

Frugality ratios and improved truthful mechanisms for vertex cover ^{*}

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1 Introduction

Situations in which one has to hire a team of agents to perform a task are quite typical in many domains. In a market-based environment, this can be achieved by means of a reverse auction: the agents submit their bids and the centre selects a team based on the agents' ability to work with each other, as well as their payment requirements. The problem is complicated by the fact that only *some* subsets of agents constitute a valid team: the task may require several skills, and each agent may possess only a subset of these skills, the agents must be able to communicate with each other, etc. In a more computing-oriented scenario, the centre may want to purchase connectivity between two points in a network that consists of independent subnetworks; in this case, the valid teams are paths between the two points. The latter problem is usually referred to as a *shortest-path auction* and has been studied extensively in the recent literature starting with the seminal paper by Nisan and Ronen [17] (see also [1, 8, 7, 5, 13, 6, 18]). Generally, this setting can be formalized by specifying (explicitly or implicitly) the *feasible* sets, which are the sets of agents that are capable of performing the overall task; consequently, auctions of this type are sometimes referred to as *set-system auctions*.

The centre and the agents have conflicting goals: the centre wants to spend as little money as possible, and the agents want to maximise their earnings. Therefore, an important characteristic of a set-system auction is the total payment of the centre, or, more precisely, the relationship between the latter and the true cost of the cheapest feasible set. Clearly, when the true costs of each agent are not publicly known, one cannot expect these two quantities to coincide: if each selected agent were to be paid its bid, the agents would attempt to extract more profit by overbidding. To put it differently, to achieve truthful bidding (which is considered to be a very desirable property of an auction mechanism), the centre must pay a carefully designed bonus to each agent. Since it is unrealistic to expect the payment of the centre to be as low as the cost of the cheapest feasible set, one needs a more realistic benchmark for measuring the overpayment of the mechanism.

The concept of *frugality* was introduced by Archer and Tardos [1] in the context of shortest-path auctions. The paper [1] proposes to measure the overpayment of a mechanism by the ratio between its total payment and the cost of the cheapest path that is disjoint from the path

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selected by the mechanism; this quantity is called the *frugality ratio*. The authors show that, for a large class of truthful mechanisms for the shortest-path problem, the frugality ratio is as large as the number of edges in the shortest path. These results were subsequently extended by [7] to hold for *all* truthful shortest-path auctions. Talwar [19] extends this definition of frugality ratio to general set systems, and studies the frugality ratio of the classical VCG mechanism [20, 3, 12] for many specific set systems, such as minimum spanning trees and set covers.

However, while the definition of frugality ratio proposed by [1] is well-motivated and has been instrumental in studying truthful mechanisms for set systems, it is not completely satisfactory. Consider, for example, the graph of Figure 1 with the costs $c_{AB} = c_{BC} = c_{CD} = 0$, $c_{AC} = c_{BD} = 1$. This graph is 2-connected and the VCG payment to the winning path ABCD is bounded. However, the graph contains no A–D path that is disjoint from ABCD, and hence the frugality ratio of VCG on this graph remains undefined. In auctions for other types of set systems, the requirement that there exists a feasible solution that is disjoint from the selected one is even more severe: for example, for vertex-cover auctions it means that the underlying graph must be bipartite. In [14], the authors suggest a better benchmark, which intuitively corresponds to the value of the cheapest Nash equilibrium. This quantity, which they denote by ν , is defined for any monopoly-free set system, i.e., a set system such that no element of the ground set belongs to all feasible sets. Based on this new definition, the authors construct new mechanisms for the shortest path problem whose overpayment is shown to be within a constant factor of optimal.

In this paper we investigate the payment bound ν proposed by Karlin et al. We show that the definition in [14] can be seen as a result of answering two independent ‘yes’/‘no’ questions. By answering these questions differently, we obtain three alternative payment bounds. All four payment bounds arise as Nash equilibria of certain games; the differences between them can be seen, respectively, as “the price of initiative” and “the price of co-operation”. We study the properties of these payment bounds as well as the relationships between them, both for general set systems and for specific problems, such as shortest-path auctions and vertex-cover auctions. This study allows us to make recommendations for selecting an appropriate payment bound for different scenarios; in particular, it suggests that the payment bound ν of [14] is not always the best possible choice.

Another important contribution of this paper is a new truthful mechanism for vertex-cover auctions, which combines polynomial-time computability and good approximation ratio with a good frugality ratio. Moreover, we show how to transform any truthful mechanism for the vertex-cover problem into a frugal one. Our proof illustrates the usefulness of the several definitions of frugality introduced in this paper: while our final result is with respect to the original definition of [14], we first bound the frugality ratio of our mechanism with respect to a weaker bound, and then bootstrap from this result to obtain the desired bound. We believe that this proof technique can be useful for more general set systems.

In the past, vertex-cover auctions have been studied by Talwar [19] and Calinescu [4]. Both of these papers are based on the definition of frugality ratio that originates in [1]; as mentioned before, this means that their results only apply to bipartite graphs. Paper [19] shows that the frugality ratio of VCG is at most Δ , where Δ is the maximum degree in the graph. However, since finding the cheapest vertex cover is an NP-hard problem, the VCG mechanism is computationally infeasible. The first (and, to the best of our knowledge, only) paper to investigate polynomial-time truthful mechanisms for vertex cover is [4]. This paper studies an auction that is based on the greedy allocation algorithm, which has an

approximation ratio of $\log n$. While the main focus of [4] is the more general set cover problem, the results of [4] imply a frugality ratio of $2\Delta^2$ for vertex cover. Our mechanism uses the local ratio algorithm [2] as its allocation rule, thus producing a vertex cover whose cost is within a factor of 2 of the optimal solution. Moreover, we show that, for all mechanisms in a large class that includes our mechanism as well as those of [19] and [4], the frugality ratio is at most 2Δ (with respect to the definition of [14]). Our results can be extended to set-cover auctions, resulting in a frugality ratio that is better than the one of [4] by a factor of k , where k is the maximum set size.

2 Preliminaries

In most of this paper, we discuss auctions for set systems. A *set system* is a pair $(\mathcal{E}, \mathcal{F})$, where \mathcal{E} is the *ground set*, $|\mathcal{E}| = n$, and \mathcal{F} is a collection of *feasible sets*, which are subsets of \mathcal{E} . Particular classes of set systems that are of interest to us are *shortest paths*, where the ground set consists of all edges of a network, and the feasible sets are paths between two specified vertices s and t , and *vertex covers*, where the elements of the ground set are the vertices of a graph, and the feasible sets are vertex covers for this graph.

In set system auctions, each element e of the ground set is owned by an independent agent and has an associated non-negative cost c_e . The goal of the centre is to select (purchase) a feasible set. Each element e in the selected set incurs a cost of c_e . The elements that are not selected incur no costs.

The auction proceeds as follows: all elements of the ground set make their bids, the centre selects a feasible set based on the bids and makes payments to the agents. Formally, an auction is defined by an *allocation rule* $A : \mathbf{R}^n \mapsto \mathcal{F}$ and a *payment rule* $P : \mathbf{R}^n \mapsto \mathbf{R}^n$. The allocation rule takes as input a vector of bids and decides which of the sets in \mathcal{F} should be selected. The payment rule also takes as input a vector of bids and decides how much to pay to each agent. The standard requirements are *individual rationality*, i.e., the payment to each agent should be at least as high as its incurred cost (0 for agents not in the selected set and c_e for agents in the selected set) and *incentive compatibility*, or *truthfulness*, i.e., each agent's dominant strategy is to bid its true cost.

