the claim number N_j cannot have a Poisson or binomial distribution, and the claim amount S_j cannot have a compound Poisson distribution. We also observe that $N_0 \geq 1$ a.s. ensures that $N_j \geq 1$ a.s., $j \geq 1$, since for any $k \geq 1$, $N_{k+1} \geq N_k$. We will see later that the positivity of the numbers N_j and amounts S_j is needed in order to guarantee that the estimators of f_j and g_j are well-defined; see p. 370.

Now we return to the question about the prediction of future aggregate claim numbers and total claim amounts given past and present observations. In this context, it is convenient to recall Lemma 5.2.2. It says that the conditional expectation

$$E(N_{j+1} | \mathcal{F}_j) = E(N_{j+1} | N_0, \dots, N_j)$$

minimizes the mean square error $E[(N_{j+1} - M)^2]$ in the class of all random variables M which have finite variance and are measurable with respect to the σ -field

$$\mathcal{F}_j = \sigma(N_0, \dots, N_j)$$

generated by N_0, \dots, N_j . Hence, if $\text{var}(N_j) < \infty$ for $j = 0, 1, \dots$, condition (11.2.2) defines the quantity $f_j N_{i,i+j}$ as the *best predictor* of $N_{i,i+j+1}$ (in the mean square sense) given the payment numbers $N_{ii}, \dots, N_{i,i+j}$ from the past and present periods $C_{ii}, \dots, C_{i,i+j}$. A similar remark applies to the best predictor of $S_{i,i+j+1}$ given $S_{ii}, \dots, S_{i,i+j}$.

Conditions such as (11.2.4) and (11.2.5) look rather ad hoc. Mack's main motivation for introducing them was to explain the form of the so-called *chain ladder estimators* which are commonly used in practice for estimating reserves in future periods; see Sections 11.2.3–11.2.5 for a discussion of these estimators. In this sense, Mack's model is dictated by an applied statistical procedure. In the framework of the general cluster process it is not straightforward to find a specification that satisfies the conditions (11.2.4) and (11.2.5). This fact is even true when the claim arrivals T_i come from a homogeneous Poisson process on $(0, \infty)$; see Section 11.3.

In general, it is impossible to calculate conditional expectations of the type $E(N_{j+1} | N_0, \dots, N_j)$ or $E(S_{j+1} | S_0, \dots, S_j)$ without specific assumptions on the underlying stochastic models. Mack's condition (4) defines these conditional expectations in a tractable way. Mack's model has the advantage that it suggests a clear program how to proceed:

- Since we do not know the parameters f_j and g_j we have to replace them by suitable estimators $\hat{f}_j^{(m)}$ and $\hat{g}_j^{(m)}$ which can be calculated from the past and present observations $N_{i,i+j}$ and $S_{i,i+j}$, $i = 1, \dots, m$, $i + j \leq m$. In this context, condition (3) is crucial: one uses the independence of the observations across different years in order to derive desirable statistical properties of the estimators such as unbiasedness, consistency and asymptotic normality. These issues are discussed in Sections 11.2.3 and 11.2.4.
- Condition (4) in Mack's model suggests surrogate predictors for $N_{i,m+1}$ and $S_{i,m+1}$: replace the unknown quantities f_{m-i} and g_{m-i} by their estimators $\hat{f}_{m-i}^{(m)}$ and $\hat{g}_{m-i}^{(m)}$ to get approximations to the *best predictors* (in the mean square sense) $f_{m-i} N_{im}$ and $g_{m-i} S_{im}$ of $N_{i,m+1}$ and $S_{i,m+1}$, respectively, given by

$$\hat{N}_{i,m+1} = \hat{f}_{m-i}^{(m)} N_{im} \quad \text{and} \quad \hat{S}_{i,m+1} = \hat{g}_{m-i}^{(m)} S_{im}.$$

Finally, estimate the mean square prediction errors

$$E[(\hat{N}_{i,m+1} - N_{i,m+1})^2] \quad \text{and} \quad E[(\hat{S}_{i,m+1} - S_{i,m+1})^2].$$

These are the objectives of Section 11.2.5, where we also treat the prediction of $N_{i,m+k}$ for $k > 1$.

In what follows, we focus on the sequences of the payment numbers $N_{i,i+j}$. The theory for the total claim amounts $S_{i,i+j}$ is completely analogous and is omitted.

11.2.3 Some Asymptotic Results in the Chain Ladder Model

In Mack's model knowledge of the unknown constants f_j is crucial for forecasting. Estimators of these quantities are suggested by standard asymptotic theory which we discuss next. Based on this theory, the *chain ladder estimator* of f_j occurs as a method of moments estimator.

Condition (3) implies that the claim histories arising from distinct years C_i have the same distribution and are mutually independent. This iid property opens the door to standard asymptotic theory when $m \rightarrow \infty$. By condition (3), the sequence $(N_{i,i+j})_{i \geq 1}$ consists of iid random variables for every fixed $j \geq 0$. In the m th year (at the present, say) only the past and present claim numbers $N_{i,i+j}$, $i = 1, \dots, m-j$, have been observed. The Glivenko-Cantelli lemma immediately implies for every fixed $j \geq 0$,

$$\sup_{x \in \mathbb{R}} \left| \frac{1}{m-j} \sum_{i=1}^{m-j} I_{[0,x]}(N_{i,i+j}) - P(N_j \leq x) \right| \xrightarrow{\text{a.s.}} 0 \quad \text{as } m \rightarrow \infty.$$

Similarly, for fixed $j \geq 0$, the strong law of large numbers implies strong consistency for the sample versions of EN_j , $\text{var}(N_j)$ and $\text{cov}(N_{j_1}, N_{j_2})$ as $m \rightarrow \infty$:

$$\bar{N}_m^{(j)} = \frac{1}{m-j} \sum_{i=1}^{m-j} N_{i,i+j} \xrightarrow{\text{a.s.}} EN_j, \quad (11.2.6)$$

$$[s_m^{(j)}]^2 = \frac{1}{m-j-1} \sum_{i=1}^{m-j} \left(N_{i,i+j} - \bar{N}_m^{(j)} \right)^2 \xrightarrow{\text{a.s.}} \text{var}(N_j), \quad (11.2.7)$$

$$\begin{aligned} \hat{\gamma}_m(j_1, j_2) &= \frac{1}{m-j_2} \sum_{i=1}^{m-j_2} (N_{i,i+j_1} - \bar{N}_m^{(j_1)}) (N_{i,i+j_2} - \bar{N}_m^{(j_2)}) \\ &\xrightarrow{\text{a.s.}} \text{cov}(N_{j_1}, N_{j_2}), \quad j_1 < j_2. \end{aligned} \quad (11.2.8)$$

Here we assumed $EN_j < \infty$ in (11.2.6) and $\text{var}(N_j) < \infty$ in (11.2.7), (11.2.8) in order to meet the conditions of the strong law of large numbers. The verification of the details in these limit relations is left as Exercise 2(a) on p. 382. In order to derive asymptotic confidence bands for these estimators, one can apply the multivariate central limit theorem for iid random vectors. The central limit theorem yields joint convergence in distribution towards an appropriate centered Gaussian vector. We encourage the reader to work out the details; see Exercises 2(b)-(e).

The asymptotics do not provide a good approximation for fixed m and if j_1 , j_2 or j are close to m or if m is small. Then the number of terms averaged in the quantities above are too small in order to apply the strong law of large numbers or the central limit theorem in a meaningful way.

By virtue of (11.2.4) we observe for $j = 0, 1, \dots$, that

$$\begin{aligned} EN_{j+1} &= E[E(N_{j+1} \mid N_0, \dots, N_j)] \\ &= f_j EN_j. \end{aligned}$$

Since we assume that N_j is positive for every j , we also have $EN_j > 0$ and therefore

$$f_j = \frac{EN_{j+1}}{EN_j}, \quad j = 0, 1, \dots \quad (11.2.9)$$

This relation and the strong law of large numbers for $\bar{N}_m^{(j)}$ in (11.2.6) suggest that the expectations in (11.2.9) should be replaced by the corresponding sample means, resulting in some method of moments estimator for f_j :

$$\hat{f}_j^{(m)} = \frac{\bar{N}_m^{(j+1)}}{\bar{N}_{m-1}^{(j)}} = \frac{\sum_{i=1}^{m-j-1} N_{i,i+j+1}}{\sum_{i=1}^{m-j-1} N_{i,i+j}}. \quad (11.2.10)$$

This quantity is the *chain ladder estimator* of f_j .

The ratio structure of $\hat{f}_j^{(m)}$ justifies the assumption in Mack's model that the random variables $N_{i,i+j}$ be positive. If $P(N_j = 0) > 0$ then there is a positive probability that the denominator $\sum_{i=1}^{m-j-1} N_{i,i+j}$ vanishes: by independence of the $N_{i,i+j}$'s,

$$\begin{aligned} P\left(\sum_{i=1}^{m-j-1} N_{i,i+j} = 0\right) &= P(N_{i,i+j} = 0, i = 1, \dots, m-j-1) \\ &= [P(N_j = 0)]^{m-j-1} > 0. \end{aligned}$$

Next we summarize some of the asymptotic properties of the chain ladder estimators.

Proposition 11.2.2 (Asymptotic properties of the chain ladder estimators)
Assume the conditions of Mack's model.

- (1) The chain ladder estimator $\hat{f}_j^{(m)}$ is a strongly consistent estimator of its deterministic counterpart f_j . This means that for fixed $j = 0, 1, \dots$,

$$\hat{f}_j^{(m)} \xrightarrow{\text{a.s.}} f_j \quad \text{as } m \rightarrow \infty.$$

- (2) If in addition $\text{var}(N_j) < \infty$ for $j = 0, 1, \dots$, then $\hat{f}_j^{(m)}$ is asymptotically normal, i.e., it satisfies the central limit theorem

$$\sqrt{m} \left(\hat{f}_j^{(m)} - f_j \right) \xrightarrow{d} N(0, (EN_j)^{-2} [E(N_{j+1}^2) - f_j^2 E(N_j^2)]).$$

Proof. (1) Consistency is straightforward from the strong law of large numbers (11.2.6) and relation (11.2.9):

$$\hat{f}_j^{(m)} = \frac{\bar{N}_m^{(j+1)}}{\bar{N}_{m-1}^{(j)}} \xrightarrow{\text{a.s.}} \frac{EN_{j+1}}{EN_j} = f_j .$$

(2) We first observe that

$$\begin{aligned} & \sqrt{m} \left[\hat{f}_j^{(m)} - f_j \right] \\ &= \sqrt{m} \left[\frac{\bar{N}_m^{(j+1)}}{\bar{N}_{m-1}^{(j)}} - f_j \right] = \frac{\sqrt{m}}{\bar{N}_{m-1}^{(j)}} \left[\bar{N}_m^{(j+1)} - f_j \bar{N}_{m-1}^{(j)} \right] \\ &= \frac{\sqrt{m/(m-j-1)}}{\bar{N}_{m-1}^{(j)}} \left[\frac{1}{\sqrt{m-j-1}} \sum_{i=1}^{m-j-1} (N_{i,i+j+1} - f_j N_{i,i+j}) \right]. \end{aligned}$$

By condition (3) in Mack's model, the sequence $(N_{i,i+j+1} - f_j N_{i,i+j})_{i \geq 1}$ is iid for fixed j and, by assumption, consists of mean zero variables with finite variance. An application of the central limit theorem in combination with the strong law of large numbers $\bar{N}_{m-1}^{(j)} \xrightarrow{\text{a.s.}} EN_j$ implies that

$$\sqrt{m} \left[\hat{f}_j^{(m)} - f_j \right] \xrightarrow{d} N(0, (EN_j)^{-2} \text{var}(N_{j+1} - f_j N_j)) . \quad (11.2.11)$$

By direct calculation,

$$\text{var}(N_{j+1} - f_j N_j) = \text{var}(N_{j+1}) + f_j^2 \text{var}(N_j) - 2 f_j \text{cov}(N_j, N_{j+1}) .$$

A conditioning argument yields

$$\begin{aligned} \text{cov}(N_j, N_{j+1}) &= E(N_{j+1} N_j) - EN_{j+1} EN_j \\ &= E[N_j E(N_{j+1} | N_j)] - E[E(N_{j+1} | N_j)] EN_j \\ &= f_j \text{var}(N_j), \\ \text{var}(N_{j+1}) &= E(N_{j+1}^2) - f_j^2 (EN_j)^2 . \end{aligned}$$

Hence the asymptotic variance in (11.2.11) is given by the quantity

$$\frac{\text{var}(N_{j+1} - f_j N_j)}{(EN_j)^2} = \frac{E(N_{j+1}^2) - f_j^2 E(N_j^2)}{(EN_j)^2} .$$

This concludes the proof. □

The asymptotic variance in the central limit theorem for $\hat{f}_j^{(m)}$ can be approximated by the method of moments estimators for $E(N_j^2)$, $E(N_{j+1}^2)$ and by $\hat{f}_j^{(m)}$.

This means that the asymptotic variance $(EN_j)^{-2}[E(N_{j+1}^2) - f_j^2 E(N_j^2)]$ is approximated by the strongly consistent estimator

$$\frac{\sum_{i=1}^{m-j-1} [N_{i,i+j+1}^2 - \hat{f}_j^{(m)}]^2 N_{i,i+j}^2}{(m-j)^{-1} \left(\sum_{i=1}^{m-j} N_{i,i+j} \right)^2}.$$

This estimator of the asymptotic variance allows one to construct asymptotic confidence bands for the estimator $\hat{f}_j^{(m)}$.

11.2.4 Moments of the Chain Ladder Estimators

The estimator $\hat{f}_j^{(m)}$ is not the exact sample version of EN_{j+1}/EN_j . This would be the quantity

$$\frac{\bar{N}_m^{(j+1)}}{\bar{N}_m^{(j)}} = \frac{m-j}{m-j-1} \frac{\sum_{i=1}^{m-j-1} N_{i,i+j+1}}{\sum_{i=1}^{m-j} N_{i,i+j}}.$$

However, it is not difficult to see that the latter estimator has the same asymptotic properties as $\hat{f}_j^{(m)}$. One of the reasons for the choice of $\hat{f}_j^{(m)}$ is the fact that it is an unbiased estimator of its deterministic counterpart f_j . We verify this property.

Lemma 11.2.3 (Expectation and covariances of the chain ladder estimators)
Assume the conditions of Mack's model. Then the following statements hold.

(1) *The chain ladder estimators are unbiased:*

$$E\hat{f}_j^{(m)} = f_j, \quad j = 0, \dots, m-2.$$

(2) *If, in addition, $\text{var}(N_j) < \infty$ for $j \geq 0$, then the chain ladder estimators are uncorrelated:*

$$\text{cov}(\hat{f}_{j_1}^{(m)}, \hat{f}_{j_2}^{(m)}) = 0, \quad 0 \leq j_1 < j_2 \leq m-2.$$

In what follows, we will often make use of the following σ -fields:³

$$\mathcal{G}_j^{(m)} = \sigma(N_{i,i+k}, 0 \leq k \leq j, 1 \leq i+k \leq m), \quad j = 0, 1, \dots,$$

$$\mathcal{F}_{i,i+j} = \sigma(N_{ii}, \dots, N_{i,i+j}), \quad i = 1, 2, \dots, j = 0, 1, \dots.$$

Proof. (1) Since $\sum_{i=1}^{m-j-1} N_{i,i+j}$ is measurable with respect to $\mathcal{G}_j^{(m)}$,

$$E(\hat{f}_j^{(m)} | \mathcal{G}_j^{(m)}) = \frac{\sum_{i=1}^{m-j-1} E(N_{i,i+j+1} | \mathcal{G}_j^{(m)})}{\sum_{i=1}^{m-j-1} N_{i,i+j}}.$$

³ For any family of random variables $(A_t)_{t \in I}$, $\sigma(A_t, t \in I)$ is the σ -field generated by this family.

By virtue of Mack's condition (3), the random variable $N_{i,i+j+1}$ is independent of $(N_{l,l+k})_{k \geq 0}$ for $l \neq i$. Since $\mathcal{F}_{i,i+j} \subset \mathcal{G}_j^{(m)}$ for $i = 1, \dots, m-j-1$ and $0 \leq j \leq m-2$, we therefore get

$$E(N_{i,i+j+1} | \mathcal{G}_j^{(m)}) = E(N_{i,i+j+1} | \mathcal{F}_{i,i+j}) = f_j N_{i,i+j}. \quad (11.2.12)$$

In the last step we used Mack's condition (4). Combining the equations above, we arrive at

$$E(\hat{f}_j^{(m)} | \mathcal{G}_j^{(m)}) = f_j. \quad (11.2.13)$$

Taking expectations, we see that $E\hat{f}_j^{(m)} = f_j$, hence $\hat{f}_j^{(m)}$ is an unbiased estimator of f_j .

(2) The uncorrelatedness of the chain ladder estimators is again seen by a conditioning argument. Assume $j_1 < j_2$. Then $\hat{f}_{j_1}^{(m)}$ is measurable with respect to $\mathcal{G}_{j_2}^{(m)}$ and hence by (11.2.13),

$$E(\hat{f}_{j_1}^{(m)} \hat{f}_{j_2}^{(m)} | \mathcal{G}_{j_2}^{(m)}) = \hat{f}_{j_1}^{(m)} E(\hat{f}_{j_2}^{(m)} | \mathcal{G}_{j_2}^{(m)}) = \hat{f}_{j_1}^{(m)} f_{j_2}.$$

Taking expectations, we see that

$$\begin{aligned} \text{cov}(\hat{f}_{j_1}^{(m)}, \hat{f}_{j_2}^{(m)}) &= E(\hat{f}_{j_1}^{(m)} \hat{f}_{j_2}^{(m)}) - E(\hat{f}_{j_1}^{(m)}) E(\hat{f}_{j_2}^{(m)}) \\ &= E(\hat{f}_{j_1}^{(m)} f_{j_2}) - f_{j_1} f_{j_2} = 0. \end{aligned}$$

□

The variance of the chain ladder estimator $\hat{f}_j^{(m)}$ is not as easily established. Problems arise because $\hat{f}_j^{(m)}$ is the ratio of two dependent random variables. For the following calculations it will be convenient to use the notation:

$$\tilde{N}_{i,i+j} = N_{i,i+j+1} - f_j N_{i,i+j}.$$

We also assume that $\text{var}(N_j) < \infty$ for $j = 0, 1, \dots$. Since $E(\hat{f}_j^{(m)} | \mathcal{G}_j^{(m)}) = f_j$, we have

$$\begin{aligned} \text{var}(\hat{f}_j^{(m)} | \mathcal{G}_j^{(m)}) &= E\left(\left[\frac{\sum_{i=1}^{m-j-1} \tilde{N}_{i,i+j}}{\sum_{i=1}^{m-j-1} N_{i,i+j}}\right]^2 \middle| \mathcal{G}_j^{(m)}\right) \\ &= \frac{\sum_{i=1}^{m-j-1} \text{var}(\tilde{N}_{i,i+j} | \mathcal{G}_j^{(m)})}{\left(\sum_{i=1}^{m-j-1} N_{i,i+j}\right)^2} \\ &\quad + \frac{\sum_{1 \leq i \neq k \leq m-j-1} \text{cov}(\tilde{N}_{i,i+j}, \tilde{N}_{k,k+j} | \mathcal{G}_j^{(m)})}{\left(\sum_{i=1}^{m-j-1} N_{i,i+j}\right)^2}. \end{aligned}$$

We observe that

$$\begin{aligned}\text{var}(\tilde{N}_{i,i+j} \mid \mathcal{G}_j^{(m)}) &= \text{var}(\tilde{N}_{i,i+j} \mid \mathcal{F}_{i,i+j}), \\ \text{cov}(\tilde{N}_{i,i+j}, \tilde{N}_{k,k+j} \mid \mathcal{G}_j^{(m)}) &= E(\tilde{N}_{i,i+j} \tilde{N}_{k,k+j} \mid \mathcal{F}_{i,i+j} \vee \mathcal{F}_{k,k+j}),\end{aligned}$$

where $\mathcal{F}_{i,i+j} \vee \mathcal{F}_{k,k+j}$ is the smallest σ -field containing both $\mathcal{F}_{i,i+j}$ and $\mathcal{F}_{k,k+j}$, $i, k = 1, \dots, m-j-1$. The covariances vanish for $i < k$ by a conditioning argument:

$$\begin{aligned}E(\tilde{N}_{i,i+j} \tilde{N}_{k,k+j} \mid \mathcal{F}_{i,i+j} \vee \mathcal{F}_{k,k+j}) \\ = E\left[E(\tilde{N}_{i,i+j} \tilde{N}_{k,k+j} \mid \mathcal{F}_{i,i+j} \vee \mathcal{F}_{k,k+j+1}) \mid \mathcal{F}_{i,i+j} \vee \mathcal{F}_{k,k+j}\right] \\ = E\left[\tilde{N}_{k,k+j} E(\tilde{N}_{i,i+j} \mid \mathcal{F}_{i,i+j} \vee \mathcal{F}_{k,k+j+1}) \mid \mathcal{F}_{i,i+j} \vee \mathcal{F}_{k,k+j}\right] \\ = E\left[\tilde{N}_{k,k+j} E(\tilde{N}_{i,i+j} \mid \mathcal{F}_{i,i+j}) \mid \mathcal{F}_{i,i+j} \vee \mathcal{F}_{k,k+j}\right] = 0,\end{aligned}$$

where we used the independence of $\tilde{N}_{i,i+j}$ and $\mathcal{F}_{k,k+j+1}$ and condition (4) in Mack's model. Thus we proved that

$$\text{var}(\tilde{f}_j^{(m)} \mid \mathcal{G}_j^{(m)}) = \frac{\sum_{i=1}^{m-j-1} \text{var}(\tilde{N}_{i,i+j} \mid \mathcal{F}_{i,i+j})}{\left(\sum_{i=1}^{m-j-1} N_{i,i+j}\right)^2}. \quad (11.2.14)$$

At this point we realize that the conditions of Mack's model do not suffice in order to evaluate these conditional variances. This is not surprising since condition (4) only concerns the conditional expectations of the $N_{i,i+j}$'s, not their second moment structure.

