

Tromino Tilings of Domino Deficient Rectangles

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Abstract

In this paper I have settled the open problem, posed by J. Marshall Ash and S. Golomb in [4], of tiling an $m \times n$ rectangle with L -shaped trominoes, with the condition that $3 \mid (mn - 2)$ and a domino is removed from the given rectangle. It turns out that for any given $m, n \geq 7$, the only pairs of squares which prevent a tiling are $\{(1,2), (2,2)\}$, $\{(2,1), (2,2)\}$, $\{(2,3), (2,4)\}$, $\{(3,2), (4,2)\}$ and their symmetric counterparts. For all other cases, the existence of a tiling is shown. Some results on tiling the general case of 2-deficiency are also discussed.

1 Introduction

Trominoes, which are part of the more general class *Polyominoes*, were first introduced by Golomb [3] as part of recreational mathematics. A tromino is a shape made up of three 1×1 squares assembled in an L -shape as shown in the following figure:

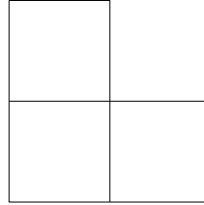


Figure 1: A tromino.

In this paper, integer-sided rectangles with each dimension at least 2, and with two 1×1 squares removed, such that $3 \mid (mn - 2)$, are considered. Our primary concern is whether these shapes can be tiled by trominoes or not, such that each tromino covers exactly one 1×1 square. As stated in [4] if one square is removed, the resultant shape is called a *deficient* rectangle. Similarly, if two such squares are missing, the resultant shape is called a *2-deficient* rectangle. If the removed square was a corner square, the resulting shape is called a *dog-eared* rectangle. Extending this notation, if the cornermost square and the square adjacent to it are missing, the resulting shape will be called a *2-deficient dog-eared* rectangle. The area of a tromino is 3, so only shapes whose areas are a multiple of 3 can be tiled by

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trominoes. Here, we consider a special class of 2-deficient rectangles, the *domino deficient rectangles*. In these rectangles the missing squares share a common edge, that is, rectangles from which a *domino* has been removed. This paper determines which of these rectangles with area divisible by 3 are tileable and which are not. It turns out that for any given $m, n \geq 7$, the only pairs of squares which prevent a tiling are $\{(1,2), (2,2)\}$, $\{(2,1), (2,2)\}$, $\{(2,3), (2,4)\}$, $\{(3,2), (4,2)\}$ and their symmetric counterparts. For all other cases, the existence of a tiling is shown. Some results on tiling the general case of 2-deficiency in rectangles are also discussed.

1.1 Definitions and notation

We use conventions similar to those used in [4]. However, for the sake of continuity, we review some notions. We denote a rectangle with i rows and j columns by $R(i, j)$. Decompositions into non-overlapping subrectangles are indicated by means of an additive notation. For instance, a $3i \times 2j$ rectangle can be decomposed into ij 3×2 subrectangles and this fact is written as $R(3i, 2j) = \sum_{a=1}^i \sum_{b=1}^j R(3, 2) = ijR(3, 2)$. It follows that any $3i \times 2j$ and $2i \times 3j$ rectangle can be tiled by trominoes. We denote the 1×1 square lying in row i and column j as (i, j) . Domino deficient rectangles with i rows and j columns are denoted in general by $R(i, j)^{--}$. The reader is referred to [5] for details. For convenience, trominoes are denoted in the rest of the paper as a composition of two lines forming an L -shape across an actual tromino. This is illustrated in the following figure:

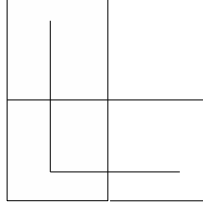


Figure 2: Notation for a tromino.

The missing squares are shown as dark-shaded squares in all the figures. Any pair of missing squares that does not allow a tiling of the resultant structure is referred to as a *bad pair*.

1.2 Organization of the Paper

In section 2 the *2-deficient dog-eared rectangle theorem* is presented, which states that any 2-deficient dog-eared rectangle with dimensions greater than 3 is tileable. In section 3 a simple case for domino deficiency, when one of the dimensions is 2, is discussed. Sections 4 and 5 characterize bad pairs when one dimension of the given rectangle is 4, 7 and 10 respectively. Section 6 presents the proof of the major result, the *2-deficient rectangle theorem*. Section 7 classifies the bad pairs when one dimension is 5. Finally, some results on the general case of 2-deficiency are discussed in Section 8.

2 The 2-Deficient Dog-Eared Rectangle Theorem

In this section we develop a mathematical technique for solving a special case of our more general problem. This technique is further useful in simplifying and solving the more complex cases dealt in later sections.

Theorem 1 *If $mn \equiv 2 \pmod{3}$ where $m, n \geq 4$, and a corner square and the square adjacent to it are both removed from an $R(m, n)$, then what remains can always be tiled by trominoes.*

Proof: Suppose $m = 3j+1$, $n = 3k+2$ where $j, k \geq 1$. In each of the cases shown we will break the given rectangles into subrectangles, which can either be tiled by the Chu-Johnsonbaugh Theorem or by the Figures 3-5. Each of the full subrectangles obtained can be tiled by the Chu-Johnsonbaugh Theorem [2].