An allocation rule is *monotone* if an agent cannot increase his chance of getting selected by raising its bid. Formally, for any bid vector \mathbf{b} and any $e \in \mathcal{E}$, if $e \notin A(\mathbf{b})$ then $e \notin A(b_1, \dots, b'_e, \dots, b_n)$ for any $b'_e > b_e$. Given a monotone allocation rule A and a bid vector \mathbf{b} , the *threshold bid* t_e of an agent $e \in A(\mathbf{b})$ is the highest bid of this agent that still wins the auction, given that the bids of other participants remain the same. Formally, $t_e = \sup\{b'_e \in \mathbf{R} \mid e \in A(b_1, \dots, b'_e, \dots, b_n)\}$.

It is well known (see, e.g. [17, 11]) that any auction that has a monotone allocation rule and pays each agent its threshold bid is truthful.

The VCG mechanism is a truthful mechanism that maximizes the “social welfare” and pays 0 to the losing agents. For set system auctions, this simply means picking a cheapest feasible set, paying each agent in the selected set its threshold bid, and paying 0 to all other agents.

If U is a set of agents, $c(U)$ denotes $\sum_{w \in U} c_w$. Similarly, $b(U)$ denotes $\sum_{w \in U} b_w$.

3 Frugality ratios

We start by reproducing the definition of the quantity ν from [14, Definition 4].

Let $(\mathcal{E}, \mathcal{F})$ be a set system and let S be the cheapest feasible set with respect to the true costs c_e where ties are broken lexicographically.

Then $\nu(\mathbf{c})$ is the solution to the following optimisation problem.

Minimise $B = \sum_{e \in S} b_e$ subject to

- (1) $b_e \geq c_e$ for all $e \in \mathcal{E}$
- (2) $\sum_{e \in S \setminus T} b_e \leq \sum_{e \in T \setminus S} c_e$ for all $T \in \mathcal{F}$
- (3) for every $e \in S$, there is a $T_e \in \mathcal{F}$ such that $e \notin T_e$ and $\sum_{e' \in S \setminus T_e} b_{e'} = \sum_{e' \in T_e \setminus S} c_{e'}$

This bound can be seen as an outcome of a two-stage process, where first each agent $e \in S$ makes a bid b_e stating how much it wants to be paid, and then the centre decides whether to accept these bids. The behaviour of both parties is affected by the following considerations. From the centre's point of view, the set S must remain the most attractive choice, i.e., it must be among the cheapest feasible sets under the new costs $c'_e = c_e$ for $e \notin S$, $c'_e = b_e$ for $e \in S$ (condition (2)). The reason for that is that if (2) is violated for some set T , the centre would prefer T to S . On the other hand, no agent would agree to a payment that does not cover its costs (condition (1)), and moreover, each agent tries to maximise its profit by bidding as high as possible, i.e., none of the agents can increase its bid without violating condition (2) (condition (3)). The centre wants to minimise the total payout, so $\nu(\mathbf{c})$ corresponds to the best possible outcome from the centre's point of view.

This definition captures many important aspects of our intuition about ‘fair’ payments in this setting. However, it can be modified in two ways, both of which are still quite natural, but result in different payment bounds.

First, we can consider the worst rather than the best possible outcome for the centre, i.e., the maximum total payment the agents can extract by jointly selecting their bids subject to (1), (2), and (3). This corresponds to maximising B subject to (1), (2), and (3) rather than minimising it. Indeed, if it is the agents in S who make the bids, this kind of bidding behaviour is, in fact, more plausible. On the other hand, one is likely to observe a total payment of ν if the centre is allowed to move first, i.e., in a game where the centre proposes payments to the agents in S and the agents accept them as long as these payments satisfy (1), (2), and (3). Hence, the difference between these two definitions can be seen as “the price of initiative”.

Second, the agents may be able to make payments to each other. In this case, if they can extract more money from the centre by agreeing on a vector on bids that violates individual rationality (i.e., condition (1)) for some bidders, they might be willing to do so, as the agents who are paid below their costs will be compensated by other members of the group. The bids must still be realistic, i.e., they have to satisfy $b_e \geq 0$. This corresponds to replacing condition (1) with the following weaker condition (1*):

$$b_e \geq 0 \text{ for all } e \in \mathcal{E}. \quad (1^*)$$

By considering all possible combinations of these modifications, we obtain four different payment bounds, namely

- $\text{TUmin}(\mathbf{c})$, which is the solution to the optimisation problem “Minimise B ” subject to (1*), (2), and (3).

- $\text{TUmax}(\mathbf{c})$, which is the solution to the optimisation problem “Maximise B ” subject to (1^*) , (2) , and (3) .
- $\text{NTUmin}(\mathbf{c})$, which is the solution to the optimisation problem “Minimise B ” subject to (1) , (2) , and (3) .
- $\text{NTUmax}(\mathbf{c})$, which is the solution to the optimisation problem “Maximise B ” subject to (1) , (2) , (3) .

The abbreviations TU and NTU correspond, respectively, to transferable utility and non-transferable utility, i.e., the agents’ ability/inability to make payments to each other. Note that quantity $\nu(c)$ from [14] is $\text{NTUmin}(\mathbf{c})$.

The second modification is more intuitively appealing in the context of the maximisation problem, as both assume some degree of co-operation between the agents. While it can also be made independently, the resulting payment bound $\text{TUmin}(\mathbf{c})$ is too strong to be a realistic benchmark, at least for general set systems. In particular, it can be smaller than the total cost of the cheapest solution (see Section 6). However, we provide the definition and some results about $\text{TUmin}(\mathbf{c})$ in the paper, both for completeness and because we believe that it may help to understand which properties of the payment bounds are important for our proofs. Another possibility would be to introduce an additional constraint $\sum_{e \in S} b_e \geq \sum_{e \in S} c_e$ in the definition of $\text{TUmin}(\mathbf{c})$ (note that this condition holds automatically for $\text{TUmax}(\mathbf{c})$, as $\text{TUmax}(\mathbf{c}) \geq \text{NTUmax}(\mathbf{c})$); however, such definition would have no direct economic interpretation, and some of our results (in particular, the ones in Section 4) would no longer be true.

Remark 1. *For the payment bounds that are derived from maximisation problems, i.e., $\text{TUmax}(\mathbf{c})$ and $\text{NTUmax}(\mathbf{c})$, constraints of type (3) are redundant and can be dropped. Hence, $\text{TUmax}(\mathbf{c})$ and $\text{NTUmax}(\mathbf{c})$ are solutions to linear programs, and therefore can be computed in polynomial time as long as we have a separation oracle for constraints in (2). This is in sharp contrast with $\text{NTUmin}(\mathbf{c})$, which can be NP-hard to compute even if the number of constraints in \mathcal{F} is polynomial (see Section 6).*

The first and third inequalities in the following observation follow from the fact that condition (1^*) is strictly weaker than condition (1) .

