UNIVERSITY OF LJUBLJANA
FACULTY OF MATHEMATICS AND PHYSICS

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Anej Rozman, Tanja Luštrek

Rich-Neighbor Edge Colorings

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Long Presentation

Advisers: Assistant Professor Janoš Vidali,
Professor Riste Škrekovski

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1. Introduction

In this paper we set out to analyse an open conjecture in a modern graph theory problem known as rich-neighbor edge coloring.

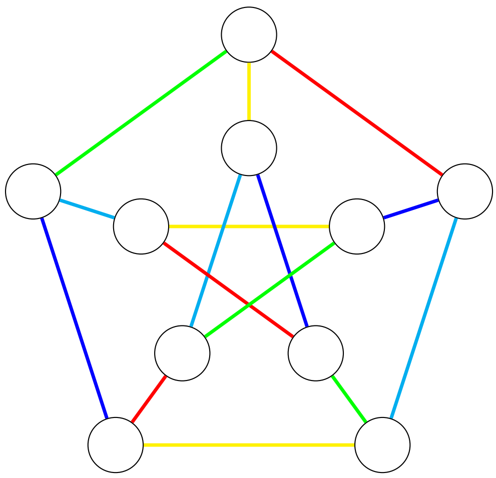
Definition 1.1. In an edge coloring, an edge e is called *rich* if all edges adjacent to e have different colors. An edge coloring is called a *rich-neighbor edge coloring* if every edge is adjacent to some rich edge.

Definition 1.2. $X'_{rn}(G)$ denotes the smallest number of colors for which there exists a rich-neighbor edge coloring.

Conjecture 1.3. *For every graph G of maximum degree Δ, $X'_{rn}(G) \leq 2\Delta - 1$ holds.*

In the paper we focus on analysing the conjecture only for regular graphs. We plan to do this by implementing an integer programming model that finds a rich-neighbor edge coloring for a given graph using the smallest number of colors possible.

Example 1.4. Let's take a look at the Petersen graph and an example of a rich-neighbor edge coloring.



We can see that for the Petersen graph (which is 3-regular) we can find an appropriate coloring with 5 colors so $X'_{rn} \leq 5 \leq 2 \cdot 3 - 1 = 5$. This shows that the conjecture holds for this graph. ♢

2. Algorithms

2.1. Integer Programming

Using SageMath we implement an integer programming model that finds a rich-neighbor edge coloring for a given graph using the smallest number of colors possible. Writen matematically, our interger program looks like this:

minimize t — we minimize the number of colors we need

subject to $\forall e: \quad \sum_{i=1}^{2\Delta-1} x_{ei} = 1$ — each edge is exactly one color

$\forall i\ \forall u\ \forall v, w \sim u, v \neq u: \quad x_{uv,i} + x_{uw,i} \leq 1$ — edges with the same vertex are a different color

$\forall e\ \forall i: \quad x_{ei} \cdot i \leq t$ — we use less or equal to t colors

$\forall i\ \forall uv\ \forall w \sim u, w \neq v\ \forall z \sim v, z \neq u, w: \quad x_{uw,i} + x_{vz,i} + y_{uv} \leq 2$ — uv is a rich edge $\Leftrightarrow$ all adjacent edges are a different color

$\forall e: \quad \sum_{f \sim e} y_f \geq 1$ — every edge is adjacent to some rich edge

$\forall e: \quad 0 \leq y_e \leq 1,\ y_e \in \mathbb{Z}$ — solutions are binary variables

$\forall e\ \forall i: \quad 0 \leq x_{ei} \leq 1,\ x_{ei} \in \mathbb{Z},$

where

$$x_{ei} = \begin{cases} 1, & \text{if edge } e \text{ is color } i \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad y_e = \begin{cases} 1, & \text{if edge } e \text{ is rich} \\ 0, & \text{otherwise.} \end{cases}$$

In our implementation of the ILP we fix the number of colors to $2\Delta - 1$ by adding the following constraint $t = 2\Delta - 1$, since the coloring can't be made with less colors (every edge has $2\Delta - 2$ neighboring edges and the edge itself has to be a different color) and any more colors do not satisfie the conjecture. Therefore we only check if the program has a solution that satisfies all the constraints. In theory this doesn't make the program faster, but in our practice tests it did make a significant differnece.

