

Minimum Pseudo-Weight and Minimum Pseudo-Codewords of LDPC Codes *

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Abstract

In this correspondence, we study the minimum pseudo-weight and minimum pseudo-codewords of low-density parity-check (LDPC) codes under linear programming (LP) decoding. First, we show that the lower bound of Kelly, Sridhara, Xu and Rosenthal on the pseudo-weight of a pseudo-codeword of an LDPC code with girth greater than 4 is tight if and only if this pseudo-codeword is a real multiple of a codeword. Then, we show that the lower bound of Kashyap and Vardy on the stopping distance of an LDPC code is also a lower bound on the pseudo-weight of a pseudo-codeword of this LDPC code with girth 4, and this lower bound is tight if and only if this pseudo-codeword is a real multiple of a codeword. Using these results we further show that for some LDPC codes, there are no other minimum pseudo-codewords except the real multiples of minimum codewords. This means that the LP decoding for these LDPC codes is asymptotically optimal in the sense that the ratio of the probabilities of decoding errors of LP decoding and maximum-likelihood decoding approaches to 1 as the signal-to-noise ratio leads to infinity. Finally, some LDPC codes are listed to illustrate these results.

Index Terms: LDPC codes, linear programming (LP) decoding, fundamental cone, pseudo-codewords, pseudo-weight, stopping sets.

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I Introduction

In the study of iterative decoding of low-density parity-check (LDPC) codes, Wiberg [28], Koetter and Vontobel [12] showed that the performance of an LDPC code under iterative decoding algorithm is closely related to the pseudo-codewords, especially the pseudo-codewords with minimum pseudo-weight, of this LDPC code. Koetter and Vontobel [12] presented explanation of pseudo-codewords in iterative decoding based on graph covering and showed that the set of pseudo-codewords can be described as a so-called fundamental polytope. Recently, linear programming (LP) decoding algorithm of linear codes was introduced by Feldman, Wainwright and Karger [3][4]. The feasible region of the linear programming problem in LP decoding algorithm [3][4] agrees with the fundamental polytope. It is known that the performance of a linear code under LP decoding is also closely related to the pseudo-codewords, especially the pseudo-codewords with minimum pseudo-weight, of this linear code. In [2], Di *et al.* showed that the performance of an LDPC code under message passing decoding algorithm over a binary erasure channel (BEC) is closely related to the stopping sets in the factor graph. Since the support of any pseudo-codeword is a stopping set [12], there are some relations between the minimum pseudo-codewords and the nonempty stopping sets of smallest size [6][20][29].

Recently, pseudo-codewords and minimum pseudo-weights of binary linear codes have been studied in [1], [3], [4], [8]-[14], [21], [25]-[27], [29] and [30]. Chaichanavong and Siegel [1, Theorem 3] gave a lower bound on the pseudo-weight of a pseudo-codeword of an LDPC code. Xia and Fu [29] showed that the lower bound of Chaichanavong and Siegel on the pseudo-weight of a pseudo-codeword is tight if and only if this pseudo-codeword is a real multiple of a codeword. Using this result they further showed that for some LDPC codes, e.g., Euclidean plane and projective plane LDPC codes [15], there are no other minimum pseudo-codewords except the real multiples of minimum codewords. Recently, Kelly, Sridhara, Xu and Rosenthal [8][10] presented a lower bound on the pseudo-weight of a pseudo-codeword of an LDPC code, which includes the Chaichanavong-Siegel bound as a special case. In [6], Kashyap and Vardy gave a lower bound on the stopping distance of an LDPC code. In this correspondence, we

study the minimum pseudo-weight and minimum pseudo-codewords of LDPC codes under LP decoding. First, we show that the lower bound of Kelly, Sridhara, Xu and Rosenthal on the pseudo-weight of a pseudo-codeword of an LDPC code with girth greater than 4 is tight if and only if this pseudo-codeword is a real multiple of a codeword. Then, we show that the lower bound of Kashyap and Vardy [6] on the stopping distance of an LDPC code is also a lower bound on the pseudo-weight of a pseudo-codeword of this LDPC code with girth 4, and this lower bound is tight if and only if this pseudo-codeword is a real multiple of a codeword. Using these results we further show that for some LDPC codes, there are no other minimum pseudo-codewords except the real multiples of minimum codewords. This means that the LP decoding for these LDPC codes is asymptotically optimal in the sense that the ratio of the probabilities of decoding errors of LP decoding and maximum-likelihood (ML) decoding approaches to 1 as the signal-to-noise ratio (SNR) leads to infinity. Finally, some LDPC codes are listed to illustrate these results. The rest of this correspondence is organized as follows. In Section II, we briefly review the LP decoding of linear codes, the concepts of pseudo-codewords of a binary linear codes, the pseudo-weight of a pseudo-codeword, the minimum pseudo-weight and minimum pseudo-codewords of a binary linear codes. In Section III, the main results of this correspondence are given. Furthermore, some LDPC codes are listed to illustrate these results. In Sections IV, the proofs of the main results are given. In Section V we end with some concluding remarks.

II Preliminaries

In this section, we briefly review the LP decoding of linear codes, the concepts of pseudo-codewords of a binary linear codes, the pseudo-weight of a pseudo-codeword, the minimum pseudo-weight and minimum pseudo-codewords of a binary linear codes.

Let C be a binary $[n, k, d]$ linear code with length n , dimension k and minimum distance d . The codewords with minimum weight d are called *minimum codewords* of C . Let A_i be the number of codewords of Hamming weight i . It is well known that the minimum distance d and A_d play an important role to evaluate the performance of

the linear code C with ML decoding.

