Uncertain Stock, Resource Extraction, and Externalities

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Outlines

- Introduction
- Consumption/extraction of exhaustible resources review
- Proposed model with two state variables
- Necessary conditions and existence of unique solution
- Observations
- Discussion
- Future work

Introduction

- Many authors (Koopman, 1973, 1974; Kemp 1976; Kumar 2005) have studied the consumption of exhaustible resources using the cup cake concept.
- Develop a model of optimal nonrenewable resource extraction when there is negative externality.
- Compare and make some statements about difference of optimal time horizon between single control variable and two state variable cases (resource stock and pollution).
- Study optimal depletion of the resources under uncertainty and compare it against a certainty case.

Consumption of exhaustible resources review

- A set up with a single state variable:
  $Q(t)$ be the exhaustible resources, $c(t)$ be it's extraction rate, $u(c)$ be utility function, T_1 be the terminal time for depletion, ρ is a discount rate of utility function

$$\max \int_0^{T_1} u(c)e^{-\rho t} dt$$

$$\dot{Q} = -c \qquad Q(0) > 0, \qquad c > 0 \tag{1}$$

- Koopman: if consumption(extraction) is more than some subsistence level, then the time horizon T_1 is finite
- Kemp: derived his result of non monotony of optimal extraction rate by assuming the infinite time horizon
- Vousden(1973): noted that if $u(c) > k$ for some lower bound k, exhaustion time is finite even if the planning horizon is infinite
- Epstein (1983) and Kumar (2005)extended couple of results regarding time horizon to the cases about uncertainty of resource stocks

Proposed model with two state variables

- We consider an exhaustible resource Q that results in negative externality represented by a stock variable Z that accumulates at rate αc and decays at rate kZ. The problem set up of maximization of utility function $u(c)$ with negatively affected stock variables is:

$$V = \max \int_0^{T_2} (u(c) - v(Z))e^{-\rho t} dt$$

$$\begin{aligned} \dot{Q} &= -c \qquad Q(0) > 0, \qquad c > 0 \\ \dot{Z} &= \alpha c - kZ \qquad Z(0) = 0, \qquad k > 0,\ 0 < \alpha < 1 \end{aligned} \tag{2}$$

  where $v(Z)$ is a convex function of Z, and $k > 0$ is natural decay rate of the pollution stock.

- To determine T_2
- To advance some propositions regarding the sign of $T_1 - T_2$
- Optimal depletion when resource stock is unknown

Necessary conditions

- Hamiltonian present value

$$\hat{H} = (u(c) - v(Z))e^{-\rho t} - \hat{\lambda} c + \hat{\mu}(\alpha c - kZ)$$

- Hamiltonian current value

$$H = \hat{H} e^{\rho t}, \quad \lambda = \hat{\lambda} e^{\rho t}, \quad \mu = \hat{\mu} e^{\rho t}$$

$$H = (u(c) - v(Z)) - \lambda c + \mu(\alpha c - kZ)$$

- Necessary conditions from Pontryagin Maximum Principle

$$u_c(c) - \lambda + \alpha\mu \leq 0 \quad \text{with equality if} \quad c > 0 \tag{3}$$

$$\dot{\lambda} = \rho\lambda \tag{4}$$

$$\dot{\mu} = (\rho + k)\mu + v_z Z \tag{5}$$

$$\lim_{t \to \infty} e^{-\rho t} \lambda(t) Q(t) = 0 \tag{6}$$

$$\lim_{t \to \infty} e^{-\rho t} \mu(t) Z(t) = 0 \tag{7}$$

$$u_{cc}(c) < 0 \quad \text{and} \quad v_{zz}(Z) \geq 0$$

Existence of optimal solution

Let $x = (Q, Z)$ be states, $c \in U$ is a control, and (x, c) be admissible solution of the problem (2) We use the Mangasarian sufficiency theorem* for existence and uniqueness of the optimal solution (x^*, c^*).
*[Atle Seierstad and Knut Sydsaster]

Theorem

If all necessary conditions (3-7) are satisfied and since $u_{cc} < 0$ *along with* $v_{zz} \geq 0$*, then there exists unique solution* (x^*, c^*) *of the problem* (2).

Observation for T_2

- Differentiating eq(3) with respect to t where $c > 0$

$$\dot{c} = -\left(\frac{u_c}{u_{cc}}\right)\left[-\rho + \left(\frac{\alpha(k\mu + v_z)}{u_c}\right)\right] \tag{8}$$

  $\Rightarrow c \uparrow$ if $\alpha(k\mu + v_z)/\rho > u_c$ and $c \downarrow$ if $\alpha(k\mu + v_z)/\rho < u_c$
  $\Rightarrow$ extraction rate increases if marginal utility is less than the discounted value of net damage from pollution due to the marginal consumption. Similarly other way for decreasing.
- Lemma 1:The resource Q eventually exhausts, i.e. $Q(T) = 0$ for terminal time T, only if

$$\int_0^T \left(\frac{v_z(\tau)}{k} - \alpha\mu\right)\frac{dc(\tau)}{dQ(T)}e^{-\rho\tau}d\tau \le \lambda_0 \text{ where } \int_0^T \frac{dc(\tau)}{dQ(T)}d\tau = -1$$

  $--\ >$We get above condition starting from $\frac{dV}{d(Q(T))}$; $V(.)$ is maximized at $Q(T) = 0$
- Lemma 2: if $\lim_{c\to 0} u_c(c) \to \infty$(finite), and the condition for $Q(T) = 0$ is satisfied, then terminal time T is infinite (finite).

