Fluctuations in Growth Processes

Angad Yuvraj and Dominic Yates

Supervised by Alex Adamou, Ole Peters and Rosalba García Millán

Introduction

- Classical view holds that fluctuations are irrelevant to growth of a system.
- Recent advancements demonstrate that fluctuations do play a role when considering the growth rate of the system.
- It has been shown that individuals achieve boost in growth rate by completely pooling and equally splitting their resources.
- We extend this concept to consider more practical forms of sharing than this type of “cooperation”.
- We consider the effects of two types of resource sharing on growth rate, and the impact of fluctuations.

Geometric Brownian Motion

- Throughout this project we study the stochastic differential equation:

$$dy = y(\ \mu dt + \ \sigma dW\ )$$

- Where μ is the drift and σ the amplitude of fluctuation.
- dW is a Wiener increment satisfying $\mathrm{W}_t = \int_0^t \mathrm{dW}$ is a Wiener process.
- W_t has independent Gaussian increments: $\mathrm{W}_{t+u} - \mathrm{W}_t \sim N(0, u)$
- Throughout we use Itô's interpretation of this equation, giving the general solution:

$$y(t) = y(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

Geometric Brownian Motion

- Ensemble Average Growth Rate:

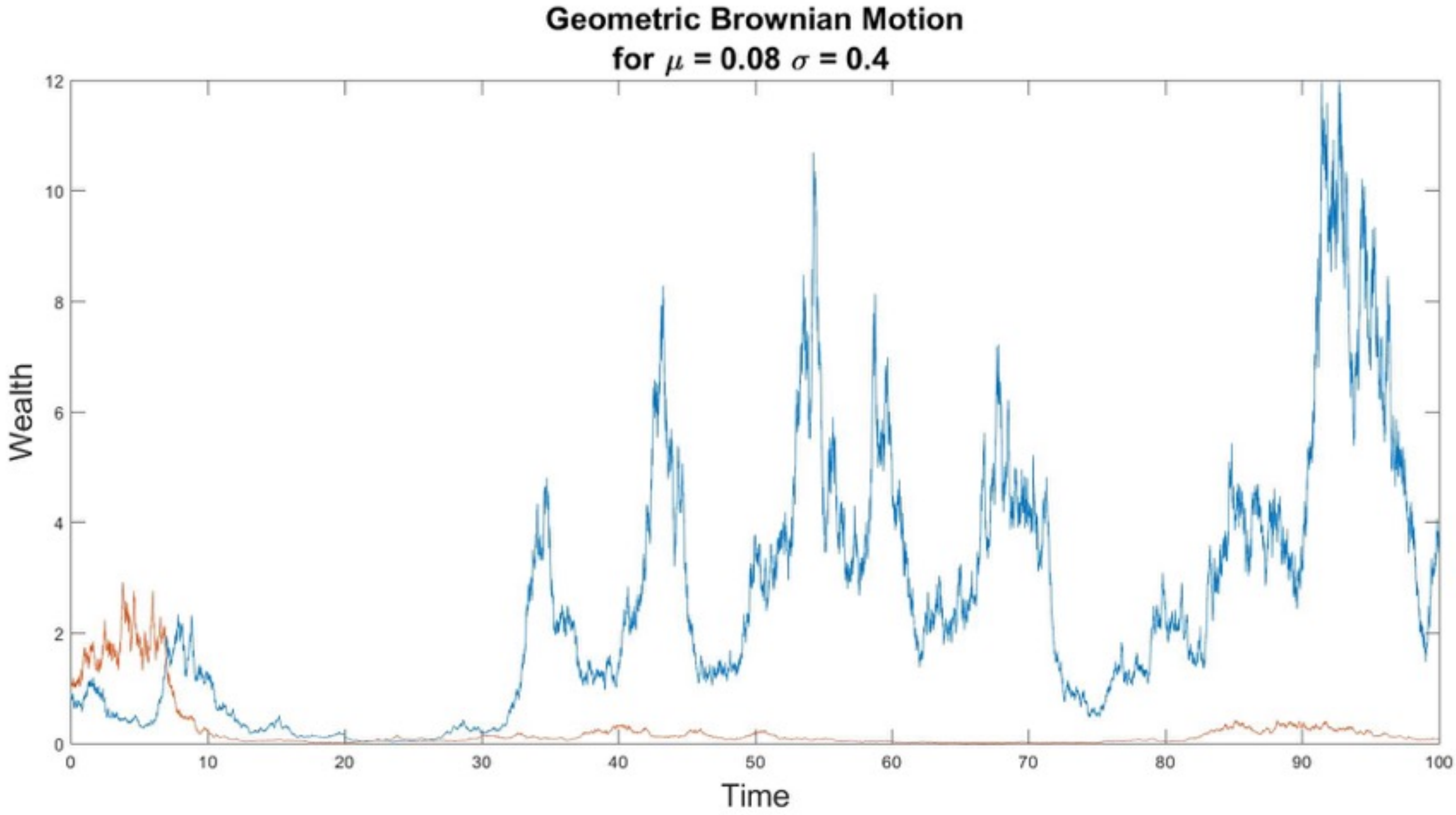
$$g_{<>} = \mu$$

- Time average growth rate is defined by $g = \lim_{T\to\infty}(g_T)$, where

$$g_T = \frac{1}{T}\ln\frac{y(T)}{y(0)}$$

- For a Geometric Brownian motion

$$g = \mu - \frac{\sigma^2}{2}$$


Geometric Brownian Motion
for μ = 0.08 σ = 0.4
Wealth
Time
0
2
4
6
8
10
12
0
10
20
30
40
50
60
70
80
90
100

Cooperation

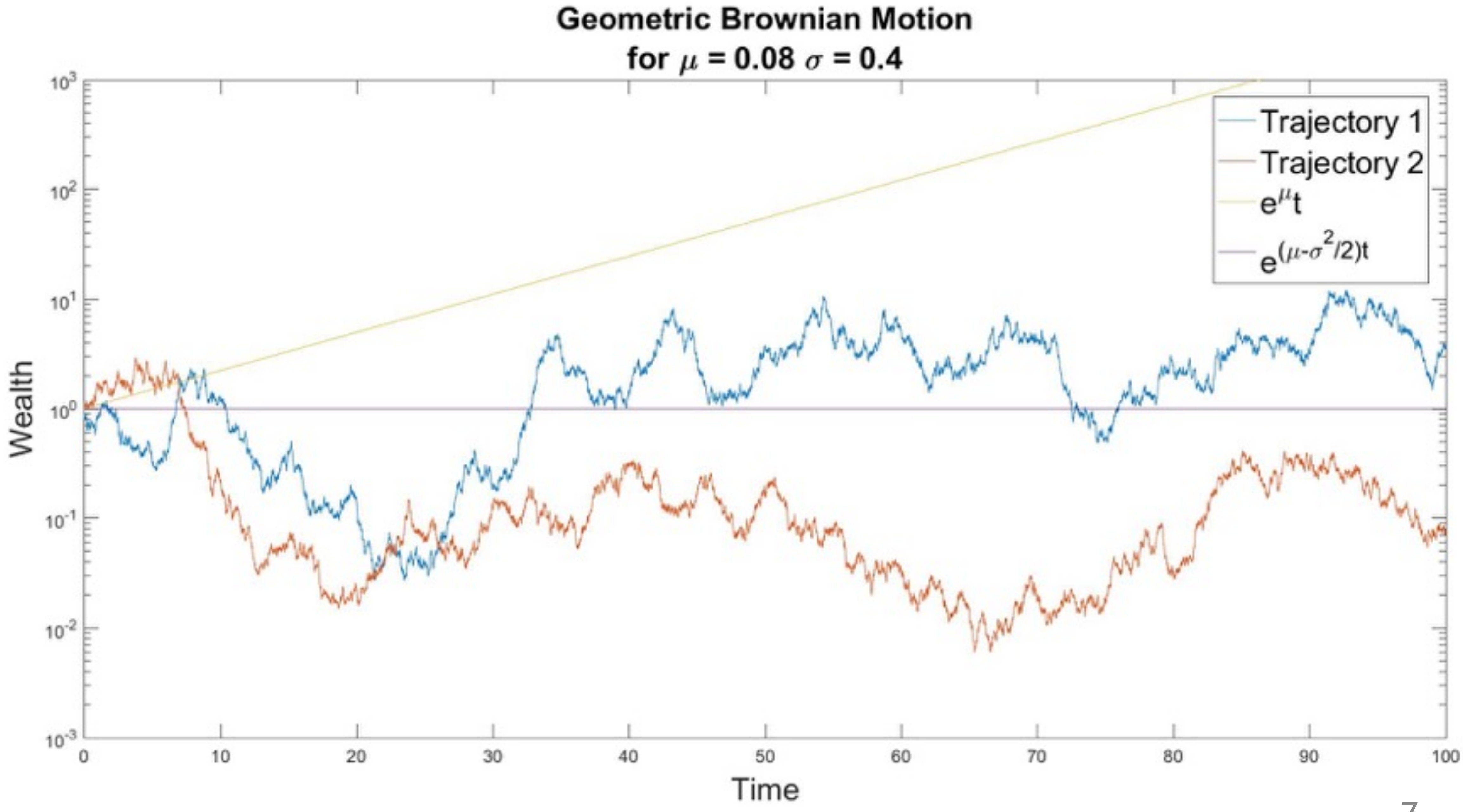
- Consider a system of two individuals continually pooling their resources and dividing them equally.
- Adamou and Peters show that for such a system of individuals executing Geometric Brownian motion, the growth rate of each individual is given by:

$$g = \mu - \frac{\sigma^2}{4}$$

- As opposed to:

$$g = \mu - \frac{\sigma^2}{2}$$

Which is the growth rate for individuals growing independently.

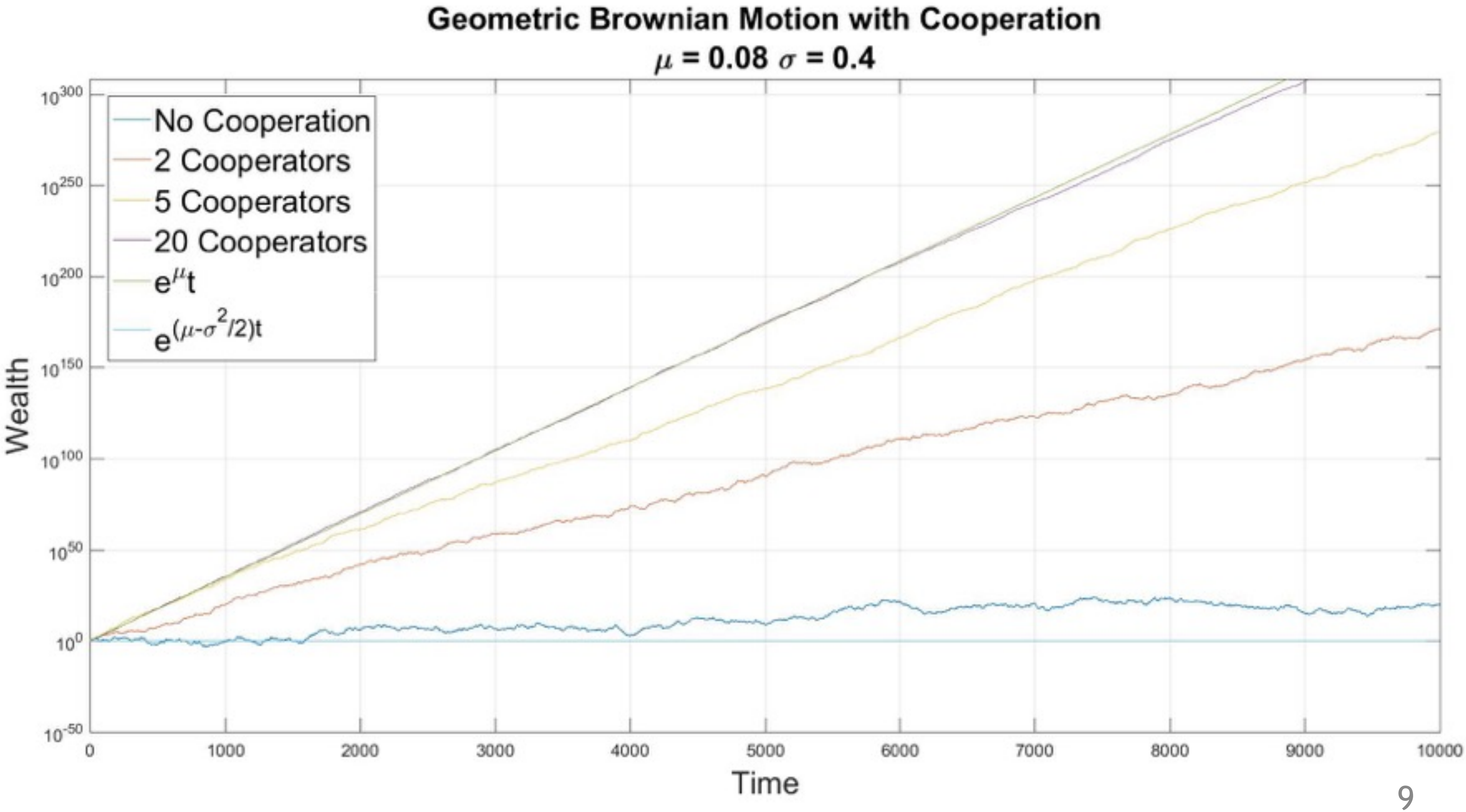


Geometric Brownian Motion
for μ = 0.08 σ = 0.4
Trajectory 1
Trajectory 2
$e^{\mu}t$
$e^{(\mu-\sigma^2/2)t}$
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
Wealth
0
10
20
30
40
50
60
70
80
90
100
Time