Let H be an $m \times n$ parity-check matrix of C , where the rows of H may be dependent. Let $I = \{1, 2, \dots, n\}$ and $J = \{1, 2, \dots, m\}$ denote the sets of column indices and row indices of H , respectively. The *Tanner graph* G_H corresponding to H is a bipartite graph comprising of n variable nodes labelled by elements of I , m check nodes labelled by elements of J , and the edge set $E \subseteq \{(i, j) : i \in I, j \in J\}$, where there is an edge $(i, j) \in E$ if and only if $h_{ji} = 1$. The girth g of G_H , or briefly the girth of H , is defined as the minimum length of circles in G_H . A *stopping set* S is a subset of I such that the projection of H on S does not contain a row of weight one. In other words, S is a subset of variable nodes in G_H such that all the neighbors of S are connected to S at least twice. The smallest size of a nonempty stopping set, denoted by $s(H)$, is called the *stopping distance* of C . The stopping sets with size $s(H)$ are called the *smallest stopping set*. The number of smallest stopping sets is denoted by $T_s(H)$.

Suppose a codeword \mathbf{c} is transmitted over a binary-input memoryless channel and \mathbf{y} is the output of the channel. The log-likelihood ratio is defined by $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ where $\lambda_i = \ln \frac{\Pr\{y_i | c_i=0\}}{\Pr\{y_i | c_i=1\}}$. Let $\text{conv}(C)$ be the convex hull of C in the real space \mathbb{R}^n . The ML decoding is equivalent to the optimization problem [3][4]: Finding $\mathbf{x} \in \text{conv}(C)$ to minimize $\lambda \mathbf{x}^T$. To decrease the decoding complexity, the region $\text{conv}(C)$ should be relaxed. For each row \mathbf{h}_j of H , $1 \leq j \leq m$, let

$$C_j = \{\mathbf{c} \in \{0, 1\}^n : \mathbf{h}_j \mathbf{c}^T = 0 \bmod 2\}.$$

The *fundamental polytope* of C is defined as

$$P(H) = \bigcap_{j=1}^m \text{conv}(C_j).$$

The LP decoding is to solve the optimization problem [3][4]: Finding $\mathbf{x} \in P(H)$ to minimize $\lambda \mathbf{x}^T$. Since $\text{conv}(C) \subset P(H)$, the LP decoding represents a sub-optimal decoder. The *support* of a real vector $\mathbf{x} \in \mathbb{R}^n$, denoted by $\text{supp}(\mathbf{x})$, is defined as the set of positions of nonzero coordinates in \mathbf{x} . For LP decoding of C , it is sufficient to consider the *fundamental cone* $K(H)$ of H , which is defined in [3][4] and [12] as the set

of vectors of $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ such that $x_i \geq 0$, $i = 1, \dots, n$ and

$$\forall 1 \leq j \leq m, \forall i \in \text{supp}(\mathbf{h}_j), \sum_{l \in \text{supp}(\mathbf{h}_j) \setminus \{i\}} x_l \geq x_i. \quad (1)$$

Without loss of generality, we can assume $\mathbf{0}$ is transmitted over the channel because of polytope symmetry [4]. Hence, the question of whether the LP decoder succeeds is equivalent to whether the following optimization problem has $\mathbf{0}$ as its optimal solution: Finding $\mathbf{x} \in K(H)$ to minimize $\sum_{i=1}^n x_i \lambda_i$. The elements of $K(H)$ are called *pseudo-codewords* of C . Two pseudo-codewords \mathbf{x}, \mathbf{y} are said to be equivalent if there exists a real number $\alpha > 0$ such that $\mathbf{y} = \alpha \mathbf{x}$. Clearly, $\mathbf{x} \in K(H) \Leftrightarrow \alpha \mathbf{x} \in K(H)$. Let $[\mathbf{x}] = \{\alpha \mathbf{x} : \alpha > 0\}$ denote the equivalence class corresponding to \mathbf{x} . A pseudo-codeword \mathbf{x} is said to be internal if there exist a real number β , $0 < \beta < 1$ and $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in K(H) \setminus [\mathbf{x}]$ such that $\mathbf{x} = \beta \mathbf{x}^{(1)} + (1 - \beta) \mathbf{x}^{(2)}$. If a nonzero pseudo-codeword \mathbf{x} is not internal, $[\mathbf{x}]$ is called an *edge* of $K(H)$. Let $M(H)$ denote the set of all edges of $K(H)$. The pseudo-codewords on edges in $M(H)$ are called *minimal pseudo-codewords*. It is known from [27] and linear programming theory [1][18] that the behavior of LP decoder is completely characterized by $M(H)$ and $|M(H)|$ must be finite for fixed C and H .

From now on, we only consider the additive white Gaussian noise (AWGN) channel. The (AWGN) *pseudo-weight* of a nonzero real vector $\mathbf{x} \in \mathbb{R}^n$ is defined as $w_P(\mathbf{x}) = \|\mathbf{x}\|_1^2 / \|\mathbf{x}\|_2^2$, where $\|\mathbf{x}\|_1 = |x_1| + \dots + |x_n|$ and $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + \dots + x_n^2}$. For the binary linear code C with parity-check matrix H , denote $d_P(H)$ the minimum pseudo-weight of nonzero pseudo-codewords of C . The pseudo-codewords with pseudo-weight $d_P(H)$ are called *minimum pseudo-codewords*. It is not difficult to see from linear programming theory [18] that minimum pseudo-codewords are also minimal pseudo-codewords. Note that the minimal pseudo-codewords in the same edge have the same pseudo-weight. We define the pseudo-weight of an edge $[\mathbf{x}] \in M(H)$ as the pseudo-weight of \mathbf{x} . The edges with minimum pseudo-weight are called *minimum edges*. The number of minimum edges is denoted by $B_P(H)$.

