Mass / Inertia Scalars

- **Mass**: measure of amount of material in a body. Units of measurement :- kg, lb.
- **Mass center**: Consider a set of Particles as shown below

The i^{th} Particle has a mass m_i

P_1 (m_1)

P_2 (m_2)

P_5 (m_5)

P_3 (m_3)

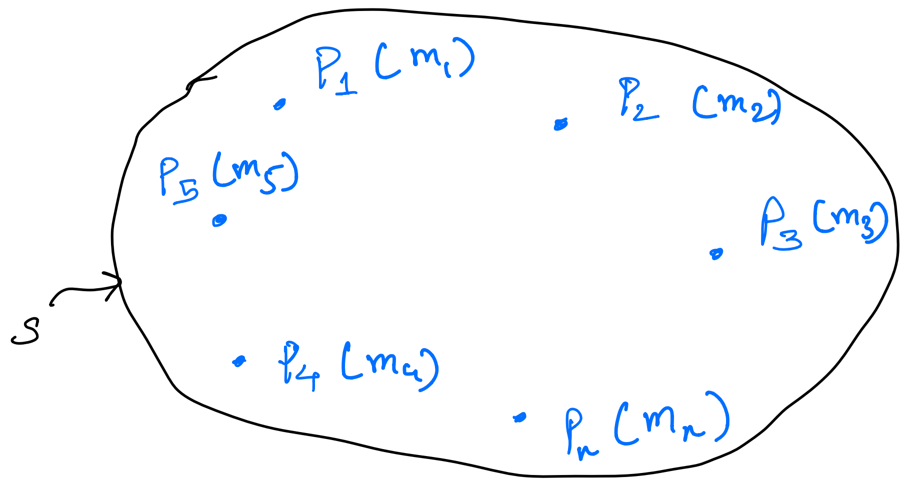
P_4 (m_4)

P_n (m_n)

Mass / Inertia Scalars

- Mass: measure of amount of material in a body.
- Mass center: Consider a set of Particles as shown below which together make up the system of particles S.

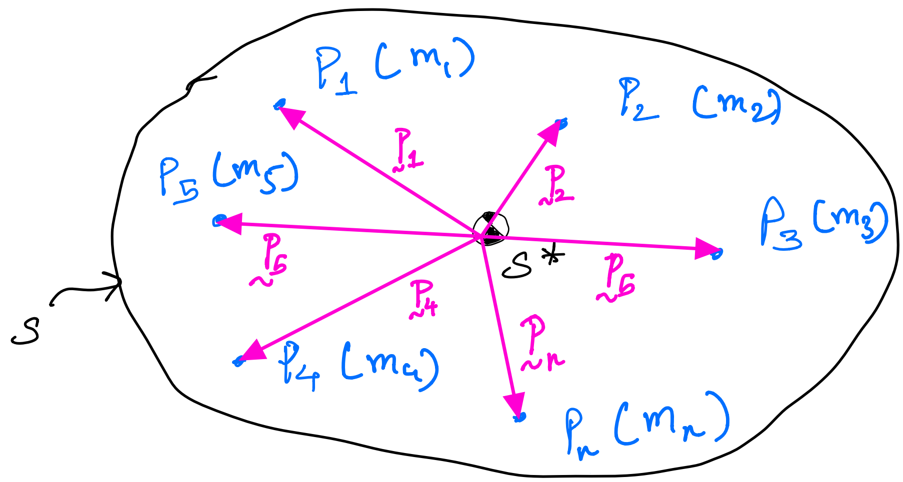
The i^{th} Particle has a mass m_i



Mass / Inertia Scalars

- **Mass**: measure of amount of material in a body.
- **Mass center**: Consider a set of particles as shown below which together make up the system of particles S.

