Key Theorems for computing $[I]^{s/o}$

There are three useful theorems to compute moments of inertia. They are:-

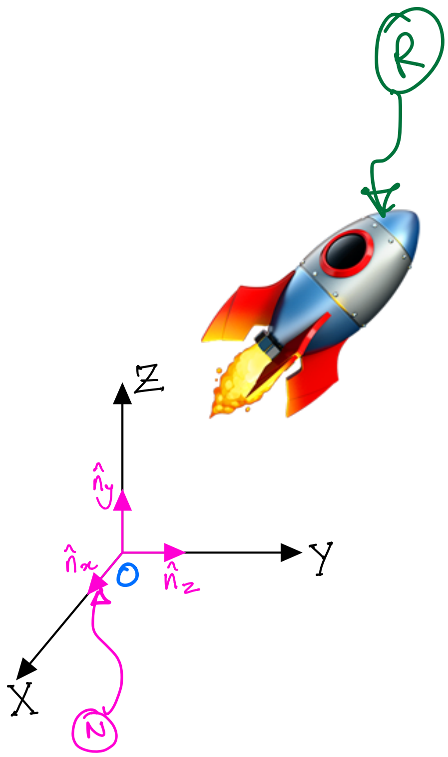
(i) Rotation theorem

(ii) Parallel axes theorem (or translation theorem)

(iii) Composite Theorem

Let us examine them in further detail.

(i) Rotation theorem:



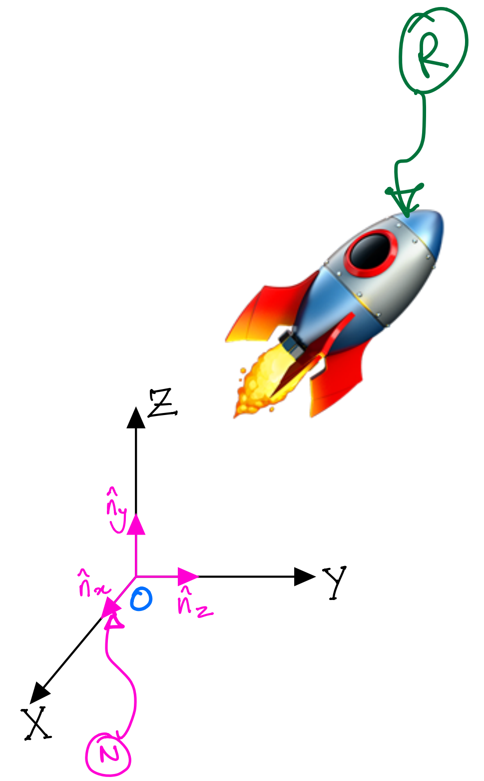
- In the figure X-Y-Z make up a cartesian coordinate system.

O is the origin.

$\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$ are unit vectors directed along X, Y, and Z respectively. The unit vectors represent a reference frame N.

- Also, the figure shows a rocket which has a reference frame R attached to it.

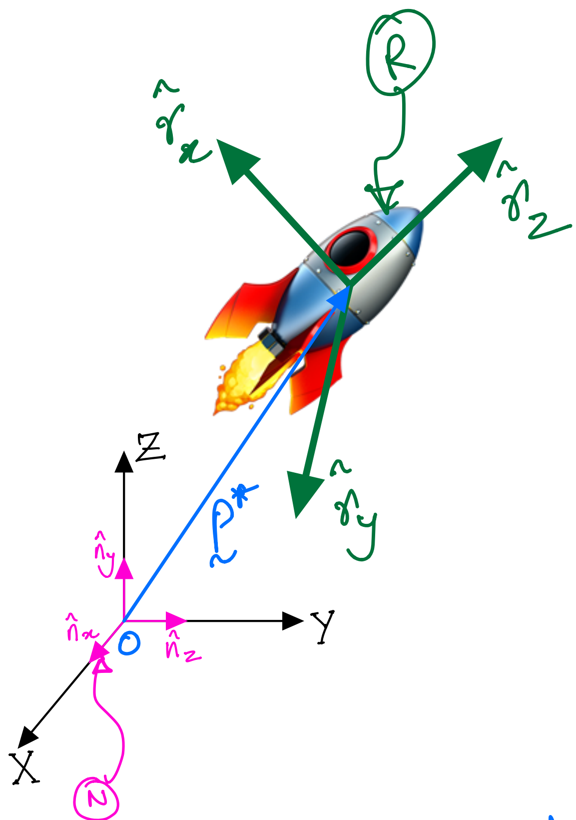
(i) <u>Rotation theorem</u>:



- In the figure X-Y-Z make up a cartesian coordinate system. O is the origin. $\hat{n}_x, \hat{n}_y, \hat{n}_z$ are unit vectors directed along X, Y, and Z respectively. The unit vectors represent a reference frame N.

