UNIVERSITY OF LJUBLJANA
FACULTY OF MATHEMATICS AND PHYSICS

Financial mathematics – 1st cycle

Anej Rozman, Tanja Luštrek

Rich-Neighbor Edge Colorings

Term Paper in Finance Lab

Long Presentation

Advisers: Assistant Professor Janoš Vidali,
Professor Riste Škrekovski

Ljubljana, 2023

Contents

1. Introduction

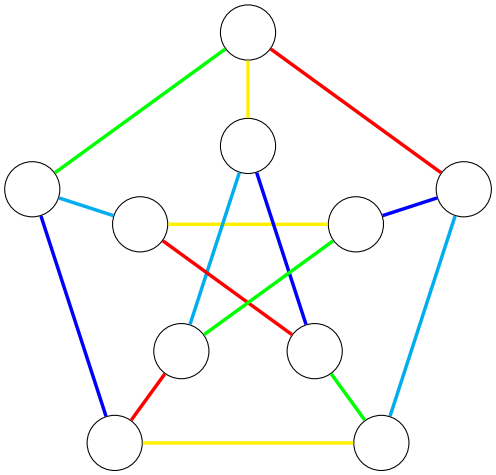
In this paper we set out to analyse an open conjecture in a modern graph theory problem known as rich-neighbor edge coloring.

Definition 1.1. In an edge coloring, an edge e is called *rich* if all edges adjacent to e have different colors. An edge coloring is called a *rich-neighbor edge coloring* if every edge is adjacent to some rich edge.

Definition 1.2. $X'_{rn}(G)$ denotes the smallest number of colors for which there exists a rich-neighbor edge coloring.

Conjecture 1.3. *For every graph G of maximum degree Δ, $X'_{rn}(G) \leq 2\Delta - 1$ holds.*

Example 1.4. Let's take a look at the Petersen graph and an example of a rich-neighbor edge coloring.



We can see that for the Petersen graph (which is 3-regular) we can find an appropriate coloring with 5 colors so $X'_{rn} \leq 5 \leq 2 \cdot 3 - 1 = 5$. This shows that the conjecture holds for this graph. ♢

2. Algorithms

2.1. Integer Programming

Using SageMath we constructed an integer programming model that finds a rich-neighbor edge coloring for a given graph. Our interger program looks like this:

$$\text{minimize } t \qquad \text{we minimize the number of colors we need}$$

$$\text{subject to } \forall e: \quad \sum_{i=1}^{k} x_{ei} = 1 \qquad \text{each edge is exactly one color}$$

$$\forall i\ \forall u\ \forall v, w \sim u, v \neq u: \quad x_{uv,i} + x_{uw,i} \leq 1$$
edges with the same vertex are a different color

$$\forall e\ \forall i: \quad x_{ei} \cdot i \leq t \qquad \text{we use less or equal to } t \text{ colors}$$

$$\forall i\ \forall uv\ \forall w \sim u, w \neq v\ \forall z \sim v, z \neq u, w: \quad x_{uw,i} + x_{vz,i} + y_{uv} \leq 2$$
uv is a rich edge $\Leftrightarrow$ all adjacent edges are a different color

$$\forall e: \quad \sum_{f \sim e} y_f \geq 1 \qquad \text{every edge is adjacent to some rich edge}$$

$$t \geq 2\Delta - 1 \qquad \text{we use} \geq 2\Delta - 1 \text{ colors}$$

$$\forall e: \quad 0 \leq y_e \leq 1,\ y_e \in \mathbb{Z}$$

$$\forall e\ \forall i: \quad 0 \leq x_{ei} \leq 1,\ x_{ei} \in \mathbb{Z},$$

where

$$x_{ei} = \begin{cases} 1, & \text{if edge } e \text{ has color } i \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad y_e = \begin{cases} 1, & \text{if edge } e \text{ is rich} \\ 0, & \text{otherwise.} \end{cases}$$

In our implementation of the ILP we fix the number of colors to $2\Delta - 1$, since the coloring can't be made with less colors (Every edge has $2\Delta - 1$ neighboring edges), and only look if the program has a solution that satisfies all the constraints. In theory this doesn't make the program faster, but in our practice tests it did make a significant differnece.

2.2. Iterative Algorithm

We also implemented an iterative algorithm that finds a rich-neighbor edge coloring for a given graph. The algorithm works by assigning a color to an edge and then checking if the coloring is valid. If it is, we move on to the next edge, otherwise we try a different color. If we can't find a valid coloring with $2\Delta - 1$ colors, we increase the number of colors by one and try again.