The i^{th} particle has a mass m_i

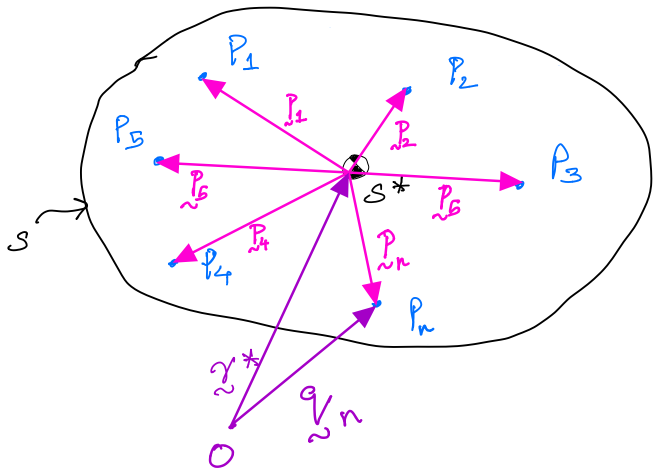


S^* is a fictitious particle such that

$$\sum_i m_i \underset{\sim}{p}_i = 0 \qquad (8.1)$$

This fictitious particle is called mass centre.

How does one locate the mass centre from a point O ?



$\underset{\sim}{r}^*$ Position vector from O to S^*

$\underset{\sim}{q}_n$ position vector from O to P_n

So, we have

$$\underset{\sim}{p}_n = \underset{\sim}{r}^* - \underset{\sim}{q}_n$$

So, expanding 8.1

$$\sum m_i \underset{\sim}{p}_i = 0$$

$$\Rightarrow \quad m_1 \underset{\sim}{p}_1 + m_2 \underset{\sim}{p}_2 + \dots + m_n \underset{\sim}{p}_n = 0$$

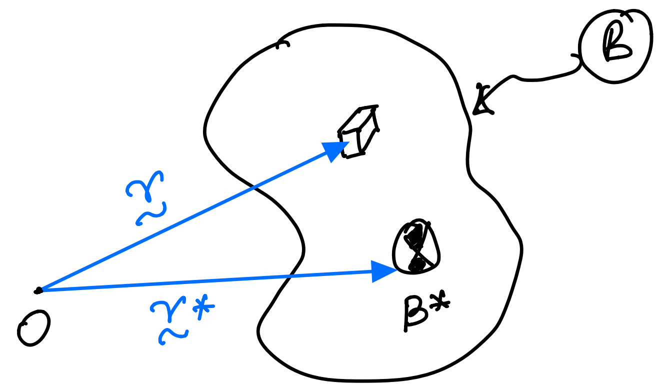
$$\Rightarrow m_1\left(\underset{\sim}{r}^* - \underset{\sim}{q}_1\right) + m_2\left(\underset{\sim}{r}^* - \underset{\sim}{q}_2\right) + \dots + m_n\left(\underset{\sim}{r}^* - \underset{\sim}{q}_n\right) = 0$$

$$\Rightarrow \underset{\sim}{r}^*\left(m_1 + m_2 + \dots + m_n\right) = m_1\underset{\sim}{q}_1 + m_2\underset{\sim}{q}_2 + \dots + m_n\underset{\sim}{q}_n$$

$$\Rightarrow \underset{\sim}{r}^* = \frac{m_1\underset{\sim}{q}_1 + m_2\underset{\sim}{q}_2 + \dots + m_n\underset{\sim}{q}_n}{\left(m_1 + m_2 + \dots + m_n\right)}$$

$$\boxed{\underset{\sim}{r}^* = \frac{\sum_i m_i \underset{\sim}{q}_i}{\sum_i m_i}} \qquad 8.2$$

