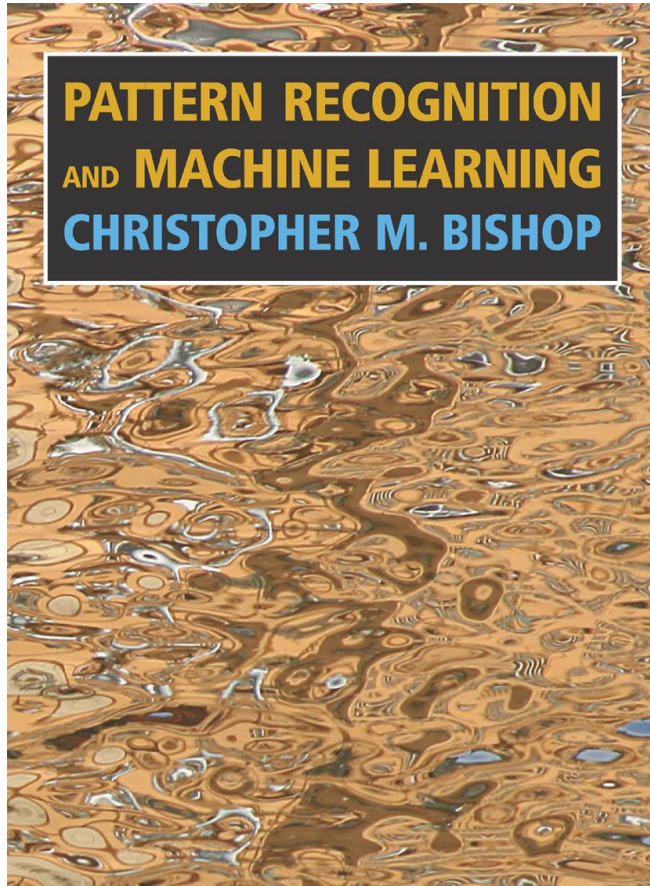


Linear Classifiers

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Office hour: Thursday 2-3pm

Reference



Today's class *roughly* follows Chapter 4-4.2.

Pattern Recognition and
Machine Learning

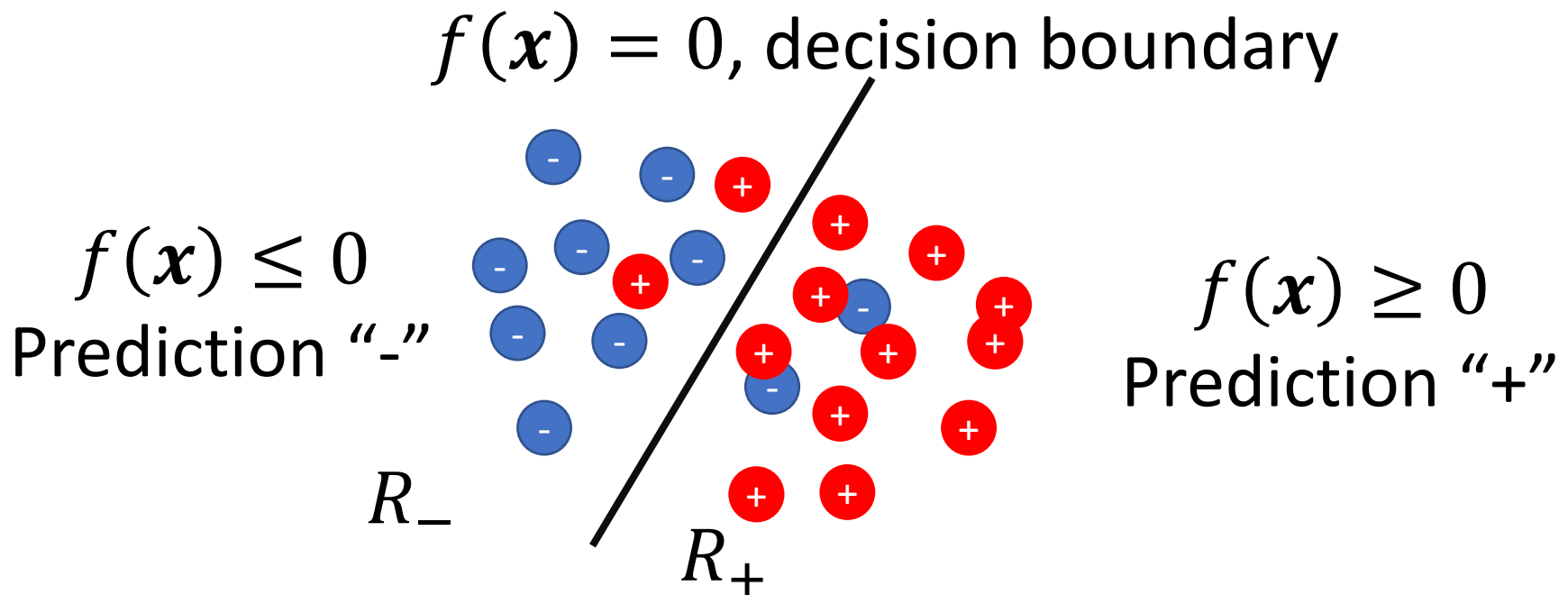
Christopher Bishop, 2006

Outline

- Geometry of decision function
- **Non probabilistic classifiers**
 - Least square classifier
 - Fisher discriminant analysis
- **Probabilistic classifiers**
 - Generative Classifiers

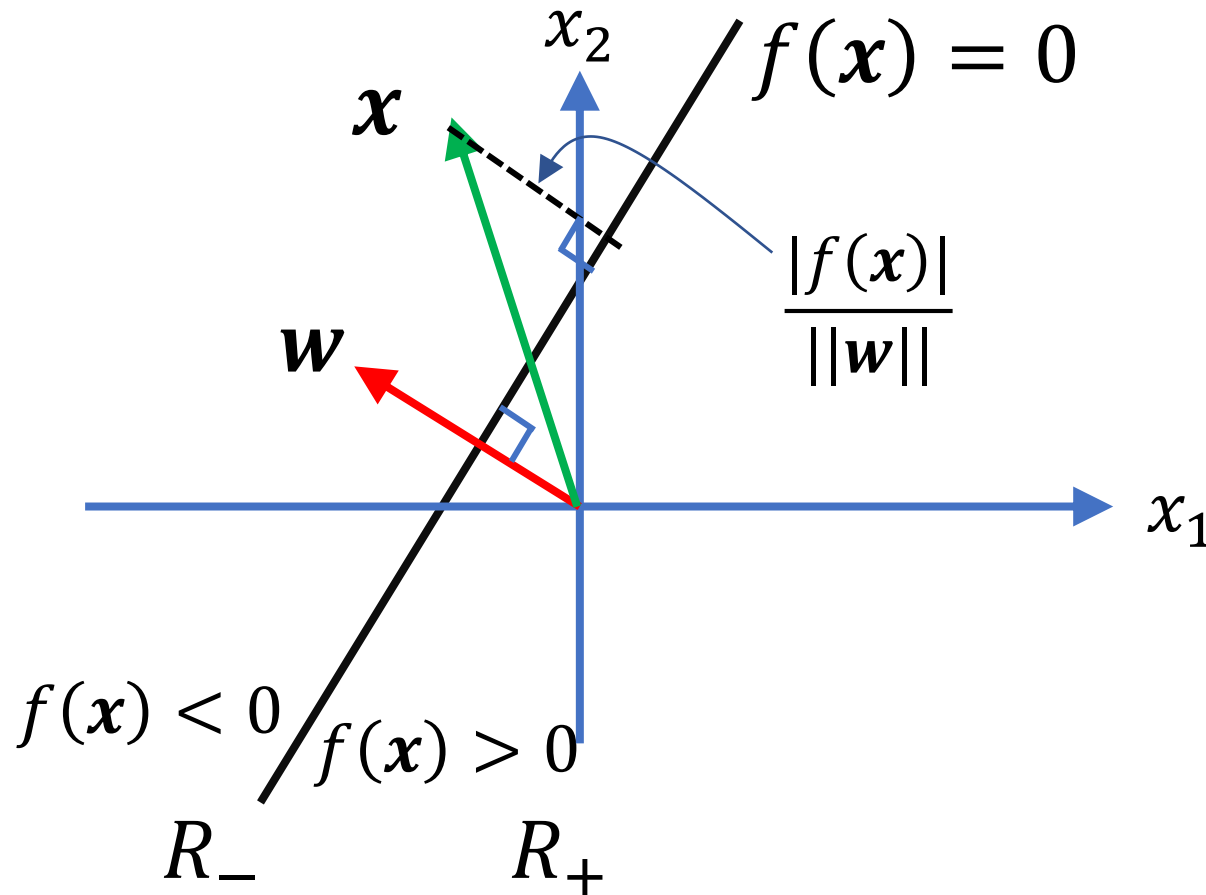
Binary Classification

- **Input:** $\{\mathbf{x}_i\}_{i=1}^n$
- **Output:** $y \in \{-1, +1\}$
- A decision boundary is defined by a function $f(\mathbf{x})$