Observation 2.

$$\text{TUmin}(\mathbf{c}) \leq \text{NTUmin}(\mathbf{c}) \leq \text{NTUmax}(\mathbf{c}) \leq \text{TUmax}(\mathbf{c}),$$

Let \mathcal{M} be a truthful mechanism for $(\mathcal{E}, \mathcal{F})$. Let $p_{\mathcal{M}}(c)$ denote the total payments of \mathcal{M} when the actual costs are c . A *frugality ratio* of \mathcal{M} with respect to a payment bound is the ratio between the payment of \mathcal{M} and this payment bound. In particular, $\phi_{\text{TUmin}}(\mathcal{M}) = \frac{\sup_c p_{\mathcal{M}}(c)}{\text{TUmin}(\mathbf{c})}$, $\phi_{\text{TUmax}}(\mathcal{M}) = \frac{\sup_c p_{\mathcal{M}}(c)}{\text{TUmax}(\mathbf{c})}$, $\phi_{\text{NTUmin}}(\mathcal{M}) = \frac{\sup_c p_{\mathcal{M}}(c)}{\text{NTUmin}(\mathbf{c})}$, and $\phi_{\text{NTUmax}}(\mathcal{M}) = \frac{\sup_c p_{\mathcal{M}}(c)}{\text{NTUmax}(\mathbf{c})}$.

We conclude this section by showing that there exist set systems and respective cost vectors, for which all four payment bounds are different. In the next section, we quantify this difference, both for general set systems, and for specific problems, such as path auctions or vertex cover.

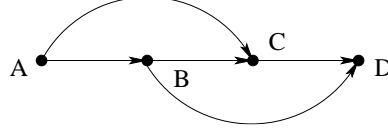


Figure 1: The diamond graph

Example 3. Consider the shortest-path auction on the graph of Figure 1. The feasible sets are all paths from A to D. It can be verified, using the reasoning of Claims 4 and 5 below, that for the cost vector $c_{AB} = c_{CD} = 2$, $c_{BC} = 1$, $c_{AC} = c_{BD} = 5$, we have

- $\text{TUmax}(\mathbf{c}) = 10$ (with the bid vector $b_{AB} = b_{CD} = 5$, $b_{BC} = 0$),
- $\text{NTUmax}(\mathbf{c}) = 9$ (with the bid vector $b_{AB} = b_{CD} = 4$, $b_{BC} = 1$),
- $\text{NTUmin}(\mathbf{c}) = 7$ (with the bid vector $b_{AB} = b_{CD} = 2$, $b_{BC} = 3$),
- $\text{TUmin}(\mathbf{c}) = 5$ (with the bid vector $b_{AB} = b_{CD} = 0$, $b_{BC} = 5$).

4 Comparing payment bounds

4.1 Path auctions

In *path auctions*, feasible sets are paths between two specified vertices in a graph. We show that for this problem, any two payment bounds can differ by a factor of 2.

Claim 4. *There is a shortest-path auction for which $\text{NTUmax}(\mathbf{c})/\text{NTUmin}(\mathbf{c}) \geq 2$.*

Proof. This construction is due to David Kempe [15]. Consider the graph of Figure 1 with the edge costs $c_{AB} = c_{BC} = c_{CD} = 0$, $c_{AC} = c_{BD} = 1$. Under these costs, ABCD is the cheapest path. The inequalities in (2) are $b_{AB} + b_{BC} \leq c_{AC} = 1$, $b_{BC} + b_{CD} \leq c_{BD} = 1$. By condition (3), both of these inequalities must be tight (the former one is the only inequality involving b_{AB} , and the latter one is the only inequality involving b_{CD}). The inequalities in (1) are $b_{AB} \geq 0$, $b_{BC} \geq 0$, $b_{CD} \geq 0$. Now, if the goal is to maximise $b_{AB} + b_{BC} + b_{CD}$, the best choice is $b_{AB} = b_{CD} = 1$, $b_{BC} = 0$, so $\text{NTUmax}(\mathbf{c}) = 2$. On the other hand, if the goal is to minimise $b_{AB} + b_{BC} + b_{CD}$, one should set $b_{AB} = b_{CD} = 0$, $b_{BC} = 1$, so $\text{NTUmin}(\mathbf{c}) = 1$. \square

Claim 5. *There is a shortest-path auction for which $\text{TUmax}(\mathbf{c})/\text{NTUmax}(\mathbf{c}) \geq 2$.*

Proof. Again, consider the graph of Figure 1. Let the edge costs be $c_{AB} = c_{CD} = 0$, $c_{BC} = 1$, $c_{AC} = c_{BD} = 1$. ABCD is the lexicographically-least cheapest path, so we can assume that $S = \{AB, BC, CD\}$. The inequalities in (2) are the same as in the previous example, and by the same argument both of them are, in fact, equalities. The inequalities in (1) are $b_{AB} \geq 0$, $b_{BC} \geq 1$, $b_{CD} \geq 0$. Our goal is to maximise $b_{AB} + b_{BC} + b_{CD}$. If we have to respect the inequalities in (1), we have to set $b_{AB} = b_{CD} = 0$, $b_{BC} = 1$, so $\text{NTUmax}(\mathbf{c}) = 1$. Otherwise, we can set $b_{AB} = b_{CD} = 1$, $b_{BC} = 0$, so $\text{TUmax}(\mathbf{c}) \geq 2$. \square

Claim 6. *There is a shortest-path auction for which $\text{NTUmin}(\mathbf{c})/\text{TUmin}(\mathbf{c}) \geq 2$.*

Proof. This construction is also based on the graph of Figure 1. The edge costs are $c_{AB} = c_{CD} = 1$, $c_{BC} = 0$, $c_{AC} = c_{BD} = 1$. $ABCD$ is the lexicographically least cheapest path, so we can assume that $S = \{AB, BC, CD\}$. Again, the inequalities in (2) are the same, and both are, in fact, equalities. The inequalities in (1) are $b_{AB} \geq 1$, $b_{BC} \geq 0$, $b_{CD} \geq 1$. Our goal is to minimise $b_{AB} + b_{BC} + b_{CD}$. If we have to respect the inequalities in (1), we have to set $b_{AB} = b_{CD} = 1$, $b_{BC} = 0$, so $\text{NTUmin}(\mathbf{c}) = 2$. Otherwise, we can set $b_{AB} = b_{CD} = 0$, $b_{BC} = 1$, so $\text{TUmin}(\mathbf{c}) \leq 1$. \square

In Section 4.4, we show that the separation results in Claim 4, 5, and 6 are optimal.

4.2 Connections between separation results

Observe that all of the separation results of the previous subsection are obtained on the same graph using very similar cost vectors. It turns out that this is not coincidental: if for some set system and a vector of costs \mathbf{c} we can separate $\text{TUmax}(\mathbf{c})$ from $\text{NTUmax}(\mathbf{c})$ by a factor of α , then for the same set system there exists a vector of costs \mathbf{c}' that separates $\text{NTUmax}(\mathbf{c}')$ from $\text{NTUmin}(\mathbf{c}')$ by the same factor, and vice versa. Similarly, separating $\text{NTUmax}(\mathbf{c})$ from $\text{NTUmin}(\mathbf{c})$ is equivalent to separating $\text{NTUmin}(\mathbf{c}')$ from $\text{TUmin}(\mathbf{c}')$. More precisely, we can prove the following theorems.

Theorem 7. *For any set system $(\mathcal{E}, \mathcal{F})$,*

$$\max_{\mathbf{c}} \frac{\text{TUmax}(\mathbf{c})}{\text{NTUmax}(\mathbf{c})} = \max_{\mathbf{c}} \frac{\text{NTUmax}(\mathbf{c})}{\text{NTUmin}(\mathbf{c})}.$$

The proof of the theorem follows directly from two lemmas.

