Notes on Geometric and Topological Aspects of Coxeter Groups and Buildings

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1.1.1. Remark

$\mathbb{S}^n$ denotes the n-dimensional unit sphere, $\mathbb{E}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{H}^n$ denotes the n-dimensional Hyperbolic space.

$\mathbb{X}^n$ denotes any of the previously mentioned n-dimensional spaces.

1.1.2. Example: Finite Dihedral Group

Given two reflections, s_1 and s_2, offset by an angle of $\frac{\pi}{m}$ across the lines l_1 and l_2 respectively, we find

$$D_{2m} = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^m = 1 \rangle$$

such that $s_1 s_2$ is a rotation by $\frac{2\pi}{m}$. This induces a tessellation on $\mathbb{S}^1$ by $2m$ closed intervals, $\mathbb{E}^1$ by $2m$ unbounded sectors, or on the unit disc with $2m$ bounded sectors. D_{2m} can also be seen as the isometry group of a regular m-gon with a vertex v being on the intersection of l_1 with $\mathbb{S}^1$ and an edge drawn from v bisected by l_2.

By letting $s = s_1$ and $r = s_1 s_2$ we get the standard presentation

$$D_{2m} = \langle s, r \mid r^m = s^2 = 1, rs = sr^{-1} \rangle.$$

1.1.3. Example: Infinite Dihedral Group

Let s_1 and s_2 be the reflection of $\mathbb{E}^1$ (real line) over 0 and 1 respectively. Note that $s_1 s_2$ is a translation by 2 units. Note that

$$D_\infty = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^\infty = 1 \rangle$$

is the infinite dihedral group. Note this can also be thought of like a finite dihedral group with the angle between the reflections to be approaching zero such that the reflections are parallel.

1.1.4. Definition: Polytopes and links

A convex polytope $P = P^n \subseteq \mathbb{X}^n$ is a convex compact intersection of a finite number of closed half-spaces in $\mathbb{X}^n$, with nonempty interior. A linke of a vertex v, $\text{link}(v)$, is the $(n-1)$-dimensional spherical polytope obtained from the intersection of P with a small sphere around v. A convex polytope P is simple if, for every vertex v of P, $\text{link}(v)$ is a simplex.

1.1.5. Example: Convex polygon

If $n = 2$ then for a convex polytope $P \subseteq \mathbb{X}^2$ a link $\text{link}(v)$ is a closed interval in $\mathbb{S}^1$ (a spherical 1-simplex) equal to the interior angle at v.

1.1.6. Definition: Fundamental Domain

Let G act on a topological space X by homeomorphisms. A fundamental domain is a closed, connected subset C of X such that $Gx \cap C \neq \emptyset$ for all $x \in X$ and $Gx \cap C = \{x\}$ for x in the interior of C. C is a strict fundamental domain if $Gx \cap C = \{x\}$ for every $x \in C$, in other words C contains exactly one point from each G-orbit.

1.1.7. Example: Fundamental Domain of D_∞

Note $[0, 1]$ is a strict fundamental domain for the action of D_∞ on $\mathbb{E}^1$.

1.1.8. *Theorem*

Let $P = P^n$ be a simple convex polytope in $\mathbb{X}^n$, where $n \geq 2$. Let $\{F_i\}_{i \in I}$ be the collection of codimension-1 faces of P^n, with each face F_i supported by the hyperplane $\mathcal{H}_i$. Suppose for all $i \neq j$, if $F_i \cap F_j \neq \emptyset$ then the dihedral angle between F_i and F_j is $\frac{\pi}{m_{ij}}$ for some integer $m_{ij} \geq 2$. Put $m_{ii} = 1$ for every $i \in I$, and $m_{ij} = \infty$ if $F_i \cap F_j = \emptyset$. For each $i \in I$, let s_i be the isometric reflection of $\mathbb{X}^n$ across the hyperplane $\mathcal{H}_i$. Let W be the group generated by the set of reflections $\{s_i\}_{i \in I}$. Then

1. the group W has presentation
$$W = \left\langle s_i \mid \left(s_i s_j\right)^{m_{ij}} = 1 \forall i, j \in I \right\rangle;$$
2. the group W is a discrete subgroup of $\mathrm{Isom}(\mathbb{X}^n)$;
3. the convex polytope P is a strict fundamental domain for the action of W on $\mathbb{X}^n$, and the action of W induces a tessellation of $\mathbb{X}^n$ by copies of P.

1.1.9. Definition: Geometric reflection Group

A group W is a geometric reflection group if W is either a finite dihedral group, an infinite dihedral group, or is as in the statement of the previous theorem. A geometric reflection group W acting on $\mathbb{X}^n$ is spherical, Euclidean or hyperbolic as $\mathbb{X}^n$ is, respectively, $\mathbb{S}^n, \mathbb{E}^n, \mathbb{H}^n$.

1.1.10. Example: Triangle groups

For a tuple of integers (p, q, r), where $2 \leq p \leq q \leq r$, there exists a triangle with interior angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$. Note this triangle can exist in spherical, Euclidean, or hyperbolic space as its sum is respectively greater than, equal to, or less than π. Note

$$W(p, q, r) = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^p = (s_2 s_3)^q = (s_3 s_1)^r = 1 \rangle.$$

1.2. Definition of a Coxeter Group

1.2.1. Definition: Coxeter System

Let I be a finite indexing set and let $S = \{s_i\}_{i \in I}$. Let $M = (m_{(ij)_{i,j \in I}}$ be a matrix such that:
- $m_{ii} = 1$ for all $i \in I$;
- $m_{ij} = m_{ji}$ for all $i, j \in I$;
- $m_{ij} \in \{2, 3, 4, ...\} \cup \{\infty\}$ for all distinct $i, j \in I$.

Then M is a Coxeter matrix. The associated Coxeter group $W = W_M$ is defined by the presentation

$$W = \left\langle S \mid \left(s_i s_j\right)^{m_{ij}} = 1, \forall i, j \in I \right\rangle.$$

The pair (W, S) is a Coxeter system and the set S is a Coxeter generating set for W.

1.2.2. Remark

Coxeter groups may have multiple distinct generating sets such that a Coxeter group's presentation is not unique.

1.2.3. Remark

Some definitions allows for Coxeter Systems (W, S) in which S is an infinite generating set.

1.3. Right-angled Coxeter groups

1.3.1. Definition: Right-angled Coxeter System

Let (W, S) be a Coxeter System with associated Coxeter matrix $M = \left(m_{ij}\right)_{i,j \in I}$. (W, S) is right-angled, and W is a right-angled Coxeter group if, for all distinct $i, j \in I$, we have $m_{ij} \in \{2, \infty\}$.

1.3.2. Remark

Note that if $m_{ij} = 2$ then $s_i s_j = s_j s_i$.

1.3.3. Example: Coxeter Group, Not Geometric Reflection Group

Consider the following group

$$W = \langle s, t, u \mid s^2 = t^2 = u^2 = (st)^2 = 1 \rangle \approx (C_2 \times C_2) * C_2.$$

Note W is not a geometric reflection group.

