Reflection Groups and Coxeter Groups - Humphreys

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1. Finite Reflection Groups

1.1. Reflections

1.1.1. Notation

$O(V)$ denotes the group of all orthogonal transformations of V

1.1.2. Remark

Each reflection s_α in W determines a reflecting hyperplane H_α and a line $L_\alpha = \mathbb{R}\alpha$ which is orthogonal to it.

1.1.3. *Proposition*

If $t \in O(V)$ and α is any nonzero vector in V, then $ts_\alpha t^{-1} = s_{t\alpha}$. In particular, if $w \in W$, then $s_{w\alpha}$ belongs to W whenever s_α does.

Proof:

□

1.2. Roots

1.2.1. Definition: Root System

Let Φ be a finite set of nonzero vectors in V satisfying:
1. $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$;
2. $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

Φ is called a **root system** and we let W be the group generated by the reflections $s_\alpha, \alpha \in \Phi$.

1.2.2. Remark

Any finite reflection group can be realized in this way (as a result of a root system). Conversely, any group W arising from a root system is finite.

Given a root system Φ and corresponding reflection group W, define Φ' to be the set of unit vectors proportional to the vectors in Φ. Then Φ' is a root sytem, with W as corresponding reflection group.

1.3. Positive and Simple Systems

1.3.1. Definition: Total Ordering of Real Vector Space

A **total ordering** of a real vector space V is a transitive relation on V ($<$) satisfying:
1. For any $\lambda, \mu \in V$ exactly one of $\lambda < \mu, \lambda = \mu, \mu < \lambda$ holds;
2. For any $\lambda, \mu, \nu \in V$, if $\mu < \nu$ then $\lambda + \mu < \lambda + \nu$;
3. If $\mu < \nu$ and c is a nonzero real number, then $c\mu < c\nu$ if $c > 0$ while $c\nu < c\mu$ if $c < 0$.

$\lambda \in V$ is **positive** if $0 < \lambda$.

1.3.2. Example

Choose arbitrary ordered basis $\lambda_1, ..., \lambda_n$ of V. Then define

$$\sum a_i \lambda_i < \sum b_i \lambda_i$$

if $a_k < b_k$ where k is the least index i for which $a_i \neq b_i$.

1.3.3. Definition: Positive System

For a root system Φ, a subset Π is a **positive system** if it consists of all those roots which are positive relative to some total ordering of V. We have that $-\Pi$ is a **negative system**.

1.3.4. Remark

Since roots come in pairs $\{\alpha, -\alpha\}$ we have that $\Phi = \Pi \sqcup (-\Pi)$.

1.3.5. Definition: Simple System

A subset Δ of a root system Φ is a simple system if Δ is a vector space basis for the $\mathbb{R}$-span of Φ in V and if moreover $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign.

The cardinality of a simple system is an invariant of Φ, since it measures the dimension of the span of Φ in V. This is called the **rank** of W.

1.3.6. ***Theorem***

1. If Δ is a simple system in Φ, then there is a unique positive system containing Δ.
2. Every positive system Π in Φ contains a unique simple system; in particular, simple systems exist.

Proof:

□

1.3.7. ***Corollary***

If Δ is a simple system in Φ, then $(\alpha, \beta) \leq 0$ for all $\alpha \neq \beta$ in Δ.

Proof:

$\square$

1.4. Conjugacy of Positive and Simple Systems

1.4.1. Remark

For any simple system Δ and $w \in W$ we have that $w\Delta$ is a simple system with corresponding positive system $w\Pi$.

1.4.2. ***Proposition***

Let Δ be a simple system, contained in the positive system Π. If $\alpha \in \Delta$, then $s_\alpha(\Pi \setminus \{\alpha\}) = \Pi \setminus \{\alpha\}$.

Proof:

$\square$

1.4.3. ***Theorem***

Any two positive (resp. simple) systems in Φ are conjugate under W.

Proof:

$\square$

1.5. Generation by Simple Reflections

1.5.1. Definition: Height

If $\beta \in \Phi$ we can write $\beta = \sum\limits_{\alpha \in \Delta} c_\alpha \alpha$ uniquely. We call $\sum c_\alpha$ the **height** of β (relative to Δ) abbreviated $\operatorname{ht}(\beta)$.

1.5.2. ***Theorem***

For a fixed simple system Δ, W is generated by the reflections $s_\alpha (\alpha \in \Delta)$.

Proof:

$\square$

1.5.3. ***Corollary***

Given Δ, for every $\beta \in \Phi$ there exists $w \in W$ such that $w\beta \in \Delta$.

Proof:

$\square$

1.6. The Length Function

1.6.1. Definition: Length

For any $w \in W$ the **length** $\ell(w)$ of w (relative to Δ) is the smallest r such that

$$w = s_1 \cdots s_r$$

where $s_i = s_{\alpha_i}$ with $\alpha_i \in \Delta$. The expression is called **reduced**. By convention we have $\ell(1) = 0$.

1.6.2. Remark

Note that $\ell(w) = 1$ if and only if $w = s_\alpha$ for some $\alpha \in \Delta$. Also note that $\ell(w) = \ell(w^{-1})$ and since we have $\det(s_\alpha) = -1$ we have that $\det(w) = (-1)^{\ell(w)}$. This implies that if w is written as the product of r reflections then r and $\ell(w)$ must have the same parity. Therefore $\ell(s_\alpha w) \in \{\ell(w)+1, \ell(w)-1\}$.

1.6.3. Notation

For a fixed simple system Δ and corresponding positive system Π, define

$$n(w) := \mathrm{Card}(\Pi \cap w^{-1}(-\Pi)) = \text{number of positive roots sent to negative roots by } w.$$

Note we have $n(w^{-1}) = n(w)$.

1.6.4. ***Lemma***

Let $\alpha \in \Delta, w \in W$. Then:
1. $w\alpha > 0 \Longrightarrow n(s_\alpha w) = n(w) + 1$
2. $w\alpha < 0 \Longrightarrow n(s_\alpha w) = n(w) - 1$
1. $w^{-1}\alpha > 0 \Longrightarrow n(ws_\alpha) = n(w) + 1$
2. $w^{-1}\alpha < 0 \Longrightarrow n(ws_\alpha) = n(w) - 1$

Proof:

□

1.6.5. ***Corollary***

If $w \in W$ is written as $w = s_1 \cdots s_r$, then $n(w) \leq r$. In particular, $n(w) \leq \ell(w)$.