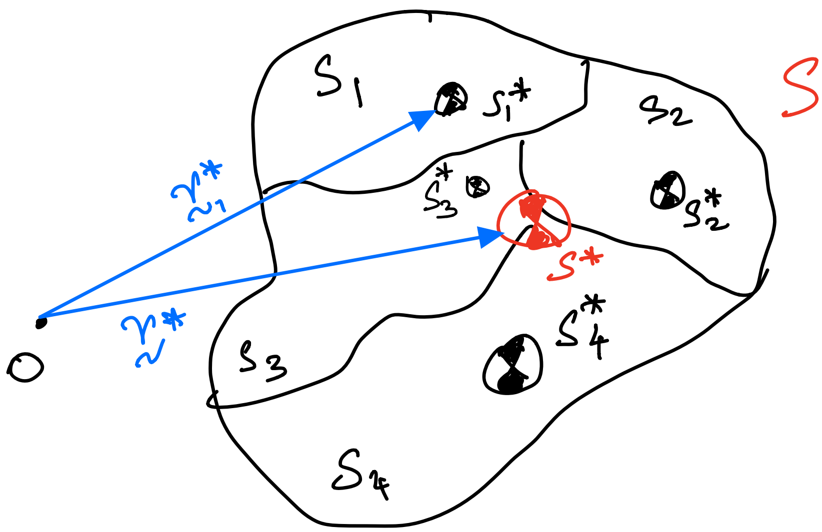
For a continua:-



$$\underset{\sim}{r}^* = \frac{\int \underset{\sim}{r}\, dm}{\int dm} = \frac{\int \rho\, \underset{\sim}{r}\, dV}{\int \rho\, dV} \qquad (8.3)$$

where dm is elemental mass that can be obtained from density ρ and the elemental volume dV

- Composite theorem for mass centre.



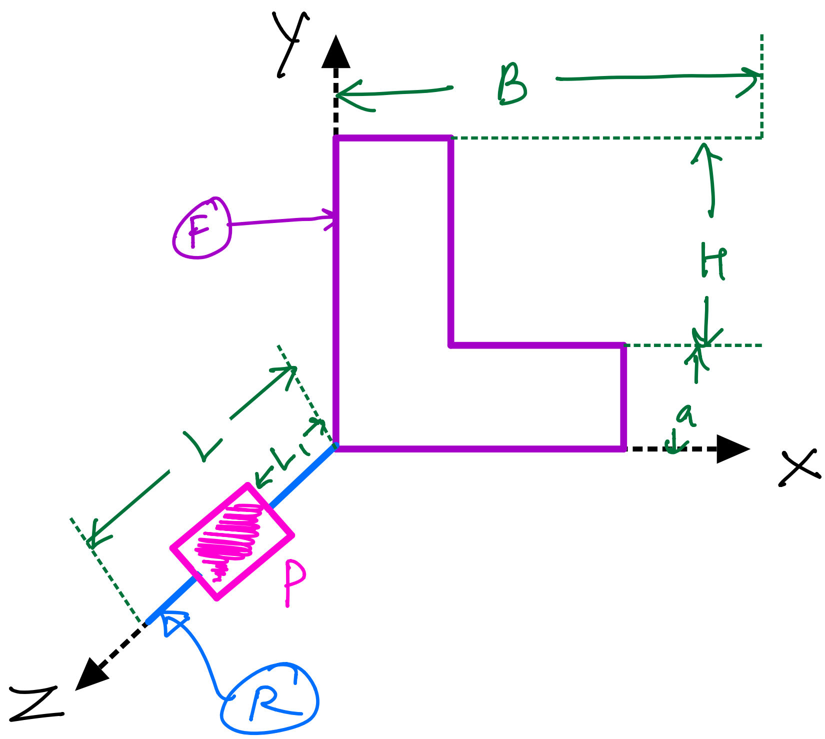
$$\underset{\sim}{r}^* = \frac{m_{s_1}\underset{\sim}{r}_1^* + m_{s_2}\underset{\sim}{r}_2^* + m_{s_3}\underset{\sim}{r}_3^* + \dots}{m_{s_1} + m_{s_2} + m_{s_3} \dots} \quad \text{—— } (8.4)$$

$\underset{\sim}{r}_i^*$ is the position vector locating the mass centre of S_i, the i^{th} system of particles

m_{s_i} is the mass of i^{th} System

$\underset{\sim}{r}^*$ is the mass centre of the composite system S

Example:



Given

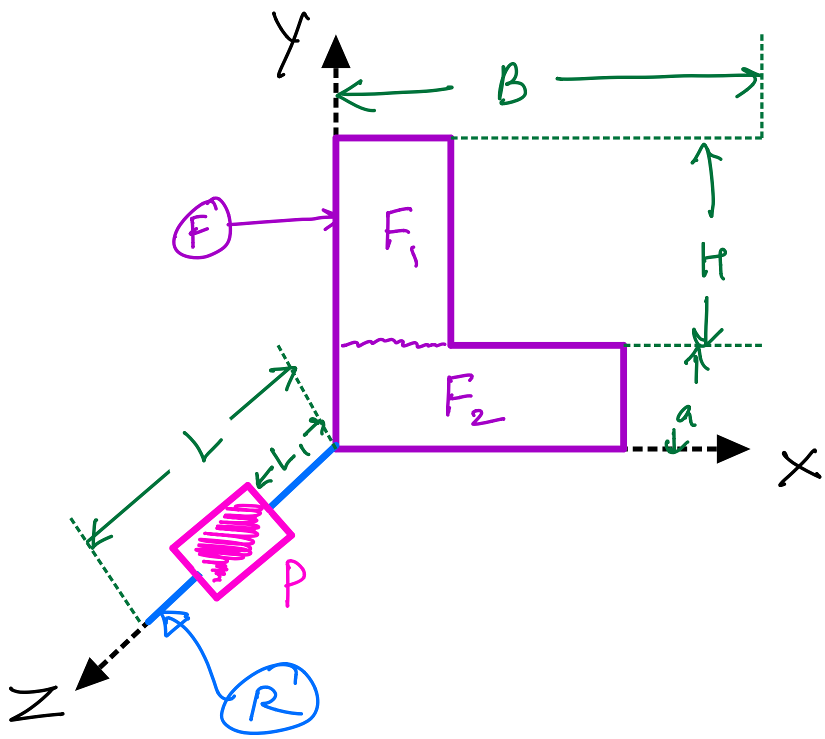
F and R are two bodies of mass density ρ kg/m^2 and σ kg/m respectively.

P is a particle of mass m.

Dimension are as shown on figure.

Find: mass center of the combined system.

Example:



F is split into two: F_1 and F_2

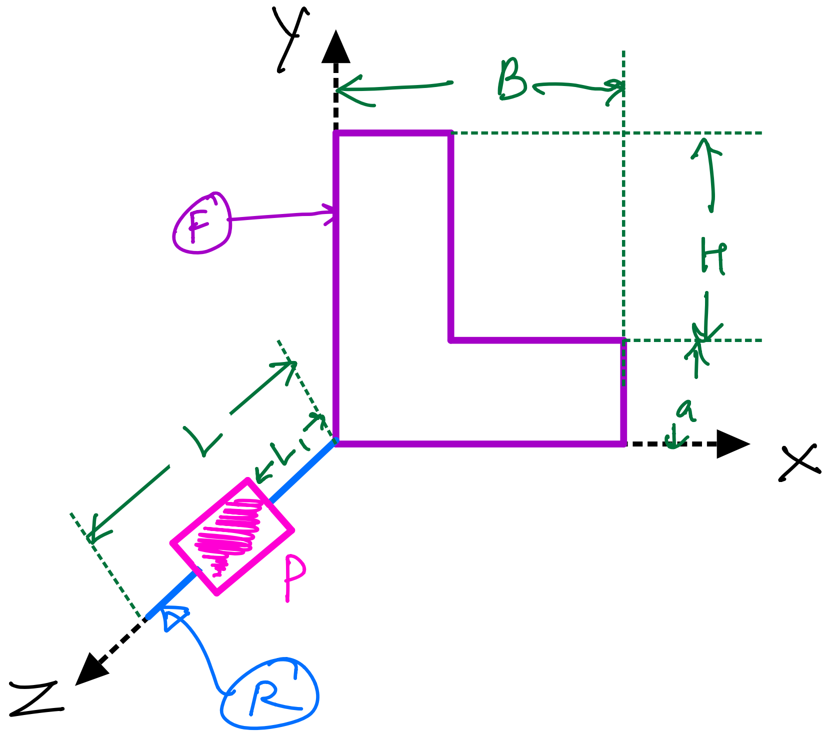
$m_{F_1} = \rho H a$ is mass of F_1

$m_{F_2} = \rho B a$ is mass of F_2

Also

$m_R = \sigma L$ is mass of R

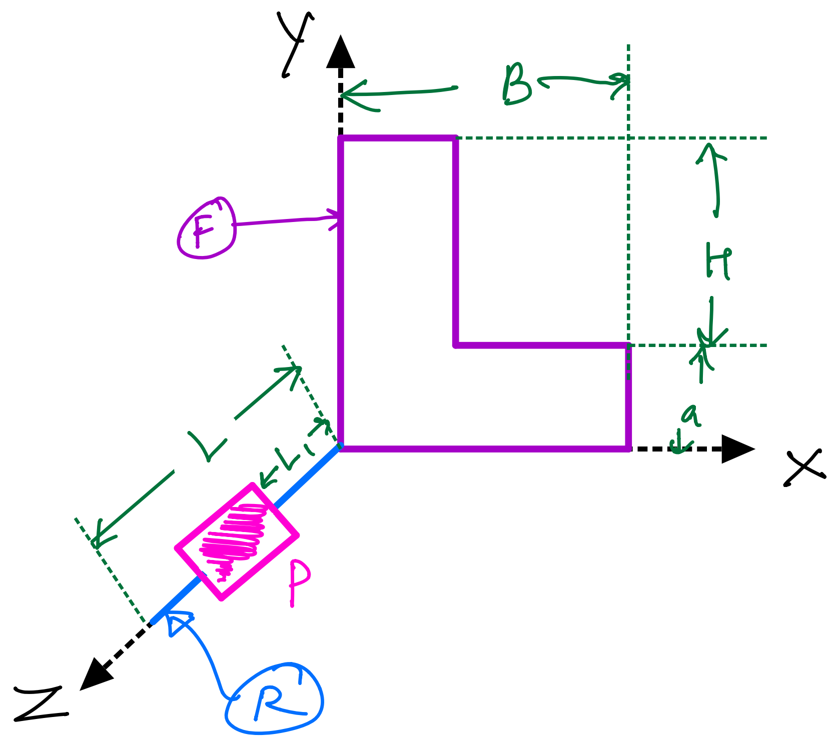
Example:



Then,

$$X^* = \frac{m_{F_1}\frac{a}{2} + m_{F_2}\frac{B}{2} + m_R(0) + m(0)}{m_{F_1} + m_{F_2} + m_R + m}$$

Example:



Then,

$$X^* = \frac{m_{F_1}\frac{a}{2} + m_{F_2}\frac{B}{2} + m_R(0) + m(0)}{m_{F_1} + m_{F_2} + m_R + m}$$

Similarly,

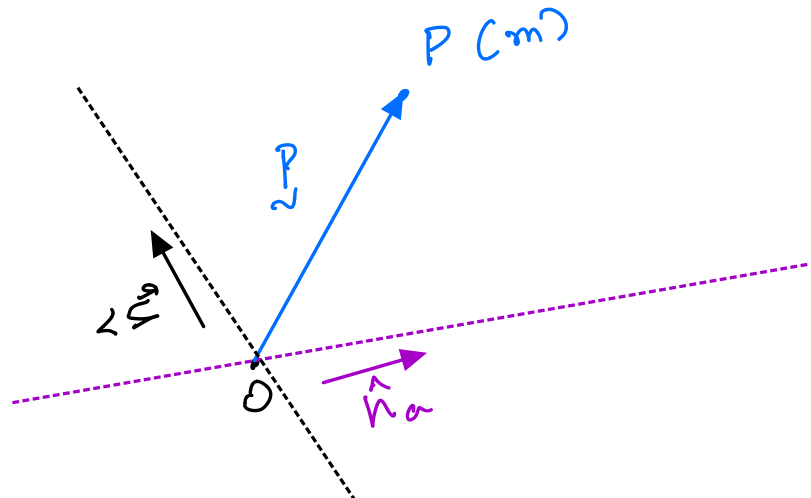
$$Y^* = \frac{m_{F_1}\left(a + \frac{H}{2}\right) + m_{F_2}\frac{a}{2}}{m_{F_1} + m_{F_2} + m_R + m}$$