1.4. Weyl groups

1.4.1. Definition: Root System

A finite collection Φ of vectors in $\mathbb{R}^n$ satisfying that Φ spans $\mathbb{R}^n$ and $\alpha \in \Phi$ then $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$. For each $\alpha \in \Phi$, let $\mathcal{H}_\alpha = \mathcal{H}_{-\alpha}$ be the hyperplane through the origin which is orthogonal to α, and let $s_\alpha = s_{-\alpha}$ be the isometric reflection in $\mathcal{H}_\alpha$. Then s_α swaps α and $-\alpha$, and the axioms for Φ ensure that $s_\alpha(\beta)$ is in Φ for all $\beta \in \Phi$, and that all compositions of reflections $s_\alpha s_\beta$ have finite order. It is also required (only for crystallographic restrction) that for any $\alpha, \beta \in \Phi$ the projection of β onto α must be an integer or half-integer multiple of α.

1.4.2. Definition: Weyl Group

The Weyl Group of a root system Φ is the group $W = W(\Phi)$ generated by the set of reflections $\{s_\alpha \mid \alpha \in \Phi\}$. It can be proved that W is a finite Coxeter group, with Coxeter generating set given by a subset of the reflections $\{s_\alpha \mid \alpha \in \Phi\}$.

1.4.3. Example: Weyl Group A_2

Let $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$ such that $W = W(\Phi)$ is generated by s_α, s_β since

$$s_{\alpha+\beta} = s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta.$$

1.4.4. Remark

D_{2m} is a Weyl group if and only if $m \in \{2, 3, 4, 6\}$.

1.4.5. Definition: Crystallographic Restriction (Coxeter Matrix)

A Coxeter matrix $M = (m_{ij})$ satisfies the crystallographic restriction if $m_{ij} \in \{2, 3, 4, 6, \infty\}$ for $i \neq j$. If the crystallographic restriction is satisfied for a Coxeter group then it is the Weyl group for some Kac-Moody group.

2. Some combinatorial theory of Coxeter groups

2.0.1. Notation

In contrast to a group product such as $s_1 \cdots s_n$, a word in S written as a finite sequence of elements of S, $(s_1, \ldots, s_n)$.

2.0.2. Definition: Word Length

The word length of some $g \in G$ with respect to the generating set S is

$$\ell_S(g) = \min\{n \in \mathbb{N} \mid \exists s_1, \ldots, s_n \in S \text{ such that } g = s_1 \cdots s_n\}$$

and $\ell_S(1) = 0$. If $\ell_S(g) = n \geq 1$ and $g = s_1 \cdots s_n$ then $(s_1, \ldots, s_n)$ is variously called a reduced expression, reduced word, or a minimal word for g. The word metric on G with respect to S is given by

$$d_S(g, h) = \ell_S(g^{-1}h).$$

2.0.3. Remark

Note that $d_S(hg, hg') = d_S(g, g')$.

2.0.4. Definition: Cayley Graph

A Cayley graph $\mathrm{Cay}(G, S)$ has vertex set G and edges given by $\{(g, gs) \mid g \in G, s \in S\}$. Note since S is a generating set the graph is connected.

2.0.5. Remark

G acts on $\mathrm{Cay}(G, S)$ on the left by graph automorphisms, preserving edge labels. Also note that gsg^{-1} is the unique group element that flips the edge $\{g, gs\}$ since $s \in S$ is an involution.

2.0.6. Remark

The word metric extends directly to the path metric; the distance between two vertices on $\mathrm{Cay}(G, S)$ where each edge is a unit length. This equivalence is found by observing $d_S(g, h) = \ell_S(g^{-1}h)$ is trivially the shortest path from $e \to g^{-1}h$. Now if g acts on $\mathrm{Cay}(G, S)$ on the left we have that this same path is now from $g \to h$. As a graph automorphisms we have that this is the shortest path.

2.0.7. *Lemma*

There is an epimorphism $\varepsilon : W \to \mathbb{Z}_2$ induced by $\varepsilon(s) = 1$ for all $s \in S$.

2.0.8. ***Corollary***

Each $s \in S$ is an involution in the group W.

2.0.9. ***Lemma***

If (W, S) is even, then for each $i \in I$, there is an epimorphism $\varepsilon_i : W \to \mathbb{Z}_2$ induced by $\varepsilon(s_i) = 1$ and $\varepsilon(s) = 0$ for all $s \in S \setminus \{s_i\}$.

2.0.10. ***Corollary***

If (W, S) is an even Coxeter system, the elements of S are pairwise distinct group elements in W.

2.0.11. ***Corollary***

In a Coxeter system (W, S), the elements of S are pairwise distinct involutions in W.

2.0.12. Definition: Simple Graph

Every edge must have distinct end points (no self-loops) and there should only exist at most one edge between an two distinct vertices.

2.0.13. ***Corollary***

Let (W, S) be a Coxeter system. Then $\mathrm{Cay}(W, S)$ is a connected simple graph.

Proof: Note by previous corollary we have that $1 \notin S$ since 1 is not an involution and therefore no self-loops can exist in $\mathrm{Cay}(W, S)$. Also note by another previous corollary that all elements in S are pairwise dinstinct group elements/involutions and therefore at most one edge may exist between two vertices. Therefore $\mathrm{Cay}(W, S)$ is simple.

$\square$

2.0.14. Definition: Pre-Reflection System

Let X be a connected simple graph that G acts on by graph automorphisms and let $R \subseteq G$ such that
1. each $r \in R$ is an involution;
2. R is closed under conjugation, that is, for all $g \in G$ and all $r \in R$, we have $grg^{-1} \in R$;
3. R generates G;
4. for every edge e in X there exists a unique $r_e \in R$ which flips e, that is, r_e swaps the two end points of e;
5. for every $r \in R$ there is at least one edge e in X which is flipped by r.

(X, R) is a pre-reflection system for a group G and for each $r \in R$, the wall H_r is the set of midpoints of edhes which are flipped by r.

2.0.15. *Lemma*

Let (W, S) be any Coxeter system. Let $X = \mathrm{Cay}(W, S)$ and let

$$R = \{wsw^{-1} \mid w \in W, s \in S\}.$$

Then (X, R) is a pre-reflection system.

Proof: By the previous corollary we have that $\mathrm{Cay}(W, S)$ is a connected simple graph. Note that wsw^{-1} flips $\{w, ws\}$ and no other group element flips this edge.

□

2.0.16. Definition: Reflection System

A pre-reflection system (X, R) is a reflection system if in addition for every $r \in R$ we have that $X \setminus H_r$ has exactly two components. Then R is the set of reflections.

2.0.17. Definition: Deletion Condition

If $(s_1, ..., s_k)$ is a word in S with $\ell(s_1 \cdots s_k) < k$, then there are indices $i < j$ such that

$$s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$$

where $\hat{s}_i$ means we delete this letter.