Proof:

□

1.7. Deletion and Exchange Conditions

1.7.1. ***Theorem***

Fix a simple system Δ. Let $w = s_1 \cdots s_r$ be any expression of $w \in W$ as a product of simple reflections ($s_i = s_{\alpha_i}$ with repetitions permitted). Suppose $n(w) < r$. Then there exist indices $1 \leq i < j \leq r$ satisfying:
1. $\alpha_i = (s_{i+1} \cdots s_{j-1})\alpha_j$;
2. $s_{i+1}s_{i+2} \cdots s_j = s_i s_{i+1} \cdots s_{j-1}$;
3. $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$ where hat denotes omission (**deletion condition**).

Proof:

□

1.7.2. ***Corollary***

If $w \in W$, then $n(w) = \ell(w)$.

Proof:

□

1.7.3. Remark

Let $\Pi(w) \coloneqq \Pi \cap w^{-1}(-\Pi)$ and let

$$\beta_i \coloneqq s_r s_{r-1} \cdots s_{i+1}(\alpha_i), \quad \beta_r \coloneqq \alpha_r$$

where $w = s_1 \cdots s_r$ is a reduced expression and $s_i = s_{\alpha_i}$. Then $\Pi(w) = \{\beta_1, \ldots, \beta_r\}$ where β_i is distinct.

1.7.4. *Proposition:* Exchange Condition

Let $w = s_1 \cdots s_r$, where each s_i is a simple reflection. If $\ell(ws) < \ell(w)$ for some simple reflection $s = s_\alpha$, then there exists an index i for which $ws = s_1 \cdots \hat{s}_i \cdots s_r$. In particular, w has a reduced expression ending in s if and only if $\ell(ws) < \ell(w)$.

Proof:

□

1.8. Simple Transitivity and the Longest Element

1.8.1. *Theorem*

Let Δ be a simple system, Π the corresponding positive system. The following conditions on $w \in W$ are equivalent:

1. $w\Pi = \Pi$;
2. $w\Delta = \Delta$;
3. $n(w) = 0$;
4. $\ell(w) = 0$;
5. $w = 1$.

Proof:

□

1.9. Generators and Relations

1.9.1. Notation

For any roots α, β we say $m(\alpha, \beta)$ is the order of $s_\alpha s_\beta$ in W.

1.9.2. ***Theorem***

Fix a simple system Δ in Φ. Then W is generated by the set $S := \{s_\alpha, \alpha \in \Delta\}$, subject only to the relations:

$$\left(s_\alpha s_\beta\right)^{m(\alpha,\beta)} = 1.$$

Proof:

$\square$

1.9.3. Definition: Coxeter System

The tuple (W, S) is called a **Coxeter system** where W is the group generated by the set S and is therefore called a **Coxeter group**.

It is required that $m(\alpha, \alpha) = 1$ but a relation $\left(s_\alpha s_\beta\right)^{m(\alpha,\beta)} = 1$ may be omitted to allow infinite order.

1.10. Parabolic subgroups and Minimal Coset Representatives

1.10.1. Definition: Parabolic Subgroup

For a given simple system Δ let S be the set of simple reflections $s_\alpha, \alpha \in \Delta$. For any subset $I \subseteq S$ define W_I to be the subgroup of W generated by all $s_\alpha \in I$ and let $\Delta_I := \{\alpha \in \Delta \mid s_\alpha \in I\}$. W_I is called a **parabolic subgroup**.

Note that $\Delta \to w\Delta$ then we have $W_I \to wW_I w^{-1}$.

1.10.2. ***Proposition***

Fix a simple system Δ and the corresponding set S of simple reflections. Let $I \subseteq S$, and define Φ_I to be the intersection of Φ with the $\mathbb{R}$-span V_I of Δ_I in V.

1. Φ_I is a root system in V (resp. V_I) with simplie system Δ_I and with corresponding reflection group W_I (resp W_I restricted to V_I).
2. Viewing W_I as a reflection group, with length function ℓ_I relative to the simple system Δ_I, we have $\ell = \ell_I$ on W_I.
3. Define $W^I := \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in I\}$. Given $w \in W$, there is a unique $u \in W^I$ and a unique $v \in W_I$ such that $w = uv$. Their lengths satisfy $\ell(w) = \ell(u) + \ell(v)$. Moreover, u is the unique element of smallest length in the coset wW_I.

Proof:

$\square$

1.10.3. Definition: Minimal Coset Representatives

The distinguished coset representatives W^I may be called **minimal coset** representatives.

1.11. Poincare Polynomials

1.11.1. Definition: Poincare Polynomial

Define the following sequence

$$a_n := \text{Card } \{w \in W \mid \ell(w) = n\}.$$

Then define the polynomial

$$W(t) := \sum_{n \geq 0} a_n t^n = \sum_{w \in W} t^{\ell(w)}.$$

$W(t)$ is called the **Poincare polynomial** of W.

1.11.2. Example

Let $W = S_3$ such that $W(t) = 1 + 2t + 2t^2 + t^3$.

1.11.3. Remark

Note we have that

$$W(t) = W_I(t) W^I(t).$$

This can be used to derive $W(t)$ via an algorithm by induction on $|S|$.

1.11.4. Notation

Let $(-1)^I = (-1)^{|I|}$.

1.11.5. ***Proposition***

$$\sum_{I \subseteq S} (-1)^I \frac{W(t)}{W_I(t)} = \sum_{I \subseteq S} (-1)^I W^I(t) = t^N$$

where $N := |\Pi|$ is the length of the longest element in W.