Just like the minimum Hamming weight d and A_d of a linear code for the performance of ML decoding, the minimum pseudo-weight $d_P(H)$ and $B_P(H)$ are crucial for the performance of LP decoding. In order to obtain better performance, we should try

to find a suitable parity-check matrix H to maximize $d_P(H)$ and then minimize $B_P(H)$. Since every codeword is a pseudo-codeword and the support of any pseudo-codeword is a stopping set [12], it is known that $d_P(H) \leq s(H) \leq d$ regardless of the choice of H , and $B_P(H) \geq T_s(H) \geq A_d$ for any H such that $d_P(H) = s(H) = d$. It is well known that the LP decoding is asymptotically optimal, in the sense that the ratio of the probabilities of decoding errors of LP decoding and ML decoding approaches to 1 as the SNR leads to infinity, if and only if $d_P(H) = d$ and $B_P(H) = A_d$.

III Main Results

In this section, the main results of this correspondence are given. Furthermore, some LDPC codes are listed to illustrate these results.

Let C be a binary $[n, k, d]$ linear code with parity-check matrix H . If the Tanner graph G_H has the girth $g \geq 6$ and H has the uniform column weight γ , Tanner [22] showed that the minimum distance $d \geq d_L$, where

$$d_L = \begin{cases} 1 + \gamma + \cdots + \gamma(\gamma - 1)^{\frac{g-6}{4}}, & \frac{g}{2} \text{ odd}, \\ 1 + \gamma + \cdots + \gamma(\gamma - 1)^{\frac{g-8}{4}} + (\gamma - 1)^{\frac{g-4}{4}}, & \frac{g}{2} \text{ even}. \end{cases} \quad (2)$$

Orlitsky, Urbanke, Vishwanathan and Zhang [19] further showed that the stopping distance $s(H) \geq d_L$. Recently, Kelly, Sridhara, Xu and Rosenthal [8][10] showed that the minimum pseudo-weight $d_P(H) \geq d_L$, and the bound still holds when H has the minimum column weight γ . In the next theorem which will be proved in section IV, we give the necessary and sufficient condition of $w_P(\mathbf{x}) = d_L$ for a nonzero pseudo-codeword $\mathbf{x} \in K(H)$.

Theorem 1 *Let C be a binary linear code with length n . Let H be a parity-check matrix of C with girth $g \geq 6$ and minimum column weight γ . Then $w_P(\mathbf{x}) \geq d_L$ for any nonzero pseudo-codeword $\mathbf{x} = (x_1, \dots, x_n) \in K(H)$, where d_L is defined in (2). Moreover, the equality holds if and only if $d_L \mathbf{x} / \sum_{l=1}^n x_l$ is a codeword of C with Hamming weight d_L .*

It is easy to check that

$$d_L = \begin{cases} \beta^{(g-2)/4} + 2 \left(\frac{\beta^{(g-2)/4} - 1}{\beta - 1} \right), & g/2 \text{ odd}, \\ 2 \left(\frac{\beta^{g/4} - 1}{\beta - 1} \right), & g/2 \text{ even}, \end{cases} \quad (3)$$

where $\beta = \gamma - 1$. In particular,

$$d_L = \begin{cases} \beta + 2, & g = 6, \\ 2(\beta + 1), & g = 8, \\ 2(\beta^2 + \beta + 1), & g = 12, \\ 2(\beta^3 + \beta^2 + \beta + 1), & g = 16, \end{cases} \quad (4)$$

where $\beta = \gamma - 1$. The next theorem can be easily obtained from Theorem 1.

Theorem 2 *Let C be a binary $[n, k, d]$ linear code with length n , dimension k and minimum distance d . Let H be a parity-check matrix of C with girth g ($g \geq 6$) and minimum column weight γ . If $d = d_L$, where d_L is defined in (2), then $d_P(H) = s(H) = d$ and $B_P(H) = T_s(H) = A_d$, where $d_P(H)$ is the minimum pseudo-weight, $s(H)$ is the stopping distance, A_d is the number of minimum codewords, $T_s(H)$ is the number of smallest stopping sets, and $B_P(H)$ is the number of minimum edges.*

Proof: It is known that the support of a codeword is a stopping set [3][6][20] and every stopping set S corresponds to one pseudo-codeword with pseudo-weight $|S|$ [12]. Hence, $d \geq s(H) \geq d_P(H)$. It follows from Theorem 1 that $d_P(H) \geq d_L$. Hence, $d_P(H) = s(H) = d$ if $d = d_L$. Furthermore, by Theorem 1, $B_P(H) = T_s(H) = A_d$. \square

Remark 1 *For any binary linear code C satisfying the conditions of Theorem 2, the minimum codewords, the nonempty stopping sets of smallest size and the minimum edges are all equivalent, which implies that the LP decoding is asymptotically optimal for C .*

Note that [29, Theorems 1 and 2] are the special cases of $g = 6$ of Theorems 1 and 2, respectively. In [29], it is shown that two classes of finite geometry LDPC codes, i.e., the projective plane LDPC codes and Euclidean plane LDPC codes [15], meet the conditions of Theorem 2. Thus, the LP decoding is asymptotically optimal for the finite plane LDPC codes. Below we give some examples of LDPC codes satisfying the conditions of Theorem 2.