2.0.18. Definition: Exchange Condition

If $(s_1, ..., s_k)$ is a reduced expression for $w \in W$, the for any $s \in S$, either $\ell(sw) = k + 1$, or there is an index i such that

$$w = ss_1 \cdots \hat{s}_i \cdots s_k.$$

2.0.19. *Theorem*

Suppose a group W is generated by a set of distinct involutions S. Then the following are equivalent:
1. The pair (W, S) is a Coxeter system.
2. If $X = \mathrm{Cay}(W, S)$ and $R = \{wsw^{-1} \mid w \in W, s \in S\}$, then (X, R) is a reflection system.
3. The pair (W, S) satisfies the deletion condition.
4. The pair (W, S) satisfies the exchange condition.

2.0.20. *Lemma*

Suppose a group W is generated by a set of distinct involutions S. Let $X = \mathrm{Cay}(W, S)$ and let $R = \{wsw^{-1} \mid w \in W, s \in S\}$. Let $(s_1, \ldots, s_k)$ be a word in S with associated sequence $(r_1, \ldots, r_k)$ of elements of R. If $r_i = r_j$ for some $1 \le i < j \le k$, then in the group W,

$$s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k.$$

Proof: Let $r = r_i = r_j$ and for $1 \le p \le k$ let $w_p = s_1 \cdots s_p$. Let γ be the associated path in X, so γ has successive vertices $1, w_1, \ldots, w_k$. Now apply r to the subpath of γ from $w_i \to w_{j-1}$. The image is the path from $w_{i-1} \to w_j$. The new path generated is given by

$$(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{j-1}, s_{j+1}, \ldots, s_k).$$

$\square$

2.0.21. *Lemma*

With the same assumptions as in the previous lemma for each $r \in R$ we have that $X \setminus H_r$ has at most two connected components.

2.0.22. *Lemma*

Assume that (W, S) is a coxeter system. Then for all $w \in W$ and $r \in R$, any word for w crosses H_r the same number of times mod 2.

2.0.23. *Corollary*

Let (W, S) be a Coxeter system, let $X = \mathrm{Cay}(W, S)$ and let

$$R = \{wsw^{-1} \mid w \in W, s \in S\}.$$

Then (X, R) is a reflection system.

2.0.24. *Corollary*

If $\underline{s} = (s_1, ..., s_k)$ is a word in S with associated reflections $(r_1, ..., r_k)$, then $\underline{s}$ is a reduced expression if and only if r_i are pairwise distinct.

2.0.25. Definition: Braid Move

Let $s, t \in S$ be distinct involutions and let m_{st} is the order of st in W. If m_{st} is finite, a braid move on a word in S swaps a subword $(s, t, s, ...)$ containing m_{st} letters with a subword $(t, s, t, ...)$ containing m_{st} letters.

2.0.26. *Theorem:* Tits

Suppose a group W is generated by a set of distinct involutions S and the exchange condition holds. Then

1. a word $(s_1, ..., s_k)$ in S is reduced if and only if it cannot be shortened by a sequence of
   - deleting a subword (s, s), $s \in S$, or
   - carrying out a braid move;
2. two reduced expressions in S represent the same group element $w \in W$ if and only if they are related by a finite sequence of braid moves.

2.0.27. Example: Pre-Reflection System

$D_\infty \rtimes \mathbb{Z}_2$

2.0.28. Remark

$B_3 = \langle a, b \mid aba = bab \rangle$ is torsion free.

3. The Tits representation

3.0.1. ***Theorem:*** **Tits Representation**

Let I be a finite indexing set, let $S = \{s_i\}_{i \in I}$ and let $M = \{m_{ij}\}_{i,j \in I}$ be a Coxeter matrix. Let

$$W = \left\langle S \mid \left(s_i s_j\right)^{m_{ij}} = 1, \forall i,j \in I \right\rangle$$

be the associated Coxeter group. Then there is a faithful representation

$$\rho : W \to \mathrm{GL}_n(\mathbb{R}),$$

where $n = |S| = |I|$, such that:

- for each $i \in I$, $\rho(s_i) = \sigma_i$ is a linear involution with fixed set a hyperplane;
- for all $i \neq j$, the product $\sigma_i \sigma_j$ has order m_{ij}.

3.0.2. Example: Tits Representation

Let $I = \{1, \dots, n\}$ and let $\{e_1, \dots, e_n\}$ be a basis for V, an n-dimensional vector space over $\mathbb{R}$. Define the symmetric bilinear form

$$B(e_i, e_j) = \begin{cases} -\cos\left(\frac{\pi}{m_{ij}}\right) & m_{ij} < \infty \\ -1 & m_{ij} = \infty \end{cases}.$$

Note $B(e_i, e_i) = 1$ and $B(e_i, e_j) \leq 0$ if $i \neq j$. Consider the following

$$\begin{aligned} H_i &= \{v \in V \mid B(e_i, v) = 0\} \\ \sigma_i(v) &= v - 2B(e_i, v)e_i. \end{aligned}$$

Note σ_i swaps e_i and $-e_i$, fixes H_i, $\sigma_i^2 = \mathrm{id}$, and that σ_i preserves the bilinear form B.

3.0.3. ***Proposition***

1. The product $\sigma_i \sigma_j$ has order m_{ij} for all distinct $i, j \in I$.
2. The map $s_i \mapsto \sigma_i$ extends to a homomorphism $\rho : W \to \mathrm{GL}(V)$.

3.0.4. ***Corollary***

Let (W, S) be a Coxeter system. Then the selements of S are pairwise distinct involutions of W.

3.0.5. ***Corollary***

Let (W, S) be a Coxeter sustem with Coxeter matrix $M = (m_{ij})$. Then for all distinct $i, j \in I$, the product $s_i s_j$ has order m_{ij} in W.

3.0.6. ***Proposition***

ρ is faithful.

3.0.7. Definition: Chamber of Tits Representation

The chamber C associated to the Tits representation is a the subset of V^* given by

$$C = \{\varphi \in V^* \mid \varphi(e_i) \geq 0, \forall i \in I\}.$$

This chamber is the "simplicial cone" cut out by the hyperplanes H_i^*.

We define the iterior of C as

$$\mathring{C} = \text{interior}(C) = \{\varphi \in V^* \mid \varphi(e_i) > 0, \forall i \in I\}.$$

3.0.8. ***Theorem***

Let $w \in W$, If $w\mathring{C}$. If $w\mathring{C} \cap \mathring{C} \neq \emptyset$, then $w = 1$.

3.0.9. Definition: Tits Cone

The Tits cone of W is the subset of V^* given by $\bigcup_{w\in W} wC$.

3.0.10. ***Corollary***

A Coxeter group W may be identified with a discrete subgroup of $\mathrm{GL}_n(\mathbb{R})$.

3.0.11. ***Corollary***

Coxeter groups and their subgroups are linear.

3.0.12. ***Theorem:*** **Selberg's Lemma**

Finitely generated linear groups are virtually torsion-free.