Therefore we introduce an additional condition proposed by Mack [100]:

(5) *In addition to conditions (3) and (4) of Mack's model, $\text{var}(N_j) < \infty$, $j = 0, 1, \dots$, and there exist non-negative real numbers σ_j^2 such that*

$$\begin{aligned}\text{var}(N_{j+1} \mid N_0, \dots, N_j) &= E[(N_{j+1} - f_j N_j)^2 \mid N_0, \dots, N_j] \\ &= \sigma_j^2 N_j, \quad j = 0, 1, \dots\end{aligned} \quad (11.2.15)$$

The rationale for this condition is again convenience. Recall that condition (4) in Mack's model had only one purpose: it was chosen in order to explain the form of the chain ladder estimators $\tilde{f}_j^{(m)}$ used by practitioners for predicting future claim numbers. Hence condition (4) was chosen for pragmatic reasons. As a pleasant byproduct, as we have seen in the above calculations, conditions (3) and (4) enable one to prove asymptotic results, unbiasedness and uncorrelatedness of the chain ladder estimators.

Assuming condition (5), we are now able to continue the calculations for $\text{var}(f_j^{(m)})$. Here is the final result.

Lemma 11.2.4 (Variance of chain ladder estimator $\tilde{f}_j^{(m)}$)

Under the conditions of Mack's model with the additional condition (5), we have for every $j = 0, \dots, m - 2$,

$$\text{var}(\tilde{f}_j^{(m)}) = \sigma_j^2 E \left[\left(\sum_{i=1}^{m-j-1} N_{i,i+j} \right)^{-1} \right]. \quad (11.2.16)$$

Proof. From condition (11.2.15) we immediately get

$$\text{var}(\tilde{N}_{i,i+j} \mid \mathcal{F}_{i,i+j}) = \sigma_j^2 N_{i,i+j}.$$

Then we obtain from (11.2.14)

$$\text{var}(\tilde{f}_j^{(m)} \mid \mathcal{G}_j^{(m)}) = \frac{\sum_{i=1}^{m-j-1} \sigma_j^2 N_{i,i+j}}{\left(\sum_{i=1}^{m-j-1} N_{i,i+j} \right)^2} = \frac{\sigma_j^2}{\sum_{i=1}^{m-j-1} N_{i,i+j}}. \quad (11.2.17)$$

The unconditional variance (11.2.16) can now be evaluated by an application of the relation

$$\begin{aligned} \text{var}(\tilde{f}_j^{(m)}) &= E[\text{var}(\tilde{f}_j^{(m)} \mid \mathcal{G}_j^{(m)})] + \text{var}(E(\tilde{f}_j^{(m)} \mid \mathcal{G}_j^{(m)})) \\ &= E[\text{var}(\tilde{f}_j^{(m)} \mid \mathcal{G}_j^{(m)})], \end{aligned}$$

where we used that $E(\tilde{f}_j^{(m)} \mid \mathcal{G}_j^{(m)}) = f_j$; see (11.2.13). \square

The unconditional variance of $\tilde{f}_j^{(m)}$ given in (11.2.16) is in general difficult to study. Here is an elementary bound:

$$\text{var}(\tilde{f}_j^{(m)}) \leq \frac{\sigma_j^2}{m-j-1} \left[1 - \int_1^\infty y^{-2} [P(y \leq N_j)]^{m-j-1} dy \right]. \quad (11.2.18)$$

An immediate consequence is the estimate

$$\text{var}(\tilde{f}_j^{(m)}) \leq \frac{\sigma_j^2}{(m-j-1)k},$$

provided $N_j \geq k$ a.s. for some integer $k \geq 1$. In particular, we have $\text{var}(\tilde{f}_j^{(m)}) \rightarrow 0$ as $m \rightarrow \infty$. We leave the verification of (11.2.18) as Exercise 6 on p. 383.

Estimators of the quantities σ_j^2 can again be derived by the method of moments. Taking expectations in the equation (11.2.15) defining σ_j^2 , we obtain

$$E[(N_{j+1} - f_j N_j)^2] = \sigma_j^2 E N_j.$$

Now replace the expectations on both sides by sample versions:

$$\frac{1}{m-j-1} \sum_{i=1}^{m-j-1} (N_{i,i+j+1} - f_j N_{i,i+j})^2 = [\hat{\sigma}_j^{(m)}]^2 \frac{1}{m-j-1} \sum_{i=1}^{m-j-1} N_{i,i+j}.$$

Finally, replacing in the last equation f_j by its estimator $\hat{f}_j^{(m)}$, we obtain

$$[\hat{\sigma}_j^{(m)}]^2 = \frac{\sum_{i=1}^{m-j-1} (N_{i,i+j+1} - \hat{f}_j^{(m)} N_{i,i+j})^2}{\sum_{i=1}^{m-j-1} N_{i,i+j}}. \quad (11.2.19)$$

These estimators are not unbiased due to the replacement of the quantities f_j by their chain ladder estimators. However, they are asymptotically unbiased; we leave the verification of these properties as Exercise 4 on p. 382.

11.2.5 Prediction in Mack's Model

One-Step Ahead Prediction

We assume Mack's model with the additional condition (5). Interpreting m as the present time, our first aim is to predict the future payment numbers $N_{i,m+1}$ for $i = 1, \dots, m$, i.e., one period ahead. Natural predictors arise from condition (4) in Mack's model: replace the unknown quantities f_j in the 1-year ahead predictors $E(N_{i,m+1} | \mathcal{F}_{im}) = f_{m-i} N_{im}$ by their chain ladder estimators $\hat{f}_j^{(m)}$. This procedure yields the following predictors:

$$\hat{N}_{i,m+1} = \hat{f}_{m-i}^{(m)} N_{im}, \quad i = 2, \dots, m. \quad (11.2.20)$$

Due to the replacement of the unknown parameter f_{m-i} by an estimator the surrogate quantity $\hat{N}_{i,m+1}$ loses the property as minimizer of the mean square error $E[(N_{i,m+1} - M)^2]$ over the class of all random variables M which have finite variance and are measurable with respect to the σ -field \mathcal{F}_{im} . In view of condition (5) the optimal prediction variance is given by the quantity

$$\begin{aligned} E[(N_{i,m+1} - f_{m-i} N_{im})^2] &= E[\text{var}(N_{i,m+1} | \mathcal{F}_{im})] \\ &= E[\sigma_{m-i}^2 N_{im}] \\ &= \sigma_{m-i}^2 E N_{m-i}. \end{aligned} \quad (11.2.21)$$

The variance $\sigma_{m-i}^2 E N_{m-i}$ is a minimum for the prediction error of the quantity $\hat{N}_{i,m+1}$. The mean square prediction error of $\hat{N}_{i,m+1}$ is the quantity

$$\text{err}_N^{(m)} = E \left[(\hat{N}_{i,m+1} - N_{i,m+1})^2 \right].$$

We observe that

$$\begin{aligned} \text{err}_N^{(m)} &= E [(N_{i,m+1} - f_{m-i} N_{im})^2] + E [(\hat{f}_{m-i}^{(m)} - f_{m-i})^2 N_{im}^2] \\ &= \sigma_{m-i}^2 E N_{m-i} + \text{var}(\hat{f}_{m-i}^{(m)}) E(N_{m-i}^2). \end{aligned}$$

Here we used (11.2.21) and the independence of $(N_{im}, N_{i,m+1})$ and $\hat{f}_{m-i}^{(m)}$. Indeed, by its definition the chain ladder estimator $\hat{f}_{m-i}^{(m)}$ only involves the quantities $N_{k,k+(m-i)+1}$, $N_{k,k+(m-i)}$, $k \leq i-1$, which are independent of $(N_{im}, N_{i,m+1})$.

Together with the formula for $\text{var}(\hat{f}_{m-i}^{(m)})$ from Lemma 11.2.4 we get the following result for the prediction error.

Lemma 11.2.5 (Mean square error for the one-step ahead prediction error)
Assume the conditions of Mack's model with the additional assumption (5). The 1-step ahead mean square error of the predictor $\hat{N}_{i,m+1}$ defined in (11.2.20) is given by

$$\begin{aligned}\text{err}_N^{(m)} &= \sigma_{m-i}^2 EN_{m-i} + \text{var}(\hat{f}_{m-i}^{(m)}) E(N_{m-i}^2) \\ &= \sigma_{m-i}^2 \left[EN_{m-i} + E\left[\left(\sum_{k=1}^{i-1} N_{k,k+m-i}\right)^{-1}\right] E(N_{m-i}^2) \right].\end{aligned}$$

A comparison of the variance of the best predictor of $N_{i,m+1}$ given in (11.2.21) and of the prediction error of $\hat{N}_{i,m+1}$ shows that the additional term $\text{var}(\hat{f}_{m-i}^{(m)}) E(N_{m-i}^2)$ appears which is due to the uncertainty of the parameter estimator $\hat{f}_{m-i}^{(m)}$.

The quantities EN_{m-i} and $E(N_{m-i}^2)$ can further be expanded by using conditions (4) and (5). For example,

$$EN_{m-i} = f_{m-i-1} \cdots f_0 EN_0, \quad (11.2.22)$$

and writing $\mathcal{F}_j = \sigma(N_0, \dots, N_j)$, we obtain

$$\begin{aligned}E(N_{m-i}^2) &= E[\text{var}(N_{m-i} | \mathcal{F}_{m-i-1})] + E[E((f_{m-i-1} N_{m-i-1})^2 | \mathcal{F}_{m-i-1})] \\ &= \sigma_{m-i-1}^2 EN_{m-i-1} + f_{m-i-1}^2 E(N_{m-i-1}^2) \\ &= \sigma_{m-i-1}^2 f_{m-i-2} \cdots f_0 EN_0 + f_{m-i-1}^2 E(N_{m-i-1}^2).\end{aligned} \quad (11.2.23)$$

The latter recursion can be further iterated and yields a variety of expressions for $E(N_{m-i}^2)$ in terms of f_j 's and σ_j^2 's; see Exercise 7 on p. 383.

In order to bound the prediction error $\text{err}_N^{(m)}$ one needs to replace the quantities σ_{m-i}^2 , EN_{m-i} and $E(N_{m-i}^2)$ by suitable estimators, for example by the method of moments estimators proposed in the previous sections. By virtue of the strong law of large numbers these estimators are strongly consistent estimators of their deterministic counterparts. On the other hand, the strong law of large numbers is an asymptotic result, i.e., one needs to average a large number of observations in order to get reliable estimates. By construction of these estimators this implies that $\text{err}_N^{(m)}$ can only be estimated in a reasonable way if i and m are large.

Since the quantities EN_{m-i} and $E(N_{m-i}^2)$ appearing in $\text{err}_N^{(m)}$ satisfy the recursive relations (11.2.22) and (11.2.23), one might be tempted to replace EN_{m-i} and $E(N_{m-i}^2)$ by these recursions and then estimate the parameters f_k , σ_k^2 and the moments EN_0 and $E(N_0^2)$ constituting those formulae. This approach does not avoid the problem of too small a sample size mentioned above. Additional parameter estimators also yield more statistical uncertainty in the estimated prediction error.

Finally, one also needs to estimate the quantity $E\left[\left(\sum_{k=1}^{i-1} N_{k,k+m-i}\right)^{-1}\right]$ appearing in $\text{err}_N^{(m)}$. This is difficult as discussed at the end of Section 11.2.4, where we got bounds for the variance of $\hat{f}_{m-i}^{(m)}$. Since we assume $N_j \geq 1$ a.s. a trivial upper bound is $(i-1)^{-1}$. This bound again indicates that one needs i large in order to make the prediction error small.

Since precise bounds for $\text{var}(\hat{f}_{m-i}^{(m)})$ are difficult to get, various authors searched for surrogate quantities of the prediction error of $\hat{N}_{i,m+1}$. A popular one is the “conditional mean square error”

$$\begin{aligned}\widetilde{\text{err}}_N^{(m)} &= E((\hat{N}_{i,m+1} - N_{i,m+1})^2 | \mathcal{G}_{m-i}^{(m)}) \\ &= E((N_{i,m+1} - f_{m-i} N_{im})^2 | \mathcal{F}_{im}) + N_{im}^2 \text{var}(\hat{f}_{m-i}^{(m)} | \mathcal{G}_{m-i}^{(m)}) \\ &= \sigma_{m-i}^2 \left[N_{im} + \frac{N_{im}^2}{\sum_{k=1}^{i-1} N_{k,k+m-i}} \right],\end{aligned}$$

where we applied condition (5) and (11.2.17) in the last step. In contrast to the unconditional error measure $\text{err}_N^{(m)}$, the quantity $\widetilde{\text{err}}_N^{(m)}$ does not have a straightforward interpretation as the error in the mean square prediction problem for $N_{i,m+1}$. Of course, $\widetilde{\text{err}}_N^{(m)}$ minimizes⁴ the mean square error $E[((\hat{N}_{i,m+1} - N_{i,m+1})^2 - M)^2]$ in the class of all finite variance random variables M which are measurable with respect to the σ -field $\mathcal{G}_{m-i}^{(m)}$ representing the information gathered from the observations $N_{l,l+k}$, $0 \leq k \leq m-i$, $1 \leq l+k \leq m$, from the past and present.

***k*-Steps Ahead Prediction**

Condition (4) in Mack’s model does not only specify the form of the one-step ahead predictor $E(N_{j+1} | N_0, \dots, N_j) = f_j N_j$ of N_{j+1} but it also defines the best predictor (in the mean square sense) of N_{j+k} given N_0, \dots, N_j for any $k \geq 1$. Using the properties of conditional expectations and induction, the *k*-steps ahead predictor or the *best predictor* of N_{j+k} is given by

⁴ Here one also needs to assume that the fourth moment of $\hat{N}_{i,m+1} - N_{i,m+1}$ is finite.

$$\begin{aligned}
E[N_{j+k} \mid N_0, \dots, N_j] &= E\left[E(N_{j+k} \mid N_0, \dots, N_{j+k-1}) \mid N_0, \dots, N_j\right] \\
&= f_{j+k-1} E[N_{j+k-1} \mid N_0, \dots, N_j] \\
&= f_{j+k-1} \cdots f_j N_j.
\end{aligned} \tag{11.2.24}$$

The corresponding prediction error is then calculated by a conditioning argument:

$$\begin{aligned}
E[(N_{j+k} - f_{j+k-1} \cdots f_j N_j)^2] &\tag{11.2.25} \\
= E[(N_{j+k} - f_{j+k-1} N_{j+k-1})^2] &+ f_{j+k-1}^2 E[(N_{j+k-1} - f_{j+k-2} N_{j+k-2})^2] \\
+ \cdots + f_{j+k-1}^2 \cdots f_{j+1}^2 E[(N_{j+1} - f_j N_j)^2].
\end{aligned}$$

The reader is encouraged to go through the steps of this derivation; see Exercise 8 on p. 384. Now, assuming conditions (4) and (5), we obtain

$$\begin{aligned}
E[(N_{j+k} - f_{j+k-1} \cdots f_j N_j)^2] & \\
= \sigma_{j+k-1}^2 EN_{j+k-1} &+ f_{j+k-1}^2 \sigma_{j+k-2}^2 EN_{j+k-2} + \cdots + f_{j+k-1}^2 \cdots f_{j+1}^2 \sigma_j^2 EN_j \\
= EN_j \sum_{l=0}^{k-1} f_j \cdots f_{j+l-1} \sigma_{j+l}^2 [f_{j+l+1} \cdots f_{j+k-1}]^2. &
\end{aligned} \tag{11.2.26}$$

We take relation (11.2.24) as a starting point for finding an approximation to the best predictor of $N_{i,m+k}$ (in the mean square sense) given the past and present observations N_{ii}, \dots, N_{im} . The best predictor is the quantity

$$E(N_{i,m+k} \mid \mathcal{F}_{im}) = f_{m-i+k-1} \cdots f_{m-i} N_{im}, \quad k \geq 1.$$

A candidate for an approximation is obtained by replacing the unknown parameters f_l by their chain ladder estimators:⁵

$$\widehat{N}_{i,m+k} = \widehat{f}_{m-i+k-1}^{(m)} \cdots \widehat{f}_{m-i}^{(m)} N_{im}. \tag{11.2.27}$$

As we explained in the context of one-step ahead prediction, $\widehat{N}_{i,m+k}$ does not have the property of best predictor of $N_{i,m+k}$ (in the mean square sense) given the past and present observations $N_{i,i+j}$, $i+j \leq m$. The additional uncertainty in the chain ladder estimators implies that the prediction error of $\widehat{N}_{i,m+k}$ is larger than the minimal one. This is also seen from the decomposition (11.2.28).

⁵ Here we implicitly assume that the chain ladder estimators on the right-hand side are well-defined. Indeed, an inspection of their definition in (11.2.10) shows that one needs to require that $k \geq i-1$ in order to avoid summation over empty index sets.

Lemma 11.2.6 (Mean square error for the k -steps ahead prediction error)
Assume the conditions of Mack's model with the additional assumption (5). The mean square error $\text{err}_N^{(m)}$ of the k -steps ahead predictor $\widehat{N}_{i,m+k}$ defined in (11.2.27) is given by

$$\begin{aligned} \text{err}_N^{(m)} &= E[(N_{i,m+k} - \widehat{N}_{i,m+k})^2] \\ &= E[(N_{i,m+k} - E(N_{i,m+k} | \mathcal{F}_{im}))^2] + E[(E(N_{i,m+k} | \mathcal{F}_{im}) - \widehat{N}_{i,m+k})^2], \end{aligned} \quad (11.2.28)$$

where

$$\begin{aligned} &E[(N_{i,m+k} - E(N_{i,m+k} | \mathcal{F}_{im}))^2] \\ &= EN_{m-i} \sum_{l=m-i}^{m+k-i-1} [f_{m-i+k-1} \cdots f_{l+1}]^2 \sigma_l^2 f_{l-1} \cdots f_{m-i}. \end{aligned} \quad (11.2.29)$$

The second term in (11.2.28) has the following lower and upper bounds:

$$\begin{aligned} &\sum_{l=m-i}^{m+k-i-1} [f_{m-i+k-1} \cdots f_{l+1}]^2 \sigma_l^2 f_{l-1} \cdots f_{m-i} E\left[\left(\sum_{r=1}^{i-1} N_{r,r+m-i}\right)^{-1}\right] \\ &\leq \frac{E[(E(N_{i,m+k} | \mathcal{F}_{im}) - \widehat{N}_{i,m+k})^2]}{E(N_{m-i}^2)} \\ &\leq \sum_{l=m-i}^{m+k-i-1} [f_{m-i+k-1} \cdots f_{l+1}]^2 a_l, \end{aligned}$$

where

$$\begin{aligned} a_l &= \frac{\sigma_l^2}{m-l-1} \left[\frac{\sigma_{l-1}^2}{m-l} + f_{l-1}^2 \right] \cdots \left[\frac{\sigma_{m-i+1}^2}{i-2} + f_{m-i+1}^2 \right] \times \\ &\quad \times \left[\sigma_{m-i}^2 E\left[\left(\sum_{r=1}^{i-1} N_{r,r+m-i}\right)^{-1}\right] + f_{m-i}^2 \right]. \end{aligned}$$

A sketch of the proof is given as Exercise 10 on p. 384. The interested reader may fill in the gaps in the proof. Of course, the proof of (11.2.29) follows from (11.2.26) by setting $j = m - i$.

Lemma 11.2.6 is not very informative since it does not tell one much about the order of magnitude of the prediction error. The results indicate that the prediction of the future values $N_{i,m+k}$ for $k \geq 2$ is not an easy matter if one uses the naive chain ladder predictors $\widehat{N}_{i,m+k}$. The prediction error is hard to evaluate. It also depends on a large number of parameters which have to be estimated.

Comments

The *chain ladder method* is a popular technique for estimating claim reserves based on historical information about the annual payment numbers and total claim amounts. It consists of a *statistical procedure* which determines the form of the chain ladder estimators $\hat{f}_j^{(m)}$ given in (11.2.10) and suggests predicting future numbers of payments $N_{i,m+k}$, $k \geq 1$, by the quantity $\hat{N}_{i,m+k} = N_{im} \hat{f}_{m-i} \cdots \hat{f}_{m-i+k-1}$. The chain ladder estimators $\hat{f}_j^{(m)}$ and predictors $\hat{N}_{i,m+k}$ have been used in claims reserving problems as an “industrial standard” without theoretical justification.

In order to give some theoretical basis for the chain ladder estimators and associated prediction Schnieper [133] and Mack [100, 101, 102] assumed restrictions on conditional moments such as conditions (4) and (5) which are commonly referred to as *Mack’s model*. These conditions explain the form of the chain ladder estimators and predictors in a natural way. Since chain ladder prediction is based only on aggregated (annual, say) information about the progression of payments arising from claims in one period it is useful as a rough approximation to future aggregated claim amounts and numbers of payments. The predictions can be used for building reserves in a simple way.

