Gaussian Process Modulated Cox Processes under Linear Inequality Constraints

Andrés F. López-Lopera,[1] ST John[2] and Nicolas Durrande[2]
[1]Mines Saint-Étienne, France. [2]PROWLER.io, UK.

Motivation

- GP-modulated Cox processes for modelling point patterns.
- Existing approaches require mapping functions from GP to positive intensity, and closed-form solutions are restricted to specific kernels.
- **Here:** approximation of Cox processes where positiveness can be imposed directly on the GP, with no restrictions on the kernel.
- We can ensure **any** linear inequality constraint (e.g. monotonicity, convexity).

The likelihood of an inhomogeneous Poisson process with intensity $\lambda(\cdot)$ is:

$$f_{(N,X_1,\cdots,X_n)}(n,\mathbf{x}_1,\cdots,\mathbf{x}_n) = \frac{\exp(-\mu)}{n!}\prod_{i=1}^{n}\lambda(\mathbf{x}_i), \quad \text{with} \quad \mu = \int_{\mathcal{S}}\lambda(\mathbf{s})\,d\mathbf{s}, \qquad (1)$$

μ denotes the intensity measure (overall intensity).

Cox process: extension of (1) with $\lambda(\cdot)$ sampled from a non-negative process $\Lambda(\cdot)$.

Finite approximation under linear inequality constraints

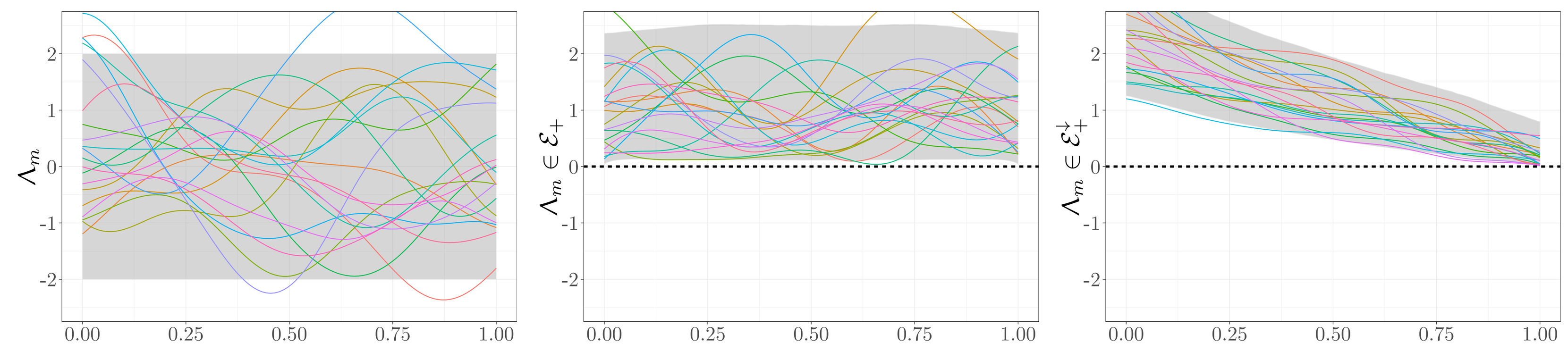
Define $\Lambda_m(\cdot)$ as piecewise-linear interpolation of $\Lambda(\cdot)$ at knots $t_1,\cdots,t_m$ [1]:

$$\Lambda_m(x) = \sum_{j=1}^{m}\phi_j(x)\xi_j, \qquad (2)$$

where $\xi_j := \Lambda(t_j)$ for $j=1,\cdots,m$, and $\phi_1,\cdots,\phi_m$ are hat basis functions. Hence any linear inequality constraint $\Lambda_m(\cdot)\in\mathcal{E}$ can be ensured by constraining ξ [1]:

$$\Lambda_m(\cdot)\in\mathcal{E} \;\Leftrightarrow\; \xi\in\mathcal{C}, \qquad (3)$$

where $\mathcal{C}$ is a convex set such as $\mathcal{C}_+ := \{c\in\mathbb{R}^m; \forall j=1,...,m : c_j \geq 0\}$. Adding a Gaussian prior on ξ gives



Effect of different constraints over Λ_m. Samples generated via Hamiltonian Monte Carlo (HMC) [2]. Shaded area: 95% confidence interval.