Cooperation

- The result can be generalized for N individuals sharing their resources equally, to:

$$g = \mu - \frac{\sigma^2}{2N}\ .$$


Geometric Brownian Motion with Cooperation
μ = 0.08 σ = 0.4
No Cooperation
2 Cooperators
5 Cooperators
20 Cooperators
$e^{\mu}t$
$e^{(\mu-\sigma^2/2)t}$
Wealth
Time
10^{300}
10^{250}
10^{200}
10^{150}
10^{100}
10^{50}
10^{0}
10^{-50}
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Our Model

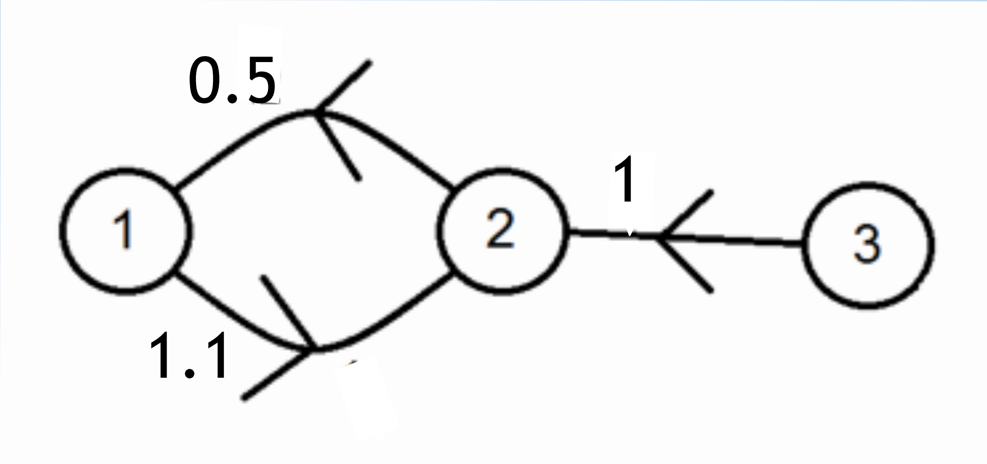
- The motivation behind our model came after studying a paper by Jain and Krishna on the evolution of a complex system.
- The system comprised of elements that catalysed the growth of other elements.
- We noted that this represented a system in which individuals were cooperating, except that there was no cost to the catalysis.
- This meant that the system didn’t display the same conservation present in the previous cooperation model, in which individuals pooled and split resources.
- Their guiding equation helped us to generalise the cooperation model.

Our Model

- We consider a population of N individuals which interact through the giving and taking of resources, not necessarily in a symmetric manner.
- We visualise this concept through graphs where each node represented an individual and directed edges represented the flow of resources.
- We do not allow a node to have a directed edge to itself, and each node may have at most one directed edge towards each other node.
- We allow edges to have different weights, corresponding to different rates of resource sharing.

The Weighted Adjacency Matrix

- For a graph of N nodes we consider the weighted adjacency matrix C defined by:
- $C_{ij} = \begin{cases} k, & \textit{if there is an edge of weight } k \textit{ from } j \textit{ to } i \\ 0, & \textit{otherwise} \end{cases}$
- For example for the graph below, we have a weighted adjacency matrix, $C = \begin{pmatrix} 0 & 0.5 & 0 \\ 1.1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$



Our Model

- For a graph with N nodes and weighted adjacency matrix C, the governing equation for the growth of the ith individual in the system is given by:

$$dx_i(t) = x_i(t)(\mu\, dt + \sigma\, dW_i(t)) + \sum_{j=1}^{N}\left(C_{ij}x_j(t) - C_{ji}x_i(t)\right) dt$$

- The first bracket is the individual's underlying growth, given by GBM.
- The second bracket consists of two sums. The first can be seen as the receiving term for the ith node, the second as the giving term.

Our Model

- $$dx_i(t) = x_i(t)(\mu\, dt + \sigma\, dW_i(t)) + \left(\underbrace{\sum_{j=1}^{N} C_{ij} x_j(t)}_{\text{Receiving Term}} - \underbrace{\sum_{j=1}^{N} C_{ji} x_i(t)}_{\text{Giving Term}} \right) dt$$
  - The receiving term is a sum over all incoming links to the i^th node, with each term being proportional to the weight of the link as well as the wealth of the node from which the link is incoming.
  - The giving term is a sum over all outgoing links from the i^th node, with each term proportional to the weight of the link as well as the wealth of the i^th node.

Our Model

- μ is irrelevant to the dynamics of the system except as a background growth parameter.

- If we consider a rescaled wealth $z_i(t) = e^{-\mu t} x_i(t)$, then:

$$dz_i(t) = z_i(t)(\sigma\, dW_i(t)) + \sum_{j=1}^{N} \Big(C_{ij} z_j(t) - C_{ji} z_i(t) \Big)\, dt$$

- i.e. $x_i(t)$ grows like $e^{\mu t} z_i(t)$, where $z_i(t)$ is in the same system with $\mu = 0$.

Our Model

- For example, in the case of the graph from before, we have

$C = \begin{pmatrix} 0 & 0.5 & 0 \\ 1.1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and so the governing equations are:

$$dx_1 = x_1(\mu\, dt + \sigma\, dW_1) + (0.5x_2 - 1.1x_1)dt$$
$$dx_2 = x_2(\mu\, dt + \sigma\, dW_2) + (1.1x_1 + x_3 - 0.5x_2)dt$$
$$dx_3 = x_3(\mu\, dt + \sigma\, dW_3) - x_3 dt$$

- Note that for a general weighted adjacency matrix, the reallocation of resources is conservative, i.e.

$\sum_{i=1}^{N}\sum_{j=1}^{N}\left(C_{ij}x_j - C_{ji}x_i\right) dt = 0$ at all times t.

The Deterministic Case

- When $\sigma = 0$, we have:

$$dx_i(t) = \mu\, x_i(t)\, dt + \sum_{j=1}^{N} \left(C_{ij} x_j(t) - C_{ji} x_i(t) \right) dt$$

- Then $d\left(\sum_{i=1}^{N} x_i(t)\right) = \mu \sum_{i=1}^{N} x_i(t)\, dt$, i.e. $\sum_{i=1}^{N} x_i(t) = \sum_{i=1}^{N} x_i(0) e^{\mu t}$
- So the evolution of the total wealth of the system is independent of any giving or receiving.
- The growth rate of a node with no incoming or outgoing links would be μ and there is no arrangement in which an individual can achieve a higher growth rate.

Questions

- What happens when individuals only engage in partial sharing?
- What rate of sharing does it take to achieve maximum growth rate?
- To what extent does growth rate depend on the way resources are distributed?
- Does any of the above depend on how noisy the system is?

Complete Graph on N nodes

- We consider the case in which every individual shares at the rate τ. We call τ the Cooperation Parameter.

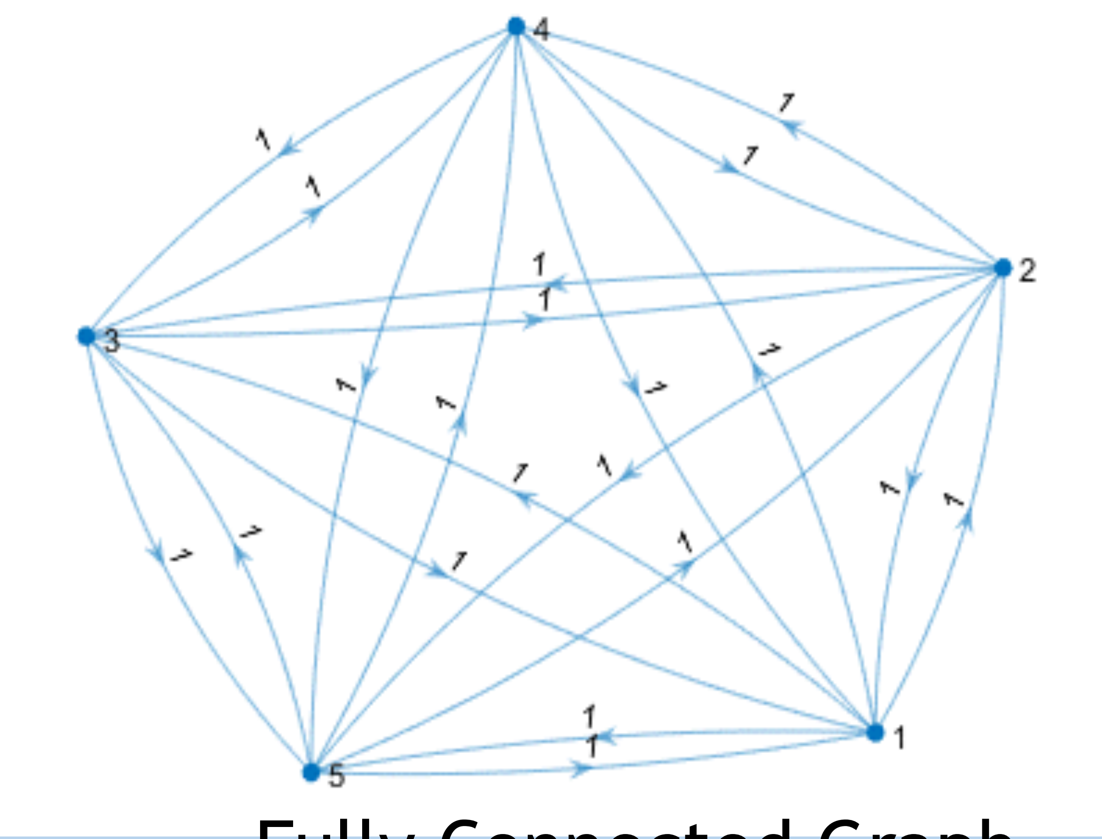
- A Complete Graph on N-nodes is a structure in which all of the individuals share their resources with all of the other individuals.

- The share which each individual contributes is distributed equally among the rest.

Complete Graph on N nodes

- The matrix for a complete graph on N nodes is:

$$C_{ij} = \begin{cases} \dfrac{\tau}{N-1}, & i \neq j \\ 0\ , & i = j \end{cases}$$



Fully Connected Graph for N = 5

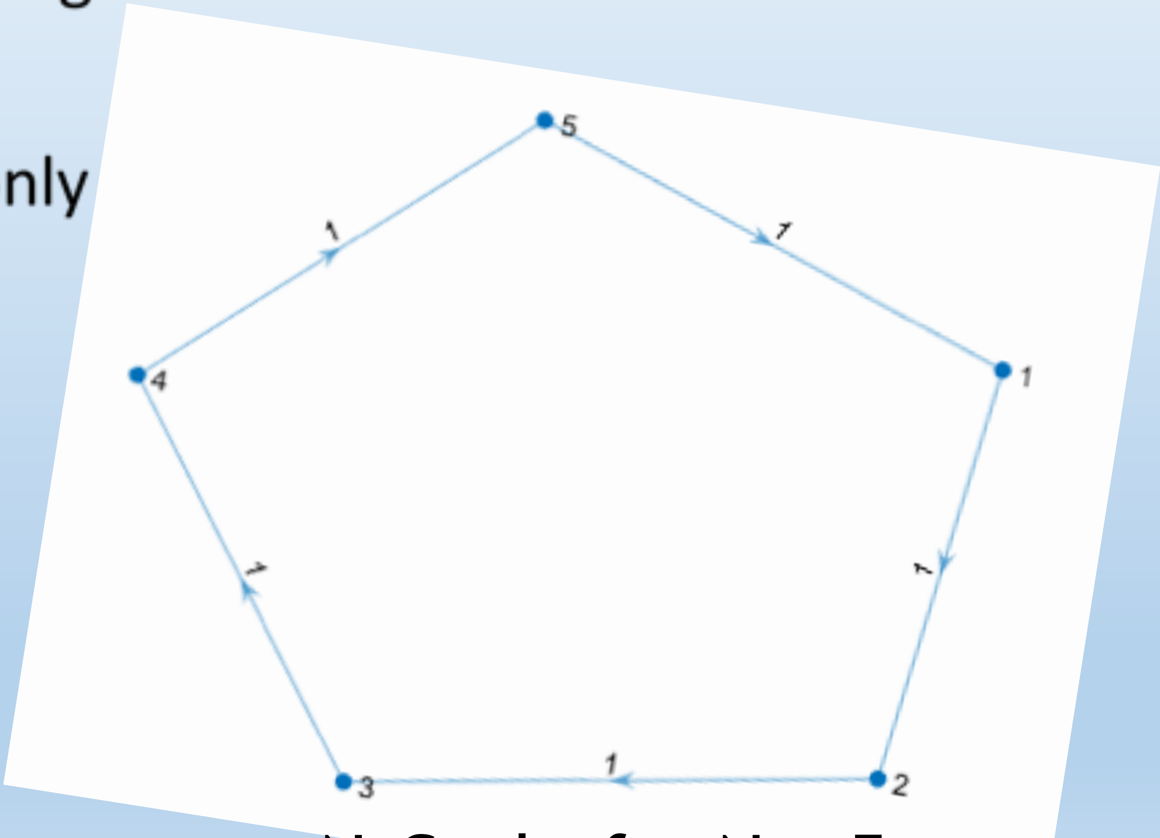
N-Cycle

- In an N-Cycle, each individual *still* gives at a rate τ.