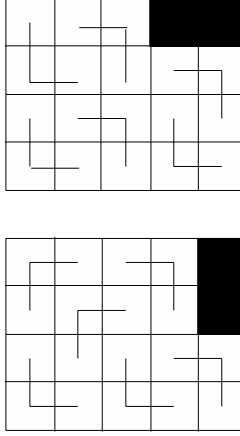


Figure 3: Tiling of $R(4, 5)^{-}$.

CASE 1: $m = 4$, $n = 3k+5$, $k \geq 0$.

Here we can split the given $m \times n$ rectangle as:

$$R(4, 3k+5)^{-} = R(4, 3k) + R(4, 5)^{-}$$

CASE 2: $m = 7$, $n = 3k+5$, $k \geq 0$.

Since the dimension m is odd, we divide the above case into two subcases accordingly as k is even or odd.

Subcase 1: $k = 2l$, $l \geq 0$.

$$R(7, 6l+5)^{-} = R(7, 6l) + R(7, 5)^{-}$$

Subcase 2: $k = 2l+1$, $l \geq 0$.

$$\begin{aligned} R(7, 6l+8)^{-} &= R(7, 6l) + R(7, 8)^{-} \\ &= R(7, 6l) + R(3, 8) + R(4, 3) + R(4, 5)^{-} \end{aligned}$$

CASE 3: $m = 3j+4$, $n = 3k+5$ where $j \geq 2$, $k \geq 0$.

Here again the dimension m can be odd, hence we consider two subcases accordingly

as k is odd or even.

Subcase 1: $k = 2l, l \geq 0$.

$$\begin{aligned} R(3j+4, 6l+5)^{-} &= R(3j+4, 6l) + R(3j+4, 5)^{-} \\ &= R(3j+4, 6l) + R(3j, 5) + R(4, 5)^{-} \end{aligned}$$

Subcase 2: $k = 2l+1, l \geq 0$.

$$\begin{aligned} R(3j+4, 6l+8)^{-} &= R(3j+4, 6l) + R(3j+4, 8)^{-} \\ &= R(3j+4, 6l) + R(3j, 8) + R(4, 3) + R(4, 5)^{-} \end{aligned}$$

□

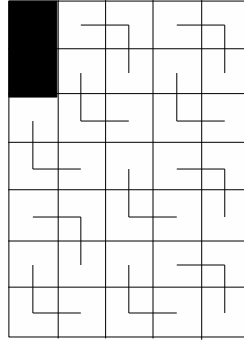


Figure 4: Tiling of $R(7, 5)^{-}$ when a vertical domino is removed.

3 A Simple Result for the case $m = 2$

In order to simplify the presentation of more general cases in Sections 4-7, we isolate the case where one dimension of the given rectangle is 2. For this case we have the following result:

Lemma 1 *When a vertical domino is removed from $R(2, 3j+4)$, $j \geq 0$, the only horizontal position which allows a tiling is when the column number $x = 3k+1$, where $k \geq 0$.*

Proof: Only in this case will the area on both sides of the missing domino be divisible by 3. The rest follows from Chu-Johnsonbaugh Theorem [2]. □

Lemma 2 *When a horizontal domino is removed from $R(2, 3j+4)$, $j \geq 0$, the only position allowing a tiling is the pair of columns $(3x+2, 3x+3)$, where $x \geq 0$.*

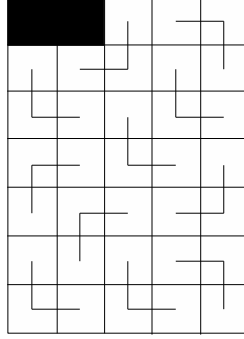


Figure 5: Tiling of $R(7, 5)^{--}$ when a horizontal domino is removed.

Proof: It is easy to see that only in this case the area on both sides is divisible by 3, so both sides can be tiled by the Chu-Johnsonbaugh Theorem. \square

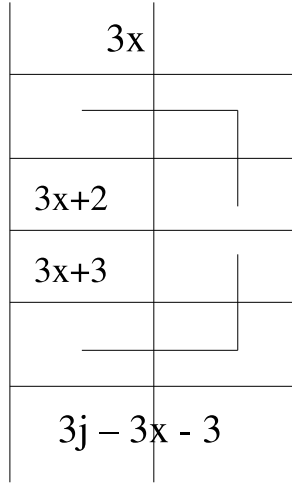


Figure 6: Tiling of $R(2, 3j + 4)^{--}$ when a horizontal domino is removed.

4 Tilings of $R(4, 3j + 8)^{--}$

We will now proceed towards building the background for proving the major result. We will deal with the case $n = 5$ separately at the end of the paper. For now it will be assumed that $n \geq 8$. All the bad pairs for $R(4, 8)^{--}$ are illustrated in Figure 7 (their symmetric counterparts are excluded for clarity).