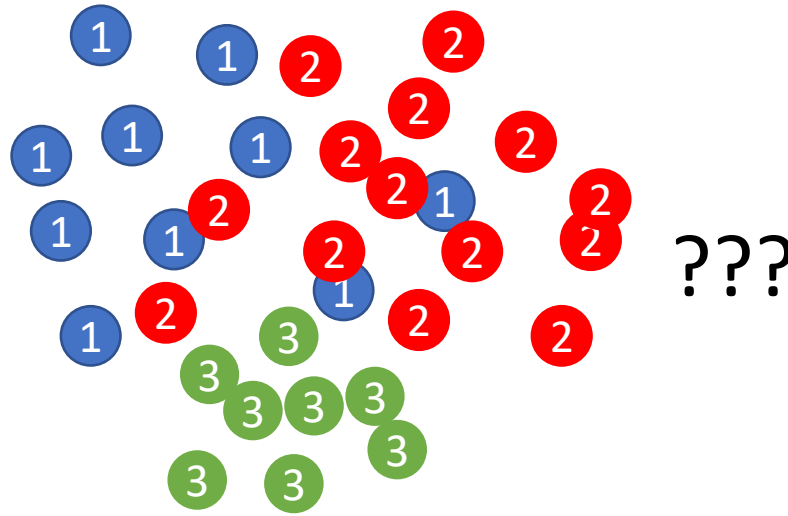
Geometry of Binary Classification

Suppose $f(\mathbf{x}) = \langle \mathbf{w}, \mathbf{x} \rangle + w_0$



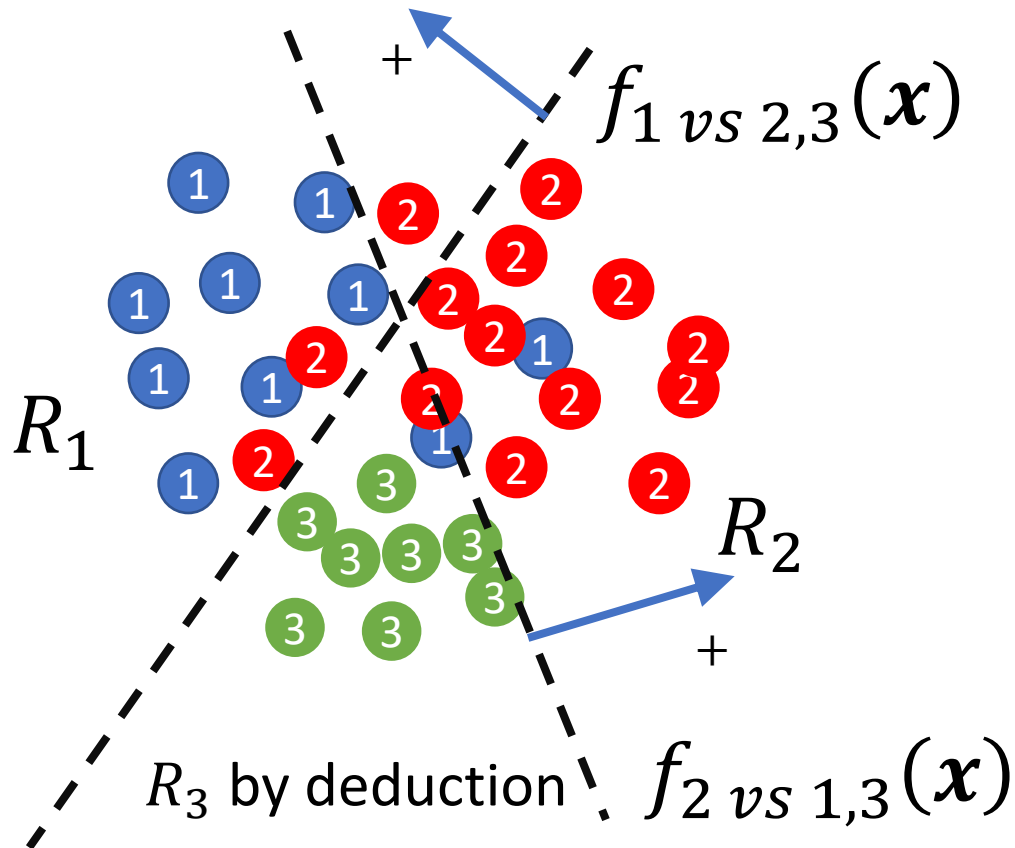
Multi-class Classification

- **Input:** $\{\mathbf{x}_i\}_{i=1}^n$
- **Output:** $y \in \{1 \dots K\}$
- The geometry gets a lot more complicated...
 - Cannot simply check the sign of a single $f(\mathbf{x})$.



