Purdue University

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Formula Sheet for Midterm Exam

1. Introduction

- Present Value Formula:

$$P_t = \mathsf{E}\left[\frac{\tilde{D}_{t+1}}{1+r} + \frac{\tilde{D}_{t+2}}{(1+r)^2} + \frac{\tilde{D}_{t+3}}{(1+r)^3} + \cdots\right]$$

- Arithmetic Average:

$$\bar{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$$

- Geometric Average:

$$\bar{r}_{geo} = [(1+r_1)(1+r_2)\dots(1+r_T)]^{\frac{1}{T}} - 1$$

- Real vs. Nominal Returns (π is the rate of inflation):

$$\begin{aligned} 1 + r_{\text{real}} &= \frac{1 + r_{\text{nominal}}}{1+\pi} \\ r_{\text{real}} &\approx r_{\text{nominal}} - \pi \end{aligned}$$

2. Fixed Income

- Bond Prices

$$P = \frac{B_1}{\$100}C + \frac{B_2}{\$100}C + \cdots + \frac{B_T}{\$100}(FV + C)$$

- If rates are compounded m times a year, the present value formula changes to

$$\text{PV} = \frac{FV}{1+r} \quad \text{to} \quad \text{PV} = \frac{FV}{\left(1+\frac{r}{m}\right)^m}$$

- Zero Bond Prices and Spot Rates

$$P = \frac{FV}{(1+r_t)^t} \quad \Rightarrow \quad r_t = \left(\frac{FV}{P}\right)^{\frac{1}{t}} - 1$$

- Duration

$$\text{Duration} = \frac{\text{PV}(\text{CF}_1)}{P}\times 1 + \frac{\text{PV}(\text{CF}_2)}{P}\times 2 + \cdots + \frac{\text{PV}(\text{CF}_T)}{P}\times T$$

- Approximate Bond Price Change (using duration)

$$\Delta P \approx -\frac{1}{1+y} \times \text{Duration} \times P \times \Delta y$$

- Immunization using Duration

$$\Delta P_{old} + \Delta P_{new} = 0 \Leftrightarrow -\frac{1}{1+y} \times \text{Duration}_{old} \times P_{old} \times \Delta y - \frac{1}{1+y} \times \text{Duration}_{new} \times P_{new} \times \Delta y = 0$$

3. Modern Portfolio Theory

- Unless stated otherwise, we always assume $\sum_{i=1}^{N} w_i = 1$.

- Portfolio expected return:

$$\mathsf{E}[\tilde{r}_p] = w_1 \mathsf{E}[\tilde{r}_1] + w_2 \mathsf{E}[\tilde{r}_2] + ... + \mathsf{E}[\tilde{r}_N]$$

- Portfolio variance:

$$\mathsf{var}(\tilde{r}_p) = \sum_{i=1}^{N} \sum_{j=1}^{N} = w_i w_j \sigma_{i,j} = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{1 \leq i < j \leq N} w_i w_j \sigma_{i,j}$$

- Minimum Variance Portfolio (two assets):

$$w_1 = \frac{\sigma_2^2 - \sigma_{1,2}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{1,2}}$$

- Tangency portfolio ($N = 2$):

$$w_1^* = \frac{(\mathsf{E}(\tilde{r}_1) - r_f)\sigma_2^2 - (\mathsf{E}(\tilde{r}_2) - r_f)\sigma_{1,2}}{(\mathsf{E}(\tilde{r}_1) - r_f)\sigma_2^2 + (\mathsf{E}(\tilde{r}_2) - r_f)\sigma_1^2 - (\mathsf{E}(\tilde{r}_1) + \mathsf{E}(\tilde{r}_2) - 2r_f)\sigma_{1,2}}$$

- Optimal portfolio for quadratic utility investor (two asset case): $U(\mu, \sigma) = \mu - \gamma\sigma^2 \Rightarrow w^* = \frac{\mu - r_f}{2\gamma\sigma^2}$.