Clearly, in some of the above cases shown, the cornermost square becomes inaccessible to a tromino. The badness of the rest of the cases is illustrated through Figures 8 and 9. For the pair $\{(2, 3), (3, 3)\}$, we see that in one case the cornermost square $(1, 1)$ or $(5, 1)$ becomes inaccessible, while in the other case we have isolated

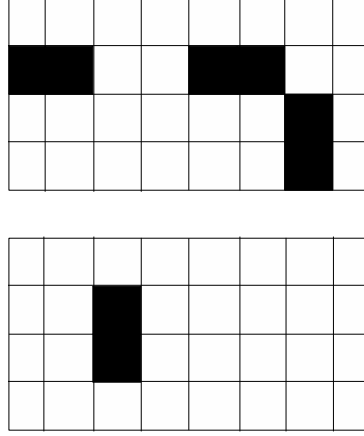


Figure 7: Bad squares for $R(4, 8)^{--}$.

$R(2, 8)$ which, by Chu-Johnsonbaugh Theorem, is not tileable by trominoes. The next figure shows the 2nd bad case. In this case, the only possible way to cover the square (3,1) is as shown. This again renders the cornermost square (1, 1) inaccessible. Therefore, in neither of the above cases enumerated is a tiling by trominoes possible.

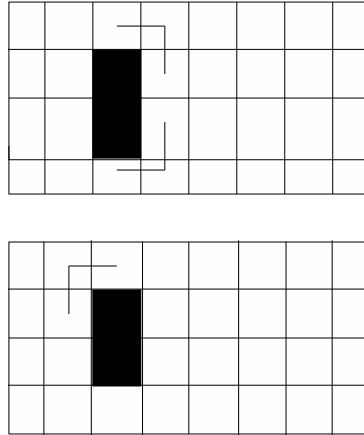


Figure 8: 1st bad case for $R(4, 8)^{--}$.

We note that only in the case dealt in Figure 8 is the position of the bad squares symmetric with respect to a shift of 3 columns, i.e., if we shift the missing domino 3 places to the right, we will get the same untileable configuration. We deal with this case in Figure 10 by joining $R(4, 8)^{--}$ with the adjacent $R(4, 3)$, and present an approach for tiling the general case $R(4, 3j + 8)^{--}$. The boundary between the appended $R(4, 3)$ and $R(4, 8)^{--}$ is shown by an extended line.

Lemma 3 *The bad pairs for $R(4, 3j+8)^{--}$ are $\{(2, 1), (2, 2)\}$, $\{(1, 2), (2, 2)\}$, $\{(2, 3), (2, 4)\}$, $\{(2, 3), (3, 3)\}$, and their symmetric counterparts.*

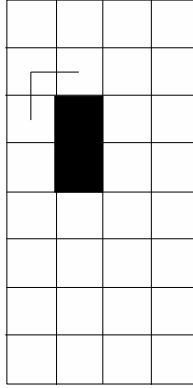


Figure 9: 2nd bad case for $R(4, 8)^{--}$.

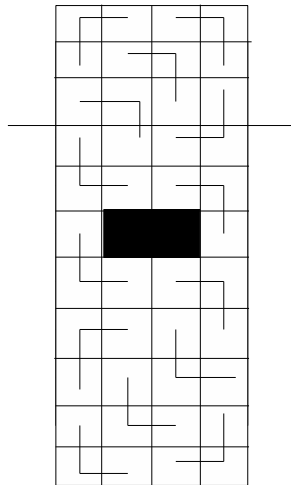


Figure 10: Tiling for the only symmetric bad case.

Proof: The proof will be in the forward direction of the claim, namely, that if the missing domino does not come under the above cases then a tiling exists. We note that $R(4, 3j + 8)^{-} = jR(4, 3) + R(4, 8)^{-}$. If this $R(4, 8)^{-}$ can be tiled, we are done. Else, we attach the left(right) $R(4, 3)$ and remove the right(left) $R(4, 3)$ from $R(4, 8)^{-}$ so as to keep the missing squares inside $R(4, 8)^{-}$. If the new $R(4, 8)^{-}$ can be tiled, we are done. If a tiling is still not possible, then we have encountered the bad pair shown in Figure 8, since the missing squares are symmetrically placed with respect to a shift of three columns. In this situation, we join the adjacent $R(4, 3)$ to place these missing squares in the center and apply the tiling of $R(4, 11)^{-}$ shown in Figure 10. \square

5 Tilings of $R(7, 3j + 8)^{-}$ and $R(10, 3j + 8)^{-}$

Since 7 is odd, we divide this case into two subcases.

Subcase 1: $j = 2l$, where $l \geq 0$.

The bad pairs for $R(7, 8)^{-}$ are given in the following figure:

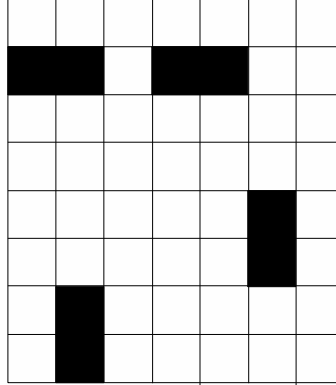


Figure 11: Bad pairs of squares for $R(7, 8)^{-}$.