3.0.13. ***Theorem:*** **Malcev's Theorem**

Finitely generated linear groups are residually finite.

3.0.14. ***Corollary***

Coxeter groups are virtually torsion-free and residually finite.

3.0.15. Definition: Reducible/Irreducible

A Coxeter system (W, S) is reducible if $S = S' \sqcup S''$ with S' and S'' nonempty, such that $m_{ij} = 2$ for all $s_i \in S'$ and $s_j \in S''$. A Coxeter system (W, S) is irreducible if it is not reducible.

3.0.16. Remark

A reducible Coxeter group $W = \langle S' \rangle \times \langle S'' \rangle$ is a clear direct product of generated groups.

3.0.17. ***Theorem***

Suppose (W, S) is irreducible and $n = |S|$. Then we have:

1. The bilinear form B is positive definite if and only if W is finite. In this case, W is a spherical geometric reflection group generated by the set $S = \{s_i\}$ of reflections in the codimension-1 faces of a simplex in $\mathbb{S}^{n-1}$, with s_i the reflection in face F_i, so that faces F_i and F_j meet at dihedral angle $\frac{\pi}{m_{ij}}$.
2. The bilinear form B is positive semidefinite of corank 1 if and only if W is a Euclidean geometric reflection group. In this case, W is generated by the set $S = \{s_i\}$ of reflections in the codimension-1 faces of either
   - if $n = 2$, an interval in $\mathbb{E}^1$ (this is the case $W = D_\infty$);
   - if $n > 3$, a simplex in $\mathbb{E}^{n-1}$, with s_i the reflection in the codimension-1 face F_i, so that faces F_i and F_j meet at dihedral angle $\frac{\pi}{m_{ij}}$.

3.0.18. Definition: Coxeter Polytope

Let W be a finite Coxeter group. A Coxeter polytope for W is the convex hull of the W-orbit on V^* of a point $x \in \mathring{C}$.

3.0.19. Definition: Special Subgroup

For each $T \subseteq S$, the special subgroup W_T of W is $W_T = \langle T \rangle$. If $T = \emptyset$, we define $W_\emptyset$ to be the trivial group. Note these subgroups are also called (standard) parabolic subgroups or visual subgroups.

3.0.20. Notation

If $J \subseteq I$ then $W_J = \langle s_j \mid j \in J \rangle$.

3.0.21. ***Theorem***

Let (W, S) be a Coxeter system.

1. The pair (W_T, T) is a Coxeter system for each $T \subseteq S$.
2. For all $T \subseteq S$ and $w \in W_T$, we have $\ell_T(w) = \ell_S(w)$, and any reduced expression for w uses only letters in T. Hence $\mathrm{Cay}(W_T, T)$ embeds isometrically as a convex subgraph of $\mathrm{Cay}(W, S)$.
3. If $T, T' \subseteq S$, then $W_T \cap W_{T'} = W_{T \cap T'}$ and $\langle W_T, W_{T'} \rangle = W_{T \cup T'}$.
4. The map $T \mapsto W_T$ is a bijection

$$\{\text{subsets of } S\} \longrightarrow \{\text{special subgroups of } W\}$$

which preserves the partial order given by inclusion.

4. The basic construction of a geometric realisation

4.0.1. Definition: Abstract Simplicial Complex

An abstract simplicial complex consists of a set V, possibly infinite, called the vertex set, and a collection X of finite subsets of V, such that
1. $\{v\} \in X$ for all $v \in V$;
2. if $\Delta \in X$ and $\Delta' \subseteq \Delta$, then $\Delta' \in X$.

An element of X is called a simplex. If Δ is a simplex and $\Delta' \subsetneq \Delta$ then Δ' is a face of Δ.

Δ is a k-simplex if $\dim \Delta = |\Delta| - 1 = k$.

The k-skeleton $X^{(k)}$ is the union of all simpleces of dimension at most k (note this is a simplicial complex).

The dimension of a simplicial complex X is $\dim X = \max\{\dim(\Delta) \mid \Delta \in X\}$.

X is pure if all its maximal simplicies have the same definition.

4.0.2. Example: Standard n-simplex Δ^n

The standard n-simplex, Δ^n, is the convex hull of $(n+1)$ points $(1, \ldots), \ldots, (0, \ldots, 1) \in \mathbb{R}^{n+1}$.

4.0.3. Remark

If X is a simplicial cell complex and $\Delta \subseteq X^{(0)}$ then $\Delta \in X$ if and only if Δ spans a copy of Δ^n in X.

4.0.4. Definition: Mirror Structure

Let (W, S) be any Coxeter system and let X be a connected, Hausdorff topological space. A mirror structure on X over S is a collection $(X_s)_{s \in S}$ where each X_s is a nonempty, closed subset of X. For each $s \in S$, we call X_s the s-mirror of X.

4.0.5. Notation

Define $S(x) := \{s \in S \mid x \in X_s\}$.

4.0.6. Remark

We define a relation $\sim$ on $W \times X$ by

$$(w, x) \sim (w', x') \text{ if and only if } x = x' \wedge w^{-1}w' \in W_{S(x)}$$

where W_T is the special subground $\langle T \rangle$. Note W is a Coxeter group and X is some connect, Hausdorff, topological space.

4.0.7. Definition: Basic Construction

The basic construction is the quotient

$$\mathcal{U}(W, X) = W \times X / \sim$$

equipped with the quotient topology (where W is equipped with the discrete topology). $[w, x]$ is the equvialence class of $(w, x) \in \mathcal{U}(W, X)$.

$wX = \{w\} \times X$ is called a chamber. Note there is an embedding $X \to wX, x \mapsto [w, x]$.

4.0.8. ***Lemma***

We have that $\mathcal{U}(W, X)$ is a connected topological space.

4.0.9. Definition: Locally Finite

We say $\mathcal{U}(W, X)$ is locally finite if for every $[w, x] \in \mathcal{U}(W, X)$ there is an open neighborhood of $[w, x]$ which meets only finitely many chambers.

4.0.10. ***Lemma***

The following are equivalent

1. The basic construction $\mathcal{U}(W, X)$ is locally finite.
2. For all $x \in X$, the special subgroup $W_{S(x)}$ is finite.
3. For all $T \subseteq S$ such that W_T is infinite, $\bigcap_{x \in T} X_t = \emptyset$.

4.0.11. ***Lemma***

The fundamental chamber X is a strict fundamental domain for the action W on $\mathcal{U}(W, X)$. Hence $\mathcal{U}(W, X)/W$ is the Hausdorff space X.

4.0.12. *Lemma*

The stabiliser in W of $[w, x] \in \mathcal{U}(W, X)$ is $wW_{S(x)}w^{-1}$. In particular, up to conjugacy every point stabiliser subgroup of W.

