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Financial mathematics – 1st cycle

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Rich-Neighbor Edge Colorings

Term Paper in Finance Lab
Long Presentation

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1. INTRODUCTION

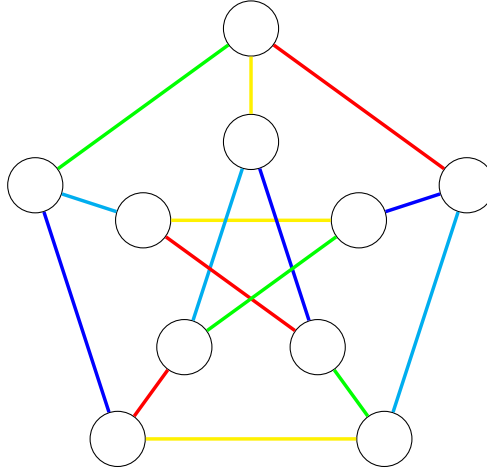
In this paper we set out to analyse an open conjecture in a modern graph theory problem known as rich-neighbor edge coloring.

Definition 1.1. In an edge coloring, an edge e is called *rich* if all edges adjacent to e have different colors. An edge coloring is called a *rich-neighbor edge coloring* if every edge is adjacent to some rich edge.

Definition 1.2. $X'_{rn}(G)$ denotes the smallest number of colors for which there exists a rich-neighbor edge coloring.

Conjecture 1.3. For every graph G of maximum degree Δ , $X'_{rn}(G) \leq 2\Delta - 1$ holds.

Example 1.4. Let's take a look at the Petersen graph and an example of a rich-neighbor edge coloring.



We can see that for the Petersen graph (which is 3-regular) we can find a correct coloring with 5 colors so $X'_{rn} \leq 5 \leq 2 \cdot 3 - 1 = 5$. This shows that the conjecture holds for this graph. \diamond

2. PLAN

Our plan is to create an integer program that “proves” the conjecture for (small) regular graphs of degree 4 or more (it finds a rich-neighbor edge coloring for every k -regular graph on n vertices) and to make a random search algorithm for checking classes of graphs that are too large to be checked individually.

2.1. Integer Programming. Using SageMath we plan to construct an integer programming model that finds a rich-neighbor edge coloring for a given graph. Our integer program looks like this:

$$\begin{array}{ll}
 \text{minimize } t & \text{we minimize the number of colors we need} \\
 \\
 \text{subject to } \forall e : \sum_{i=1}^k x_{ei} = 1 & \text{each edge is exactly one color} \\
 \\
 \forall i \forall u \forall v, w \sim u, v \neq u : x_{uv,i} + x_{uw,i} \leq 1 & \text{edges with the same vertex are a different color} \\
 \\
 \forall e \forall i : x_{ei} \cdot i \leq t & \text{we use less or equal to } t \text{ colors} \\
 \\
 \forall i \forall uv \forall w \sim u, w \neq v \forall z \sim v, z \neq u, w : x_{uw,i} + x_{vz,i} + y_{uv} \leq 2 & uv \text{ is a rich edge} \Leftrightarrow \text{all adjacent edges are a different color} \\
 \\
 \forall e : \sum_{f \sim e} y_f \geq 1 & \text{every edge is adjacent to some rich edge} \\
 \\
 t \geq 2\Delta - 1 & \text{we use } \geq 2\Delta - 1 \text{ colors} \\
 \\
 \forall e : 0 \leq y_e \leq 1, y_e \in \mathbb{Z} \\
 \\
 \forall e \forall i : 0 \leq x_{ei} \leq 1, x_{ei} \in \mathbb{Z},
 \end{array}$$

where

$$x_{ei} = \begin{cases} 1, & \text{if edge } e \text{ has color } i \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad y_e = \begin{cases} 1, & \text{if edge } e \text{ is rich} \\ 0, & \text{otherwise.} \end{cases}$$

We plan to look for appropriate graph colorings with $\geq 2\Delta - 1$ colors because, if we find a coloring with $2\Delta - 1$ colors then the smallest possible number of colors is smaller or equal to that and the conjecture holds for that graph. If we need more than $2\Delta - 1$ colors then it does not hold. The computation is also a lot quicker this way.

Next we will determine at what point the computation of rich-neighbor edge coloring becomes too intense for this technique and we will switch to the random search algorithm.

2.2. Random Search. We will construct a random search algorithm that will check if the conjecture holds for a class of graphs that are too large to be checked individually.