Lemma 8. *If there is a cost vector \mathbf{c} for $(\mathcal{E}, \mathcal{F})$ such that $\text{TUmax}(\mathbf{c})/\text{NTUmax}(\mathbf{c}) = \alpha$, then there is another cost vector \mathbf{c}' such that $\text{NTUmax}(\mathbf{c}')/\text{NTUmin}(\mathbf{c}') \geq \alpha$.*

Proof. Suppose that there is a cost vector \mathbf{c} such that $\text{TUmax}(\mathbf{c}) = X$, $\text{NTUmax}(\mathbf{c}) = Y$, $X/Y = \alpha$. Assume without loss of generality that the winning set S consists of elements $1, \dots, k$, and let $\mathbf{b}^1 = (b_1^1, \dots, b_k^1)$ and $\mathbf{b}^2 = (b_1^2, \dots, b_k^2)$ be the bid vectors that correspond to $\text{TUmax}(\mathbf{c})$ and $\text{NTUmax}(\mathbf{c})$, respectively.

Construct the cost vector \mathbf{c}' by setting $c'_i = c_i$ for $i \notin S$, $c'_i = \min\{c_i, b_i^1\}$ for $i \in S$. Clearly, S is a cheapest set under \mathbf{c}' . Moreover, as the costs of elements outside of S remained the same, the right-hand sides of all constraints in (2) did not change, so any bid vector that satisfies (2) and (3) with respect to \mathbf{c} , also satisfies them with respect to \mathbf{c}' . We will construct two bid vectors \mathbf{b}^3 and \mathbf{b}^4 that satisfy conditions (1), (2), and (3) for the cost vector \mathbf{c}' , and have $\sum_{i \in S} b_i^3 = X$, $\sum_{i \in S} b_i^4 = Y$. As $\text{NTUmax}(\mathbf{c}') \geq X$ and $\text{NTUmin}(\mathbf{c}') \leq Y$, this implies the lemma.

We can set $b_i^3 = b_i^1$: this bid vector satisfies conditions (2) and (3) since \mathbf{b}^1 does, and we have $b_i^1 \geq \min\{c_i, b_i^1\} = c'_i$, which means that \mathbf{b}^3 satisfies condition (1). Furthermore, we can set $b_i^4 = b_i^2$. Again, \mathbf{b}^4 satisfies conditions (2) and (3) since \mathbf{b}^2 does, and since \mathbf{b}^2 satisfies condition (1), we have $b_i^2 \geq c_i \geq c'_i$, which means that \mathbf{b}^4 satisfies condition (1). \square

Lemma 9. *If there is a cost vector \mathbf{c} for $(\mathcal{E}, \mathcal{F})$ such that $\text{NTUmax}(\mathbf{c})/\text{NTUmin}(\mathbf{c}) = \alpha$, then there is another cost vector \mathbf{c}' such that $\text{TUmax}(\mathbf{c}')/\text{NTUmax}(\mathbf{c}') \geq \alpha$.*

Proof. Suppose that there is a cost vector \mathbf{c} such that $\text{NTUmax}(\mathbf{c}) = X$, $\text{NTUmin}(\mathbf{c}) = Y$, $X/Y = \alpha$. Again, assume that the winning set S consists of elements $1, \dots, k$, and let $\mathbf{b}^1 = (b_1^1, \dots, b_k^1)$ and $\mathbf{b}^2 = (b_1^2, \dots, b_k^2)$ be the bid vectors that correspond to $\text{NTUmax}(\mathbf{c})$ and $\text{NTUmin}(\mathbf{c})$, respectively.

Construct the cost vector \mathbf{c}' by setting $c'_i = c_i$ for $i \notin S$, $c'_i = b_i^2$ for $i \in S$. As \mathbf{b}^2 satisfies condition (2), S is a cheapest set under \mathbf{c}' . As in the previous construction, the right-hand sides of all constraints in (2) did not change. Let \mathbf{b}^3 be a bid vector that corresponds to $\text{NTUmax}(\mathbf{c}')$. Let us prove that $\text{NTUmax}(\mathbf{c}') = \sum_{i \in S} b_i^3 = Y$. Indeed, the bid vector \mathbf{b}^3 must satisfy $b_i^3 \geq c'_i = b_i^2$ for $i = 1, \dots, k$ (condition (1)). Suppose that $b_i^3 > c'_i$ for some $i = 1, \dots, k$, and consider the constraint in (2) that is tight for b_i^2 . There is such a constraint, as \mathbf{b}^2 satisfies condition (3). Namely, for some T not containing i ,

$$\sum_{j \in S \setminus T} b_j^2 = \sum_{j \in T \setminus S} c_j.$$

For every j appearing in the left-side of this constraint, we have $b_j^3 \geq b_j^2$ but $b_i^3 > b_i^2$, so the bid vector \mathbf{b}^3 violates this constraint. Hence, $b_i^3 = c'_i = b_i^2$ for all i and therefore $\text{NTUmax}(\mathbf{c}') = \sum_{i \in S} b_i^3 = Y$.

On the other hand, we can construct a bid vector \mathbf{b}^4 that satisfies conditions (2) and (3) with respect to \mathbf{c}' and has $\sum_{i \in S} b_i^4 = X$. Namely, we can set $b_i^4 = b_i^1$: as \mathbf{b}^1 satisfies conditions (2) and (3), so does \mathbf{b}^4 . As $\text{TUmax}(\mathbf{c}') \geq \sum_{i \in S} b_i^4$, this proves the lemma. \square

Theorem 10. *For any set system $(\mathcal{E}, \mathcal{F})$,*

$$\max_{\mathbf{c}} \frac{\text{NTUmax}(\mathbf{c})}{\text{NTUmin}(\mathbf{c})} = \max_{\mathbf{c}} \frac{\text{NTUmin}(\mathbf{c})}{\text{TUmin}(\mathbf{c})}.$$

Again, the proof of the theorem follows from two lemmas.

Lemma 11. *If there is a cost vector \mathbf{c} for $(\mathcal{E}, \mathcal{F})$ such that $\text{NTUmax}(\mathbf{c})/\text{NTUmin}(\mathbf{c}) = \alpha$, then there is another cost vector \mathbf{c}' such that $\text{NTUmin}(\mathbf{c}')/\text{TUmin}(\mathbf{c}') \geq \alpha$.*

Proof. Suppose that there is a cost vector \mathbf{c} such that $\text{NTUmax}(\mathbf{c}) = X$, $\text{NTUmin}(\mathbf{c}) = Y$, $X/Y = \alpha$. Again, assume that the winning set S consists of elements $1, \dots, k$, and let $\mathbf{b}^1 = (b_1^1, \dots, b_k^1)$ and $\mathbf{b}^2 = (b_1^2, \dots, b_k^2)$ be the bid vectors that correspond to $\text{NTUmax}(\mathbf{c})$ and $\text{NTUmin}(\mathbf{c})$, respectively.

The cost vector \mathbf{c}' is obtained by setting $c'_i = c_i$ for $i \notin S$, $c'_i = b_i^1$ for $i \in S$. Since \mathbf{b}^1 satisfies condition (2), S is a cheapest set under \mathbf{c}' , and the right-hand sides of all constraints in (2) did not change.