- Also, the figure shows a rocket which has a reference frame R attached to it. You are given the inertia matrix of the rocket R about O along the unit vectors of frame N:

$$[I]^{R/O} = \begin{bmatrix} I_{xx}^{R/O} & I_{xy}^{R/O} & I_{xz}^{R/O} \\ I_{yx}^{R/O} & I_{yy}^{R/O} & I_{yz}^{R/O} \\ I_{zx}^{R/O} & I_{zy}^{R/O} & I_{zz}^{R/O} \end{bmatrix}$$



- So far, given:-

$$[I]^{R/O} = \begin{bmatrix} I_{xx}^{R/O} & I_{xy}^{R/O} & I_{xz}^{R/O} \\ I_{yx}^{R/O} & I_{yy}^{R/O} & I_{yz}^{R/O} \\ I_{zx}^{R/O} & I_{zy}^{R/O} & I_{zz}^{R/O} \end{bmatrix}$$

- Also, the figure shows a rocket which has a reference frame R attached to it. Naturally, this reference frame also has 3 mutually orthogonal unit vectors: $\hat{r}_x$ $\hat{r}_y$ $\hat{r}_z$

- Now, we can define another inertia matrix for R about O along the newly introduced rotating reference frame's unit vectors $\hat{r}_x$, $\hat{r}_y$ and $\hat{r}_z$. This matrix and it's elements are:-

$$[J]^{R/O} = \begin{bmatrix} J_{xx}^{R/O} & J_{xy}^{R/O} & J_{xz}^{R/O} \\ J_{yx}^{R/O} & J_{yy}^{R/O} & J_{yz}^{R/O} \\ J_{zx}^{R/O} & J_{zy}^{R/O} & J_{zz}^{R/O} \end{bmatrix}$$

Question: How are the two matrices related?

- We begin by assuming that R is a particle of mass, m_R.
- Let us consider a product of inertia from each matrix.

$$I_{xy}^{R/O} \triangleq m_R\left(\underset{\sim}{p}^* \times \hat{n}_x\right) \cdot \left(\underset{\sim}{p}^* \times \hat{n}_y\right)$$

$$J_{xy}^{R/O} \triangleq m_R\left(\underset{\sim}{p}^* \times \hat{r}_x\right) \cdot \left(\underset{\sim}{p}^* \times \hat{r}_y\right)$$

- $\underset{\sim}{p}^*$ is a position vector from O to the origin of the frame R.

We know from our discussion on direction cosine matrices that the unit vectors of R can be related to the unit vectors of N as:-

$$\begin{bmatrix} \hat{r}_x \\ \hat{r}_y \\ \hat{r}_z \end{bmatrix} = {}^{R}C^{N} \begin{bmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{bmatrix}$$

${}^{R}C^{N}$ is the direction cosine matrix of R in N. It is a 3-by-3 matrix, which we shall assume has the following elements.

$${}^{R}C^{N} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

So, we can now easily derive:-

$$\hat{r}_x = C_{11}\hat{n}_x + C_{12}\hat{n}_y + C_{13}\hat{n}_z$$

$$\hat{r}_y = C_{21}\hat{n}_x + C_{22}\hat{n}_y + C_{23}\hat{n}_z$$

And then, we can rewrite the product of inertia $J_{xy}^{R/O}$ as:-

$$J_{xy}^{R/O} \triangleq m_R\left(\underset{\sim}{p}^* \times \hat{r}_x\right)\cdot\left(\underset{\sim}{p}^* \times \hat{r}_y\right)$$

$$J_{xy}^{R/O} \triangleq m_R\left(\underset{\sim}{p}^* \times \left(C_{11}\hat{n}_x + C_{12}\hat{n}_y + C_{13}\hat{n}_z\right)\right)\cdot\left(\underset{\sim}{p}^* \times \left(C_{21}\hat{n}_x + C_{22}\hat{n}_y + C_{23}\hat{n}_z\right)\right)$$

