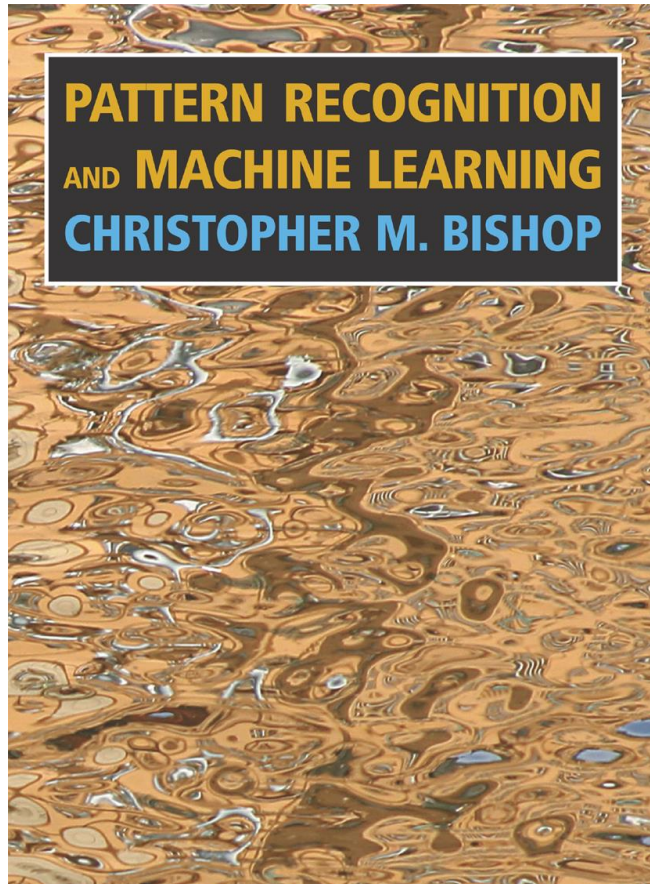


Probabilistic Model Selection in Regression

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Reference



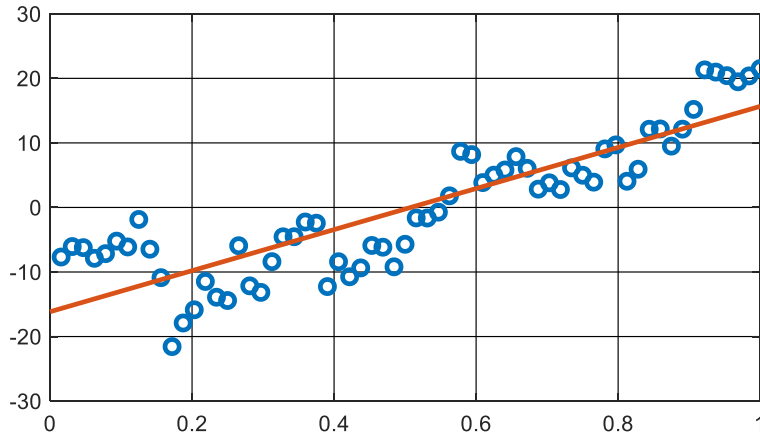
Today's class *roughly* follows Chapter 3.4-3.52.