Conditions such as (4) and (5) do not explain the dynamics of the underlying claim arrival and payment processes in a satisfactory way. This is not surprising since only the first and second conditional moments of the annual dynamics are specified. Given only these conditions, one cannot simulate the process $(N_j)_{j \geq 0}$. Additional conditions on the underlying point process structure are needed.

Such conditions will be introduced in Section 11.3, where we study some models in the context of the general cluster point process. The underlying claim arrival process (T_n) will be assumed to be Poisson. Even in this “simple” Poissonian case, one cannot reproduce the characteristics described by conditions (4) and (5) in Mack’s model.

Chain ladder estimation and prediction have received much attention in the actuarial literature since the seminal paper by Mack [100] appeared. A recent discussion can be found in Buchwalder et al. [26] and the related paper by Mack et al. [103]. Both papers and the references therein describe the current state of stochastic modeling, estimation and prediction for the chain ladder model.

Exercises

Section 11.2.2

- (1) Show that one can rewrite conditions (11.2.4) and (11.2.5) as requirements on the annual increments $N_{k+1} - N_k$ and $S_{k+1} - S_k$:

$$E(N_{j+1} - N_j | N_0, N_1 - N_0, \dots, N_j - N_{j-1}) = (f_j - 1) N_j,$$

$$E(S_{j+1} - S_j | S_0, S_1 - S_0, \dots, S_j - S_{j-1}) = (g_j - 1) S_j.$$

Show that the right-hand expressions are non-negative, i.e., $f_j \geq 1$ and $g_j \geq 1$.

Section 11.2.3

- (2) (a) Find appropriate moment conditions on N_j , $j = 0, 1, \dots$, such that the strong laws of large numbers (11.2.6)–(11.2.8) hold.
- (b) Find appropriate moment conditions on N_j , $j = 0, 1, \dots$, such that the central limit theorem holds for each of the quantities $\bar{N}_m^{(j)}$, $[s_m^{(j)}]^2$ and $\hat{\gamma}_m(j_1, j_2)$ defined in (11.2.6)–(11.2.8).
- (c) Under appropriate moment conditions on N_j , $j = 0, 1, 2, \dots$, apply the multivariate central limit theorem in order to show joint asymptotic normality for a finite number of covariance estimators $\hat{\gamma}_m(j_1, j_2)$, $0 \leq j_1 < j_2 \leq j_0$, for any $j_0 \geq 1$.
- (d) Consider the quantities

$$\hat{\rho}_m(j_1, j_2) = \frac{\hat{\gamma}_m(j_1, j_2)}{s_m^{(j_1)} s_m^{(j_2)}}, \quad 0 \leq j_1 < j_2 \leq j_0,$$

for any $j_0 \geq 1$. Under the conditions of (a), show that

$$\hat{\rho}_m(j_1, j_2) \xrightarrow{\text{a.s.}} \text{corr}(N_{j_1}, N_{j_2}), \quad m \rightarrow \infty.$$

- (e) Under the conditions of (c), apply the multivariate central limit theorem, the strong law of large numbers and the continuous mapping theorem in order to prove joint asymptotic normality of the quantities

$$\sqrt{m}(\hat{\rho}_m(j_1, j_2) - \text{corr}(N_{j_1}, N_{j_2})), \quad 0 \leq j_1 < j_2 \leq j_0,$$

for any $j_0 \geq 1$.

- (3) Assume the additional Mack condition (5) on p. 374. Show that the asymptotic variance in (11.2.11) is then given by σ_j^2/EN_j .

Section 11.2.4

- (4) Assume Mack's model with the additional condition (5).
 - (a) Consider the estimator $[\hat{\sigma}_j^{(m)}]^2$ of σ_j^2 defined in (11.2.19). Calculate the expectation of this estimator and show that it is asymptotically unbiased.
 - (b) A method of moments estimator alternative to $[\hat{\sigma}_j^{(m)}]^2$ can be obtained from the relation

$$E[(N_{j+1} - f_j N_j)^2/N_j] = \sigma_j^2. \quad (11.2.30)$$

This relation is obtained from condition (5) by taking expectations:

$$E((N_{j+1} - f_j N_j)^2/N_j \mid N_0, \dots, N_j) = \sigma_j^2,$$

Derive this moment estimator, denoted by $[\hat{\sigma}_j^{(m)}]^2$, by first replacing the expectation in (11.2.30) by a sample version and then replacing f_j by $\hat{f}_j^{(m)}$.

- (c) Show that the estimator $[\tilde{\sigma}_j^{(m)}]^2$ from (b) satisfies the relation

$$E([\tilde{\sigma}_j^{(m)}]^2) = \sigma_j^2 + \text{var}(\hat{f}_j^{(m)}) EN_j.$$

Show that the estimator $[\tilde{\sigma}_j^{(m)}]^2$ is asymptotically unbiased.

- (5) Let (Ω, \mathcal{F}, P) be a probability space. A sequence of σ -fields $\mathcal{F}_n \subset \mathcal{F}$, $n = 0, \dots, k$, such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for $0 \leq n \leq k-1$ is called a *filtration*. A sequence $(M_n)_{0 \leq n \leq k}$ of random variables defined on (Ω, \mathcal{F}, P) is called a *martingale* relative to the filtration $(\mathcal{F}_n)_{0 \leq n \leq k}$ if the following three properties hold (see Williams [145], p. 94):

- (i) $E|M_n| < \infty$ for $n = 0, \dots, k$.
- (ii) M_n is measurable with respect to \mathcal{F}_n for every $n = 0, \dots, k$, i.e., the σ -field $\sigma(M_n)$ generated by M_n is contained in \mathcal{F}_n .
- (iii) $E(M_{n+1} | \mathcal{F}_n) = M_n$ for $0 \leq n \leq k-1$.

Consider the sequence of chain ladder estimators $\hat{f}_j^{(m)}$ of f_j for fixed m and define $\tilde{f}_j^{(m)} = \hat{f}_j^{(m)} / f_j$. Assume the conditions of Mack's model. Define

$$M_n = \tilde{f}_0^{(m)} \cdots \tilde{f}_n^{(m)}$$

and $\mathcal{G}_n^{(m)} = \sigma(\{N_{i,i+k} : 0 \leq k \leq n, i+k \leq m\})$, $n = 0, \dots, m-2$.

- (a) Show that $(M_n)_{n=0,\dots,m-2}$ constitutes a martingale relative to the filtration $(\mathcal{G}_n^{(m)})_{n=0,\dots,m-2}$.
- (b) Show that $E(M_{n+k} | \mathcal{G}_n^{(m)}) = M_n$ a.s. for $k = 0, 1, \dots$, such that $n+k \leq m-2$.
- (c) Show that $EM_n = 1$ for $n = 0, \dots, m-2$.
- (d) Show that the *martingale difference sequence* $X_n = M_n - M_{n-1}$, $n = 1, \dots, m-2$, consists of uncorrelated random variables provided that $E(X_n^2) < \infty$, $n = 1, \dots, m-2$.
- (e) Give conditions on N_j , $j = 0, 1, \dots$, ensuring that $E(X_n^2) < \infty$ for $n = 1, \dots, m-2$.
- (f) Assume that $E(X_n^2) < \infty$ for $n = 1, \dots, m-2$. Show that

$$\hat{f}_0^{(m)} \cdots \hat{f}_n^{(m)} f_{n+1} \cdots f_{n+k} \quad \text{for } n+k \leq m-2$$

minimizes the mean square error $E[(\hat{f}_0 \cdots \hat{f}_{n+k} - M)^2]$ in the class of all finite variance random variables M which are measurable with respect to $\mathcal{G}_n^{(m)}$.

- (6) Prove the inequality (11.2.18). Hint: use the elementary inequality

$$(m-j-1) \min(N_{1,1+j}, \dots, N_{m-j-1,m-1}) \leq \sum_{i=1}^{m-j-1} N_{i,i+j},$$

and the relation $EA = \int_0^\infty P(A > x) dx$ which is valid for any random variable $A \geq 0$ a.s.

Section 11.2.5

- (7) Use the recursive formula (11.2.23) in order to express $E(N_{m-i}^2)$ as a function of EN_0 , $E(N_0^2)$ and the parameters f_k and σ_k^2 .

- (8) Verify the decomposition of the k -steps ahead prediction error $E[(N_{j+k} - f_{j+k-1} \cdots f_j N_j)^2]$ in (11.2.25).
 Hint: Observe that the random variables $N_{j+1} - f_j N_j$, $j = 0, 1, \dots$ are uncorrelated.
- (9) Prove relation (11.2.33).
- (10) Below we give a sketch of the proof of Lemma 11.2.6. The interested reader is encouraged to go through the steps of this proof and to fill in the details. We start by observing that

$$\begin{aligned} \text{err}_N^{(m)} &= E[(N_{i,m+k} - \hat{N}_{i,m+k})^2] \\ &= E[(N_{i,m+k} - E(N_{i,m+k} | \mathcal{F}_{im}))^2] + E[(E(N_{i,m+k} | \mathcal{F}_{im}) - \hat{N}_{i,m+k})^2], \end{aligned} \quad (11.2.31)$$

where we used the independence of $\hat{f}_{m-i+k-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)}$ and $(N_{im}, N_{i,m+k})$:

$$\begin{aligned} &E\left[(N_{i,m+k} - E(N_{i,m+k} | \mathcal{F}_{im})) (E(N_{i,m+k} | \mathcal{F}_{im}) - \hat{N}_{i,m+k})\right] \\ &= E\left[(N_{i,m+k} - E(N_{i,m+k} | \mathcal{F}_{im})) \times \right. \\ &\quad \left. \times (f_{m-i+k-1} \cdots f_{m-i} - \hat{f}_{m-i+k-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)}) N_{im}\right] \\ &= E\left[(N_{i,m+k} - E(N_{i,m+k} | \mathcal{F}_{im})) N_{im}\right] \times \\ &\quad \times E[f_{m-i+k-1} \cdots f_{m-i} - \hat{f}_{m-i+k-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)}]. \end{aligned}$$

The first expectation on the right-hand side vanishes since it can be written in the form

$$E\left[E\left[N_{i,m+k} - E(N_{i,m+k} | \mathcal{F}_{im}) | \mathcal{F}_{im}\right] N_{im}\right] = 0.$$

The first term in (11.2.31) can be read off from (11.2.26) by setting $j = m-i$: it is the mean square error of the best predictor of $N_{i,m+k}$ given \mathcal{F}_{im} .

We proceed by bounding $E[(E(N_{i,m+k} | \mathcal{F}_{im}) - \hat{N}_{i,m+k})^2]$ from above and below. Since N_{im} is independent of $\hat{f}_{m+k-i-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)}$ we obtain

$$\begin{aligned} &E[(E(N_{i,m+k} | \mathcal{F}_{im}) - \hat{N}_{i,m+k})^2] \\ &= E(N_{im}^2) E[(\hat{f}_{m+k-i-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)} - f_{m+k-i-1} \cdots f_{m-i})^2] \\ &= E(N_{im}^2) E\left[\left(\sum_{l=m-i}^{m+k-i-1} \hat{f}_{m-i}^{(m)} \cdots \hat{f}_{l-1}^{(m)} (\hat{f}_l^{(m)} - f_l) f_{l+1} \cdots f_{m+k-i-1}\right)^2\right]. \end{aligned} \quad (11.2.32)$$

In the last step we used the elementary equality

$$\prod_{r=1}^n a_r - \prod_{r=1}^n b_r = \sum_{r=1}^n a_1 \cdots a_{r-1} (a_r - b_r) b_{r+1} \cdots b_n \quad (11.2.33)$$

which is valid for all real $a_i, b_i, i = 1, \dots, n$, and $n \geq 1$. A conditioning argument shows that the quantities $(f_l - \hat{f}_l^{(m)})\hat{f}_{l-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)}$ in (11.2.32) are uncorrelated and therefore

$$\begin{aligned} & E[(E(N_{i,m+k} | \mathcal{F}_{im}) - \hat{N}_{i,m+k})^2] \\ &= E(N_{im}^2) \sum_{l=m-i}^{m+k-i-1} [f_{m+k-i-1} \cdots f_{l+1}]^2 E[(f_l - \hat{f}_l^{(m)})^2 [\hat{f}_{l-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)}]^2]. \end{aligned}$$

Now consider a typical term in the sum above. Condition on $\mathcal{G}_l^{(m)}$ and apply (11.2.17) to obtain

$$\begin{aligned} & E \left[(f_l - \hat{f}_l^{(m)})^2 [\hat{f}_{l-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)}]^2 \middle| \mathcal{G}_l^{(m)} \right] \\ &= E[(f_l - \hat{f}_l^{(m)})^2 | \mathcal{G}_l^{(m)}] [\hat{f}_{l-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)}]^2 \\ &= \sigma_l^2 \left(\sum_{r=1}^{m-l-1} N_{r,r+l} \right)^{-1} [\hat{f}_{l-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)}]^2. \end{aligned} \quad (11.2.34)$$

So far we have not used inequalities. However, it seems difficult to calculate the expected value of the right-hand expression. Therefore we leave the path of precise calculations and first apply lower bounds. We observe that

$$\begin{aligned} & E \left[\left(\sum_{r=1}^{m-l-1} N_{r,r+l} \right)^{-1} [\hat{f}_{l-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)}]^2 \middle| \mathcal{G}_{l-1}^{(m)} \right] \\ &= E \left(\frac{\left(\sum_{r=1}^{m-l} N_{r,r+l} \right)^2}{\sum_{r=1}^{m-l-1} N_{r,r+l}} \middle| \mathcal{G}_{l-1}^{(m)} \right) \left(\sum_{r=1}^{m-l} N_{r,r+l-1} \right)^{-2} [\hat{f}_{l-2}^{(m)} \cdots \hat{f}_{m-i}^{(m)}]^2 \\ &\geq E \left(\sum_{r=1}^{m-l} N_{r,r+l} \middle| \mathcal{G}_{l-1}^{(m)} \right) \left(\sum_{r=1}^{m-l} N_{r,r+l-1} \right)^{-2} [\hat{f}_{l-2}^{(m)} \cdots \hat{f}_{m-i}^{(m)}]^2 \\ &= f_{l-1} \left(\sum_{r=1}^{m-l} N_{r,r+l-1} \right)^{-1} [\hat{f}_{l-2}^{(m)} \cdots \hat{f}_{m-i}^{(m)}]^2. \end{aligned}$$

In the last step we used (11.2.12). Taking expectations in the above bounds and using induction, we obtain the lower bound

$$E \left[\frac{[\hat{f}_{l-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)}]^2}{\sum_{r=1}^{m-l-1} N_{r,r+l}} \right] \geq f_{l-1} \cdots f_{m-i} E \left[\frac{1}{\sum_{r=1}^{i-1} N_{r,r+m-i}} \right].$$

A combination of the estimates above yields the desired lower bound of the prediction error.

For the upper bound we proceed in a similar way. We start from (11.2.34) and take expectations. Since $N_j \geq 1$ a.s. for every $j \geq 0$ we get the obvious bound

$$\begin{aligned}
& E \left[(f_l - \hat{f}_l^{(m)})^2 [\hat{f}_{l-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)}]^2 \right] \\
& \leq \frac{\sigma_l^2}{m-l-1} E \left[[\hat{f}_{l-1}^{(m)} \cdots \hat{f}_{m-i}^{(m)}]^2 \right] \\
& = \frac{\sigma_l^2}{m-l-1} E \left[[(\hat{f}_{l-1}^{(m)} - f_{l-1})^2 + f_{l-1}^2] [\hat{f}_{l-2}^{(m)} \cdots \hat{f}_{m-i}^{(m)}]^2 \right].
\end{aligned}$$

In the last equality we used a conditioning argument. Multiple use of (11.2.34) and induction yield a bound for the right-hand side

$$\begin{aligned}
& \leq \frac{\sigma_l^2}{m-l-1} \left[\frac{\sigma_{l-1}^2}{m-l} + f_{l-1}^2 \right] E \left[[\hat{f}_{l-2}^{(m)} \cdots \hat{f}_{m-i}^{(m)}]^2 \right] \\
& \leq \frac{\sigma_l^2}{m-l-1} \left[\frac{\sigma_{l-1}^2}{m-l} + f_{l-1}^2 \right] \cdots \left[\frac{\sigma_{m-i+1}^2}{i-2} + f_{m-i+1}^2 \right] \times \\
& \quad \times \left[\sigma_{m-i}^2 E \left[\frac{1}{\sum_{r=1}^{i-1} N_{r,r+m+i}} \right] + f_{m-i}^2 \right].
\end{aligned}$$

Collecting the above bounds, we arrive at the desired upper bound for the prediction error.

11.3 An Informal Discussion of a Cluster Model with Poisson Arrivals

11.3.1 Specification of the Model

Recall the general cluster model given in Definition 11.1.1. The model describes the arrival times and the payment process (number of payments and corresponding total amount) for claims in a portfolio. In this context, the arrival time T_i of the i th claim is marked by the random element

$$A_i = ((Y_{ik})_{k \geq 1}, (X_{ik})_{k \geq 1}, K_i), \quad i = 1, 2, \dots.$$

These points assume values in the mark space

$$E_A = (0, \infty)^\infty \times (0, \infty)^\infty \times \mathbb{N}.$$

For elements $x = (x_k)$ and $y = (y_k)$ in the sequence space $(0, \infty)^\infty$ the metric

$$\rho(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}$$

makes $(0, \infty)^\infty$ a complete separable metric space; see Billingsley [17], Appendix 1. The product spaces E_A and

$$E = (0, \infty) \times E_A$$

are then also complete separable metric spaces. Therefore we can define open sets in E and E_A which generate the corresponding Borel σ -fields; see Billingsley [17] for an introduction to probability theory on metric spaces.

In addition to the assumptions (1)–(2) of the general cluster model on p. 363 we require the following conditions:

- (3) *The arrival times $0 < T_1 < T_2 < \dots$ constitute a homogeneous Poisson process on $(0, \infty)$ with intensity $\lambda > 0$.*
- (4) *The sequence (A_i) consists of iid random elements with common distribution F_A on E_A . This means that the claim histories corresponding to different claims are mutually independent and have the same distribution.*
- (5) *The sequences (A_i) and (T_i) are independent.*
- (6) *The sequence $(X_{ik})_{k \geq 1}$ consists of iid positive claim sizes with distribution F .*
- (7) *For each $i \geq 1$, K_i and the sequences $(X_{ik})_{k \geq 1}$, $(Y_{ik})_{k \geq 1}$ are mutually independent.*

In what follows, we write K, A, X, \dots for generic elements of the iid sequences $(K_i), (A_i), (X_{ik}), \dots$, and F_K, F_A, F_X, \dots , for their common distributions.

The random variable K_i is an integer-valued random variable describing the number of payments X_{ij} , $j = 1, \dots, K_i$, from the insurer to the insured for the i th claim in the portfolio. At the instant of time $T_{ij} = T_i + \sum_{k=1}^j Y_{ik}$ the amount X_{ij} is paid. We will also consider the case when $K_i = \infty$ a.s., for example, when the points $\sum_{k=1}^j Y_{ik}$, $j = 1, 2, \dots$, constitute a Poisson process; see Section 11.3.3.

We interpret (T_i, A_i) , $i = 1, 2, \dots$, as the points of a marked PRM with state space E . As a matter of fact, E is not a subset of a Euclidean space. However, the point process theory can be extended to state spaces which are complete separable metric spaces. In this treatment we do not want to struggle with the abstract point process theory. We will proceed without checking all the details of the general theory but we mention that the results in Chapter 7 remain valid for complete separable metric state spaces E . The interested reader is referred to rigorous treatments as given in Daley and Vere-Jones [38, 39, 40].

The distribution of the PRM N is determined by its mean measure on E . Since N is a marked PRM the mean measure has product form

$$\lambda \text{Leb} \times F_A,$$

and since the components of A are independent the mark distribution F_A has product structure as well:

$$F_A = F_Y \times F_X \times F_K = F_Y \times F^\infty \times F_K.$$

The distribution $F_X = F^\infty = F \times F \times \dots$ is an infinite product measure on $(0, \infty)^\infty$ since the sequence X consists of iid random variables with common distribution F ; see for example Billingsley [18].

Thus we have collected all ingredients of the marked PRM($\lambda \text{Leb} \times F_A$) with representation

$$N = \sum_{i=1}^{\infty} \varepsilon_{(T_i, A_i)}.$$

For this point process we can define Poisson integrals, and these integrals have the same properties as for a state space $E \subset \mathbb{R}^d$. In what follows, we will use Poisson integrals with respect to N in order to describe the payment numbers and total claim amounts in the general cluster model and to determine some of their distributional and moment properties.