This leads to

$$\begin{aligned} J_{xy}^{R/O} \triangleq\ & C_{11}\left(I_{xx}^{R/O}C_{21} + I_{xy}^{R/O}C_{22} + I_{xz}^{R/O}C_{23}\right) \\ & + C_{12}\left(I_{yx}^{R/O}C_{21} + I_{yy}^{R/O}C_{22} + I_{yz}^{R/O}C_{23}\right) \\ & + C_{13}\left[I_{zx}^{R/O}C_{21} + I_{zy}^{R/O}C_{22} + I_{zz}^{R/O}C_{23}\right) \end{aligned}$$

or in terms of matrix multiplication.

$$J_{xy}^{R/O} \triangleq \begin{bmatrix} C_{11} & C_{12} & C_{13} \end{bmatrix} \begin{bmatrix} I_{xx}^{R/O} & I_{xy}^{R/O} & I_{xz}^{R/O} \\ I_{yx}^{R/O} & I_{yy}^{R/O} & I_{yz}^{R/O} \\ I_{zx}^{R/O} & I_{zy}^{R/O} & I_{zz}^{R/O} \end{bmatrix} \begin{bmatrix} C_{21} \\ C_{22} \\ C_{23} \end{bmatrix}$$

$$J_{xy}^{R/O} \triangleq \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} \end{bmatrix}}_{\text{first row of } {}^{R}C^{N}} \underbrace{\begin{bmatrix} I_{xx}^{R/O} & I_{xy}^{R/O} & I_{xz}^{R/O} \\ I_{yx}^{R/O} & I_{yy}^{R/O} & I_{yz}^{R/O} \\ I_{zx}^{R/O} & I_{zy}^{R/O} & I_{zz}^{R/O} \end{bmatrix}}_{[I]^{R/O}} \underbrace{\begin{bmatrix} C_{21} \\ C_{22} \\ C_{23} \end{bmatrix}}_{\text{transpose of second row of } {}^{R}C^{N}}$$

More generally, it can be shown that

$$\boxed{[J]^{R/O} = {}^{R}C^{N} \, [I]^{R/O} \, \underbrace{\left({}^{R}C^{N}\right)^{T}}_{\text{transpose of } {}^{R}C^{N}}}$$

Rotation theorem.

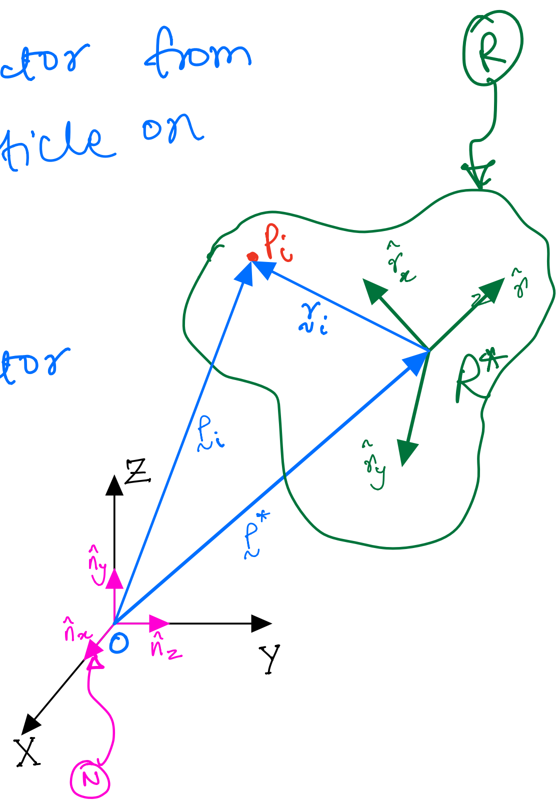
teorema huygens

(ii) Parallel axes theorem :-

In the previous section we presumed that the rocket was a point mass.
What happens if this assumption is relaxed and rockets are better approximated as a system of particles?

- $\underset{\sim}{p}^*$ is a position vector from O to R^*, the origin of the frame R.
- R^* is the mass center of the Rocket.
- $\underset{\sim}{p}_i$ is a position vector from O to P_i, the i^{th} particle on the rocket body.
- $\underset{\sim}{r}_i$ is a position vector from R^* to P_i. So,

$$\underset{\sim}{p}_i = \underset{\sim}{p}^* + \underset{\sim}{r}_i$$



I_rr* è dato dall'intera massa m che moltiplica il prodotto dot (cross(p*,O.x), cross(p*,O.x))   per Ixx .....

esattamente come in huygens Ia = Ib + mL^2;   intera massa per l *asse l

Now, we know that the product of inertia for this system of particles along the $\hat{n}_x$ and $\hat{n}_y$ directions is given by:-

$$I_{xy}^{R/O} \triangleq \sum_i m_i (\underset{\sim}{p}_i \times \hat{n}_x) \cdot (\underset{\sim}{p}_i \times \hat{n}_y)$$

or...