Let \mathbf{b}^3 be a bid vector that corresponds to $\text{NTUmin}(\mathbf{c}')$. It is easy to see that $\text{NTUmin}(\mathbf{c}') = \sum_{i \in S} b_i^3 = X$, since the bid vector \mathbf{b}^3 must satisfy $b_i^3 \geq c'_i = b_i^1$ for $i = 1, \dots, k$ (condition (1)), and $\sum_{i \in S} b_i^1 = \text{NTUmax}(\mathbf{c}) = X$. On the other hand, we can construct a bid vector \mathbf{b}^4 that satisfies conditions (2) and (3) with respect to \mathbf{c}' and has $\sum_{i \in S} b_i^4 = Y$. Namely, we can set $b_i^4 = b_i^2$: as \mathbf{b}^2 satisfies conditions (2) and (3), so does \mathbf{b}^4 . As $\text{TUmin}(\mathbf{c}') \leq \sum_{i \in S} b_i^4$, this proves the lemma. \square

Lemma 12. *If there is a cost vector \mathbf{c} for $(\mathcal{E}, \mathcal{F})$ such that $\text{NTUmin}(\mathbf{c})/\text{TUmin}(\mathbf{c}) = \alpha$, then there is another cost vector \mathbf{c}' such that $\text{NTUmax}(\mathbf{c}')/\text{NTUmin}(\mathbf{c}') \geq \alpha$.*

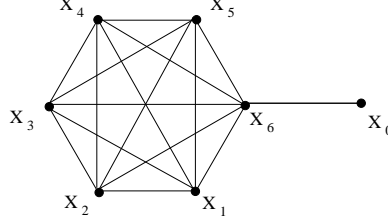


Figure 2: Graph that separates payment bounds for vertex cover, $n = 7$

Proof. Suppose that there is a cost vector \mathbf{c} such that $\text{NTUmin}(\mathbf{c}) = X$, $\text{TUmin}(\mathbf{c}) = Y$, $X/Y = \alpha$. Again, assume that the winning set S consists of elements $1, \dots, k$, and let $\mathbf{b}^1 = (b_1^1, \dots, b_k^1)$ and $\mathbf{b}^2 = (b_1^2, \dots, b_k^2)$ be the bid vectors that correspond to $\text{NTUmin}(\mathbf{c})$ and $\text{TUmin}(\mathbf{c})$, respectively.

Construct the cost vector \mathbf{c}' by setting $c'_i = c_i$ for $i \notin S$, $c'_i = \min\{c_i, b_i^2\}$ for $i \in S$. Clearly, S is a cheapest set under \mathbf{c}' . Moreover, as the costs of elements outside of S remained the same, the right-hand sides of all constraints in (2) did not change.

We will construct two bid vectors \mathbf{b}^3 and \mathbf{b}^4 that satisfy conditions (1), (2), and (3) for the cost vector \mathbf{c}' , and have $\sum_{i \in S} b_i^3 = X$, $\sum_{i \in S} b_i^4 = Y$. As $\text{NTUmax}(\mathbf{c}') \geq X$ and $\text{NTUmin}(\mathbf{c}') \leq Y$, this implies the lemma.

We can set $b_i^3 = b_i^1$. Indeed, the vector \mathbf{b}^3 satisfies conditions (2) and (3) since \mathbf{b}^1 does. Also, since \mathbf{b}^1 satisfies condition (1), we have $b_i^1 \geq c_i \geq c'_i$, i.e., \mathbf{b}^3 satisfies condition (1) with respect to \mathbf{c}' . On the other hand, we can set $b_i^4 = b_i^2$: the vector \mathbf{b}^4 satisfies conditions (2) and (3) since \mathbf{b}^2 does, and it satisfies condition (1), since $b_i^2 \geq c'_i$. \square

4.3 Vertex-Cover auctions

In *vertex-cover auctions*, we are given a graph $G = (V, E)$, $|V| = n$, and feasible sets are sets of vertices that cover all edges. In contrast to the case of path auctions, the gap between $\text{NTUmin}(\mathbf{c})$ and $\text{NTUmax}(\mathbf{c})$ (and hence between $\text{NTUmax}(\mathbf{c})$ and $\text{TUmax}(\mathbf{c})$, and between $\text{TUmin}(\mathbf{c})$ and $\text{NTUmin}(\mathbf{c})$) can be proportional to the size of the graph.

Claim 13. *For any $n \geq 3$, there is a vertex-cover auction on an n -vertex graph for which $\text{TUmax}(\mathbf{c})/\text{NTUmax}(\mathbf{c}) \geq n - 2$.*

Proof. The underlying graph consists of an $(n - 1)$ -clique with the vertex set X_1, \dots, X_{n-1} , and an extra vertex X_0 adjacent to X_{n-1} . The costs are $c_{X_1} = c_{X_2} = \dots = c_{X_{n-2}} = 0$, $c_{X_0} = c_{X_{n-1}} = 1$. We can assume that $S = \{X_0, X_1, \dots, X_{n-2}\}$ (this is the lexicographically first vertex cover of cost 1). For this set system, the constraints in (2) are $b_{X_i} + b_{X_0} \leq c_{X_{n-1}} = 1$ for $i = 1, \dots, n - 2$. Clearly, we can satisfy conditions (2) and (3) by setting $b_{X_i} = 1$ for $i = 1, \dots, n - 2$, $b_{X_0} = 0$. Hence, $\text{TUmax}(\mathbf{c}) \geq n - 2$. For $\text{NTUmax}(\mathbf{c})$, there is an additional constraint $b_{X_0} \geq 1$, so the best we can do is to set $b_{X_i} = 0$ for $i = 1, \dots, n - 2$, $b_{X_0} = 1$, which implies $\text{NTUmax}(\mathbf{c}) = 1$. \square

Combining Claim 13 with Lemmas 8 and 11, we derive the following corollaries.

Corollary 14. *For any $n \geq 3$, there is a vertex-cover auction on an n -vertex graph for which $\text{NTUmax}(\mathbf{c})/\text{NTUmin}(\mathbf{c}) \geq n - 2$.*

Corollary 15. *For any $n \geq 3$, there is a vertex-cover auction on an n -vertex graph for which $\text{NTUmin}(\mathbf{c})/\text{TUmin}(\mathbf{c}) \geq n - 2$.*

4.4 Upper bounds

We can show an upper bound for *all* set systems that matches the lower bounds proven in the previous subsection. Namely, we prove that the gap between $\text{TUmax}(\mathbf{c})$ and $\text{TUmin}(\mathbf{c})$ is at most n ; as $\text{TUmin}(\mathbf{c}) \leq \text{NTUmin}(\mathbf{c}) \leq \text{NTUmax}(\mathbf{c}) \leq \text{TUmin}(\mathbf{c})$, this bound applies to any pair of payment bounds.

Theorem 16. *For any set system and any cost vector \mathbf{c} , we have $\text{TUmax}(\mathbf{c})/\text{TUmin}(\mathbf{c}) \leq n$.*

Proof. Assume without loss of generality that the winning set S consists of elements $1, \dots, k$. Let c_1, \dots, c_k be the true costs of elements in S , let b'_1, \dots, b'_k be their bids that correspond to $\text{TUmin}(\mathbf{c})$, and let b''_1, \dots, b''_k be their bids that correspond to $\text{TUmax}(\mathbf{c})$.