GP-Modulated Cox Processes & Cox Process Inference

- By considering $\Lambda_m(\cdot)$, the likelihood in (1) becomes

$$f_{(N,X_1,\cdots,X_n)|\{\xi_1,\cdots,\xi_m\}}(n,x_1,\cdots,x_n) = \frac{1}{n!}\exp\left(-\sum_{j=1}^{m}c_j\xi_j\right)\prod_{i=1}^{n}\sum_{j=1}^{m}\phi_j(x_i)\xi_j. \qquad (4)$$

  - Since (4) depends on $\xi_1,\cdots,\xi_m$, it can be approximated using samples of ξ.
  - Parameters $\boldsymbol{\theta}$ can be estimated by maximising (4) with a stochastic global optimisation algorithm.
- The posterior of ξ conditioned on a point pattern $(N=n, X_1=x_1,\cdots,X_n=x_n)$ is

$$f_{\boldsymbol{\xi}|\{N=n,X_1=x_1,\cdots,X_n=x_n\}}(\boldsymbol{\chi}) \propto f_{(N,X_1,\cdots,X_n)|\{\boldsymbol{\xi}=\boldsymbol{\chi}\}}(n,x_1,\cdots,x_n)\; f_{\boldsymbol{\xi}}(\boldsymbol{\chi}), \qquad (5)$$

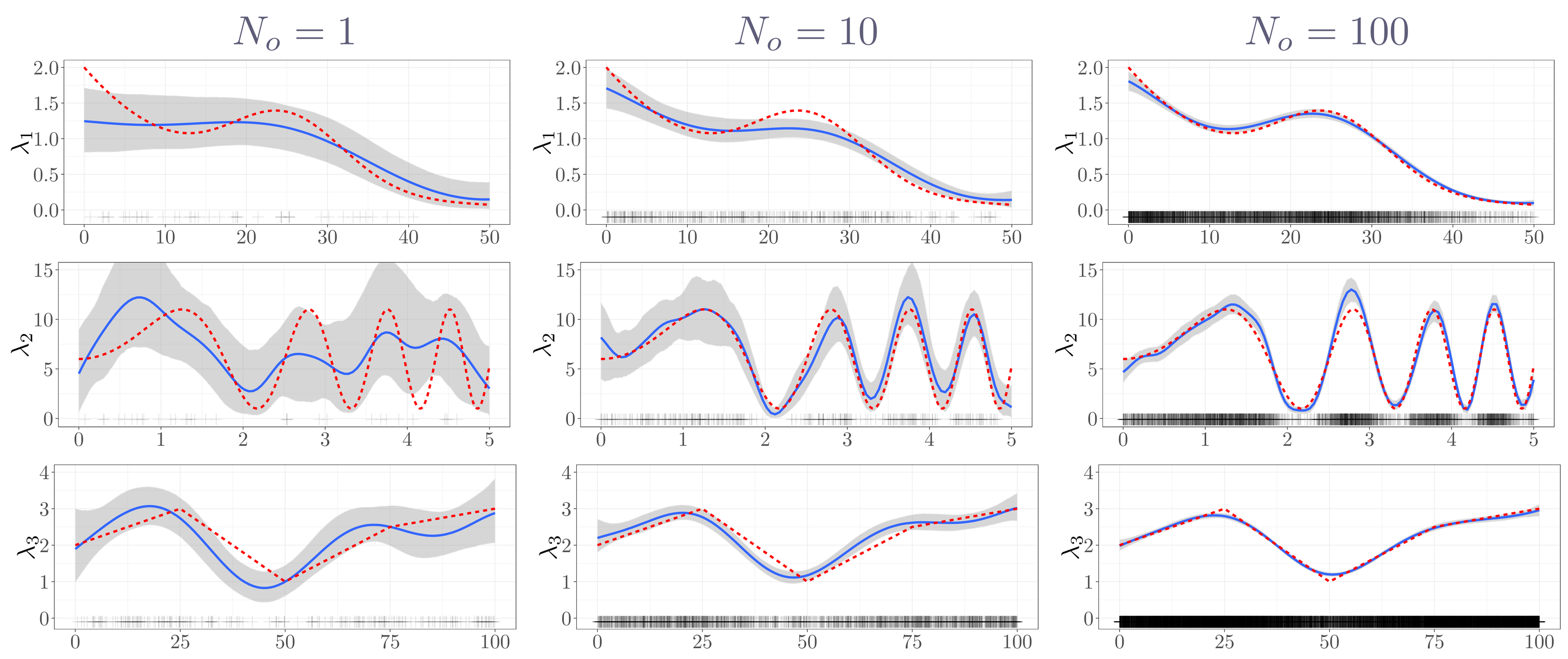
with likelihood (4) and truncated Gaussian density $f_{\boldsymbol{\xi}}(\boldsymbol{\chi})$,

