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Rich-Neighbor Edge Colorings

Term Paper in Finance Lab

Long Presentation

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1. Introduction

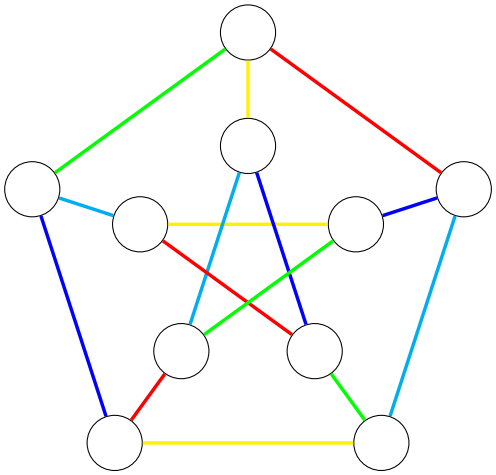
In this paper we set out to analyse an open conjecture in a modern graph theory problem known as rich-neighbor edge coloring.

Definition 1.1. In an edge coloring, an edge e is called *rich* if all edges adjacent to e have different colors. An edge coloring is called a *rich-neighbor edge coloring* if every edge is adjacent to some rich edge.

Definition 1.2. $X'_{rn}(G)$ denotes the smallest number of colors for which there exists a rich-neighbor edge coloring.

Conjecture 1.3. *For every graph G of maximum degree Δ, $X'_{rn}(G) \leq 2\Delta - 1$ holds.*

Example 1.4. Let's take a look at the Petersen graph and an example of a rich-neighbor edge coloring.



We can see that for the Petersen graph (which is 3-regular) we can find an appropriate coloring with 5 colors so $X'_{rn} \leq 5 \leq 2 \cdot 3 - 1 = 5$. This shows that the conjecture holds for this graph. ♢

2. Algorithms

2.1. Integer Programming

Using SageMath we constructed an integer programming model that finds a rich-neighbor edge coloring for a given graph using the smallest number of colors possible. Our interger program looks like this:

$$\text{minimize } t \qquad \text{we minimize the number of colors we need}$$

$$\text{subject to } \forall e: \quad \sum_{i=1}^{2\Delta-1} x_{ei} = 1 \qquad \text{each edge is exactly one color}$$

$$\forall i\ \forall u\ \forall v, w \sim u, v \neq u: \quad x_{uv,i} + x_{uw,i} \leq 1$$
edges with the same vertex are a different color

$$\forall e\ \forall i: \quad x_{ei} \cdot i \leq t \qquad \text{we use less or equal to } t \text{ colors}$$

$$\forall i\ \forall uv\ \forall w \sim u, w \neq v\ \forall z \sim v, z \neq u, w: \quad x_{uw,i} + x_{vz,i} + y_{uv} \leq 2$$
uv is a rich edge $\Leftrightarrow$ all adjacent edges are a different color

$$\forall e: \quad \sum_{f \sim e} y_f \geq 1 \qquad \text{every edge is adjacent to some rich edge}$$

$$\forall e: \quad 0 \leq y_e \leq 1,\ y_e \in \mathbb{Z}$$

$$\forall e\ \forall i: \quad 0 \leq x_{ei} \leq 1,\ x_{ei} \in \mathbb{Z},$$

where

$$x_{ei} = \begin{cases} 1, & \text{if edge } e \text{ has color } i \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad y_e = \begin{cases} 1, & \text{if edge } e \text{ is rich} \\ 0, & \text{otherwise.} \end{cases}$$

In our implementation of the ILP we fix the number of colors to $2\Delta - 1$ by adding the following constraint

$$t = 2\Delta - 1,$$

since the coloring can't be made with less colors (every edge has $2\Delta - 2$ neighboring edges and the edge itself has to be a different color) and any more colors do not satisfie the conjecture. Therefore we only check if the program has a solution that satisfies all the constraints and the conjecture. In theory this doesn't make the program faster, but in our practice tests it did make a significant differnece.

Our implementation of the IPL is demonstrated in the following code.

Algorithm 1: richNeighbor

Input: Graph G
Output: Colors *colors*, Rich edges *richEdges*

$p \leftarrow$ MixedIntegerLinearProgram(maximization = False)
$x \leftarrow p.new\_variable(binary = True)$
$y \leftarrow p.new\_variable(binary = True)$
$t \leftarrow p.new\_variable(integer = True)$
$p.set\_objective(t[0])$
$\text{maxCol} \leftarrow 2 \cdot G.degree()[0] - 1$
$p.add\_constraint(t[0] = \text{maxCol})$
for *e in G.edges(labels = False)* **do**
  $p.add\_constraint(\sum_{i=1}^{\text{maxCol}} x[\text{Set}(e), i] = 1)$
for (u, v) *in G.edges(labels = False)* **do**
  $p.add\_constraint(\sum_{j \in G[u]} y[\text{Set}((u, j))] + \sum_{l \in G[v]} y[\text{Set}((l, v))] - 2y[\text{Set}((u, v))] \geq 1)$
for *e in G.edges(labels = False)* **do**
  for *i in 1 to maxCol* **do**
    $p.add\_constraint(i \cdot x[\text{Set}(e), i] \leq t[0])$
for *i in 1 to maxCol* **do**
  for (u, v) *in G.edges(labels = False)* **do**
    for *w in G[u]* **do**
      if $w = v$ **then**
        continue
      $p.add\_constraint(x[\text{Set}((u, v)), i] + x[\text{Set}((u, w)), i] \leq 1)$
    for *z in G[v]* **do**
      if $z = u$ **then**
        continue
      $p.add\_constraint(x[\text{Set}((u, v)), i] + x[\text{Set}((v, z)), i] \leq 1)$
for (u, v) *in G.edges(labels = False)* **do**
  for *w in G.neighbors(u)* **do**
    for *z in G.neighbors(v)* **do**
      if $w = v$ *or* $z = u$ **then**
        continue
      for *i in 1 to maxCol* **do**
        $p.add\_constraint(x[\text{Set}((u, w)), i] + x[\text{Set}((v, z)), i] + y[\text{Set}((u, v))] \leq 2)$
return *colors*, *richEdges*