Consider the conditions (2) and (3) for S . One can pick a subset \mathcal{L} of at most k inequalities in (2) so that for each $i = 1, \dots, k$ there is at least one inequality in \mathcal{L} that is tight for b'_i . Suppose that the j th inequality in \mathcal{L} is of the form

$$b_{i_1} + \dots + b_{i_t} \leq c(T_j \setminus S).$$

For b'_i , all inequalities in \mathcal{L} are, in fact equalities. Hence, by adding up all of them we obtain

$$k \sum_{i=1, \dots, k} b'_i \geq \sum_{j=1, \dots, k} c(T_j \setminus S).$$

On the other hand, all these inequalities appear in condition (2), so they must hold for b''_i , i.e.,

$$\sum_{i=1, \dots, k} b''_i \leq \sum_{j=1, \dots, k} c(T_j \setminus S).$$

Combining these two inequalities shows that $n\text{TUmin}(\mathbf{c}) \geq k\text{TUmin}(\mathbf{c}) \geq \text{TUmax}(\mathbf{c})$. \square

Remark 17. *The final line of the proof of Theorem 16 demonstrates that the upper bound can be strengthened to the size of the winning set, k . Note that in Claim 13, as well as in Corollaries 14 and 15, the size of the winning set is $n - 1$, so these results do not contradict each other.*

For path auctions, this upper bound can be improved to 2, matching the lower bounds of Section 4.1.

Theorem 18. *For any network (G, s, t) and any cost vector \mathbf{c} , $\text{TUmax}(\mathbf{c}) \leq 2\text{TUmin}(\mathbf{c})$.*

Proof. Assume without loss of generality that the lexicographically-least cheapest s - t path, P , in G is $\{e_1, \dots, e_k\}$, where $e_1 = (s, v_1), e_2 = (v_1, v_2), \dots, e_k = (v_{k-1}, t)$. To simplify notation, let c_1, \dots, c_k be the true costs of e_1, \dots, e_k , let $\mathbf{b}' = (b'_1, \dots, b'_k)$ be a bid vector that corresponds to $\text{TUmin}(\mathbf{c})$, and let $\mathbf{b}'' = (b''_1, \dots, b''_k)$ be a bid vector that correspond to $\text{TUmax}(\mathbf{c})$.

For any $i = 1, \dots, k$, there is a constraint in (2) that is tight for b'_i with respect to the bid vector \mathbf{b}' , i.e., an s - t path P_i that avoids e_i and satisfies

$$b(P \setminus P_i) = c(P_i \setminus P).$$

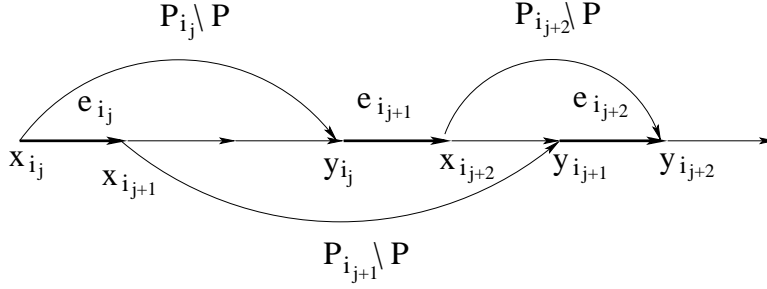


Figure 3: Proof of Theorem 18: constraints for \hat{P}_{i_j} and $\hat{P}_{i_{j+2}}$ do not overlap

We can assume without loss of generality that P_i coincides with P up to some vertex x_i , then deviates from P to avoid e_i , and finally returns to P at a vertex y_i and coincides with P from then on (clearly, it might happen that $s = x_i$ or $t = y_i$). Indeed, if P_i deviates from P more than once, one of these deviations is not necessary to avoid e_i and can be replaced with the respective segment of P without increasing the cost of P_i . Among all paths of this form, let \hat{P}_i the one with the largest value of y_i , i.e., the “rightmost” one. This path corresponds to an inequality I_i of the form

$$b'_{x_{i+1}} + \dots + b'_{y_i} \leq c(\hat{P}_i \setminus P).$$

As in the proof of Theorem 16, we construct a set of tight constraints \mathcal{L} such that every variable b'_i appears in at least one of these constraints. \mathcal{L} is constructed inductively as follows. Start by setting $\mathcal{L} = \{I_1\}$. At the j th step, suppose that all variables up to (but not including) b'_{i_j} appear in at least one inequality in \mathcal{L} . Add I_{i_j} to \mathcal{L} .

Note that for any j we have $y_{i_{j+1}} > y_{i_j}$. This is because the inequalities added to \mathcal{L} during steps 1– j did not cover $b'_{i_{j+1}}$. See Figure 3. Since $y_{i_{j+2}} > y_{i_{j+1}}$, we must also have $x_{i_{j+2}} > y_{i_j}$: otherwise, $\hat{P}_{i_{j+1}}$ would not be the “rightmost” constraint for $b'_{i_{j+1}}$. Therefore, the variables in $I_{i_{j+2}}$ and I_{i_j} , do not overlap, and hence no b'_i can appear in more than two inequalities in \mathcal{L} .

Now we follow the argument of the proof of Theorem 16 to finish. By adding up all of the (tight) inequalities in \mathcal{L} for b'_i we obtain

$$2 \sum_{i=1, \dots, k} b'_i \geq \sum_{j=1, \dots, k} c(\hat{P}_j \setminus P).$$

On the other hand, all these inequalities appear in condition (2), so they must hold for b''_i , i.e.,

$$\sum_{i=1, \dots, k} b''_i \leq \sum_{j=1, \dots, k} c(\hat{P}_j \setminus P),$$

so $\text{TUmax}(\mathbf{c}) \leq 2\text{TUmin}(\mathbf{c})$. □

5 Truthful mechanisms for vertex cover

Recall that for a vertex-cover auction on a graph $G = (V, E)$, an *allocation rule* is an algorithm that takes as input a bid b_v for each vertex and returns a vertex cover \hat{S} of G . We will assume

that G is connected, since a separate auction can be run on each connected component.

As explained in Section 2, we can combine a monotone allocation rule with threshold payments to obtain a truthful auction.

An allocation rule is *locally optimal* if the following is true: if $b_v > \sum_{w \sim v} b_w$, then vertex v will not be chosen.

The *local ratio* approximation algorithm of [2] is monotone and is locally optimal as long as the edges are considered in a fixed order; its approximation ratio is 2. So are the greedy approximation algorithm, whose approximation ratio is $\log n$, and any algorithm that computes the optimal vertex cover. Hence, any of them can be used as a basis for a truthful auction.

We show in this section that any locally optimal and monotone allocation algorithm for vertex cover yields an auction \mathcal{M} with frugality ratio $\phi_{\text{NTUmin}}(\mathcal{M}) \leq 2\Delta$, where Δ is the maximum degree of the graph. The result naturally extends to the smaller frugality ratios that we consider, i.e., $\phi_{\text{NTUmax}}(\mathcal{M})$ and $\phi_{\text{TUmax}}(\mathcal{M})$. It does not extend to the larger frugality ratio $\phi_{\text{TUmin}}(\mathcal{M})$ because the bound $\text{TUmin}(\mathbf{c})$ is inappropriately low – we show in Section 6.1 that $\text{TUmin}(\mathbf{c})$ can be smaller than the total cost of the smallest vertex cover. We start with the following lemma.