Pattern Recognition and
Machine Learning

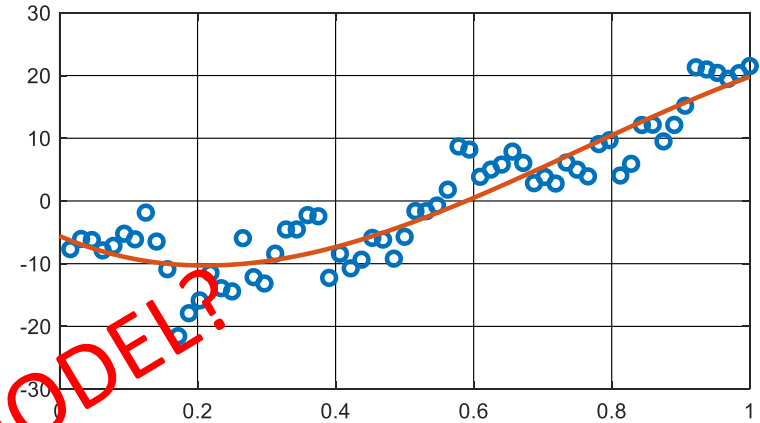
Christopher Bishop, 2006

Apple Stock Price Jul – Sep

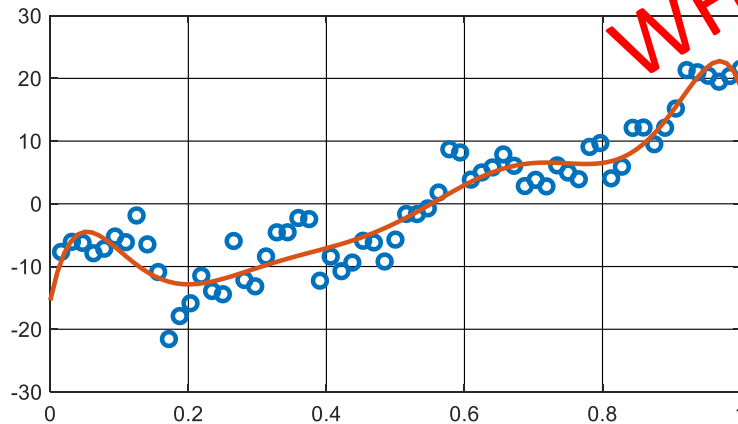
Linear Model



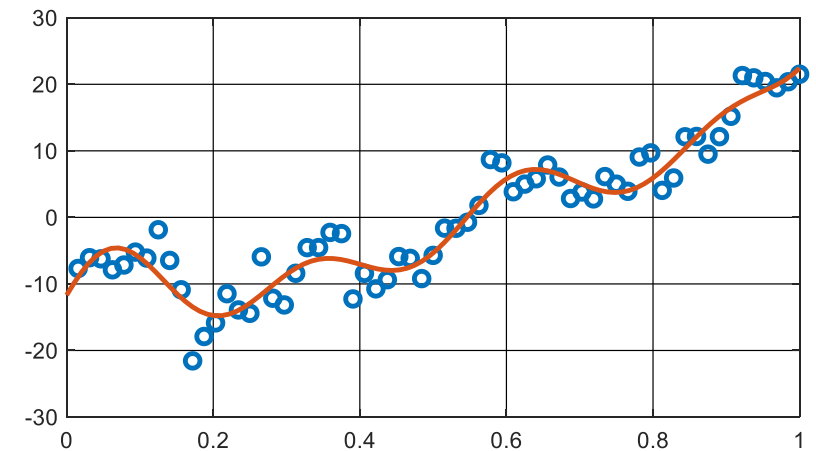
Polynomial



Trigonometric



RBF basis



WHICH MODEL?

Frequentist Model Selection

- We want to minimize the expected (squared) loss:
- $\sum_{i=1..n} \mathbb{E}_D [[y - f(\mathbf{x}_i; \mathbf{w})]^2 | \mathbf{x}_i]$, over our dataset D .
- $\mathbb{E}_D [[y - f(\mathbf{x}_i; \mathbf{w})]^2 | \mathbf{x}_i]$ is **minimized**
 - when **bias** and **variance** is balanced
- This cannot be done in practice as, $\mathbb{E}_D [[y - f(\mathbf{x}_i; \mathbf{w})]^2 | \mathbf{x}_i]$ cannot be calculated.
 - We cannot generate different D easily!
- Use out sample error (approximated by testing error):
 - $\mathbb{E}_x [\mathbb{E}_D [[y - f(\mathbf{x}_i; \mathbf{w})]^2 | \mathbf{x}]] \approx \frac{1}{n'} \sum_{i \in D'} [y - f(\mathbf{x}_i; \mathbf{w}_{LS})]^2$

