

EVIDENCE THAT $P \neq NP$

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Abstract: The question of whether the class of decision problems that can be solved by deterministic polynomial-time algorithms, P , is equal to the class of decision problems that can be solved by nondeterministic polynomial-time algorithms, NP , has been open since it was first formulated by Cook, Karp, and Levin in 1971. In this paper, we give evidence that they are not equal by examining the SUBSET-SUM problem.

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Let P be the class of decision problems that can be solved by deterministic polynomial-time algorithms and NP be the class of decision problems that can be solved by nondeterministic polynomial-time algorithms. It has been an open question since the early 1970's whether or not $P = NP$. In this note, we give evidence that they are not equal. The argument in this note is put forward in a purely intuitive spirit and should not be interpreted as a rigorous proof; however, the author is by no means claiming that a rigorous version of the argument presented here is impossible. And the author welcomes and challenges anyone to produce a rigorous version. Let us start out by considering the following commonly known NP problem:

SUBSET-SUM: *Given $n \in \mathbb{N}$, vector $\mathbf{a} \in \mathbb{Z}^n$, and scalar $b \in \mathbb{Z}$ (each represented in binary), determine whether there exists a vector, $\mathbf{x} \in \{0, 1\}^n$, such that $\mathbf{a} \cdot \mathbf{x} = b$.*

Let $S = \{\mathbf{a} \cdot \mathbf{x} : \mathbf{x} \in \{0, 1\}^n\}$, so the SUBSET-SUM problem is to determine whether $b \in S$. And consider the following algorithm, which we shall call algorithm \mathcal{A} , found in a paper by G.J. Woeginger (2003) and brought to the attention of the author by R.B. Lyngsoe, which runs in $O(2^{\frac{n}{2}})$ time (assuming that the algorithm can perform arithmetic in constant-time and that the algorithm can sort in linear-time) and can be described as follows:

Sort sets $b - S^- = b - \{\mathbf{a} \cdot \mathbf{x} \in S : x_1 = \dots = x_{\lceil \frac{n}{2} \rceil} = 0\}$ and $S^+ = \{\mathbf{a} \cdot \mathbf{x} \in S : x_{\lceil \frac{n}{2} \rceil + 1} = \dots = x_n = 0\}$ in ascending order. Compare the first two elements in each of the lists. If there is a match, then stop and output that there is a solution. If not, then compare the greater element with the next element on the other list. Continue this process until there is a match, in which case there is a solution, or until one of the lists runs out of elements, in which case there is no solution.

We now state and argue two propositions:

Proposition 1: *Algorithm \mathcal{A} has the best running-time (with respect to $n \geq N$ for large N) of all algorithms which solve SUBSET-SUM, assuming that the algorithms can perform arithmetic in constant-time and that the algorithms can sort in linear-time.*

Argument: We use induction on n : We shall leave it to the reader to find a large enough N to verify the basis step. Let us assume true for n and prove true for $n + 1$: Any algorithm that solves SUBSET-SUM, given input $([\mathbf{a} : a_{n+1}], b)$, equivalently solves two subproblems by determining whether $b - a_{n+1} \in S$ or $b \in S$ (where S is defined as above for problems of size n). Now, if these two subproblems were

completely unrelated to one another, then the fastest algorithm that solves SUBSET-SUM for problems of size $n + 1$ would solve both subproblems individually using algorithm \mathcal{A} , by the induction hypothesis; however, the two subproblems are in fact related to one another in that they both involve set S , so information obtained from solving any one of the subproblems may be used to solve the other subproblem - Notice that if solving one of the subproblems takes 6 units of time, 3 units to sort lists S^- and S^+ and 3 units to go through the lists, then it is possible to solve the other subproblem in only 3 units of time, since the lists are already sorted from solving the first subproblem. Such a procedure takes a total of 9 units of time, instead of the 12 units of time that it would take to solve both subproblems individually with algorithm \mathcal{A} .

And it is possible to make even more use of information obtained from solving any one of the subproblems to save time solving the other subproblem: If n is odd and the algorithm sorts sets $S^- \cup (S^- + a_{n+1})$ and S^+ instead of sorting sets S^- and S^+ , then the algorithm will take 4 units of time to sort the lists and 4 units of time to go through the lists, a total of 8 units of time instead of the 12 units of time that the algorithm would take to solve both subproblems individually with algorithm \mathcal{A} . And when n is even, the running-time of such an improved strategy will not differ from that of the first intelligent strategy that we mentioned, which takes 9 units of time. When this improved strategy is implemented, the algorithm not only solves each subproblem in the fastest way possible by the induction hypothesis, but also the algorithm makes use of any information that is obtained from solving any one of the subproblems to save time solving the other subproblem whenever it is possible to do such. Therefore, such an algorithm will, for problems of size $n + 1$, have the best running-time of all algorithms which solve SUBSET-SUM. Since this procedure is, in fact, descriptive of how algorithm \mathcal{A} works on problems of size $n + 1$, we have evidence that algorithm \mathcal{A} has the best running-time of all algorithms which solve SUBSET-SUM. \square

Proposition 2: *Algorithm \mathcal{A} has the best asymptotic running-time (with respect to n) of all algorithms that solve SUBSET-SUM when the input is restricted so that each $|a_i| < 2^n$ and $|b| < n \cdot 2^n$.*

Argument: Since set S can still be of size 2^n when the input is restricted so that each $|a_i| < 2^n$ and $|b| < n \cdot 2^n$, roughly the same arguments in Proposition 1 which show that \mathcal{A} is the best algorithm to solve the unrestricted SUBSET-SUM problem apply also to the restricted

SUBSET-SUM problem where each $|a_i| < 2^n$ and $|b| < n \cdot 2^n$. So since SUBSET-SUM is in NP and \mathcal{A} runs in super-polynomial time with respect to n for input of polynomial size $O(n^2)$ (when expressed in binary), we have evidence that $P \neq NP$. \square

Acknowledgements: I would like to thank G-d, my wife Aliza, my parents, my grandparents, and all my true friends for giving me the inspiration to write this paper.

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