4. CAPM

- The tangency portfolio T is the market portfolio, with weights given by the market capitalization of each asset.
- The only thing that matters for equilibrium returns is market risk, measured by beta

$$\beta_i = \frac{\mathsf{cov}(\tilde{r}_i, \tilde{r}_m)}{\sigma_m^2}$$

- The relationship between expected returns and beta is the Security Market Line (SML)

$$\mathsf{E}[\tilde{r}_i] = r_f + \beta_i[\mathsf{E}(\tilde{r}_m) - r_f]$$

- Beta can be estimated as the regression coefficient in the CAPM regression

$$\tilde{r}_{i,t} - r_f = \alpha_i + \beta_i(\tilde{r}_{m,t} - r_f) + \tilde{\varepsilon}_{i,t}$$

- The total risk (variance) of asset i can be decomposed

$$\underbrace{\mathsf{var}(\tilde{r}_i)}_{\text{total risk}} = \underbrace{\beta_i^2 \mathsf{var}(\tilde{r}_m)}_{\text{systematic risk}} + \underbrace{\mathsf{var}(\tilde{\varepsilon}_i)}_{\text{idiosyncratic risk}}$$

- The R-squared of the regression is the ratio of systematic risk to the total risk

$$R_i^2 = \frac{\text{systematic risk}}{\text{total risk}} = \frac{\beta_i^2 \mathsf{var}(\tilde{r}_m)}{\mathsf{var}(\tilde{r}_i)}$$

5. Performance Measures

- Sharpe ratio:

$$SR_{\text{fund}} = \frac{\bar{r}_{\text{fund}} - r_f}{\sigma_{\text{fund}}}$$

- Treynor's measure:

$$T_{\text{fund}} = \frac{\bar{r}_{\text{fund}} - r_f}{\beta_{\text{fund}}}$$

- Jensen's alpha (CAPM):

$$\hat{\alpha} = \bar{r}_{\text{fund}} - \Big(r_f + \hat{\beta}^{\text{mkt}}(\bar{r}_{\text{mkt}} - r_f)\Big)$$

6. Statistics

- Expectation ($\tilde{r}$ discrete):

$$\mathsf{E}[\tilde{r}] = \sum_{s=1}^{S} P(\tilde{r}_s = r_s) \times r_s$$

- Variance:

$$\mathsf{var}(\tilde{r}) = \sigma^2(\tilde{r}) = \mathsf{E}[(\tilde{r} - \mathsf{E}(\tilde{r}))^2]$$

- Standard deviation:

$$\sigma(\tilde{r}) = \sqrt{\sigma^2(\tilde{r})}$$

- Covariance:

$$\mathsf{cov}(\tilde{r}_1, \tilde{r}_2) = \mathsf{E}[(\tilde{r}_1 - \mathsf{E}(\tilde{r}_1))(\tilde{r}_2 - \mathsf{E}(\tilde{r}_2))]$$

- Correlation:

$$\rho_{1,2} = \frac{\mathsf{cov}(\tilde{r}_1, \tilde{r}_2)}{\sigma_1 \sigma_2}$$

- Univariate Regression:
  By regressing the *dependent* variable Y on the *independent* (or *explanatory*) variable X, one gets the regression line:

  $$Y_t = \alpha + \beta X_t + \varepsilon_t$$

  where α is the intercept, β is the slope, and ε_t is the error term. One typically assumes $\mathsf{E}(\varepsilon_t) = 0$, and $\mathsf{cov}(X_t, \varepsilon_t) = 0$. The slope β is given by $\beta = \mathsf{cov}(X, Y)/\mathsf{var}(X)$. The variance of Y decomposes as $\mathsf{var}(Y) = \beta^2 \mathsf{var}(X) + \mathsf{var}(\varepsilon)$. The goodness of fit of the regression is measured by $R^2 = \beta^2 \mathsf{var}(X)/\mathsf{var}(Y)$.