MAE223-L3-01

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Kinematics

Particle:
1) position vector
2) velocity vector
3) acceleration vector

} linear motion

Rigid Body:

all points on RB have particle Kinematics

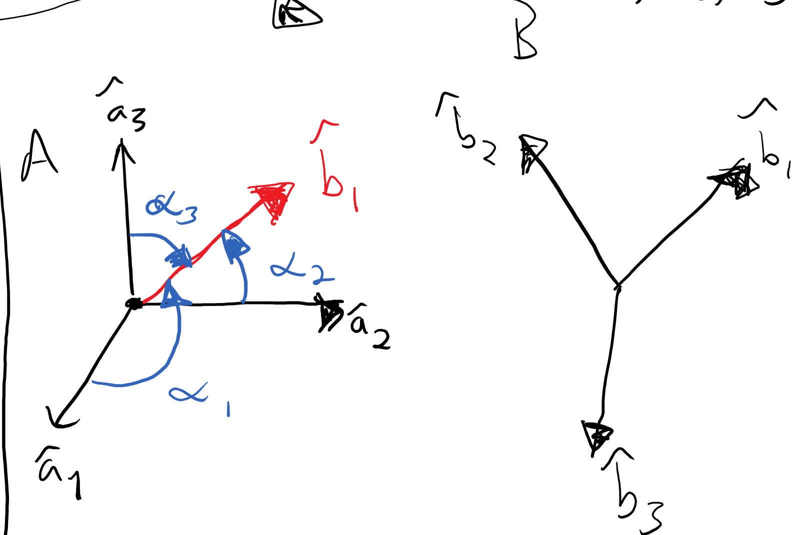
1) angular position (orientation)
2) angular velocity
3) angular acceleration

Rigid Body Orientation (attitude)

Suppose two RF A, B with fixed coordinate systems $\hat{a}_1, \hat{a}_2, \hat{a}_3$
$\hat{b}_1, \hat{b}_2, \hat{b}_3$

How can we describe the orientation of B relative to A.

use angles between $\hat{b}_1, \hat{b}_2, \hat{b}_3$ wrt $\hat{a}_1, \hat{a}_2, \hat{a}_3$



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$$\hat{b}_1 = \cos\alpha_1 \hat{a}_1 + \cos\alpha_2 \hat{a}_2 + \cos\alpha_3 \hat{a}_3$$

$$= (\hat{b}_1 \cdot \hat{a}_1)\hat{a}_1 + (\hat{b}_1 \cdot \hat{a}_2)\hat{a}_2 + (\hat{b}_1 \cdot \hat{a}_3)\hat{a}_3$$

$$\hat{b}_2 = \cos\beta_1 \hat{a}_1 + \cos\beta_2 \hat{a}_2 + \cos\beta_3 \hat{a}_3$$

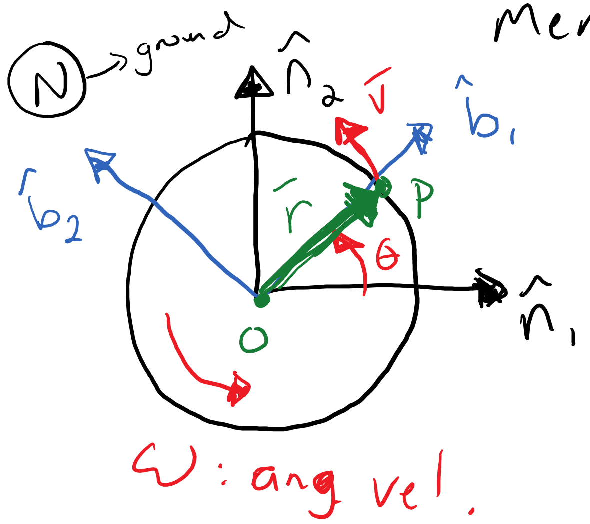
$$\hat{b}_3 = \cos\gamma_1 \hat{a}_1 + \cos\gamma_2 \hat{a}_2 + \cos\gamma_3 \hat{a}_3$$