Example 11.3.1 (The chain ladder revisited)

We reconsider the payment numbers and total claim amounts arising from disjoint intervals of time in the framework of the chain ladder model with more concrete distributional assumptions of the marked Poisson process; see Section 11.2.1. For this reason, we consider the periods of time (years say)

$$C_i = (i-1, i], \quad i = 1, 2, \dots$$

and the aggregated periods

$$C_{i,i+j} = C_i \cup \dots \cup C_{i+j}, \quad j = 0, 1, \dots$$

The number of payments for the claims which occur in year C_i and for which payments are executed in the period $C_{i,i+j}$ for some $j \geq 0$ is given by the Poisson integral

$$N_{i,i+j} = \int_{C_i \times E_A} \sum_{l=1}^k I_{\{t+z_l \in C_{i,i+j}\}} N(dt, d(y_r), d(x_r), dk) \quad (11.3.35)$$

$$= \int_E \sum_{l=1}^k I_{\{t \in C_i, t+z_l \in C_{i,i+j}\}} N(dt, d(y_r), d(x_r), dk), \quad j \geq 0,$$

where $z_l = y_1 + \dots + y_l$, $l \geq 1$. Writing for $n \geq 1$

$$Z_{nl} = Y_{n1} + \dots + Y_{nl}, \quad T_{nl} = T_n + Z_{nl}, \quad l \geq 1,$$

we get an alternative expression which is more compact and clear:

$$N_{i,i+j} = \sum_{n=1}^{\infty} \sum_{l=1}^{K_n} I_{\{T_n \in C_i, T_{nl} \in C_{i,i+j}\}}, \quad j \geq 0.$$

Similarly, the total amount of payments for those claims which occurred in year C_i and triggered payments in $C_{i,i+j}$ is given by the Poisson integral

$$\begin{aligned}
S_{i,i+j} &= \int_{C_i \times E_A} \sum_{l=1}^k x_l I_{\{t+z_l \in C_{i,i+j}\}} N(dt, d(y_r), d(x_r), dk) \\
&= \sum_{n=1}^{\infty} \sum_{l=1}^{K_n} X_{nl} I_{\{T_n \in C_i, T_{nl} \in C_{i,i+j}\}}, \quad j \geq 0. \tag{11.3.36}
\end{aligned}$$

The sequences $((N_{i,i+j}, S_{i,i+j}))_{j \geq 0}$, $i = 1, 2, \dots$, are mutually independent. This is due to the fact that the integration areas $C_i \times E_A$ in the Poisson integrals (11.3.35) and (11.3.36) are disjoint for distinct years C_i . This follows from Lemma 7.2.10 adapted to the abstract state space E . Moreover, $(N_{i,i+j})_{j \geq 0} \stackrel{d}{=} (N_{1,1+j})_{j \geq 0}$ for $i \geq 1$, and a similar remark applies to $(S_{i,i+j})_{j \geq 0}$. This follows by a conditioning argument in view of the mutual independence of (T_n) , (A_n) and the homogeneity of the Poisson arrival process with points T_n . The verification of this property is left as Exercise 1 on p. 402. Therefore the sequences of the Poisson integrals $((N_{i,i+j}, S_{i,i+j}))_{j \geq 0}$, $i = 1, 2, \dots$, are iid. Hence condition (3) in Mack's model on p. 366 is satisfied.

For a given year C_i , the integrands in the Poisson integrals $N_{i,i+j}, S_{i,i+j}$, $j = 0, 1, \dots$, do not, in general, have disjoint support and therefore these integrals are dependent.

Whereas it is easy to verify Mack's condition (3) we could not find a specification of the PRM N which guarantees the validity of Mack's condition (4). In Section 11.3.3, where we assume that the cluster processes starting at the Poisson points T_i are also homogeneous Poisson processes, we can show that condition (4) is not satisfied. \square

11.3.2 An Analysis of the First and Second Moments

In this section we consider the marked point process specified in Section 11.3.1 without the claim size component $(X_{ir})_{r \geq 1}$. Abusing notation, we use the same symbols for the point process, the marks and the state space as in the previous section: $N = \sum_{i=1}^{\infty} \varepsilon_{(T_i, A_i)}$ denotes the marked PRM($\lambda \text{Leb} \times F_A$) on $E = (0, \infty) \times E_A$ with $E_A = (0, \infty)^{\infty} \times \mathbb{N}$ and

$$A_i = ((Y_{ir})_{r \geq 1}, K_i).$$

In what follows, we determine the expected value, variance and covariance structure of the process

$$M(b) = \int_{(0,1] \times E_A} \sum_{l=1}^k I_{\{t+z_l \in (0,b]\}} N(dt, d(y_r), dk), \quad b \geq 1. \tag{11.3.37}$$

For fixed $b \geq 1$ the random variable $M(b)$ describes the number of payments in $(0, b]$ for claims which occurred in the first year $(0, 1]$. We can study the corresponding total claim amounts but these calculations are more involved than

the following ones; the interested reader may go through them as an exercise. The results for $b < 1$ yield slightly different formulae but the calculations are completely analogous.

The first and second moments of the process M give us some insight into the order of magnitude of the random variables $M(b)$, $b \geq 1$, and their dependence structure. We will apply Lemmas 7.2.7 and 7.2.12 which remain valid for the marked PRM N with the metric state space E .

Lemma 11.3.2 *Let U be a uniform $U(0, 1)$ random variable, independent of the generic sequence $(Y_n) = (Y_{1n})$. Write $Z_l = Y_1 + \dots + Y_l$, $l \geq 1$, and $\bar{F}_K = 1 - F_K$.*

- (1) *The Poisson integrals $M(b)$, $b \geq 1$, are finite with probability 1.*
- (2) *If $EK < \infty$ then $M(b)$ has finite expected value given by*

$$EM(b) = \lambda \sum_{l=1}^{\infty} \bar{F}_K(l-1) P(U + Z_l \leq b), \quad b \geq 1. \quad (11.3.38)$$

- (3) *If in addition $E(K^2) < \infty$ then for $1 \leq b_1 \leq b_2$,*

$$\begin{aligned} & \text{cov}(M(b_1), M(b_2)) \\ &= \lambda \sum_{l=1}^{\infty} \bar{F}_K(l-1) l P(U + Z_l \leq b_1) \\ & \quad + \lambda \sum_{l_2=2}^{\infty} \bar{F}_K(l_2-1) \sum_{l_1=1}^{l_2-1} P(U + Z_{l_1} \leq b_1, U + Z_{l_2} \leq b_2), \end{aligned} \quad (11.3.39)$$

In particular, setting $b_1 = b_2 = b \geq 1$,

$$\text{var}(M(b)) = \lambda \sum_{l=1}^{\infty} (2l-1) \bar{F}_K(l-1) P(U + Z_l \leq b) < \infty. \quad (11.3.40)$$

Proof. (1) Consider the function

$$g_b(t, y, k) = \sum_{l=1}^k I_{\{t \in (0, 1], t+z_l \in (0, b]\}}, \quad b \geq 1,$$

where $(t, y, k) \in (0, 1] \times E_A$ and $z_l = y_1 + \dots + y_l$, $l \geq 1$, for $y = (y_r)_{r \geq 1}$. Since g_b assumes values in the set of non-negative integers,

$$\int_{(0,1] \times E_A} \min(g_b(t, y, k), 1) dt F_A(dy, dk) \leq 1 < \infty.$$

According to Lemma 7.2.7(2), this condition implies that the Poisson integral $M(b) = \int_E g_b dN$ is finite a.s.

(2) By construction of g_b , $g_b \leq k$. Therefore

$$M(b) = \int_E g_b(t, y, k) N(dt, dy, dk) \leq \int_E k N(dt, dy, dk).$$

By virtue of Lemma 7.2.12(1), the right-hand side has expectation given by

$$\lambda \int_{(0,1] \times E_A} k dt F_A(dy, dk) = \lambda \sum_{k=1}^{\infty} k P(K=k) = \lambda EK.$$

Since we assume $EK < \infty$, $M(b)$ has finite mean. From Lemma 7.2.12(1) we also have that

$$\begin{aligned} EM(b) &= \lambda \int_{(0,1] \times E_A} g_b(t, y, k) dt F_A(dy, dk) \\ &= \lambda \int_0^1 E \left[\sum_{l=1}^K I_{\{t+Z_l \leq b\}} \right] dt. \end{aligned}$$

A conditioning argument (on K) yields that

$$\begin{aligned} EM(b) &= \lambda \int_0^1 E \left[\sum_{l=1}^K P(t + Z_l \leq b) \right] dt \\ &= \lambda E \left[\sum_{l=1}^K P(U + Z_l \leq b) \right], \end{aligned}$$

where U is a uniform $U(0, 1)$ random variable, and K , U and (Z_l) are independent. An interchange of the order of summation in

$$EM(b) = \lambda \sum_{k=1}^{\infty} P(K=k) \sum_{l=1}^k P(U + Z_l \leq b)$$

yields the desired formula (11.3.38).

(3) Since g_b assumes the non-negative integers as values, an appeal to Lemma 7.2.12(2) implies that the condition

$$\int_E [g_b(t, y, k)]^2 dt F_A(dy, dk) < \infty \quad (11.3.41)$$

is necessary and sufficient for $\text{var}(M(b)) < \infty$. An argument similar to the one above shows that $E(K^2) < \infty$ is a sufficient condition for (11.3.41) to hold. The verification is left as Exercise 2 on p. 402.

We know from Lemma 7.2.12(3) that for $1 \leq b_1 \leq b_2$,

$$c(g_{b_1}, g_{b_2}) = \text{cov}(M(b_1), M(b_2)) = \lambda \int_E g_{b_1}(t, y, k) g_{b_2}(t, y, k) dt F_A(dy, dk).$$

Hence, for independent K , (Z_l) , U as above, a conditioning argument and Fubini's theorem yield

$$\begin{aligned} c(g_{b_1}, g_{b_2}) &= \lambda E \left(\sum_{l_1=1}^K \sum_{l_2=1}^K P(U + Z_{l_1} \leq b_1, U + Z_{l_2} \leq b_2) \right) \\ &= \lambda \sum_{k=1}^{\infty} P(K = k) \sum_{l_1=1}^k \sum_{l_2=1}^k P(U + Z_{l_1} \leq b_1, U + Z_{l_2} \leq b_2) \\ &= \lambda \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} P(U + Z_{l_1} \leq b_1, U + Z_{l_2} \leq b_2) P(K \geq \max(l_1, l_2)). \end{aligned}$$

Now observe that $\{U + Z_{l_1} \leq b_1\} \subset \{U + Z_{l_2} \leq b_2\}$ for $l_1 \geq l_2$. Thus we get the simpler expression

$$\begin{aligned} c(g_{b_1}, g_{b_2}) &= \lambda \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{l_1} P(U + Z_{l_1} \leq b_1) \bar{F}_K(l_1 - 1) \\ &\quad + \lambda \sum_{l_1=1}^{\infty} \sum_{l_2=l_1+1}^{\infty} P(U + Z_{l_1} \leq b_1, U + Z_{l_2} \leq b_2) \bar{F}_K(l_2 - 1) \\ &= \lambda \sum_{l=1}^{\infty} l P(U + Z_l \leq b_1) \bar{F}_K(l - 1) \\ &\quad + \lambda \sum_{l_2=2}^{\infty} \bar{F}_K(l_2 - 1) \sum_{l_1=1}^{l_2-1} P(U + Z_{l_1} \leq b_1, U + Z_{l_2} \leq b_2). \end{aligned}$$

This is the desired expression (11.3.39). In particular, for $b_1 = b_2 = b$ since $\{U + Z_{l_2} \leq b\} \subset \{U + Z_{l_1} \leq b\}$ for $l_2 \geq l_1$,

$$\text{var}(M(b)) = \lambda \sum_{l=1}^{\infty} \bar{F}_K(l - 1) (2l - 1) P(U + Z_l \leq b).$$

□

We continue by considering some particular cases.

Example 11.3.3 (The number of payments is constant)

If one assumes that the number of payments K_i for the i th claim in the first year is a known integer $k \geq 1$ the formulae in Lemma 11.3.2 simplify. For example,

$$EM(b) = \lambda \sum_{l=1}^k P(U + Z_l \leq b),$$

$$\text{var}(M(b)) = \lambda \sum_{l=1}^k (2l - 1) P(U + Z_l \leq b).$$

□

Example 11.3.4 (Payments at the arrivals of a stopped Poisson process)

Assume that at each claim arrival T_i a homogeneous Poisson process with points $T_{ij} - T_i$ and intensity $\gamma > 0$ starts. This means that $Z_l = Y_1 + \dots + Y_l$ for an iid exponential $\text{Exp}(\gamma)$ sequence (Y_k) . In particular, Z_l has a $\Gamma(l, \gamma)$ distribution with density

$$f_{Z_l}(s) = e^{-\gamma s} \gamma^l \frac{s^{l-1}}{(l-1)!}, \quad s \geq 0.$$

We obtain

$$\begin{aligned} EM(b) &= \lambda \sum_{l=1}^{\infty} \bar{F}_K(l-1) P(Z_l \leq b - U) \\ &= \lambda \sum_{l=1}^{\infty} \bar{F}_K(l-1) \int_{b-1}^b P(Z_l \leq t) dt \\ &= \lambda \sum_{l=1}^{\infty} \bar{F}_K(l-1) \int_{b-1}^b \int_0^t e^{-\gamma s} \gamma^l \frac{s^{l-1}}{(l-1)!} ds dt \\ &= \lambda \gamma \int_{b-1}^b \int_0^t \left[\sum_{l=0}^{\infty} \bar{F}_K(l) e^{-\gamma s} \frac{(\gamma s)^l}{l!} \right] ds dt, \end{aligned} \tag{11.3.42}$$

where we used Fubini's theorem in the last step. Write $(\tilde{N}(s))_{s \geq 0}$ for a homogeneous Poisson process on $(0, \infty)$ with intensity γ which is independent of K . Then a conditioning argument and another application of Fubini's theorem show that

$$\begin{aligned} EM(b) &= \lambda \gamma \int_{b-1}^b \int_0^t P(K \geq \tilde{N}(s) + 1) ds dt \\ &= \lambda \gamma \int_0^b \min(b-s, 1) P(K \geq \tilde{N}(s) + 1) ds. \end{aligned}$$

If K has a geometric distribution given by

$$P(K = k) = p q^{k-1}, \quad k = 1, 2, \dots.$$

for some $p \in (0, 1)$, $q = 1 - p$, one can evaluate $EM(b)$ explicitly:

$$EM(b) = \frac{\lambda}{p} \left(1 - \frac{e^{-\gamma p b} (e^{\gamma p} - 1)}{\gamma p} \right). \quad (11.3.43)$$

For an evaluation of $\text{var}(M(b))$, see Exercise 8 on p. 403. \square

Example 11.3.5 (Payments at iid instants of time)

Here we assume that the payments for the i th claim after the Poisson arrival T_i are made at the times $T_i + Z_{ij}$, $j = 1, \dots, K_i$, where $(Z_{ij})_{j \geq 1}$ constitutes an iid sequence of positive random variables. Moreover, the sequences $((Z_{ij})_{j \geq 1}, K_i)$ are iid and independent of (T_i) , and K_i and $(Z_{ij})_{j \geq 1}$ are independent for $i = 1, 2, \dots$. This means that the times Z_{ij} , $j = 1, \dots, K_i$, constitute a random sample, in particular they are not ordered and therefore it seems that they do not fit into the model considered in this section. This is however not correct since the distribution of $(M(b))_{b \geq 1}$ does not change if Z_{ij} , $j = 1, \dots, K_i$ is replaced by its ordered sample.

For calculations of moments it is convenient to go through the steps of the proof of Lemma 11.3.2 which become much simpler with iid random variables Z_{ij} . For example, one can derive the following moments. In what follows, Z , Z_1 and Z_2 are iid with the same distribution as Z_{ij} , independent of the uniform $U(0, 1)$ variable U .

$$EM(b) = \lambda P(U + Z \leq b) EK,$$

$$\text{var}(M(b)) = \lambda P(U + Z \leq b) EK$$

$$+ \lambda P(U + Z_1 \leq b, U + Z_2 \leq b) [E(K^2) - EK].$$

\square

11.3.3 A Model when Clusters are Poisson Processes

In this section we modify the model of Section 11.3.1 in a different way. We assume $K_i = \infty$ a.s. and let the points $T_{ij} - T_i = \sum_{k=1}^j Y_{ik}$ constitute a Poisson process on $(0, \infty)$ with mean measure μ . In this case it is reasonable to consider the PRM N with points (T_i, A_i) , where

$$A_i = ((Y_{ik})_{k \geq 1}, (X_{ik})_{k \geq 1}), \quad i = 1, 2, \dots,$$

and with state space $E = (0, \infty) \times E_A$, $E_A = (0, \infty)^\infty \times (0, \infty)^\infty$. This means that we lose K_i as an additional mark of T_i . Under this assumption various calculations will become simpler than in Section 11.3.2.

We are interested in the stochastic processes

$$\begin{aligned}
M(b) &= \int_{(0,1] \times E_A} \sum_{l=1}^{\infty} I_{\{t+z_l \in (0,b]\}} N(dt, d(y_r), d(x_r)) \\
&= \sum_{i=1}^{\infty} \sum_{l=1}^{\infty} I_{\{T_i \in (0,1], T_{il} \in (0,b]\}} , \quad b \geq 1 , \\
S(b) &= \int_{(0,1] \times E_A} \sum_{l=1}^{\infty} x_l I_{\{t+z_l \in (0,b]\}} N(dt, d(y_r), d(x_r)) \\
&= \sum_{i=1}^{\infty} \sum_{l=1}^{\infty} X_{il} I_{\{T_i \in (0,1], T_{il} \in (0,b]\}} , \quad b \geq 1 .
\end{aligned}$$

The quantity $M(b)$ represents the number of payments from the insurer to the insured in $(0, b]$ and $S(b)$ is the corresponding total amount of payments for those claims that occurred in the period $(0, 1]$. We again have Poisson integral representations:

$$\begin{aligned}
M(b) &= \int_{(0,1] \times E_A} g_b dN \quad \text{with} \quad g_b(t, y, x) = \sum_{l=1}^{\infty} I_{\{t+z_l \in (0,b]\}} , \\
S(b) &= \int_{(0,1] \times E_A} h_b dN \quad \text{with} \quad h_b(t, y, x) = \sum_{l=1}^{\infty} x_l I_{\{t+z_l \in (0,b]\}} .
\end{aligned}$$

In what follows, we write $\mu(t) = \mu[0, t]$ for $t \geq 0$.

Lemma 11.3.6 (Expectation, variance and covariance structure of $M(b)$ and $S(b)$)

(1) *We have for any $b \geq 1$,*

$$EM(b) = \lambda \int_{b-1}^b \mu(t) dt \quad \text{and} \quad ES(b) = \lambda EX_{11} \int_{b-1}^b \mu(t) dt ,$$

where we assume in the latter case that $EX_{11} < \infty$.

(2) *We have for $1 \leq b_1 \leq b_2$,*

$$\begin{aligned}
\text{cov}(M(b_1), M(b_2)) &= \lambda \int_{b_1-1}^{b_1} \mu(t) [1 + \mu(b_2 - b_1 + t)] dt , \\
\text{cov}(S(b_1), S(b_2)) &= \lambda \int_{b_1-1}^{b_1} \mu(t) [E(X_{11}^2) + (EX_{11})^2 \mu(b_2 - b_1 + t)] dt ,
\end{aligned}$$

where we assume in the latter case that $\text{var}(X_{11}) < \infty$.

Proof. By assumption,

$$\Pi(t) = \sum_{l=1}^{\infty} I_{\{Z_l \leq t\}} , \quad t > 0 ,$$

constitutes a Poisson process on $(0, \infty)$ with mean measure μ . Then we immediately conclude from Lemma 7.2.12(1) and the Poisson integral representation of $M(b)$ for $b \geq 1$ that

$$\begin{aligned} EM(b) &= \lambda \int_{(0,1] \times E_A} g_b(t, y, x) dt F_A(dy, dx) \\ &= \lambda \int_0^1 E \left(\sum_{l=1}^{\infty} I_{\{t+Z_l \in (0,b]\}} \right) dt \\ &= \lambda \int_0^1 E\Pi(b-t) dt = \lambda \int_{b-1}^b \mu(t) dt. \end{aligned}$$

Similarly, for the iid sequence $(X_l) = (X_{1l})$ which is independent of $(Y_l) = (Y_{1l})$,

$$\begin{aligned} ES(b) &= \lambda \int_{(0,1] \times E_A} h_b(t, y, x) dt F_A(dy, dx) \\ &= \lambda \int_0^1 E \left(\sum_{l=1}^{\infty} X_l I_{\{t+Z_l \in (0,b]\}} \right) dt \\ &= \lambda EX_{11} \int_0^1 \sum_{l=1}^{\infty} P(Z_l \leq b-t) dt \\ &= \lambda EX_{11} \int_0^1 E\Pi(b-t) dt = \lambda EX_{11} \int_{b-1}^b \mu(t) dt. \end{aligned}$$

Now assume $1 \leq b_1 \leq b_2$. The independent increments of the Poisson process Π imply that $\text{cov}(\Pi(s), \Pi(t)) = \text{var}(\Pi(s)) = \mu(s)$ for $s \leq t$. The covariance function can be evaluated by using Lemma 7.2.12(3):

$$\begin{aligned} \text{cov}(M(b_1), M(b_2)) &= \lambda \int_{(0,1] \times E_A} g_{b_1}(t, y, x) g_{b_2}(t, y, x) dt F_A(dy, dx) \\ &= \lambda \int_0^1 E \left(\sum_{l_1=1}^{\infty} I_{\{t+Z_{l_1} \leq b_1\}} \sum_{l_2=1}^{\infty} I_{\{t+Z_{l_2} \leq b_2\}} \right) dt \\ &= \lambda \int_0^1 [\text{cov}(\Pi(b_1 - t), \Pi(b_2 - t)) + E\Pi(b_1 - t) E\Pi(b_2 - t)] dt \\ &= \lambda \int_0^1 \mu(b_1 - t) [1 + \mu(b_2 - t)] dt. \end{aligned}$$

The substitution $z = b_1 - t$ yields the desired result.