The proof of their badness is similar to that used in $R(4, 3j + 8)^{-}$ so it is left for the reader. A similar approach for tiling $R(7, 6l + 8)^{-}$ will be adopted as in case of $R(4, 3j + 8)^{-}$. However, there are a few differences. For two bad pairs, namely, $\{(2, 1), (2, 2)\}$ and $\{(2, 3), (2, 4)\}$, a slight different approach is followed as the former case is symmetric with respect to a shift of 6 columns, while the latter goes out of range. Thus, we join $R(7, 8)$ with the adjacent $R(7, 6)$. The tiling of these cases is shown in the Figure 12.

Lemma 4 *The bad pairs for $R(7, 3j + 8)^{-}$ are $\{(2, 1), (2, 2)\}$, $\{(1, 2), (2, 2)\}$, $\{(2, 3), (2, 4)\}$, $\{(3, 2), (4, 2)\}$, and their symmetric counterparts.*

Proof: Once again we adopt the approach of forward direction for proof. $R(7, 6l + 8)^{-} = lR(7, 6) + R(7, 8)^{-}$. If this $R(7, 8)^{-}$ can be tiled, we are done. Else, we attach the left(right) $R(7, 6)$ and remove the right(left) $R(7, 6)$ from $R(7, 8)^{-}$ so as to keep the missing squares inside $R(7, 8)^{-}$. If the new $R(7, 8)^{-}$ can be tiled, we are done. If the above approach could not have been followed, then the missing

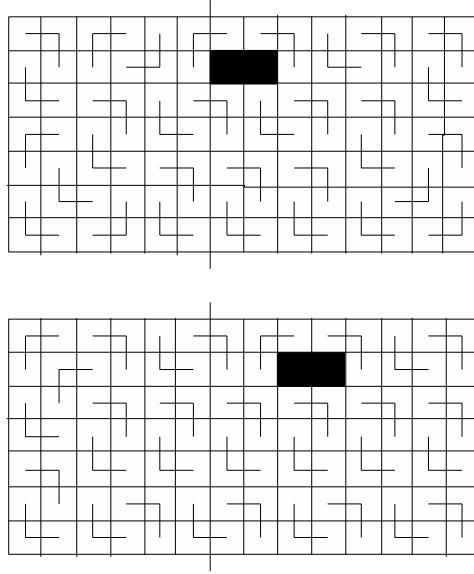


Figure 12: Tilings for exceptional cases in $R(7, 8)^{--}$.

squares are one of the pairs $\{(2, 1), (2, 2)\}$ and $\{(2, 3), (2, 4)\}$. In these cases we join the adjacent $R(7, 6)$ to $R(7, 8)^{--}$ so as to convert this case into the ones shown in Figure 12 and apply the tiling shown. \square

Subcase 2: $j = 2l+1$, where $l \geq 0$.

The bad pairs for $R(7, 11)^{--}$ are same as those for $R(7, 8)^{--}$. In this case, none of the bad pairs are symmetrically placed with respect to a shift of 6 columns, and we note that $R(7, 6l + 11)^{--} = lR(7, 6) + R(7, 11)^{--}$.

The bad pairs for $R(10, 8)^{--}$ are the same as those enumerated for the previous two cases. Moreover, none of these pairs are symmetric with respect to a shift of 3 columns, and we have $R(10, 3j + 8)^{--} = jR(10, 3) + R(10, 8)^{--}$.

6 The 2-Deficient Rectangle Theorem

We now state our major result.

Theorem 2 *For any two given integers $m, n \geq 7$, such that $3 \nmid (mn-2)$, if a domino is removed from $R(m, n)$, the resulting structure can always be tiled with trominoes provided the domino does not occupy the pairs of squares $\{(2, 1), (2, 2)\}$, $\{(1, 2), (2, 2)\}$, $\{(2, 3), (2, 4)\}$, $\{(3, 2), (4, 2)\}$ or their symmetric counterparts.*

Proof: We treat the cases $m = 7, 10$ individually, and then proceed inductively. If $m \geq 13$, then $m - 6 \geq 6$. So, we slice a full rectangle of height 6 off of either the top or the bottom of $R(m, n)^{--}$, that is, $R(m, n)^{--} = R(m - 6, n)^{--} + R(6, n)$. Since the last term is tileable by the Chu-Johnsonbaugh Theorem, this first reduces

the cases $m \in [13, 17]$ to the cases $m \in [7, 11]$, then the cases $m \in [19, 23]$ to the cases $m \in [13, 17]$, and so on.

If the missing squares in the resulting $R(7, 3j + 8)^{--}$ or $R(10, 3j + 8)^{--}$ that we finally get by the above approach are not one among $\{(2, 1), (2, 2)\}$, $\{(1, 2), (2, 2)\}$, $\{(2, 3), (2, 4)\}$, $\{(3, 2), (4, 2)\}$ or their symmetric counterparts, then the subrectangle can be tiled with trominoes following the approaches discussed above. If the missing squares are among the ones enumerated above, then we join accordingly $R(6, n)$ from above or below and will then tile the resulting composite subrectangle. We need only consider $R(7, 8)^{--}$, $R(7, 11)^{--}$ or $R(10, 8)^{--}$ joined from above or below with $R(6, 8)$ or $R(6, 11)$. The other half of the composite subrectangle is tileable by the Chu-Johnsonbaugh Theorem. Each of these cases is discussed in the subcases below, and their corresponding tilings are shown. The convention for figures will be changed slightly. The blank portion denotes an area of the rectangle which can be tiled either by the Chu-Johnsonbaugh Theorem, the Dog-Eared Rectangle Theorem, or the 2-Deficient Dog-Eared Rectangle Theorem.