Observation for T_2

- Exhaustion theorem for two state variable case:

Theorem

The resource is not exhausted in finite time if the following conditions are satisfied:

$$\int_0^T \left(\frac{v_z(\tau)}{k}\right) \frac{dc(\tau)}{dQ(T)} e^{-\rho\tau} d\tau \leq \lambda_0 - \alpha\hat{\mu} \text{ where } \int_0^T \frac{dc(\tau)}{dQ(T)} d\tau = -1$$

and

$$\lim_{c \to 0} u_c(c) \to \infty$$

If $\lim_{c \to 0} u_c(c) < \infty$, *the resources exhausted in finite time.*

- Corollary: If $\lim_{c \to 0} u_c(c) \to \infty$. If $v(.)$ is constant and if the shadow value of pollution $\hat{\mu} < \infty$, then the resource is exhausted in finite time.If $v(Z)$ is linear, the resource is exhausted if slope of $v(Z)$ is less than $k(\lambda_0 - \hat{\mu})$.

Observation for sign of $(T_1 - T_2)$

- We are comparing the sign of following two cases with terminal time T_1 and T_2
  - $$V_1 = \max \int_0^{T_1} u(c)e^{-\rho t}dt$$
    $$\dot{Q} = -c \qquad Q(0) > 0, \qquad c > 0 \tag{9}$$
  - $$V_2 = \max \int_0^{T_2} (u(c) - v(Z))e^{-\rho t}dt$$
    $$\begin{aligned} &\dot{Q} = -c \qquad Q(0) > 0, \qquad c > 0 \\ &\dot{Z} = \alpha c - kZ \qquad Z(0) > 0, \qquad k > 0 \end{aligned} \tag{10}$$
- From current Hamiltonian, if $c > 0$
  - $u_c - \lambda_1 = 0$
  - $u_c - \lambda + \alpha\mu = 0$
- Define $W = (u_c)^{-1}$ in a neighbourhood of λ_1 such that $\lambda - \alpha\mu \in B(\lambda_1)$, $\alpha\mu = \mu'$

Observation for sign of $(T_1 - T_2)$

We make statement of sign of $(T_1 - T_2)$ with the help of following expression: $T_1 - T_2 < 0$ will require

$$\int_0^{T_1} (W(\lambda_1) - W(\lambda - \mu'))dt < 0$$

Theorem

sign $(T_1 - T_2) = sign((\mu' + \lambda_1 - \lambda)W_{\lambda_1})$

Since $W_{\lambda_1} < 0$,

Corollary

If $sign(\mu' + \lambda_1 - \lambda) \geq 0$, *for all* t, *then* $T_1 \leq T_2$. *And, If* $sign(\mu' + \lambda_1 - \lambda) < 0$, *for all* t, *then* $T_1 > T_2$.

Observation for uncertainty of resources

$Q(t)$ is the cumulative source extraction(=consumption)

- Say $0 \leq \pi(Q(t)) \leq 1$ be the probability distribution of exhaustible resources stock before extraction
- Maximizing the expected value of utility function with control of uncertainty

$$V_3 = E\left(\int_0^T (u(c) - v(Z))e^{-\rho t} dt\right)$$

$$V_3 = \max\left(\int_0^T (u(c) - v(Z))\pi(Q(t)e^{-\rho t} dt\right)$$

$$\begin{aligned} \dot{Q} &= -c \qquad Q(0) > 0, \qquad c > 0 \\ \dot{Z} &= \alpha c - kZ \qquad Z(0) > 0, \qquad k > 0,\ 0 < \alpha < 1 \end{aligned} \tag{11}$$

- Hamiltonian present value with resources uncertainty

$$H_1 = (u(c) - v(Z))\pi(Q(t))e^{-\rho t} - \lambda_1 c + \mu_1(\alpha c - kZ) \tag{12}$$

- Hamiltonian present value with resources certainty

$$\hat{H} = (u(c) - v(Z))e^{-\rho t} - \hat{\lambda} c + \hat{\mu}(\alpha c - kZ) \tag{13}$$

Observation for uncertainty of resources

We compare the standard means $(\frac{\dot{c}}{c})$ of optimal depletion of exhaustible resources between uncertainty and certainty of resources. $u_{cc}(c) < 0$

- From (12):uncertainty

$$\frac{\dot{c}}{c} = \frac{u_c(c)}{u_{cc}(c)}\frac{\rho}{c} - \frac{\alpha v_z(Z)}{cu_{cc}(c)} - \frac{\alpha \mu_1 k e^{\rho t}}{cu_{cc}(c)\pi(Q(t))} + \frac{u_c(c)}{u_{cc}(c)}\frac{\pi_Q(Q(t))}{\pi(Q(t))} - \frac{(u(c) - v(Z))\pi_Q(Q(t))}{cu_{cc}(c)\pi(Q(t))}$$

- From (13):certainty

$$\frac{\dot{c}}{c} = \frac{u_c(c)}{u_{cc}(c)}\frac{\rho}{c} - \frac{\alpha v_z(Z)}{cu_{cc}(c)} - \frac{\alpha \hat{\mu} k e^{\rho t}}{cu_{cc}(c)}$$

Discussion

- The resource can be partitioned and consumed in any size. However, all such partitions, no matter how minuscule, inflict a negative side effect.
- This work extends the optimal time horizon results of cake eating problems to the case involving two states.
- This work also extends the optimal depletion of an exhaustible stock of uncertain size to the case of two state variables and compare with optimal depletion with certainty.
- The intended application of this discussion is in the context of climate change caused by fossil fuel consumption.

Future Work

- Numerical analysis of the work by employing reliable data
- Study sensitivity of rate of extraction with respect to rate of uncertainty.

References

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THANK YOU!

Questions and suggestions are appreciated !!