The algorithm is not guaranteed to find a coloring with $2\Delta - 1$ colors, but it is guaranteed to find a coloring with 2Δ colors.

Algorithm 1: richNeighbor

Input: Graph G

Output: Colors *colors*, Rich edges *richEdges*

$p \leftarrow$ MixedIntegerLinearProgram(maximization = False)
$x \leftarrow p.new\_variable(binary = True)$
$y \leftarrow p.new\_variable(binary = True)$
$t \leftarrow p.new\_variable(integer = True)$
$p.set\_objective(t[0])$
maxCol $\leftarrow 2 \cdot G.degree()[0] - 1$
$p.add\_constraint(t[0] \geq \text{maxCol})$
for *e in G.edges(labels = False)* **do**
    $p.add\_constraint(\sum_{i=1}^{\text{maxCol}} x[\text{Set}(e), i] = 1)$
for *(u, v) in G.edges(labels = False)* **do**
    $p.add\_constraint(\sum_{j \in G[u]} y[\text{Set}((u, j))] + \sum_{l \in G[v]} y[\text{Set}((l, v))] - 2y[\text{Set}((u, v))] \geq 1)$
for *e in G.edges(labels = False)* **do**
    for *i in 1 to maxCol* **do**
        $p.add\_constraint(i \cdot x[\text{Set}(e), i] \leq t[0])$
for *i in 1 to maxCol* **do**
    for *(u, v) in G.edges(labels = False)* **do**
        for *w in G[u]* **do**
            if *w = v* **then**
                continue
            $p.add\_constraint(x[\text{Set}((u, v)), i] + x[\text{Set}((u, w)), i] \leq 1)$
        for *z in G[v]* **do**
            if *z = u* **then**
                continue
            $p.add\_constraint(x[\text{Set}((u, v)), i] + x[\text{Set}((v, z)), i] \leq 1)$
for *(u, v) in G.edges(labels = False)* **do**
    for *w in G.neighbors(u)* **do**
        for *z in G.neighbors(v)* **do**
            if *w = v or z = u* **then**
                continue
            for *i in 1 to maxCol* **do**
                $p.add\_constraint(x[\text{Set}((u, w)), i] + x[\text{Set}((v, z)), i] + y[\text{Set}((u, v))] \leq 2)$
return *colors*, *richEdges*

The algorithm is very fast for small graphs, but it becomes too slow for graphs with more than 10 vertices. Ta tekst so smeti ma tle bi pc mogli mal blefirat da je iterativn algoritm pocasnejsi od ILPja.

3. Complete search

Since the number of k-regular graphs on n verices increases exponentially with n, we can't check all of them. We can however check all of them for small values of n.

3.1. Graph generation

In this part of the project we would like to thank professor Riste Škrekovski for providing us with access to multiple servers and supercomputers across Slovenia, where we ran our files.

4. Random Search

Along with testing the hypothesis for smaller graphs, we were also interested in checking if the conjecture holds for larger graphs. Since we can't even begin to check all for exmaple 20-regular graphs on 100 vertices, a random search algorytm seemed as a natural continuation of our problem. And since the we aren't looking for an optimum solution (anything more than the minimum $2\Delta - 1$ disproves the conjecture), we opted for a modification approach. We start with a random graph, check if the conjecture holds, and then modify it in a way that preserves the regularity and connectedness. We repeat this process indefinetly. Since .....dodaj razlog....., on every iteration, there is a small probability that we generate a completely new random graph and start again.

Algorithm 2: tweak

Input: Graph G
Output: Tweaked graph T

$T \leftarrow$ graph.copy()
$e_1 \leftarrow T$.random_edge()
$u_1, v_1, \text{extra}_1 \leftarrow e_1$
T.delete_edge(e_1)
$e_2 \leftarrow T$.random_edge()
$u_2, v_2, \text{extra}_2 \leftarrow e_2$
T.delete_edge(e_2)
$p \leftarrow$ random()
if $p < 0.5$ **then**
    T.add_edge(u_1, u_2)
    T.add_edge(v_1, v_2)
else
    T.add_edge(u_1, v_2)
    T.add_edge(v_1, u_2)
if *not* T.*is_connected*() **then**
    return tweakGraph(graph)
return T

4.1. Graph generation

Since we only need one graph to start our iterative process, we can generate it randomly using the built in sage function `graphs.RandomRegular`.