Theorem 22.5 in Greene (2002)

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In this note I give a proof of Theorem 22.5 on p.781 in Greene (2002). The same theorem also appears in Greene (2018) as Theorem 19.5 on p.950. This theorem is fundamental to the derivation of Heckman's two-step estimator and a bias term expression cited in Breen et al. (2015).

The theorem states that if y and z have a bivariate normal distribution with means μ_y and μ_z, standard deviations σ_y and σ_z and correlation ρ, then

$$\mathbb{E}\left[Y|Z>a\right] = \mu_y + \rho\sigma_y\lambda(\tilde{a}),$$

where $\tilde{a} = (a-\mu_z)/\sigma_z$, and $\lambda(\tilde{z}) = \phi(\tilde{z})/[1-\Phi(\tilde{z})]$.

A key use of integration by parts is drawn from Weiller (1959).

We start by noting

$$f(y,z|Z>a) = \frac{f(y,z)}{P(Z>a)}.$$

Therefore,

$$\begin{aligned}
&f(y|Z>a) \\
=&\frac{1}{P(Z>a)}\int_a^\infty f(y,z)dz \\
=&\frac{1}{P(Z>a)}\int_a^\infty \frac{1}{2\pi\sigma_y\sigma_z\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{z-\mu_z}{\sigma_z}\right)^2 - 2\rho\left(\frac{z-\mu_z}{\sigma_z}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}dz
\end{aligned}$$

Simply denoting $(z-\mu_z)/\sigma_z$ by $\tilde{z}$ and similarly for $\tilde{y}$, but actual change-of-variable does not happen here.

$$\begin{aligned}
=&\frac{1}{P(Z>a)}\frac{1}{2\pi\sigma_y\sigma_z\sqrt{1-\rho^2}}\int_a^\infty \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}\right]\right\}\exp\left\{-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right\}dz \\
=&\frac{1}{P(Z>a)}\frac{1}{2\pi\sigma_y\sigma_z\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\int_a^\infty \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}\right]\right\}dz
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}(Y|Z>a) \\
=&\int_{-\infty}^\infty yf(y|Z>a)dy \\
=&\frac{1}{P(Z>a)}\frac{1}{2\pi\sigma_y\sigma_z\sqrt{1-\rho^2}}\int_{-\infty}^\infty y\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\int_a^\infty \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}\right]\right\}dzdy
\end{aligned}$$

Using change of variable, with $|J| = \sigma_y \sigma_z$, and letting $\tilde{a} = (a - \mu_z)/\sigma_z$,

$$= \frac{1}{P(Z > a)} \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} (\sigma_y \tilde{y} + \mu_y) \exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right] \int_{\tilde{a}}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2 - 2\rho\tilde{z}\tilde{y}\right]\right\} d\tilde{z}d\tilde{y}$$

Now note that

$$\begin{aligned}
&\frac{1}{P(Z > a)} \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \mu_y \exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right] \int_{\tilde{a}}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2 - 2\rho\tilde{z}\tilde{y}\right]\right\} d\tilde{z}d\tilde{y} \\
&= \mu_y \frac{1}{P(Z > a)} \int_{-\infty}^{\infty} \int_{\tilde{a}}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2 - 2\rho\tilde{z}\tilde{y} + \tilde{y}^2\right]\right\} d\tilde{z}d\tilde{y}
\end{aligned}$$

Letting $\tilde{Z} = (Z - \mu_z)/\sigma_z$,

$$\begin{aligned}
&= \mu_y \frac{1}{P(Z > a)} P(\tilde{Z} > \tilde{a}) \\
&= \mu_y
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}(Y|Z > a) \\
&= \mu_y + \frac{1}{P(Z > a)} \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \sigma_y \tilde{y} \exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right] \int_{\tilde{a}}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2 - 2\rho\tilde{z}\tilde{y}\right]\right\} d\tilde{z}d\tilde{y}
\end{aligned}$$

Using integraion by parts below (with regard to $\tilde{y}$),

$$= \mu_y - \frac{1}{P(Z > a)} \frac{\sigma_y(1-\rho^2)}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \left\{\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\right\}' \int_{\tilde{a}}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2 - 2\rho\tilde{z}\tilde{y}\right]\right\} d\tilde{z}d\tilde{y}$$

And

$$\begin{aligned}
&\frac{1}{P(Z > a)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left\{\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\right\} \int_{\tilde{a}}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2 - 2\rho\tilde{z}\tilde{y}\right]\right\} d\tilde{z}\Bigg|_{-\infty}^{\infty} \\
&= \frac{1}{P(\tilde{Z} > \tilde{a})} \int_{\tilde{a}}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2 - 2\rho\tilde{z}\tilde{y} + \tilde{y}^2\right]\right\} d\tilde{z}\Bigg|_{-\infty}^{\infty}
\end{aligned}$$

Letting $\tilde{Y} = (Y - \mu_y)/\sigma_y$, hence $\tilde{Y} \sim N(0, 1)$,

$$\begin{aligned}
&= f_{\tilde{Y}}(\tilde{y}|\tilde{Z} > \tilde{a})\Bigg|_{-\infty}^{\infty} \\
&= 0
\end{aligned}$$

Hence,

$$
\begin{aligned}
&\mathbb{E}(Y|Z>a)\\
&=\mu_y+\frac{1}{P(Z>a)}\frac{\sigma_y(1-\rho^2)}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\frac{\partial}{\partial\tilde{y}}\int_{\tilde{a}}^{\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}\right]\right\}d\tilde{z}d\tilde{y}\\
&=\mu_y+\frac{1}{P(Z>a)}\frac{\sigma_y(1-\rho^2)}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\int_{\tilde{a}}^{\infty}\frac{\partial}{\partial\tilde{y}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}\right]\right\}d\tilde{z}d\tilde{y}\\
&=\mu_y+\frac{1}{P(Z>a)}\frac{\sigma_y(1-\rho^2)}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\int_{\tilde{a}}^{\infty}\frac{\rho}{1-\rho^2}\tilde{z}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}\right]\right\}d\tilde{z}d\tilde{y}\\
&=\mu_y+\frac{\rho\sigma_y}{P(Z>a)}\int_{-\infty}^{\infty}\int_{\tilde{a}}^{\infty}\tilde{z}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}+\tilde{y}^2\right]\right\}d\tilde{z}d\tilde{y}
\end{aligned}
$$

Changing the order of integration by Fubini's theorem,

$$
=\mu_y+\frac{\rho\sigma_y}{P(Z>a)}\int_{\tilde{a}}^{\infty}\tilde{z}\int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}+\tilde{y}^2\right]\right\}d\tilde{y}d\tilde{z}
$$

Recognizing the inner integral as obtaining marginal density from joint standard normal density,

$$
\begin{aligned}
&=\mu_y+\frac{\rho\sigma_y}{P(Z>a)}\int_{\tilde{a}}^{\infty}\tilde{z}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\tilde{z}^2}{2}\right)d\tilde{z}\\
&=\mu_y+\frac{\rho\sigma_y}{P(Z>a)}\left[-\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\tilde{z}^2}{2}\right)\right]\Bigg|_{\tilde{a}}^{\infty}\\
&=\mu_y+\frac{\rho\sigma_y}{P(Z>a)}\left[\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\tilde{a}^2}{2}\right)\right]\\
&=\mu_y+\rho\sigma_y\frac{\phi(\tilde{a})}{1-\Phi(\tilde{a})}\\
&=\mu_y+\rho\sigma_y\lambda(\tilde{a})
\end{aligned}
$$

Q.E.D. And note that $\lambda(\cdot)$ is called the inverse Mills ratio.

References

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