$$I_{xy}^{R/O} \triangleq \sum_i m_i \left((\underset{\sim}{p}^* + \underset{\sim}{r}_i) \times \hat{n}_x \right) \cdot \left((\underset{\sim}{p}^* + \underset{\sim}{r}_i) \times \hat{n}_y \right)$$

or...

$$I_{xy}^{R/O} = \sum_i m_i (\underset{\sim}{p}^* \times \hat{n}_x) \cdot (\underset{\sim}{p}^* \times \hat{n}_y) + (\underset{\sim}{p}^* \times \hat{n}_x) \cdot \left(\sum_i m_i \underset{\sim}{r}_i \times \hat{n}_y \right) + \left(\sum_i m_i \underset{\sim}{r}_i \times \hat{n}_x \right) \cdot (\underset{\sim}{p}^* \times \hat{n}_y) + \sum_i m_i (\underset{\sim}{r}_i \times \hat{n}_x) \cdot (\underset{\sim}{r}_i \times \hat{n}_y)$$

The terms highlighted in green go to zero because $\sum_i m_i \underset{\sim}{r}_i = 0$ by the definition of mass centers.

$$\boxed{I_{xy}^{R/O} = I_{xy}^{R/R^*} + I_{xy}^{R^*/O}}$$

I_{xy}^{R/R^*} is the product of inertia of R about the mass center R^*.

$I_{xy}^{R^*/O}$ is the product of inertia of R^* about O.

Some comments:-

1. This result extrapolates to the moment of inertia scalars. For example,

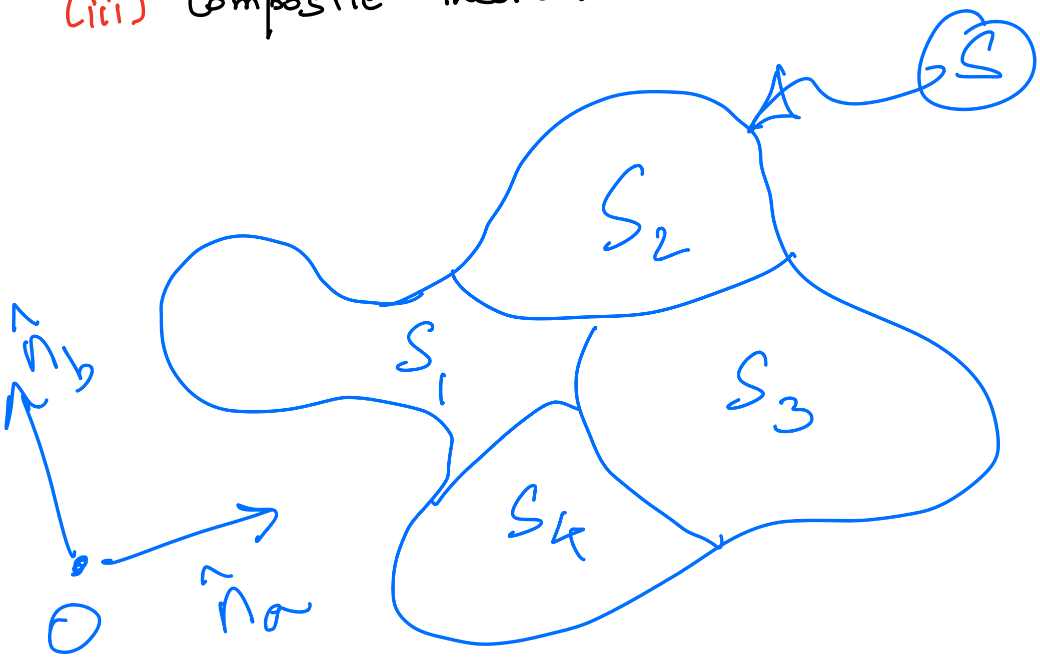
$$\underline{I}_{xx}^{R/O} = \underline{I}_{xx}^{R/R^*} + \underline{I}_{xx}^{R^*/O}$$

2. This result extrapolates to the inertia matrix.

$$[\underline{I}]^{R/O} = [\underline{I}]^{R/R^*} + [\underline{I}]^{R^*/O}$$

3. This rule is only valid when going through the mass centre.

(iii) Composite Theorem



$$I_{ab}^{S/O} = \sum_i I_{ab}^{S_i/O}$$