$\boxed{C_{ij} \text{ is the cosine of the angle between } \hat{b}_i \text{ and } \hat{a}_i}$

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}}_{\text{direction cosine matrix}} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix}$$

<u>direction</u> <u>cosine</u> <u>matrix</u> of B relative to A

Ex

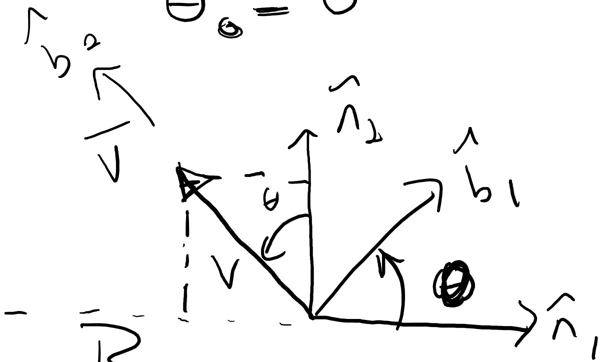


Merry go round

$$\|\bar r\| = r$$

ω is constant, so

$$\theta = \omega t \qquad \theta_0 = 0$$



$$\bar r = r\cos\theta\,\hat n_1 + r\sin\theta\,\hat n_2$$

$\bar r = r\,\hat b_1 \longrightarrow$ r is fixed in B

$$\bar V = V\,\hat b_2$$

$$\bar V = V(-\sin\theta\,\hat n_1 + \cos\theta\,\hat n_2)$$

$$^{B}\frac{d\bar r}{dt} \neq {}^{N}\frac{d\bar r}{dt}$$

$$^{B}\frac{d\bar r}{dt} = 0$$

$$^{N}\frac{d\bar r}{dt} = \bar V = V\hat b_2 = -V\sin\theta\,\hat n_1 + V\cos\theta\,\hat n_2$$

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angular velocity of rigid body B in RF A has to do with the rate of change of orientation of B in A.

Suppose $\hat{b}_1, \hat{b}_2, \hat{b}_3$ are RH set of mutually perpendicular units vectors <u>fixed</u> in B.

$$^A\underline{\omega}^B \triangleq \text{angular velocity of B in A}$$

$$\triangleq \hat{b}_1\left(\frac{^A d\hat{b}_2}{dt}\cdot\hat{b}_3\right) + b_2\left(\frac{^A d\hat{b}_3}{dt}\cdot\hat{b}_1\right) + \hat{b}_3\left(\frac{^A d\hat{b}_1}{dt}\cdot b_2\right)$$

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$${}^A\bar{\omega}^B \triangleq$$ angular velocity of B in A

$$\hat{b}_1\left(\frac{{}^A d\hat{b}_2}{dt}\cdot\hat{b}_3\right) + b_2\,\frac{}{dt}\cdot\hat{b}_1\Big) +$$

$$\hat{b}_3\left(\frac{{}^A d\hat{b}_1}{dt}\cdot\hat{b}_2\right)$$

$${}^A\bar{\omega}^B \times \hat{b}_1 = 0 + \underbrace{\hat{b}_2\times\hat{b}_1}_{-\hat{b}_3}\left(\frac{{}^A d\hat{b}_3}{dt}\cdot\hat{b}_1\right) + \underbrace{\hat{b}_3\times\hat{b}_1}_{\hat{b}_2}\left(\frac{{}^A d\hat{b}_1}{dt}\cdot\hat{b}_2\right)$$

$$\dot{\hat{b}}_1\cdot\hat{b}_1 = ?$$ remember that $\|\hat{b}_1\| = 1$

$$= 0$$

$$\hat{b}_1\cdot\hat{b}_3 = 0 \xRightarrow{d/dt} \dot{\hat{b}}_1\cdot\hat{b}_3 + \hat{b}_1\cdot\dot{\hat{b}}_3 = 0 \Rightarrow \dot{\hat{b}}_1\cdot\hat{b}_3 = -\hat{b}_1\cdot\dot{\hat{b}}_3$$

$${}^A\bar{\omega}^B \times \hat{b}_1 = -\hat{b}_3\left(\dot{\hat{b}}_3\cdot\hat{b}_1\right) + \hat{b}_2\left(\dot{\hat{b}}_1\cdot\hat{b}_2\right)$$

