Proof of Theorem 22.5 in Greene (2002)

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In this note I give a proof of Theorem 22.5 on p.781 in Greene (2002). The same theorem also appears in Greene (2018) as Theorem 19.5 on p.950. This theorem is fundamental to the derivation of Heckman's two-step estimator and a bias term expression cited in Breen et al. (2015).

The theorem states that if y and z have a bivariate normal distribution with means μ_y and μ_z, standard deviations σ_y and σ_z and correlation ρ, then

$$\mathbb{E}\left[Y|Z > a\right] = \mu_y + \rho\sigma_y\lambda(\tilde{a}),$$

where $\tilde{a} = (a - \mu_z)/\sigma_z$, and $\lambda(\tilde{a}) = \phi(\tilde{a})/[1 - \Phi(\tilde{a})]$.

A key use of integration by parts is drawn from Weiller (1959).

We start by noting

$$f(y, z|Z > a) = \frac{f(y, z)}{P(Z > a)}.$$

Therefore,

$$\begin{aligned}
& f(y|Z > a) \\
=& \frac{1}{P(Z > a)} \int_a^\infty f(y, z)dz \\
=& \frac{1}{P(Z > a)} \int_a^\infty \frac{1}{2\pi\sigma_y\sigma_z\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{z-\mu_z}{\sigma_z}\right)^2 - 2\rho\left(\frac{z-\mu_z}{\sigma_z}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\} dz
\end{aligned}$$

Simply denoting $(z - \mu_z)/\sigma_z$ by $\tilde{z}$ and similarly for $\tilde{y}$, but actual change-of-variable does not happen here.

$$\begin{aligned}
=& \frac{1}{P(Z > a)} \frac{1}{2\pi\sigma_y\sigma_z\sqrt{1-\rho^2}} \int_a^\infty \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2 - 2\rho\tilde{z}\tilde{y}\right]\right\} \exp\left\{-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right\} dz \\
=& \frac{1}{P(Z > a)} \frac{1}{2\pi\sigma_y\sigma_z\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right] \int_a^\infty \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2 - 2\rho\tilde{z}\tilde{y}\right]\right\} dz
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E}(Y|Z > a) \\
=& \int_{-\infty}^\infty y f(y|Z > a)dy \\
=& \frac{1}{P(Z > a)} \frac{1}{2\pi\sigma_y\sigma_z\sqrt{1-\rho^2}} \int_{-\infty}^\infty y \exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right] \int_a^\infty \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2 - 2\rho\tilde{z}\tilde{y}\right]\right\} dzdy
\end{aligned}$$

Using change-of-variable to make $\tilde{y}$ and $\tilde{z}$ the variables, with $|J| = \sigma_y\sigma_z$, and letting $\tilde{a} = (a-\mu_z)/\sigma_z$,

$$= \frac{1}{P(Z>a)}\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}(\sigma_y\tilde{y}+\mu_y)\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\int_{\tilde{a}}^{\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}\right]\right\}d\tilde{z}d\tilde{y}$$

Now note that

$$\begin{aligned}
&\frac{1}{P(Z>a)}\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\mu_y\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\int_{\tilde{a}}^{\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}\right]\right\}d\tilde{z}d\tilde{y}\\
&=\mu_y\frac{1}{P(Z>a)}\int_{-\infty}^{\infty}\int_{\tilde{a}}^{\infty}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}+\tilde{y}^2\right]\right\}d\tilde{z}d\tilde{y}
\end{aligned}$$

Letting $\tilde{Z} = (Z-\mu_z)/\sigma_z$,

$$\begin{aligned}
&=\mu_y\frac{1}{P(Z>a)}P(\tilde{Z}>\tilde{a})\\
&=\mu_y
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}(Y|Z>a)\\
&=\mu_y+\frac{1}{P(Z>a)}\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\sigma_y\tilde{y}\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\int_{\tilde{a}}^{\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}\right]\right\}d\tilde{z}d\tilde{y}
\end{aligned}$$

Using integraion by parts below (with regard to $\tilde{y}$),

$$=\mu_y-\frac{1}{P(Z>a)}\frac{\sigma_y(1-\rho^2)}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\left\{\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\right\}'\int_{\tilde{a}}^{\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}\right]\right\}d\tilde{z}d\tilde{y}$$