Lemma 19. *Consider a graph $G = (V, E)$ with maximum degree Δ . Let \mathcal{M} be a vertex-cover auction on G that has a locally optimal and monotone allocation rule and pays each agent its threshold bid. Then for any cost vector \mathbf{c} , the total payment of \mathcal{M} satisfies $p_{\mathcal{M}}(\mathbf{c}) \leq \Delta(c(V))$.*

Proof. First note that any such auction is truthful, so we can assume that each agent's bid is equal to its cost. Let \hat{S} be the vertex cover selected by the allocation rule. Note that

$$p_{\mathcal{M}}(\mathbf{c}) = \sum_{v \in \hat{S}} t_v \leq \sum_{v \in \hat{S}} \sum_{w \sim v} c_w \leq \sum_{w \in V} \Delta c_w = \Delta c(V).$$

□

We now give some lower bounds on the payment bounds, building on each other.

Lemma 20. *For a vertex cover instance $G = (V, E)$ in which S is the lexicographically-least minimum vertex cover, $\text{TUmax}(\mathbf{c}) \geq c(V \setminus S)$*

Proof. For a vertex w with at least one neighbour in S , let $d(w)$ denote the number of neighbours that w has in S . Consider the bid vector \mathbf{b} in which, for each $v \in S$, $b_v = \sum_{w \sim v, w \notin S} c_w / d(w)$. Then $\sum_{v \in S} b_v = \sum_{v \in S} \sum_{w \sim v, w \notin S} c_w / d(w) = \sum_{w \notin S} c_w = c(V \setminus S)$. To finish we want to show that \mathbf{b} is feasible in the sense that it satisfies (2). Consider a vertex cover T , and extend the bid vector \mathbf{b} by assigning $b_v = c_v$ for $v \notin S$. Then

$$b(T) = c(T \setminus S) + b(S \cap T) \geq c(T \setminus S) + \sum_{v \in S \cap T} \sum_{w \in \bar{S} \cap \bar{T}: w \sim v} c_w / d(w),$$

and since all edges between $\bar{S} \cap \bar{T}$ and S go to $S \cap T$, the right-hand-side is equal to

$$c(T \setminus S) + \sum_{w \in \bar{S} \cap \bar{T}} c_w = c(T \setminus S) + c(\bar{S} \cap \bar{T}) = c(V \setminus S) = b(S).$$

□

Lemma 21. *For a vertex cover instance $G = (V, E)$ in which S is the lexicographically-least minimum vertex cover, $\text{NTUmax}(\mathbf{c}) \geq c(V \setminus S)$*

Proof. If $c(S) \geq c(V \setminus S)$, by condition (1) we are done. Therefore, for the rest of the proof we assume that $c(S) < c(V \setminus S)$. We show how to construct a bid vector $(b_e)_{e \in S}$ that satisfies conditions (1) and (2) such that $b(S) \geq c(V \setminus S)$; clearly, this implies $\text{NTUmax}(\mathbf{c}) \geq c(V \setminus S)$.

Recall that a network flow problem is described by a directed graph $\Gamma = (V_\Gamma, E_\Gamma)$, a source node $s \in V_\Gamma$, a sink node $t \in V_\Gamma$, and a vector of capacity constraints $a_e, e \in E_\Gamma$. Consider a network (V_Γ, E_Γ) such that $V_\Gamma = V \cup \{s, t\}$, $E_\Gamma = E_1 \cup E_2 \cup E_3$, where $E_1 = \{(s, v) \mid v \in S\}$, $E_2 = \{(v, w) \mid v \in S, w \in V \setminus S, (v, w) \in E\}$, $E_3 = \{(w, t) \mid w \in V \setminus S\}$. Observe that the graph (V, E_2) is obtained from G by deleting the edges that have both endpoints in S .

Set the capacity constraints for $e \in E_\Gamma$ as follows: $a_{(s,v)} = c_v$, $a_{(w,t)} = c_w$, $a_{(v,w)} = +\infty$ for all $v \in S, w \in V \setminus S$.

Recall that a *cut* is a partition of the vertices in V_Γ into two sets C_1 and C_2 so that $s \in C_1, t \in C_2$; we denote such cut by $C = (C_1, C_2)$. Abusing notation, we write $e = (u, v) \in C$ if $u \in C_1, v \in C_2$ or $u \in C_2, v \in C_1$, and say that an edge $e = (u, v)$ *crosses* the cut C . The *size* of a cut C is computed as $s(C) = \sum_{(v,w) \in C} a_{(v,w)}$.

Let $C_{\min} = (\{s\} \cup S' \cup W', \{t\} \cup S'' \cup W'')$ be a minimum cut in Γ , where $S', S'' \subseteq S, W', W'' \subseteq V \setminus S$. As we can upperbound $s(C)$ by $c(S) < +\infty$, and any edge in E_2 has infinite capacity, no edge $(u, v) \in E_2$ crosses C .

Consider the network $\Gamma' = (V_{\Gamma'}, E_{\Gamma'})$, where $V_{\Gamma'} = \{s\} \cup S' \cup W' \cup \{t\}$, $E_{\Gamma'} = \{(u, v) \in E_\Gamma \mid u, v \in V_{\Gamma'}\}$. Clearly, $C' = (\{s\} \cup S' \cup W', \{t\})$ is a minimum cut in Γ' (otherwise, there would exist a smaller cut for Γ). As $s(C') = c(W')$, we have $c(S') \geq c(W')$.

Now, consider the network $\Gamma'' = (V_{\Gamma''}, E_{\Gamma''})$, where $V_{\Gamma''} = \{s\} \cup S'' \cup W'' \cup \{t\}$, $E_{\Gamma''} = \{(u, v) \in E_\Gamma \mid u, v \in V_{\Gamma''}\}$. Similarly, $C'' = (\{s\}, S'' \cup W'' \cup \{t\})$ is a minimum cut in Γ'' , $s(C'') = c(S'')$. As the size of a maximum flow from s to t is equal to that of a minimum cut separating s and t , there exists a flow $\mathcal{F} = (f_e)_{e \in E_{\Gamma''}}$ of size $c(S'')$. This flow has to saturate all edges between s and S'' , i.e., $f_{(s,v)} = c_v$ for all $v \in S''$. Now, increase the capacities of all edges between s and S'' to $+\infty$. In the modified network, the size of a minimum cut (and hence a maximum flow) is $c(W'')$, and a maximum flow $\mathcal{F}' = (f'_e)_{e \in E_{\Gamma''}}$ can be constructed by greedily augmenting \mathcal{F} .

Set $b_v = c_v$ for all $v \in S'$, $b_v = f'_{(s,v)}$ for all $v \in S''$. As \mathcal{F}' is constructed by augmenting \mathcal{F} , we have $b_v \geq c_v$ for all $v \in S$, i.e., condition (1) is satisfied.

Now, let us check that no vertex cover $T \subseteq V$ can violate condition (2). Set $T_1 = T \cap S'$, $T_2 = T \cap S''$, $T_3 = T \cap W'$, $T_4 = T \cap W''$; our goal is to show that $b(S' \setminus T_1) + b(S'' \setminus T_2) \leq c(T_3) + c(T_4)$.