$$f_{\boldsymbol{\xi}}(\boldsymbol{\chi}) = \frac{\exp\left\{-\frac{1}{2}\boldsymbol{\chi}^\top\boldsymbol{\Gamma}^{-1}\boldsymbol{\chi}\right\}}{\int_0^\infty \exp\left\{-\frac{1}{2}\mathbf{s}^\top\boldsymbol{\Gamma}^{-1}\mathbf{s}\right\}d\mathbf{s}}, \quad \text{for} \;\; \boldsymbol{\chi}\geq\mathbf{0}. \qquad (6)$$

  - Approximate (5) using samples of ξ, then infer $\Lambda_m(\cdot)$ via Metropolis-Hastings.
- Can be extended to higher dimensions ($d\geq 2$) with tensorisation (see paper).

Illustrations

Examples with Multiple Observations



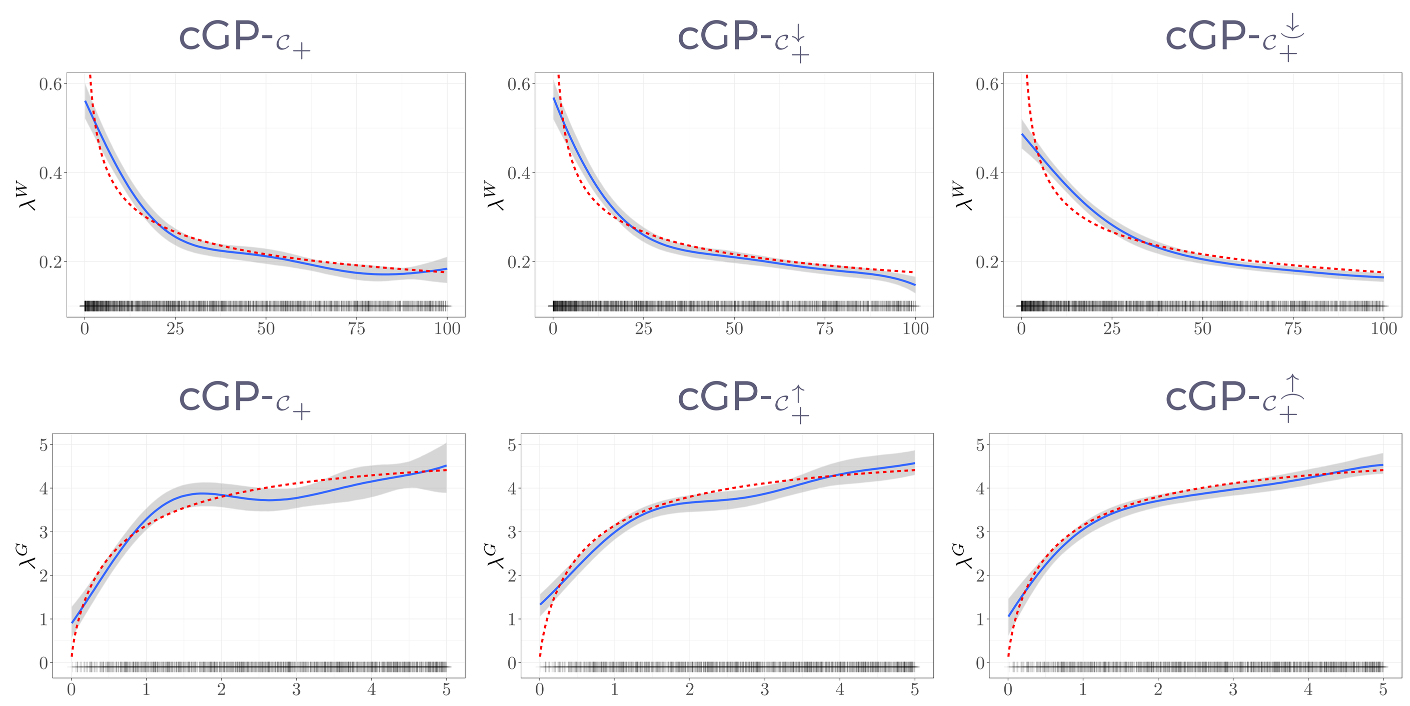
+ point patterns ■ λ ■ $\hat{\lambda}$ ■ 90% confidence interval

Inference with multiple observations using the toy examples from [3].

$Q^2 = 1 - \text{SMSE}$ results averaged over 20 (†10) replicates. Our method (cGP-c_+) is compared to log-GPs [4] and Variational Bayes for Point Processes (VBPP) [5].

Toy	N_o	Q^2 $(\mu\pm\sigma)$ [%] log-GP	VBPP	cGP-c_+
λ_1	1	51.2±30.1	51.9±26.1	**65.7±14.3**
	10	95.1± 3.9	94.6± 3.7	**95.4± 2.3**
	100	**99.5± 0.2**	**99.5± 0.3**	**99.5± 0.3**
λ_2	1	-35.2±43.4	-1.1±28.8	**0.7±24.0**
	10	72.6± 9.1	71.7±10.4	**81.9± 7.4**
	100	95.4± 0.7	92.1± 3.9	**97.8± 0.6**
λ_3	1	49.2±22.6	49.5±29.9	**58.1±21.4**
	10	91.7± 4.4	93.8± 2.8	**94.3± 2.5**
	100	98.4± 0.4	**98.9±0.3**†	**98.8± 0.3**