Proof:

□

1.12. Fundamental Domains

1.12.1. Definition: Fundamental Domain

Let Π be a positive system and H_α be a hyperplane with respect to some root α. There exists corresponding open half-spaces A_α and $-A_\alpha$ where

$$A_\alpha := \{\lambda \in V \mid (\lambda, \alpha) > 0\}.$$

Then define

$$C := \bigcap_{\alpha \in \Delta} A_\alpha$$

which is open and convex since A_α is open and convex. Note C is also a cone (closed under positive scalar multiples). Then let $D = \overline{C}$ be the intersection of closed half-spaces $H_\alpha \cup A_\alpha$. So

$$D = \{\lambda \in V \mid (\lambda, \alpha) \geq 0 \text{ for all } \alpha \in \Delta\}.$$

D is a closed convex cone. D is a **fundamental domain** for the action of W on V, i.e., each $\lambda \in V$ is conjugate under W to one and only one point in D.

1.12.2. *Lemma*

Each $\lambda \in V$ is W-conjugate to some $\mu \in D$. Moreover, $\mu - \lambda$ is a nonnegative $\mathbb{R}$-linear combination of Δ.

Proof:

□

1.12.3. Definition: Isotropy Group

The **isotropy group** (or **stabilizer**) of an element $\mu \in V$ is defined by $\{w \in W \mid w\mu = \mu\}$.

1.12.4. *Theorem*

Fix $\Pi \supseteq \Delta$ (hence D), as above.
1. If $w\lambda = \mu$ for $\lambda, \mu \in D$, then $\lambda = \mu$ and w is a product of simple reflections fixing λ. In particular, if $\lambda \in C$, then the isotropy group of λ is trivial.
2. D is a fundamental domain for the action of W on V.
3. If $\lambda \in V$, the isotropy group of λ if generated by those reflections $s_\alpha (\alpha \in \Phi)$ which it contains.
4. If U is any subset of V, then the subgroup of W fixing U pointwise is generated by those reflections s_α which it contains.

Proof:

□

1.12.5. Remark

W exhibits a simply transitive action on a family of open sets. This family is the connected components of the complement in V of $\bigcup_\alpha H_\alpha$ and are called **chambers**.

Given a chamber C corresponding to a simple system Δ, its **walls** are defined to be the hyperplanes $H_\alpha(\alpha \in \Delta)$. Each wall has a 'positive' or 'negative' side (with C lying on the positive side). Then the roots in Δ can be characterized as those roots which are orthogonal to some wall of C and positively directed.

1.13. The Latiice of Parabolic Subgroups

1.13.1. *Proposition*

Under the correspondence $I \mapsto W_I$, the collection of parabolic subgroups $W_I(I \subseteq S)$ is isomorphic to the lattice of subsets of S.

Proof:

□

1.14. Reflections in W

1.14.1. *Proposition*

Every reflection in W is of the form s_α for some $\alpha \in \Phi$.

Proof:

□

1.15. The Coxeter Complex

1.15.1. Definition: Coxeter Complex

Fix a simple system Δ and the corresponding set S of simple reflections. For any subset I of S define

$$C_I = \{\lambda \in D \mid (\lambda, \alpha) = 0 \text{ for all } \alpha \in \Delta_I, (\lambda, \alpha) > 0 \text{ for all } \alpha \in \Delta \setminus \Delta_I\}$$

where D is a fundamental domain. The sets C_I partition D, with $C_\emptyset = C$ and $C_S = \{0\}$. V is partitioned by $\mathcal{C} = wC_I(w \in W, I \subseteq S)$. wC_I and $w'C_I$ are disjoint unless w and w' lie in the same left coset in W/W_I in which case they are equal. For distinct I and J we have wC_I and $w'C_J$ are always disjoint. We call $\mathcal{C}$ the **Coxeter complex** of W. Any set wC_I is called a **facet** of type I.

1.15.2. ***Proposition***

For each $I \subseteq S$, the isotropy group of the facet C_I of $\mathcal{C}$ is precisely W_I. Thus the parabolic subgroups of W are the isotropy groups of the elements of $\mathcal{C}$.

Proof:

□

1.15.3. Remark

Note you can interpret $\mathcal{C}$ as an abstract simplicial complex where the vertices are the left cosets wW_I, where I is maximal in S. A finite set of vertices determines a 'simplex' if these vertices have a nonempty intersection.

1.16. An Alternating Sum Formula

1.16.1. Note

Let $H_1, …, H_r$ be an arbitrary collection of hyperplanes in V (dimension n). Each hyperplane $H = H^0$ defines a positive half-space H^+ and a negative half-space H^-. An element of the complex $\mathcal{K}$ is a nonempty intersection of the form

$$K = \bigcap H_i^{\varepsilon_i}, \quad \varepsilon_i \in \{0, +, -\}.$$

dim $K = i$ if the linear span has dimension i, where the linear span L is the intersection of all H_i^0 which occur in the definition of K.

1.16.2. ***Lemma***

Denote by n_i the number of elements of $\mathcal{K}$ having dimension i. Then

$$\sum_i (-1)^i n_i = (-1)^n.$$

Proof:

□

1.16.3. ***Proposition***

$$\sum_{I \subseteq S} (-1)^I f_I(w) = \det(w).$$

Proof:

$\square$

2. Classification of Finite Reflection Groups

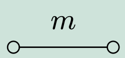
2.1. Isomorphisms

2.1.1. Definition: Coxeter Graph

For a Coxeter group W generated by a simple system Δ, let Γ be a graph with vertex set $V \cong \Delta$. Edges are given for roots $\alpha \neq \beta$ where $m(\alpha, \beta) \geq 3$ labeled with $m(\alpha, \beta)$. Note since simple systems are conjugate, Γ does not depend on the choice Δ.

2.1.2. Example

For the dihedral group $W = D_m$ we have



For $W = S_{n+1}$ then with n vertices we have

2.1.3. *Proposition*

For $i = 1, 2$ let W_i be a finite reflection group acting on the euclidean space V_i. Assume W_i is essential. If W_1 and W_2 have the same Coxeter graph, then there is an isometry of V_1 onto V_2 inducing an isomorphism of W_1 onto W_2. (In particular, if $V_1 = V_2$, the subgroups W_1 and W_2 are conjugate in $O(V)$.)

Proof:

$\square$

2.2. Irreducible Components

2.2.1. Definition: Irreducible

A Coxeter system (W, S) is **irreducible** if the Coxeter graph Γ is connected (Φ is also called irreducible in this case).