- Now the resources are passed only to the next node in the cycle.

- The matrix is

$$C_{ij} = \begin{cases} \tau, & j \equiv i - 1\ (mod\ N) \\ 0\,, & \text{otherwise} \end{cases}$$



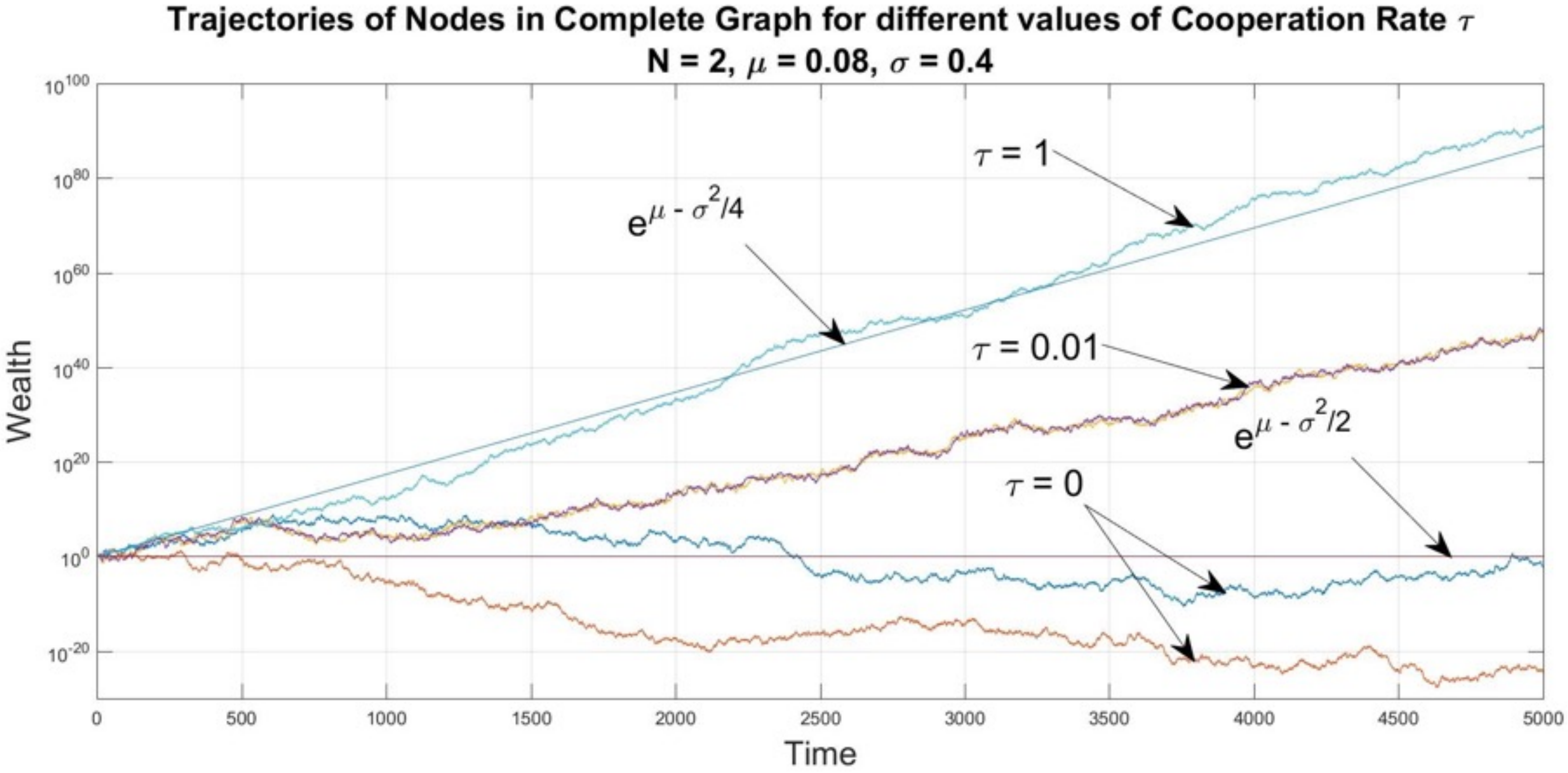
N-Cycle for N = 5

The Complete Graph for N = 2

- In the case N = 2 both of the previous structures are the same.
- Each node gives to (and receives from) the other at a rate τ.
- The equations are:

$$dx_1 = x_1(\mu\, dt + \sigma\, dW_1) + \tau(x_2 - x_1)dt$$

$$dx_2 = x_2(\mu\, dt + \sigma\, dW_2) + \tau(x_1 - x_2)dt$$


Trajectories of Nodes in Complete Graph for different values of Cooperation Rate τ
N = 2, μ = 0.08, σ = 0.4
Wealth
Time
τ = 1
$e^{\mu - \sigma^2/4}$
τ = 0.01
$e^{\mu - \sigma^2/2}$
τ = 0
10^{100}
10^{80}
10^{60}
10^{40}
10^{20}
10^{0}
10^{-20}
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000

Normalised Growth, G

- We see that the growth rates of the trajectories lie between

$$g_{min} = \mu - {\sigma^2}/{2} \text{ and } g_{max} = \mu - {\sigma^2}/{4}.$$

- We normalise g by considering instead:

$$G = 2\left(\frac{g - \left(\mu - \frac{\sigma^2}{2}\right)}{\sigma^2/2}\right)$$

- For no cooperation, $G = 0$ and for full cooperation $G = 1$.

Normalised Growth, G

- More generally, for a system of N nodes, we have

$$G = \left(\frac{N}{N-1}\right)\left(\frac{g-\left(\mu-\frac{\sigma^2}{2}\right)}{\sigma^2/_2}\right)$$

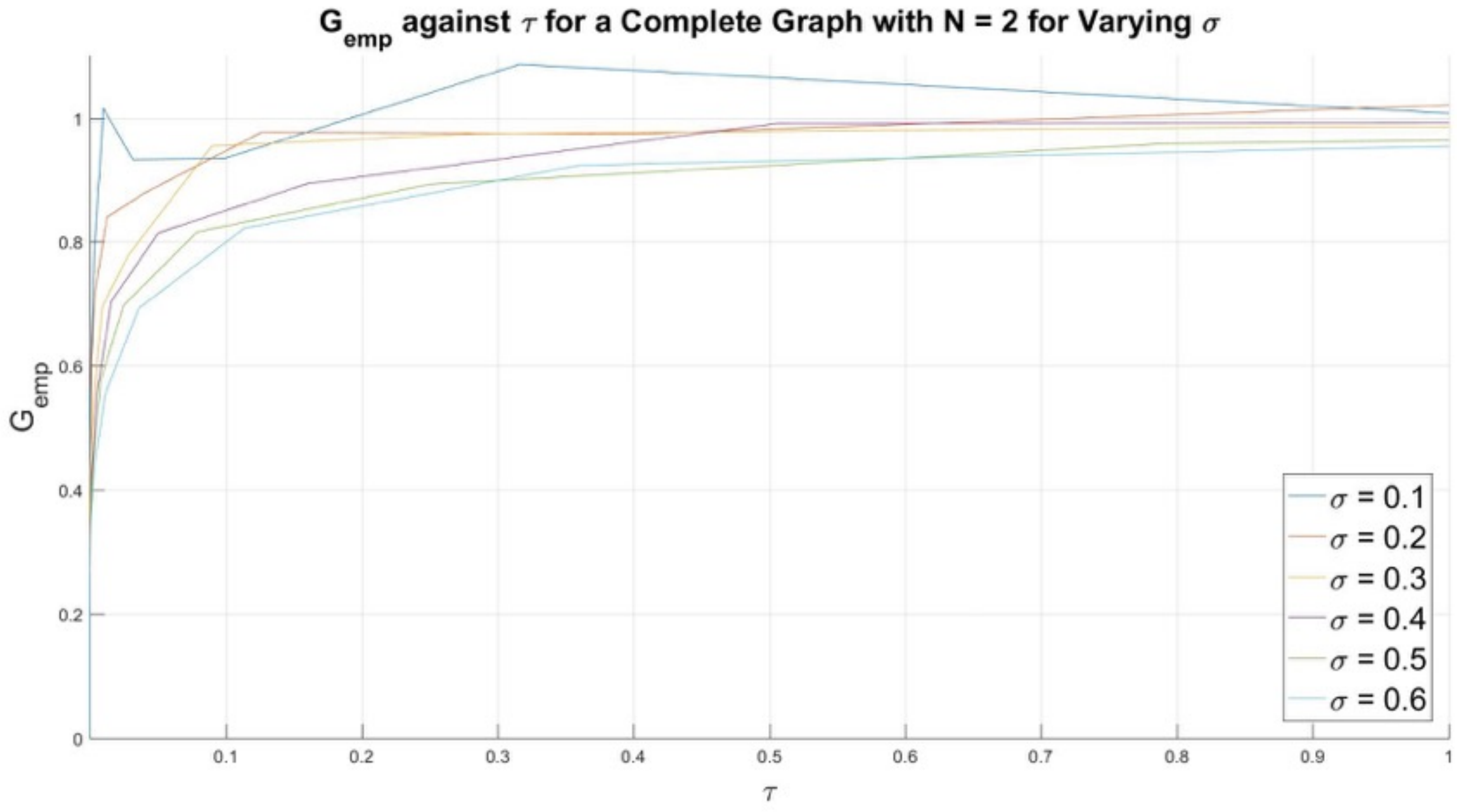
- Again, $G = 0$ for no cooperation, and $G = 1$ for full cooperation.
- This is a more standard parameter by which we can measure the effects of cooperation on a system of N nodes.
- It acts as a measure to compare systems with different values of μ and σ.

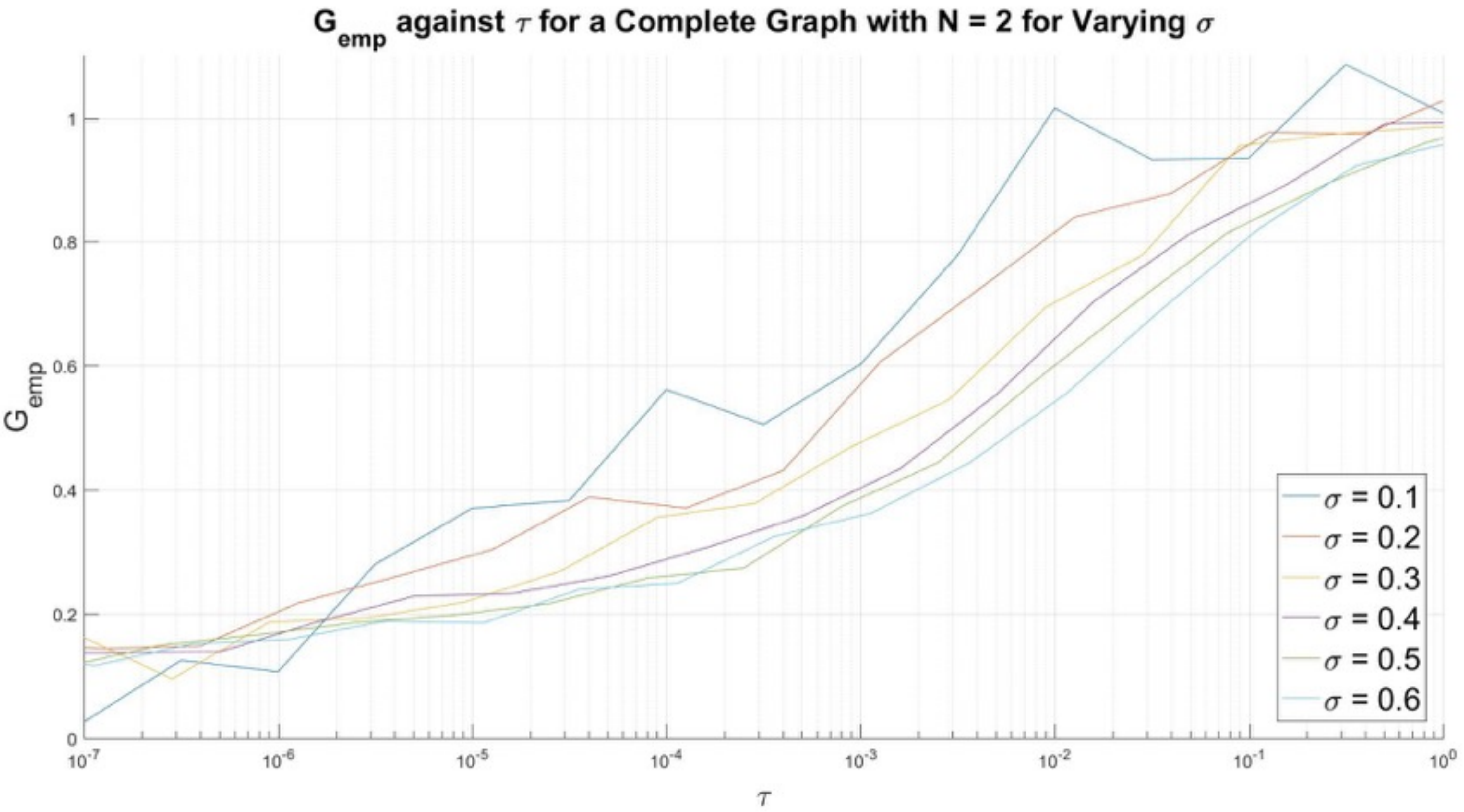
Normalised Growth, G

- Recall $g_T = \frac{1}{T} \ln \frac{y(T)}{y(0)}$.