Subcase 1: The subrectangle obtained finally is $R(7, 8)^{--}$. Four cases will arise. The various tilings are shown in the Figures 13-16.

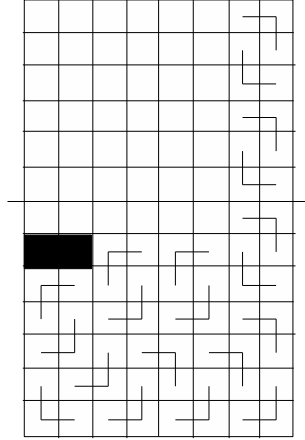


Figure 13: Tiling when missing squares are $\{(2, 1), (2, 2)\}$.

Subcase 2: The subrectangle obtained finally is $R(7, 11)^{--}$. Again, four cases will arise. The various tilings are shown in the Figures 17-20. The gray area shown in Figure 20 is tileable by the Chu-Johnsonbaugh Theorem. The other two areas by its sides are tileable by the 2-Deficient Dog-Eared Rectangle Theorem.

Subcase 3: The subrectangle obtained finally is $R(10, 8)^{--}$. The situation is slightly different in this case. For the pairs $\{(2, 3), (2, 4)\}$ and $\{(2, 1), (2, 2)\}$, the bottommost $R(6, 8)$ is sliced off while the upper $R(6, 8)$ is appended to form a new $R(10, 8)^{--}$. In the new 2-deficient rectangle obtained, the missing squares are no longer among the bad squares, hence, the structure is tileable with trominoes. The pairs $\{(1, 2), (2, 2)\}$ and $\{(3, 2), (4, 2)\}$ actually become symmetrical when we ap-

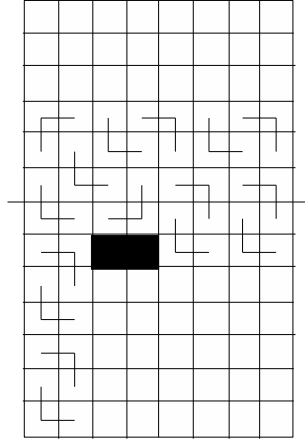


Figure 14: Tiling when missing squares are $\{(2, 3), (2, 4)\}$.

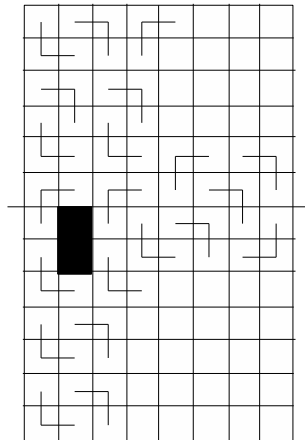


Figure 15: Tiling when missing squares are $\{(1, 2), (2, 2)\}$.

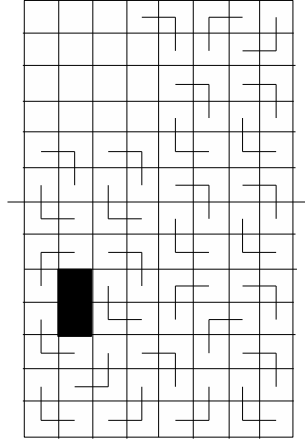


Figure 16: Tiling when missing squares are $\{(3, 2), (4, 2)\}$.

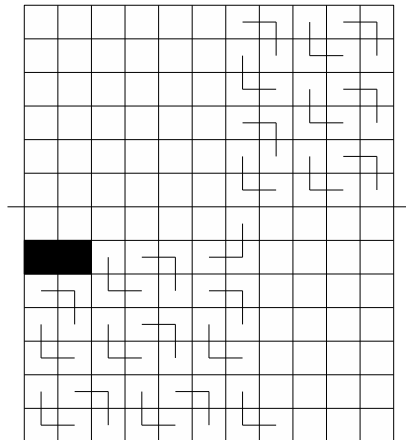


Figure 17: Tiling when missing squares are $\{(2, 1), (2, 2)\}$.

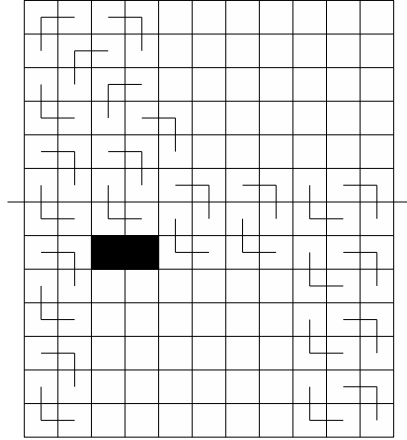


Figure 18: Tiling when missing squares are $\{(2, 3), (2, 4)\}$.

