UNIVERSITY OF LJUBLJANA
FACULTY OF MATHEMATICS AND PHYSICS

Financial mathematics – 1st cycle

Anej Rozman, Tanja Luštrek

Rich-Neighbor Edge Colorings

Term Paper in Finance Lab

Long Presentation

Advisers: Assistant Professor Janoš Vidali,
Professor Riste Škrekovski

Ljubljana, 2023

Contents

1. Introduction

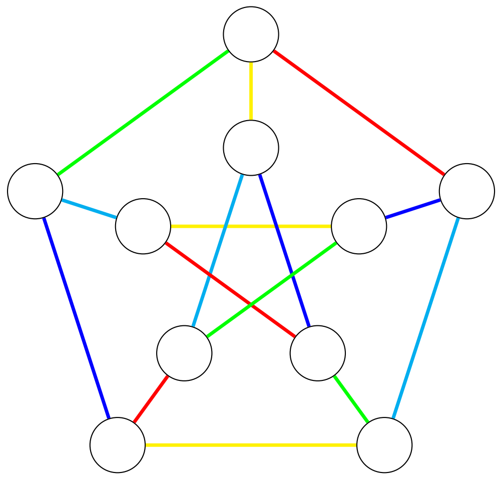
In this paper we set out to analyse an open conjecture in a modern graph theory problem known as rich-neighbor edge coloring.

Definition 1.1. In an edge coloring, an edge e is called *rich* if all edges adjacent to e have different colors. An edge coloring is called a *rich-neighbor edge coloring* if every edge is adjacent to some rich edge.

Definition 1.2. $X'_{rn}(G)$ denotes the smallest number of colors for which there exists a rich-neighbor edge coloring.

Conjecture 1.3. *For every graph G of maximum degree Δ, $X'_{rn}(G) \leq 2\Delta - 1$ holds.*

Example 1.4. Let's take a look at the Petersen graph and an example of a rich-neighbor edge coloring.



We can see that for the Petersen graph (which is 3-regular) we can find an appropriate coloring with 5 colors so $X'_{rn} \leq 5 \leq 2 \cdot 3 - 1 = 5$. This shows that the conjecture holds for this graph. $\diamondsuit$

2. Generating Graphs

3. Algorithms

3.1. Integer Programming

Using SageMath we constructed an integer programming model that finds a rich-neighbor edge coloring for a given graph. Our interger program looks like this:

minimize t — we minimize the number of colors we need

subject to $\forall e: \quad \sum_{i=1}^{k} x_{ei} = 1$ — each edge is exactly one color

$\forall i \; \forall u \; \forall v, w \sim u, v \neq u: \quad x_{uv,i} + x_{uw,i} \leq 1$ — edges with the same vertex are a different color

$\forall e \; \forall i: \quad x_{ei} \cdot i \leq t$ — we use less or equal to t colors

$\forall i \; \forall uv \; \forall w \sim u, w \neq v \; \forall z \sim v, z \neq u, w: \quad x_{uw,i} + x_{vz,i} + y_{uv} \leq 2$ — uv is a rich edge $\Leftrightarrow$ all adjacent edges are a different color

$\forall e: \quad \sum_{f \sim e} y_f \geq 1$ — every edge is adjacent to some rich edge

$t \geq 2\Delta - 1$ — we use $\geq 2\Delta - 1$ colors

$\forall e: \quad 0 \leq y_e \leq 1, \, y_e \in \mathbb{Z}$

$\forall e \; \forall i: \quad 0 \leq x_{ei} \leq 1, \, x_{ei} \in \mathbb{Z},$

where

$$x_{ei} = \begin{cases} 1, & \text{if edge } e \text{ has color } i \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad y_e = \begin{cases} 1, & \text{if edge } e \text{ is rich} \\ 0, & \text{otherwise.} \end{cases}$$

We only looked for appropriate graph colorings with $\geq 2\Delta - 1$ colors because, if we find a coloring with $2\Delta - 1$ colors then the smallest possible number of colors also satisfies the conjecture for that graph. If we need more than $2\Delta - 1$ colors then the conjecture does not hold. The computation is also a lot quicker this way.

3.2. Iterative Algorithm