Frequentist Model Selection

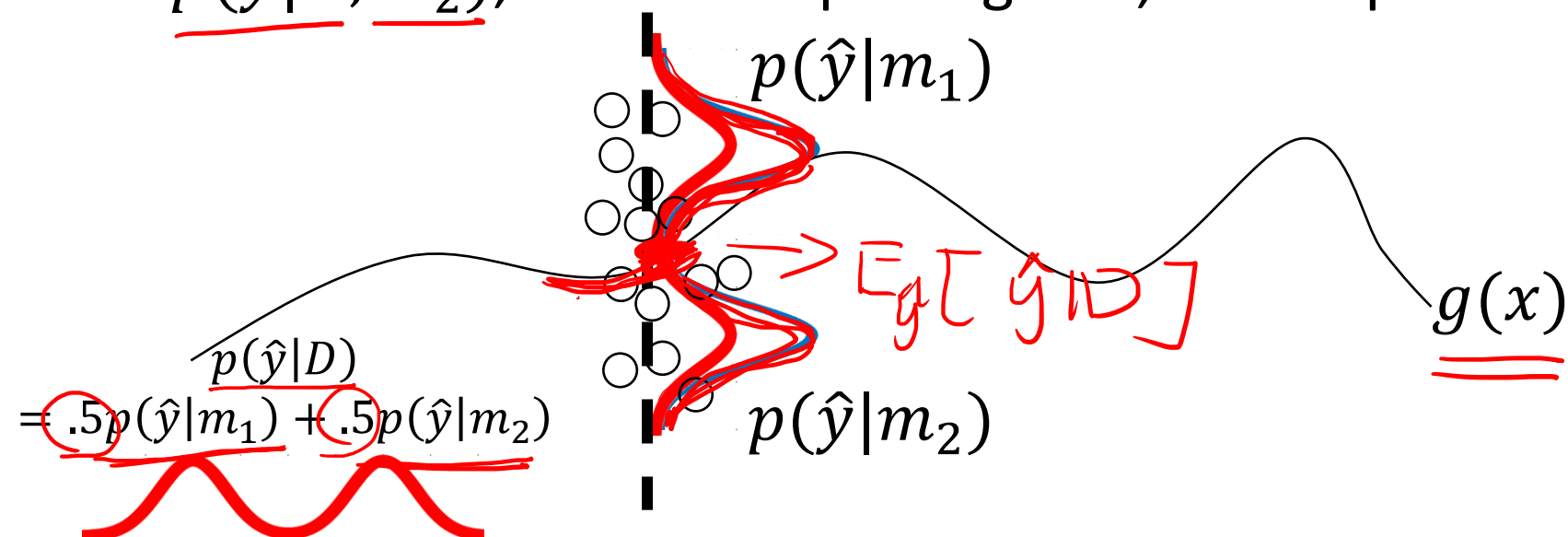
- There are issues regarding this model selection approach.
- This frequentist approach requires us to hold out sample during training.
 - We lose information in part of our dataset.
 - CV helps, but calculation is heavy.
 - Our dataset may not be IID.
- How would we select a model if we adopt a probabilistic view?

Probabilistic Model "Selection"


- Build uncertainty of models using priors over models:
- Let $m \in \{m_1 \dots m_K\}$,
- If we choose $p(m)$ as a model prior.
1 2 3 ... 5
uniform
- Then we can write posterior of model using Bayes rule:
- $p(m|D) \propto p(D|m)p(m)$
- This express the preference over models given D .
- How do we choose a model for prediction?

Probabilistic Model Average

- Bayesians never choose, they marginalize:
- $p(\hat{y}|D) = \sum_{m \in \{\dots\}} p(\hat{y}|D, m) p(m|D)$
 - a weighted sum
 - If $p(\hat{y}|D, m_1)$ gives a different prediction than $p(\hat{y}|D, m_2)$, instead of picking one, we keep both.



Probabilistic Model Average

- $p(\hat{y}|D) = \sum_{m \in \{\dots\}} p(\hat{y}|D, m) p(m|D)$ 
 - Probabilistic model sel.: Using the most probable models given by $p(m|D)$ to approx. $p(\hat{y}|D)$.


- In comparison with frequentist model sel.



$$\hat{m} = \operatorname{argmin}_{\underline{m}} \sum_{i=1..n} \mathbb{E}_D \left[[y - f(\mathbf{x}_i; \mathbf{w}, m)]^2 | \mathbf{x}_i \right]$$

- We can see:

- Frequentist minimizes, Bayesian marginalizes.

Probabilistic Model Selection

- $p(\hat{y}|D) = \sum_{m \in \{\dots\}} \underline{p(\hat{y}|D, m)} \underline{p(m|D)}$ 
- How can you calculate $p(m|D)$?

$$\boxed{p(m|D)} \propto \underbrace{p(D|m)}_{\text{model evidence}} \underbrace{p(m)}_{\text{prior}}$$


Model Evidence

- Suppose your model m is governed a set of parameters \mathbf{w}
- Then $p(D|m) = \int p(D|\mathbf{w}, m) p(\mathbf{w}|m) d\mathbf{w}$
- **Note:** model evidence is the normalizer of para. posterior
- $p(\mathbf{w}|D, m) = \frac{p(D|\mathbf{w}, m) p(\mathbf{w}|m)}{p(D|m)}$



