

# Proof of Homework

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*Proof.*  $p(\mathbf{w}|D) \propto p(D|\mathbf{w})p(\mathbf{w})$ , under our assumption we have

$$p(\mathbf{w}|D) \propto \left[ \prod N_{y_i}(f(\mathbf{x}_i, \mathbf{w}), \sigma^2) \right] N_{\mathbf{w}}(\mathbf{0}, \sigma_w^2 \mathbf{I})$$

Take the logarithm, we have

$$\begin{aligned} \log p(\mathbf{w}|D) &= \left[ \sum_{i=1}^n -\frac{\|y_i - f(\mathbf{x}_i; \mathbf{w})\|^2}{2\sigma^2} \right] + \frac{-\|\mathbf{w}\|^2}{2\sigma_w^2} + \text{const.} \\ &= \left[ \sum_{i=1}^n -\frac{\|y_i - \langle \mathbf{w}, \mathbf{x} \rangle\|^2}{2\sigma^2} \right] + \frac{-\|\mathbf{w}\|^2}{2\sigma_w^2} + \text{const.} \\ &= -\frac{\mathbf{w}^\top \mathbf{w}}{2\sigma_w^2} - \frac{\mathbf{w}^\top \mathbf{X} \mathbf{X}^\top \mathbf{w}}{2\sigma^2} + \frac{2\mathbf{w}^\top \mathbf{X} \mathbf{y}}{2\sigma^2} - \frac{\mathbf{y} \mathbf{y}^\top}{2\sigma^2} + \text{const.} \end{aligned}$$

Let us ignore all the terms that are not related to  $\mathbf{w}$ .

$$\begin{aligned} &-\frac{\mathbf{w}^\top \mathbf{w}}{2\sigma_w^2} - \frac{\mathbf{w}^\top \mathbf{X} \mathbf{X}^\top \mathbf{w}}{2\sigma^2} + \frac{2\mathbf{w}^\top \mathbf{X} \mathbf{y}}{2\sigma^2} \\ &= -\frac{1}{2\sigma^2} \left\{ \mathbf{w}^\top \left[ \frac{\sigma^2}{\sigma_w^2} \mathbf{I} + \mathbf{X} \mathbf{X}^\top \right] \mathbf{w} - 2\mathbf{w}^\top \mathbf{X} \mathbf{y} \right\} \\ &= -\frac{1}{2\sigma^2} \left\{ \mathbf{t}^\top \left[ \frac{\sigma^2}{\sigma_w^2} \mathbf{I} + \mathbf{X} \mathbf{X}^\top \right] \mathbf{t} \right\} + \text{const.}, \end{aligned}$$

where  $\mathbf{t} = \mathbf{w} - \left[ \frac{\sigma^2}{\sigma_w^2} \mathbf{I} + \mathbf{X} \mathbf{X}^\top \right]^{-1} \mathbf{X} \mathbf{y}$ .

We can see that  $\log p(\mathbf{w}|D) = -\frac{1}{2\sigma^2} \left\{ \mathbf{t}^\top \left[ \frac{\sigma^2}{\sigma_w^2} \mathbf{I} + \mathbf{X} \mathbf{X}^\top \right] \mathbf{t} \right\} + \text{const.}$ , so

$$p(\mathbf{w}|D) = N_{\mathbf{w}} \left( \left[ \frac{\sigma^2}{\sigma_w^2} \mathbf{I} + \mathbf{X} \mathbf{X}^\top \right]^{-1} \mathbf{X} \mathbf{y}, \sigma^2 \left[ \frac{\sigma^2}{\sigma_w^2} \mathbf{I} + \mathbf{X} \mathbf{X}^\top \right]^{-1} \right).$$

The second part of the proof follows Bishop's book, 2.115. We have

$$p(\hat{y}|\mathbf{x}; \mathbf{w}) = N_{\hat{y}}(f(\mathbf{x}; \mathbf{w}), \sigma^2),$$

and

$$p(\mathbf{w}|D) = N_{\mathbf{w}} \left( \left[ \frac{\sigma^2}{\sigma_w^2} \mathbf{I} + \mathbf{X} \mathbf{X}^\top \right]^{-1} \mathbf{X} \mathbf{y}, \sigma^2 \left[ \frac{\sigma^2}{\sigma_w^2} \mathbf{I} + \mathbf{X} \mathbf{X}^\top \right]^{-1} \right),$$

2.115 tells us

$$p(\hat{y}|\mathbf{x}, \mathbf{w}) = N_{\hat{y}} \left[ \mathbf{x}^\top \left[ \frac{\sigma^2}{\sigma_w^2} \mathbf{I} + \mathbf{X} \mathbf{X}^\top \right]^{-1} \mathbf{X} \mathbf{y}, \sigma^2 + \sigma^2 \mathbf{x}^\top \left[ \frac{\sigma^2}{\sigma_w^2} \mathbf{I} + \mathbf{X} \mathbf{X}^\top \right]^{-1} \mathbf{x} \right].$$

□