Example 1 A class of regular LDPC codes called $LU(3, q)$ codes were constructed in [7], where q is a prime power. $LU(3, q)$ codes have the following parameters, where n is the code length, d is the minimum distance, m is the number of rows of the parity-check matrix, ρ is the uniform row weight of the parity-check matrix, γ is the uniform column weight of the parity-check matrix, and g is the girth of the Tanner graph.

$$n = q^3, \quad m = q^3, \quad d = 2q, \quad \rho = q, \quad \gamma = q, \quad g = 8.$$

This class of LDPC codes meet the conditions of Theorem 2. Thus the LP decoding is asymptotically optimal for $LU(3, q)$ codes.

Example 2 In [17], regular LDPC codes were constructed from generalized polygons. In [17, Table 1], for a prime power q , LDPC codes $W(q)$, $H(3, q^2)$, $H(q)$, $T(q^3, q)$ and $\bar{O}(q)$ have the following parameters, where n is the code length, d is the minimum distance, γ is the uniform column weight of the parity-check matrix, and g is the girth of the Tanner graph.

- (i) $W(q)$: $n = (q + 1)(q^2 + 1)$, $d = 2(q + 1)$, $\gamma = q + 1$, $g = 8$;
- (ii) $H(3, q^2)$: $n = (q^2 + 1)(q^3 + 1)$, $d = 2(q + 1)$, $\gamma = q + 1$, $g = 8$;
- (iii) $H(q)$: $n = (q + 1)(q^4 + q^2 + 1)$, $d = 2(q^2 + q + 1)$, $\gamma = q + 1$, $g = 12$;
- (iv) $T(q^3, q)$: $n = (q^3 + 1)(q^8 + q^4 + 1)$, $d = 2(q^2 + q + 1)$, $\gamma = q + 1$, $g = 12$;
- (v) $\bar{O}(q)$, $q = 2^{2e+1}$: $n = (q^2 + 1)(q^3 + 1)(q^6 + 1)$, $d = 2(q^3 + q^2 + q + 1)$, $\gamma = q + 1$, $g = 16$.

By (4), it is obvious that these LDPC codes meet the conditions of Theorem 2. Thus the LP decoding is asymptotically optimal for them.

Example 3 In [21], some LDPC codes with $d_P(H) = d$ were constructed by enumerating a regular tree for a fixed number l of layers and employing a connection algorithm based on mutually orthogonal Latin squares to close the tree.

- (i) *Type-I* A construction [21]: It is known that if l or $g/2$ is odd, then $d = d_L$. Hence,

these LDPC codes with odd $g/2$ meet the conditions of Theorem 2 and the LP decoding is asymptotically optimal for them.

(ii) *Type-II* construction [21]: For the binary case and $l = 3$, the Type II construction yields exactly the projective plane LDPC code [23][29]. For the binary case and $l = 4$, it is conjectured that $d = d_L$ in [21]. Clearly, if this conjectured is true, then these LDPC codes meet the conditions of Theorem 2 and the LP decoding is asymptotically optimal for them. In particular, it is known from [21] and [24] that this is true for the $(2, 2)$ -Finite-Generalized-Quadrangles-based LDPC codes.

The next theorem shows that the lower bound of Kashyap and Vardy [6] on the stopping distance of an LDPC code is also a lower bound on the pseudo-weight of a pseudo-codeword of this LDPC code, and this lower bound is tight if and only if this pseudo-codeword is a real multiple of a codeword. The proof of this theorem will be given in section IV.

Theorem 3 *Let C be a binary linear code with length n . Let H be an $m \times n$ parity-check matrix of C with minimum column weight γ . If any two distinct columns of H have at most λ common 1's and γ/λ is an integer, then $w_P(\mathbf{x}) \geq \frac{\gamma}{\lambda} + 1$ for any nonzero pseudo-codeword $\mathbf{x} = (x_1, \dots, x_n) \in K(H)$. Moreover, the equality holds if and only if $(\frac{\gamma}{\lambda} + 1)\mathbf{x} / \sum_{i=1}^n x_i$ is a codeword of C with Hamming weight $\frac{\gamma}{\lambda} + 1$.*

Remark 2 *If H has uniform column weight γ , Kashyap and Vardy [6, Theorem 1] showed that the stopping distance $s(H) \geq \frac{\gamma}{\lambda} + 1$. Since $d_P(H) \leq s(H)$, Theorem 3 implies the Kashyap-Vardy lower bound on the stopping distance.*

The next theorem can be easily obtained from Theorem 3.

Theorem 4 *Let C be a binary $[n, k, d]$ linear code with length n , dimension k and minimum distance d . Let H be a parity-check matrix of C with minimum column weight γ . If any two distinct columns of H have at most λ common 1's and γ/λ is an integer, and if $d = \frac{\gamma}{\lambda} + 1$, then $d_P(H) = s(H) = d$ and $B_P(H) = T_s(H) = A_d$, where $d_P(H)$ is the minimum pseudo-weight, $s(H)$ is the stopping distance, A_d is the number of minimum codewords, $T_s(H)$ is the number of smallest stopping sets, and $B_P(H)$ is the number of minimum edges.*

Proof: It is known that $d \geq s(H) \geq d_P(H)$. By Theorem 3, $d_P(H) \geq \frac{\gamma}{\lambda} + 1$. Hence, $d_P(H) = s(H) = d$ if $d = \frac{\gamma}{\lambda} + 1$. Furthermore, by Theorem 3, $B_P(H) = T_s(H) = A_d$. \square

Remark 3 In Theorems 3 and 4, the girth of the Tanner graph G_H is at least 6 if $\lambda = 1$ and 4 if $\lambda > 1$. The special cases of $\lambda = 1$ of Theorems 3 and 4 are exactly [29, Theorems 1 and 2] and the special cases of $g = 6$ of Theorems 1 and 2.