Our actual implementation of the IPL is demonstrated in the following code.

Algorithm 1: richNeighbor

Input: graph G

Output: colors of edges $colors$, rich edges $richEdges$

```
p ← MixedIntegerLinearProgram(maximization = False)
x ← p.new_variable(binary = True)
y ← p.new_variable(binary = True)
t ← p.new_variable(integer = True)
p.set_objective(t[0])
maxCol ← 2 · G.degree()[0] − 1
p.add_constraint(t[0] = maxCol)
for e in G.edges(labels = False) do
    p.add_constraint(∑_{i=1}^{maxCol} x[Set(e), i] = 1)
for (u, v) in G.edges(labels = False) do
    p.add_constraint(∑_{j∈G[u]} y[Set((u, j))] + ∑_{l∈G[v]} y[Set((l, v))] −
      2y[Set((u, v))] ≥ 1)
for e in G.edges(labels = False) do
    for i in 1 to maxCol do
        p.add_constraint(i · x[Set(e), i] ≤ t[0])
for i in 1 to maxCol do
    for (u, v) in G.edges(labels = False) do
        for w in G[u] do
            if w = v then
                continue
            p.add_constraint(x[Set((u, v)), i] + x[Set((u, w)), i] ≤ 1)
        for z in G[v] do
            if z = u then
                continue
            p.add_constraint(x[Set((u, v)), i] + x[Set((v, z)), i] ≤ 1)
for (u, v) in G.edges(labels = False) do
    for w in G.neighbors(u) do
        for z in G.neighbors(v) do
            if w = v or z = u then
                continue
            for i in 1 to maxCol do
                p.add_constraint(x[Set((u, w)), i] + x[Set((v, z)), i] +
                  y[Set((u, v))] ≤ 2)
return colors, richEdges
```

Example 2.1. Using the `richNeighbor` algorithm on the Petersen graph gives us the following result.

$$\begin{aligned} colors = \{ & (\{0,1\},1) : 0.0, (\{0,1\},2) : 0.0, (\{0,1\},3) : 0.0, (\{0,1\},4) : 0.0, \\ & (\{0,1\},5) : 1.0, (\{0,4\},1) : 0.0, (\{0,4\},2) : 0.0, (\{0,4\},3) : 1.0, \\ & \dots \\ & (\{9,6\},3) : 0.0, (\{9,6\},4) : 0.0, (\{9,6\},5) : 0.0, (\{9,7\},1) : 0.0, \\ & (\{9,7\},2) : 0.0, (\{9,7\},3) : 0.0, (\{9,7\},4) : 1.0, (\{9,7\},5) : 0.0\} \end{aligned}$$

$$\begin{aligned} richEdges = \{ & \{0,1\} : 1.0, \{0,4\} : 0.0, \{0,5\} : 0.0, \{1,2\} : 1.0, \{1,6\} : 0.0, \\ & \{3,4\} : 0.0, \{9,4\} : 1.0, \{5,7\} : 0.0, \{8,5\} : 1.0, \{2,3\} : 1.0, \\ & \{2,7\} : 0.0, \{8,6\} : 0.0, \{9,6\} : 0.0, \{8,3\} : 1.0, \{9,7\} : 1.0\} \end{aligned}$$

Both *colors* and *richEdges* are dictionaries and their key-value pairs are structured like

$$(e,i) : \begin{cases} 1.0, & \text{if edge } e \text{ is color } i \\ 0.0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad e : \begin{cases} 1.0, & \text{if edge } e \text{ is rich} \\ 0.0, & \text{otherwise.} \end{cases}$$

$\diamondsuit$

2.2. Iterative Algorithm

We debated implementing an iterative algorythm that finds a rich neighbor edge coloring, but we were not able to come up with anything that runs in polynomial time. From [1] we know that integer linear programs are NP-complete problems, so it did't make much sense implementing another slow algorythm, since it wouldn't make much of a difference in the number of graphs we would be able to check.

3. Complete search

As illustrated in Table 1, the number of k-regular graphs on n vertices grows exponentially with n. This poses a computational challenge, as checking all graphs becomes impossible for larger values of n. However, we can still check all graphs for smaller values of n up to 15.