2.2.2. *Proposition*

Let (W, S) have Coxeter graph Γ, with connected components $\Gamma_1, ..., \Gamma_r$, and let $S_1, ..., S_r$ be the corresponding subsets of S. Then W is the direct product of the parabolic subgroups $W_{S_1}, ..., W_{S_r}$, and each Coxeter system $\left(W_{S_i}, S_i\right)$ is irreducible.

Proof:

□

2.3. Coxeter Graphs and Associated Bilinear Forms

2.3.1. Remark

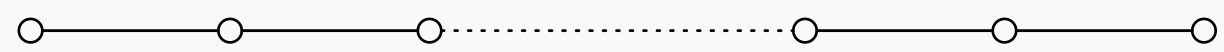
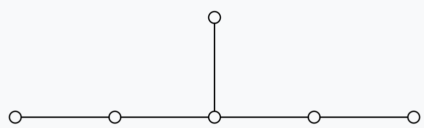
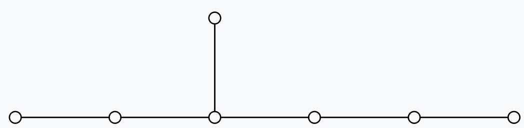
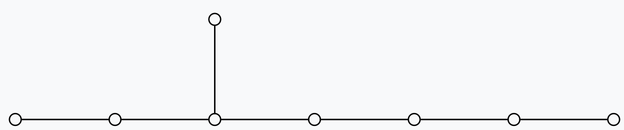
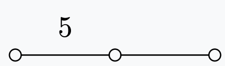
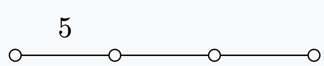
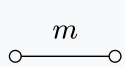
We associate to a Coxeter graph Γ with vertex set S of cardinality n a symmetric $n \times n$ matrix A by setting $a(s, s') := -\cos\left(\frac{\pi}{m(s,s')}\right)$. This defines a bilinear form $x^t A y$ for any $x, y \in \mathbb{R}^n$.

We call Γ positive definite or positive semidefinite when A has the corresponding property. Also note that A is positive definite (resp. semidefinite) if and only if all its principala minors are positive (resp. nonnegative).

When Γ is derived from a finite reflection group W, the matrix A is positive definite since it represents the standard euclidean inner product relative to the basis Δ of V.

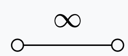
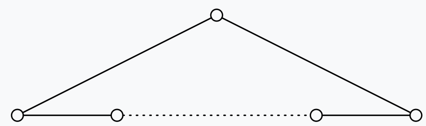
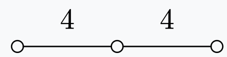
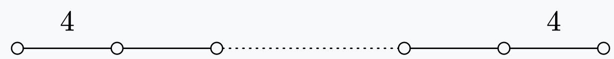
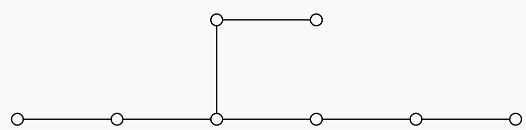
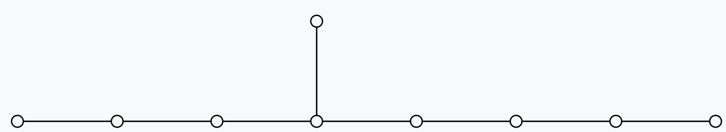
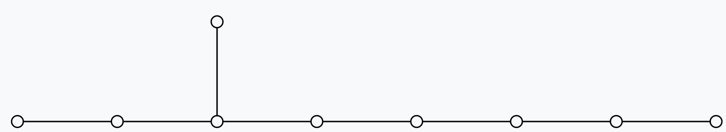
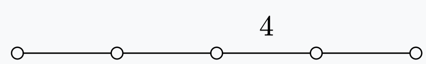
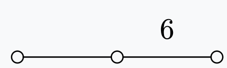
2.4. Some Positive Definite Graphs

2.4.1. Note

$A_n (n \geq 1)$	
$B_n (n \geq 2)$	
$D_n (n \geq 4)$	
E_6	
E_7	
E_8	
F_4	
H_3	
H_4	
$I_2(m)$	

2.5. Some Positive Semidefinite Graphs

2.5.1. Note

$\widetilde{A}_1$	
$\widetilde{A}_n (n \geq 2)$	
$\widetilde{B}_2 = \widetilde{C}_2$	
$\widetilde{B}_n (n \geq 3)$	
$\widetilde{C}_n (n \geq 3)$	
$\widetilde{D}_n (n \geq 4)$	
$\widetilde{E}_6$	
$\widetilde{E}_7$	
$\widetilde{E}_8$	
$\tilde{F}_4$	
$\widetilde{G}_2$	

2.6. Subgraphs

2.6.1. Definition: Subgraph

A subgraph of a Coxeter graph Γ is a graph with some vertices omitted, some of the edges label's decremented, or both.

2.6.2. Definition: Indecomposable

A real $n \times n$ matrix A is **indecomposable** if there is no partition on the index set into nonempty subsets I, J such that $a_{ij} = 0$ whenever $i \in I, j \in J$.

2.6.3. Remark

Coxeter graph is indecomposable precisely when the graph is connected.

2.6.4. ***Proposition***

Let A be a real symmetric $n \times n$ matrix which is positive semidefinite and indecomposable (in particular, the eigenvalues of A are real and nonnegative). Assume $a_{ij} \leq 0$ whenever $i \neq j$. Then:

1. $N := \{x \in \mathbb{R}^n \mid x^t A x = 0\}$ coincides with the nullspace of A and has dimension ≤ 1.
2. The smallest eigenvalue of A has multiplicity 1, and has an eigenvector whose coordinates are strictly positive.

Proof:

□

2.6.5. ***Corollary***

If Γ is a connected Coxeter graph of positive type, then every (proper) subgraph is positive definite.

Proof:

□

2.7. Classification of Graphs of Positive Type

2.7.1. ***Theorem***

The previous positive definite and positive semidefinite graphs are the only connected Coxeter graphs of positive type.

Proof:

□

2.8. Crystallographic Groups

2.8.1. Definition: Crystallographic Group

A subgroup G of $\mathrm{GL}(V)$ is crystallographic if it stabilizes a lattice in V (the $\mathbb{Z}$-span of a basis of V): $gL \subseteq L$ for all $g \in G$.