Example 4 Consider the binary $[2^r - 1, 2^r - r - 1, 3]$ Hamming code. Let H be a parity-check matrix which consists of all the non-zero codewords of the binary $[2^r - 1, r, 2^{r-1}]$ simplex code. It is easy to see that $\gamma = 2^{r-1}$ and $\lambda = 2^{r-2}$. Hence, by Theorems 3 and 4, $d_P(H) = 3$ and any minimum pseudo-codeword must be the real multiple of some minimum codeword.

Let $q = 2^s$ and $EG(m, q)$ be the m -dimensional Euclidean Geometry over $GF(q)$. It is known from [16] and [23] that there are q^m points and $q(q^m - 1)/(q - 1)$ hyperplanes in $EG(m, q)$. By removing a point of $EG(m, q)$ together with the $(q^m - 1)/(q - 1)$ hyperplanes containing this point, we obtain a slightly modified incidence matrix H of points and hyperplanes in $EG(m, q)$. Suppose the rows of H indicate the hyperplanes. The point-hyperplane Euclidean geometry LDPC code C with the parity-check matrix H has the following parameters: length $n = q^m - 1$, uniform column weight of H $\gamma = q^{m-1}$, uniform row weight $\rho = q^{m-1}$, girth of Tanner graph $g = 4$ if $m > 2$. It is easy to see that any two distinct columns of H have at most $\lambda = q^{m-2}$ common 1's. By Theorem 3, we have that $d \geq s(H) \geq d_P \geq \gamma/\lambda + 1 = q + 1$. From [16] and [23], we know that H can be put in cyclic form and the generator polynomial $g(x)$ can be determined. By [16, p. 315, (8.33)], it is known that the dimension $k = 2^{sm} - (m + 1)^s$. The following examples show that $d = q + 1$ in some cases.

Example 5 Let $m = 3$ and $s = 2$. Then C is a binary $[63, 48]$ code with generator polynomial $g(x) = 1 + x^2 + x^4 + x^{11} + x^{13} + x^{14} + x^{15}$ [16, pp. 310-311]. By Theorem 3, we know that $d \geq s(H) \geq d_P \geq q + 1 = 5$. In fact, it is easy to calculate by computer that d does equal 5. For example, $1 + x^{23} + x^{33} + x^{36} + x^{37}$ is a weight 5 codeword.

Hence, by Theorem 4, we have that $d = s(H) = d_P$ and $A_d = T_s(H) = B_P(H)$. Hence, the LP decoding for C is asymptotically optimal.

Example 6 Let $m = 3$ and $s = 3$. Then C is a binary $[511, 448]$ code [23, Example 1]. The girth of the Tanner graph is 4 and C performs very good under iterative decoding [23]. By Theorem 3, we know that $d \geq s(H) \geq d_P \geq q + 1 = 9$. In fact, it can be calculated by the method in [5] that $d = 9$. Hence, by Theorem 4, we have that $d = s(H) = d_P$ and $A_d = T_s(H) = B_P(H)$. Hence, the LP decoding for C is asymptotically optimal.

IV Proofs of Theorems 1 and 3

In this section, we prove Theorems 1 and 3. Chaichanavong and Siegel [1, Proposition 2] gave a lower bound on the pseudo-weight of a real vector. In [29], the necessary and sufficient condition for this bound being tight is discussed. Let u be a positive integer. Denote \mathcal{F}_u the set of vectors $\mathbf{y} \in [0, 1/u]^n$ such that $\sum_{i=1}^n y_i = 1$.

Lemma 1 [29] For any $\mathbf{y} \in \mathcal{F}_u$ we have $w_P(\mathbf{y}) \geq u$. The equality holds if and only if \mathbf{y} has exactly u nonzero components with value $1/u$.

A. Proof of Theorem 1

Kelly, Sridhara, Xu and Rosenthal [8][10] showed that $d_P(H) \geq d_L$. From the proof of [10, Theorem 3.1], we know that for any nonzero pseudo-codeword $\mathbf{x} = (x_1, \dots, x_n) \in K(H)$,

$$d_L x_i \leq \sum_{j=1}^n x_j, \quad \text{for any } i \in \text{supp}(\mathbf{x}). \quad (5)$$

Let $\mathbf{y} = \mathbf{x} / \sum_{i=1}^n x_i$. Since $\mathbf{x} \in K(H)$, then $\mathbf{y} \in K(H)$ and $\mathbf{y} \in \mathcal{F}_{d_L}$ by (5). Hence, by Lemma 1, $w_P(\mathbf{x}) = w_P(\mathbf{y}) \geq d_L$. Furthermore, $w_P(\mathbf{x}) = d_L$ if and only if $\mathbf{c} = d_L \mathbf{x} / \sum_{l=1}^n x_l$ is a binary vector with Hamming weight d_L . Now, we show that $w_P(\mathbf{x}) = d_L$ if and only if \mathbf{c} is a codeword with Hamming weight d_L of C . If \mathbf{c} is a codeword with Hamming weight d_L of C , then $w_P(\mathbf{x}) = w_P(\mathbf{c}) = w_H(\mathbf{c}) = d_L$. On the other hand, if