Model Evidence Approximation

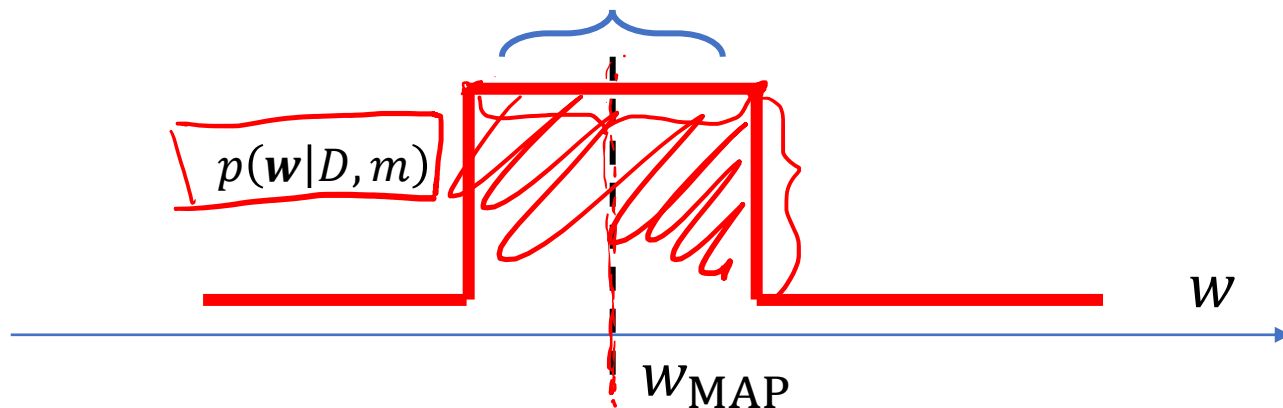
- Let us consider the simplest approximation of

- $p(D|m) = \int p(D|\mathbf{w}, m) p(\mathbf{w}|m) d\mathbf{w}$

- Note: $p(\mathbf{w}|D, m) \propto p(D|\mathbf{w}, m) p(\mathbf{w}|m)$

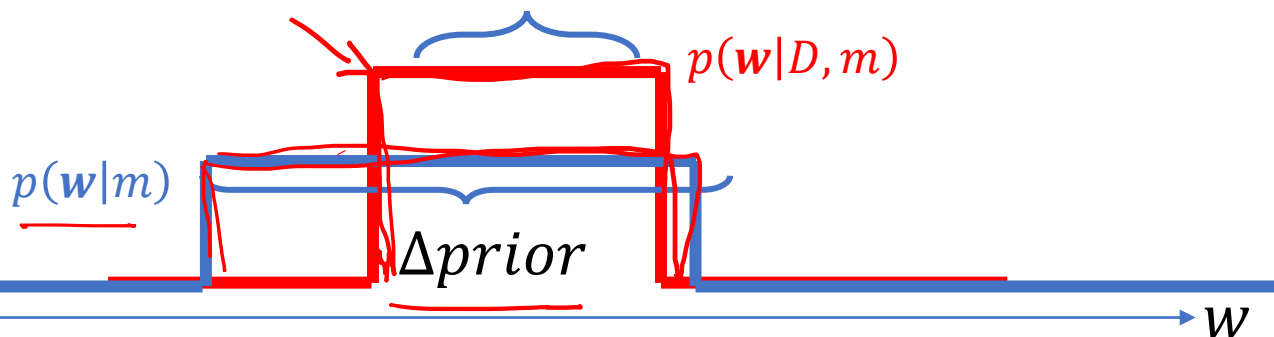
- Suppose $p(\mathbf{w}|D, m)$ plateaus at w_{MAP}

$\Delta_{\text{posterior}}$



Model Evidence Approximation

- Then $\int p(D|\mathbf{w}, m)p(\mathbf{w}|m)d\mathbf{w} \approx p(D|\mathbf{w}_{\text{MAP}}, m)p(\mathbf{w}_{\text{MAP}}|m) \cdot \Delta\text{posterior}$
 - as $\int f(x)dx \approx f(x_0) \cdot \Delta x$, if f can be approx. by a step function with “length” Δx peaks at x_0
 - If $p(\mathbf{w}|m) = \frac{1}{\Delta\text{prior}}$ is flat as well, then
- $p(D|m) \approx p(D|\mathbf{w}_{\text{MAP}}, m) \frac{\Delta\text{posterior}}{\Delta\text{prior}}$



Model Evidence Approximation

- $\log p(D|m) \approx \log p(D|w_{\text{MAP}}, m) + \log \frac{\Delta_{\text{posterior}}}{\Delta_{\text{prior}}}$

- As posterior is almost always sharper than prior,

$$\frac{\Delta_{\text{posterior}}}{\Delta_{\text{prior}}} < 1.$$

$P(w|D)$

$P(w)$

- The second term is always negative. In fact, **the sharper** our posterior is, **more negative** it is.