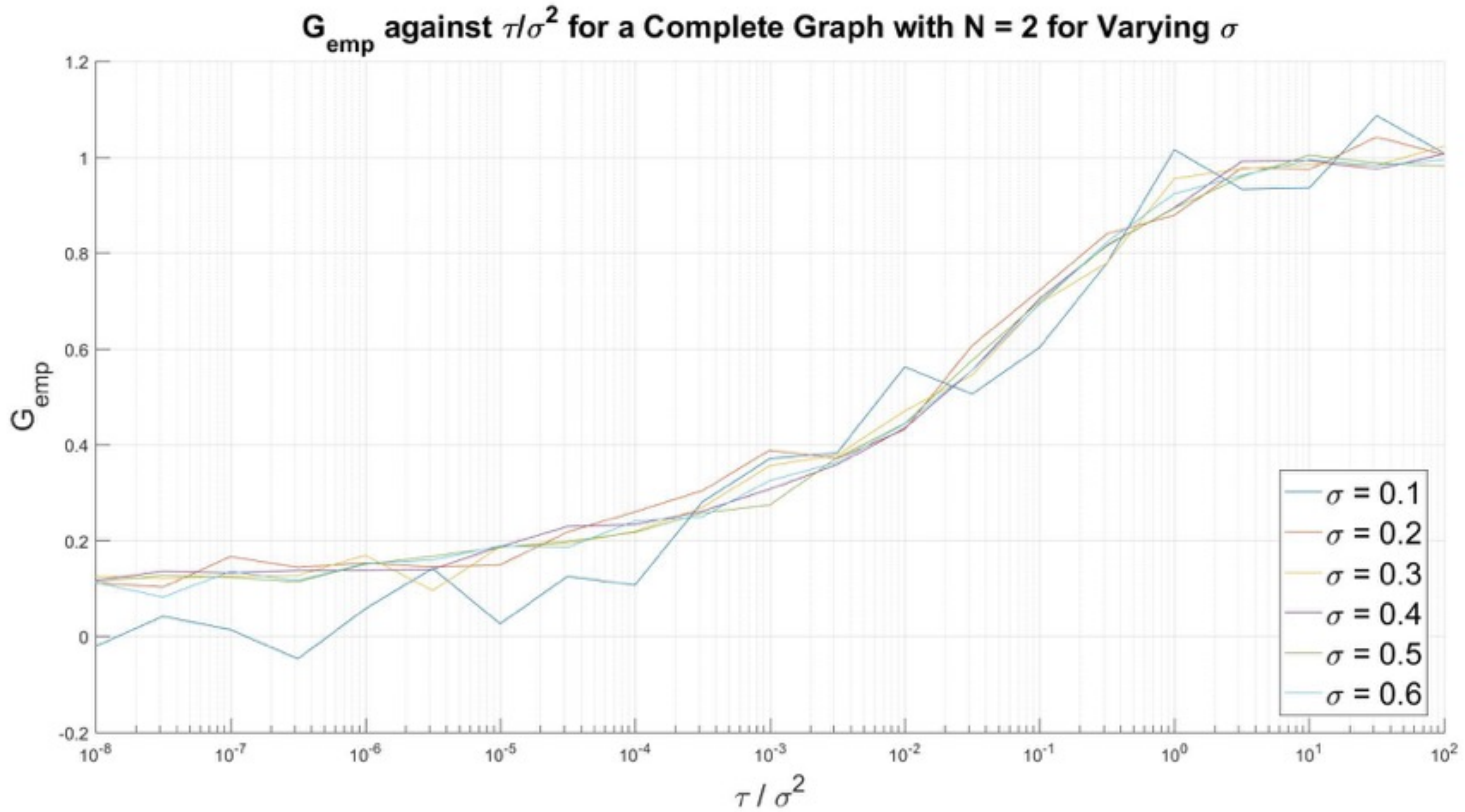
- We estimate g by simulating the growth of the system and calculate the average over many realisations of the observed value of g_T.

- We call this average g_{emp} and compute the corresponding G_{emp}.


G_{emp} against τ for a Complete Graph with N = 2 for Varying σ
G_{emp}
τ
0
0.2
0.4
0.6
0.8
1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
σ = 0.1
σ = 0.2
σ = 0.3
σ = 0.4
σ = 0.5
σ = 0.6



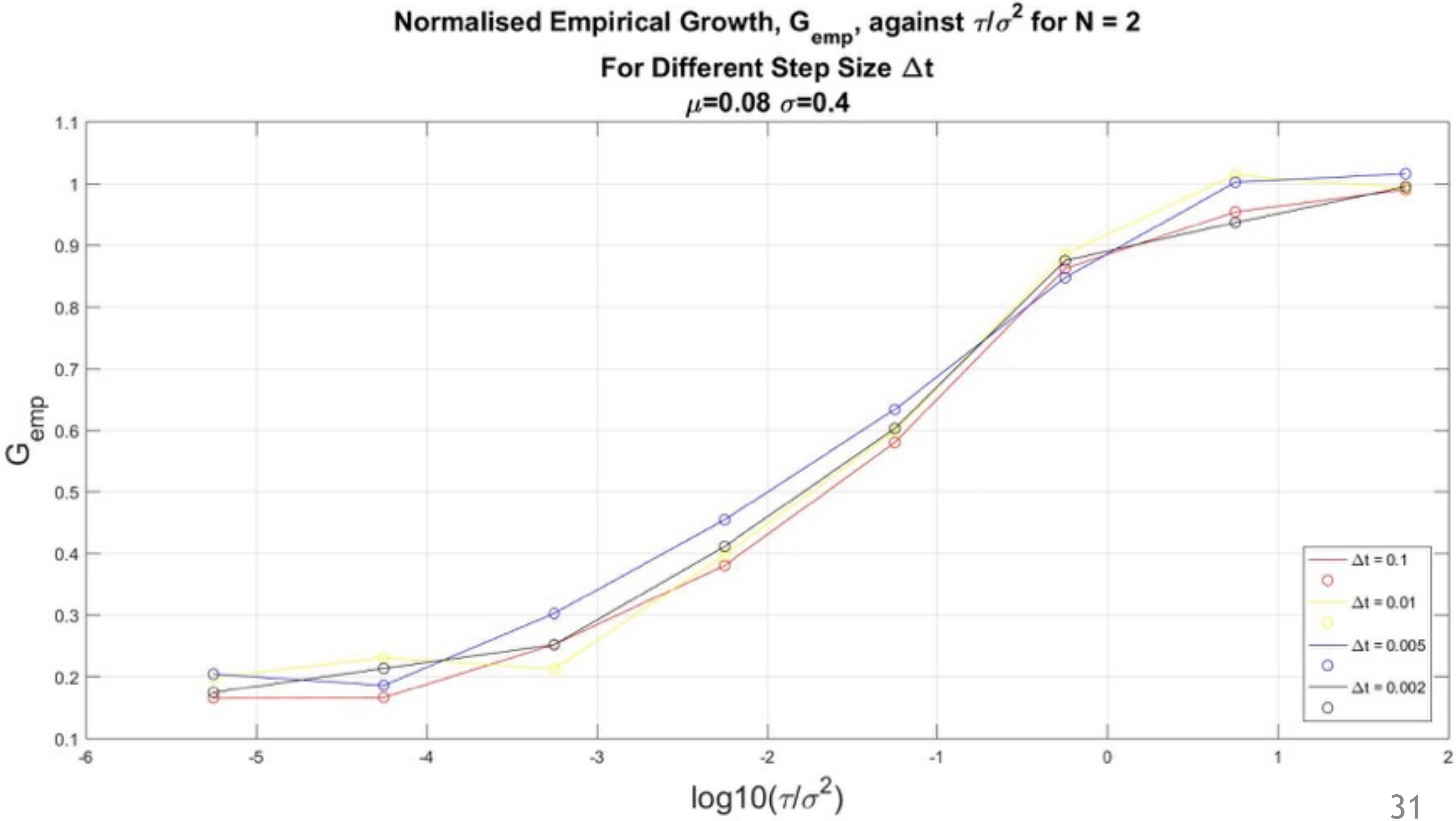
G_{emp} against τ for a Complete Graph with N = 2 for Varying σ
G_{emp}
τ
0
0.2
0.4
0.6
0.8
1
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
σ = 0.1
σ = 0.2
σ = 0.3
σ = 0.4
σ = 0.5
σ = 0.6



G_emp against τ/σ² for a Complete Graph with N = 2 for Varying σ
G_emp
τ / σ²
σ = 0.1
σ = 0.2
σ = 0.3
σ = 0.4
σ = 0.5
σ = 0.6
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
10^-8
10^-7
10^-6
10^-5
10^-4
10^-3
10^-2
10^-1
10^0
10^1
10^2

Interpretation of Results for N = 2

- We see that $\tau/_{\sigma^2}$ is the relevant parameter when considering the effect of cooperation on normalised growth.
- Full cooperation (all individuals have the same amount at all times) is achieved as $\tau \to \infty$.
- However maximised growth G ≈ 1 is observed even for relatively small values of τ.
- We see that even partial sharing can result in a significantly higher growth rate.
- Further we observe that G ≈ 1 even for $\tau \ll \dfrac{1}{\Delta t}$.


Normalised Empirical Growth, G_{emp}, against τ/σ^2 for N = 2
For Different Step Size Δt
μ=0.08 σ=0.4
G_{emp}
$\log10(\tau/\sigma^2)$
Δt = 0.1
Δt = 0.01
Δt = 0.005
Δt = 0.002

Questions

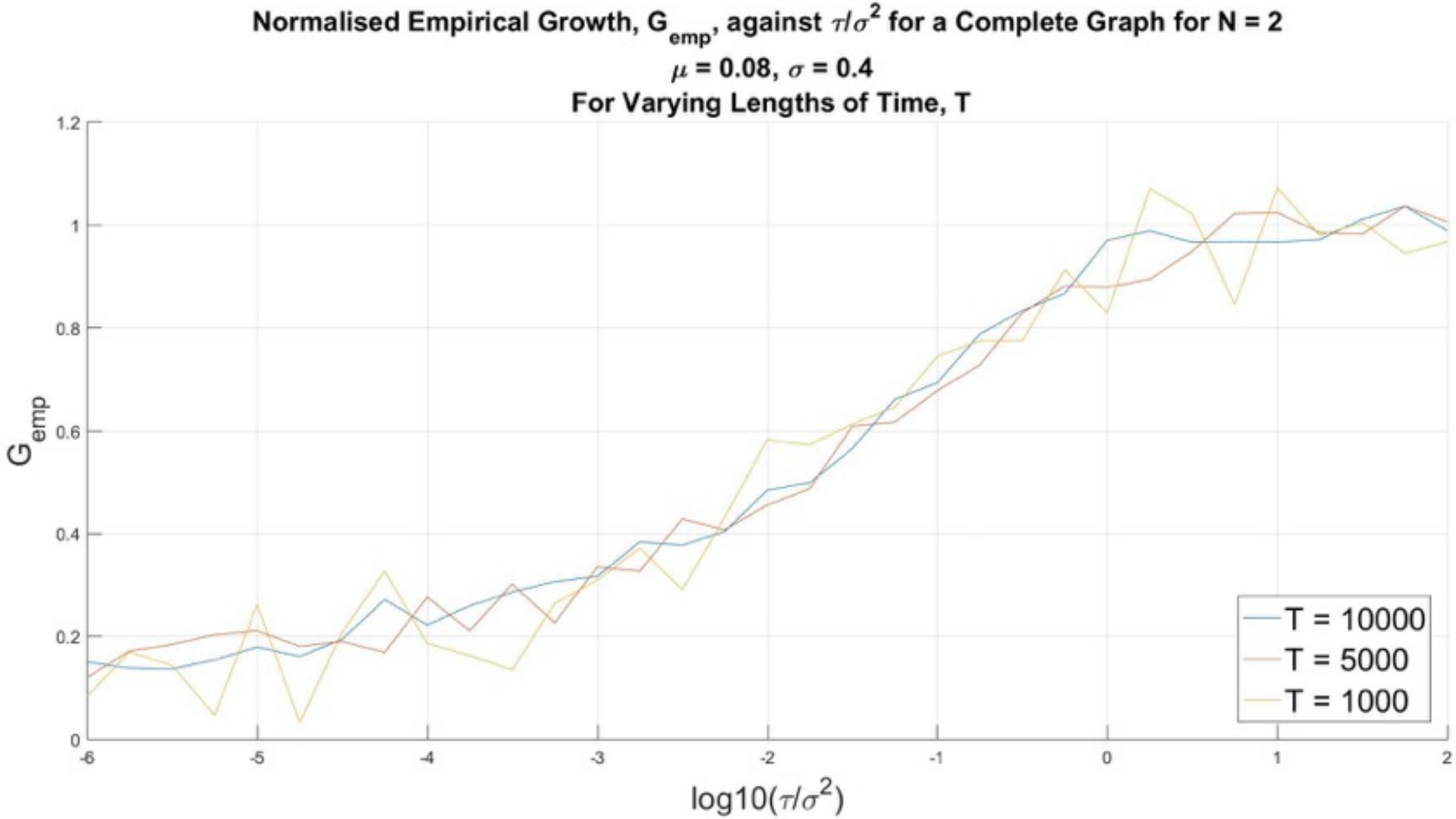
- **What happens when individuals only engage in partial sharing?**
- What rate of sharing does it take to achieve a higher growth rate?
- To what extent does growth rate depend on the way resources are distributed?
- Does any of the above depend on how noisy the system is?