Now introduce the compound Poisson process

$$C(t) = \sum_{l=1}^{\Pi(t)} X_l = \sum_{l=1}^{\infty} X_l I_{\{Z_l \leq t\}}, \quad t \geq 0.$$

It has independent increments and therefore its covariance function is given by $\text{cov}(C(s), C(t)) = \text{var}(C(s)) = E(X_{11}^2) \mu(s)$ for $s \leq t$. Having this in mind, we obtain

$$\begin{aligned} & \text{cov}(S(0, b_1], S(0, b_2]) \\ &= \lambda \int_{(0,1] \times E_A} h_{b_1}(t, y, x) h_{b_2}(t, y, x) dt F_A(dy, dx) \\ &= \lambda \int_0^1 E \left(\sum_{l_1=1}^{\infty} X_{l_1} I_{\{t+Z_{l_1} \leq b_1\}} \sum_{l_2=1}^{\infty} X_{l_2} I_{\{t+Z_{l_2} \leq b_2\}} \right) dt \\ &= \lambda \int_0^1 [\text{cov}(C(b_1 - t), C(b_2 - t)) + EC(b_1 - t)EC(b_2 - t)] dt \\ &= \lambda \int_{b_1-1}^{b_1} \mu(t)[E(X_{11}^2) + (EX_{11})^2 \mu(b_2 - b_1 + t)] dt. \end{aligned}$$

This concludes the proof. \square

Prediction of Claim Numbers and Total Claim Amounts

We intend to calculate the conditional expectations

$$E(M(b, b+x] | M(b)) \text{ and } E(S(b, b+x] | M(b)) \quad \text{for } b \geq 1,$$

where as usual $M(b, b+x]$ and $S(b, b+x]$ are the increments of the processes $(M(s))_{s \geq 1}$ and $(S(s))_{s \geq 1}$, respectively, on the interval $(b, b+x]$ for some positive x . This means that we want to predict (in the mean square sense) the number of payments in $(b, b+x]$ and the corresponding total amount for those claims which occurred in $(0, 1]$ and for which payments have been executed in $(0, b]$.

The following representation of $M(b)$ is crucial in this section:

$$M(b) = \sum_{k=1}^{\infty} I_{\{T_k \leq 1\}} \Pi_k(b - T_k) = \sum_{k=1}^{\tilde{N}(1)} \Pi_k(b - T_k), \quad (11.3.44)$$

where

$$\Pi_k(t) = \sum_{l=1}^{\infty} I_{\{Z_{kl} \leq t\}}, \quad \tilde{N}(t) = \sum_{k=1}^{\infty} I_{\{T_k \leq t\}}, \quad t \geq 0.$$

The sequence (Π_k) consists of iid Poisson processes on $(0, \infty)$ with common mean measure μ , independent of the homogeneous Poisson process \tilde{N} with intensity λ .

By virtue of (11.3.44) the σ -field generated by $M(b)$ is contained in the σ -field which is generated by the sequences $(\Pi_i(b - T_i))$ and (T_i) . Therefore we have

$$\begin{aligned} & E(M(b, b+x] \mid M(b)) \\ &= \sum_{k=1}^{\infty} E \left[E \left[I_{\{T_k \leq 1\}} \Pi_k(b - T_k, b + x - T_k) \mid (\Pi_i(b - T_i)), (T_i) \right] \middle| M(b) \right] \\ &= \sum_{k=1}^{\infty} E \left[I_{\{T_k \leq 1\}} E \left(\Pi_k(b - T_k, b + x - T_k) \mid \Pi_k(b - T_k), T_k \right) \middle| M(b) \right] \\ &= \sum_{k=1}^{\infty} E \left[I_{\{T_k \leq 1\}} \mu(b - T_k, b + x - T_k) \middle| M(b) \right]. \end{aligned} \tag{11.3.45}$$

In the last step we used the independent increments of Π_k .

Similarly, by a conditioning argument we obtain

$$\begin{aligned} & E(S(b, b+x] \mid M(b)) \\ &= \sum_{k=1}^{\infty} E \left[E \left(\sum_{l=1}^{\infty} X_{kl} I_{\{T_k \leq 1, Z_{kl} \in (b-T_k, b+x-T_k]\}} \mid (Z_{kl})_{k,l=1,2,\dots}, (T_i) \right) \middle| M(b) \right] \\ &= EX_{11} E(M(b, b+x] \mid M(b)). \end{aligned} \tag{11.3.46}$$

We consider the special case when Π is a homogeneous Poisson process. In this case, one can evaluate $E(M(b, b+x] \mid M(b))$ explicitly.

Proposition 11.3.7 (Prediction of $M(b, b+x]$ and $(S, b, b+x]$ given $M(b)$)
Assume that Π is a homogeneous Poisson process on $(0, \infty)$ with intensity $\gamma > 0$. Then for any $m = 0, 1, \dots$, and $b > 1$,

$$E(M(b, b+x] \mid M(b) = m) = \lambda \gamma x \frac{\phi_1^{(m)}(\gamma)}{\phi_2^{(m)}(\gamma)},$$

$$E(S(b, b+x] \mid M(b) = m) = \lambda \gamma x EX_{11} \frac{\phi_1^{(m)}(\gamma)}{\phi_2^{(m)}(\gamma)},$$

where $\phi_1^{(m)}$, $\phi_2^{(m)}$ are the m th derivatives of the Laplace-Stieltjes transforms of the random variables $\sum_{i=1}^{L+1}(b - U_i)$, $\sum_{i=1}^L(b - U_i)$, respectively, and L is a $\text{Pois}(\lambda)$ distributed random variable independent of the iid uniform $\text{U}(0, 1)$ sequence (U_i) .

Proof. It follows from (11.3.45) and $\mu = \gamma \text{Leb}$ that for any $m \geq 0$,

$$\begin{aligned} E(M(b, b+x] \mid M(b) = m) &= \gamma x E\left[\tilde{N}(1) \mid M(b) = m\right] \\ &= \gamma x \sum_{r=0}^{\infty} r P(\tilde{N}(1) = r \mid M(b) = m). \end{aligned} \quad (11.3.47)$$

In view of (11.3.44) we observe that

$$\begin{aligned} P(\tilde{N}(1) = r \mid M(b) = m) &= \frac{P\left(\tilde{N}(1) = r, \sum_{k=1}^{\tilde{N}(1)} \Pi_k(b - T_k) = m\right)}{P\left(\sum_{k=1}^{\tilde{N}(1)} \Pi_k(b - T_k) = m\right)} \\ &= \frac{P\left(\tilde{N}(1) = r, \Pi_1\left(\sum_{k=1}^{\tilde{N}(1)} (b - T_k)\right) = m\right)}{P\left(\Pi_1\left(\sum_{k=1}^{\tilde{N}(1)} (b - T_k)\right) = m\right)}. \end{aligned}$$

Here we used the independence of (T_k) and (Π_k) and the Poisson property of the independent processes Π_k . Conditionally on $\tilde{N}(1)$, we can use the order statistics property of the homogeneous Poisson process:

$$P\left(\sum_{k=1}^{\tilde{N}(1)} (b - T_k) \in \cdot \mid \tilde{N}(1) = r\right) = P(R_r(b) \in \cdot), \quad r = 0, 1, \dots,$$

where

$$R_r(b) = \sum_{i=1}^r (b - U_i), \quad r = 0, 1, \dots,$$

for an iid $\text{U}(0, 1)$ sequence (U_i) , and Π_1 , \tilde{N} and (U_i) are mutually independent. We conclude that

$$P(\tilde{N}(1) = r \mid M(b) = m) = \frac{P(\tilde{N}(1) = r) P(\Pi_1(R_r(b)) = m)}{P(\Pi_1(R_{\tilde{N}(1)}(b)) = m)}.$$

Therefore we conclude from (11.3.47) that

$$E(M(b, b+x] \mid M(b) = m) = \gamma x \frac{\sum_{r=0}^{\infty} r P(\tilde{N}(1) = r) P(\Pi_1(R_r(b)) = m)}{\sum_{s=0}^{\infty} P(\tilde{N}(1) = s) P(\Pi_1(R_s(b)) = m)}. \quad (11.3.48)$$

We have by a conditioning argument,

$$\begin{aligned}
& \sum_{s=0}^{\infty} P(\tilde{N}(1) = s) P(\Pi_1(R_s(b)) = m) \\
&= \sum_{s=0}^{\infty} P(\tilde{N}(1) = s) E \left[e^{-\gamma R_s(b)} \frac{(\gamma R_s(b))^m}{m!} \right] \\
&= E \left[e^{-\gamma R_{\tilde{N}(1)}(b)} \frac{(\gamma R_{\tilde{N}(1)}(b))^m}{m!} \right]. \tag{11.3.49}
\end{aligned}$$

For any non-negative random variable A the m th derivative of its Laplace-Stieltjes transform $\phi_A(h) = \exp\{-hA\}$ is given by

$$\phi_A^{(m)}(h) = (-1)^m E[e^{-hA} A^m], \quad h > 0. \tag{11.3.50}$$

By virtue of (11.3.49) and (11.3.50) we get

$$\sum_{s=0}^{\infty} P(\tilde{N}(1) = s) P(\Pi_1(R_s(b)) = m) = \frac{(-\gamma)^m}{m!} \phi_{R_{\tilde{N}(1)}(b)}^{(m)}(\gamma). \tag{11.3.51}$$

In a similar fashion we derive

$$\begin{aligned}
& \sum_{r=0}^{\infty} r P(\tilde{N}(1) = r) P(\Pi_1(R_r(b)) = m) \\
&= \lambda \sum_{r=0}^{\infty} e^{-\lambda} \frac{\lambda^r}{r!} P(\Pi_1(R_{r+1}(b)) = m) \\
&= \lambda \sum_{r=0}^{\infty} e^{-\lambda} \frac{\lambda^r}{r!} E \left[e^{-\gamma R_{r+1}(b)} \frac{(\gamma R_{r+1}(b))^m}{m!} \right] \\
&= \lambda E \left[e^{-\gamma R_{\tilde{N}(1)+1}(b)} \frac{(\gamma R_{\tilde{N}(1)+1}(b))^m}{m!} \right]. \tag{11.3.52}
\end{aligned}$$

By virtue of (11.3.52) and (11.3.50) we finally get

$$\sum_{r=0}^{\infty} r P(\tilde{N}(1) = r) P(\Pi_1(R_r(b)) = m) = \lambda \frac{(-\gamma)^m}{m!} \phi_{R_{\tilde{N}(1)+1}(b)}^{(m)}(\gamma). \tag{11.3.53}$$

A combination of (11.3.48), (11.3.51) and (11.3.53) yields the desired result:

$$E(M(b, b+x] \mid M(b) = m) = \lambda \gamma x \frac{\phi_{R_{\tilde{N}(1)+1}(b)}^{(m)}(\gamma)}{\phi_{R_{\tilde{N}(1)}(b)}^{(m)}(\gamma)}. \quad (11.3.54)$$

The result for $E(S(b, b+x] \mid M(b) = m)$ follows from (11.3.46). \square

Noticing that $R_{\tilde{N}(1)}(b)$ is a compound Poisson sum, the Laplace-Stieltjes transforms of the random variables $R_{\tilde{N}(1)}(b)$ and $R_{\tilde{N}(1)+1}(b)$ can be calculated explicitly for $h > 0$:

$$\begin{aligned}\phi_{R_{\tilde{N}(1)}(b)}(h) &= e^{-\lambda(1-\phi_{b-U}(h))}, \\ \phi_{R_{\tilde{N}(1)+1}(b)}(h) &= \phi_{R_{\tilde{N}(1)}(b)}(h) \phi_{b-U}(h), \\ \phi_{b-U}(h) &= h^{-1} e^{-h b} (e^h - 1).\end{aligned}$$

We also mention that

$$\phi_{R_{\tilde{N}(1)+1}(b)}^{(m)}(\gamma) = \sum_{k=0}^m \binom{m}{k} \phi_{R_{\tilde{N}(1)}(b)}^{(k)}(\gamma) \phi_{b-U}^{(m-k)}(\gamma).$$

The right-hand side of (11.3.54) is a ratio of polynomials of the functions $\phi_{b-U}^{(k)}(\gamma)$, $k = 0, 1, \dots, m$. In particular, the predictor $E(M(b, b+x] \mid M(b))$ is a highly non-linear function of $M(b)$. Similar calculations as above show that

$$\begin{aligned}E(M(b, b+x] \mid M(1), \dots, M(b)) &= E(M(b, b+x] \mid M(b)), \\ b &= 1, 2, \dots, x = 1, 2, \dots.\end{aligned}$$

Using the notation of Mack's model, Mack's condition (4) on p. 367 requires that

$$E(M(j, j+1] \mid M(1), \dots, M(j)) = (f_j - 1) M(j), \quad j = 1, 2, \dots.$$

This means that the predictor of $M(j, j+1]$ given $M(1), \dots, M(j)$ would be a linear function of $M(j)$. Under the conditions of the model chosen in this section, this goal cannot be achieved.

Comments

The cluster Poisson models in this section are inspired by models which have been used in teletraffic applications. In this context, the Poisson point T_i describes the arrival of a packet in a large computer network. The packet triggers activities in the network at the instants of time $T_{il} = T_i + \sum_{k=1}^l Y_{ik}$ with corresponding amounts of work described by the variable X_{il} which can be interpreted as a random reward or as the amount of work brought into the system. Such models are tractable because of the underlying Poisson structure.

For example, one can calculate moments and derive asymptotic results for the workload process describing the global behavior of a large stochastic network over a large time horizon. Results of this kind can be found in Faÿ et al. [50] and Mikosch and Samorodnitsky [111], where also much more general cluster processes were considered. Cluster models have also been used in the engineering literature on large computer network; see Hohn and Veitch [72], Hohn et al. [73] and Hernandez-Campos et al. [69]. Instead of modeling arrivals of packets by a cluster Poisson process, they applied them to connections and sessions in such networks.

Cluster processes have a long history in applied probability theory. Among others, they have been used for modeling bunching in motor traffic or computer failure patterns; see Daley and Vere-Jones [38].

Exercises

Section 11.3.1

- (1) Show that the sequences $(N_{i,i+j})_{j \geq 0}$, $i = 1, 2, \dots$, defined in (11.3.35) are iid. Give a similar argument for the sequences $(S_{i,i+j})_{j \geq 0}$, $i = 1, 2, \dots$

Section 11.3.2

- (2) Show that $E(K^2) < \infty$ is a sufficient condition for $\text{var}(M(b)) < \infty$, $b \geq 1$, where $M(b)$ is defined in (11.3.37).
- (3) Let $K \geq 1$ be an integer-valued random variable with $\text{var}(K) < \infty$ and $P(K > 1) > 0$.
- Show that $E(K^2) - EK$ is positive.
 - Prove that $2 \sum_{k=1}^{\infty} k P(K \geq k) = E(K^2) + EK$.
- (4) Assume $EK < \infty$ and let \tilde{K} be a random variable with distribution⁶

$$F_{\tilde{K}}(\{k\}) = (EK)^{-1} P(K \geq k), \quad k = 1, 2, \dots$$

Also assume that \tilde{K} , a uniform $U(0, 1)$ random variable U and (Z_l) are independent. Prove that $EM(b)$, $b \geq 1$, in (11.3.38) can be written in the form

$$EM(b) = \lambda EK P(U + Z_{\tilde{K}} \leq b).$$

- (5) Let $K \geq 1$ be an integer-valued random variable with $P(K > 1) > 0$ and $E(K^2) < \infty$. Then the quantities

$$F_{\hat{K}}(\{k\}) = (E(K^2))^{-1} (2k - 1) P(K \geq k), \quad k = 1, 2, \dots,$$

constitute the distribution of a random variable \hat{K} with values in $\{1, 2, \dots\}$; see Exercise 3. Assuming \hat{K} , the uniform $U(0, 1)$ random variable U and (Z_l) independent, show that $\text{var}(M(b))$, $b \geq 1$, in (11.3.40) can be written in the form

$$\text{var}(M(b)) = \lambda E(K^2) P(U + Z_{\hat{K}} \leq b).$$

⁶ The distribution $F_{\tilde{K}}$ can be interpreted as the *integrated tail distribution* of F_K . The integrated tail distribution appeared in the context of bounds on the ruin probability; see p. 163.

- (6) The payment numbers $M(b) = \int_{(0,1] \times E_A} g_b dN$, $b \geq 1$, are Poisson integrals and therefore they have a compound Poisson representation; see Corollary 7.2.8.
- Show that $EN((0,1] \times E_A) = \lambda$.
 - Show that the distribution F_Z in the $CP(\lambda, F_Z)$ representation of $M(b)$ is given by

$$F_Z(\{m\}) = P\left(\sum_{l=1}^K I_{\{U+Z_l \in (0,b]\}} = m\right) \quad m = 0, 1, \dots$$

Here U is $U(0,1)$ distributed and U , K and (Z_l) are independent.

- (7) Assume the conditions of Example 11.3.4 and that K has a geometric distribution given by

$$P(K = k) = pq^{k-1}, \quad k = 1, 2, \dots,$$

for some $p \in (0,1)$, $q = 1 - p$.

- Evaluate the tail $P(K \geq k)$, $k = 1, 2, \dots$
 - Evaluate $EM(b)$ given in (11.3.43) by using (11.3.42).
- (8) Consider the model of Example 11.3.4. Verify that the variance of $M(b)$, $b \geq 1$, is given by

$$\text{var}(M(b)) = \lambda \gamma \int_{b-1}^b \int_0^t E\left[P(K \geq \tilde{N}(s) + 1 \mid \tilde{N}(s)) (2\tilde{N}(s) - 1)\right] ds dt,$$

where K, \tilde{N} are defined in the example.

- (9) Verify the formulae for $EM(b)$ and $\text{var}(M(b))$ in Example 11.3.5.

Section 11.3.3

- (10) Consider the Poisson cluster model of Section 11.3.3.
- Calculate the covariance of two increments $M(b_1, b_2]$ and $M(b_3, b_4]$ for any $b_1, b_3 \geq 1$ and $b_2 \geq b_1$ and $b_4 \geq b_3$.
 - Give conditions under which the increments specified under (a) are independent.
 - Specify the results of (a) and (b) for a homogeneous Poisson cluster process, i.e., $\mu(t) = \gamma t$ for some $\gamma > 0$.

References

Each reference is followed, in square brackets, by a list of the page numbers where this reference is cited.

1. AARSEN, K. AND HAAN, L. DE (1994) On the maximal life span of humans. *Mathematical Population Studies* **4**, 259–281. [312]
2. ALSMEYER, G. (1991) *Erneuerungstheorie*. Teubner, Stuttgart. [65]
3. ANDERSEN, P.K., BORGAN, Ø., GILL, R.D. AND KEIDING, N. (1993) *Statistical Models Based on Counting Processes*. Springer, New York. [5]
4. APPLEBAUM, D. (2004) *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge (UK). [360]
5. ASMUSSEN, S. (2000) *Ruin Probabilities*. World Scientific, Singapore. [125,137, 162,177]
6. ASMUSSEN, S. (2003) *Applied Probability and Queues*. Springer, Berlin. [65]
7. ASMUSSEN, S., BINSWANGER, K. AND HØJGAARD, B. (2000) Rare event simulation for heavy-tailed distributions. *Bernoulli* **6**, 303–322. [138]
8. ASMUSSEN, S. AND GLYNN, P.W. (2007) *Stochastic Simulation. Algorithms and Analysis*. Springer, New York. [131,137]
9. ASMUSSEN, S. AND RUBINSTEIN, R.Y. (1995) Steady-state rare events simulation in queueing models and its complexity properties. In: DSHALALOW, J. (Ed.) *Advances in Queueing: Models, Methods and Problems*, pp. 429–466. CRC Press, Boca Raton. [138]
10. BACCELLI, F. AND BREMAUD, P. (2003) *Elements of Queueing Theory*. 2nd Edition. Springer, Berlin. [224]
11. BARBOUR, A.D. , HOLST, L. AND JANSON, S. (1992) *Poisson Approximation*. Oxford University Press, New York. [132,242]
12. BARNDORFF-NIELSEN, O.E., MIKOSCH, T. AND RESNICK, S.I. (Eds.) (2002) *Lévy Processes: Theory and Applications*. Birkhäuser, Boston. [12,360]
13. BEIRLANT, J., GOEGEBEUR, Y., TEUGELS, J.L. AND SEGERS, J. (2004) *Statistics of Extremes. Theory and Applications*. Wiley, Chichester. [321]

