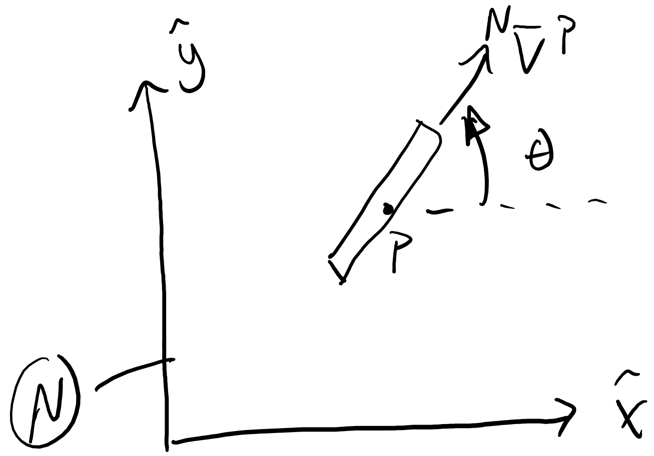


$$\tan\theta = \frac{\dot{y}}{\dot{x}}$$

$$\dot{x}\tan\theta - \dot{y} = 0$$

is the a nonholonomic constraint

or a

holonomic constraint

$\hookrightarrow f($ coordinates, constants, time$) = 0$

$$\frac{d f}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial f}{\partial t}$$

$$\frac{\partial f}{\partial x} = \tan\theta \qquad \frac{\partial f}{\partial \theta} = 0$$

$$\frac{\partial f}{\partial y} = -1 \qquad \frac{\partial f}{\partial t} = 0$$

$\ast$ all mixed partial derivatives have to commute if $\frac{df}{dt}$ is integrable $\ast$

$$\frac{\partial^2 f}{\partial x \partial \theta} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \theta}\right) = 0$$

$$\frac{\partial^2 f}{\partial \theta \partial x} = \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right) = \sec^2\theta$$

not equal!, thus nonintegrable $\Rightarrow$ so $\dot{x}\tan\theta - \dot{y} = 0$ is an essential nonholonomic constraint

MAE223-L7-02

Wednesday, October 18, 2017 10:13 AM

3 G.C.'s : x, y, θ and 1 motion (nonholonomic) constraint

only two of the G.C.s can move independently

let m = # nonholonomic constraints

n = minimum # of G.C.s (extraneous G.C.s. already eliminated with holonomic constraints)

$p = n - m$

→ # of degrees of freedom (DoF)

for the skate: $p = 3 - 1 = \underline{\underline{2}}$

number of velocities that can change independently at any one instant or the # of velocities needed to characterize the motion

Suppose that there are p independent velocities then the remaining m velocities can be calculated with:

$$u_r = \sum_{s=1}^{p} A_{rs} u_s + B_r$$

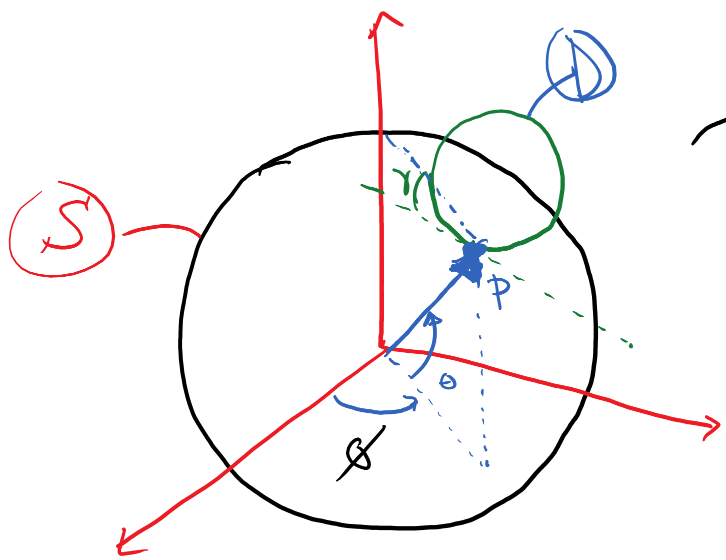
u_r: vector of dep. velocities

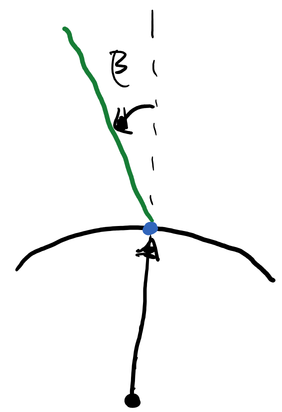
B_r: vector of ind. velocities

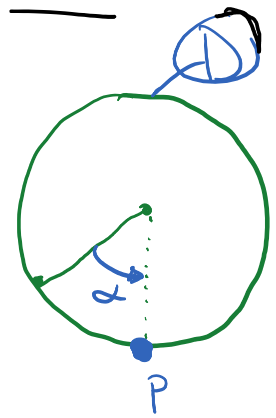
$u_1, \ldots, u_n$ total # vels.