2.8.2. ***Proposition***

If W is crystallographic, then each integer $m(\alpha, \beta)$ must be 2, 3, 4, or 6 when $\alpha \neq \beta$ in Δ.

Proof:

□

2.8.3. Remark

Note this implies groups of type H_3 and H_4 as well of D_n for $n = 2, 4, 6, 8, 12$ are not crystallographic.

2.9. Crystallographic Root Systems and Weyl Groups

2.9.1. Definition: Weyl Group

A root system Φ is **crystallographic** if it satisfies the additional requirement:

$$\frac{2(\alpha,\beta)}{\beta,\beta} \in \mathbb{Z}, \forall \alpha, \beta \in \Phi.$$

These integers are called **Cartan integers**. The group W generated by all reflections $s_\alpha(\alpha \in \Phi)$ is known as the **Weyl group** of Φ.

This requirement ensures all roots are $\mathbb{Z}$-linear combinations of Δ, and that the $\mathbb{Z}$-span of Δ in V is a W-stable lattice.

2.9.2. Definition: Dual Root System

Let $a^\vee := 2\alpha/(\alpha,\alpha)$ and let $\Phi^\vee$ of all **coroots** $\alpha^\vee(\alpha \in \Phi)$. $\Phi^\vee$ is also a crystallographic root system in V, with simple system $\Delta^\vee := \{\alpha^\vee \mid \alpha \in \Delta\}$. This is also called the inverse or dual root system.

Short roots α in a system Φ of type B_n give rise to long roots $\alpha^\vee$ in the system $\Phi^\vee$ of type C_n (and vice versa).

2.9.3. Definition: Root Lattice

For a root system Φ in V the $\mathbb{Z}$-span $L(\Phi)$ is called the **root lattice**. The **coroot lattice** is defined as $L(\Phi^\vee)$. Both are W-stable.

The **weight lattice** and **coweight lattice** are defined as

$$\hat{L}(\Phi) := \{\lambda \in V \mid (\lambda, \alpha^\vee) \in \mathbb{Z}, \forall \alpha \in \Phi\}$$
$$\hat{L}(\Phi^\vee) := \{\lambda \in V \mid (\lambda, \alpha) \in \mathbb{Z}, \forall \alpha \in \Phi\}$$

2.9.4. Remark

$\hat{L}(\Phi)$ is a subgroup of $L(\Phi)$ with finite index f and similarly $\hat{L}(\Phi^\vee)$ is a subgroup of $L(\Phi^\vee)$. Where f is the determinant of the matrix of Cartan integers $(\alpha, \beta^\vee)$ for any $\alpha, \beta \in \Delta$. f is also called the **index of connection** in Lie theory.

$\hat{L}/L$ is isomorphic to the fundamental group of a compact Lie group of adjoint type having W as Weyl group.

2.9.5. Remark

A partial ordering on V (if Δ is fixed) exists: $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a nonnegative $\mathbb{Z}$-linear combination of Δ.

If Φ is irreducible then there exists a unique highest long root $\tilde{\alpha}$ and unique highest short root.

2.10. Construction of Root Systems

2.10.1. Note

$(A_n, n \geq 1)$ Let V be a hyperplan in $\mathbb{R}^{n+1}$ consisting of vectors whose coordinates add up to 0. Φ is the set $\{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n+1\}$ and $\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, ..., \varepsilon_n - \varepsilon_{n+1}\}$. Then $\tilde{\alpha} = \varepsilon_1 - \varepsilon_{n+1}$. W is S_{n+1} which acts by permutting ε_i.

2.11. Computing the Order of W

2.11.1. Note

Long roots form a single W-orbit (when W is irreducible). $(\tilde{\alpha}, \alpha) \geq 0$ for any $\alpha \in \Delta$ thus $\tilde{\alpha}$ lies in the fundamental domain D. This gives an inductive method for calculating $\operatorname{ord}(W)$ via the Orbit-Stabilizer theorem.

2.12. Exceptional Weyl Groups

2.12.1. Note

F_4 is the group of symmetries of a regular solid in $\mathbb{R}^4$ having 24 faces which are octahedra.

E_6 is the group of automorphisms of the configuration of 27 lines on a cubic surface.

E_7 has the following interesting properties: $L(\Phi)/2L(\Phi)$ is a seven-dimensional vector space of $\mathbb{F}_2$, $L(\Phi)/2\hat{L}(\Phi)$ is a six-dimensional vector space over F_2. Both have interesting inner product relations.

E_8 has $L(\Phi)/2L(\Phi)$ is an eight-dimensional vector space over $\mathbb{F}_2$. The inner product has interesting properties as well.

2.13. Groups of Types H_3 and H_4

2.13.1. Note

H_3 is the symmetry group of the icosahedron (20 triangular faces) in $\mathbb{R}^3$ (order of 120)

H_4 is the symmetry group of a regular 120-sided solid (with dodecadegral faces) in $\mathbb{R}^4$ (order of 14400).

2.13.2. ***Lemma***

Any finite subgroup G of even order in $\mathbb{H}$ is a root system (when regarded as a subset of $\mathbb{R}^4$).

Proof:

$\square$

3. Polynomial Invariants of Finite Reflection Groups

4. Affine Reflection Groups

4.1. Affine Reflections

4.1.1. Definition: Affine Reflection

The **affine group** of V, denoted $\mathrm{Aff}(V)$, is the semidirect product of $\mathrm{GL}(V)$ and the group of translations by elements of V.

$$g t(\lambda) g^{-1} = t(g\lambda)$$

for any $g \in \mathrm{GL}(V)$, $\lambda \in V$, and t any translation. This shows the group of translations is normalized by $\mathrm{GL}(V)$.

Define the affine hyperplane for a root α and integer k

$$H_{\alpha,k} = \{\lambda \in V \mid (\lambda, \alpha) = k\}.$$

$H_{\alpha,k}$ can be attained by translating the hyperplane H_α by $\frac{k}{2}\alpha^\vee$. Define $\mathcal{H}$ to be the collection of $H_{\alpha,k}$ for any $\alpha \in \Phi, k \in \mathbb{Z}$.