14. BEIRLANT, J., TEUGELS, J.L. AND VYNCKIER, P. (1996) *Practical Analysis of Extreme Values*. Leuven University Press, Leuven. [91,93,321]
15. BERTOIN, J. (1998) *Lévy Processes*. Cambridge University Press, Cambridge (UK). [224,241,360]
16. BICKEL, P. AND FREEDMAN, D. (1981) Some asymptotic theory for the bootstrap. *Ann. Statist.* **9**, 1196–1217. [136,141]
17. BILLINGSLEY, P. (1968) *Convergence of Probability Measures*. Wiley, New York. [9,141,157,283,293,321,329,387]
18. BILLINGSLEY, P. (1986) *Probability and Measure*. 2nd Edition. Wiley, New York. [8,21,24,39,76,84,133,147,170,222,238,316,351,387]
19. BINGHAM, N.H., GOLDFIE, C.M. AND TEUGELS, J.L. (1987) *Regular Variation*. Cambridge University Press, Cambridge (UK). [100,103,173,311,323,340]
20. BJÖRK, T. (1999) *Arbitrage Theory in Continuous Time*. Oxford University Press, Oxford (UK). [143,288]
21. BONDESSON, L. (1992) *Generalized Gamma Convolutions and Related Classes of Distributions and Densities. Lecture Notes in Statistics* **76**. Springer, Berlin, New York. [347]
22. BREIMAN, L. (1992) *Probability*. SIAM Classics in Applied Mathematics No 7. [230]
23. BREMAUD, P. (2005) *Point Processes and Queues: Martingale Dynamics*. Springer, Berlin. [224]
24. BROCKWELL, P.J. AND DAVIS, R.A. (1991) *Time Series: Theory and Methods*. 2nd Edition. Springer, New York. [32,170,195,283,288]
25. BROCKWELL, P.J. AND DAVIS, R.A. (1996) *Introduction to Time Series and Forecasting*. Springer, New York. [288]
26. BUCHWALDER, M., BÜHLMANN, H., MERZ, M. AND WÜTHRICH, M.V. (2006) The mean square error of prediction in the chain ladder reserving method (Mack and Murphy revisited). *ASTIN Bulletin* **36**, 521–542. [381]
27. BÜHLMANN, H. (1967) Experience rating and credibility I. *ASTIN Bulletin* **4**, 199–207. [185,195]
28. BÜHLMANN, H. (1969) Experience rating and credibility II. *ASTIN Bulletin* **5**, 157–165. [185]
29. BÜHLMANN, H. (1970) *Mathematical Methods in Risk Theory*. Springer, Berlin. [79,185,208,211]
30. BÜHLMANN, H. AND GISLER, A. (2005) *A Course in Credibility Theory and Its Applications*. Springer, Berlin. [211]
31. BÜHLMANN, H. AND STRAUB, E. (1970) Glaubwürdigkeit für Schadensätze. *Mittl. Ver. Schw. Vers. Math.* **73**, 205–216. [209]
32. CHAMBERS, J.M. (1977) *Computational Methods for Data Analysis*. Wiley, New York. Wadsworth, Belmont Ca., Duxbury Press, Boston. [84]
33. CHOSSY, R. VON AND RAPPL, G. (1983) Some approximation methods for the distribution of random sums. *Insurance: Math. Econom.* **2**, 251–270. [127]
34. COLES, S. *An Introduction to Statistical Modeling of Extreme Values*. Springer, London. [321]
35. CONT, R. AND TANKOV, P. (2003) *Financial Modelling with Jump Processes*. Chapman and Hall, New York. [360]

36. CRAMÉR, H. (1930) *On the mathematical theory of risk*. Skandia Jubilee Volume, Stockholm. Reprinted in: Martin-Löf, A. (Ed.) Cramér, H. (1994) *Collected Works*. Springer, Berlin. [151,162]
37. CRESSIE, N.A.C. (1993) *Statistics for Spatial Data*. Wiley, New York. [42,195,224]
38. DALEY, D. AND VERE-JONES, D. (1988) *An Introduction to the Theory of Point Processes*. Springer, Berlin. [46,223,228,242,256,283–299,321,387,402]
39. DALEY, D. AND VERE-JONES, D. (2003) *An Introduction to the Theory of Point Processes. Elementary Theory and Methods*. Vol. 1. Springer, New York. [223,242,256,285,321,387]
40. DALEY, D. AND VERE-JONES, D. (2008) *An Introduction to the Theory of Point Processes*. Vol. 2. Springer, New York. [223,242,256,285,321,387]
41. DOUKHAN, P., OPPENHEIM, G. AND TAQQU, M.S. (Eds.) (2003) *Long Range Dependence*. Birkhäuser, Boston. [283]
42. EBERLEIN, E. (2001) Applications of generalized hyperbolic Lévy motions to finance. In: BARNDORFF-NIELSEN, O.E., MIKOSCH, T. AND RESNICK, S.I. (Eds.) *Lévy Processes: Theory and Applications*, pp. 283–318. Birkhäuser, Boston. [360]
43. EFRON, B. (1979) Bootstrap methods: another look at the jackknife. *Ann. Statist.* **7**, 1–26. [132,136]
44. EFRON, B. AND TIBSHIRANI, R.J. (1993) *An Introduction to the Bootstrap*. Chapman and Hall, New York. [136,286]
45. EMBRECHTS, P., GRÜBEL, R. AND PITTS, S.M. (1993) Some applications of the fast Fourier transform algorithm in insurance mathematics. *Statist. Neerlandica* **47**, 59–75. [125]
46. EMBRECHTS, P., KLÜPPELBERG, C. AND MIKOSCH, T. (1997) *Modelling Extremal Events for Insurance and Finance*. Springer, Heidelberg. [46,60,76,96–107,130,137,144,149,167,171,176,224,299,308–309,321–328]
47. EMBRECHTS, P. AND VERAVERBEKE, N. (1982) Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance: Math. Econom.* **1**, 55–72. [174]
48. FALK, M., HÜSLER, J. AND REISS, R.-D. (2004) *Laws of Small Numbers: Extremes and Rare Events*. 2nd Edition. Birkhäuser, Basel. [321]
49. FAN, J. AND GIJBELS, I. (1996) *Local Polynomial Modelling and Its Applications*. Chapman and Hall, London. [34]
50. FAÝ, G., GONZALEZ-AREVALO, B., MIKOSCH, T. AND SAMORODNITSKY, G. (2006) Modeling teletraffic arrivals by a Poisson cluster process. *QUESTA* **54**, 121–140. [284,402]
51. FELLER, W. (1971) *An Introduction to Probability Theory and Its Applications II*. Wiley, New York. [50,65,103,173,179,181,295,342,347,351,358,360]
52. GALAMBOS, J. (1987) *Asymptotic Theory of Extreme Order Statistics*. 2nd Edition. Krieger, Malabar (Florida). [321]
53. GASSER, T., ENGEL, J. AND SEIFERT, B. (1993) Nonparametric function estimation. In: *Computational Statistics. Handbook of Statistics*. Vol. 9, pp. 423–465. North-Holland, Amsterdam. [34]

54. GIHMAN, I.I. AND SKOROHOD, A.V. (1974) *The Theory of Stochastic Processes I*. Springer, Berlin. [339,360]
55. GIHMAN, I.I. AND SKOROHOD, A.V. (1974) *The Theory of Stochastic Processes II*. Springer, Berlin. [360]
56. GLASSERMAN, P. (2004) *Monte Carlo Methods in Financial Engineering*. Springer, New York. [131,137]
57. GNEDENKO, B.V. AND KOLMOGOROV, A.N. (1954) *Limit Theorems for Sums of Independent Random Variables*. Addison-Wesley, Cambridge (Mass.) [343,358,360]
58. GOLDIE, C.M. (1991) Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.* **1**, 126–166. [171]
59. GOLDIE, C.M. AND KLÜPPELBERG, C. (1996) Subexponential distributions. In: ADLER, R., FELDMAN, R. AND TAQQU, M.S. (Eds.) *A Practical Guide to Heavy Tails: Statistical Techniques for Analysing Heavy-Tailed Distributions*, pp. 435–460. Birkhäuser, Boston. [105]
60. GRANDELL, J. (1991) *Aspects of Risk Theory*. Springer, Berlin. [176]
61. GRANDELL, J. (1997) *Mixed Poisson Processes*. Chapman and Hall, London. [69]
62. GRÜBEL, R. AND HERMESMEIER, R. (1999) Computation of compound distributions I: aliasing errors and exponential tilting. *ASTIN Bulletin* **29**, 197–214. [125]
63. GRÜBEL, R. AND HERMESMEIER, R. (2000) Computation of compound distributions II: discretization errors and Richardson extrapolation. *ASTIN Bulletin* **30**, 309–331. [125]
64. GUMBEL, E.J. (1958) *Statistics of Extremes*. Columbia University Press, New York. [321]
65. GUT, A. (1988) *Stopped Random Walks*. Springer, Berlin. [60,65]
66. HAAN, L. DE AND FERREIRA, A. (2006) *Extreme Value Theory. An Introduction*. Springer, New York. [321]
67. HALL, P.G. (1992) *The Bootstrap and Edgeworth Expansion*. Springer, New York. [126,136]
68. HEIDELBERGER, P. (1995) Fast simulation of rare events in queueing and reliability models. *ACM TOMACS* **6**, 43–85. [138]
69. HERNANDEZ-CAMPOS, F., NOBEL, A.B., SMITH, F.D. AND JEFFAY, K. (2005) Understanding patterns of TCP connection usage with statistical clustering. In: *Proceedings of the 13th IEEE International Symposium on Modeling, Analysis, and Simulation of Computer and Telecommunication Systems*, pp. 35–44. IEEE Computer Society, Los Alamitos (USA). [402]
70. HESS, K.T., LIEWALD, A. AND SCHMIDT, K.D. (2002) An extension of Panjer's recursion. *ASTIN Bulletin* **32**, 283–298. [125]
71. HOGG, R.V. AND KLUGMAN, S.A. (1984) *Loss Distributions*. Wiley, New York. [93]
72. HOHN, N. AND VEITCH, D. (2003) Inverting sampled traffic. In: *Proceedings of the ACM/SIGCOMM Internet Measurement Conference, Miami, USA, November 2003*, pp. 222–233. ACM Press, New York. [402]

73. HOHN, N., VEITCH, D. AND ABRY, P. (2003) Cluster processes: a natural language for network traffic. *IEEE Trans. Signal Process.* **51**, 2229–2244. [402]
74. JACOD, J. AND SHIRYAEV, A.N. (2002) *Limit Theorems for Stochastic Processes*. 2nd Edition. Springer, Berlin. [223,360]
75. JANICKI, A. AND WERON, A. (1993) *Simulation and Chaotic Behaviour of α -Stable Stochastic Processes*. Marcel Dekker, New York. [360]
76. JENSEN, J. (1995) *Saddlepoint Approximations*. Oxford University Press, Oxford. [126]
77. KAAS, R., GOOVAERTS, M., DHAENE, J. AND DENUIT, M. (2001) *Modern Actuarial Risk Theory*. Kluwer, Boston. [79,81,125,195,211]
78. KALLENBERG, O. (1973) Characterization and convergence of random measures and point processes. *Z. Wahrscheinlichkeitstheorie verw. Geb.* **27**, 9–21. [69]
79. KALLENBERG, O. (1983) *Random Measures*. 3rd Edition. Akademie-Verlag, Berlin. [46,223,242,256,293,299,321]
80. KALLENBERG, O. (1997) *Foundations of Modern Probability Theory*. Springer, Berlin. [223,233]
81. KLÜPPELBERG, C. AND MIKOSCH, T. (1995) Explosive shot noise processes with applications to risk retention. *Bernoulli* **1**, 125–147. [285]
82. KLÜPPELBERG, C. AND MIKOSCH, T. (1995) Delay in claim settlement and ruin probability approximations. *Scand. Actuar. J.* 154–168. [285]
83. KLÜPPELBERG, C., MIKOSCH, T. AND SCHÄRF, A. (2003) Regular variation in the mean and stable limits for Poisson shot noise. *Bernoulli* **9**, 467–496. [285]
84. KESTEN, H. (1973) Random difference equations and renewal theory for products of random matrices. *Acta Math.* **131**, 207–248. [171]
85. KINGMAN, J.F.C. (1996) *Poisson Processes*. Oxford University Press, Oxford (UK). [46,242]
86. KLUGMAN, S.A., PANJER, H.H. AND WILLMOT, G.E. (1998) *Loss Models. From Data to Decisions*. Wiley, New York. [80,195,211]
87. KOLLER, M. (2000) *Stochastische Modelle in der Lebensversicherung*. Springer, Berlin. [13]
88. KOTZ, S. AND NADARAJAH, S. (2000) *Extreme Value Distributions. Theory and Applications*. Imperial College Press, London. [321]
89. KRENGEL, U. (1985) *Ergodic Theorems*. Walter de Gruyter & Co., Berlin. [283]
90. KYPRIANOU, A.E. (2006) *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, Berlin. [224,241,360]
91. KYPRIANOU, A.E., SCHOUTENS, W. AND WILMOTT, P. (2005) *Exotic Option Pricing and Advanced Lévy Models*. Wiley, New York. [360]
92. LEADBETTER, M.R., LINDGREN, G. AND ROOTZÉN, H. (1983) *Extremes and Related Properties of Random Sequences and Processes*. Springer, Berlin. [224,299,321]
93. LEHMANN, E.L. (1986) *Testing Statistical Hypotheses*. Springer, New York. [195]
94. LINNIK, YU.V. (1964) *Decomposition of Probability Distributions*. Dover Publications, Inc., New York; Oliver and Boyd Ltd., Edinburgh-London. [347,360]

95. LINNIK, YU.V. AND OSTROVSKII, I.V. (1977) *Decomposition of Random Variables and Vectors*. American Mathematical Society, Providence, R.I. [347,360]
96. LIPTSER, R.SH. AND SHIRYAYEV, A.N. (1989) *Theory of Martingales*. Kluwer, Dordrecht. [360]
97. LUKACS, E. (1970) *Characteristic Functions*. 2nd Edition. Griffin, London. [47,140,341,343,347,360]
98. LUKACS, E. (1983) *Developments in Characteristic Function Theory*. Macmillan Co., New York. [347,360]
99. LUNDBERG, F. (1903) *Approximerad framställning av sannolikhetsfunktionen. Återförsäkring av kollektivrisker*. Akad. Afhandling. Almqvist och Wiksell, Uppsala. [3,5,224,241,360]
100. MACK, T. (1993) Distribution-free calculation of the standard error of chain ladder reserve estimates. *ASTIN Bulletin* **23**, 213–225. [366,374,381]
101. MACK, T. (1994) Which stochastic model is underlying the chain ladder method? *Insurance: Math. Econom.* **15**, 133–138. [381]
102. MACK, T. (1994) Measuring the variability of chain ladder reserve estimates. In: *Proceedings of the Casualty Actuarial Society Forum, Spring 1994*, Vol. **1**, pp. 101–182. [381]
103. MACK, T., QUARG, G. AND BRAUN, C. (2006) The mean square error of prediction in the chain ladder reserving method. A comment. *ASTIN Bulletin* **36**, 543–553. [381]
104. MAMMEN, E. (1992) *When Does Bootstrap Work? Asymptotic Results and Simulations. Lecture Notes in Statistics* **77**. Springer, New York. [136]
105. MATTHES, K., KERSTAN, J. AND MECKE, J. (1978) *Infinitely Divisible Point Processes*. Wiley, Chichester, New York. [223,321]
106. MEYN, S. AND TWEEDIE, R. (1996) *Markov Chains and Stochastic Stability*. Springer, Berlin. [230]
107. MIKOSCH, T. (1998) *Elementary Stochastic Calculus with Finance in View*. World Scientific, Singapore. [12,143]
108. MIKOSCH, T. (2003) Modeling dependence and tails of financial time series. In: FINKENSTÄDT, B. AND ROOTZÉN, H. (Eds.) *Extreme Values in Finance, Telecommunications and the Environment*, pp. 185–286. CRC Press, Boca Raton. [43,171]
109. MIKOSCH, T. AND NORVAIŠA, R. (2000) Stochastic integral equations without probability. *Bernoulli* **6**, 401–434. [288]
110. MIKOSCH, T., RESNICK, S., ROOTZÉN, H. AND STEGEMAN, A. (2002) Is network traffic approximated by stable Lévy motion or fractional Brownian motion? *Ann. Appl. Probab.* **12**, 23–68. [284]
111. MIKOSCH, T. AND SAMORODNITSKY, G. (2007) Scaling limits for cumulative input processes. *Math. Oper. Res.* **32**, 890–919. [284,402]
112. MØLLER, T. AND STEFFENSEN, M. (2007) *Market-Valuation Methods in Life and Pension Insurance*. Cambridge University Press, Cambridge (UK). [11,13]
113. MÜLLER, H.-G. AND STADTMÜLLER, U. (1987) Estimation of heteroscedasticity in regression analysis. *Ann. Statist.* **15**, 610–625. [34]
114. NORBERG, R. (1993) Prediction of outstanding liabilities in non-life insurance. *ASTIN Bulletin* **23**, 95–116. [224,259,265,286]

115. PANJER, H.H. (1981) Recursive evaluation of a family of compound distributions. *ASTIN Bulletin* **11**, 22–26. [121]
116. PETROV, V.V. (1975) *Sums of Independent Random Variables*. Springer, Berlin. [127,340,351,360]
117. PETROV, V.V. (1995) *Limit Theorems of Probability Theory*. Oxford University Press, Oxford. [340,360]
118. PITMAN, E.J.G. (1980) Subexponential distribution functions. *J. Austral. Math. Soc. Ser. A* **29**, 337–347. [108]
119. PRIESTLEY, M.B. (1981) *Spectral Analysis and Time Series*. Vols. I and II. Academic Press, New York. [32]
120. REISS, R.-D. AND THOMAS, M. (2007) *Statistical Analysis of Extreme Values with Applications to Insurance, Finance, Hydrology and Other Fields*. 3rd Edition. Birkhäuser, Basel, 2007. [321]
121. RESNICK, S.I. (1986) Point processes, regular variation and weak convergence. *Adv. Appl. Probab.* **18**, 66–138. [299,321]
122. RESNICK, S.I. (1987) *Extreme Values, Regular Variation, and Point Processes*. Springer, New York. [43,46,84,103,217,223,230,242,246,256,293–298,309,321–323]
123. RESNICK, S.I. (1992) *Adventures in Stochastic Processes*. Birkhäuser, Boston. [13,17,46,57,62–65,157,173,223,242,256,321]
124. RESNICK, S.I. (2007) *Heavy Tail Phenomena; Probabilistic and Statistical Modeling*. Springer, Berlin. [223,241,284,299,321,345,350]
125. RIPLEY, B.D. (2004) *Spatial Statistics*. Wiley, New York. [224]
126. ROGERS, L.C.G. AND WILLIAMS, D. (2000) *Diffusions, Markov Processes and Martingales*. Vol. 1. Cambridge University Press, Cambridge (UK). [8,12,13]
127. ROLSKI, T., SCHMIDLI, H., SCHMIDT, V. AND TEUGELS, J. (1999) *Stochastic Processes for Insurance and Finance*. Wiley, New York. [65,125,162,176]
128. ROOTZÉN, H. AND TAJVIDI, N. (1997) Extreme value statistics and wind storm losses: a case study. *Scand. Actuar. J.* 70–94. [91]
129. RYTGAARD, M. (1996) Simulation experiments on the mean residual life function $m(x)$. In: *Proceedings of the XXVII ASTIN Colloquium, Copenhagen, Denmark*. Vol. 1, pp. 59–81. [91]
130. SAMORODNITSKY, G. (2006) Long range dependence. *Foundations and Trends in Stochastic Systems* **1**, 163–257. [283]
131. SAMORODNITSKY, G. AND TAQQU, M.S. (1994) *Stable Non-Gaussian Random Processes. Stochastic Models with Infinite Variance*. Chapman and Hall, London. [51,224,241,283,288,360]
132. SATO, K.-I. (1999) *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge (UK). [9,12,140,224,241,336–360]
133. SCHNIEPER, R. (1991) Separating true IBNR and IBNER claims. *ASTIN Bulletin* **21**, 111–127. [381]
134. SCHOUTENS, W. (2003) *Lévy Processes in Finance: Pricing Financial Derivatives*. Wiley, New York. [360]
135. SCHROEDER, M. (1990) *Fractals, Chaos, Power Laws*. Freeman, New York. [100]

136. SIGMA (2008) Tables on the major losses 1970-2007. *Sigma Publication No 1*, Swiss Re, Zürich, p. 40. The publication is available under www.swissre.com. [96]
137. SKOROKHOD, A.V. (1964) *Random Processes with Independent Increments*. Nauka, Moscow. (In Russian) English translation 1991, Kluwer, Dordrecht. [360]
138. SPITZER, F. (1976) *Principles of Random Walk*. 2nd Edition. Springer, Berlin. [155]
139. STEUTEL, F.W. AND HARN, K. VAN (2003) *Infinite Divisibility of Probability Distributions on the Real Line*. Marcel Dekker, New York. [347,360]
140. STOYAN, D., KENDALL, W.S. AND MECKE, J. (1996) *Stochastic Geometry and Its Applications*. 2nd Edition. Wiley, New York. [224]
141. STRAUB, E. (1988) *Non-Life Insurance Mathematics*. Springer, New York. [195,209,211]
142. SUNDT, B. (1999) *An Introduction to Non-Life Insurance Mathematics*. 4th Edition. VVW Karlsruhe. [195,211]
143. SUNDT, B. (1999) On multivariate Panjer recursions. *ASTIN Bulletin* **29**, 29–46. [125]
144. VERAVERBEKE, N. (1993) Asymptotic estimates for the probability of ruin in a Poisson model with diffusion. *Insurance: Math. Econom.* **13**, 57–62. [360]
145. WILLIAMS, D. (1991) *Probability with Martingales*. Cambridge University Press, Cambridge (UK). [57,130,193,383]
146. WILLINGER, W., TAQQU, M.S., LELAND, M. AND WILSON. D. (1995) Self-similarity through high variability: statistical analysis of Ethernet LAN traffic at the source level. In: *Proceedings of the ACM/SIGCOMM'95, Cambridge, MA. Computer Communications Review* **25**, 100–113. [60]
147. WILLMOT, G.E. AND LIN, X.S. (2001) *Lundberg Approximations for Compound Distributions with Insurance Applications*. Springer, Berlin. [125]
148. ZOLOTAREV, V.M. (1986) *One-Dimensional Stable Distributions*. American Mathematical Society, Providence, R.I. [342]