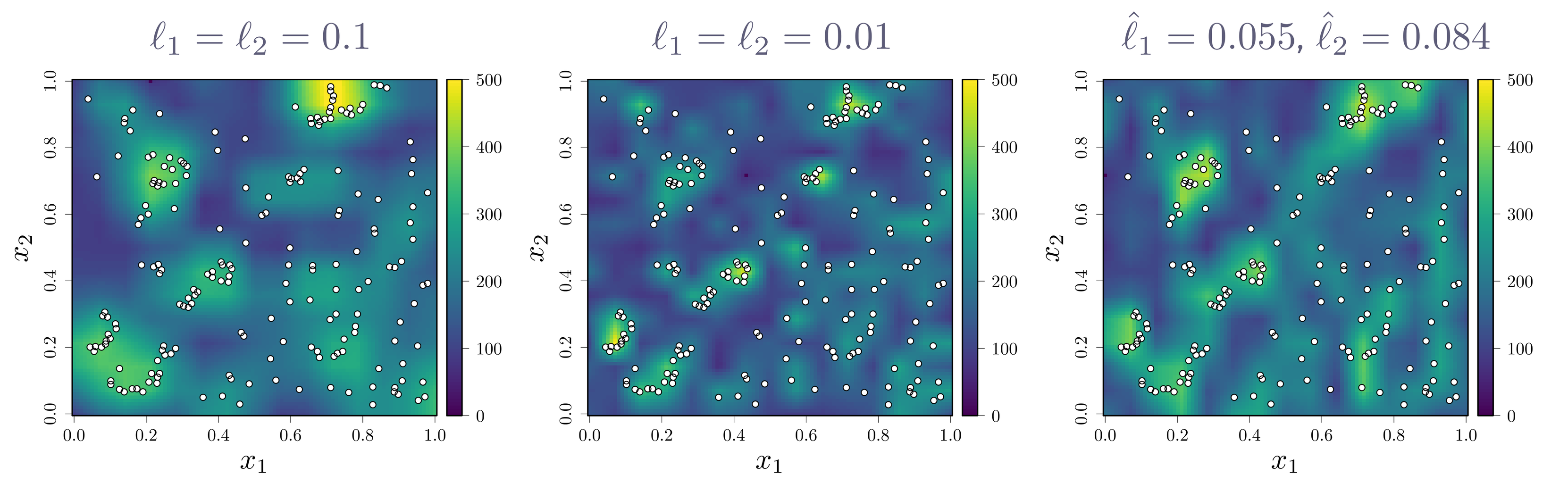
Modelling Hazard Rates in Renewal Processes



+ point patterns ■ λ ■ $\hat{\lambda}$ ■ 90% confidence interval

Inference for a Weibull process (top row) and a Gamma process (bottom row) under constraints: (left) positivity, (centre) + monotonicity, (right) + convexity.

2D Redwoods Data



Estimated intensity for the redwoods dataset [6] with different length-scale parameters (ℓ_1,ℓ_2).

Conclusions

- The proposed GP-modulated Cox process model is based on a finite-dimensional approximation of a GP that obeys linear inequality constraints.
- It allows us to compute analytically (1) and the intensity measure μ, which is not always the case when using a link function.
- Our dedicated Metropolis-Hastings algorithm with truncated Gaussian proposals allows inferring Λ_m with high acceptance and effective sampling rates.

Further discussion

The main limitation regarding the scaling with the number of dimensions lies in the construction by tensorisation. We believe that this downside is not inherent to the model and that other types of design of the basis functions could be used in high dimensions (e.g. sparse designs).

References

[1] A. F. López-Lopera, F. Bachoc, N. Durrande, and O. Roustant. Finite-dimensional Gaussian approximation with linear inequality constraints. *SIAM/ASA JUQ*, 2018.

[2] A. Pakman and L. Paninski. Exact Hamiltonian Monte Carlo for truncated multivariate Gaussians. *J. Comput. Graph. Stat.*, 2014.

[3] R. P. Adams, I. Murray, and D. J.C. MacKay. Tractable nonparametric Bayesian inference in Poisson processes with Gaussian process intensities. In *ICML*, 2009.

[4] J. Møller, A. Syversveen, and R. Waagepetersen. Log Gaussian Cox processes. *Scand. J. Stat.*, 2001.

[5] C. M. Lloyd, T. Gunter, M. A. Osborne, and S. J. Roberts. Variational inference for Gaussian process modulated Poisson processes. In *ICML*, 2015.

[6] A. Baddeley, E. Rubak, and R. Turner. *Spatial Point Patterns: Methodology and Applications with R*. Chapman & Hall/CRC Inter. Statistics. 2015.

Contact Information

Email: andres-felipe.lopez@emse.fr.
Website: https://sites.google.com/site/anfelopera/