And

$$\begin{aligned}
&\frac{1}{P(Z>a)}\frac{1}{2\pi\sqrt{1-\rho^2}}\left\{\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\right\}\int_{\tilde{a}}^{\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}\right]\right\}d\tilde{z}\Bigg|_{-\infty}^{\infty}\\
&=\frac{1}{P(\tilde{Z}>\tilde{a})}\int_{\tilde{a}}^{\infty}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}+\tilde{y}^2\right]\right\}d\tilde{z}\Bigg|_{-\infty}^{\infty}
\end{aligned}$$

Letting $\tilde{Y} = (Y-\mu_y)/\sigma_y$, hence $(\tilde{Y},\tilde{Z}) \sim N(0,0,1,1,\rho)$. And recognizing the density of joint standard normal density,

$$\begin{aligned}
&=\int_{\tilde{a}}^{\infty}\frac{1}{P(\tilde{Z}>\tilde{a})}f_{\tilde{Z},\tilde{Y}}(\tilde{z},\tilde{y})\,d\tilde{z}\Bigg|_{-\infty}^{\infty}\\
&=f_{\tilde{Y}}(\tilde{y})\Bigg|_{-\infty}^{\infty}\\
&=0
\end{aligned}$$

Hence,

$$
\begin{aligned}
&\mathbb{E}(Y|Z>a)\\
&= \mu_y + \frac{1}{P(Z>a)}\frac{\sigma_y(1-\rho^2)}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\frac{\partial}{\partial\tilde{y}}\int_{\tilde{a}}^{\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}\right]\right\}d\tilde{z}d\tilde{y}\\
&= \mu_y + \frac{1}{P(Z>a)}\frac{\sigma_y(1-\rho^2)}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\int_{\tilde{a}}^{\infty}\frac{\partial}{\partial\tilde{y}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}\right]\right\}d\tilde{z}d\tilde{y}\\
&= \mu_y + \frac{1}{P(Z>a)}\frac{\sigma_y(1-\rho^2)}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2(1-\rho^2)}\tilde{y}^2\right]\int_{\tilde{a}}^{\infty}\frac{\rho}{1-\rho^2}\tilde{z}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}\right]\right\}d\tilde{z}d\tilde{y}\\
&= \mu_y + \frac{\rho\sigma_y}{P(Z>a)}\int_{-\infty}^{\infty}\int_{\tilde{a}}^{\infty}\tilde{z}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}+\tilde{y}^2\right]\right\}d\tilde{z}d\tilde{y}
\end{aligned}
$$

Changing the order of integration by Fubini's theorem,

$$
= \mu_y + \frac{\rho\sigma_y}{P(Z>a)}\int_{\tilde{a}}^{\infty}\tilde{z}\int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\tilde{z}^2-2\rho\tilde{z}\tilde{y}+\tilde{y}^2\right]\right\}d\tilde{y}d\tilde{z}
$$

Recognizing the inner integral as obtaining marginal density from joint standard normal density,

$$
\begin{aligned}
&= \mu_y + \frac{\rho\sigma_y}{P(Z>a)}\int_{\tilde{a}}^{\infty}\tilde{z}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\tilde{z}^2}{2}\right)d\tilde{z}\\
&= \mu_y + \frac{\rho\sigma_y}{P(Z>a)}\left[-\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\tilde{z}^2}{2}\right)\right]\Bigg|_{\tilde{a}}^{\infty}\\
&= \mu_y + \frac{\rho\sigma_y}{P(Z>a)}\left[\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\tilde{a}^2}{2}\right)\right]
\end{aligned}
$$

Letting $\phi(\cdot)$ and $\Phi(\cdot)$ be the standard normal pdf and cdf,

$$
\begin{aligned}
&= \mu_y + \rho\sigma_y\frac{\phi(\tilde{a})}{1-\Phi(\tilde{a})}\\
&= \mu_y + \rho\sigma_y\lambda(\tilde{a})
\end{aligned}
$$

Q.E.D. And note that $\lambda(\cdot)$ is called the inverse Mills ratio.

References

Richard Breen, Seungsoo Choi, and Anders Holm. Heterogeneous Causal Effects and Sample Selection Bias. *Sociological Science*, 2:351–369, 2015. doi: 10.15195/v2.a17.

William Greene. *Econometric Analysis*. Prentice Hall, 5th edition, 2002.

William Greene. *Econometric Analysis*. Pearson, 8th edition, 2018.

H Weiller. Means and Standard Deviations of a Truncated Normal Bivariate Distribution. *Australian Journal of Statistics*, 1(3):73–81, November 1959. doi: 10.1111/j.1467-842X.1959.tb00277.x.