and

$$Z^* = \frac{m_R\frac{L}{2} + m L_1}{m_{F_1} + m_{F_2} + m_R + m}$$

Inertia Scalar

For a particle, P, of mass m, we can define a parameter called the inertia scalar. This is defined relative to an arbitrary point, O. There are two such inertia scalars:-



1. Product of inertia

Notation: $I_{ab}^{P/O}$ is the product of inertia of P along two lines thru point O that are parallel to unit vectors $\hat{n}_a$ and $\hat{n}_b$.

$$\boxed{I_{ab}^{P/O} \triangleq m\left(\underset{\sim}{p} \times \hat{n}_a\right)\cdot\left(\underset{\sim}{p} \times \hat{n}_b\right)} \quad (8.5)$$

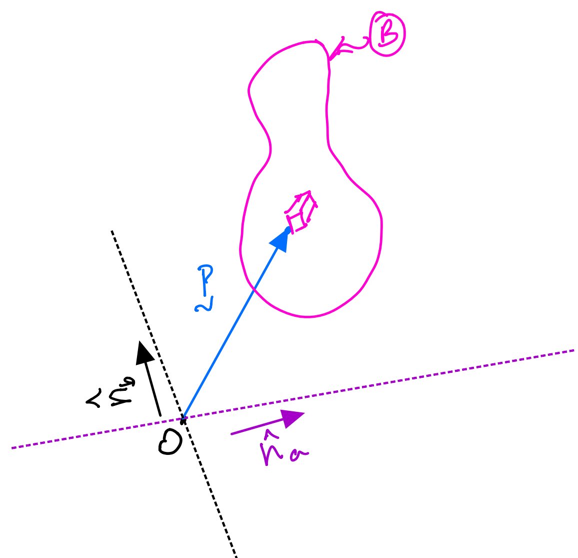
8.5 can be extended for both systems of particles and continua.

- Product of inertia of system of particles.

$$I_{ab}^{S/O} \triangleq \sum_i m_i (\underset{\sim}{p}_i \times \hat{n}_a) \cdot (\underset{\sim}{p}_i \times \hat{n}_b) \quad \text{— (8.6)}$$

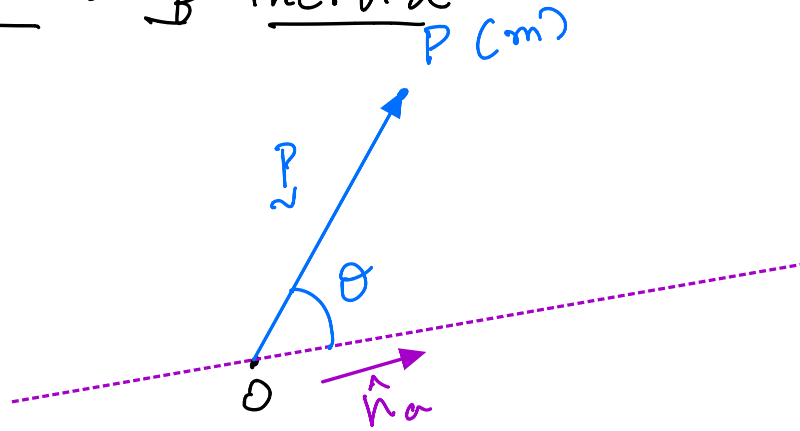
- Product of inertia of continua.

$$I_{ab}^{B/O} \triangleq \int dm (\underset{\sim}{p} \times \hat{n}_a) \cdot (\underset{\sim}{p} \times \hat{n}_b) \quad \text{— (8.7)}$$



NOTE - In all cases, $I_{ab} = I_{ba}$ because the formula relies on the dot product of vectors.

