Corollaries of BV Decomposition

Cor. 1 Assume additive noise ;

if $\exists\, w^*,\ g(x) = f(x; w^*) = \langle w^*, x \rangle$

then $(g(x) - E_\epsilon[f_{LS} | x_0])^2 = 0$

$$g^2 - 2g E_\epsilon[f_{LS}|x_0] + E_\epsilon^2[f_{LS}|x_0]$$

$$= g^2 - 2g E_\epsilon[x_0(XX^T)^{-1}X(g+\epsilon)] + E_\epsilon^2[x_0(XX^T)^{-1}X(g+\epsilon)]$$

$$= g^2 - 2g E_\epsilon[x_0(XX^T)^{-1}Xg] + E_\epsilon^2[x_0(XX^T)^{-1}Xg]$$

$$= g^2 - 2g E_\epsilon[x_0\underbrace{(XX^T)^{-1}XX^T}_{I}w^*] + E_\epsilon^2[x_0\underbrace{(XX^T)^{-1}XX^T}_{I}w^*]$$

$$= g^2 - 2g E_\epsilon[g] + E_\epsilon^2[g]$$

$$= 0$$

OR : show $g(x) - E_\epsilon[f_{LS}|x_0] = 0$ directly

Cor 2. $\mathrm{Var}[f_{LS}|x_0] = \langle h, h \rangle \sigma^2$

$h = x_0(XX^T)^{-1}X$

$$\mathrm{Var}[f_{LS}|x_0] = E_\epsilon[f_{LS}^2] - E_\epsilon^2[f_{LS}]$$

$$= E_\epsilon[\langle h, y\rangle^2] - E_\epsilon^2[hy^T]$$

$$= E_\epsilon[\langle h, g+\epsilon\rangle^2] - E_\epsilon^2[hy^T]$$

$$= E_\epsilon[\langle h, g\rangle^2 + \langle h, \epsilon\rangle^2 + \langle h, g\rangle\cdot\langle h, \epsilon\rangle] - E_\epsilon^2[\langle$$

$$= \langle h, g\rangle^2 + E_\epsilon[\langle h, \epsilon\rangle^2] + 0 - E_\epsilon^2[\langle h, g+\epsilon\rangle$$

$$= \langle h, g\rangle^2 - E_\epsilon^2[\langle h, g\rangle] + E_\epsilon[\langle h, \epsilon\rangle^2] + 0$$

$$= 0 + E_\epsilon[\langle h, \epsilon\rangle^2] + 0$$

$$= 0 + E_\epsilon[\mathrm{tr}[h^T\epsilon\epsilon^T h]]$$

$$= E_\epsilon[\mathrm{tr}[hh^T\epsilon\epsilon^T]] \Leftarrow \mathrm{tr}(\vec{a}\vec{b}) = \mathrm{tr}(\vec{b}\vec{a}^T)$$

$$= \mathrm{tr}[E_\epsilon[hh^T \cdot \epsilon^T\epsilon]]$$

$$= tr[E_\epsilon[hh^T \epsilon^T \epsilon]]$$

$$= tr[hh^T E_\epsilon[\epsilon^T \epsilon]]$$

$$= tr[hh^T \cdot \sigma^2 I] \quad \Leftarrow \text{quadratic rule}$$

$$I \in R^{n \times n}$$

$$= \sigma^2\, tr[h^T h] \quad \Leftarrow tr(ab^T) = tr[b^T a]$$

$$= \sigma^2 \cdot \langle h, h \rangle$$

Or, $$E_\epsilon[\langle h, \epsilon \rangle^2] = E_\epsilon[h^T \epsilon \epsilon^T h]$$

$$= h^T E_\epsilon[\epsilon \epsilon^T] h$$

$$= h^T \cdot h \cdot \sigma^2 = \langle h, h \rangle \cdot \sigma^2$$