Index

A

(a, b)-condition 122
Adjustment coefficient 158
Affinely transformed maxima and order statistics 305
criterion for convergence in distribution 307
extreme value distribution 306
generalized extreme value distribution 309
Jenkinson-von Mises representation 309
max-stable distribution 306
Age process of a renewal process 63
see backward recurrence time
Aggregate claim amount process 4, 71
see total claim amount process
Aggregation of claim sizes
regularly varying claim sizes 101, 102
subexponential claim sizes 103
Aggregation of Poisson random measures 254
of marked Poisson random measures 254
ARMA process 288
AR(1) process 288
relation with Ornstein-Uhlenbeck process 288
Arrivals, arrival times 3
of the Danish fire insurance data 32
of a homogeneous Poisson process 16
inspection paradox 19

of an inhomogeneous Poisson process 21
joint distribution 21
Asymptotic expansion in the central limit theorem 126
Autoregressive process of order one 288
relation with Ornstein-Uhlenbeck process 288
Auxiliary function 314
examples 315
relation with hazard rate 108
and maximum domain of attraction of the Gumbel distribution 313
and mean excess function 314
and von Mises function 314
properties 316

B

Backward recurrence time
of a homogeneous Poisson process 19
of a renewal process 63
Basic decomposition of time-claim size space 270
of claim number process 271
IBNR claims 271
claim number 272
total claim amount 273
RBNS claims 270
claim number 272
outstanding 276
settled 274
total claim amount 274

- settled claims 270
 - of total claim amount 273
 - Basic model 268
 - basic decomposition 270
 - time-claim size space 270
 - Bayes estimation 187
 - in the heterogeneity model 187, 189
 - linear Bayes estimation 199
 - minimum risk estimator 190
 - risk 190
 - Benktander distributions 98
 - Berry-Esséen inequality 127
 - Blackwell's renewal theorem 61
 - Brownian motion 10
 - in the Lévy-Itô representation 353
 - as a Lévy process 336
 - reflection principle 157
 - Bühlmann model 204
 - credibility estimator 208
 - credibility weight 208
 - linear Bayes estimation 206
 - Bühlmann-Straub model 209
 - linear Bayes estimation 210
 - Burr distribution 98
- C**
- Càdlàg sample paths 8
 - of a Lévy process 336
 - Skorokhod space 9
 - Chain ladder estimator 370
 - asymptotic normality 370
 - consistency 370
 - expectation 372
 - variance 375
 - Chain ladder model 365
 - chain ladder estimator 370
 - Mack's model 366
 - prediction 376
 - Characteristic function of a Lévy process 344
 - Characteristic triplet 344
 - examples 346
 - Lévy-Khintchine formula 344
 - Cauchy distribution 347
 - as infinitely divisible distribution 347
 - as stable distribution 341
 - Cauchy functional equation 340
 - Cauchy process 357
- α -stable Lévy motion 356
 - Central limit theorem
 - asymptotic expansions 126
 - Berry-Esséen inequality 127
 - conditional 127
 - for a mixed Poisson process does not hold 70
 - for a renewal process 60
 - saddle point approximation 126
 - for the total claim amount process in the renewal model 75
 - error bounds 125
 - Claim arrival, arrival time 3
 - see arrivals
 - Claim number process 3, 7
 - basic decomposition in basic model 271
 - models 7
 - basic model 268
 - mixed Poisson process 66
 - Poisson process 7
 - renewal process 53
 - Claim severity 3
 - see claim size
 - Claim size 3
 - and claim times in a joint PRM 39
 - Claim size distributions 82
 - large claims 98
 - regularly varying claim sizes 99
 - subexponential claim sizes 103
 - small claim condition 158
 - small claims 96
 - Claim time 3
 - see arrivals
 - Cluster point process 363
 - Collective risk model 3
 - aggregate claim amount process 4, 71
 - arrivals, arrival times 3
 - claim arrival, arrival time 3
 - claim number process 3, 7
 - mixed Poisson process 66
 - models 7
 - Poisson process 7
 - renewal process 53
 - claim severity, size 3
 - distributions 82
 - claim time 3
 - compound sum process 4
 - compound geometric process 111

- compound Poisson process 12
 - portfolio 3
 - homogeneous 3
 - total claim amount process 4, 71
 - Compound geometric sum 111
 - characteristic function 111
 - as a mixture distribution 112
 - and ruin probability 172
 - for exponential claim sizes 174
 - Compound Poisson process 12, 113
 - characteristic function 111
 - and Cramér-Lundberg model 12
 - and decomposition of time and claim size space 115
 - in the Cramér-Lundberg model 118
 - and infinitely divisible distributions 140, 342
 - as a Lévy process 12, 336
 - sums of independent compound Poisson sums 113
 - Compound Poisson representation of a Poisson integral 235
 - Compound sum process 4
 - characteristic function 110
 - compound geometric process 111
 - compound Poisson process 12, 115
 - Consumer price index (CPI) 261
 - Continuity, stochastic 335
 - Convergence to Types Theorem 323
 - Cox process 67
 - CPI
 - see consumer price index
 - Cramér-Lundberg model 12
 - and basic model 268
 - and central limit theorem 75
 - compound Poisson property 114
 - as a marked Poisson random measure 249
 - mean of the total claim amount 73
 - multivariate claim sizes 251
 - and shot noise 28
 - and strong law of large numbers 75
 - variance of the total claim amount 74
 - Cramér's ruin bound 162
 - defective renewal equation 166
 - Esscher transform 166
 - for exponential claim sizes 167, 174
 - integral equation 163
 - Smith's key renewal theorem 166
 - Credibility estimator 208
 - credibility weight 208
 - linear Bayes estimator 206
 - Credibility theory
 - see experience rating
 - Credibility weight 208
- ## D
- Danish fire insurance data 1980–1990 in terms of 1985 prices
 - arrival times 32
 - claim sizes 91
 - Danish fire insurance data 1980–2002 in terms of 2002 prices
 - annual claim numbers and claim amounts 261
 - Decomposition of time and claim size space for a compound Poisson process 115
 - in the Cramér-Lundberg model 118
 - Decomposition of time-claim size space
 - in the basic model 259
 - by claim size 259
 - by year of occurrence 261
 - by year of reporting 263
 - Deductible in excess-of-loss reinsurance 143, 260
 - Defective renewal equation 166
 - Delay in reporting 268
 - Dirac measure 216
 - Direct Riemann integrability 61
 - and Cramér's ruin bound 167
 - Distribution of a point process 222
 - finite-dimensional distributions of a point process 222
 - Laplace functional 222
- ## E
- ECOMOR (Excédent du coût moyen relatif) reinsurance 144
 - asymptotic theory for claim size distributions in the maximum domain of attraction of the Gumbel distribution 328
 - for exponential claim sizes 145
 - Elementary renewal theorem 57
 - Empirical distribution function 83

- empirical quantile function 84
- Empirical mean excess function 91
 - mean excess plot 91
- Empirical quantile function 84
 - empirical distribution function 83
 - QQ-plot 84
- Equivalence premium principle 78
- Erlang distribution 16
- Esscher transform 166
- Exact asymptotics for the ruin probability
- compound geometric representation of the ruin probability 172
- Cramér's ruin bound 162
 - defective renewal equation 166
 - Esscher transform 166
 - Smith's key renewal theorem 166
- for exponential claim sizes 174
- integral equation 163
- integrated tail distribution 163
- large claim case 174
- small claim case 162
- Excess life of a renewal process 63
 - see forward recurrence time
- Excess-of-loss reinsurance 143
 - deductible 143, 260
- Expected shortfall 88
 - see mean excess function
- Expected value premium principle 79
 - safety loading 79
- Experience rating 185
 - Bayes estimation 187, 189
 - heterogeneity model 187
 - minimum risk estimator 190
 - risk 190
 - linear Bayes estimation 199
 - Bühlmann model 204
 - Bühlmann-Straub model 209
 - normal equations 204
- Exponentially tilted distribution 166
- Exponential premium principle 81
- Extremal variate 328
- Extremes
 - convergence 300
 - in a random sample 318
 - convergence in distribution under affine transformations 305
 - criterion for convergence 307
- maxima and order statistics of iid random variables 300
- and point process of exceedances 300
- and point process of exceedances at the times of a renewal process 316
- Extreme value distribution 149, 306
 - convergence in distribution of affinely transformed maxima and order statistics 305
 - Fisher-Tippett theorem 308
 - Fréchet distribution 149, 306
 - maximum domain of attraction 311
 - regular variation 311
 - generalized extreme value distribution 309
 - Jenkinson-von Mises representation 309
 - Gumbel distribution 149, 306
 - auxiliary function 314
 - maximum domain of attraction 313
 - von Mises function 314
 - maximum domain of attraction (MDA) 309
 - characterization 310
 - as max-stable distributions 306
 - and its type 306
 - Weibull distribution 149, 306
 - maximum domain of attraction 312

F

- Filtration 383
 - martingale 383
- Finite-dimensional distributions of a point process 222
 - distribution of a point process 222
 - and weak convergence of point processes 293
 - and convergence of Laplace functionals 295
- Fisher-Tippett theorem 308
 - convergence in distribution of affinely transformed maxima and order statistics 305
- extreme value distribution 306
 - Fréchet distribution 306
 - Gumbel distribution 306
 - Weibull distribution 306

- generalized extreme value distribution 309
 Jenkinson-von Mises representation 309
 Forgetfulness property of the exponential distribution 20, 48, 89
 Forward recurrence time
 of a homogeneous Poisson process 19
 of a renewal process 63
 Fractional Brownian motion 289
 fractional Gaussian noise 289
 long range dependence 283, 289
 self-similar process 289
 Fréchet distribution 149
 extremal variate 328
 as extreme value distribution 306
 Fisher-Tippett theorem 308
 generalized extreme value distribution 309
 Jenkinson-von Mises representation 309
 maximum domain of attraction 311
 as max-stable distribution 306
 examples 312
 and regular variation 311
 and its type 306
- G**
- Gamma distribution 16
 Erlang distribution 16
 as infinitely divisible distribution 342
 Gamma process 356
 Gaussian process 336
 Brownian motion 336
 fractional Brownian motion 289
 fractional Gaussian noise 289
 General cluster process 363
 Generalized extreme value distribution (GEV) 309
 Jenkinson-von Mises representation 309
 maximum domain of attraction (MDA) 309
 characterization 310
 Generalized inverse
 of a distribution function 82
 of a non-decreasing function 231, 242,
 245
- Generalized Pareto distribution 107
 Generalized Poisson process 35, 227
 order statistics property 52
 Poisson random measure 39, 227
 GEV
 see generalized extreme value distribution
 Glivenko-Cantelli lemma 84
 Gumbel distribution 149
 extremal variate 328
 as extreme value distribution 306
 Fisher-Tippett theorem 308
 generalized extreme value distribution 309
 Jenkinson-von Mises representation 309
 maximum domain of attraction 313
 auxiliary function 314
 examples 317
 rapid variation of the tail 316
 von Mises function 314
 as max-stable distribution 306
 and its type 306
- H**
- Hazard rate function 108
 relation with the auxiliary function 315
 Heavy-tailed distribution 86, 89
 large claim distribution 98
 regularly varying distribution 99
 and ruin probability 174
 subexponential distribution 103
 Heterogeneity model 187
 Bayes estimation 189
 minimum risk estimator 190
 risk 190
 and the strong law of large numbers 196
 Homogeneous Poisson process 9
 arrival times 16
 joint distribution 21
 compound Poisson process 12, 113
 independent increments 8
 inspection paradox 19
 intensity 9
 inter-arrival times 19
 joint distribution 21

- Kallenberg's theorem 293
 as a Lévy process 10, 336
 order statistics property 26
 record sequence of an iid exponential sequence 231
 relations with inhomogeneous Poisson process 14
 as a renewal process 16
 standard homogeneous Poisson process 9
 stationary increments 10
 strong law of large numbers 56
 transformation to inhomogeneous Poisson process by time change 15
- H**
 Homogeneous Poisson random measure 227
 intensity 228
 Kallenberg's theorem 293
 simple point process 218
- Homogeneous portfolio 3
 Hurst coefficient 283
 fractional Gaussian noise 289
 long range dependence, long memory 283
- I**
- IBNR claim
 see incurred but not reported claim
- Importance sampling 131
- Increment of a stochastic process
 independent increments 8, 335
 Brownian motion 9, 336
 compound Poisson process 118
 Lévy process 10, 335
 Poisson process 7
 stationary increments 10, 335
- Incurred but not reported (IBNR)
 claim 43, 251, 271
 in the basic model 271
 claim number 272
 total claim amount 273
 and Poisson random measure 251
 relation with teletraffic model 278
 stationary claim number process 280
- Independent increments
 of a stochastic process 8
 Brownian motion 9, 336
 compound Poisson process 118
- Lévy process 10, 335
 Poisson process 7
- I**ndex of regular variation 99
Individual model 187
 Bühlmann model 204
 Bühlmann-Straub model 209
 heterogeneity model 187, 188
 risk 190
- Industrial fire data (US) 91
- Infinitely divisible distribution 140, 341
 characteristic triplet 344
 examples 341
 Cauchy distribution 347
 compound Poisson distribution 342
 gamma distribution 342
 log-normal distribution 342
 normal distribution 341
 Pareto distribution 342
 Poisson distribution 342
 stable distribution 341
 student distribution 342
 Lévy-Khintchine formula 344
 characteristic triplet 344
 Lévy measure 350
 and Lévy process 341
- Inhomogeneous Poisson process 9
 arrival times 21
 joint distribution 21
 inter-arrival times
 joint distribution 21
 transformation to homogeneous Poisson process by time change 15
 transformation of the points of a Poisson random measure 245
- Initial capital in the risk process 152
- Inspection paradox of the homogeneous Poisson process 19
- Integrated tail distribution 163
 and subexponentiality 176
- Intensity, intensity function
 of a Poisson process 9
 of a Poisson random measure 228
 relation with the Markov intensities 13
- Inter-arrival times
 of the homogeneous Poisson process 19
 inspection paradox 19

of the inhomogeneous Poisson process 21
joint distribution 21
of the renewal process 53

J

Jenkinson-von Mises representation of the generalized extreme value distribution 309
Jump measure of a Lévy process
see Lévy measure

K

Kallenberg's theorem 293
simple point process 218
Karamata's theorem 181
Key renewal theorem 62
and Cramér's ruin bound 166
Kolmogorov's consistency theorem 8, 341

L

Laplace functional of a point process 222
distribution of a point process 222
finite-dimensional distributions of a point process 222
relation with Laplace-Stieltjes transform 223
of a Poisson random measure 233
and weak convergence of point processes 295
Laplace-Stieltjes transform 110
relation with Laplace functional 223
multivariate 223
of a positive stable random variable 50
properties 178
and ruin probability 173
Large claim distribution 98
regularly varying distribution 99
and ruin probability 174
subexponential distribution 103
Largest claim sizes
joint distribution 325
Largest claims reinsurance 144
asymptotic theory for claim size distributions in the maximum domain of attraction of the Gumbel distribution 330
for exponential claim sizes 145
joint distribution of largest claim sizes 325
Largest (most costly) insured losses 1970–2007 97
Lévy-Itô representation of a Lévy process 349, 353
Lévy-Khintchine formula 344
and characteristic function of a Lévy process 342
characteristic triplet 344
examples 346
infinitely divisible distribution 341
Lévy measure 350
Lévy measure 350
Lévy-Khintchine formula 344
Lévy process 10, 335
Lévy process 10, 335
Brownian motion 10, 336
characteristic function 343
characteristic triplet 344
infinite divisibility 341
Lévy-Khintchine formula 344
compound Poisson process 12, 336
gamma process 356
expectation 338
homogeneous Poisson process 9, 336
independent increments 8, 335
and infinite divisibility of the marginal distributions 341
Lévy-Itô representation 349, 353
Lévy-Khintchine formula 344
Lévy measure 350
pure jump Lévy process 354
spectrally positive Lévy process 355
stable Lévy process 356
 α -stable Lévy motion 357
Cauchy process 357
stationary increments 10
stochastic continuity 335
subordinator 355
variance 338
Light-tailed distribution 86, 89
small claim condition 158
small claim distribution 96
Linear Bayes estimation 199, 200
in the Bühlmann model 206

- credibility estimator 208
 - in the Bühlmann-Straub model 210
 - normal equations 204
 - Logarithmic distribution** 139
 - and the negative binomial distribution as a compound Poisson sum 140
 - Log-gamma distribution** 98
 - Log-normal distribution** 98
 - as infinitely divisible distribution 342
 - Long range dependence, long memory** 283
 - fractional Gaussian noise 289
 - Hurst coefficient 283
 - stationary IBNR claim number process 280
 - teletraffic model 278
 - Lundberg coefficient** 158
 - for exponential claim sizes 160
 - Lundberg's inequality** 157, 159
 - adjustment coefficient 158
 - for exponential claim sizes 160
 - Lundberg coefficient 158
- ## M
- Mack's model** 366
 - chain ladder model 365
 - chain ladder estimator 370
 - prediction 376
 - Mark distribution of a marked Poisson random measure** 246
 - Marked Poisson random measure** 246
 - Cramér-Lundberg model as marked Poisson random measure 249
 - thinning 247
 - mark sequence 246
 - mark distribution 246
 - Mark sequence of a marked Poisson random measure** 246
 - Markov property of the Poisson process** 12
 - intensities 13
 - transition probabilities 13
 - Martingale** 178, 383
 - filtration 383
 - Maxima of iid random variables and aggregation**
 - of regularly varying random variables 102
 - of subexponential random variables 103
 - convergence in distribution under affine transformations** 305
 - criterion for convergence 307
 - Fisher-Tippett theorem 308
 - convergence of maxima and order statistics** 300
 - of a random sample 318
 - extreme value distribution** 149
 - Fréchet distribution 149
 - Gumbel distribution 149
 - maximum domain of attraction (MDA) 309
 - Weibull distribution 149
 - Maximum domain of attraction (MDA) of an extreme value distribution** 309
 - characterization** 310
 - of Fréchet distribution 311
 - examples 312
 - and regular variation 311
 - of Gumbel distribution** 313
 - and auxiliary function 314
 - examples 317
 - rapid variation of the tail 316
 - von Mises function 314
 - of Weibull distribution** 312
 - examples 313
 - Max-stable distributions** 306
 - as extreme value distributions 306
 - MDA**
 - see maximum domain of attraction
 - Mean excess function** 88
 - and auxiliary function 314
 - empirical mean excess function 91
 - of the generalized Pareto distribution 107
 - mean excess loss function 88
 - table of important examples 90
 - Mean excess loss function** 88
 - see mean excess function
 - Mean excess plot** 88, 91
 - empirical mean excess function 91
 - of heavy-tailed distributions 89
 - of light-tailed distributions 89

- Mean measure of a Poisson random measure (PRM) 39, 227
 Radon measure 227
- Mean residual life function 88
 see mean excess function
- Mean value function of a Poisson process 8
- Memory in a stationary process 281
- Mill's ratio 87
- Minimum linear risk estimator
 see linear Bayes estimation
- Minimum risk estimator
 see Bayes estimation
- Mixed Poisson process 66
 as a Cox process 67
 definition 66
 mixing variable 66
 negative binomial process 66
 order statistics property 68
 overdispersion 68
 strong law of large numbers 70
- Mixing variable of a mixed Poisson process 66
- Mixture distribution 110
 aggregation of independent marked Poisson random measures 254
 characteristic function 113
 compound geometric sum 112
 and ruin probability 172, 173
 definition 112
 sum of compound Poisson random variables 113
- Moment generating function 110
- Monte Carlo approximation to the total claim amount 130
 importance sampling 131
- Multiplicity of a point 218
- N**
- Negative binomial distribution 66
 as a compound Poisson distribution 140
 and logarithmic distribution 139
- Negative binomial process
 as a mixed Poisson process 66
- Net premium principle 78
- Net profit condition (NPC) 155
 and premium calculation principles 156
 safety loading 156
- Normal distribution as infinitely divisible distribution 341
- Normal equations 204
 linear Bayes estimator 204
 in the Bühlmann model 206
 in the Bühlmann-Straub model 210
- No ties in the sample 23
- NPC
 see net profit condition
- O**
- Operational time 8, 9, 15
- Order statistics, ordered sample 22, 300
 convergence 300
 convergence in distribution under affine transformations 305
 criterion for convergence 307
 criterion for a random sample 318
 joint density 22
 joint limit distribution of upper order statistics 328
 and maxima 300
 of a random sample 318
 no ties in the sample 23
- order statistics property
 of a generalized Poisson process (Poisson random measure) 52
 of a mixed Poisson process 68
 of a Poisson process 22
- relation with point process of exceedances 300
 relation with point process of exceedances at the times of a renewal process 316
- representation of an exponential ordered sample via iid exponential random variables 49
- representation of a uniform ordered sample via iid exponential random variables 49
- Order statistics property
 of a generalized Poisson process (Poisson random measure) 52
 of the mixed Poisson process 68
 of the Poisson process 22, 24

