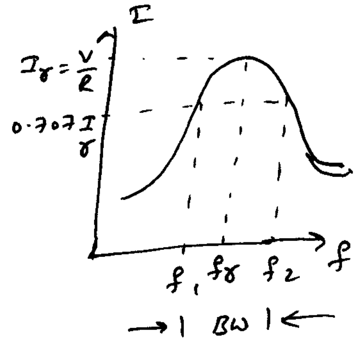
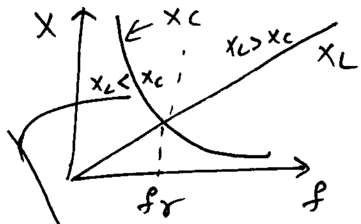
$f_r = \sqrt{f_1 f_2}$ proof :-



at half power frequencies, the circuit current becomes $\frac{1}{\sqrt{2}} I_r$.

circuit impedance at resonance $= R$.



$\therefore$ circuit impedance at half power frequencies is

$\sqrt{2} R \quad (\because Z = \frac{V}{I})$

$$\Rightarrow \sqrt{2} R = \sqrt{R^2 + X^2} \Rightarrow R = X \text{ at } f_1 \,\&\, f_2.$$

at f_1 :-

$$X_1 = \omega_1 L - \frac{1}{\omega_1 C} = -R \quad \because X_L < X_C \text{ at } f_1$$

$$\omega_1 L + R - \frac{1}{\omega_1 C} = 0 \Rightarrow \omega_1^2 + \frac{R}{L}\omega_1 - \frac{1}{LC} = 0$$

$$\Rightarrow \omega_1 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = -\alpha \pm \sqrt{\alpha^2 + \omega_r^2} \rightarrow (1)$$

where $\alpha = \frac{R}{2L}$ & $\omega_r = \frac{1}{\sqrt{LC}}$

as -ve freq is not possible

$$(1) \Rightarrow -\alpha + \sqrt{\alpha^2 + \omega_r^2} \rightarrow (2)$$

at f_2 :-

$$X_2 = \omega_2 L - \frac{1}{\omega_2 C} = +R \Rightarrow \omega_2 L - R - \frac{1}{\omega_2 C} = 0$$

$$\Rightarrow \omega_2^2 - \frac{R}{L}\omega_2 - \frac{1}{LC} = 0 \Rightarrow \omega_2 = \alpha + \sqrt{\alpha^2 + \omega_r^2} \rightarrow (3)$$

$$(2) \times (3) \Rightarrow \omega_1 \omega_2 = \alpha^2 + \omega_r^2 - \alpha^2 \Rightarrow \omega_1 \omega_2 = \omega_r^2$$

$$\Rightarrow (2\pi f_1)(2\pi f_2) = (2\pi f_r)^2 \Rightarrow \boxed{f_r = \sqrt{f_1 f_2}}$$