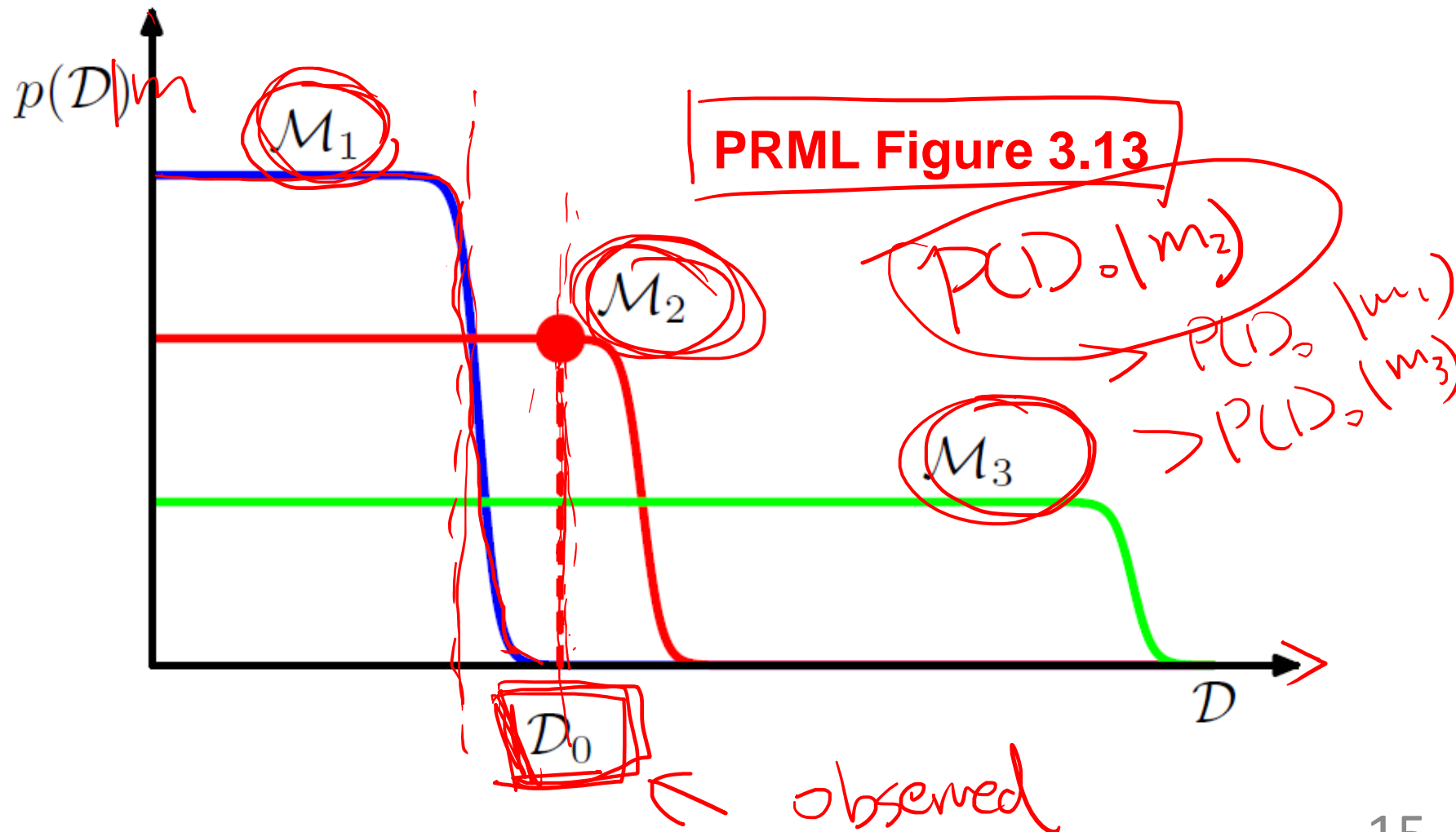
- Trade-off is made between $\log p(D|w_{\text{MAP}}, m)$ and

$$\log \frac{\Delta_{\text{posterior}}}{\Delta_{\text{prior}}}$$

Model Evidence Approximation

- Now, analyze a model with b parameters:
- Assuming $\frac{\Delta_{\text{posterior}}}{\Delta_{\text{prior}}}$ is the same for all w_i and w_i are independent
- $\log p(D|m) \approx \log p(D|w_{\text{MAP}}, m) + b \log \frac{\Delta_{\text{posterior}}}{\Delta_{\text{prior}}}$
- Why? Prove this.
- If too many parameters in a model, $b \log \frac{\Delta_{\text{posterior}}}{\Delta_{\text{prior}}}$ decreases!
 - $\log p(D|w_{\text{MAP}}, m)$ increases (why?).
- Model evidence prefers intermediate model complexity.