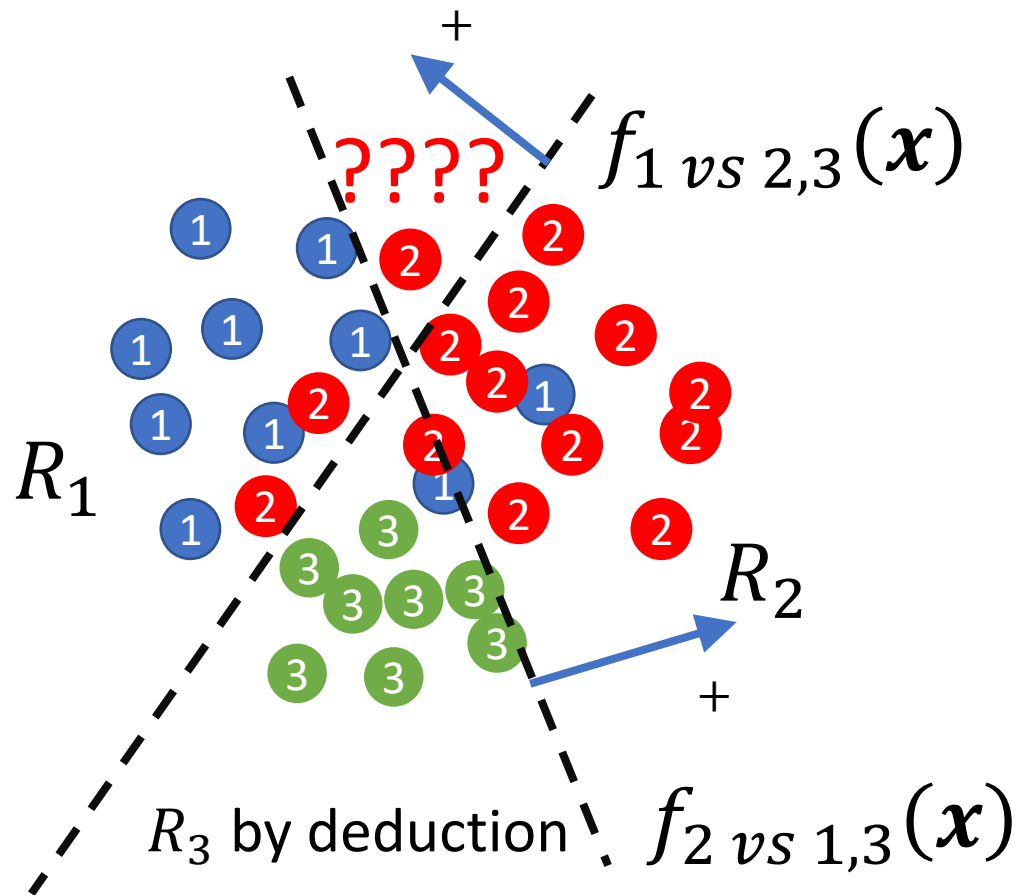
One versus The Other

- Construct $K - 1$ classifier
- Classify Class k vs. the rest of classes



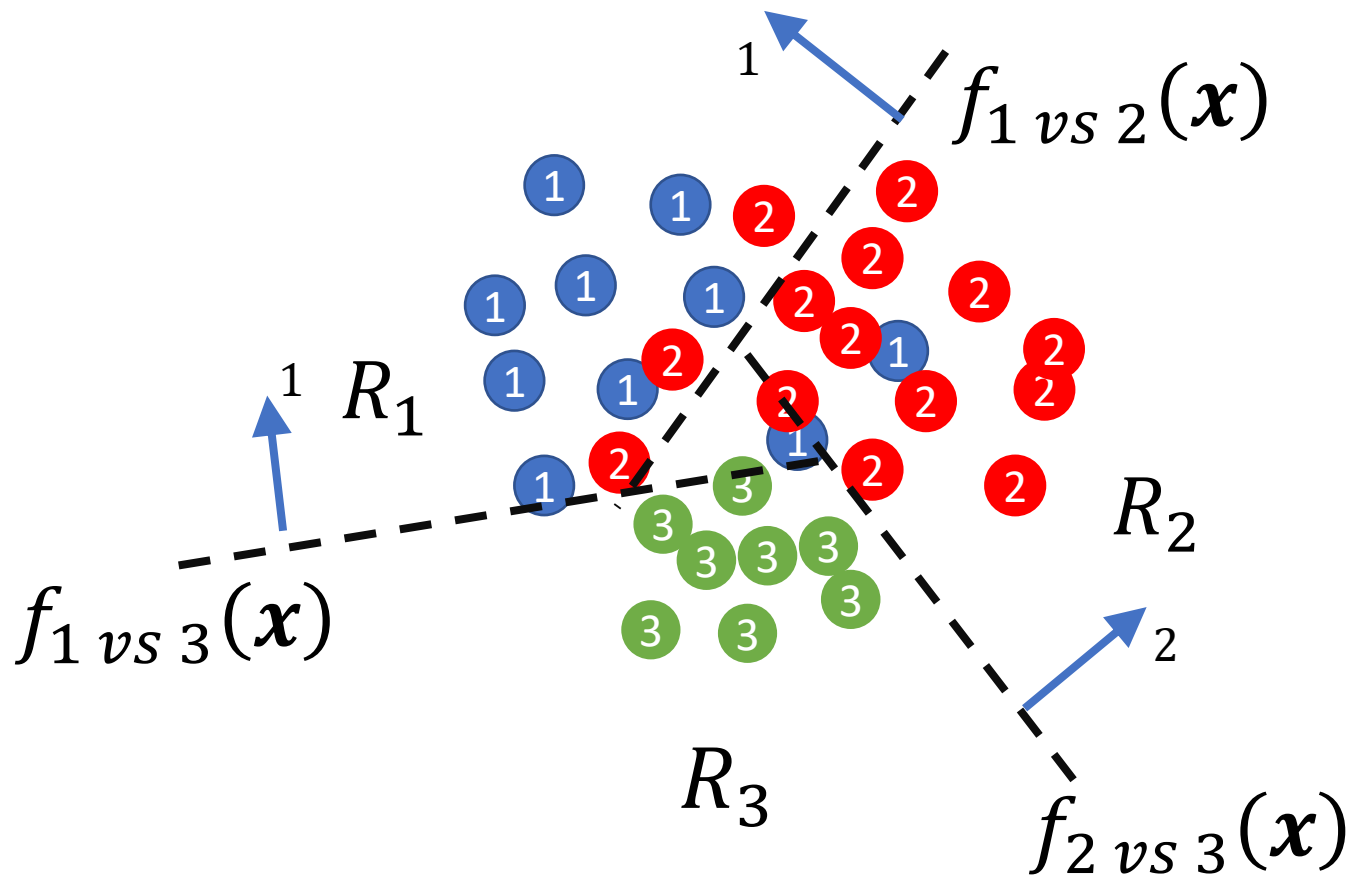
One versus The Other

- One versus the other also creates confusion!



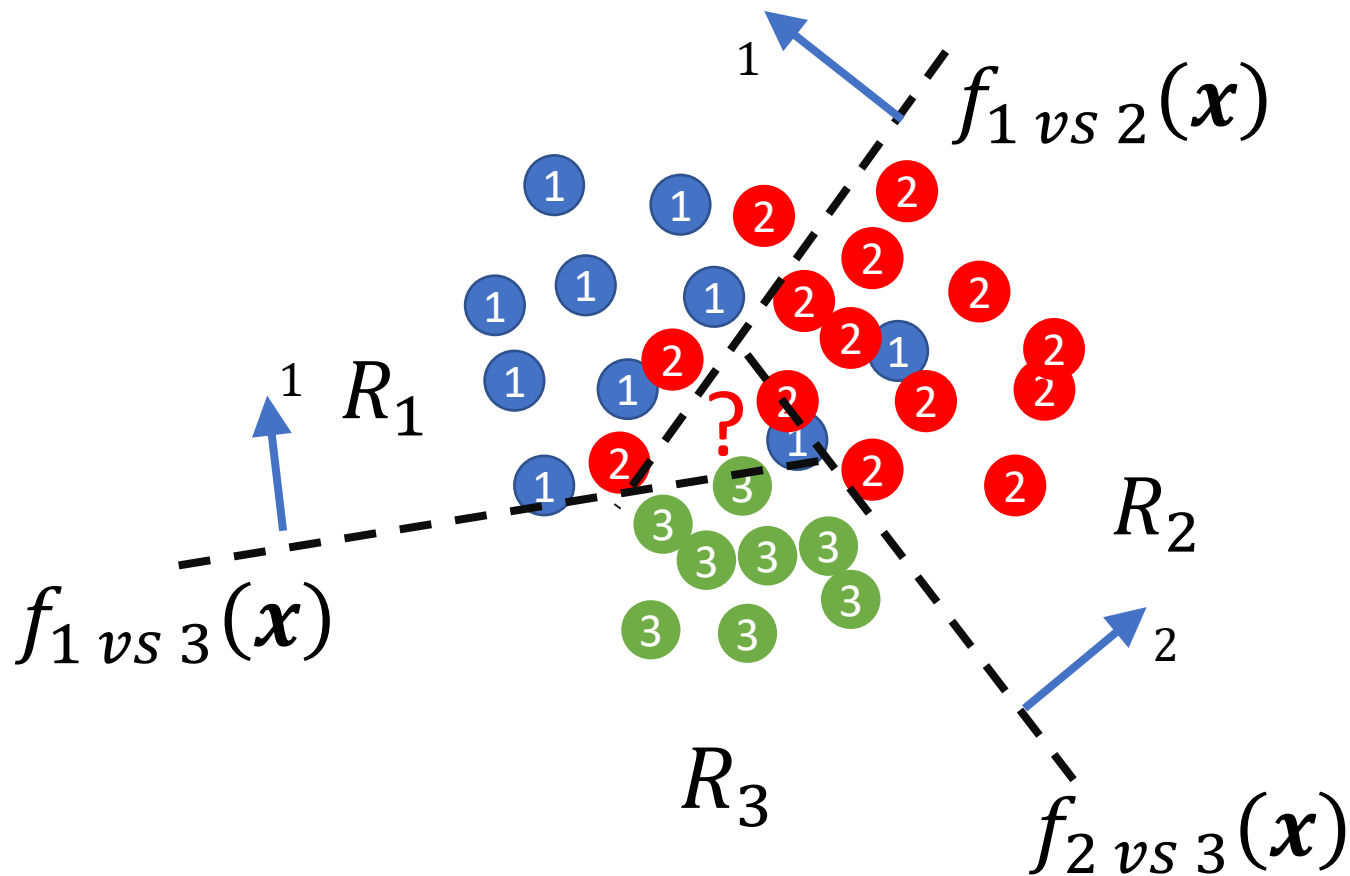
One versus One

- We can create pairwise classifiers and check majority vote.



One versus One

- One versus one creates confusion as well...



Multi-class Classification

- Rather than relying on sign of f to make predictions, we estimate K functions:
- $\{f_k(\mathbf{x}; \mathbf{w}_k)\}_{k=1}^K$
- Given an \mathbf{x} , prediction is $\hat{k} = \max_k f_k(\mathbf{x}; \mathbf{w}_j)$,
- **Problem:** f_k does not have a simple geometry interpretation anymore.
- However, f_k does have probabilistic interpretation.