Proof: Consider the following

$$\begin{aligned}\mathrm{Stab}_W([w, x]) &= \{w' \in W \mid w' \cdot (w, x) \sim (w, x)\} \\ &= \{w' \in W \mid (w'w, x) \sim (w, x)\} \\ &= \left\{w' \in W \mid (w'w)^{-1}w \in W_{S(x)}\right\} \\ &= \left\{w' \in W \mid w^{-1}w'w \in W_{S(x)}\right\} \\ &= wW_{S(x)}w^{-1}.\end{aligned}$$

□

4.0.13. *Lemma*

The space $\mathcal{U}(W, X)$ is Hausdorff.

4.0.14. Definition: Properly Discontinuous

Let G be a discrete group and let Y be a Hausdorff space. An action by homeomorphisms of G on Y is properly discontinuous if

1. the quotient Y/G is Hausdorff;
2. for all $y \in Y$, the group $G_u = \mathrm{Stab}_G(y)$ is finite;
3. for all $y \in Y$, there is an open neighborhood U_y of y which is stabilised by G_y such that $gU_y \cap U_y = \emptyset$ for all $g \in G \setminus G_y$.

4.0.15. *Lemma*

The W-action on $\mathcal{U}(W, X)$ is properly discontinuous if and only if $W_{S(x)}$ is finite for every $x \in X$.

4.0.16. *Theorem:* Vinberg

Let (W, S) be any Coxeter system. Suppose W acts by homeomorphisms on a connected Hausdorff space Y such that every $s \in S$, the fixed point set Y^s of s is nonempty. Suppose further that X is a connected Hausdorff space iwth mirror structure $(X_s)_{s\in S}$. Then if $f : X \to Y$ is a continuous map such that $f(X_s) \subseteq Y^s$ for all $s \in S$, there is a unique extension of f to a W-equivariant map $\tilde{f} : \mathcal{U}(W, X) \to Y$ given by

$$\tilde{f}([w, x]) = w \cdot f(x).$$

4.0.17. *Theorem*

Let $X = P^n$ be a simple convex polytope in $\mathbb{X}^n$ for $n \geq 2$, with codimension-1 faces $\{F_i\}_{i \in I}$. Assume that if $i \neq j$ and $F_i \cap F_j \neq \emptyset$, then the dihedral angle between F_i and F_j is $\frac{\pi}{m_{ij}}$ where $m_{ij} \in \{2, 3, \dots\}$ is finite. Put $m_{ii} = 1$ and $m_{ij} = \infty$ if $F_i \cap F_j = \emptyset$.

Let $S = \{s_i\}_{i \in I}$ and let (W, S) be the Coxeter system with Coxeter matrix $\left(m_{ij}\right)_{i,j \in I}$. Define a mirror structure on X by $X_{s_i} = F_i$. For each $i \in I$, let $\overline{s}_i \in \operatorname{Isom}(\mathbb{X}^n)$ be the reflection in F_i. Let $\overline{W}$ be the subgroup of $\operatorname{Isom}(\mathbb{X}^n)$ generated by the set $\overline{s}_i$. Then

1. there is an isomorphism $\varphi : W \to \overline{W}$ induced by $s_i \mapsto \overline{s}_i$;
2. the induced map $\mathcal{U}(W, P^n) \to \mathbb{X}^n$ is a homeomorphism;
3. the Coxeter group W acts properly discontinuously on $\mathbb{X}^n$ with strict fundamental domain P^n, hence W is a discrete subgroup of $\operatorname{Isom}(\mathbb{X}^n)$ and $\mathbb{X}^n$ is tessellated by copies of P^n.

4.0.18. Definition: Atlas

A n-dimensional topological manifold M^n has an $\mathbb{X}^n$-strcture if it has an atlas of charts $\{\psi_\alpha : U_\alpha \to \mathbb{X}^n\}_{\alpha \in A}$ such that

- $(U_\alpha)_{\alpha \in A}$ is an open cover of M^n;
- each ψ_α is a homeomorphism onto its image;
- for all $\alpha, \beta \in A$ the map

$$\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha\left(U_\alpha \cap U_\beta\right) \to \psi_\beta\left(U_\alpha \cap U_\beta\right)$$

  is the restriction of an element of $\operatorname{Isom}(\mathbb{X}^n)$.

In particular an $\mathbb{X}^n$-structure turns M^n into a smooth Riemannian manifold. We will use the following

- An $\mathbb{X}^n$-structure on M^n induces one on its universal cover $\widetilde{M}^n$.
- There is a developing map $D : \widetilde{M}^n \to \mathbb{X}^n$ given by analytic continuation along paths.
- If M^n is metrically complete, D is a covering map.

5. The Davis complex

5.0.1. Notation

The Davis complex generated by the Coxeter system (W, S) is denoted by $\Sigma = \Sigma(W, S)$.

5.0.2. Definition: Spherical Subsets and Spherical Special Subgroup

A subset $T \subseteq S$ is a **spherical** if the special subgroup W_T is finite, in which case we say that W_T is a **sphecial special subgroup** (this is due to W_T acting naturally on a sphere if it is irreducible).

5.0.3. Remark

Finite irreducible Coxeter systems are classified.

5.0.4. Definition: Nerve

The nerve of (W, S), denoted $L = L(W, S)$, is the simplicial complex with a simplex σ_T for each $T \subseteq S$ such that $T \neq \emptyset$ and W_T is finite.

Note this is a simplicial complex (although it encludes the empty simplex).

5.0.5. Example: Nerve of Triangle Group

If W is the $(3, 3, 3)$-triangle group we have

$$W = \langle s,t,u \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (us)^3 = 1 \rangle.$$

Then the nerve L is a triangle with vertices s, t, u with no interior since W is infinite.

5.0.6. Remark

Fora given Euclidean or hyperbolic geometric reflection group W with strict fundamental domain P the nerve L can be identified with the boundary of P^* (the dual polytope of P).

5.0.7. Example: Right-angled Defining Graph

Let Γ be a finite simple graph with vertex set $S = V(\Gamma)$ and edge set $E(\Gamma)$. Then we have

$$\begin{aligned} W_\Gamma &= \langle S \mid s^2 = 1, \forall s \in S, st = ts \iff \{s, t\} \in E(\Gamma) \rangle \\ &= \langle S \mid s^2 = 1, \forall s \in S, (st)^2 = 1 \iff \{s, t\} \in E(\Gamma) \rangle \end{aligned}$$

is the associated right-angled Coxeter group. Γ is called the defining graph of W_Γ.

Note that if S has no spherical subsets T with $|T| > 2$ then $L(W_\Gamma, S) = \Gamma$.

5.0.8. Definition: Flag Complex

A simplicial complex L is a flag complex if a finite nonempty set of vertices T in L spans a simplex in L if and only if any two elements of T span an edge in L.

5.0.9. *Lemma*

If (W, S) is a right-angled Coxeter system, then $L(W, S)$ is a flag complex.

Proof: Suppose $T \subseteq S, T \neq \emptyset$ and any two vertices in T are connected by an edge in L. Then $W_T \approx (C_2)^{|T|}$ is finite, so T is spherical and σ_T is in L.

