CENG 223

Discrete Computational Structures

Fall 2022-2023

Take Home Exam 2

Due date: November 20, 23:55

Question 1 (20 pts)

For each of the following functions, show whether they are **a)** surjective or not **b)** injective or not.

- $f_1 : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$
- $f_2 : \bar{\mathbb{R}}^+ \to \mathbb{R}, \quad f(x) = x^2$
- $f_3 : \mathbb{R} \to \bar{\mathbb{R}}^+, \quad f(x) = x^2$
- $f_4 : \bar{\mathbb{R}}^+ \to \bar{\mathbb{R}}^+, \quad f(x) = x^2$

Note: $\bar{\mathbb{R}}^+$ denotes the set of nonnegative real numbers.

Question 2 (20 pts)

A function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be **continuous at** $x_0 \in A$ if

$$\forall \varepsilon \in \mathbb{R}^+ \, \exists \delta \in \mathbb{R}^+ \, \forall x \in A \;\; (\|x - x_0\| < \delta \to \|f(x) - f(x_0)\| < \varepsilon)$$

where $\|x\|$ represents the Euclidean norm, *i.e.* for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the norm is given as $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$. If f is continuous at every $x \in A$, f is **continuous**. Use this definition to

a) show that every function $f : A \subset \mathbb{Z} \to \mathbb{R}$ is continuous.

b) show that a necessary and sufficient condition for a function $f : \mathbb{R} \to \mathbb{Z}$ to be continuous is that f is a constant function.

Question 3 (20 pts)

a) Show that a finite Cartesian product of countable sets, *i.e.* $X_n = A_1 \times A_2 \times \ldots \times A_n$ for all $n \geq 2$, is countable.

b) Show that an infinite countable product of the set $X = \{0, 1\}$ with itself is uncountable.

Note: You can use the following without proving them: *i)* the set of positive integers $\mathbb{Z}^+$, $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ have the same cardinality, *ii)* a set A is countable if and only if there exists some $f : \mathbb{Z} \to A$ that is surjective.

Question 4 (25 pts)

Arrange the following functions so that each function is big-O of the next function. Show your work.

$$2^n, \quad n^{50}, \quad (\log n)^2, \quad \sqrt{n}\log n, \quad 5^n, \quad (n!)^2, \quad n^{51} + n^{49}$$

Note: You can use calculus in this question.

Question 5 (15 pts)

a) Use the Euclidean algorithm to find $\mathbf{gcd}(94, 134)$.

b) Goldbach's conjecture states that every even integer greater than 2 is the sum of two primes. Show that this statement is equivalent to the statement that every integer greater than 5 is the sum of three primes.

Question 6 (ungraded)

Let $f : A \to B$ be a function, $A_0 \subset A$, $B_0 \subset B$ and f^{-1} denote the preimage of B_0 under f defined by

$$f^{-1}(B_0) = \{a \mid f(a) \in B_0\}$$

a) Show that $A_0 \subseteq f^{-1}(f(A_0))$ and that equality holds if f is injective.

b) Show that $f(f^{-1}(B_0)) \subseteq B_0$ and that equality holds if f is surjective.

Question 7 (ungraded)

a) A real number is said to be **algebraic** over the rationals if it satisfies some polynomial equation of positive degree

$$x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$$

where $a_i \in \mathbb{Q}$. Assuming that each polynomial has only finitely many roots, show that the set of algebraic numbers is countable.

b) A real number is said to be **transcendental** if it is not algebraic. Show that the set of transcendental numbers are uncountable.

Regulations

1. Your submission should be a single vector-based PDF document with the name `the2.pdf`.
2. **Late Submission:** Not allowed.
3. **Cheating: We have zero tolerance policy for cheating**. People involved in cheating will be punished according to the university regulations.
4. **Updates & Announces:** You must follow the odtuclass for discussions and possible updates. You can ask your questions in the Student Forum on the course page in odtuclass.
5. **Evaluation:** Your `.pdf` file will be checked for plagiarism automatically using "black-box" technique and manually by assistants, so make sure to obey the specifications.

Submission

Submission will be done via odtuclass. For those who prefer to use LaTeX to generate the vector-based PDF file, a template answer file `the2.tex` is provided in odtuclass. You need to compile the filled template yourselves and submit the generated `.pdf` file only.