Finiteness of Time Length T

- Is this just an equilibration issue?

- There is a question of whether all trajectories, for any value of $\tau/_{\sigma^2}$, will reach the optimum growth of G = 1 in the long time limit.

- If this were true we would expect to see some systematic shift in the curve G when increasing the observation time T.



Normalised Empirical Growth, G_{emp}, against τ/σ^2 for a Complete Graph for N = 2
μ = 0.08, σ = 0.4
For Varying Lengths of Time, T
G_{emp}
$\log10(\tau/\sigma^2)$
T = 10000
T = 5000
T = 1000
1.2
1
0.8
0.6
0.4
0.2
0
-6
-5
-4
-3
-2
-1
0
1
2

Questions

- What happens when individuals only engage in partial sharing?

- **What rate of sharing does it take to achieve maximum growth rate?**

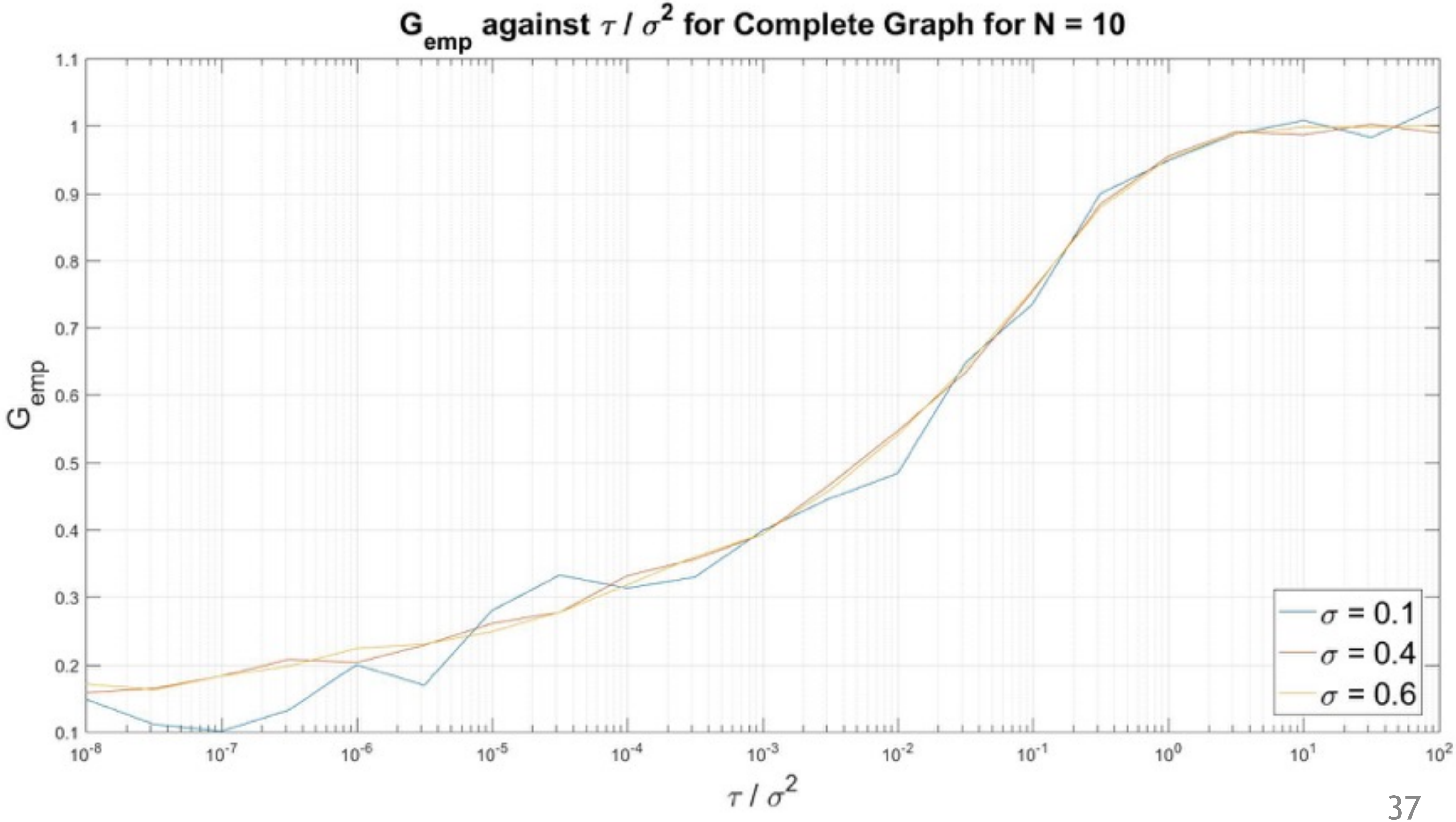
- To what extent does growth rate depend on the way resources are distributed?

- Does any of the above depend on how noisy the system is?

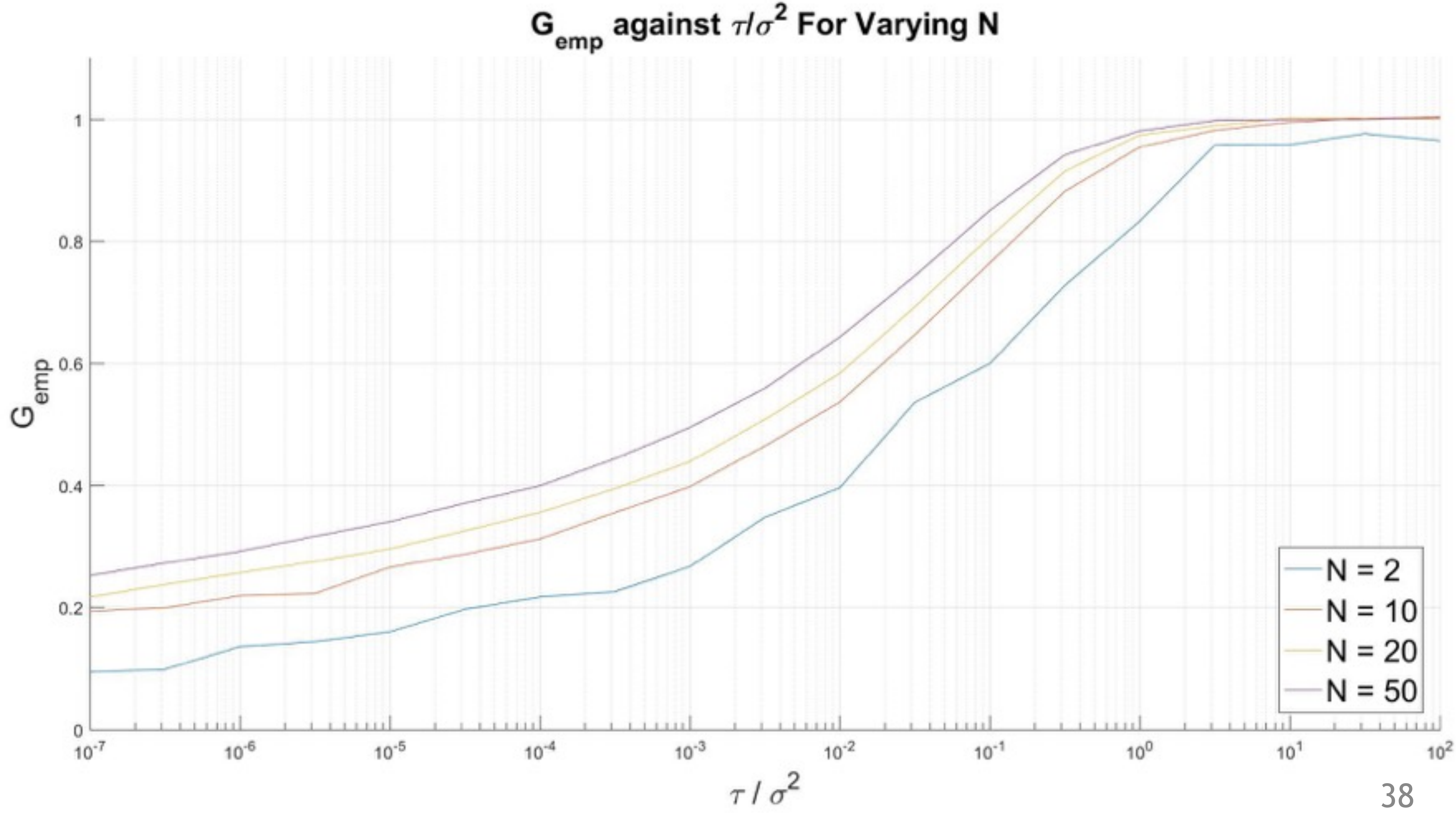
Fully Connected Graph for Different N

-
  - As previously remarked, $C_{ij} = \begin{cases} \frac{\tau}{N-1}, & i \neq j \\ 0\ , & i = j \end{cases}$

  - We consider how the growth G scales with increasing N.



G_{emp} against τ / σ^2 for Complete Graph for N = 10
G_{emp}
τ / σ^2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
σ = 0.1
σ = 0.4
σ = 0.6



G_{emp} against τ/σ^2 For Varying N
G_{emp}
τ / σ^2
N = 2
N = 10
N = 20
N = 50
0
0.2
0.4
0.6
0.8
1
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}

Fully Connected Graph for Different N

- The governing equations are:

$$dx_i(t) = x_i(t)(\mu\, dt + \sigma\, dW_i(t)) + \frac{\tau}{N-1}\sum_{\substack{j=1 \\ j\neq i}}^{N}\Big(x_j(t) - x_i(t)\Big)\, dt$$

$$= x_i(t)(\mu\, dt + \sigma\, dW_i(t)) + \frac{\tau}{N-1}\sum_{j=1}^{N}\Big(x_j(t) - x_i(t)\Big)\, dt$$

$$= x_i(t)(\mu\, dt + \sigma\, dW_i(t)) + \frac{N\tau}{N-1}\Big(< x(t) >_N - x_i(t)\Big) dt$$

$$dx_i(t) = x_i(t)(\mu\, dt + \sigma\, dW_i(t)) + \begin{cases} -\tau x_i(t)dt + \dfrac{\tau}{N-1}\displaystyle\sum_{\substack{j=1 \\ j\neq i}}^{N} x_j(t)dt & \text{Form One} \\ -\dfrac{N\tau}{N-1}(x_i(t) - \langle x(t) \rangle_N)dt & \text{Form Two} \end{cases}$$