□

5.0.10. Definition: Chamber of Davis Complex

The chamber K is the cone on the barycentric subdivision L' of the nerve $L = L(W, S)$. For each $s \in S$, define $K_s \subset K$ to be the closed star in L' of the vertext s.

5.0.11. *Lemma*

For all $x \in K$, the set $S(X) = \{s \in S \mid x \in K_s\}$ is spherical, so each $W_{S(x)}$ is finite. Moreover, the collection $\{S(x) \mid x \in K\}$ is exactly the collection of spherical subsets of S.

5.0.12. Example: Chamber of Polytope

Let W be a Euclidean or hyperbolic geometric reflection group with fundamental domain P and let P^* be its dual polytope. Then $L = \partial P^*$ such that $L' = (\partial P^*)' = (\partial P)'$. Thus K is the cone on the barycentric subdivision of ∂P. The mirrors are the barycentric subdivisions of the codimension-1 faces of P.

5.0.13. Definition: Davis Complex (Basic Construction)

The **Davis complex** $\Sigma = \Sigma(W, S)$ is the basic construction

$$\Sigma = \mathcal{U}(W, K) = W \times K / \sim,$$

where the chamber K with mirror structure $(K_s)_{s \in S}$ is as before.

5.0.14. *Corollary*

The Davis complex $\Sigma = \mathcal{U}(W, K)$ is connected, Hausdorff, and locally finite. The W-action on Σ is properly discontinuous with quotien K, and all point stabilisers are conjugates of finite special subgroups of W.

5.0.15. *Theorem*

The Davis complex $\Sigma = \Sigma(W, S)$ is contractible.

5.0.16. Definition

For $w \in W$ define

$$\begin{aligned} \mathrm{In}(w) &= \{s \in S \mid \ell(ws) < \ell(w)\} \\ &= \{s \in S \mid \text{a reduced expression for } w \text{ can end in } s\} \\ \mathrm{Out}(w) &= \{s \in S \mid \ell(ws) > \ell(w)\}. \end{aligned}$$

Since $\ell(ws) = \ell(w) \pm 1$, we have $S = \mathrm{In}(w) \sqcup \mathrm{Out}(w)$.

5.0.17. *Proposition*

For all $w \in W$, $\mathrm{In}(w)$ is a spherical subset, that is, $W_{\mathrm{In}(w)}$ is finite.

5.0.18. *Lemma*

Suppose there is a $w_0 \in W$ such that $\ell(w_0 s) < \ell(w_0)$ for all $s \in S$. Then W is finite.

5.0.19. *Lemma*

Let $T \subseteq S$ and suppose w is a minimal length element in the left coset wW_T. Then any $w' \in wW_T$ can be written as $w' = wa'$ where $a' \in W_T$ is such that $\ell(w') = \ell(w) + \ell(a')$. Moreover, each coset wW_T contains a unique element of minimal length.

5.0.20. *Lemma*

The chamber K is contractible, and for all spherical $T \subseteq S$, the union of mirrors

$$K^T = \bigcup_{t \in T} K_t$$

is contractible.

5.0.21. *Lemma*

The chamber K defined previously is the geometric realisation of the post $\{T \subseteq S \mid W_T \text{ is finite}\}$ order by inclusion.

5.0.22. *Corollary*

The chamber K defined previously is the geometric realisation of the poset $\{W_T \mid T \subseteq S \text{ and } W_T \text{ is finite}\}$ ordered by inclusion.

5.0.23. *Theorem*

We can identify the Davis complex Σ with the geometric realisation of the poset

$$\{wW_T \mid w \in W, T \subseteq S \text{ and } W_T \text{ is finite}\},$$

ordered by inclusion.

5.0.24. *Theorem*

We can identify the (barycentric subdivision of the) Coxeter complex with the geometric realisation of the poset

$$\{wW_T \mid w \in W, T \subsetneq S\},$$

ordered by inclusion.

5.0.25. *Theorem*

The Davis complex Σ may be identified with the CW complex which has 1-skeleton $\mathrm{Cay}(W, S)$, and a cell with vertex set $U \subseteq W$ whenever $U = wW_T$ for some $w \in W$ and $T \subseteq S$ with W_T finite.

5.0.26. *Lemma*

The Davis complex Σ is simply connect.

5.0.27. Definition: Polyhedral Complex

A **polyhedral complex** is a finite-dimensional CW complex in which each n-cell is metrised as convex polytope in $\mathbb{X}^n$ (the same $\mathbb{X}$ for each cell), and the restrictions of the attaching maps to codimension-1 faces are isometries. A polyhedral complex is spherical, Euclidean or hyperbolic as $\mathbb{X}^n$ is $\mathbb{S}^n$, $\mathbb{E}^n$, or $\mathbb{H}^n$, respectively.

5.0.28. Definition: Geodesic Space

For $x, y \in X$ a geodesic from x to y is a map γ from $[a, b] \subseteq \mathbb{R}$ to X so that $\gamma(a) = x, \gamma(b) = y$ and $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $i, t' \in [a, b]$. (X, d) is a geodesic space if every pair of points in X is connected by a geodesic.

5.0.29. *Theorem*

If a connected polyhedral complex X has finitely many isometry types of cells, then X is a complete geodesic space.

5.0.30. *Theorem*

When equipped with the piecewise Euclidean metric determined by $\underline{d} = (d_s)_{s \in S}$ as below, $\Sigma = \Sigma(\underline{d})$ is a complete $\mathrm{CAT}(0)$ space.

$\underline{d}$ is a sequence where $d_s > 0$. Let $\rho_T : W_T \to O(|T|, \mathbb{R})$ for any finite W_T where $\rho_T(t)$ is the hyperplane H_t with unit normal vector e_t. H_t and $H_{t'}$ meet at dihedral angle $\frac{\pi}{m}$ where $\langle t,t' \rangle \approx D_{2m}$. Let the chamber

$$C_T = \{x \in \mathbb{R}^n \mid \langle x,e_t \rangle \geq 0, \forall t \in T\}.$$

There exists a unique point $x_T = x_{T(\underline{d})}$ in the interior of C_T such that $d(x_T, H_t) = d_t > 0$ for all $t \in T$. Each cell is metrised with vertex set wW_T.

5.0.31. Definition: Comparison Triangle

Let $[xy]$ be the geodesic segment from x to y is the geodesic space (X, d). Given a geodesic triangle

$$\Delta = [x_1 x_2] \cup [x_2 x_3] \cup [x_3 x_1]$$

in X, there is a comparison triangle

$$\overline{\Delta} = [\overline{x}_1 \overline{x}_2] \cup [\overline{x}_2 \overline{x}_3] \cup [\overline{x}_3 \overline{x}_1]$$

with $d_X(x_i, x_j) = d_{\mathbb{E}^2}(\overline{x}_i \overline{x}_j)$ for any $i \neq j$. For any $p \in [x_i x_j]$ there is a unique comparison point $\overline{p} \in [\overline{x}_i \overline{x}_j]$ such that $d_X(x_i p) = d_{\mathbb{E}^2}(\overline{x}_i \overline{p})$.