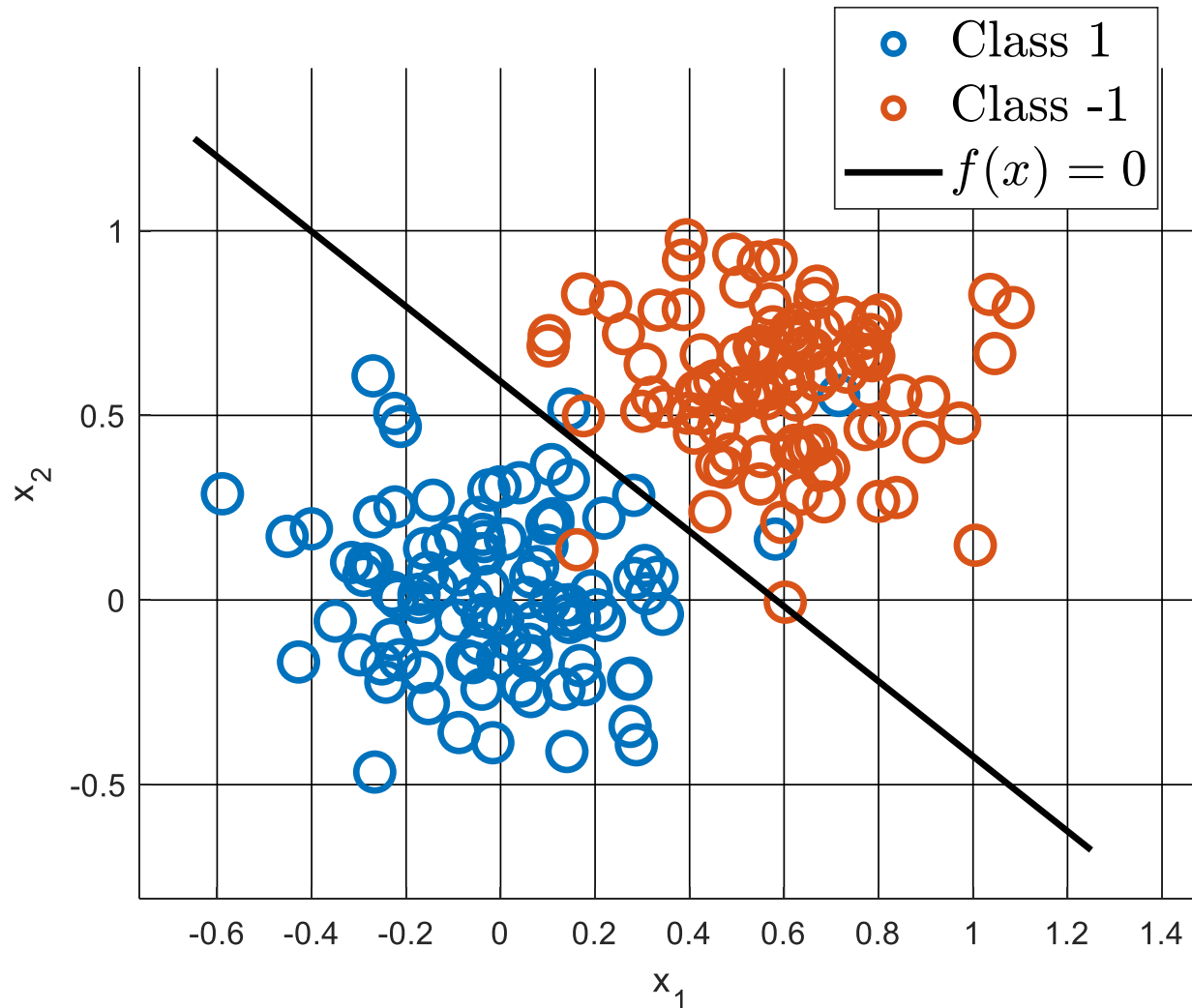
How do we obtain the
prediction function f ?

Can we use least squares?

Least Square Classifier

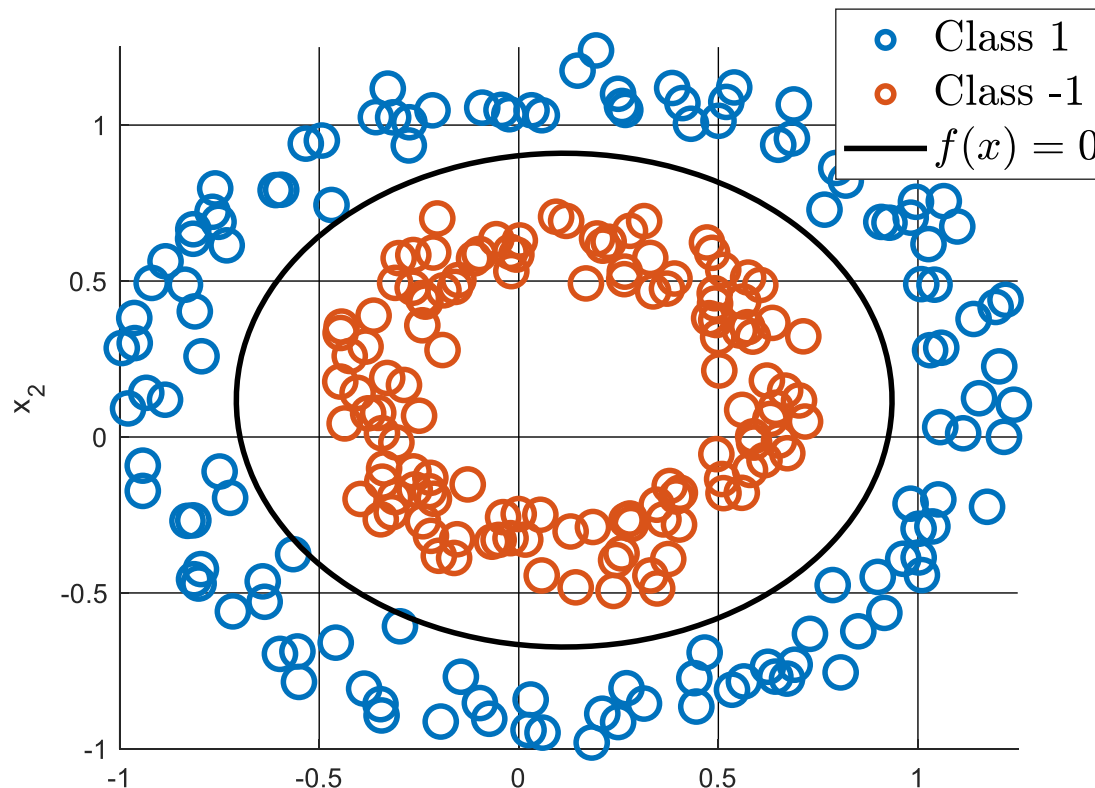
- For binary classification, import D in LS.
- $\mathbf{w}_{\text{LS}} := \underset{\mathbf{w}}{\operatorname{argmin}} \sum_{i \in D} [y_i - f(\mathbf{x}_i; \mathbf{w})]^2$
 - Now y_i takes binary value 1 or -1
- Prediction function $f(\mathbf{x}_i; \mathbf{w}_{\text{LS}})$.
- The predicted label $\hat{y} := \operatorname{sign}(f(\mathbf{x}_i; \mathbf{w}_{\text{LS}}))$