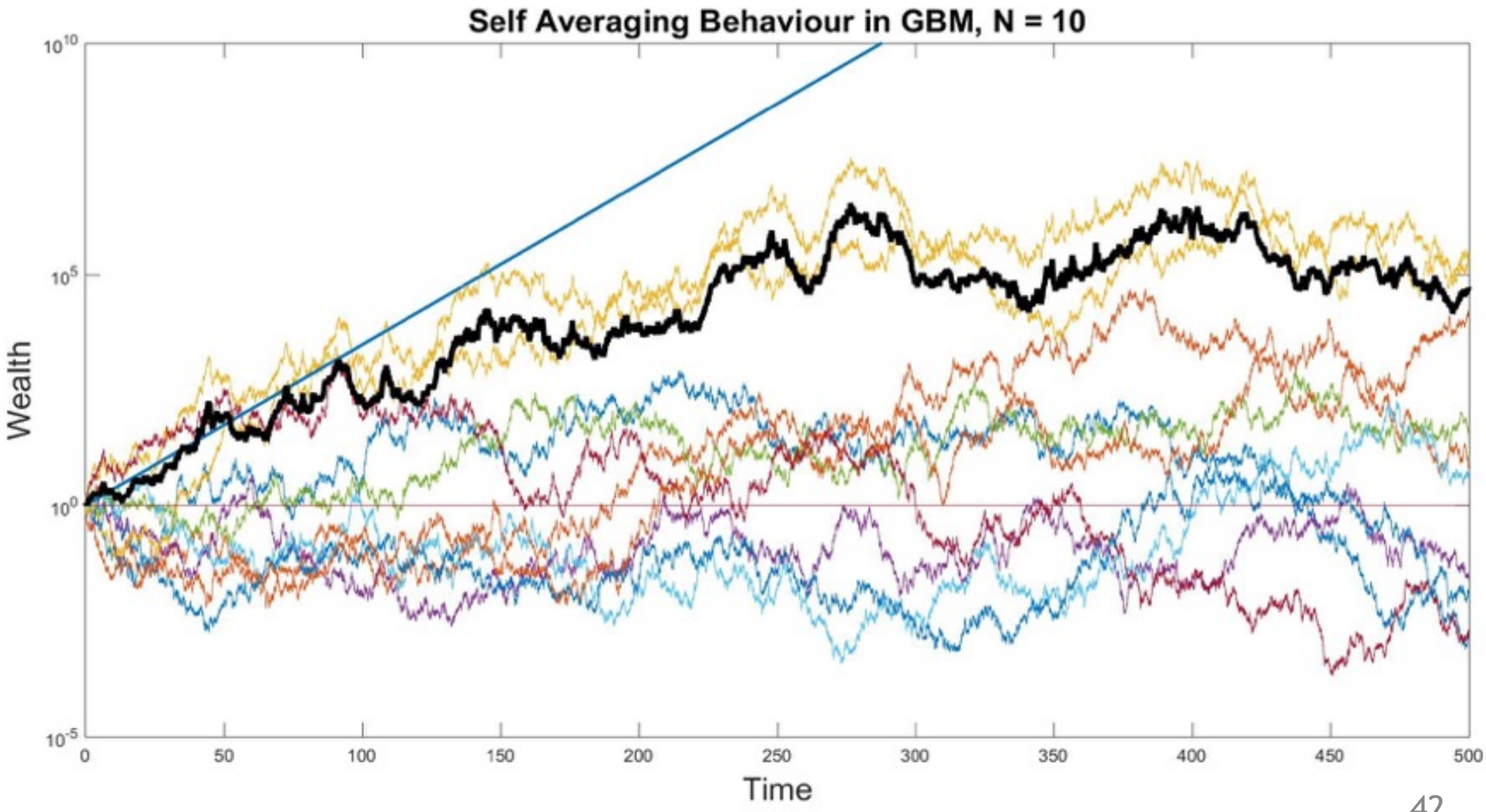
- In form one, each individual gives at a rate τ. The contributions of this node are split equally amongst the other N-1 nodes.
- In form two, each individual contributes to a central pool at a rate $\frac{N\tau}{N-1}$ and the pool is instantaneously shared equally between all N nodes.

$$dx_i(t) = x_i(t)(\mu\, dt + \sigma\, dW_i(t)) + \begin{cases} -\boldsymbol{\tau x_i(t)dt + \frac{\tau}{N-1}\sum_{\substack{j=1 \\ j\neq i}}^{N} x_j(t)dt} & \text{Form One} \\ -\boldsymbol{\frac{N\tau}{N-1}(x_i(t) - < x(t) >_N)dt} & \text{Form Two} \end{cases}$$

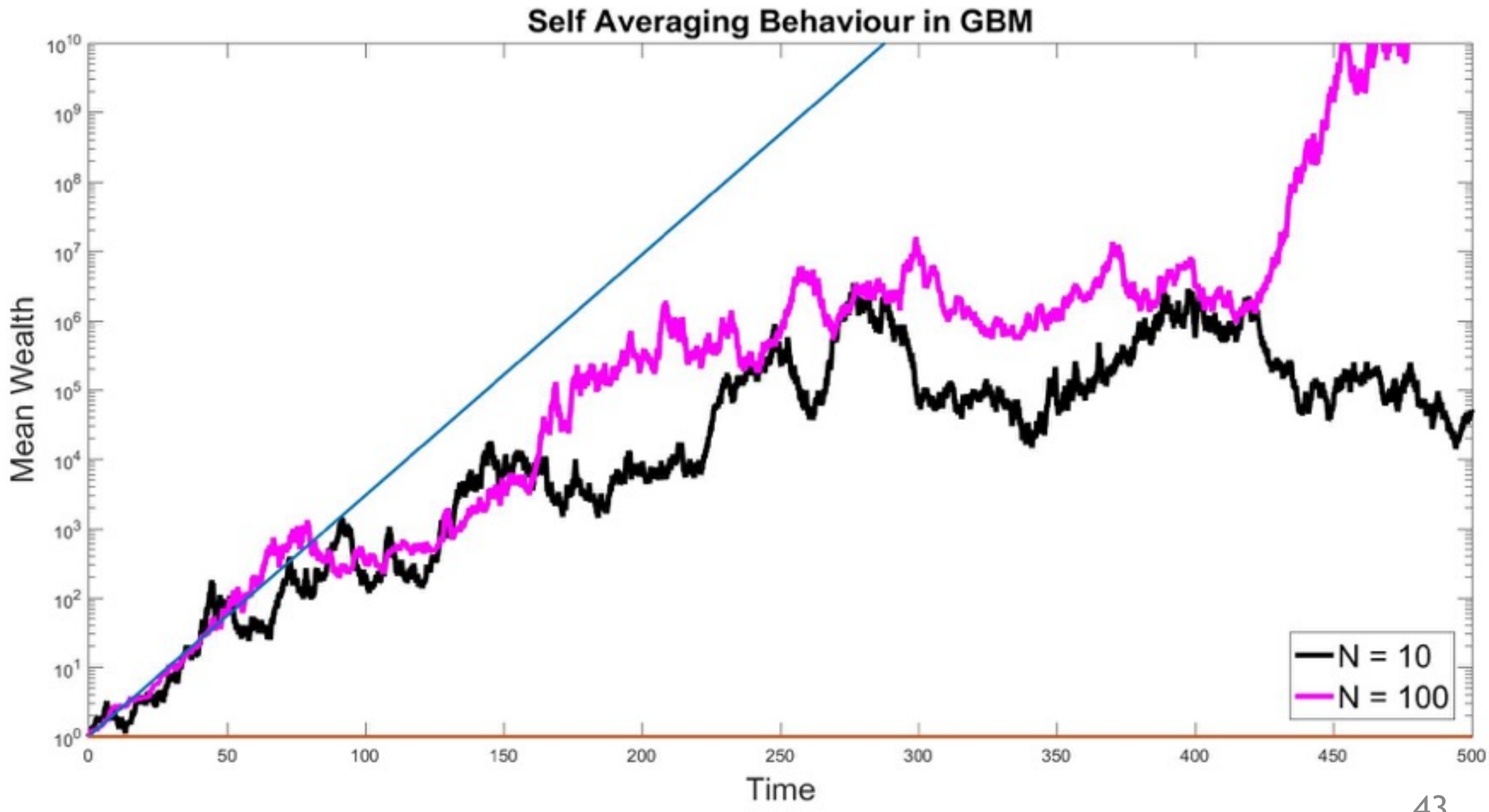
- For convenience we introduce the notation J, where:

$$J = \frac{N\tau}{N-1}$$

- There is a result by Bouchaud which says that in the **large N** limit, for $\frac{J \ln N}{\sigma^2} \gg \frac{1}{2}$, the system achieves growth rates of $g \approx \mu$, i.e. $G \approx 1$.



Self Averaging Behaviour in GBM, N = 10
Wealth
Time
10^{10}
10^{5}
10^{0}
10^{-5}
0
50
100
150
200
250
300
350
400
450
500



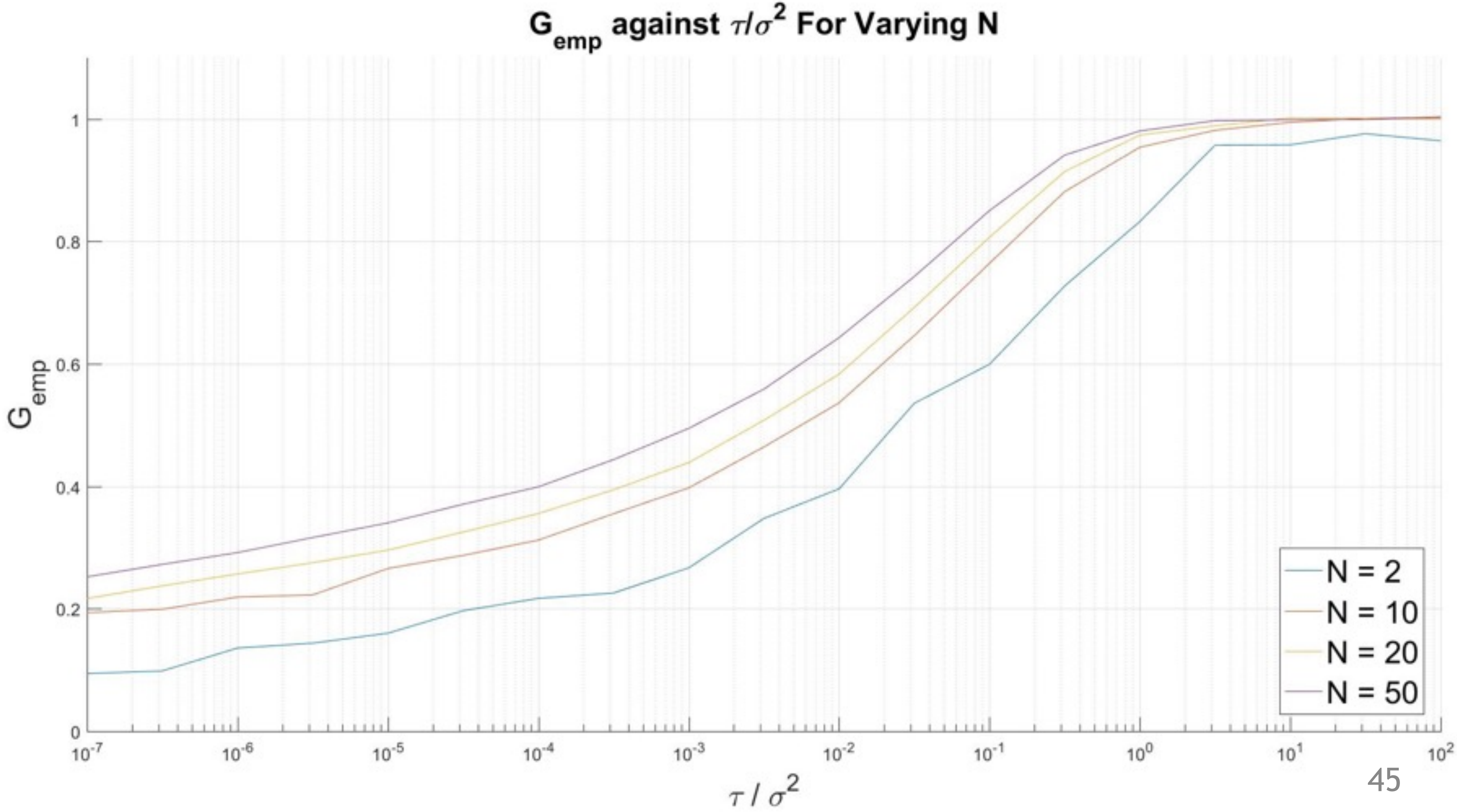
Self Averaging Behaviour in GBM
Mean Wealth
Time
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
10^{7}
10^{8}
10^{9}
10^{10}
0
50
100
150
200
250
300
350
400
450
500
N = 10
N = 100