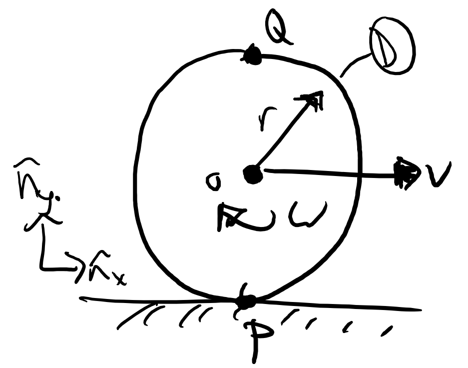
$r = \underbrace{p+1, \ldots, n}_{m}$

ex Thin disk rolling without slip on a sphere









Where is P?
at any instant of time point P is fixed in D

$V^Q = ?$ $2v$

$V^P = ?$ 0

$\omega = \frac{v}{r}$

$^{N}\bar{V}^{o} = v\,\hat{n}_x$

$^{N}\bar{V}^{Q} = {}^{N}\bar{V}^{o} + {}^{N}\bar{\omega}^{D} \times r^{Q/o}$

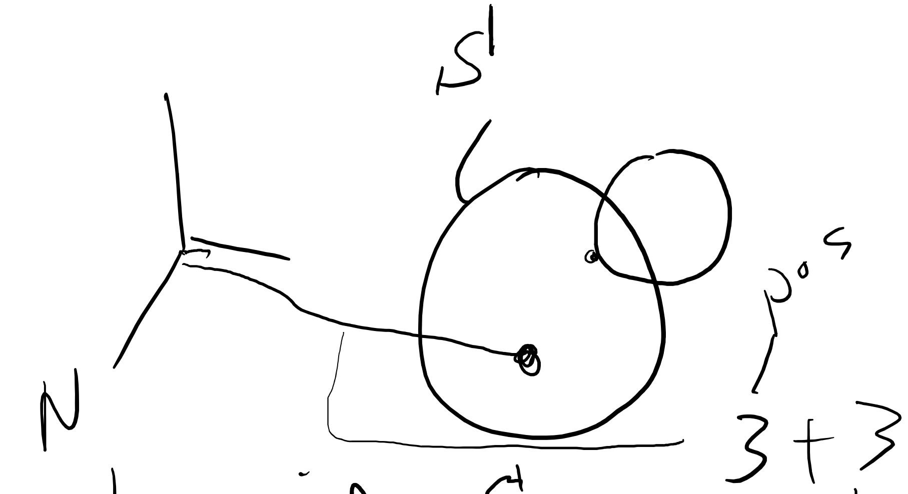
$= v\hat{n}_x + \omega r \hat{n}_x$

$= v\hat{n}_x + v\hat{n}_x = \boxed{2v\hat{n}_x}$

$\left.\begin{array}{l} \boxed{{}^{S}\bar{V}^{P} = 0} \\ \text{potential motion constraint} \end{array}\right\}$ 2 motion constraints

5 G.Cs : $\phi, \theta, \gamma, \beta, \alpha$

$m = 2$ $n = 5$ $\boxed{p = 3}$ # DoF



What if S can not fully move and rotate in N?

What is m, n, p?

$m = 2$

$n = 11$

$\boxed{p = 9}$

What happens to system as time progresses?

G.C's are functions of time

$\bar{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$ Newton's Law $\bar{F} = m\bar{a}$ ↳ related to $\ddot{\bar{q}}$

$\bar{q}(t)$ have to integrate twice: $\ddot{\bar{q}} \to \dot{\bar{q}} \to \bar{q}$

we need keep track of n independent G.Cs, $\bar{q}$, and p independent velocities, $\dot{\bar{q}}$.

$p \leq n$

It may seem obvious to use $\dot{\bar{q}}$ to track the motion, but it turns out not to be so desireable in all cases.

Rather we can use

Generalized Speeds.

equivalent to $\dot{\bar{q}}$

Consider system S whose configuration in N is completely defined by minimum set of G.C's, $q_1, \ldots, q_n$. We the define a set of n scalars $u_1, \ldots, u_n$ as the generalized speeds of S in N if:

a) each u_i is a linear combination of each $\dot{q}_i$

$$u_i = \sum_{j=1}^{n} Y_{ij} \dot{q}_j + Z_i \quad (i = 1, \ldots, n)$$

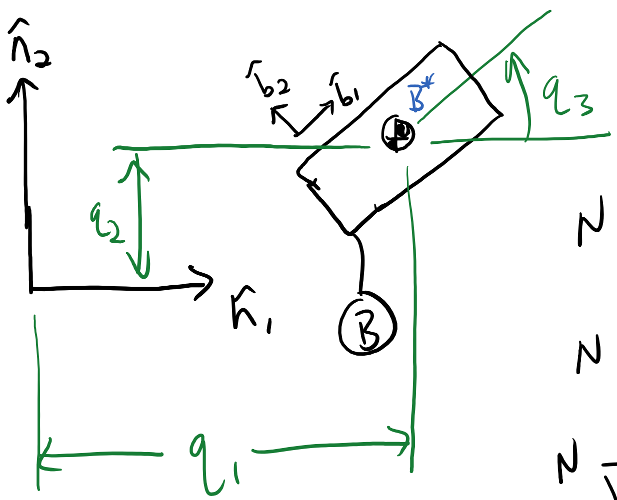
$Y_{ij}, Z_i = f(q_i, t)$ and constants

b) it must be possible to solve for $\dot{q}_i$'s given the u_i's.