2.2. Iterative Algorithm

We also implemented an iterative algorithm that finds a rich-neighbor edge coloring for a given graph. The algorithm works by assigning a color to an edge and then checking if the coloring is valid. If it is, we move on to the next edge, otherwise we try a different color. If we can't find a valid coloring with $2\Delta - 1$ colors, we increase the number of colors by one and try again.

The algorithm is not guaranteed to find a coloring with $2\Delta - 1$ colors, but it is guaranteed to find a coloring with 2Δ colors.

The algorithm is very fast for footnotesize graphs, but it becomes too slow for graphs with more than 10 vertices. Ta tekst so smeti ma tle bi pc mogli mal blefirat da je iterativn algoritm

pocasnejsi od ILPja.

3. Complete search

As illustrated in Table 1, the number of k-regular graphs on n vertices is of exponential growth. This poses a computational challenge, as exhaustively examining all such graphs becomes impractical for larger values of n. The sheer scale of potential combinations makes a complete search unfeasible within reasonable timeframes.

However, a thorough exploration of all k-regular graphs remains viable and efficient for footnotesize values of n.

Vertices	Degree 4	Degree 5	Degree 6	Degree 7
5	1	0	0	0
6	1	1	0	0
7	2	0	1	0
8	6	3	1	1
9	16	0	4	0
10	59	60	21	5
11	265	0	266	0
12	1544	7848	7849	1547
13	10778	0	367860	0
14	88168	3459383	21609300	21609301
15	805491	0	1470293675	0
16	8037418	2585136675	113314233808	733351105934

Table 1. Number of k-regular graphs on n vertices

3.1. Graph generation

Graphs for the complete search were taken from a collection of `.tex` files, provided by Professor Skrekovski. The files contain all k-regular graphs on n vertices up to 15 vertices.

4. Random Search

In addition to examining the hypothesis for smaller graphs, we were also interested in checking if the conjecture holds for larger graphs. The challenge is, as said before, the pure impracticality of systematically testing all potential scenarios of colorings in graphs with many vertices.

Recognizing the enormity of this task, a random search algoritm seemed a natural continuation of our problem. Here we opted for a modification approach.

We start with a random graph, check if the conjecture holds and then modify it in a way that preserves the regularity and connectedness. We repeat this process indefinetly. Since, if a rich-neighbor edge coloring exists in a given graph there is a bigger probability that it also exists in similar graphs, therefore, on every iteration, there is a footnotesize probability that we generate a completely new random graph and start again.

Written below is the algorithm for modifying graphs. First it selects two random edges and deletes them. Then it assigns a random probability to p and based on the value of p it adds back to different edges we did not have before. It repeats this until it gets a connected modified graph.

Algorithm 2: tweak

Input: Graph G
Output: Tweaked graph T
$T \leftarrow$ graph.copy()
$e_1 \leftarrow T$.random_edge()
$u_1, v_1, \text{extra}_1 \leftarrow e_1$
T.delete_edge(e_1)
$e_2 \leftarrow T$.random_edge()
$u_2, v_2, \text{extra}_2 \leftarrow e_2$
T.delete_edge(e_2)
$p \leftarrow$ random()
if $p < 0.5$ **then**
    T.add_edge(u_1, u_2)
    T.add_edge(v_1, v_2)
else
    T.add_edge(u_1, v_2)
    T.add_edge(v_1, u_2)
if *not* T.*is_connected*() **then**
    return tweakGraph(graph)
return T

4.1. Graph generation

Since we only need one graph to start our iterative process, we can generate it randomly using the built in sage function `graphs.RandomRegular`.

5. Checking the coloring

dodaj še opisa algoritmov

Algorithm 3: Check Coloring

Input: Graph G, Coloring *coloring*
Output: Boolean indicating proper coloring
for v *in* G.*vertices*() **do**
    col $\leftarrow$ set()
    for w *in* G.*neighbors*(v) **do**
        for i *in range*$(1, 2 \cdot G.degree()[0])$ **do**
            if *coloring*$[(Set((v, w)), i)] = 1$ **then**
                col.add(i)
    if *len*(*col*) $\neq$ *len*(G.*neighbors*(v)) **then**
        return False
return True

Algorithm 4: Check Richness

Input: Graph G, Rich edges $richEdges$
Output: Boolean indicating proper rich-neighbor edge coloring

```
for (u, v) in G.edges(labels = False) do
    S ← 0
    for w in G.neighbors(u) do
        if w = v then
            S ← S + richEdges[Set((u, w))]
    for z in G.neighbors(v) do
        if z = u then
            S ← S + richEdges[Set((v, z))]
    if S = 0 then
        return False
return True
```

6. Findings

7. Conclusion