Model Evidence Prefers Intermediate Model Complexity



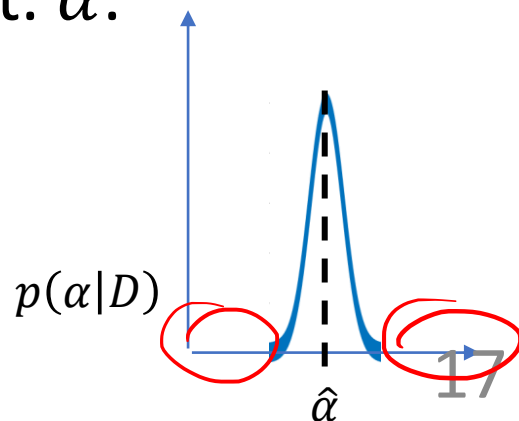
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- 126

Tuning Hyper Parameters

- 2 is hyper Para* $P(m|D)$
- We would like to calculate the predictive distribution:
 - $p(\hat{y}|D) = \int p(\hat{y}|D, \alpha) p(\alpha|D) d\alpha$
 $= \int \int p(\hat{y}|\mathbf{w}, \alpha) p(\mathbf{w}|D, \alpha) p(\alpha|D) d\mathbf{w} d\alpha$
 - However, integral w.r.t. α may not be easy (“intractable”).

- If $p(\alpha|D)$ is super “pointy” at $\hat{\alpha}$, we only need to use one parameter to approximate the integral w.r.t. α .

$$\int \int p(\hat{y}|\mathbf{w}, \alpha) p(\mathbf{w}|D, \alpha) p(\alpha|D) d\mathbf{w} d\alpha \approx \int p(\hat{y}|\mathbf{w}, \hat{\alpha}) p(\mathbf{w}|D, \hat{\alpha}) d\mathbf{w}$$



Model Evidence Approximation with Hyper Parameters

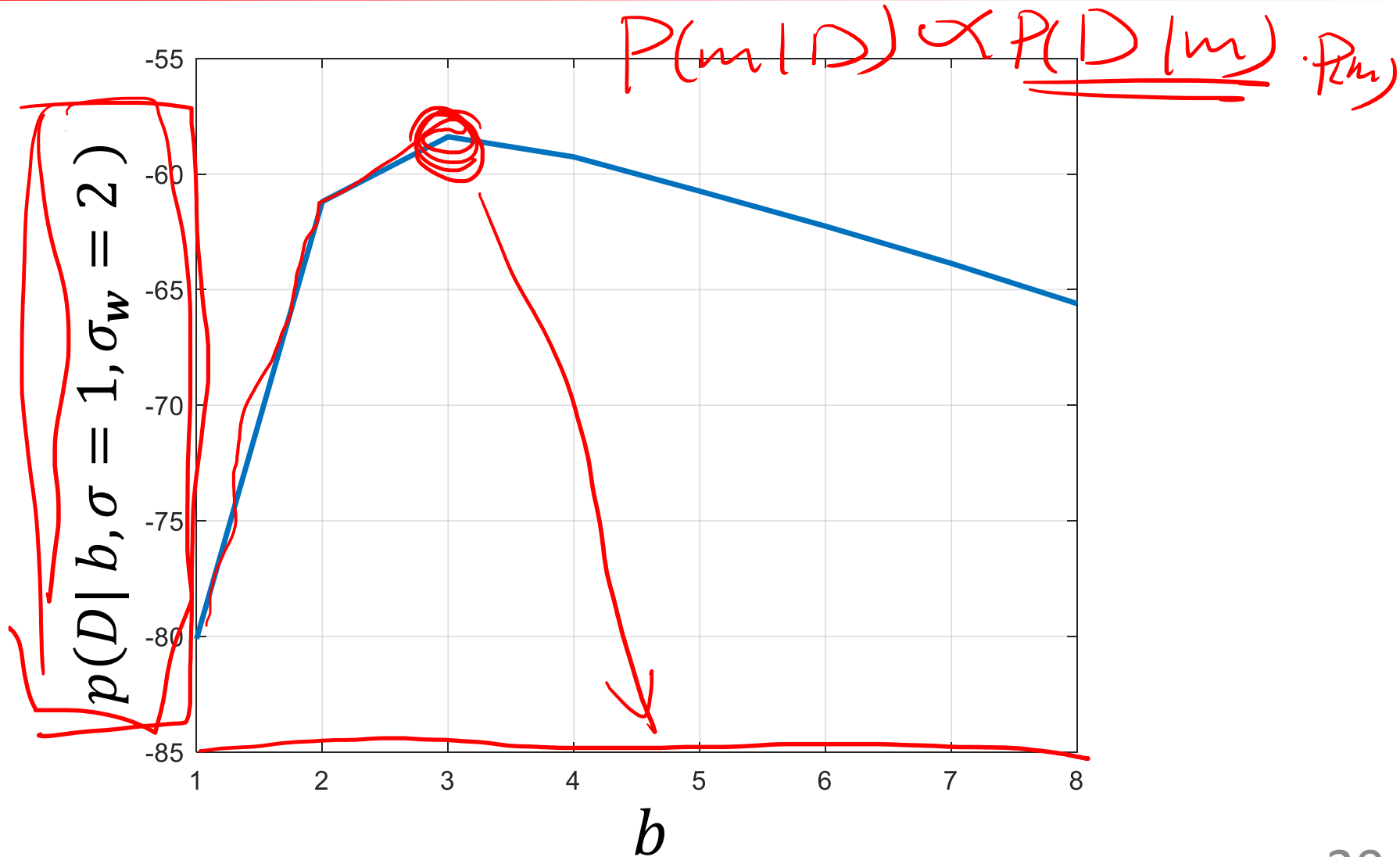
- To find $\hat{\alpha}$ at the peak, we need to maximize $p(\alpha|D)$
- $p(\alpha|D) \propto \underbrace{p(D|\alpha)p(\alpha)}_{\text{Model Evidence!}} = p(\alpha) \int p(D|\mathbf{w}, \alpha) p(\mathbf{w}|\alpha) d\mathbf{w}$
- If $p(\alpha)$ is relatively flat, we just
- $\hat{\alpha} := \underset{\alpha}{\operatorname{argmax}} \int p(D|\mathbf{w}, \alpha) p(\mathbf{w}|\alpha) d\mathbf{w}$
- “Marginalized Likelihood Maximization”
- Or “Evidence Approximation”