Least Square Classifier



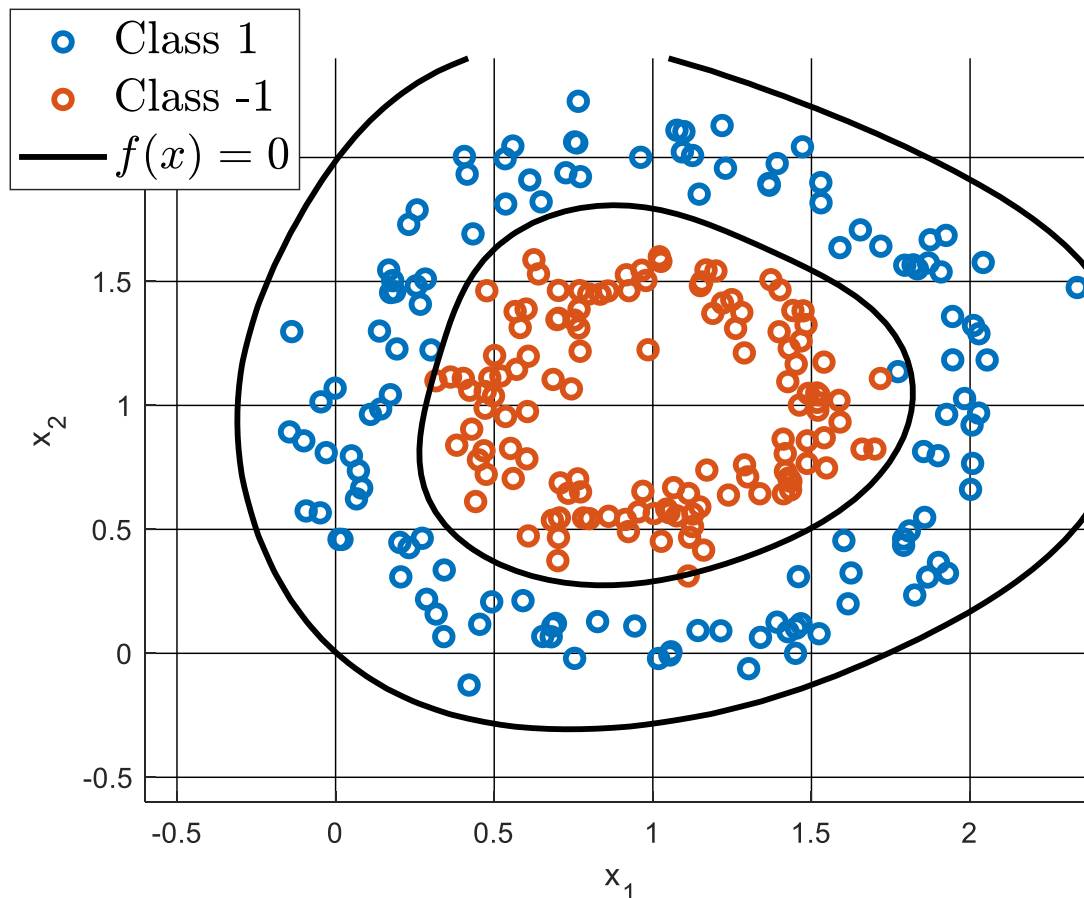
Least Square Classifier

- You can use feature transform ϕ for f as well.
- $f(x; w) := \langle w, \phi(x) \rangle$,
- e.g. poly., trigonometric, RBF, kernel.



Least Square Classifier

Data may not be separable in the original space but can be separable in the **feature space** created by ϕ !



Multi-class LS classification

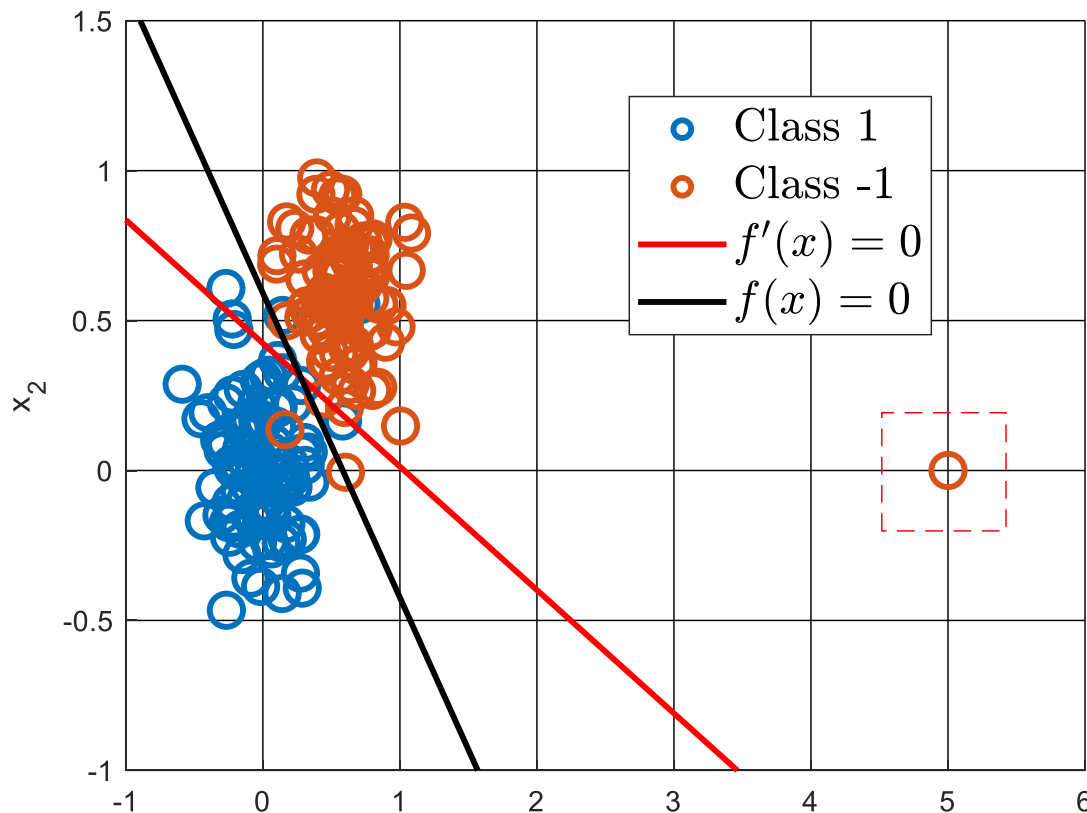
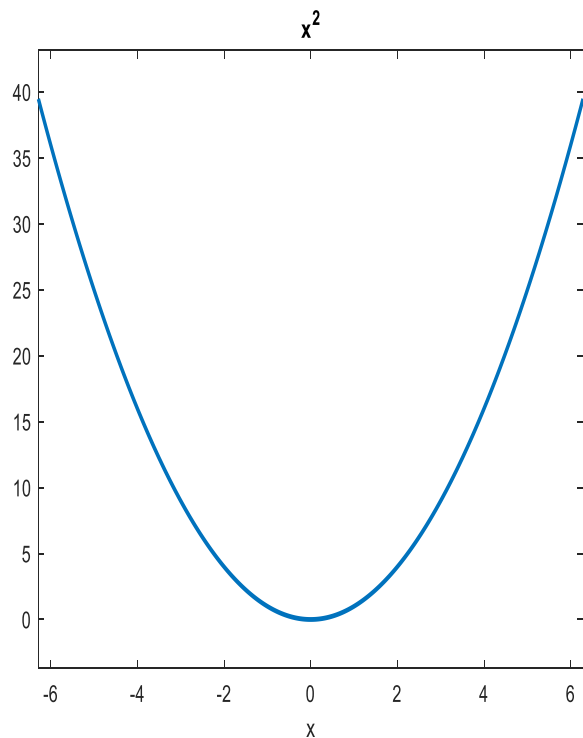
- LS can be adapted to multi-class classification.
- Suppose output $y \in \{1 \dots K\}$
- Replace $y_i = k$ in D with $\mathbf{t}_i \in \{0,1\}^K$.
 - $t_i^{(k)} = 1$.
 - $t_i^{(j)} = 0, \forall j \neq k$
- **“One-hot encoding”**

- $\mathbf{W}_{\text{LS}} := \operatorname{argmin}_{\mathbf{W}} \sum_{i \in D} \|\mathbf{t}_i - \mathbf{W}^\top \tilde{\mathbf{X}}_i\|^2$

- $\mathbf{W}_{\text{LS}} \in R^{(d+1) \times K}, \tilde{\mathbf{X}} := [\mathbf{X}_i^\top, \mathbf{1}]^\top, \mathbf{X}_i := [\mathbf{x}_i, \mathbf{x}_i \dots] \in R^{d \times K}$

Why not to use LS Classifier?

- Square loss does not make sense in classification tasks.
- Data point far away from decision boundary can influence the decision boundary by a lot.



Why not to use LS Classifier?