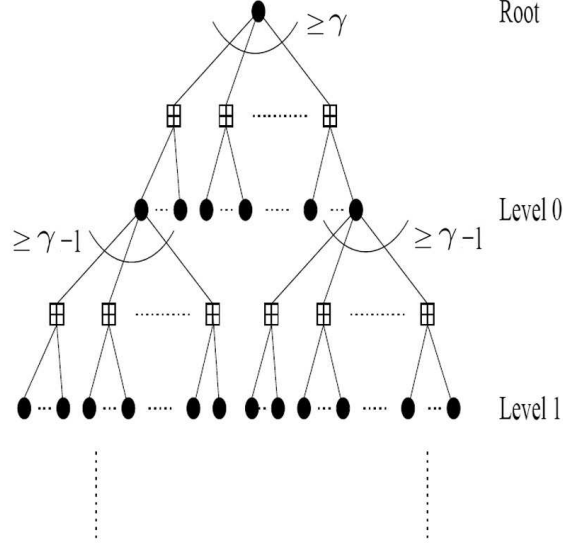


Figure 1: Local tree structure for a $\geq \gamma$ -left graph

$w_P(\mathbf{x}) = d_L$, then the pseudo-codeword \mathbf{c} is a binary vector with Hamming weight d_L . Next, we show that \mathbf{c} must be a codeword of C .

Clearly, $S = \text{supp}(\mathbf{c})$ is a stopping set with size d_L since \mathbf{c} is a pseudo-codeword. For any fixed $i \in S$, we construct a local tree of i (see Figure 1) as in the proof of [10, Theorem 3.1]. For the sake of convenience, we briefly describe the construction procedure as follows. Below we use f, e to denote check nodes and i, j, l to denote variable nodes of the Tanner graph G_H . In this local tree of i in the Tanner graph G_H , i is the root of this local tree. A check node f which is connected to i by an edge, is called a son of i , and a variable node j which is connected to f by an edge, except its parent i , is called a son of f or a grandson of i , and a check node e which is connected to j by an edge, except its parent f , is called a son of j , and so on. Let $\text{child}(i)$ denote the set of all children of i . Since there are at least γ 1's in the i -th column of H ,

$$|\text{child}(i)| \geq \gamma. \quad (6)$$

All grandchildren of i , denoted by $L_0(i)$, are called *Level 0* variable nodes, i.e.,

$$L_0(i) = \bigcup_{f \in \text{child}(i)} \text{child}(f). \quad (7)$$

Now we fix a check node $f^* \in \text{child}(i)$. Denote

$$N_0(f^*) = \text{child}(f^*) \subseteq L_0(i). \quad (8)$$

For each Level 0 variable node $j \in L_0(i)$, it is obvious that

$$|\text{child}(j)| \geq \gamma - 1. \quad (9)$$

For any variable node j in this local tree of i , let

$$M(j) = \bigcup_{f \in \text{child}(j)} \text{child}(f) \quad (10)$$

be the set of all grandchildren of j . The union of $M(j)$ where $j \in L_0(i)$ form the set of *Level 1* variable nodes, denoted by $L_1(i)$. In other words,

$$L_1(i) = \bigcup_{j \in L_0(i)} M(j). \quad (11)$$

Define

$$N_1(f^*) = \bigcup_{j \in N_0(f^*)} M(j) \subseteq L_1(i) \quad (12)$$

where f^* is the same fixed check node as in (8). Repeating the above steps, we obtain the local tree of i . Let $t = \lfloor (g - 6)/4 \rfloor \geq 0$ be the largest integer not greater than $(g - 6)/4$. Then $g = 4t + 6$ for odd $g/2$ and $g = 4t + 8$ for even $g/2$. For $1 \leq m \leq t$, denote

$$L_m(i) = \bigcup_{j \in L_{m-1}(i)} M(j), \quad (13)$$

$$N_m(f^*) = \bigcup_{j \in N_{m-1}(f^*)} M(j) \subseteq L_m(i), \quad (14)$$

and

$$N_{t+1}(f^*) = \bigcup_{j \in N_t(f^*)} M(j), \quad \text{for even } g/2. \quad (15)$$

Note that for $1 \leq m \leq t + 1$,

$$|\text{child}(j)| \geq \gamma - 1, \quad \forall j \in L_{m-1}(i). \quad (16)$$

The local tree of i satisfies the following *pairwise disjoint properties* by the definition of girth.

- (i) Since $g \geq 6$, all $child(f)$ in the union of (10) are pairwise disjoint; for $1 \leq m \leq t$, all $M(j)$ in the union of (13) are pairwise disjoint; $\{i\}, L_0(i), L_1(i), \dots, L_t(i)$ are pairwise disjoint.

These conclusions can be verified as follows. Let $j_1 \in L_{m_1}(i)$ and $j_2 \in L_{m_2}(i)$, where $0 \leq m_1 \leq m_2 \leq t$. From the local tree, it is easy to see that the length of the path from i to j_1 is exactly $2m_1 + 2$ and the length of the path from i to j_2 is exactly $2m_2 + 2$. Hence, there is a path from j_1 to j_2 with length at most $2(m_1 + m_2) + 4 \leq 4t + 4$. Since $g \geq 4t + 6$, there is no closed path with length at most $4t + 4$, which implies the conclusions.

- (ii) If $g/2$ is even, i.e., $g = 4t + 8$, we have the further properties: all $M(j)$ in the union of (15) are pairwise disjoint; $\{i\}, L_0(i), L_1(i), \dots, L_t(i), N_{t+1}(f^*)$ are pairwise disjoint.

