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<u>Mass</u>

zeroth moment $\triangleq$ how much mass there is

first moment $\triangleq \sum_{i=1}^{\nu} m_i \bar{r}_i = \int_{mass} \bar{r}\, dm$

mass center $\bar{r}_{cm} = \dfrac{\sum_{i=1}^{\nu} m_i \bar{r}_i}{\underbrace{\sum_{i=1}^{\nu} m_i}_{\text{total mass}}} = \dfrac{\int_{mass} \bar{r}\, dm}{\int_{mass} dm} = m_{TOT}$

Second moment of mass = "Inertia"

Inertia matrix $\begin{bmatrix} I_{xx} & \cdot & \cdot \\ \cdot & I_{yy} & \cdot \\ \cdot & \cdot & I_{zz} \end{bmatrix}$

<u>Inertia Vector</u>

$\bar{I}_a$ is the inertia vector of a set of S particles relative to a point O for $\hat{n}_a$.

$$\bar{I}_a \triangleq \sum_{i=1}^{\nu} m_i \bar{p}_i \times (\hat{n}_a \times \bar{p}_i)$$

$\bar{I}_a$ contains all the moments and products of inertia of S relative to point O

where $\bar{p}_i$ = position vector from O to m_i.

S : set of particles

$\hat{n}_a$ = unit vector is parallel to a line L_a which passes through O.

Inertia scalar

I_{ab} inertia scalar of S relative to O for $\hat{n}_a$ and $\hat{n}_b$ where $\hat{n}_b$ is another (different in general) unit vector.

$$I_{ab} \triangleq \bar{I}_a \cdot \hat{n}_b$$

[interpret this as the component of $\bar{I}_a$ in the $\hat{n}_b$ direction]

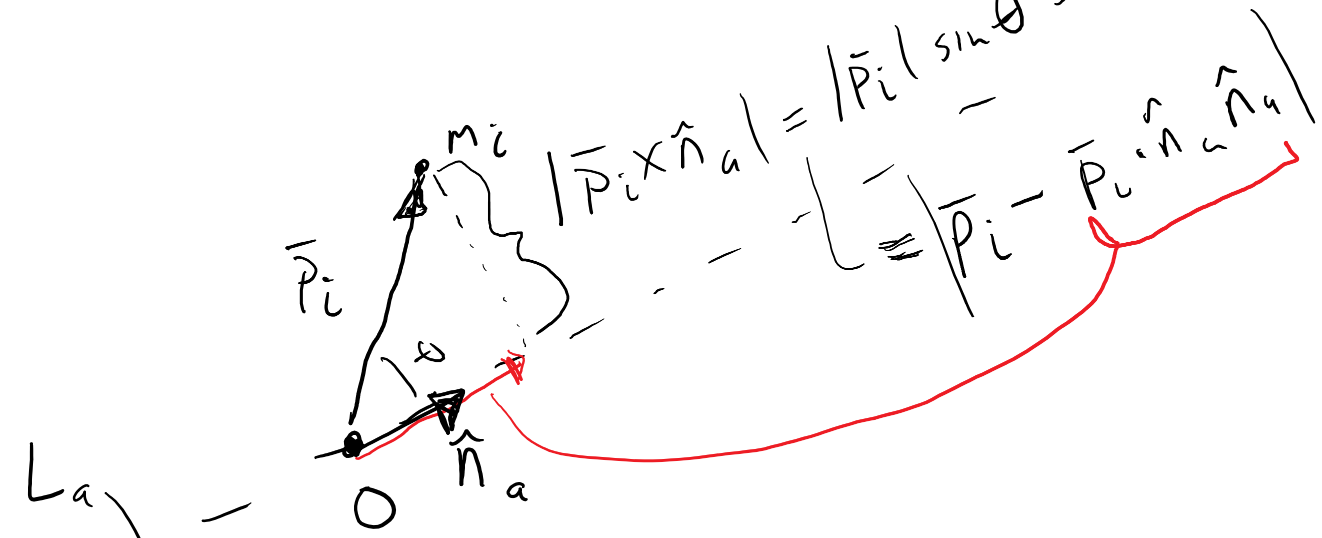
recall that $\bar{a} \cdot \bar{b} \times c = \bar{a} \times b \cdot c$

$$I_{ab} = \sum_{i=1}^{\nu} m_i (\bar{p}_i \times \hat{n}_a) \cdot (\bar{p}_i \times \hat{n}_b) = I_{ba} = \bar{I}_b \cdot \hat{n}_a$$