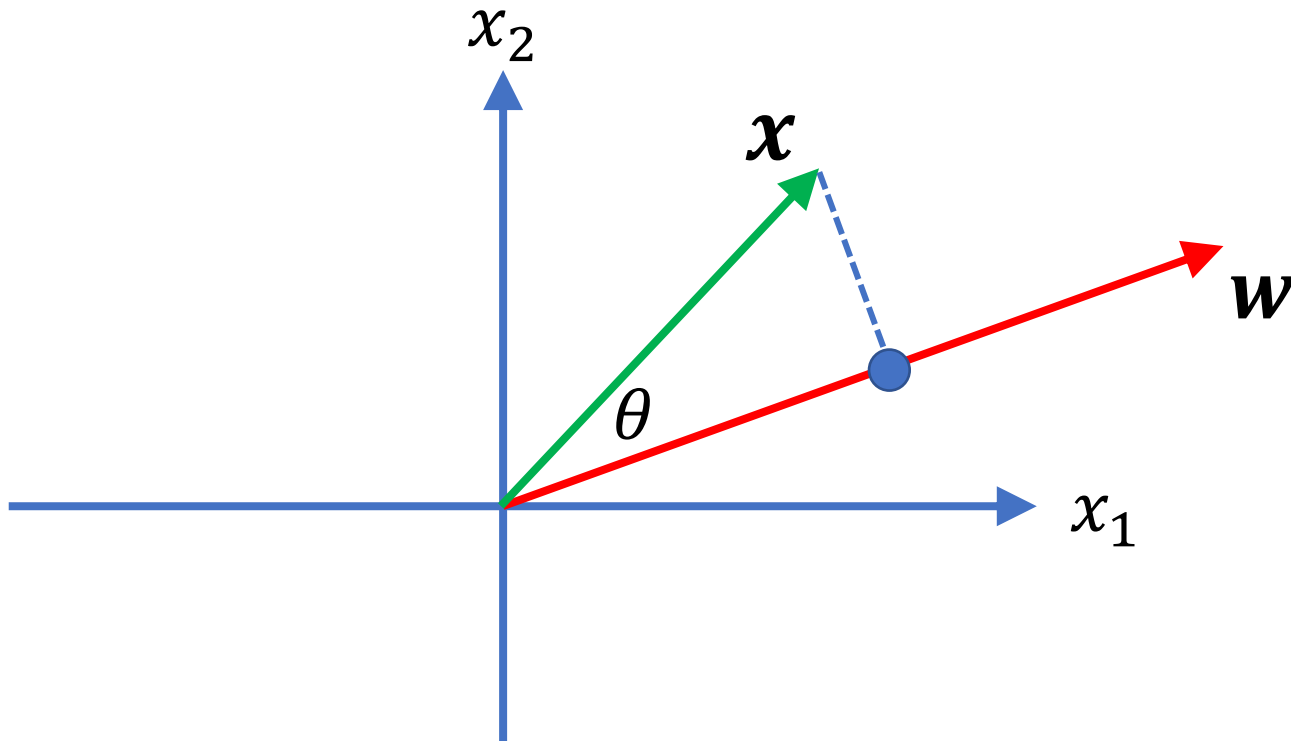
- Unlike LS regression,
- LS classification lacks a probabilistic interpretation.
- It cannot be interpreted as Maximum Likelihood of some probabilistic model on D .

Fisher Discriminant Analysis (FDA)



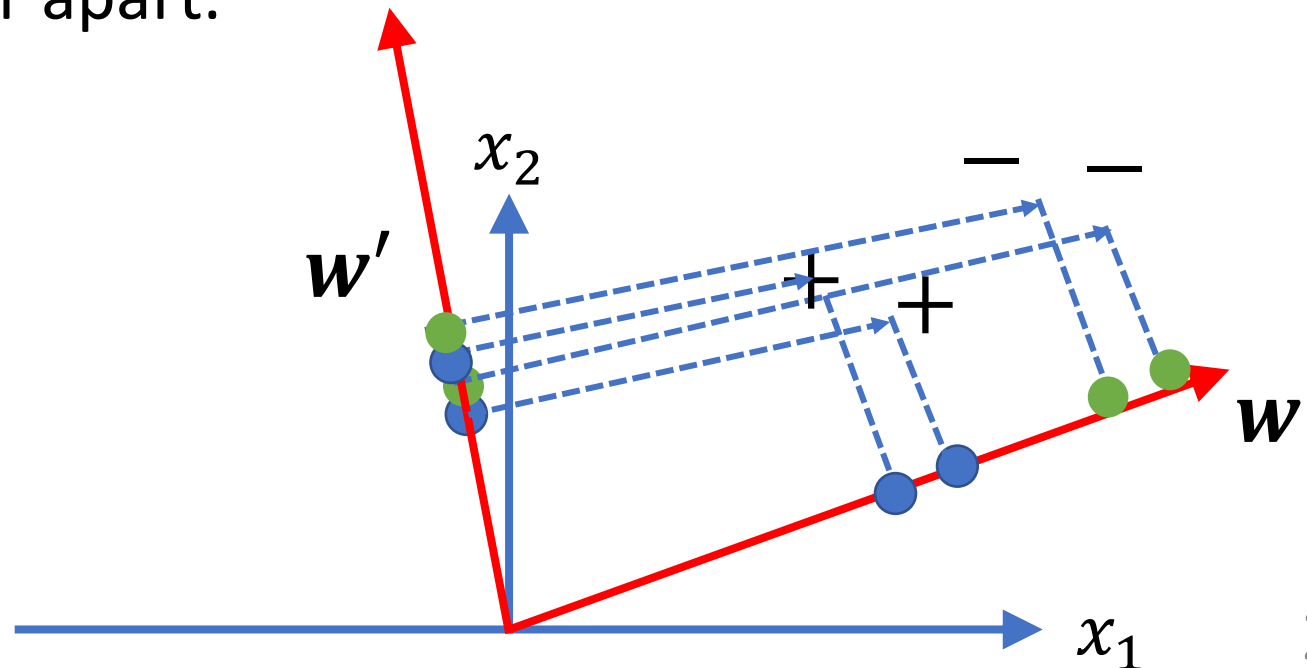
Embedding by Inner Product

- The inner product $\langle \mathbf{w}, \mathbf{x} \rangle$ “embeds” \mathbf{x} , onto a one-dimensional line along \mathbf{w} direction.