2. Moment of inertia



Notation: $I_{aa}^{P/O}$ is the moment of inertia of P about a line through point O which is parallel to the unit vector $\hat{n}_a$.

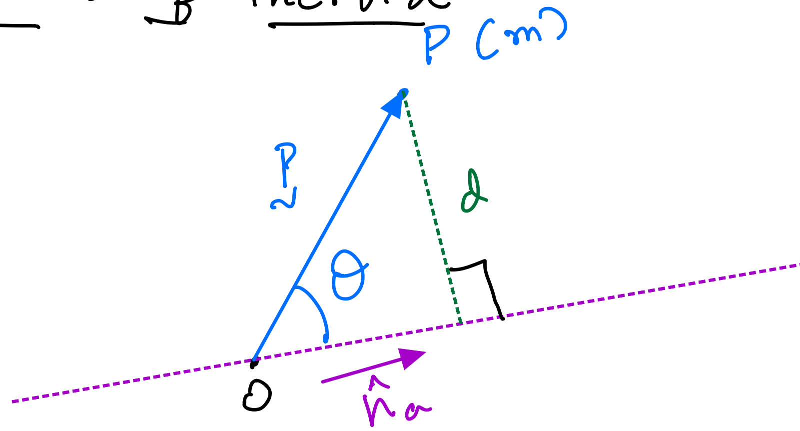
$$\boxed{I_{aa}^{P/O} \triangleq m\,(\underset{\sim}{p} \times \hat{n}_a) \cdot (\underset{\sim}{p} \times \hat{n}_a)} \quad (8.8)$$

$$I_{aa}^{P/O} \triangleq m \left| \underset{\sim}{p} \times \hat{n}_a \right|^2$$

$$= m \left| |\underset{\sim}{p}| \, |\hat{n}_a| \sin\theta \right|^2$$

$$= m \left| |\underset{\sim}{p}| \; 1 \; \sin\theta \right|^2$$

2. <u>Moment</u> of <u>inertia</u>



Notation: $I_{aa}^{P/O}$ is the moment of inertia of P about a line through point O which is parallel to the unit vector $\hat{n}_a$.

$$\boxed{I_a^{P/O} \triangleq m\,(\underset{\sim}{p} \times \hat{n}_a)\cdot(\underset{\sim}{p} \times \hat{n}_a)} \quad (8.8)$$

$$I_{aa}^{P/O} \triangleq m\,\left|\underset{\sim}{p} \times \hat{n}_a\right|^2$$

$$= m\left|\,|\underset{\sim}{p}|\;|\hat{n}_a|\,\sin\theta\right|^2$$

$$= m\left|\,|\underset{\sim}{p}|\;1\;\sin\theta\right|^2$$

$$= m\,d^2$$

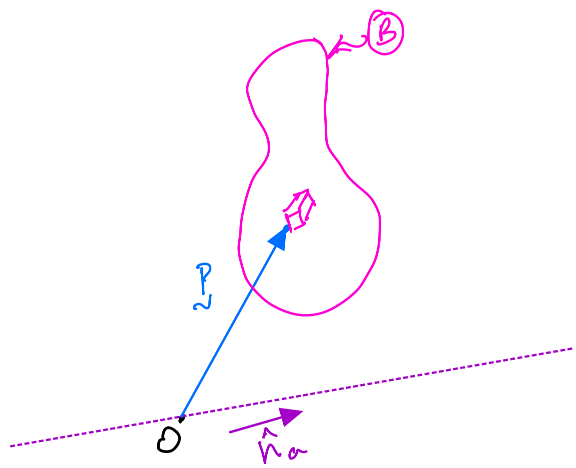
8.8 can be extended to both systems of particles and continua.