- of the homogeneous Poisson process 26
 - and shot noise 27
 - and symmetric functions 26, 28
 - Ornstein-Uhlenbeck process 288
 - driven by a homogeneous Poisson process 288
 - relation with AR(1) process 288
 - Outstanding RBNS claim 276
 - settled RBNS claim 274
 - Overdispersion of a mixed Poisson process 68
- P**
- Panjer recursion 120
 - (a, b)-condition 122
 - recursion scheme 122
 - for stop-loss contract 124
 - Pareto distribution 98
 - as infinitely divisible distribution 342
 - Partial sum process 4
 - Peter-and-Paul distribution 101
 - Point measure 216
 - point process 216
 - simple 218
 - Point process 216
 - cluster point process 363
 - distribution 222
 - finite-dimensional distributions 222
 - Laplace functional 222
 - point measure 216
 - point process of exceedances 218, 300
 - and convergence of maxima and order statistics 300
 - weak convergence to a Poisson random measure 300
 - point process of exceedances at the times of a renewal process 316
 - and convergence of maxima and order statistics of a random sample 318
 - random measure 223
 - simple 218
 - Kallenberg's theorem 293
 - renewal model 218
 - renewal process 218
 - point process of exceedances 218, 300
 - state space 216
 - weak convergence 293
 - convergence of the finite-dimensional distributions 293
 - convergence of the Laplace functionals 295
 - Point process of exceedances 218, 300
 - and convergence of maxima and order statistics 300
 - at the times of a renewal process 316
 - weak convergence to a Poisson random measure 300, 318
 - Poisson distribution 7
 - characteristic function 38
 - as infinitely divisible distribution 342
 - Raikov's theorem 47
 - Poisson integral 232
 - compound Poisson representation 235
 - covariance 239
 - existence and finiteness 233, 237
 - expectation 239
 - independence for disjoint supports 237
 - uncorrelatedness and independence 240
 - variance 239
 - Poisson process 7
 - arrival times
 - joint distribution 21
 - càdlàg sample paths 8
 - definition 7
 - finite-dimensional distributions 8
 - generalized Poisson process 35
 - see also Poisson random measure 39, 227
 - homogeneous 9
 - as a Lévy process
 - as a renewal process 16
 - stationary increments 10
 - transformation to inhomogeneous Poisson process by time change 15
 - independent increments 8
 - inhomogeneous 9, 245
 - transformation to homogeneous Poisson process by time change 15
 - intensity, intensity function 9
 - inter-arrival times
 - joint distribution 21
 - Markov property 13

- relation with the intensity function 13
- mean value function 8
operational time 8, 9, 15
- mixed Poisson process 66
- order statistics property 22, 24
- planar 44
- Poisson random measure (PRM) 39, 227
mean measure of PRM 39
state space 40
- rate, rate function 9
- transformed Poisson process 35, 41
- Poisson random measure (PRM) 39, 227
aggregation 254
definition 227
distribution 222
finite-dimensional distributions 222
- generalized Poisson process 35
- homogeneous 227
intensity 228
- inhomogeneous 245
- intensity function 228
- Laplace functional 233
- marked 246
Cramér-Lundberg model as marked
Poisson random measure 249
- mark distribution 246
- mark sequence 246
- thinning 247
- mean measure of PRM 39, 227
- Radon measure 227
- Raikov's theorem 258
- representation if mean measure is finite 227
- transformation of the points 41, 244
inhomogeneous Poisson process 245
- order statistics property 52
- rate function 228
- state space 40, 227
- thinning 247
- weak convergence 294, 298
vague convergence of mean measures 298
- Portfolio 3
homogeneous 3
inhomogeneous in the Bühlmann-Straub model 209
- Prediction and reserves in Mack's model 376
- Premium
and experience rating 189
in the risk process 152
premium rate 152
- Premium calculation principles 78
equivalence premium principle 78
expected value premium principle 79
exponential premium principle 81
net premium principle 78
and net profit condition (NPC) 156
and safety loading 78, 79
standard deviation premium principle 79
theoretical requirements 81
variance premium principle 79
- Premium rate 152
- PRM
see Poisson random measure
- Probability of ruin
see ruin probability 153
- Proportional reinsurance 143, 261
- Pure jump Lévy process 354
- Q**
- QQ-plot
see quantile-quantile plot
- Quadratic risk
in Bayes estimation 190
in linear Bayes estimation 200
normal equations 204
- Quantile of a distribution 82
- Quantile function 82
empirical quantile function 84
generalized inverse of a distribution function 82
- Quantile-quantile plot (QQ-plot) 82, 84
empirical quantile function 84
and Glivenko-Cantelli lemma 84
for heavy-tailed distribution 86
for light-tailed distribution 86
- R**
- Radon measure 227
- Raikov's theorem
for Poisson distribution 47

- for Poisson random measure 258
- Random measure 223
- Range of dependence in a stationary process 281
- Rapid variation 316
 - and maximum domain of attraction of the Gumbel distribution 316
- Rate, rate function
 - of a Poisson process 9
 - of a Poisson random measure 228
- RBNS claim
 - see reported but not settled claim
- Record, record time of an iid sequence 53, 228
 - of an iid exponential sequence 231
 - as a homogeneous Poisson process 231
 - as a Markov chain 230
 - number of records in a sample 231
 - expectation and variance 232
- Recurrence time of a renewal process 63
 - backward recurrence time 63
 - of a homogeneous Poisson process 19
 - forward recurrence time 63
 - of a homogeneous Poisson process 19
- Reflection principle of Brownian motion 157
- Regularly varying distribution 99
 - aggregation of regularly varying random variables 101, 102
 - convolution closure 101, 102
 - examples 99
 - and maxima 102
 - and maximum domain of attraction of the Fréchet distribution 311
 - examples 312
 - and maximum domain of attraction of the Weibull distribution 312
 - examples 313
 - moments 100
 - and ruin probability 174
 - and subexponential distribution 103
 - tail index 99
- Regularly varying function 99
 - index 99
 - Karamata's theorem 181
 - regularly varying distribution 99
 - slowly varying function 99
- Reinsurance treaties 142
- of extreme value type
 - ECOMOR reinsurance 144
 - largest claims reinsurance 144
 - asymptotic theory 324
- joint distribution of the largest claim sizes 325
- of random walk type
 - excess-of-loss reinsurance 143
 - proportional reinsurance 143, 261
 - stop-loss reinsurance 143
- Renewal equation 62
 - defective 166
 - and renewal function 62
 - and ruin probability 166
- Renewal function 60
 - satisfies the renewal equation 62
- Renewal model for the total claim amount 71
 - central limit theorem 75
 - mean of the total claim amount process 73
 - as a simple point process 218
- Sparre-Anderson model 71
 - strong law of large numbers 75
- variance of the total claim amount process 74
- Renewal process 53
 - backward recurrence time 63
 - of a homogeneous Poisson process 19
 - central limit theorem 60
 - elementary renewal theorem 57
 - forward recurrence time 63
 - of a homogeneous Poisson process 19
 - homogeneous Poisson process as a renewal process 16
 - recurrence time 63
 - renewal sequence 53
 - as a simple point process 218
 - strong law of large numbers 56
 - variance, asymptotic behavior 60
- Renewal sequence 53
 - of a homogeneous Poisson process 16
- Renewal theory
 - Blackwell's renewal theorem 61
 - direct Riemann integrability 61
 - and Cramér's ruin bound 167
 - elementary renewal theorem 57
 - renewal equation 62
 - renewal function 60

- Smith's key renewal theorem 62
and Cramér's ruin bound 166
- Reported but not settled (RBNS)
claim 270
claim number 272
outstanding 276
settled 274
total claim amount 274
- Residual life of a renewal process 63
see forward recurrence time
- Retention level in stop-loss reinsurance 143
- Risk (quadratic) in the individual model
in the Bühlmann model 199
in the Bühlmann-Straub model 210
in the heterogeneity model 190
in linear Bayes estimation 200
normal equations 204
- Risk models (collective)
Cramér-Lundberg model 12
renewal model 71
- Risk process 152
initial capital 152
net profit condition (NPC) 155
premium, premium rate 152
ruin 152
ruin probability 153
adjustment coefficient 158
compound geometric representation 172, 173
Cramér's ruin bound 162
for exponential claim sizes 174
integral equation 163
integrated tail distribution 163
large claim case 174
Lundberg coefficient 158
Lundberg's inequality 157, 159
net profit condition (NPC) 155
skeleton process 154
small claim condition 158
the small claim case 162
- ruin time 153
safety loading 79
surplus process 152
- Risk theory 3
- Ruin 152
- Ruin probability 153
adjustment coefficient 158
- compound geometric representation 172, 173
Cramér's ruin bound 162
and defective renewal equation 166
and Esscher transform 166
integral equation 163
and Smith's key renewal theorem 166
- exact asymptotics
large claim case 174
small claim case 162
- for exponential claim sizes 174
- integral equation 163
integrated tail distribution 163
- Lundberg coefficient 158
- Lundberg's inequality 157, 159
for exponential claim sizes 160
- net profit condition (NPC) 155
- safety loading 79
- skeleton process 154
- small claim condition 158
and tail of the distribution of a stochastic recurrence equation 171
- Ruin time 153
- S**
- Saddle point approximation 126
- Safety loading 78
and expected value premium calculation principle 79
and net profit condition (NPC) 156
- Self-similar process 289, 358
fractional Brownian motion 289
symmetric α -stable Lévy motion 358
- Settled RBNS claim 274
outstanding RBNS claim 276
- Settlement of claim payments 268
duration 269
- settled claim in basic model 270
claim number 272
total claim amount 273
- Shot noise 27, 28
and the Cramér-Lundberg model 31
- Simple point measure 218
- Simple point process 218
Kallenberg's theorem 293
point process of exceedances 218

- point process of exceedances at the times of a renewal process 316
 renewal model 218
 renewal process 218
 Skeleton process for probability of ruin 154
 Skorokhod space 9
 càdlàg sample paths 8
 Slowly varying function 99
 Karamata's theorem 181
 regularly varying function 99
 representation 99
 Small claim condition 158
 Small claim distribution 96
 Smith's key renewal theorem 62
 and Cramér's ruin bound 166
 Sparre-Anderson model 71
 see renewal model
 Spectral representation of an α -stable distribution 357
 Spectrally positive Lévy process 355
 subordinator 355
 Stable distribution 50, 98
 Cauchy distribution 341, 347
 characteristic function 341
 as a large claim distribution 98
 series representation via Poisson process 50
 Stable Lévy process 356
 α -stable Lévy motion 357
 Cauchy process 357
 self-similarity 358
 spectral representation 357
 Standard deviation premium principle 79
 Standard homogeneous Poisson process 9
 State space
 of a point process 216
 of a Poisson random measure 40, 227
 Stationary claim number process 280
 Stationary increments of a stochastic process 10
 Lévy process 335
 Stochastic continuity 335
 Lévy process 335
 and càdlàg sample paths 336
 Stochastic recurrence equation 167
 and ruin probability 171
 Stop-loss reinsurance 143
 Panjer recursion for stop-loss contract 124
 retention level 143
 Stopping time 57
 Wald's identity 58
 Strictly stationary process 280
 autocovariance, autocorrelation 282
 Strong law of large numbers
 in the heterogeneity model 196
 for the mixed Poisson process 70
 for the renewal process 56
 for the total claim amount process in the renewal model 75
 Subexponential distribution 103
 aggregation of subexponential claim sizes 103
 basic properties 103
 examples 105
 and hazard rate function 108
 and maxima of iid random variables 103
 regularly varying distribution 99
 and maximum domain of attraction of the Fréchet distribution 311
 and ruin probability 174
 tail of the total claim amount distribution 129
 Subordinator 355
 gamma process 356
 spectrally positive Lévy process 355
 Surplus process 152
 see risk process
- ## T
- Tail equivalence of distributions 314
 Tail index of a regularly varying distribution 99
 Teletraffic model 278
 and stationary IBNR claim number process 280
 Thinning of a Poisson random measure 247
 Ties in the sample 23
 Time-claim size space 259, 270
 basic decomposition of basic model 270
 of claim number 271

- of total claim amount 273
decomposition
 by claim size 259
 by year of occurrence 261
 by year of reporting 263
Total claim amount process 4, 71
approximation to distribution
 by central limit theorem 125
 conditional 127
 error bounds 125
 by Monte Carlo methods 130
 tail for subexponential claim
 sizes 129
basic decomposition in basic
 model 273
characteristic function 110
Cramér-Lundberg model 12
 central limit theorem 75
compound Poisson representation
 on a subspace 253
mean 73
strong law of large numbers 75
variance 74
order of magnitude 72
Panjer recursion 120
renewal model 71
 central limit theorem 75
mean 73
Sparre-Anderson model 71
strong law of large numbers 75
variance 74
Transformation of the points of a
 Poisson random measure 244
 inhomogeneous Poisson process 245
Transition probabilities
 of the Poisson process as a Markov
 process 13
 intensities 13
Truncated normal distribution 86
Type of a distribution 306
 Convergence to Types Theorem 323
- U
- US industrial fire data 91
- V
- Vague convergence 298
Variance premium principle 79
Von Mises function 314
 properties 316
 rapid variation of the tail 316
 tail equivalence with distributions in
 the maximum domain of attraction
 of the Gumbel distribution 314
- W
- Wald's identity 58
stopping time 57
Weak convergence of point processes 293
 convergence of the finite-dimensional
 distributions 293
 convergence of the Laplace function-
 als 295
Kallenberg's theorem 293
 simple point process 218
of Poisson random measures 294, 298
and vague convergence of mean
 measures 298
Weibull (claim size) distribution 96, 98
Weibull (extreme value) distribution 149
 extremal variate 328
 as extreme value distribution 306
 Fisher-Tippett theorem 308
 generalized extreme value distribu-
 tion 309
 Jenkinson-von Mises representa-
 tion 309
maximum domain of attraction 312
 examples 313
 as max-stable distribution 306
 and its type 306

List of Abbreviations and Symbols

We have tried as much as possible to use uniquely defined abbreviations and symbols. In various cases, however, symbols can have different meanings in different sections. The list below gives the most typical usage. Commonly used mathematical symbols are not explained here.

Abbreviation or Symbol	Explanation	p.
a.s.	almost sure, almost surely, with probability 1	
a.e.	almost everywhere, almost every	
$\mathcal{B}(A)$	Borel σ -field on A	
$\text{Bin}(n, p)$	binomial distribution with parameters (n, p) : $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, \dots, n$	
\mathbb{C}	set of the complex numbers	
\mathbb{C}_K^+	space of non-negative continuous functions with compact support	223
$\text{corr}(X, Y)$	correlation between the random variables X and Y	
$\text{cov}(X, Y)$	covariance between the random variables X and Y	
$\text{CP}(\lambda, F)$	compound Poisson distribution with Poisson intensity λ and distribution F of the summands	235
E	expectation or state space of a point process	216
$E_F X$	expectation of X with respect to the distribution F	
$e_F(u)$	mean excess function	88
\mathcal{E}	Borel σ -field on state space E	216
ε_x	Dirac measure at x	216
$\text{Exp}(\lambda)$	exponential distribution with parameter λ : $F(x) = 1 - e^{-\lambda x}$, $x > 0$	
F	distribution function/distribution of a random variable	
\mathcal{F}	σ -field	
F_A	distribution function/distribution of the random variable A	
F_I	integrated tail distribution:	

$F_I(x) = (E_F X)^{-1} \int_0^x \bar{F}(y) dy, x \geq 0$	163
F_n	empirical (sample) distribution function
$F^{\leftarrow}(p)$	p -quantile/quantile function of F
$F_n^{\leftarrow}(p)$	empirical p -quantile
\bar{F}	tail of the distribution function F : $\bar{F} = 1 - F$
F^{n*}	n -fold convolution of the distribution function/distribution F
\hat{f}_X	Laplace-Stieltjes transform of the random variable X :
	$\hat{f}_X(s) = E e^{-sX}, s > 0$
GEV	generalized extreme value distribution
Γ	gamma function : $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$
$\Gamma(\gamma, \beta)$	gamma distribution with parameters γ and β :
	gamma density $f(x) = \beta^\gamma (\Gamma(\gamma))^{-1} x^{\gamma-1} e^{-\beta x}, x > 0$
H	often an extreme value distribution
$H_{\xi; \mu, \sigma}$	generalized extreme value distribution with parameter (ξ, μ, σ) , $\xi \in \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0$
IBNR	incurred but not reported claim
I_A	indicator function of the set (event) A
iid	independent, identically distributed
λ	intensity or intensity function of a Poisson process
Λ	Gumbel distribution: $\Lambda(x) = \exp\{-e^{-x}\}, x \in \mathbb{R}$
Leb	Lebesgue measure
$\log x$	logarithm with basis e
$\log^+ x$	$\log^+ x = \max(\log x, 0)$
$L(x)$	slowly varying function
MDA(H)	maximum domain of attraction of the extreme value distribution H
M_n	maximum of X_1, \dots, X_n
$M_p(E)$	space of all point measures on E
$\mathcal{M}_p(E)$	σ -field on $M_p(E)$
$\mu(t)$	mean value function of a Poisson process on $[0, \infty)$
μ	mean measure of a Poisson random measure
	or Levy measure
\mathbb{N}	set of the positive integers
\mathbb{N}_0	set of the non-negative integers
$N, N(t)$	claim number or claim number process
$N, N_X, N_{X,Y}$	point processes
\tilde{N}	often a homogeneous Poisson process
$N(\mu, \sigma^2)$	Gaussian (normal) distribution with mean μ , variance σ^2
$N(0, 1)$	standard normal distribution
$N(\mu, \Sigma)$	multivariate Gaussian (normal) distribution with mean vector μ and covariance matrix Σ
NPC	net profit condition
$o(1)$	$h(x) = o(1)$ as $x \rightarrow x_0 \in [-\infty, \infty]$ means that

	$\lim_{x \rightarrow x_0} h(x) = 0$	14
$O(1)$	$h(x) = O(1)$ as $x \rightarrow x_0 \in [-\infty, \infty]$ means that	
	$\limsup_{x \rightarrow x_0} h(x) < \infty$	
ω	$\omega \in \Omega$ random outcome	
(Ω, \mathcal{F}, P)	probability space	
P_N	distribution of the point process N	222
$\phi_X(t)$	characteristic function of the random variable X :	
	$\phi_X(t) = Ee^{itX}, t \in \mathbb{R}$	
Φ	standard normal distribution/distribution function	
Φ_α	Frechet distribution: $\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}, x, \alpha > 0$	149
$\text{Pois}(\lambda)$	Poisson distribution with parameter $\lambda > 0$:	
	$p(n) = e^{-\lambda} \lambda^n / n!, n \in \mathbb{N}_0$	
PRM	Poisson random measure	
PRM(μ)	Poisson random measure with mean measure μ	227
$\psi(u)$	ruin probability	153
Ψ_α	Weibull (extreme value) distribution:	
	$\Psi_\alpha(x) = \exp\{-(-x)^\alpha\}, x < 0, \alpha > 0$	149
$\Psi_N(g)$	Laplace functional of the point process N	222
\mathbb{R}, \mathbb{R}^1	real line	
\mathbb{R}_+	$\mathbb{R}_+ = (0, \infty)$	
\mathbb{R}^d	d -dimensional Euclidean space	
$\overline{\mathbb{R}}$	$\mathbb{R} \cup \{\infty, -\infty\}$	
RBNS	reported but not settled claim	270
ρ	safety loading	79
$\rho(\hat{\mu})$	(quadratic) Bayes or linear Bayes risk of $\hat{\mu}$	190
\mathcal{S}	class of the subexponential distributions	103
$\text{sign}(a)$	sign of the real number a	
S_n	cumulative sum of X_1, \dots, X_n	
$S, S(t)$	total, aggregate claim amount process	4
t	time, index of a stochastic process	
T_i	arrival times of a claim number process	3
u	initial capital	152
$U(a, b)$	uniform distribution on (a, b)	
$U(t)$	risk process	152
$\text{var}(X)$	variance of the random variable X	
$\text{var}_F(X)$	variance of a random variable X with distribution F	
x_F	right endpoint of the distribution F on \mathbb{R} :	
	$x_F = \sup\{x \in \mathbb{R} : F(x) < 1\}.$	
X_n	claim size	3
$X_{(n-i+1)}$	i th largest order statistic in the sample X_1, \dots, X_n	22
\bar{X}_n	sample mean	
ξ	often shape parameter of the generalized extreme value distribution	308
\mathbb{Z}	set of the integers	
\mathbb{Z}_0	$\mathbb{Z} \setminus \{0\}$	

\sim	$X \sim F$: X has distribution F
	$f(x) \sim g(x)$ as $x \rightarrow x_0$ for positive functions f, g means that $f(x) = g(x)(1 + o(1))$ as $x \rightarrow x_0$
\approx	$a(x) \approx b(x)$ as $x \rightarrow x_0$ means that $a(x)$ is approximately (roughly) of the same order as $b(x)$ as $x \rightarrow x_0$. It is only used in a heuristic sense.
*	convolution or bootstrapped quantity
$\#B$	cardinality of the set B
$\ \cdot\ $	$\ x\ $ norm of x
$[.]$	$[x]$ integer part of x
$\{\cdot\}$	$\{x\}$ fractional part of x
x_+	positive part of a real number: $x_+ = \max(0, x)$
x_-	negative part of a real number: $x_- = \max(0, -x)$
B^c	complement of the set B
\overline{B}	closure of the set B
∂B	boundary of the set B
$ B $	Lebesgue measure of the set B
$\xrightarrow{\text{a.s.}}$	$A_n \xrightarrow{\text{a.s.}} A$: a.s. convergence
\xrightarrow{d}	$A_n \xrightarrow{d} A$: convergence in distribution
\xrightarrow{P}	$A_n \xrightarrow{P} A$: convergence in probability
\xrightarrow{v}	$\mu_n \xrightarrow{v} \mu$: vague convergence of measures
$\stackrel{d}{=}$	$A \stackrel{d}{=} B$: A and B have the same distribution. If A and B are stochastic processes with index set $C \subset \mathbb{R}$ then $A \stackrel{d}{=} B$ refers to the equality of the finite-dimensional distributions

298

For a function f on \mathbb{R} and intervals $(a, b]$, $a < b$, we write $f(a, b] = f(b) - f(a)$.

For measures μ and point processes N on $E \subset \mathbb{R}$ we write $\mu(a, b] = \mu((a, b])$, $N(a, b] = N((a, b])$, $\mu[a, b] = \mu([a, b])$, $N[a, b] = N([a, b])$, etc.