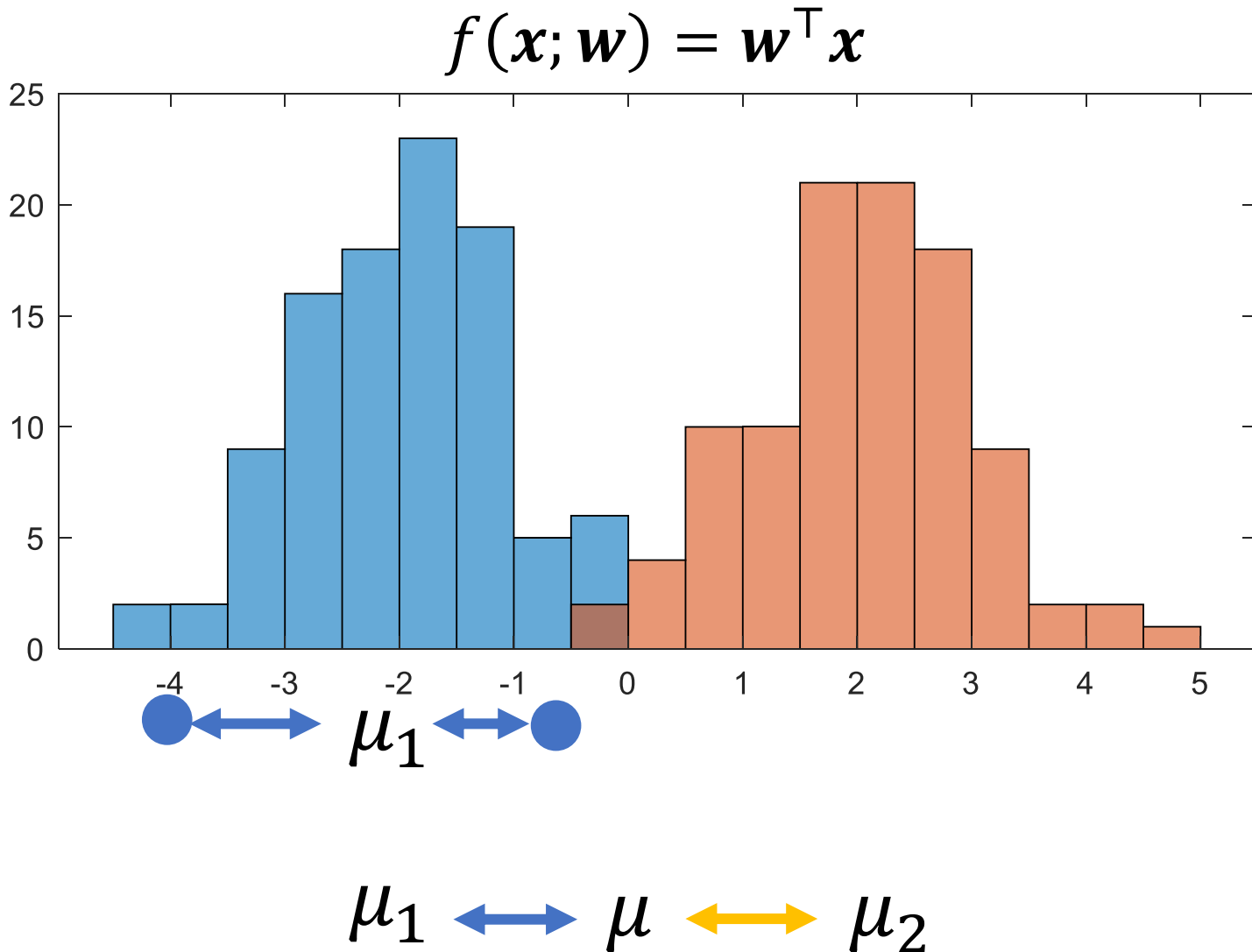


Embedding by Inner Product

- What would be a good embedding?
- Clearly, we prefer \mathbf{w} to \mathbf{w}' , as the embedding is **more separated** between $+$ and $-$.
- We want points within the class close, but points between two classes far apart.



Within Class and Between Class Scatterness



Within-class Scatterness

- Embedding is $\mathbf{w}^\top \mathbf{x}$.
- Embedded center for class k :
 - $\hat{\mu}_k = \frac{1}{n_k} \sum_{i, \mathbf{y}_i=k} \mathbf{w}^\top \mathbf{x}_i$
- Within class scatterness of class k :
- $S_{\mathbf{w},k} = \sum_{i, \mathbf{y}_i=k} (\mathbf{w}^\top \mathbf{x}_i - \hat{\mu}_k)^2$
 - Sum over points in **individual** classes.

Between-class Scatterness

- Embedded dataset center:

- $\hat{\mu} = \frac{1}{n} \sum_{i=1}^n \mathbf{w}^\top \mathbf{x}_i$

- Between-class scatterness

- $s_{b,k} = n_k (\hat{\mu}_k - \hat{\mu})^2$

- n_k is needed to make $s_{b,k}$ at the same scale with $s_{w,k}$.

Fisher Discriminant Analysis

- **Maximizing** between class scatterness \forall_k .
- **Minimize** within class scatterness \forall_k .
- $\max_w \boxed{\sum_k s_{b,k}} / \boxed{\sum_k s_{w,k}}$
- If $K = 2$, this has a simple solution that
- $\mathbf{w} := \mathbf{S}_w^{-1}(\boldsymbol{\mu}_+ - \boldsymbol{\mu}_-)$, $\mathbf{S}_w := \sum_{k=1}^K \mathbf{S}_k$
- \mathbf{S}_k is **sample covariance** of class k times n_k .
- Read PRML 4.14 for its derivation

Fisher Discriminant Analysis

- However, FDA does not learn a decision function f .
- $f(\mathbf{x}; \mathbf{w}_{\text{FDA}}) = \langle \mathbf{w}_{\text{FDA}}, \mathbf{x} \rangle$ obtained by FDA cannot be directly used for making a prediction:
- In general, $f(\mathbf{x}; \mathbf{w}_{\text{FDA}}) > 0$ does not mean \mathbf{x} is predicted as positive or negative data point: FDA does not care about classification accuracy, a.k.a., minimizing FP or FN.

Probabilistic Generative Classifiers

Probabilistic Classification

- How to put classification problem under a prob. framework?

- **Minimize Expected Loss:**

$$\hat{y} := \operatorname{argmin}_{y_0} \mathbb{E}_{p(y|\mathbf{x})} [L(y, y_0) | \mathbf{x}]$$

- We need: $p(y|\mathbf{x})$, $y \in \{1, \dots, K\}$
- **Discriminative:** Infer $p(y|\mathbf{x})$ directly.
- **Generative:** Infer $p(y|\mathbf{x}) \propto p(\mathbf{x}|y)p(y)$, infer $p(\mathbf{x}|y)$!

Continuous Input Variable

- To infer $p(\mathbf{x}|y)$, we need a model.
- If \mathbf{x} is continuous, MVN is a natural choice for $p(\mathbf{x}|y)$.
- **Model** $p(\mathbf{x}|y = k; \mathbf{w}) := N_{\mathbf{x}}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$
- Assuming IID-ness over all \mathbf{x}_i , and shared covariance $\boldsymbol{\Sigma}$
- **Write down the likelihood** over D :
- $$p(D|\mathbf{w}) = \prod_{i \in D} p(\mathbf{x}_i, y_i | \mathbf{w}) = \prod_{i \in D} p(\mathbf{x}_i | y_i; \mathbf{w}) p(y_i)$$
$$= \prod_{i \in D} N_{\mathbf{x}}(\boldsymbol{\mu}_{y_i}; \boldsymbol{\Sigma}) p(y_i)$$

Continuous Input Variable

$$\bullet \hat{\boldsymbol{\mu}}_{1\dots K}, \hat{\boldsymbol{\Sigma}} := \arg \max_{\boldsymbol{\mu}_{1\dots K}, \boldsymbol{\Sigma}} \sum_{i \in D} \log N_{\mathbf{x}}(\boldsymbol{\mu}_{y_i}; \boldsymbol{\Sigma}) p(y_i)$$

1. Plug in estimates for $p(y_i)$, which is $\frac{n_k}{n}$.

2. Now work out the MLE for $\hat{\boldsymbol{\mu}}_k := \frac{1}{n_k} \sum_{i \in D, y_i=k} \mathbf{x}_i$

3. Plug in $\hat{\boldsymbol{\mu}}_k$ to work out

$$\hat{\boldsymbol{\Sigma}} := \sum_{k=1\dots K} \frac{n_k}{n} \underbrace{\frac{1}{n_k} \sum_{i \in D, y_i=k} (\mathbf{x}_i - \hat{\boldsymbol{\mu}}_k)(\mathbf{x}_i - \hat{\boldsymbol{\mu}}_k)^\top}_{\text{MLE of covariance of individual classes!}}$$

MLE of covariance of individual classes!