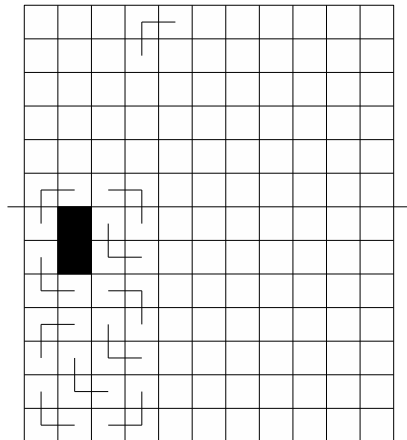


Figure 19: Tiling when missing squares are $\{(1, 2), (2, 2)\}$.

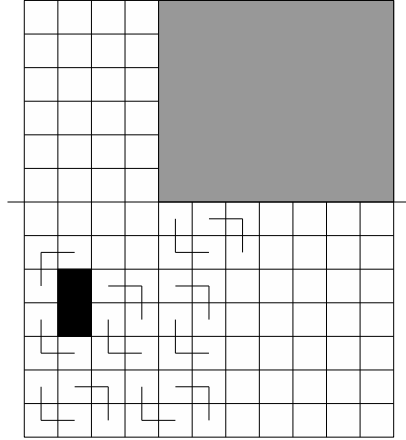


Figure 20: Tiling when missing squares are $\{(3, 2), (4, 2)\}$.

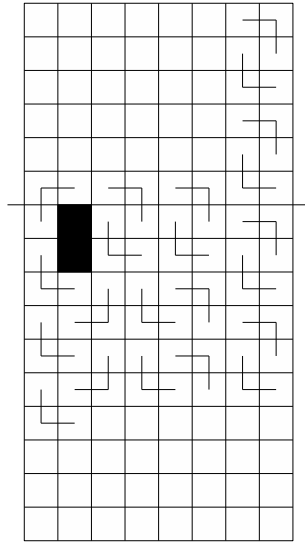


Figure 21: Tiling when missing squares are $\{(3, 2), (4, 2)\}$ or $\{(1, 2), (2, 2)\}$.

pend the upper $R(6, 8)$ with $R(10, 8)^{--}$. Therefore, only one tiling is shown in Figure 21 which suffices for both the above-mentioned cases. The former configuration is actually the reflection of the latter in the horizontal axis. \square

7 Tilings when $n = 5$

We now deal with the case when one dimension of the given rectangle is 5.

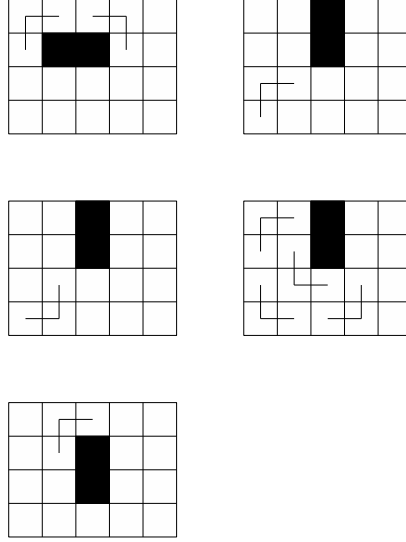


Figure 22: Bad pairs for $R(4, 5)^{--}$.

Lemma 5 *The bad pairs for $R(4, 5)^{--}$ are $\{(2, 1), (2, 2)\}$, $\{(2, 2), (2, 3)\}$, $\{(1, 2), (2, 2)\}$, $\{(1, 3), (2, 3)\}$, $\{(2, 3), (3, 3)\}$, and their symmetric counterparts.*

Proof: Out of the above five cases, the pairs $\{(2, 1), (2, 2)\}$ and $\{(1, 2), (2, 2)\}$ have already been proved to be bad for any rectangle. Hence, we will focus on the other three cases. Considering the case $\{(2, 2), (2, 3)\}$ first, the only possible way to cover the squares (1,2) and (1,3) is as shown in Figure 22. This renders the cornermost square (1,5) inaccessible by a tromino. Next we turn to the case $\{(1, 3), (2, 3)\}$. There are three configurations of the tromino covering the square (4,1). As we can see in Figure 22, in two of these configurations, the number of 1×1 squares in the portion of $R(4, 5)^{--}$ trapped between the missing squares and the given tromino are not a multiple of 3. In the third case, as can be verified by the reader, the only possible way to cover the rest of $R(4, 5)^{--}$ is the approach shown in Figure 22. But this makes the square (4,5) inaccessible. Lastly, considering the pair $\{(2, 3), (3, 3)\}$. The only possible way of covering the square (1,3) is as shown in Figure 22. However, this makes the square (1,1) inaccessible. Hence, a tiling is not possible in any of the above cases. \square

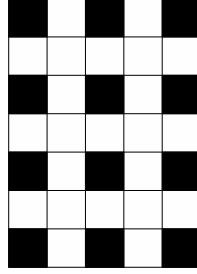


Figure 23: The rectangle $R(7,5)$.