Vertices	Degree 4	Degree 5	Degree 6	Degree 7
5	1	0	0	0
6	1	1	0	0
7	2	0	1	0
8	6	3	1	1
9	16	0	4	0
10	59	60	21	5
11	265	0	266	0
12	1544	7848	7849	1547
13	10778	0	367860	0
14	88168	3459383	21609300	21609301
15	805491	0	1470293675	0
16	8037418	2585136675	113314233808	733351105934

Table 1. Number of k-regular graphs on n vertices [2]

3.1. Graph generation

Graphs for the complete search were taken from a collection of files, provided by Professor Skrekovski. The files contain all k-regular graphs on n vertices up to 15 vertices.

4. Random Search

In addition to examining the hypothesis for smaller graphs, we were also interested in checking if the conjecture holds for larger graphs. The challenge here is testing all possible colorings in graphs with many vertices. Recognizing the enormity of this task, a random search algoritm seemed a natural continuation of our problem. Here we opted for a modification approach.

We start with a random graph, check if the conjecture holds using our ILP and then modify it in a way that preserves the regularity and connectedness. We repeat this process indefinetly, well in reality, until we stop our program. Since, if a rich-neighbor edge coloring exists in a given graph there is a bigger probability that it also exists in similar graphs, therefore, on every iteration, we keep a small probability that we generate a completely new random graph and start again.

Written below is the algorithm `tweak` for modifying graphs. First it selects two random edges and deletes them. Then it generates a random number p from $[0, 1)$ and based on the value of p it adds back two different edges in one of two possible ways which preserves the regularity of the graph. There is a chance that the newly formed graph is not connected, so we check for that and if it is not, we restart the process until we get a connected graph.

Algorithm 2: tweak

```
Input: graph G
Output: tweaked graph T
T ← graph.copy()
e_1 ← T.random_edge()
u_1, v_1, ← e_1
T.delete_edge(e_1)
e_2 ← T.random_edge()
u_2, v_2, ← e_2
T.delete_edge(e_2)
p ← random()
if p < 0.5 then
    T.add_edge(u_1, u_2)
    T.add_edge(v_1, v_2)
else
    T.add_edge(u_1, v_2)
    T.add_edge(v_1, u_2)
if not T.is_connected() then
    return tweak(G)
return T
```

4.1. Graph generation

Since we only need one graph to start our iterative process, we can generate it randomly, using the built-in SageMath function `graphs.RandomRegular`.

5. Checking the coloring

In order to make sure that our ILP implentation is without fault, we also implement the functions `checkColoring` and `checkRichness` which check if our program always returns a proper coloring, and if the coloring is actually a rich-neighbor edge coloring.

`checkColoring` takes a graph G and the output coloring of our ILP. It iterates over each vertex in the graph, checks it's neighbors and returns false if a color of the edges that include it repeats.

`checkRichness` takes a graph G and the output rich edges of our ILP. It iterates over each edge in the graph, checks it's neighbors and returns false if it doesn't find a rich edge for each edge.

Algorithm 3: checkColoring

Input: graph G, colors of edges *coloring*
Output: Boolean indicating proper coloring

```
for v in G.vertices() do
    col ← set()
    for w in G.neighbors(v) do
        for i in range(1, 2 · G.degree()[0]) do
            if coloring[(Set((v, w)), i)] = 1 then
                col.add(i)
    if len(col) ≠ len(G.neighbors(v)) then
        return False
return True
```

Algorithm 4: checkRichness

Input: graph G, rich edges *richEdges*
Output: Boolean indicating proper rich-neighbor edge coloring

```
for (u, v) in G.edges(labels = False) do
    S ← 0
    for w in G.neighbors(u) do
        if w = v then
            S ← S + richEdges[Set((u, w))]
    for z in G.neighbors(v) do
        if z = u then
            S ← S + richEdges[Set((v, z))]
    if S = 0 then
        return False
return True
```

6. Findings

7. Conclusion

References

[1] Ravindran Kannan and Clyde L. Monma. On the computational complexity of integer programming problems. In Rudolf Henn, Bernhard Korte, and Werner Oettli, editors, *Optimization and Operations Research*, pages 161–172, Berlin, Heidelberg, 1978. Springer Berlin Heidelberg.

[2] M. Meringer. Fast generation of regular graphs and construction of cages. *Journal of Graph Theory*, 30:137–146, 1999.