Define the **affine reflection** as

$$s_{\alpha,k}(\lambda) \coloneqq \lambda - ((\lambda, \alpha) - k)\alpha^\vee.$$

Note that $s_{\alpha,k} = t(k\alpha^\vee)s_\alpha$.

4.1.2. *Proposition*

1. If $w \in W$, then $wH_{\alpha,k} = H_{w\alpha,k}$ and $ws_{\alpha,k}w^{-1} = s_{w\alpha,k}$.
2. If $\lambda \in V$ satisfies $(\lambda, \alpha) \in \mathbb{Z}$ for all roots α, then $t(\lambda)H_{\alpha,k} = H_{\alpha,k+(\lambda,\alpha)}$ and $t(\lambda)s_{\alpha,k}t(-\lambda) = s_{\alpha,k+(\lambda,\alpha)}$.

Proof:

□

4.2. Affine Weyl Groups

4.2.1. Definition: Affine Weyl group

Define the **affine Weyl group** W_a to be the subgroup of $\mathrm{Aff}(V)$ generated by all affine reflections $s_{\alpha,k}$, where $\alpha \in \Phi, k \in \mathbb{Z}$.

4.2.2. Example: Infinite Dihedral group

The infinite dihedral group is the affine reflection group generated by $s_{\alpha,0} = s_\alpha$ and $s_{\alpha,1}$.

4.2.3. *Proposition*

W_a is the semidirect product of W and the translation group corresponding to the coroot lattice $L = L(\Phi^\vee)$.

Proof:

□

4.2.4. Remark

W also normalizes $\hat{L}(\Phi^\vee)$ such that we can define a semidirect product $\widehat{W_a}$ that contains W_a as a normal subgroup of finite index. $\widehat{W_a}/W_a$ is isomorphic to $\hat{L}/L$.

4.2.5. *Corollary*

If $w \in \widehat{W_a}$ and $H_{\alpha,k} \in \mathcal{H}$, then $wH_{\alpha,k} = H_{\beta,l}$ for some $\beta \in \Phi, l \in \mathbb{Z}$, and thus $ws_{\alpha,k}w^{-1} = s_{\beta,l}$.

Proof:

□

4.3. Alcoves

4.3.1. Definition: Alcoves

We define $V^\circ = V/\bigcup_{H\in\mathcal{H}} H$. The connected components in V°, $\mathcal{A}$, are called **alcoves**. $\widehat{W_a}$ permutes $\mathcal{A}$.

A specific alcove is of interest when Φ is irreducible

$$\begin{aligned} A_\circ &= \{\lambda \in V \mid 0 < \langle \lambda, \alpha \rangle < 1 \text{ for all } \alpha \in \Phi^+\} \\ &= \{\lambda \in V \mid 0 < \langle \lambda, \alpha \rangle \text{ for all } \alpha \in \Delta, \langle \lambda, \tilde{\alpha} \rangle < 1\} \end{aligned}$$

for a unique highest root such that $\tilde{\alpha} - \alpha$ is a sum of simple roots. The walls of $A_\circ$ are given by H_α for every $\alpha \in \Delta$ and $H_{\tilde{\alpha},1}$ and the corresponding reflections to be $S_a := \{s_\alpha, \alpha \in \Delta\} \cup \left\{s_{\tilde{\alpha},1}\right\}$. The walls of $wA_\circ$ can be defined as images of these hyperplanes under w for any $w \in W_a$.

4.3.2. *Proposition*

The group W_a permutes the collection $\mathcal{A}$ of all alcoves transitively, and is generated by the set S_a of reflections with respect to the walls of the alcove $A_\circ$.

Proof:

□

4.3.3. Definition: Length

S_a generates W_a so we can define the **length** $\ell(w)$ of an element $w \in W_a$ to be the smallest r for which w is a product of r elements of S_a.

4.4. Counting Hyperplanes

4.4.1. Remark

Define $\mathcal{L}(w) := \{H \in \mathcal{H} \mid H \text{ separates } A_\circ \text{ and } wA_\circ\}$ and define $n(w) = |\mathcal{L}(w)|$.

The restriction of n to W_a (instead of its domain of definition $\widehat{W_a}$) is equivalent to ℓ.

4.4.2. *Proposition*

Let $w \in \widehat{W_a}$ and fix $s \in S_a$.
1. H_s belongs to exactly one of the set $\mathcal{L}(w^{-1})$, $\mathcal{L}(sw^{-1})$.
2. $s(\mathcal{L}(w^{-1}) \setminus \{H_s\}) = \mathcal{L}(sw^{-1}) \setminus \{H_s\}$.
3. $n(ws) = n(w) - 1$ if $H_s \in \mathcal{L}(w^{-1})$, and $n(ws) = n(w) + 1$ otherwise.

Proof:

□

4.4.3. *Corollary*

For any $w \in W_a$, we have $n(w) \leq \ell(w)$.

Proof:

□

4.5. Simple Transitivity

4.5.1. ***Lemma***

If $w \neq 1$ in W_a has a reduced expression $w = s_1 \cdots s_r$, with $s_i \in S_a$, then (setting $H_i := H_{s_i}$) the hyperplanes

$$H_1, s_1 H_2, s_1 s_2 H_3, \ldots, s_1 \cdots s_{r-1} H_r$$

are all distinct.

Proof:

$\square$

4.5.2. ***Theorem***

1. Let $w \neq 1$ in W_a have a reduced expression $w = s_1 \cdots s_r$. Then we have
$$\mathcal{L} = \{H_1, s_1 H_2, s_1 s_2 H_3, \ldots, s_1 \cdots s_{r-1} H_r\}.$$
Moreover, these r hyperplanes are all distinct.
2. The function n on W_a coincides with the length function ℓ.
3. The group W_a acts simply transitively on $\mathcal{A}$.

Proof:

$\square$

4.6. Exchange Condition

4.6.1. ***Theorem:*** **Exchange Condition**

Let $w \in W_a$ have a reduced expression $w = s_1 \cdots s_r$, with $s_i \in S_a$. If $\ell(ws) < \ell(w)$ for $s \in S_a$, then there exists an index $1 \leq i \leq r$ for which $ws = s_1 \cdots \hat{s}_i \cdots s_r$.