5.0.32. Definition: $\mathrm{CAT}(0)$

A geodesic space (X, d_X) is $\mathrm{CAT}(0)$ if for every geodesic triangle Δ in X, and all points $p, q \in \Delta$,

$$d_X(p, q) \leq d_{\mathbb{E}^2}(\overline{p}, \overline{q}).$$

Similarly a geodesic is $\mathrm{CAT}(-1)$ if

$$d_X(p, q) \leq d_{\mathbb{H}^2}(\overline{p}, \overline{q})$$

or $\mathrm{CAT}(1)$ if

$$d_X(p, q) \leq d_{\mathbb{S}^2}(\overline{p}, \overline{q}).$$

5.0.33. ***Theorem***

Let X be a complete $\mathrm{CAT}(0)$ space. Then we have the following properties:
1. The space X is uniquely geodesic.
2. The space X is contractible.
3. Suppose a group G acts on X by isometries. Let H be a subgroup of G and write X^H for the fixed set of H in X. If X^H is nonempty then X^H is convex. In particular, since closed, convex subsets of complete $\mathrm{CAT}(0)$ spaces are complete $\mathrm{CAT}(0)$ spaces, every nonempty fixed set X^H is contractible by (2).
4. (Bruhat-Tits fixed point theorem). If a group G acts on X by isometries and G has a bounded orbit, then $X^G \neq \emptyset$. In particular, for every finite subgroup $H \leq G$, we have $X^H \neq \emptyset$, and so X^H is contractible by (3).
5. If a group G acts properly discontinuously and cocompactly by isometries on X then the word problem and the conjugacy problem are both solvable for G.

5.0.34. ***Corollary***

Let (W, S) be a Coxeter system and $\Sigma = \Sigma(W, S)$ be the associated Davis complex, equipped with a $\mathrm{CAT}(0)$ metric.
1. The complex Σ is contractible.
2. If $H \leq W$ is finite, then Σ^H is nonempty and contractible, and there is an element $w \in W$ and a spherical subset $T \subseteq S$ such that $H \leq wW_Tw^{-1}$.
3. The word and conjugacy problems for W are solvable.

5.0.35. ***Theorem:*** **Cartan-Hadamard theorem for $\mathrm{CAT}(0)$ spaces**

Let X be a complete, connected geodesic metric space. If X is locally $\mathrm{CAT}(0)$ then the universal cover of X is $\mathrm{CAT}(0)$.

5.0.36. ***Theorem:*** **Gromov link condition**

If X is a piecewise Euclidean polyhedral complex then X is locally $\mathrm{CAT}(0)$ if and only if for every vertex v of X, the link of v in X is $\mathrm{CAT}(1)$.

5.0.37. ***Lemma***

In Σ, the link of the vertex $v = 1$ is L, with each simplex σ_T of L metrised as the simplex Δ_T in $\mathbb{S}^{|T|-1}$ with the vertex set $\{e_t\}_{t \in T}$ (L is a nerve).

5.0.38. ***Theorem***

Suppose all simplices of a simplicial complex Δ are metrised as right-angled spherical simplices. Then Δ is $\mathrm{CAT}(1)$ if and only if Δ is flag.

5.0.39. Definition: Cube Complex

A **cube complex** is a Euclidean polyhedral complex in which each cell is metrised as Euclidean cube.

5.0.40. ***Corollary***

If W_Γ is right-angled, $\Sigma = \Sigma_\Gamma$ can be metrised as a $\mathrm{CAT}(0)$ cube complex.

5.0.41. ***Theorem***

Suppose a simplicial complex Δ is metrised as a spherical simplicial complex in which all edge lengths are $\geq \frac{\pi}{2}$. Then Δ is $\mathrm{CAT}(1)$ if and only if Δ is a metric flag complex.

5.0.42. ***Theorem***

There exists a piecewise hyperbolic structure on Σ which is $\mathrm{CAT}(-1)$ if and only if there is no subset $T \subseteq S$ such that either

1. W_T is an Euclidean geometric reflection group of dimension ≥ 2; or
2. (W_T, T) is reducible with $W_T = W_{T'} \times W_{T''}$ and $W_{T'}$ and $W_{T''}$ both infinite.

5.0.43. Definition: Word Hyperbolic

W is word hyperbolic if there exists $\delta > 0$ so that every geodesic triangle Δ in $\mathrm{Cay}(W, S)$ is δ-thin, meaning that the δ-neighborhood of any two sides of Δ contains the third. Word hyperbolic groups are sometimes also called Gromov hyperbolic groups.

5.0.44. ***Theorem***

Let (W, S) be a Coxeter system. If W is word hyperbolic then

1. W has no $\mathbb{Z} \times \mathbb{Z}$ subgroup;
2. W has a solvable word and conjugacy problem;
3. W satisfies a linear “isoperimetric inequaility”; and
4. W is automatic and biautomatic.

5.0.45. ***Corollary***

If $W = W_\Gamma$ is right-angled then W_Γ is word hyperbolic if and only if Γ has no empty squares.

5.0.46. ***Corollary***

Let (W, S) be a Coxeter system. The following are equivalent:
1. The group W is word hyperbolic.
2. The group W has no $\mathbb{Z} \times \mathbb{Z}$ subgroup.
3. There is no subset $T \subseteq S$ such that (1) and (2) in previous theorem holds.
4. The complex Σ admits a piecewise hyperbolic metric which is $\mathrm{CAT}(-1)$.

5.0.47. Definition: Classifying Space

A **classifying space** for a group G, denoted BG, is an aspherical CW complex with fundamental group G. It is also called an **Eilenberg-MacLane space** or a $K(G, 1)$. The universal cover of BG, denoted EG, is called a **universal space** for G.

5.0.48. ***Theorem***

Let (W, S) be a Coxeter system with nerve L, chamber K and Davis complex Σ. Then

$$\begin{aligned} H^i(W; \mathbb{Z}W) &\cong H^i_c(\Sigma) \\ &\cong \bigoplus_{w \in W} H^i\big(K, K^{\mathrm{Out}(w)}\big) \\ &\cong \bigoplus \big\{\big(\mathbb{Z}W^T \otimes H^i\big(K, K^{S-T}\big) \mid T \subseteq S, T \text{ is spherical}\big\} \\ &\cong \bigoplus \Big\{\Big(\mathbb{Z}W^T \otimes \overline{H^{i-1}}(L - \sigma_T) \mid T \subseteq S, T \text{ is spherical}\Big\}. \end{aligned}$$

5.0.49. Definition: Universal Space

Let G be a discrete group. A CW complex X together with a proper, cocompact, cellular G-action is a universal space for proper G-actions, denoted $\underline{E}G$, if for all finite subgroups H of G, the fixed set X^H is contractible.