$$\dot{q}_k = \sum_{j=1}^{n} y_{kj} u + z_k \quad (k = 1, \ldots, n)$$

Y_{ij} must be invertable

example: Planar motion of a rigid body



$$\left.\begin{aligned}\hat{b}_1 &= c_3\hat{n}_1 + s_3\hat{n}_2\\ \hat{b}_2 &= -s_3\hat{n}_1 + c_3\hat{n}_2\end{aligned}\right\}$$

$${}^N\bar{\omega}^B = \dot{q}_3\hat{n}_3 = \dot{q}_3\hat{b}_3$$

$${}^N\bar{V}^{B^*} = \dot{q}_1\hat{n}_1 + \dot{q}_2\hat{n}_2$$

$${}^N\bar{V}^{B^*} = (\dot{q}_1c_3 + \dot{q}_2s_3)\hat{b}_1 + (\dot{q}_2c_3 - \dot{q}_1s_3)\hat{b}_2$$

$${}^N\bar{V}^{B^*} = \underbrace{u_1\hat{b}_1 + u_2\hat{b}_2}_{\text{G.S.'s}}$$

(I choose them!)

$$u_1 = \dot{q}_1c_3 + \dot{q}_2s_3$$
$$u_2 = \dot{q}_2c_3 - \dot{q}_1s_3$$

$$\begin{bmatrix}u_1\\u_2\end{bmatrix} = \underbrace{\begin{bmatrix}c_3 & s_3\\ -s_3 & c_3\end{bmatrix}}_{Y}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}$$

$$Y^{-1} = \frac{\begin{bmatrix}c_3 & -s_3\\ s_3 & c_3\end{bmatrix}}{c_3c_3 + s_3s_3 \to 1}$$

$$\dot{q}_1 = u_1c_3 - u_2s_3$$
$$\dot{q}_2 = u_1s_3 + u_2c_3$$
$$\dot{q}_3 = u_3$$

In practice we generally choose u_i's to be components of velocities and/or angular velocities of points or bodies of physical interest ~~of~~ significance.

Note: you can always choose $u_i = \dot{q}_i$ $(i = 1, \ldots, n)$

$$\underbrace{u_i = \sum Y_{ij}\dot{q}_j + Z_j}$$

these are called the kinematical/kinematic differential equation

What is ${}^N a^{B^*} = ?$

$${}^N\bar{a}^{B^*} = \frac{{}^N d\,{}^N\bar{V}^{B^*}}{dt} = \frac{{}^B d\,{}^N\bar{V}^{B^*}}{dt} + {}^N\bar{\omega}^B \times {}^N\bar{V}^{B^*}$$

$${}^N\bar{a}^{B^*} = \dot{u}_1\hat{b}_1 + \dot{u}_2\hat{b}_2 + u_3\hat{b}_3 \times (u_1\hat{b}_1 + u_2\hat{b}_2)$$

$$= \dot{u}_1\hat{b}_1 + \dot{u}_2\hat{b}_2 + u_3u_3\hat{b}_2 - u_3u_2\hat{b}_1$$

$$= \dot{u}_1\hat{b}_1 + \dot{u}_2\hat{b}_2 + (\dot{u}_2 + u_3u_2)\hat{b}_2$$

$$= (\dot{u}_1 - u_3u_2)\hat{b}_1 + (\dot{u}_2 + u_3u_2)\hat{b}_2$$

$${}^N\bar{\alpha}^B = \dot{u}_3\hat{b}_3$$

Motion Constraints: Even if $\bar{q}$ is minimal set, the generalized speeds $u_1, \ldots, u_n$ may not be independent. (This is a property of system not the choice of u_i's). If there are no motion constraints, all n u_i's are independent and the system is a holonomic system with n DoFs. A "simple nonholonomic system" is subject to motion constraints:

$$\underset{\substack{\uparrow \\ \text{dep} \\ \text{GSs}}}{u_r} = \sum_{s=1}^{p} A_{rs} \underset{\substack{\uparrow \\ \text{Ind.} \\ \text{GSs}}}{u_s} + B_r \qquad (r = p+1, \ldots, n)$$

$\Rightarrow$ DoF: $p = n - m$

$$\bar{u}_r = \underline{A}\,\bar{u}_s + \bar{B} \Rightarrow \text{matrix form}$$