Components of the second moment vector are called moments and products inertia,

$I_{ab}^{S/O}$ called a product of inertia if $\hat{n}_a \neq \hat{n}_b$ and if $\hat{n}_a = \hat{n}_b$ then $I_{ab}^{S/O}$ is moment of inertia

Geometric interpretation



$n_b = n_a$

$$I_{aa} = \sum m_i \underbrace{(\bar{p}_i \times \hat{n}_a)^2}_{\text{scalar}}$$

$$(a_1 \hat{n}_1 + a_2 \hat{n}_2 + a_3 \hat{n}_3) \cdot (a_1 \hat{n}_1 + a_2 \hat{n}_2 + a_3 \hat{n}_3)$$

$$= a_1^2 + a_2^2 + a_3^2$$

$$|\bar{p}_i \times \hat{n}_a| = \sqrt{\underbrace{\ ^2 + \ ^2 + \ ^2}_{(\bar{p}_i \times \hat{n}_a) \cdot (\bar{p}_i \times \hat{n}_a)}}$$

$$|\bar{p}_i \times \hat{n}_a|^2 = (\bar{p}_i \times \hat{n}_a) \cdot (\bar{p}_i \times \hat{n}_a)$$

$$I_{aa} = \sum m_i l_i^2$$

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$I_{aa} = m_{TOT} k_a^2$

moment of inertia

k_a is called the radius of gyration

$k_a \sqrt{\dfrac{I_{aa}}{m_{TOT}}}$

Previous discussions and calculations of moments and products of inertia were based in some coordinate system in which to measure distances.

$$\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix} \rightarrow$$

inertia matrix
inertia tensor

If we change the coordinate system then moments and products of inertia change. Can we have a basis independent formulation of inertia??

What is a tensor?

Definition: general cartesian tensor is a set of numbers characterized by a number of indices associated which the axes of a single cartesian coordinate system and the tensor transform in the same manner as the corresponding set of products of the components of a position vector in that system.

Suppose 2 cartesian coordinate systems with 2 sets of unit vectors:

$\hat{e}_1, \hat{e}_2, \hat{e}_3$ and $\hat{e}_1', \hat{e}_2', \hat{e}_3'$

and $\alpha_{ij} = \hat{e}_i \cdot \hat{e}_j'$

$\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\qquad \bar{X}' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$

Einstein Summation Notation.

$x_j' = \alpha_{ij} X_i \implies x_j' = \alpha_{1j} X_1 + \alpha_{2j} X_2 + \alpha_{3j} X_3$

shorthand

when there is a common index we sum over that index

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another possibility:

$$\underbrace{x'_j x'_\ell}_{\text{product of measure number}} = \alpha_{ij} x_i \alpha_{k\ell} x_k = \overbrace{\alpha_{ij}\alpha_{k\ell} x_i x_k}^{9 \text{ terms!}}$$

$$x_i x_k = \alpha_{ij}\alpha_{k\ell} x'_j x'_\ell$$

$$x_i = \alpha_{ij} x'_j$$

Examples of tensors

1. zero-order tensor → all scalar values (e.g. mass)
2. first-order tensor → all vectors
   e.g. $\bar{x}, \bar{\omega}, \bar{H}$, etc.
3. second order tensors → (matrices)
   a. if $\bar{A}, \bar{B}$ are 2 vectors with a_i, b_k then nine products of components $a_i b_k$ form 2nd order tensor
   $$a'_j b'_\ell = \alpha_{ij}\alpha_{k\ell} a_i b_k$$
   "tensor product" of 2 vectors is called a "dyadic"

   b. Kronecker Delta
   $$\delta_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

   c. if $\bar{A}(\bar{r})$ is a vector field
   $$\frac{\partial a_i}{\partial x_k}$$
   2nd order tensor "tensor gradient"
   i^{th} component of $\bar{A}$ @ $\bar{r}_i$
   k^{th} component of $\bar{r}_i$

Dyadic (Dyad) is second order tensor that is a product of 2 vectors. Dyadics will let us formulate the inertia in a basis-independent form.