These conclusions can be verified as follows. Let $j_1, j_2 \in N_{t+1}(f^*)$. Then the length of the path from j_1 to f^* is exactly $2t + 3$ and the length of the path from j_2 to f^* is exactly $2t + 3$. Hence, there is a path from j_1 to j_2 with length at most $4t + 6$. On the other hand, let $j_3 \in L_m(i)$, where $0 \leq m \leq t$. Then the length of the path from j_3 through i to f^* is at most $2m + 3 \leq 2t + 3$. Hence, there is a path from j_1 to j_3 with length at most $4t + 6$. Since $g \geq 4t + 8$, there is no closed path with length at most $4t + 6$, which implies the conclusions.

For each check node $f \in child(i)$, $|child(f) \cap S| \geq 1$ since S is a stopping set including i . Hence, by (6), (8) and the pairwise disjoint property (i), and noting that $g \geq 6$, we have that

$$|L_0(i) \cap S| \geq |child(i)| \geq \gamma, \quad (17)$$

$$|N_0(f^*) \cap S| \geq 1, \quad (18)$$

where a necessary condition of $|L_0(i) \cap S| = \gamma$ is that for each check node $f \in child(i)$, $|child(f) \cap S| = 1$. In other words, for any row \mathbf{h} of H whose i -th component is 1, $w_H(\mathbf{h}_S) = 2$ where \mathbf{h}_S is the projection of \mathbf{h} on S .

For any $1 \leq m \leq t$ and $j \in L_{m-1}(i) \cap S$, we have that $|child(f) \cap S| \geq 1$ for each check node $f \in child(j)$ since S is a stopping set including j . Hence, by (10), (13),

(14), (16), and the pairwise disjoint property (i), and noting that $g \geq 4t + 6$, we have

$$\begin{aligned}
& |L_m(i) \cap S| \\
&= \left| \bigcup_{j \in L_{m-1}(i)} \bigcup_{f \in \text{child}(j)} \text{child}(f) \cap S \right| \\
&\geq \sum_{j \in L_{m-1}(i) \cap S} \sum_{f \in \text{child}(j)} 1 \\
&\geq (\gamma - 1) |L_{m-1}(i) \cap S|
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
& |N_m(f^*) \cap S| \\
&= \left| \bigcup_{j \in N_{m-1}(f^*)} \bigcup_{f \in \text{child}(j)} \text{child}(f) \cap S \right| \\
&\geq \sum_{j \in N_{m-1}(f^*) \cap S} \sum_{f \in \text{child}(j)} 1 \\
&\geq (\gamma - 1) |N_{m-1}(f^*) \cap S|.
\end{aligned} \tag{20}$$

If $g = 4t + 8$, by the pairwise disjoint property (ii), we still have

$$\begin{aligned}
& |N_{t+1}(f^*) \cap S| \\
&= \left| \bigcup_{j \in N_t(f^*)} \bigcup_{f \in \text{child}(j)} \text{child}(f) \cap S \right| \\
&\geq \sum_{j \in N_t(f^*) \cap S} \sum_{f \in \text{child}(j)} 1 \\
&\geq (\gamma - 1) |N_t(f^*) \cap S|.
\end{aligned} \tag{21}$$

Thus, by (17) and (19), for any $1 \leq m \leq t$,

$$\begin{aligned}
|L_m(i) \cap S| &\geq (\gamma - 1) |L_{m-1}(i) \cap S| \\
&\geq \cdots \geq (\gamma - 1)^m |L_0(i) \cap S| \geq (\gamma - 1)^m \gamma
\end{aligned} \tag{22}$$

and by (18), (20) and (21), if $g = 4t + 8$, then

$$\begin{aligned}
|N_{t+1}(f^*) \cap S| &\geq (\gamma - 1) |N_t(f^*) \cap S| \\
&\geq \cdots \geq (\gamma - 1)^{t+1} |N_0(f^*) \cap S| \geq (\gamma - 1)^{t+1}.
\end{aligned} \tag{23}$$

Hence, by (22) and (23), we have

$$\begin{aligned}
|\{i\} \cap S| &= 1, \\
|L_0(i) \cap S| &\geq \gamma, \\
|L_1(i) \cap S| &\geq \gamma(\gamma - 1), \\
&\vdots \quad \vdots \quad \vdots \\
|L_t(i) \cap S| &\geq \gamma(\gamma - 1)^t, \\
|N_{t+1}(f^*) \cap S| &\geq (\gamma - 1)^{t+1}, \quad \text{for even } g/2.
\end{aligned}$$

Note that $t = \lfloor (g - 6)/4 \rfloor$. Therefore $|S| \geq d_L$ by adding the above inequalities and using the pairwise disjoint properties, where a necessary condition of $|S| = d_L$ is that $|L_0(i) \cap S| = \gamma$, that is, $w_H(\mathbf{h}_S) = 2$ for any row \mathbf{h} of H whose i -th component is 1. This implies that \mathbf{c} satisfies all the parity-check equations corresponding to the rows \mathbf{h} of H whose i -th component is 1. Thus, when i varies in $S = \text{supp}(\mathbf{c})$, \mathbf{c} must satisfy every parity-check equation in H , i.e., \mathbf{c} is a codeword.