Lemma 6 (Deficient 5×7 Lemma) *If both the x or y coordinates of the domino removed from $R(5,7)$ are even, then the resulting shape is not tileable.*

Proof: We form a kind of checkerboard by marking each of the 12 squares as shown in Figure 23. If both the x or y coordinates of the removed domino are even, then any tiling of R^{--} must contain one tromino for each of the 12 marked squares, so that the tiling must have area at least $12 \cdot 3 = 36$, which is absurd since the area of R^{--} is 33. Thus, all pairs which satisfy the above criterion are bad. \square

Lemma 7 *The bad pairs for $R(5,7)^{--}$ are $\{(2,1), (2,2)\}$, $\{(2,3), (2,4)\}$, $\{(2,2), (2,3)\}$, $\{(1,2), (2,2)\}$, $\{(1,4), (2,4)\}$, $\{(3,2), (3,3)\}$, $\{(2,2), (3,2)\}$, $\{(2,4), (3,4)\}$ and their symmetric counterparts.*

Proof: Apart from $\{(3,2), (3,3)\}$, all the other pairs satisfy the criterion of the Deficient 5×7 Lemma. So turning our attention to $\{(3,2), (3,3)\}$ we find that the tromino covering square (3,1) will make either square (1,1) or the square (5,1) inaccessible. Hence, the pair $\{(3,2), (3,3)\}$ is also bad. \square

Lemma 8 *The bad pairs for $R(5,10)^{--}$ and $R(5,13)^{--}$ are $\{(2,1), (2,2)\}$, $\{(2,3), (2,4)\}$, $\{(3,2), (3,3)\}$, $\{(1,2), (2,2)\}$, $\{(2,2), (3,2)\}$ and their symmetric counterparts.*

Proof: The badness of 4 of the above configurations is already known. The proof for $\{(3,2), (3,3)\}$ is same as the above. \square

Lemma 9 *The squares enumerated in Lemma 8 are the only bad pairs for $R(5,3j+10)^{--}$.*

Proof: Since 5 is odd, we divide the proof into two cases, accordingly as j is even or odd.

Subcase 1: $j = 2l, l \geq 0$.

We adopt a procedure based on similar lines as those discussed above. Note that $R(5,6l+10)^{--} = lR(5,6) + R(5,10)^{--}$. If this $R(5,10)^{--}$ can be tiled, we are done. Else, we attach the left(right) $R(5,6)$ and remove the right(left) $R(5,6)$ from $R(5,10)^{--}$ so as to keep the missing squares inside $R(5,10)^{--}$. If $R(5,10)^{--}$ can now be tiled, we are done. Otherwise, the missing squares are one of the

pairs $\{(3, 2), (3, 3)\}$, $\{(2, 1), (2, 2)\}$ or $\{(2, 3), (2, 4)\}$. In each such case, either the missing squares are symmetrically placed with respect to a shift of 6 columns, or they convert from one into another on such a shift. In this situation, we append the adjacent $R(5, 6)$ with $R(5, 10)^{--}$ and apply one of the tilings shown in Figure 24. The remaining portions on either sides can be tiled by the Chu-Johnsonbaugh Theorem.

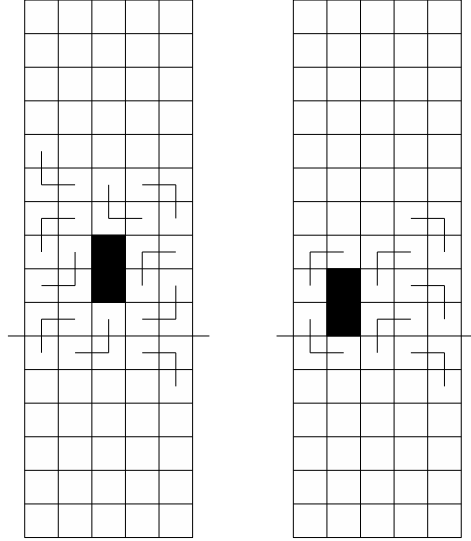


Figure 24: Tiling when missing squares are $\{(3, 2), (3, 3)\}$, $\{(2, 1), (2, 2)\}$ and $\{(2, 3), (2, 4)\}$.

Subcase 2: $j = 2l+1, l \geq 0$.

In this case none of the bad pairs are symmetric with respect to a shift of 6 columns, nor do they convert from one into another on applying such a shift. Also, we have $R(5, 6l+13)^{--} = lR(5, 6) + R(5, 13)^{--}$. \square

8 Generalization of a Special Case of 2-deficiency

We now consider a special case of 2-deficiency and present all the bad pairs when one dimension of the given rectangle is 4. For a few examples of bad pairs for the case $m = 7$ see [5].

