

The Computational Complexity of $3k$ -CLIQUE

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Abstract: In this note, we show that the fastest deterministic and exact algorithm that solves the $3k$ -CLIQUE problem must run in $\Omega(n^{2k})$ time in the worst-case scenario on a classical computer, where n is the number of vertices in the graph.

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The $3k$ -CLIQUE problem is to determine whether or not a clique of size $3k$ exists in a given undirected graph G , where k is a positive integer that is not part of the input of the problem [4]. In this note, we show that the fastest deterministic and exact algorithm that solves $3k$ -CLIQUE must run in $\Omega(n^{2k})$ time in the worst-case scenario on a classical computer, where n is the number of vertices in the graph:

Let G be an undirected graph with n vertices. For every k -clique C in G , create a corresponding vertex $v(C)$ in an auxiliary graph G' . And for every two vertices $v(C_1)$ and $v(C_2)$ in G' , create an edge connecting them in G' if and only if $C_1 \cup C_2$ forms a $2k$ -clique in G . Then G' will have $O(n^k)$ vertices and $O(n^{2k})$ edges. Note that the 3 -CLIQUE problem on G' is equivalent to the $3k$ -CLIQUE problem on G [4].

Let A be the adjacency matrix of G' . Then G' has a 3 -clique if and only if $A \circ A^2$, the Hadamard product of A and A^2 , is nonzero [2]. In general, it is impossible for an algorithm to evaluate more than one entry of $A \circ A^2$ at a time, because the formula $(A \circ A^2)_{ij} = \sum_k a_{ij} \cdot a_{ik} \cdot a_{kj}$ is different for each i, j ; therefore, in general it is impossible for an algorithm to rule out more than one entry of $A \circ A^2$ at a time as being nonzero. Hence, the fastest algorithm that determines whether $A \circ A^2$ is nonzero can do no better in the worst-case scenario, when $\Theta(n^{2k})$ entries of A are nonzero but only a constant number of entries of $A \circ A^2$ are nonzero, than to evaluate each entry of $A \circ A^2$ individually until either a nonzero entry is found or it is certain that there are no nonzero entries, which could take $\Omega(n^{2k})$ time.

Then since determining whether $A \circ A^2$ is nonzero is equivalent to the $3k$ -CLIQUE problem on G , it must also take $\Omega(n^{2k})$ time in the worst-case scenario for any deterministic and exact algorithm to solve the $3k$ -CLIQUE

problem on G . And this implies that $P \neq NP$ [1]. \square

This lower bound is confirmed by the fact that the fastest known deterministic and exact algorithm that solves $3k$ -CLIQUE was first published in 1985 and has a running-time of $\Theta(n^{\omega_k})$, where $\omega_k \geq 2$ [1, 3, 4].

References

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