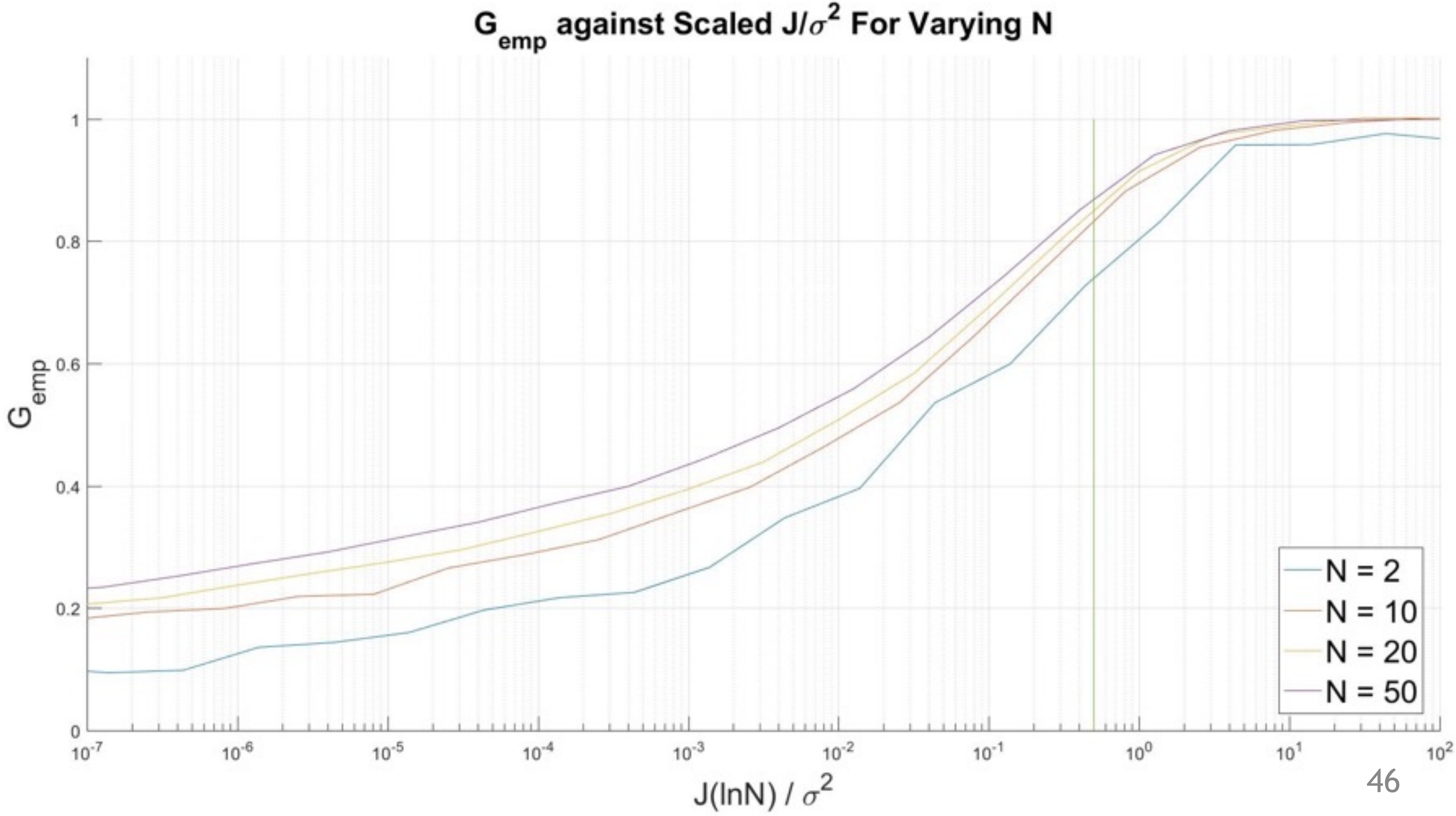
Similar Behaviour in Our Model

- Bouchaud's result implies that, for a large population, this self-averaging regime is realised for all time, provided the inequality $J^{-1} \ll t_{crit}$ holds.

- J^{-1} can be interpreted as a timescale on which the trajectories redistribute resources.

- The condition $J^{-1} \ll t_{crit}$ says that in order to realise maximum growth, the redistribution of resources must happen on a timescale within that of the self averaging regime.


G_{emp} against τ/σ^2 For Varying N
G_{emp}
0
0.2
0.4
0.6
0.8
1
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
τ / σ^2
N = 2
N = 10
N = 20
N = 50

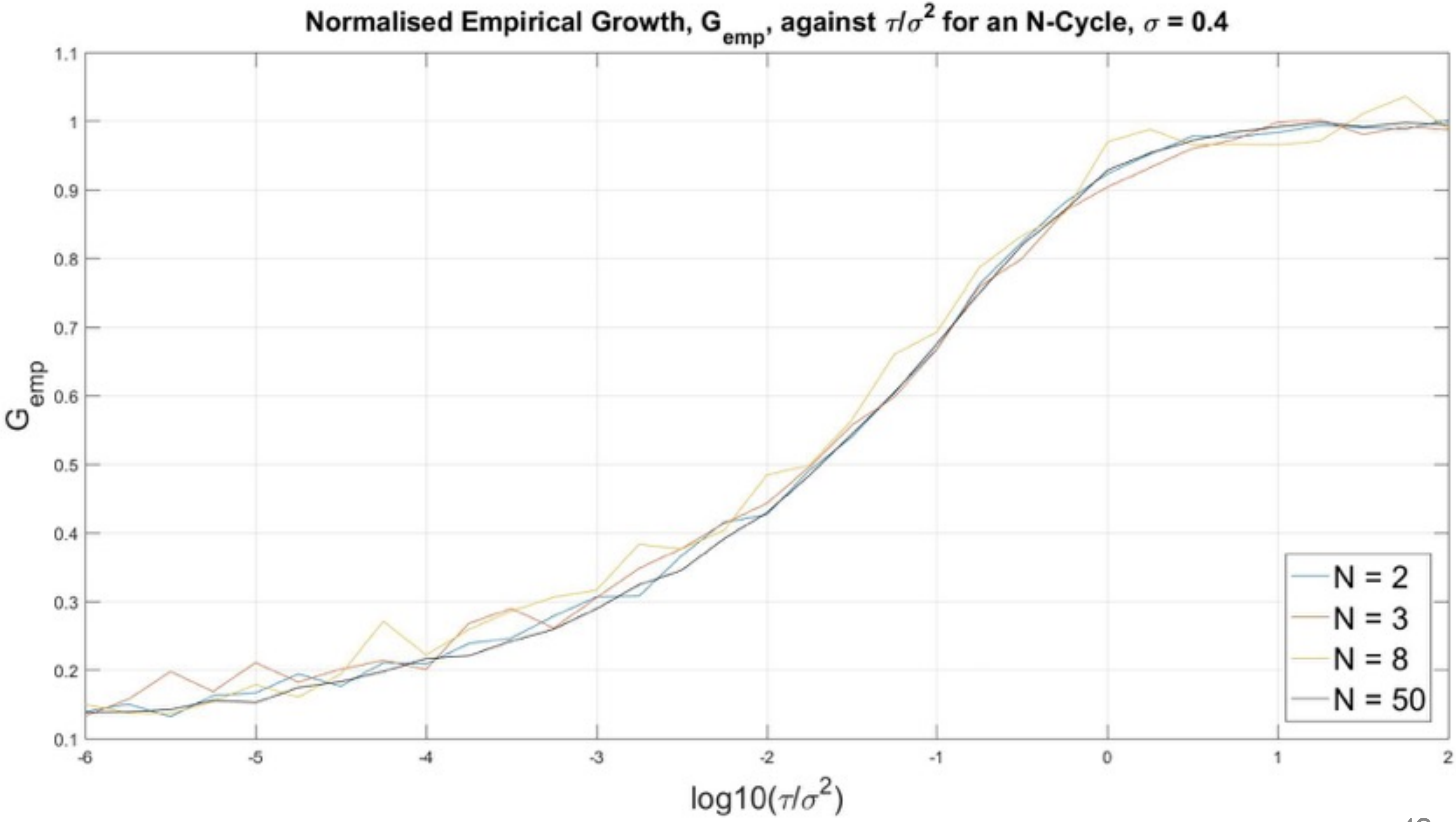

G_emp against Scaled J/σ^2 For Varying N
G_emp
0
0.2
0.4
0.6
0.8
1
10^-7
10^-6
10^-5
10^-4
10^-3
10^-2
10^-1
10^0
10^1
10^2
J(lnN) / σ^2
N = 2
N = 10
N = 20
N = 50

Questions

- What happens when individuals only engage in partial sharing?
- **What rate of sharing does it take to achieve maximum growth rate?**
- To what extent does growth rate depend on the way resources are distributed?
- Does any of the above depend on how noisy the system is?

N-Cycle

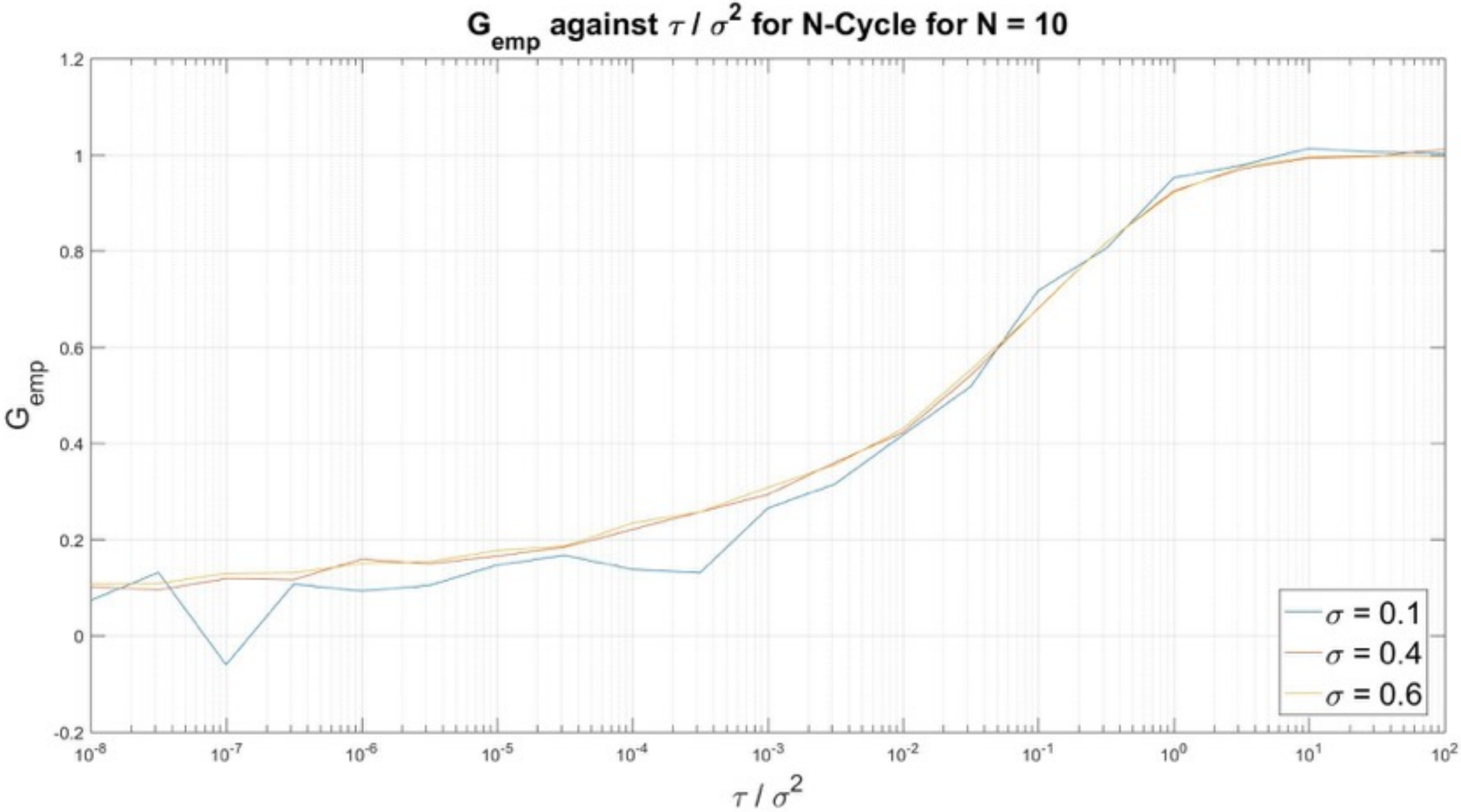
- Another generalisation of the 2 node case is to the N-Cycle, in which each node gives (at a rate τ) only to one of its nearest neighbours such that there is a unidirectional flow of resources, i.e:
- $C_{ij} = \begin{cases} \tau, & j = i - 1 \ (mod\ N) \\ 0\,, & \text{otherwise} \end{cases}$
- We explored the relationship between G, τ, σ and N for an N-Cycle.