B. Proof of Theorem 3

Let $\mathbf{y} = \mathbf{x} / \sum_{j=1}^n x_j$. Since $\mathbf{x} \in K(H)$, then $\mathbf{y} \in K(H)$. For fixed j , $1 \leq j \leq n$, let $\mathbf{h}_{q_1}, \mathbf{h}_{q_2}, \dots, \mathbf{h}_{q_\gamma}$ be the rows of H whose j -th components are 1, i.e., $\mathbf{h}_{q_i} = (h_{q_i,1}, \dots, h_{q_i,n})$ and $h_{q_i,j} = 1$ for $1 \leq i \leq \gamma$. Since $\mathbf{y} \in K(H)$,

$$y_j \leq \sum_{l \neq j} y_l h_{q_i,l}, \quad i = 1, \dots, \gamma.$$

Hence,

$$\gamma y_j \leq \sum_{i=1}^{\gamma} \sum_{l \neq j} y_l h_{q_i,l} = \sum_{l \neq j} y_l \left(\sum_{i=1}^{\gamma} h_{q_i,l} \right).$$

For any $l \neq j$, since the l -th column and j -th column of H have at most λ common 1's and $h_{q_i,j} = 1$ for $1 \leq i \leq \gamma$, we have that

$$\sum_{i=1}^{\gamma} h_{q_i,l} \leq \lambda.$$

Therefore,

$$\gamma y_j \leq \lambda \sum_{l \neq j} y_l = \lambda(1 - y_j), \quad \text{i.e.,} \quad y_j \leq \frac{\lambda}{\gamma + \lambda},$$

which implies that $\mathbf{y} \in \mathcal{F}_{\frac{\gamma}{\lambda}+1}$. Hence, by Lemma 1, $w_P(\mathbf{x}) = w_P(\mathbf{y}) \geq \frac{\gamma}{\lambda} + 1$, and the equality holds if and only if \mathbf{y} has exactly $\frac{\gamma}{\lambda} + 1$ nonzero components with value $\lambda/(\gamma + \lambda)$. This implies that $(\frac{\gamma}{\lambda} + 1)\mathbf{y} = (\frac{\gamma}{\lambda} + 1)\mathbf{x}/\sum_{j=1}^n x_j$ must be a binary vector, say \mathbf{c} , with Hamming weight $\frac{\gamma}{\lambda} + 1$.

Now, we show that \mathbf{c} must be a codeword of C . Since $\mathbf{c} \in K(H)$, $\text{supp}(\mathbf{c})$ is a stopping set of H . In other words, the projection of H onto $\text{supp}(\mathbf{c})$ is an $m \times (\frac{\gamma}{\lambda} + 1)$ matrix, denoted by $H(\mathbf{c})$, which has no rows of weight one. Note that any two distinct columns of $H(\mathbf{c})$ have at most λ common 1's. Suppose \mathbf{b} is a nonzero row of $H(\mathbf{c})$ and the j -th component of \mathbf{b} is 1 where $1 \leq j \leq \frac{\gamma}{\lambda} + 1$. Since the j -th column of $H(\mathbf{c})$ has at least γ 1's, there exists a $\gamma \times (\frac{\gamma}{\lambda} + 1)$ matrix, say $H(\mathbf{c}, j)$, consisting of \mathbf{b} and other $\gamma - 1$ rows of $H(\mathbf{c})$ such that the j -th column of $H(\mathbf{c}, j)$ is the all 1 column. Since $H(\mathbf{c}, j)$ is a submatrix of $H(\mathbf{c})$, any two distinct columns of $H(\mathbf{c}, j)$ have at most λ common 1's. Note that the j -th column of $H(\mathbf{c}, j)$ is the all 1 column, then each of the other columns has at most λ 1's. Now we count the number of 1's in $H(\mathbf{c}, j)$ in two ways. From the view of columns, the total number of 1's in $H(\mathbf{c}, j)$ is at most $\gamma + \lambda \frac{\gamma}{\lambda} = 2\gamma$. On the other hand, since $\text{supp}(\mathbf{c})$ is a stopping set, each row of $H(\mathbf{c}, j)$ has at least two 1's. Thus the total number of 1's in $H(\mathbf{c}, j)$ is at least 2γ . Hence, the total number of 1's in $H(\mathbf{c}, j)$ is exactly 2γ , and every row of $H(\mathbf{c}, j)$ has exactly two 1's. So the Hamming weight of \mathbf{b} is 2. Hence, the Hamming weights of rows of $H(\mathbf{c})$ are either 0 or 2, which implies that \mathbf{c} satisfies every parity-check equation in H and thus is a codeword.

V Conclusions

In this correspondence, we study the minimum pseudo-weight and minimum pseudo-codewords of low-density parity-check (LDPC) codes. We characterize the pseudo-codewords of an LDPC code which attain the lower bound d_L of Kelly, Sridhara, Xu and Rosenthal on the minimum pseudo-weight. That is, the pseudo-weight of a pseudo-

codeword of this LDPC code is equal to d_L if and only if this pseudo-codeword is a real multiple of a codeword with Hamming weight d_L . Furthermore, it is shown that if the minimum distance of this LDPC code is equal to d_L , then the minimum codewords, the nonempty stopping sets of smallest size and the minimum edges are all equivalent, which implies that the LP decoding is asymptotically optimal for this LDPC code. Then, we show that the lower bound of Kashyap and Vardy on the stopping distance of an LDPC code is also a lower bound on the pseudo-weight of a pseudo-codeword of this LDPC code with girth 4. The same characterization results mentioned above for the lower bound of Kelly, Sridhara, Xu and Rosenthal are also obtained for this new lower bound on the minimum pseudo-weight. Moreover, some LDPC codes are listed to illustrate these results. Finally, we pose a further research problem: For a binary LDPC code C , constructing a parity-check matrix H with minimum number of rows such that the minimum pseudo-weight of C is equal to the minimum distance of C , and the number of minimum edges is equal to the number of minimum codewords of C , i.e., the LP decoding is asymptotically optimal for this LDPC code. Until now, we even do not know whether such a parity-check matrix does exist for every binary linear code.

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