Proof:

$\square$

4.6.2. *Theorem*

The pair (W_a, S_a) is a Coxeter system.

Proof:

$\square$

4.7. Coxeter graphs and extended Dynkin diagrams

4.8. Fundamental domain

4.8.1. *Theorem*

The closure of $A_\circ$ is a fundamental domain for the action of W_a on V.

Proof:

$\square$

4.9. A formula for the order of W

4.9.1. *Theorem*

If W is an irreducible Weyl group of rank n, then

$$\operatorname{ord}(W) = n! c_1 \cdots c_n f$$

where f is the index of connection and c_i are the coefficients of the highest root where

$$\tilde{\alpha} = \sum c_i \alpha_i$$

for $\alpha_i \in \Delta$.

Proof:

$\square$

5. Coxeter Groups

5.1. Coxeter systems

5.1.1. Definition: Coxeter system

A **Coxeter system** to be a pair (W, S) consisting of a group W and a set of generators $S \subseteq W$ subject only to relations of the form $(ss')^{m(s,s')} = 1$ where $m(s,s) = 1$ and $m(s,s') = m(s',s) \geq 2$ for $s \neq s'$ in S. If no relation exists between s and s' we have $m(s,s') = \infty$.

We call $|S|$ the **rank** of (W, S). W is referred to as the **Coxeter group**. It is typically assumed that S is finite but it is not required. A **Coxeter graph** Γ is drawn by treating S as the set of vertices and a weighted edge with the weight $m(s,s')$ if $m(s,s') \geq 3$.

5.1.2. ***Proposition***

There is a unique epimorphism $\varepsilon : W \to \{1, -1\}$ sending each generator $s \in S$ to -1. In particular, each s has order 2 in W.

Proof:

□

5.2. Length function

5.2.1. Definition: Length

A **length** of $w \in W$ as the number of $s_i \in S$ such that $w = s_1 \cdots s_r$ is a **reduced expression**. Some of the following properties hold:

1. $\ell(w) = \ell(w^{-1})$
2. $\ell(w) = 1$ if and only if $w \in S$
3. $\ell(ww') \leq \ell(w) + \ell(w')$
4. $\ell(ww') \geq \ell(w) - \ell(w')$
5. $\ell(w) - 1 \leq \ell(ws) \leq \ell(w) + 1$

Also note that $\ell(1) = 0$ by convention.

5.2.2. *Proposition*

The homomorphism $\varepsilon : W \to \{\pm 1\}$ is given by $\varepsilon(w) = (-1)^{\ell(w)}$. As a result, $\ell(ws) = \ell(w) \pm 1$, for all $s \in S, w \in W$, and similarly for $\ell(sw)$.

Proof: Let $w = s_1 \cdots s_r$ be a reduced expression. Then $\varepsilon(w) = \varepsilon(s_1) \cdots \varepsilon(s_r) = (-1)^{\ell(w)}$.

□

5.3. Geometric representation of W

5.3.1. Definition: Geometric representation of W

Choose a basis of V over $\mathbb{R}$ in one-to-one correspondence with S, the impose geometry using a symmetric bilinear form B on V by $B(\alpha_s, \alpha_{s'}) = -\cos\left(\frac{\pi}{m(s,s')}\right)$. H_s, orthogonal to α_s relative to B, is complementary to the line $\mathbb{R}\alpha_s$.

We can define reflections $\sigma_s \lambda = \lambda - 2B(\alpha_s, \lambda)\alpha_s$. Note $B(\sigma_s \lambda, \sigma_s \mu) = B(\lambda, \mu)$.

5.3.2. *Proposition*

There is a unique homomorphism $\sigma : W \to \mathrm{GL}(V)$ sending s to σ_s, and the group $\sigma(W)$ preserves the form B on V. Moreover, for each pair $s, s' \in S$, the order of ss' in W is precisely $m(s, s')$.

Proof:

□

5.4. Positive and negative roots

5.4.1. Definition: Root System

Let the **root system** Φ of W consisting of all vectors $w(\alpha_s) \coloneqq \sigma(w)(\alpha_s)$ for all $w \in W$ and $s \in S$. These are unit vectors since W preserves the form B on V.

α is **positive** (resp. **negative**) if it is a linear combination of $\{\alpha_s \mid s \in S\}$ with all nonnegative (resp. nonpositive) weights.

5.4.2. Definition: Parabolic Subgroup

A **parabolic subgroup** W_I of W is the subgroup generated by a subset $I \subseteq S$.

5.4.3. ***Theorem***

Let $w \in W$ and $s \in S$. If $\ell(ws) > \ell(w)$, then $w(\alpha_s) > 0$. If $\ell(ws) < \ell(w)$, then $w(\alpha_s) < 0$.

Proof:

$\square$

5.4.4. ***Corollary***

The representation $\sigma : W \to \mathrm{GL}(V)$ is faithful.

Proof: Let $w \in \ker(\sigma)$. If $w \neq 1$, there exists $s \in S$ for which $\ell(ws) < \ell(w)$. The previous theorem states that $w(\alpha_s) < 0$. But $w(\alpha_s) = \alpha_s > 0$, which is a contradiction.

$\square$

5.5. Parabolic subgroups

5.5.1. ***Theorem***

1. For each subset I of S, the pair (W_I, I) with the given values $m(s, s')$ is a Coxeter system.
2. Let $I \subseteq S$. If $w = s_1 \cdots s_r$ is a reduced expression, and $w \in W_I$, then all $s_i \in I$. In particular, the function ℓ agrees with ℓ_I on W_I, and $W_I \cap S = I$.
3. The assignment $I \mapsto W_I$ defines the lattice isomorphism between the collection of subsets of S and the collection of subgroups W_I of W.
4. S is a minimal generating set for W.

Proof:

$\square$

5.6. Geometric interpretation of the length function

5.6.1. ***Proposition***

1. If $s \in S$, then s sends α_s to its negative, but permutes the remaining positive roots.
2. For any $w \in W$, $\ell(w)$ equals the number of positive roots sent by w to negative roots.