Linear Decision Boundary

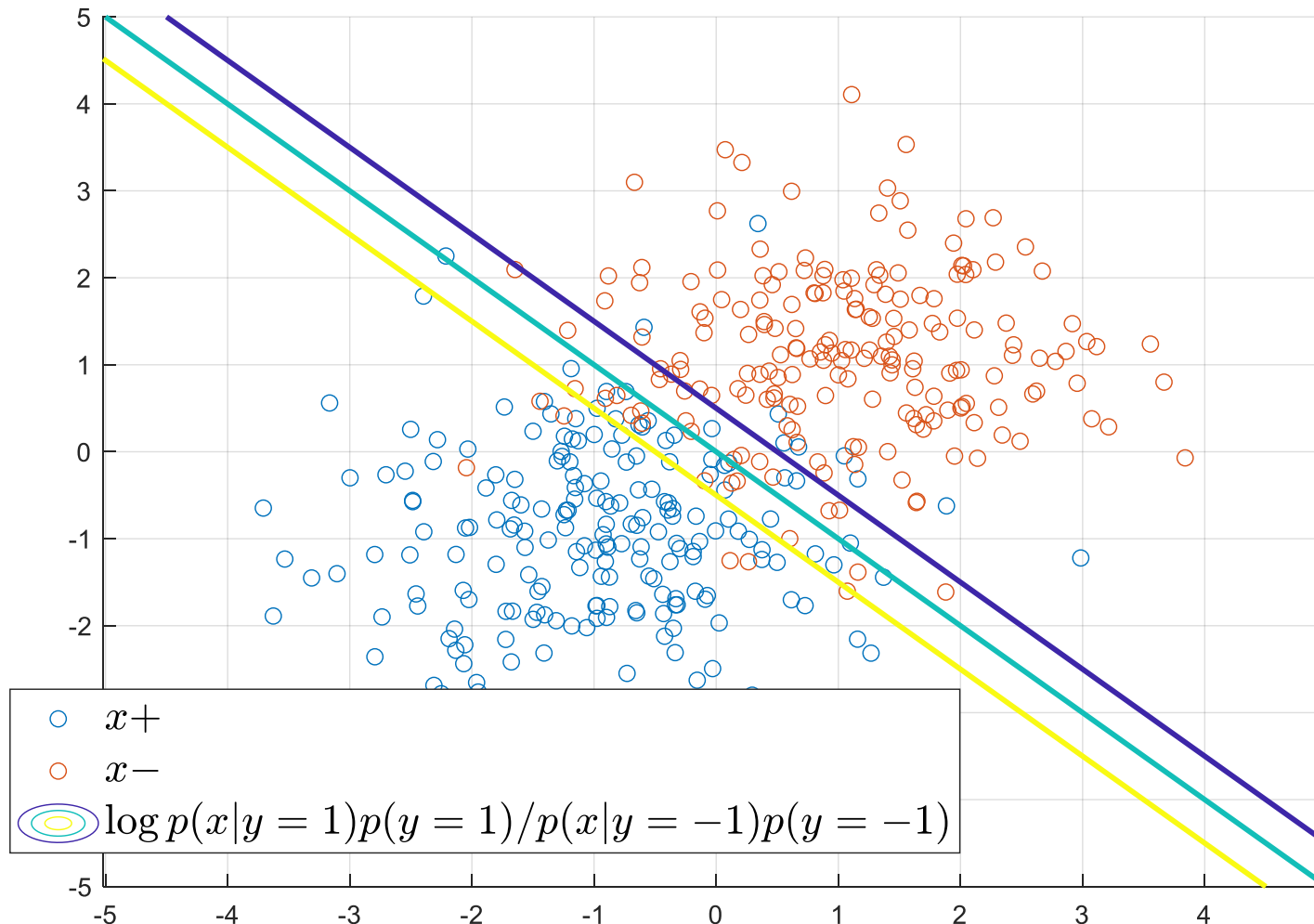
- **Prove:** when using shared covariance matrix MVN model, the decision boundary is piecewise-linear.
- **Prediction:** $\hat{y} := \operatorname{argmax}_y p(y|\mathbf{x}; \hat{\mathbf{w}}) \propto p(\mathbf{x}|y; \hat{\mathbf{w}})p(y)$
- The decision boundary is
$$\{\mathbf{x} | p(y = k|\mathbf{x}; \hat{\mathbf{w}}) = p(y = k'|\mathbf{x}; \hat{\mathbf{w}})\}$$
$$\forall k \neq k'$$

Which is the same as the set

$$\left\{ \mathbf{x} \left| \frac{p(\mathbf{x}|y = k; \hat{\mathbf{w}})p(y = k)}{p(\mathbf{x}|y = k'; \hat{\mathbf{w}})p(y = k')} = 1 \right. \right\}$$
$$\forall k \neq k'$$

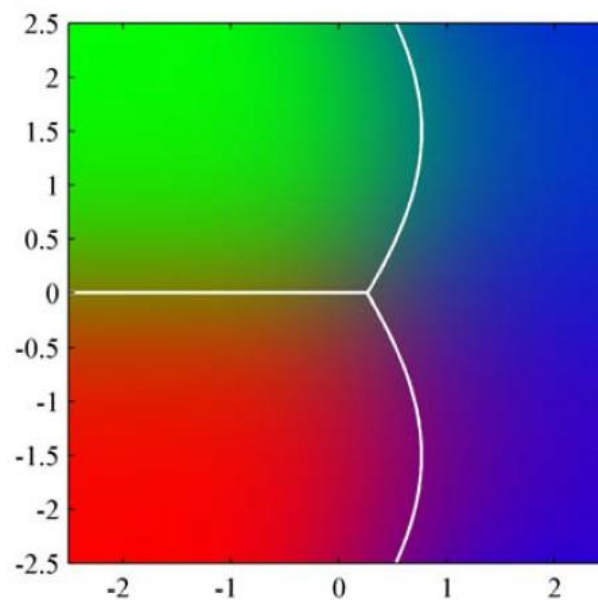
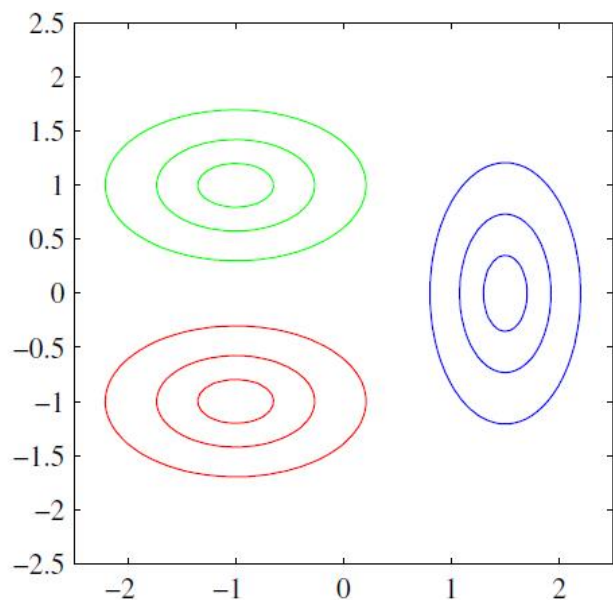
Hint: take log on both sides of the equality.

Linear Decision Boundary



Continuous Input Variable

- You can also assume for each class k , there are different covariance matrices Σ_k .
- The MLE reduces to estimating individual μ_k and Σ_k .
- **The decision boundary is no longer linear.**



Discrete Input Variable x

- In many classification tasks, we are dealing with discrete variables as x . For example, in a spam filter,
- $x := [x^{(1)}, \dots, x^{(d)}]^T$ are frequencies of words in a document. This is called “bag of words” representation.
- $y \in \{\text{spam}, \text{ham}\}$.
- For example, the document “to be or not to be”
- $x := [\text{to} = 2, \text{be} = 2, \text{or} = 1, \text{not} = 1, \text{question} = 0]^T$.
- $x^{(i)} \in \{\text{to}, \text{be}, \text{or}, \text{not}, \text{question}\}$

Naïve Bayes

- Assume $x^{(1)} \dots x^{(d)}$ are conditionally independent given y
- $p(\mathbf{x}|y) = \prod_{i=1 \dots d} p(x^{(i)}|y)$
- $p(x^{(i)}|y = k)$ is the probability of word i occurs in class k .
- It is easy to estimate:

$$p(x^{(i)}|y = k) \approx \frac{\sum_{j \in D, y_j = k} x_j^{(i)}}{\sum_{j \in D, y_j = k} \sum_{i=1}^d x_j^{(i)}}$$

- $p(\text{to}|y = \text{spam})$ is occurrences of the word “to” in “spam” emails divided by total number of words in “spam” emails in our training dataset.

Naïve Bayes

- Prediction: $\hat{y} := \operatorname{argmax}_y p(\mathbf{x} = \mathbf{x}_0 | y) p(y)$
- $p(y = k): \frac{n_k}{n}$
- $p(\mathbf{x} = \mathbf{x}_0 | y): \prod_{i=1 \dots d} p(x^{(i)} | y)^{x_0^{(i)}}$
 - $p(x^{(i)} | y)$ has been obtained by previous counting.
 - Power $x_0^{(i)}$ is used assuming occur. of words are independent.
- $p(\mathbf{x} = \text{"to be or not to be"} | y):$
 $p(to|y)^2 p(be|y)^2 p(or|y) p(not|y)$

Conclusion

- We have studied classification problem:
- Geometry of decision function
- Least square classifier
- Fisher discriminant analysis
 - Within and between scatterness
- Generative Classifiers:
 - MVN for continuous input variable
 - Naïve Bayes for discrete input variable

Homework

- Prove the statement on page 32.
- Put Naïve Bayes under an MLE framework.
- **Hint:** Assume $p(\mathbf{x}|\pi_{1\dots d})$ is a multinomial distribution with event probabilities $\pi_{1\dots d}$.
- $\frac{x^{(i)}}{\sum_{i=1}^d x^{(i)}}$ is the MLE of π_i given $D = \{\mathbf{x}\} = \left\{ \begin{bmatrix} x^{(1)} & \dots & x^{(d)} \end{bmatrix} \right\}$

Computing Lab

- Implement a version of Perception classifier: “Simplitron”
- Demo.