Lemma 10 *Apart from the pairs of squares mentioned in section 4, the other bad pairs for the general $R(4, 8)^{--}$ are shown in Figures 25 and 26.*

Proof: The proof for only some out of the 8 cases will be given. The rest are illustrated in the figures. Considering the pairs $\{(1, 2), (3, 3)\}$, $\{(2, 1), (3, 3)\}$ and $\{(2, 2), (3, 3)\}$ the tromino covering the square $(4, 3)$ will either leave an untileable $R(1, 2)$ or $R(2, 2)$. In the case $\{(1, 4), (3, 3)\}$, a tromino has to cover the square $(4, 3)$ as shown. Any tromino which covers the square $(2, 4)$ will leave an untileable

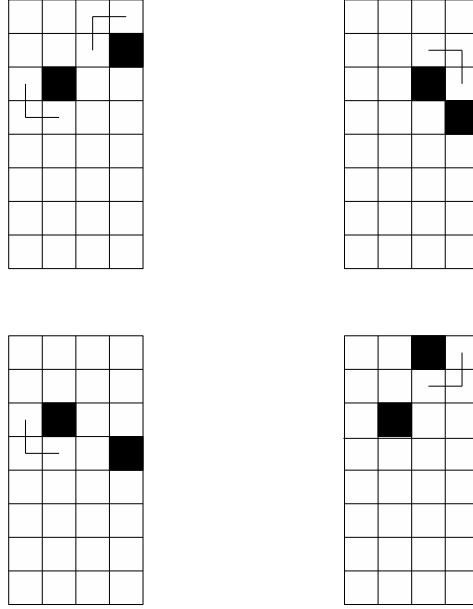


Figure 25: Bad pairs for the general $R(4,8)^{--}$.

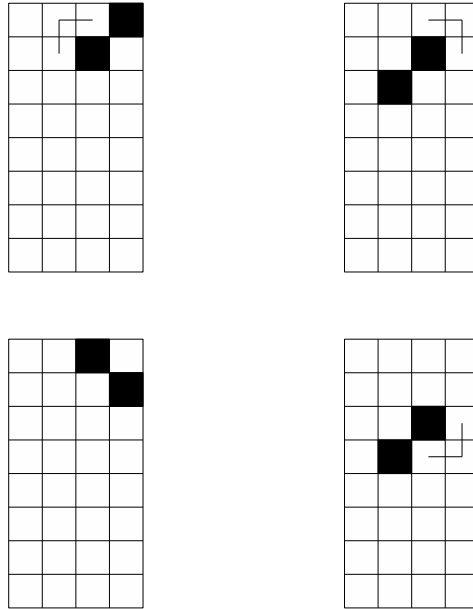


Figure 26: Bad pairs for the general $R(4,8)^{--}$.

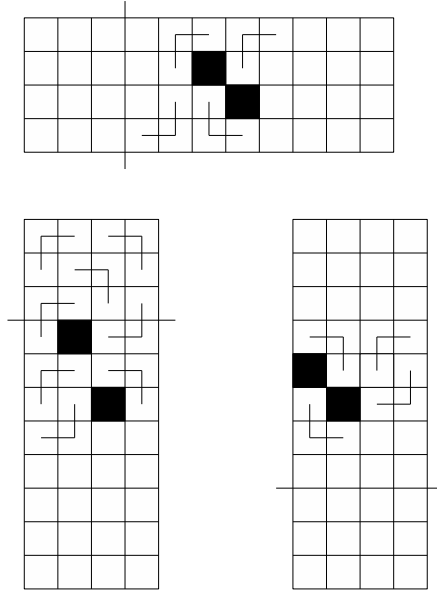


Figure 27: Tiling the general $R(4, 3j + 8)^{--}$.

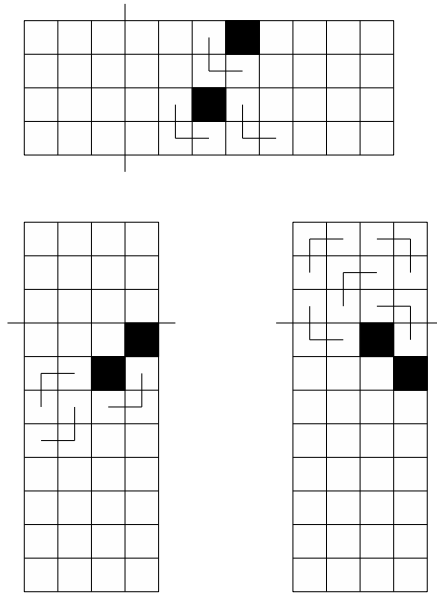


Figure 28: Tiling the general $R(4, 3j + 8)^{--}$.

area to the left (since the area is not a multiple of 3). For the case $\{(2, 3), (3, 4)\}$, the trominoes covering $(1, 3)$ and $(4, 4)$ will leave an untileable area to the left, for the same reason as above. Finally, in $\{(1, 1), (2, 2)\}$, the tromino covering $(2, 1)$ will make the square $(4, 1)$ inaccessible. \square

Lemma 11 *For the general $R(4, 3j + 8)^{--}$, the only bad pairs are those mentioned above (apart from those enumerated in section 4).*

Proof: If a single missing square can be incorporated in $R(4, 4)^-$, then the entire structure can be tiled by the Chu-Johnsonbaugh Theorem and the Deficient Rectangle Theorem as $R(4, 3j + 8)^{--} = jR(4, 3) + 2R(4, 4)^-$. If both missing squares are present in $R(4, 4)^{--}$, then note that $R(4, 3j + 8)^{--} = jR(4, 3) + R(4, 8)^{--}$. If this $R(4, 8)^{--}$ can be tiled, we are done. Else, we have encountered the bad pairs shown in Figures 25 and 26. In this case, we join the adjacent $R(4, 3)$ and apply the tiling of $R(4, 11)^{--}$ shown in Figures 27 and 28. \square

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