Example: Linear Regression

- Suppose we have a likelihood model:
- $p(\mathbf{y}_1 \dots \mathbf{y}_n | \mathbf{x}_1 \dots \mathbf{x}_n; \mathbf{w}, b) := \prod_{i \in D} N_{y_i}(\langle \mathbf{w}, \boldsymbol{\phi}_b(\mathbf{x}_i) \rangle, \sigma^2 \mathbf{I})$
- $p(\mathbf{w}; \sigma_w, b) := N_{\mathbf{w}}(\mathbf{0}, \sigma_w^2 \mathbf{I}_b)$
- Marginalized Likelihood
- $p(\mathbf{y}_1 \dots \mathbf{y}_n | \mathbf{x}_1 \dots \mathbf{x}_n; b, \sigma, \sigma_w)$
 $= \int p(\mathbf{y}_1 \dots \mathbf{y}_n | \mathbf{x}_1 \dots \mathbf{x}_n; \mathbf{w}, b, \sigma, \sigma_w) p(\mathbf{w}) d\mathbf{w}$
 $= N_y(\mathbf{0}, \sigma_w^2 \boldsymbol{\Phi}^\top \boldsymbol{\Phi} + \sigma^2 \mathbf{I})$

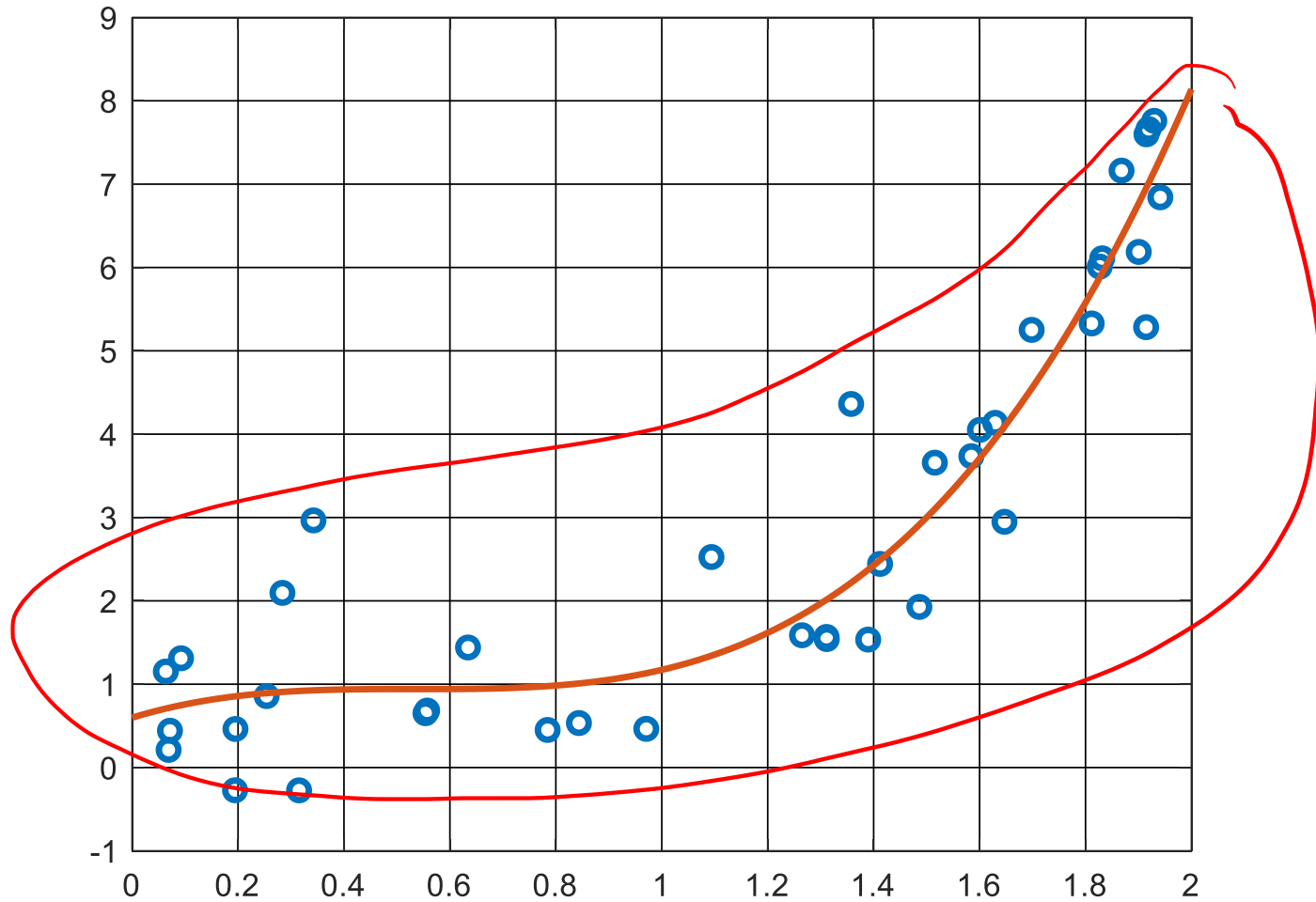
hint: use Gaussian identity!

Example: Linear Regression



Example: Linear Regression

$$f(x; w_{LS}), \underline{b = 3}$$



Conclusion

- We introduced probabilistic model selection.
