95-percent Contour Line of 2-dimensional Gaussian Distribution

Define Z as a 2-dimensional random variable

$$Z := \frac{1}{\det(2\pi\Sigma)^{\frac{1}{2}}} \exp\left(-x\Sigma^{-1}x^{\top}/2\right)$$

where x follows a 2D Gaussian distribution $\mathcal{N}(\mu, \Sigma)$.

Target Find a threshold t such that $\mathbb{P}(Z \geq t) = .95$, where $\mathbb{P}(A)$ is the probability of event A.

Hint Define

$$Z' := -x\Sigma^{-1}x^{\top}.$$

Notice that $f(z') = \frac{1}{\det(2\pi\Sigma)^{\frac{1}{2}}} \exp(-z'/2)$ is a strictly monotone decreasing function, so there exists another threshold t' such that

$$\mathbb{P}(Z \geq t) = \mathbb{P}(Z' \leq t') = .95$$

and there also exists a mapping g such that $g : t \to t'$ is a **bijection** (one to one mapping).

If x follows 2D Gaussian distribution, it is known that Z' follows χ^2 distribution with degree of freedom 2[1]. Therefore, we can work out $t' = 5.99$ using p-value table of χ^2 distribution [2]. It coinsides with squared Mahalanobis distance $5.99 \approx 6$.

The rest is finding g^{-1} so we can map t' back to t...

[1] `https://en.wikipedia.org/wiki/Chi-squared_distribution`

[2] `https://en.wikipedia.org/wiki/Chi-squared_distribution#Table_of_%CF%872_values_vs_p-values`