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$${}^{A}\bar{\omega}^{B} \times \hat{b}_1 = \hat{b}_1 \underbrace{(\dot{\hat{b}}_1 \cdot \hat{b}_1)}_{} + \hat{b}_2 (\dot{\hat{b}}_1 \cdot \hat{b}_2) + \hat{b}_3 (\dot{\hat{b}}_1 \cdot \hat{b}_3)$$

$$= \dot{\hat{b}}_1^{\,0} = \frac{{}^{A}d\hat{b}_1}{dt}$$

$${}^{A}\bar{\omega}^{B} \times \hat{b}_1 = \frac{{}^{A}d\hat{b}_1}{dt}$$

also true for an arbitrary vector $\bar{B}$

vector fixed in B

$$\boxed{\frac{{}^{A}d\bar{B}}{dt} = {}^{A}\bar{\omega}^{B} \times \bar{B}}$$

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<u>Ex</u> Have 2 RF $\hat{b}_i$ fixed in B

$\hat{a}_i$ fixed in A

measure the time histories of the projections of each $\hat{a}_i$ on each $\hat{b}_i$.

$$\alpha_i = \hat{b}_1 \cdot \hat{a}_i \quad \beta_i = \hat{b}_2 \cdot \hat{a}_i \quad \gamma_i = \hat{b}_3 \cdot \hat{a}_i$$

also $\dot{\alpha}_i, \dot{\beta}_i, \dot{\gamma}_i$ $\quad i = 1, \ldots, 3$

At any given point we know all 18 quantities

$$\hat{b}_1 = \alpha_1 \hat{a}_1 + \alpha_2 \hat{a}_2 + \alpha_3 \hat{a}_3$$

$$\hat{b}_2 = \beta_1 \hat{a}_1 + \ldots$$

$$\hat{b}_3 = \gamma_1 \hat{a}_1 + \ldots$$

$$\frac{{}^A d\hat{b}_1}{dt} = \dot{\alpha}_1 \hat{a}_1 + \dot{\alpha}_2 \hat{a}_2 + \dot{\alpha}_3 \hat{a}_3$$

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$$
{}^{A}\bar{\omega}^{B} = (\dot{\beta}_1 \gamma_1 + \dot{\beta}_2 \gamma_2 + \dot{\beta}_3 \gamma_3)\hat{b}_1 + (\dot{\gamma}_1 \alpha_1 + \dot{\gamma}_2 \alpha_2 + \dot{\gamma}_3 \alpha_3)\hat{b}_2 + (\dot{\alpha}_1 \beta_1 + \dot{\alpha}_2 \beta_2 + \dot{\alpha}_3 \beta_3)\hat{b}_3
$$

= angular velocity of B in A expressed in B

Simple angular velocity

rigid body B has a simple angular velocity in RF A if there exists for a finite time, t, a single unit vector, $\hat{k}$, whose orientation is fixed in both A and B. ($\hat{k}$ is the axis about which B is rotating in A). In this case

$${}^A\bar{\omega}^B = \omega\hat{k}$$

with $\omega = \dot{\theta}$ where θ is angle between a line L_A fixed in A and a similar line L_B fixed in B, both $\perp$ to $\hat{k}$

ω is called the angular speed of B in A.