5.0.50. ***Theorem***

For any discrete group G, an $\underline{E}G$ exists and is unique up to G-homotopy, and

$$H^*(G; \mathbb{Z}G) = H^*_c(\underline{E}G).$$

5.0.51. *Proposition*

If a discrete group G acts properly discontinuously and cocompactly on an acyclic CW complex X then

$$H^*(G; \mathbb{Z}G) = H_c^*(X).$$

5.0.52. *Corollary*

The Davis complex $\Sigma = \Sigma(W, S)$ is a finite-dimensional $\underline{E}W$. Moreover, W acts properly discontinuously and cocompactly on Σ, and Σ is acyclic (since it is contractible).

6. Buildings as unions of apartments

6.0.1. Definition: Building

Let (W, S) be a Coxeter system. A **building** of type (W, S) is a simplicial complex Δ which is a union of subcomplexes called **apartments**, with each apartment being a copy of the Coxeter complex for (W, S). The maximal simplices in Δ are called its **chambers**, and the following axioms hold:

1. Any two chambers are contained ina common aparment.
2. If A and A' are any two apartments, there is an isomorphism $A \to A'$ which fixes $A \cap A'$ pointwise.

6.0.2. Remark

A building may be the union of more than one collection of subcomplexes.

6.0.3. Remark

Am generalised m-gon is the same thing as a building of type D_{2m}. Where a generalised m-gon is a connected bipartite graph with girth $2m$ (shortest circuit) and diameter m (the maximal distance between two veritices).

7. Buildings as chamber systems

7.0.1. Definition: Chamber System

Let I be a finite set. A set C, whose elements are called chambers, is a chamber system over I if each $i \in I$ determines an equivalence relation on C, denoted $\underset{i}{\sim}$. We say that chambers x and y are i-adjacent if $x \underset{i}{\sim} y$, and that they are adjacent if $x \underset{i}{\sim} y$ for some $i \in I$.

7.0.2. Definition: Gallery

A sequence of chambers $\gamma = (c_0, ..., c_k)$ is a chamber if c_{j-1} is adjacent to c_j and $c_{j-1} \neq c_j$. γ is of type $(i_1, ..., i_k)$ where $c_{j-1} \underset{i_j}{\sim} c_j$.

7.0.3. Definition: J-residue

For any $J \subseteq I$ a J-residue is a J-connected component of C that is a maximal subset of C such that each pair of chambers in this subset is connected by a gallery with type a tuple of indexes in J.

Note that $\{i\}$-residue is an i-panel.

7.0.4. Definition: W-valued distance function

W-valued distance function is a map

$$\delta : C \times C \to W$$

such that for all reduced words $\left(s_{i_1}, ..., s_{i_k}\right)$ and all $x, y \in C$, the following holds: $\delta(x, y) = s_{i_1} \cdots s_{i_k}$ if and only if there is a gallery from x to y in C of type $(i_1, ..., i_k)$.

7.0.5. Definition: Building

Let (W, S) be a Coxeter system with $S = \{s_i \mid i \in I\}$. A building of type (W, S) is a chamber system Δ over I which is equipped with a W-valued distance function, and is such that each panel has at least two chambers.

A building is thick if each panel has at least three chambers, and thin if each panel has exactly two chambers.

7.0.6. *Proposition*

Let (W, S) be a Coxeter system and let Δ be a building of type (W, S), with W-valued distance function $\delta : \Delta \times \Delta \to W$. Then

1. Δ is connected;
2. δ maps onto W;
3. for all $x, y \in \Delta$, we have $\delta(x, y) = \delta(y, x)^{-1}$;
4. for all $x, y \in \Delta$, we have $\delta(x, y) = s_i$ if and only if $x \underset{i}{\sim} y$ and $x \neq y$;
5. if $x, y \in \Delta$ with $x \neq y$, and $x \underset{i}{\sim} y$ and $x \underset{y}{\sim} y$, then $i = j$; and
6. if $\left(s_{i_1}, ..., s_{i_k}\right)$ is reduced in (W, S), then for all chambers x and y there is at most one gallery of type $(i_1, ..., i_k)$ from x to y.

7.0.7. Definition: Minimal Gallery

A gallery in Δ is minimal if there is no shorter gallery between its end points.

7.0.8. *Lemma*

A gallery of type $(i_1, ..., i_k)$ is minimal if and only if the word $\left(s_{i_1}, ..., s_{i_k}\right)$.

7.0.9. *Proposition*

If $J \subseteq I$, then every J-residue is a building of type (W_J, J).

7.0.10. Definition: W-isometric embedding

For any subset $X \subseteq W$, a map $\alpha : X \to \Delta$ is a W-isometric embedding if, for all $x, y \in X$,

$$\delta(\alpha(x), \alpha(y)) = x^{-1}y.$$

An apartment is any image of W under a W-isometric embedding.

7.0.11. *Proposition*

For any proper subset $X \subsetneq W$, any W-isometric embedding $\alpha : X \to \Delta$ extends to a W-isometric embedding of W.

7.0.12. *Theorem*

Let Δ be a building, so that its apartments are copies of some basic construction $\mathcal{U}(W, X)$.

1. If Δ is a spherical building (its apartments are spheres tiled by the action of W) then Δ is a $\mathrm{CAT}(1)$ space.
2. If Δ is a Euclidean or affine building (its apartments are Euclidean spaces tiled by the action of W) then Δ is a $\mathrm{CAT}(0)$ space.
3. If Δ is a hyperbolic building (its apartments are hyperbolic spaces tiled by the action of W) then Δ is a $\mathrm{CAT}(-1)$ space.
4. If the apartments of Δ are Davis complexes, then Δ can be equipped with a piecewise Euclidean metric such that it is a $\mathrm{CAT}(0)$ space.
5. If the apartments of Δ are Davis complexes, and W is word hyperbolic, then Δ can be equipped with a piecewise hyperbolic metric such that it is a $\mathrm{CAT}(-1)$ space.

7.0.13. Definition: Graph Product

Let Γ be a finite simple graph with vertex set S and edge set $E(\Gamma)$. For each $s \in S$, let G_s be a nontrivial group. The graph product of the family $\{G_s\}_{s \in S}$ over Γ is the group G_Γ which is the quotient of the free product of the groups G_s by the normal subgroup generated by the commutators

$$\{[g_s, g_t] : g_s \in G_s, g_t \in G_t \text{ and } \{s, t\} \in E(\Gamma)\}.$$

That is, the graph G_Γ is generated by the vertex groups G_s, with G_s and G_t commuting in G_Γ if and only if s and t are adjacent vertices. Graph products of groups are sometimes called graph groups.

7.0.14. *Theorem:* Green

If $g = g_{1_i} \cdots g_{i_k}$ and $g' = g'_{i_1} \cdots g'_{i_k}$ are reduced expressions for $g, g' \in G_\Gamma \setminus \{1\}$, then $g = g'$ in the group G_Γ if and only if one can get from one expression to the other by "shuffling", that is, using relations of the form $\left[g_{i_j}, g_{i_{j+1}}\right] = 1$.