- The principle: Integrate over models w.r.t. model posterior.

- $p(m|D) \propto p(D|m)p(m)$

- Approximation using flat posterior and prior of \mathbf{w} .
 - $p(D|m)$ decreases as b increase.
- Approximation using marginalized likelihood.
 - Allows us to select hyper-parameters

Homework

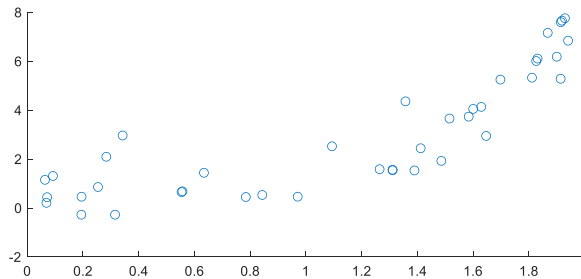
- Prove statement on page 14.
- Prove statement on page 19.
- Read PRML 3.52

Computing Lab

- Implementing least square regression with different choices of kernels:
 - Linear kernel,
 - Polynomial kernel,
 - RBF kernel.
- Apply it on prostate cancer dataset. What choice of kernel/kernel parameters minimizes the CV error?

Computing Lab

- Generate, $x \sim U(0,2)$, $y = \exp(1.5x-1) + \epsilon$, $\epsilon \sim N(0,1)$,



- Select number of basis using marginalized likelihood for different basis:
- Polynomial basis
- Trigonometric basis
- RBF basis

(Fix σ and σ_w)