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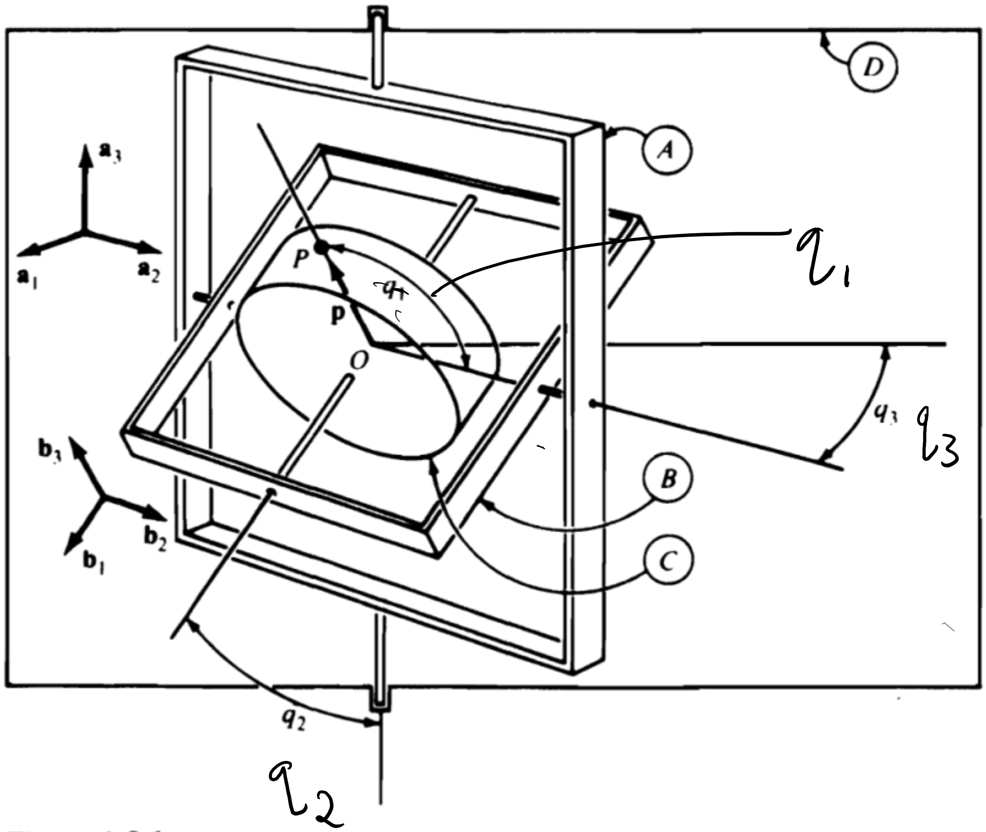

Figure 1.3.1

RF D, B, C, A

$$^D\overline{\omega}^A = -\dot{q}_3\,\hat{a}_3$$

$$^A\overline{\omega}^B = -\dot{q}_2\,\hat{a}_2 = -\dot{q}_2\,\hat{b}_2$$

$$^B\overline{\omega}^C = \dot{q}_1\,\hat{b}_1$$

Main reason ${}^A\bar{\omega}^B$ is useful is to relate the derivatives of vector $\bar{V}$ in 2 RFs A and B.

$$\boxed{\frac{{}^A d\bar{V}}{dt} = \frac{{}^B d\bar{V}}{dt} + {}^A\bar{\omega}^B \times \bar{V}}$$

Proof

Suppose $\hat{b}_i$ be a RH coordinate system fixed in B. $v_i = \bar{V}\cdot\hat{b}_i$ or $\bar{V} = \sum_{i=1}^{3} v_i \hat{b}_i$

$$\frac{{}^A d\bar{V}}{dt} = \sum_{i=1}^{3} \frac{{}^A d v_i}{dt}\hat{b}_i + \sum_{i=1}^{3} v_i \frac{{}^A d\hat{b}_i}{dt}$$

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$$ {}^A\frac{d\bar{v}}{dt} = {}^B\frac{d\bar{v}}{dt} + \sum_{i=1}^{3} v_i \, {}^A\bar{\omega}^B \times \hat{b}_i $$

dif of $\frac{d}{dt}$ in B

$$ {}^A\bar{\omega}^B \times \hat{b}_i = {}^A\frac{d b_i}{dt} $$

$$ = {}^B\frac{d\bar{v}}{dt} + \sum_{i=1}^{3} {}^A\bar{\omega}^B \times v_i \hat{b}_i $$

$$ \boxed{ {}^A\frac{d\bar{v}}{dt} = {}^B\frac{d\bar{v}}{dt} + {}^A\bar{\omega}^B \times \bar{v} } $$