Suppose vector $\bar{u} = \bar{\omega}\cdot\bar{a}\bar{b} + \bar{\omega}\cdot\bar{c}\bar{d} + \bar{\omega}\cdot\bar{e}\bar{f} + \dots$

$\bar{V} = \bar{a}\cdot\bar{b}\,\bar{\omega} + \bar{c}\cdot\bar{d}\,\bar{\omega} + \bar{e}\cdot\bar{f}\,\bar{\omega} + \dots$

factor:

$\bar{u} = \bar{\omega}(\bar{a}\bar{b} + \bar{c}\bar{d} + \dots)$

$\bar{V} = (\bar{a}\bar{b} + \bar{c}\bar{d} + \dots)\,\bar{\omega}$

Define the dyadic $\bar{Q}$ as the sum of outer products of two vectors.

$\bar{Q} = \bar{a}\bar{b} + \bar{c}\bar{d} + \dots$

$$\begin{bmatrix} a_1 \\ a_2 \\ b_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}$$

pre mu

$\bar{u} = \bar{\omega}\cdot\bar{Q}$

$\bar{V} = \bar{Q}\cdot\bar{\omega}$

post mul

pre or post multiply a vector and dyadic returns a vector

Unit Dyadic

if $\hat{a}_1, \hat{a}_2, \hat{a}_3$ are $\perp$ unit vectors

$\bar{U} = \hat{a}_1\hat{a}_1 + \hat{a}_2\hat{a}_2 + \hat{a}_3\hat{a}_3$

property:

$\bar{V}\cdot\bar{U} = \bar{V}\cdot\hat{a}_1\hat{a}_1 + \bar{V}\cdot\hat{a}_2\,\hat{a}_2 + \bar{V}\cdot\hat{a}_3\hat{a}_3$

$\bar{V} \Rightarrow V_1, V_2, V_3 = V_1\hat{a}_1 + V_2\hat{a}_2 + V_3\hat{a}_3 = \bar{V}$

$\bar{V} = v_1\hat{a}_1 + v_2\hat{a}_2 + v_3\hat{a}_3$

$\bar{U}\cdot\bar{V} = \bar{V}$

Inertia

$\bar{I}_a = \sum_{i=1}^{\nu} m_i\,\bar{p}_i \times (\hat{n}_a \times \bar{p}_i)$

Triple vector product identity

$A\times(B\times C) = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})$

$\bar{I}_a = \sum_{i=1}^{\nu} m_i(\hat{n}_a\bar{p}_i^{\,2} - \hat{n}_a\cdot\bar{p}_i\bar{p}_i) = \sum_{i=1}^{\nu} m_i(\hat{n}_a\cdot\bar{U}\,\bar{p}_i^{\,2} - \hat{n}_a\cdot\bar{p}_i\bar{p}_i)$

go to 2

2 let $\bar{I}$ be a dyadic, inertia dyadic of S relative to point O.

$\bar{I} \triangleq \sum_{i=1}^{\nu} m_i(\bar{U}\,\bar{p}_i^{\,2} - \bar{p}_i\bar{p}_i)$

$\bar{I}_a = \hat{n}_a\cdot\bar{I}$

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this definition does not involve any unit vectors like $\bar{I}_a$ or I_{ab} and $\bar{e}_i$ can be expressed in any coordinate system in any RF!!

$$\bar{I}_a = \hat{n}_a \cdot \bar{\bar{I}} \quad \text{and} \quad I_{ab} = \bar{I}_a \cdot \hat{n}_b = \hat{n}_a \cdot \bar{\bar{I}} \cdot \hat{n}_b$$

vector ↑ unit vec ↑ dyadic ↑