Proof:

$\square$

5.7. Roots and reflections

5.7.1. Remark

If we let $\alpha = w(\alpha_s)$ for some $w \in W, s \in S$ it can by shown $wsw^{-1}(\lambda) = \lambda - 2B(\lambda, \alpha)\alpha$ such that the transformation only relies on α. We denote $wsw^{-1} = s_\alpha$.

The correspondence of $\alpha \mapsto s_\alpha$ is bijective (for $\alpha \in \Pi := \Phi^+$)

5.7.2. ***Lemma***

If $\alpha, \beta \in \Phi$ and $\beta = w(\alpha)$ for some $w \in W$, then $ws_\alpha w^{-1} = s_\beta$.

Proof:

$\square$

5.7.3. ***Proposition***

Let $w \in W, \alpha \in \Pi$. Then $\ell(ws_\alpha) > \ell(w)$ if and only if $w(\alpha) > 0$.

Proof:

$\square$

5.8. Strong Exchange Condition

5.8.1. *Theorem:* Strong Exchange Condition

Let $w = s_1 \cdots s_r (s_i \in S)$, not necessarily a reduced expression. Suppose a reflection $t \in T$ satisfies $\ell(wt) < \ell(w)$. Then there is an index i for which $wt = s \cdots \hat{s}_i \cdots s_r$. If the expression w is reduced, then i is unique. Here $T = \bigcup_{w \in W} wSw^{-1}$ is the set of all reflections.

Proof:

$\square$

5.8.2. *Corollary:* Deletion Condition

1. Suppose $w = s_1 \cdots s_r (s_i \in S)$, with $\ell(w) < r$. Then there exist indices $i < j$ for which $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$.
2. If $w = s_1 \cdots s_r$, then a reduced expression for w may be obtained by omitting certain s_i (an even number).

Proof:

$\square$

5.9. Bruhat ordering

5.9.1. Definition: Bruhat ordering

Write $w' \to w$ if $w = w't$ for some $t \in T$ with $\ell(w) > \ell(w')$. Define $w' < w$ if there is a sequence

$$w' = w_0 \to w_1 \to \ldots \to w_m = w.$$

$w' \leq w$ is a partial ordering of W with 1 as the unique minimal element. This ordering is called the **Bruhat ordering**.

If we restrict $t \in s$ we get a **weak ordering** which has a one-sided nature (the Bruhat ordering can be written using either left or right multiplication by t).

5.9.2. *Proposition*

Let $w' \leq w$ and $s \in S$. Then either $w's \leq w$ or else $w's \leq ws$ (or both).

Proof:

$\square$

5.10. Subexpressions

5.10.1. Definition: Subexpressions

Given a reduced expression $w = s_1 \cdots s_r$ we have **subexpressions** of the form $s_{i_1} \cdots s_{i_q}$ where $1 \leq i_1 < \ldots < i_q \leq r$. The subexpression is formally the q-tuple obtained by selecting generators from the tuple for the reduced expression of w.

5.10.2. ***Theorem***

Let $w = s_1 \cdots s_r$ be a fixed, but arbitrary, reduced expression for w. Then $w' \leq w$ if and only if w' can be obtained as a subexpression of this reduced expression.

Proof:

$\square$

5.10.3. ***Corollary***

If $I \subseteq S$, the Bruhat ordering of W agrees on W_I with the Bruhat ordering of the Coxeter group W_I.

Proof:

$\square$

5.11. Intervals in the Bruhat ordering

5.11.1. ***Lemma***

Let $w' < w$, with $\ell(w) = \ell(w') + 1$. Suppose there exists $s \in S$ for which $w' < w's$ and $w's \neq w$. Then both $w < ws$ and $w's < ws$.

Proof:

$\square$

5.11.2. *Proposition*

Let $w' < w$. Then there exist $w_0, \ldots, w_m \in W$ such that $w' = w_0 < \ldots < w_m = w$, and $\ell(w_i) = \ell(w_{i-1}) + 1$ for $1 \leq i \leq m$.

Proof:

$\square$

5.12. Poincare series

5.12.1. Remark

$$W^I := \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in I\}.$$

5.12.2. Definition: Poincare series of W

The **Poincare series** of W is given by

$$W(t) = \sum_{n \geq 0} \mathrm{Card}(\{w \in W \mid \ell(w) = n\}) t^n.$$

5.12.3. *Proposition*

1. In the field of formal power series in t, we have the identity

$$\sum_{I \subseteq S} (-1)^I \frac{W(t)}{W_I(t)} = \sum_{I \subseteq S} (-1)^I W^I(t) = 0$$

unless W is finite, in which case the right side equals t^N.

2. $W(t)$ is an explicitly computable rational function of t.

Proof:

$\square$

5.13. Fundamental domain for W

5.13.1. Definition

We define a contragredient action $\sigma^* : W \to \mathrm{GL}(V^*)$ by $\langle w(f), w(\lambda)\rangle = \langle f, \lambda\rangle$ for $w \in W, f \in V^*, \lambda \in V$.

For any $s \in S$ we define the hyperplane $Z_s := \{f \in V^* \mid \langle f, \alpha_s\rangle = 0\}$ together with the associated half-spaces

$$A_s := \{f \in V^* \mid \langle f, \alpha_s\rangle > 0\}, \quad A'_s := \{f \in V^* \mid \langle f, \alpha_s\rangle < 0\} = s(A_s).$$

Let C be the intersection of all $A_s, s \in S$.

5.13.2. *Lemma*

Let $s \in S$ and $w \in W$. Then $\ell(sw) > \ell(w)$ if and only if $w(C) \subseteq A_s$, whereas $\ell(sw) < \ell(w)$ if and only if $w(C) \subseteq A'_s$

Proof:

□

5.13.3. *Theorem*

1. Let $w \in W$ and $I, J \subseteq S$. If $w(C_I) \cap C_J \neq \emptyset$, then $I = J$ and $w \in W_I$, so $w(C_I) = C_I$. In particular, W_I is the precise stabilizer in W of each point of C_I, and $\mathcal{C}$ is a partition of U.
2. D is a fundamental domain for the action of W on U: the W-orbit of each point of U meets D in exactly one point.
3. The cone U is convex, and every closed line segment in U meets just finitely many of the sets in the family $\mathcal{C}$.

Proof:

□