- Product of inertia of system of particles

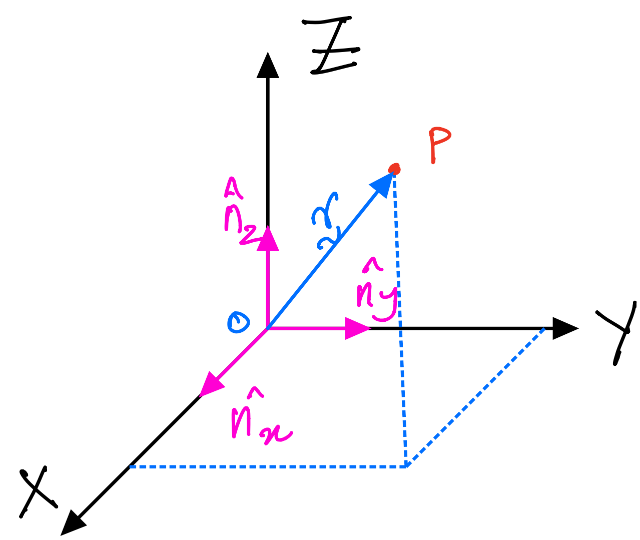
$$I_a^{S/O} \triangleq \sum_i m_i (\underset{\sim}{p}_i \times \hat{n}_a) \cdot (\underset{\sim}{p}_i \times \hat{n}_a) \quad \text{--- (8.9)}$$

- Product of inertia of continua.

$$I_a^{B/O} \triangleq \int dm (\underset{\sim}{p} \times \hat{n}_a) \cdot (\underset{\sim}{p} \times \hat{n}_a) \quad \text{--- (8.10)}$$



Example



- P is a particle of mass m
- $\hat{n}_x, \hat{n}_y, \hat{n}_z$ are unit vectors that are mutually orthogonal.
- $\underset{\sim}{r} = x\,\hat{n}_x + y\,\hat{n}_y + z\,\hat{n}_z$

Find: $I_{xx}^{P/O}$ $\quad I_{xy}^{P/O}$ $\quad I_{xz}^{P/O}$

Soln (i) $I_{xx}^{P/O} = m\,(\underset{\sim}{r} \times \hat{n}_x)\cdot(\underset{\sim}{r} \times \hat{n}_x)$

$$= m\,(z\,\hat{n}_y - y\,\hat{n}_z)\cdot(z\,\hat{n}_y - y\,\hat{n}_z)$$

$$= m\,(y^2 + z^2)$$

(ii) $I_{xy}^{P/O} = m\,(\underset{\sim}{r} \times \hat{n}_x)\cdot(\underset{\sim}{r} \times \hat{n}_y)$

$$= m\,(z\,\hat{n}_y - y\,\hat{n}_z)\cdot(-z\,\hat{n}_x + x\,\hat{n}_z)$$

$$= -m\,(xy)$$

(iii) $I_{xz}^{P/O} = -\,m\,x\,z$

From inertia scalars to inertia matrix.

- From the previous example, we now have some insight that we will be interested in computing the moments of inertia and products of inertia about a set of unit vectors that make up a reference frame.
- For this discussion, we assume that the unit vectors are:- $\hat{n}_x$, $\hat{n}_y$ and $\hat{n}_z$
- The inertia scalars can be used to define a square matrix called the **inertia matrix**

Notation:

$[I]^{S/O}$ is the inertia matrix of S, a system of particles, about the point O.

- The diagonal elements of this matrix are the moments of inertia.
- The off-diagonal elements are the products of inertia.
- So, the inertia matrix is represented as:-

$$[I]^{S/O} = \begin{bmatrix} I_{xx}^{S/O} & I_{xy}^{S/O} & I_{xz}^{S/O} \\ I_{yx}^{S/O} & I_{yy}^{S/O} & I_{yz}^{S/O} \\ I_{zx}^{S/O} & I_{zy}^{S/O} & I_{zz}^{S/O} \end{bmatrix}$$

REMINDER In all cases, $I_{ab} = I_{ba}$ because the formula relies on the dot product of vectors.

- Rigid body / continua:- The inertia scalars of a rigid body can also be arranged into an inertia matrix.