Normalised Empirical Growth, G_{emp}, against τ/σ^2 for an N-Cycle, σ = 0.4
G_{emp}
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
-6
-5
-4
-3
-2
-1
0
1
2
$\log10(\tau/\sigma^2)$
N = 2
N = 3
N = 8
N = 50

- From our simulations it seems that $G_{cycle}(\tau, \sigma, N)$=$G_{cycle}(\tau, \sigma)$
- Rearranging to give average growth rate in terms of other parameters, we obtain:

$$g = \mu + \frac{\sigma^2}{2}\left(\frac{N-1}{N} G_{cycle}(\tau, \sigma) - 1\right)$$

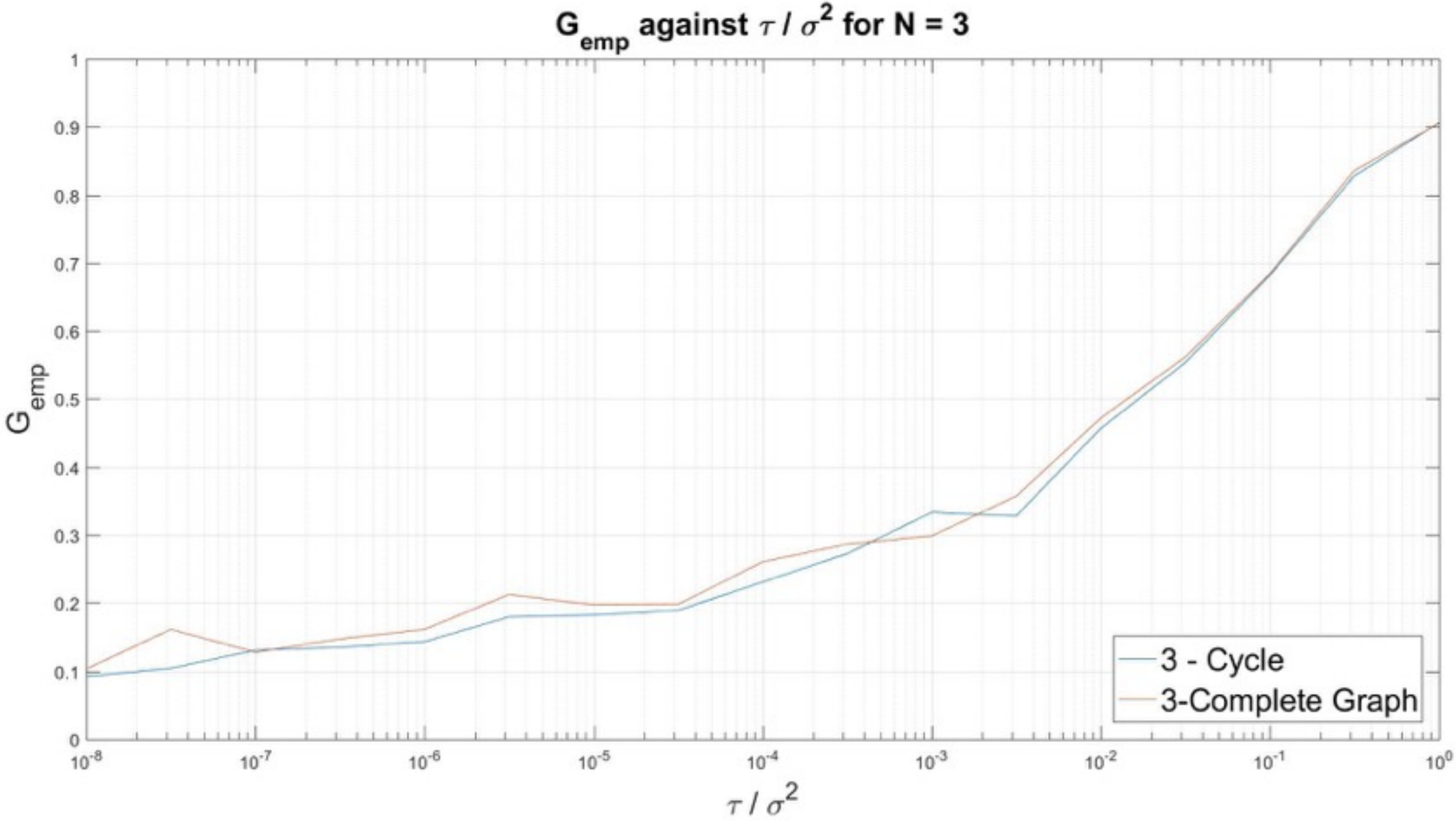
- If one can find a good estimate of $G_{cycle}(\tau, \sigma)$ for any N, one can use it to find the growth rate g for any cycle of given size with known μ, τ and σ.



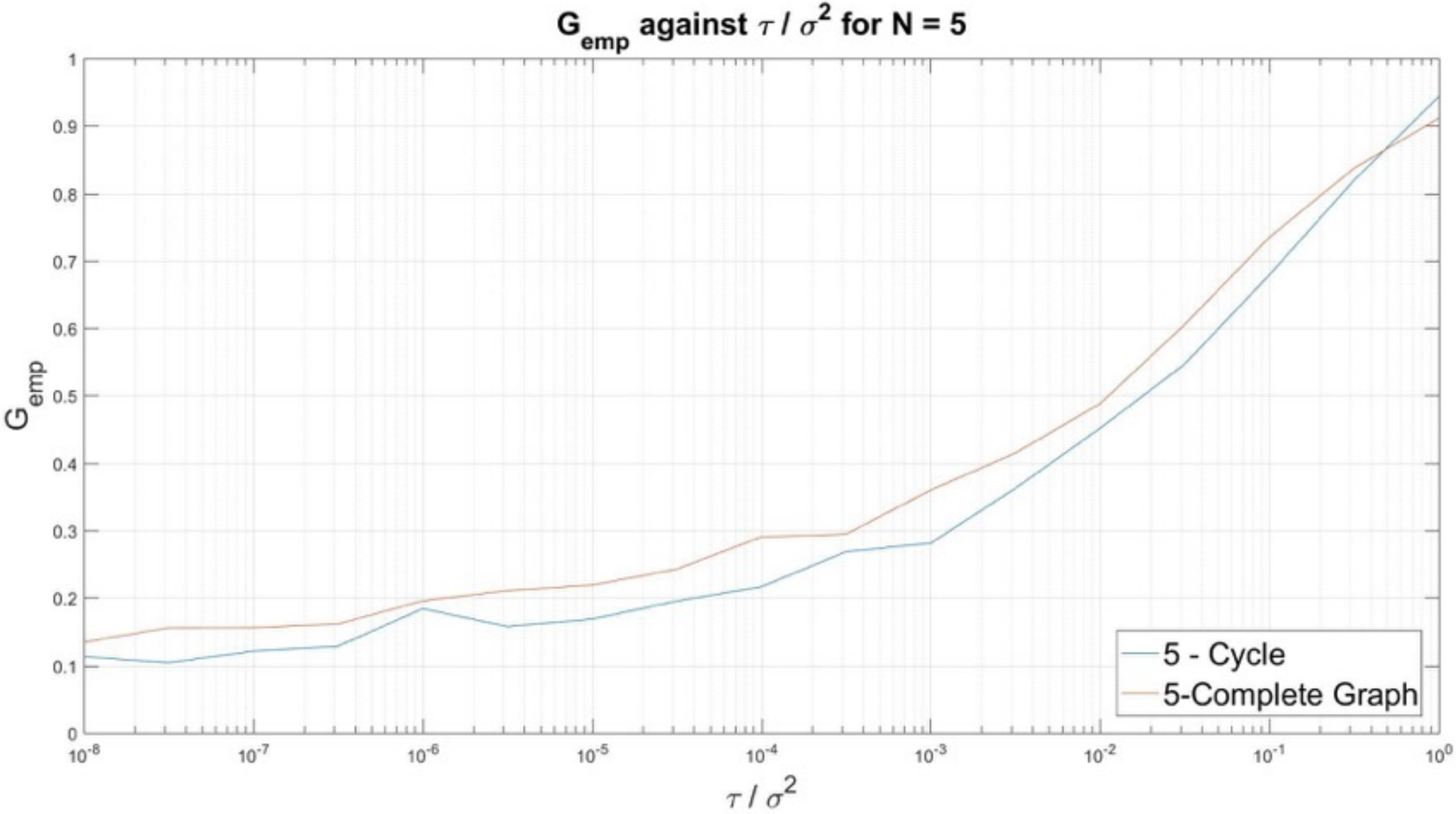
G_{emp} against τ / σ^2 for N-Cycle for N = 10
G_{emp}
τ / σ^2
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
σ = 0.1
σ = 0.4
σ = 0.6

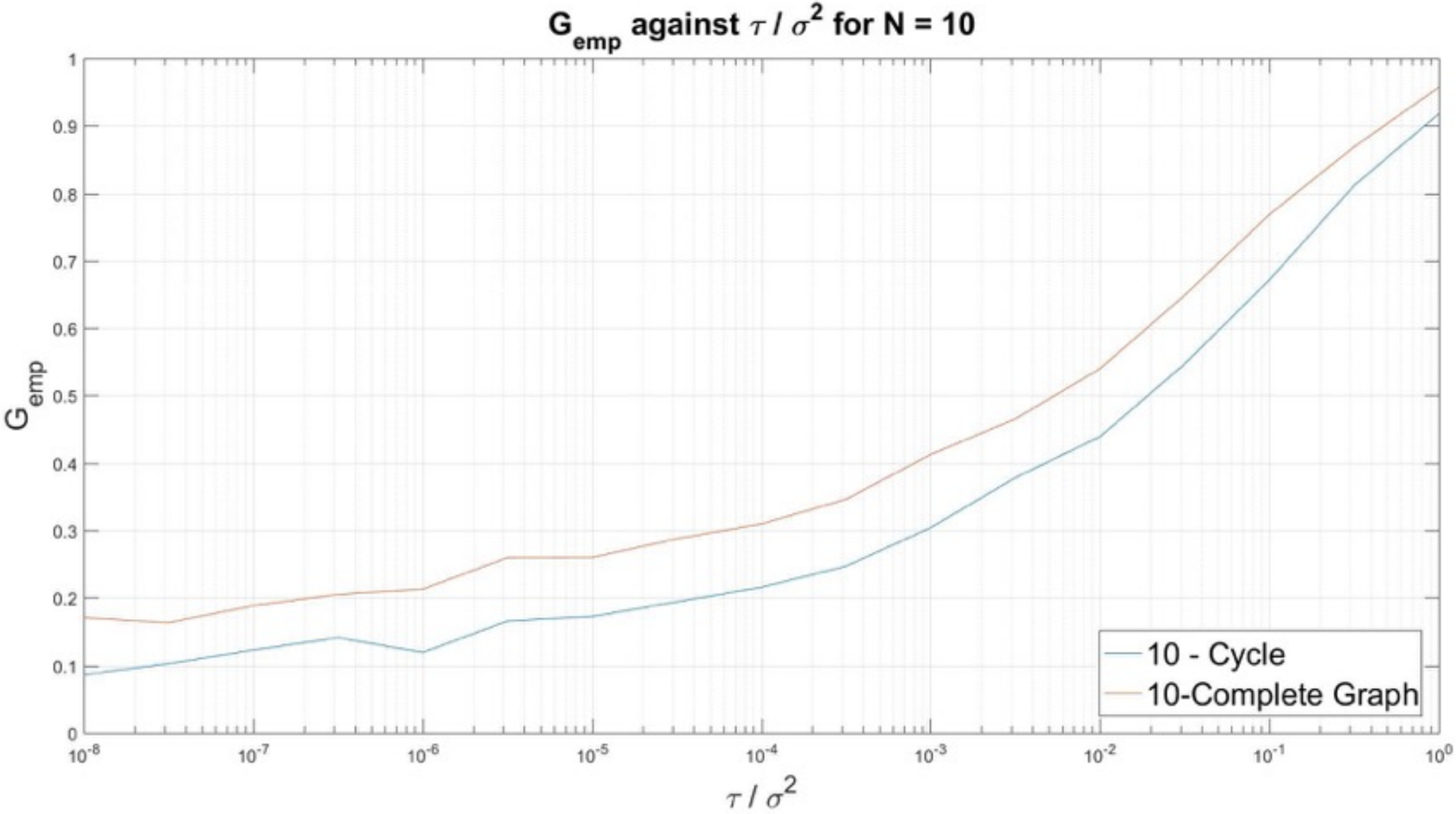
Questions

- What happens when individuals only engage in partial sharing?
- What rate of sharing does it take to achieve maximum growth rate?
- To what extent does growth rate depend on the way resources are distributed?
- **Does any of the above depend on how noisy the system is?**



G_emp against τ / σ^2 for N = 3
G_emp
τ / σ^2
3 - Cycle
3-Complete Graph
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
10^-8
10^-7
10^-6
10^-5
10^-4
10^-3
10^-2
10^-1
10^0


G_{emp} against τ / σ^2 for N = 5
G_{emp}
τ / σ^2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
5 - Cycle
5-Complete Graph


G emp against τ / σ2 for N = 10
Gemp
τ / σ2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
10-8
10-7
10-6
10-5
10-4
10-3
10-2
10-1
100
10 - Cycle
10-Complete Graph

Conclusion

- In the absence of fluctuations, there is no incentive to cooperate.
- Even a little sharing goes a long way.
- The manner of distribution does have an impact on the growth rate.

Thank you for listening.

- *References:*

1. *"The Evolutionary Advantage of Cooperation", O. Peters and A. Adamou* http://arxiv.org/abs/1506.03414

2. *"The Emergence and Growth of Complex Networks in Adaptive Systems", S. Jain and S. Krishna* Computer Phys. Comm. **121-122** (1999) 116-121.

3. *"Far from equilibrium: Wealth reallocation in the United States", Y. Berman, O. Peters and A. Adamou* http://arxiv.org/abs/1605.05631

4. *"Note on mean-field wealth models and the Random Energy Model", Jean-Phillipe Bouchard (not published)*