Consider all edges $(u, v) \in E$ such that $u \in S' \setminus T_1$. If $(u, v) \in E_2$ then $v \in T_3$ (no edge in E_2 can cross the cut), and if $u, v \in S$ then $v \in T_1 \cup T_2$. Hence, $T_1 \cup T_3 \cup S''$ is a vertex cover for G , and therefore $c(T_1) + c(T_3) + c(S'') \geq c(S) = c(T_1) + c(S' \setminus T_1) + c(S'')$. Consequently, $c(T_3) \geq c(S' \setminus T_1) = b(S' \setminus T_1)$.

Now, consider the vertices in $S'' \setminus T_2$. Any edge in E_2 that starts in one of these vertices has to end in T_4 (this edge has to be covered by T , and it cannot go across the cut). Therefore, the total flow out of $S'' \setminus T_2$ is at most the total flow out of T_4 , i.e., $b(S'' \setminus T_2) \leq c(T_4)$.

Hence, $b(S' \setminus T_1) + b(S'' \setminus T_2) \leq c(T_3) + c(T_4)$. \square

Lemma 22. *For a vertex cover instance $G = (V, E)$ in which S is the lexicographically-least minimum vertex cover, $\text{NTUmin}(\mathbf{c}) \geq c(V \setminus S)$*

Proof. Suppose for contradiction that c is a cost vector with $\text{NTUmin}(\mathbf{c}) < c(V \setminus S)$. Let b be the corresponding bid vector and let c' be a new cost vector with $c'_e = b_e$ for $e \in S$ and $c'_e = c_e$ for $e \notin S$. Equation (2) in the LP guarantees that S is an optimal solution to the cost vector c' . Now compute a bid vector b' corresponding to $\text{NTUmax}(\mathbf{c}')$. We claim that $b'_e = c'_e$ for any $e \in S$. Indeed, suppose that $b'_e > c'_e$ for some $e \in S$ ($b'_e = c'_e$ for $e \notin S$ by construction). As b satisfies conditions (1)–(3), among the inequalities in (2) there is one that is tight for e and the bid vector b . That is, $b(S \setminus T) = c(T \setminus S)$. By the construction of c' , $c'(S \setminus T) = c'(T \setminus S)$. Now since $b'_f \geq c'_f$ for all $f \in S$, $b'_e > c'_e$ implies $b'(S \setminus T) > c'(S \setminus T) = c'(T \setminus S)$. But this violates (2). So we now know $b' = c'$. Hence, we have $\text{NTUmax}(\mathbf{c}') = \sum_{e \in S} b_e = \text{NTUmin}(\mathbf{c}) < c(V \setminus S)$, giving a contradiction to the fact that $\text{NTUmax}(\mathbf{c}') \geq c'(V \setminus S)$ which we proved in Lemma 21. \square

The following observation follows by from constraint (1).

Observation 23. *For a vertex cover instance $G = (V, E)$ in which S is the lexicographically-least minimum vertex cover, $\text{NTUmin}(\mathbf{c}) \geq c(S)$*

Lemma 22 and Observation 23 imply that $\text{NTUmin}(\mathbf{c}) \geq c(V)/2$. Combining this with Lemma 19 we get the following corollary.

Corollary 24. *If the allocation rule underlying \mathcal{M} is locally optimal and monotone, then $\phi_{\text{TUmax}}(\mathcal{M}) \leq \phi_{\text{NTUmax}}(\mathcal{M}) \leq \phi_{\text{TUmin}}(\mathcal{M}) \leq 2\Delta$.*

While local ratio algorithm and the greedy algorithm (and, obviously, the optimal algorithm) are locally optimal, we can also apply our results to monotone vertex cover algorithms that do not necessarily output locally optimal solutions. To do so, we simply take the vertex cover produced by any such algorithm and transform it into a locally optimal one, considering the vertices in lexicographic order and replacing a vertex v with its neighbours whenever $b_v > \sum_{u \sim v} b_u$. Note that if a vertex u has been added to the vertex cover during this process, it means that it has a neighbour whose bid is higher than b_u , so after one pass all vertices in the vertex cover satisfy $b_v \leq \sum_{u \sim v} b_u$. This procedure is monotone in bids, and it can only decrease the cost of the vertex cover. Therefore, using it on top of a monotone allocation rule with approximation ratio α , we obtain a monotone locally optimal allocation rule with approximation ratio α . Combining it with threshold payments, we get an auction with $\phi_{\text{NTUmin}} \leq 2\Delta$.

Remark 25. *Our vertex-cover results can be extended to set cover. Namely, we can transform a set cover instance into a vertex cover instance as follows. For each set S_i , create a vertex v_i . The vertices v_i and v_j are adjacent iff the intersection of S_i and S_j is nonempty. For this vertex cover instance $\Delta = \ell M$, where ℓ is the maximum set size and M is the maximum number of sets containing any ground set element. It is easy to see that an instance of set cover is monopoly-free if a set in the set cover can be replaced with all its neighbours, so we can transform any set cover into a locally optimal one as described above. Therefore, any monotone approximation algorithm for set cover yields an auction with $\phi_{\text{NTUmin}} \leq 2\Delta$.*

6 Properties of the payment bounds

In this section we consider several desiderata for payment bounds and evaluate the four payment bounds proposed in this paper with respect to them. The particular properties

that we are interested in are monotonicity, computational hardness, and the relationship with other reasonable bounds, such as the total cost of the cheapest set, or the total VCG payment.

6.1 Comparison with total cost

Our first requirement is that a payment bound should not be less than the total cost of the selected set. Indeed, payment bounds are used to evaluate the performance of set system auctions. The latter have to satisfy individual rationality, i.e., the payment to each agent must be at least as large as its incurred costs; it is only reasonable to require the payment bound to satisfy the same.

Clearly, $\text{NTUmax}(\mathbf{c})$ and $\text{NTUmin}(\mathbf{c})$ satisfy this requirement due to condition (1), and so does $\text{TUmax}(\mathbf{c})$, since $\text{TUmax}(\mathbf{c}) \geq \text{NTUmax}(\mathbf{c})$. However, $\text{TUmin}(\mathbf{c})$ fails this test. The example of Claim 6 shows that for path auctions, $\text{TUmin}(\mathbf{c})$ can be less than the total cost by a factor of 2. Moreover, there are set systems and cost vectors for which $\text{TUmin}(\mathbf{c})$ is less than the cost of the cheapest set S by a factor of $\Omega(n)$. Consider, for example, the vertex-cover auction for the graph of Claim 13 with the costs $c_{X_1} = \dots = c_{X_{n-2}} = c_{X_{n-1}} = 1$, $c_{X_0} = 0$. The cost of a cheapest vertex cover is $n - 2$, and the lexicographically first vertex cover of cost $n - 2$ is $\{X_0, X_1, \dots, X_{n-2}\}$. The constraints in (2) are $b_{X_i} + b_{X_0} \leq c_{X_{n-1}} = 1$. Clearly, we can satisfy conditions (2) and (3) by setting $b_{X_1} = \dots = b_{X_{n-2}} = 0$, $b_{X_0} = 1$, which means that $\text{TUmin}(\mathbf{c}) \leq 1$.

This observation suggests that the payment bound $\text{TUmin}(\mathbf{c})$ is too strong to be realistic. In particular, it explains why we cannot expect the results of the previous section to hold with respect to $\text{TUmin}(\mathbf{c})$: for the instance of vertex cover described above, we have $\phi_{\text{TUmin}}(\mathcal{M}) \geq n - 2$ for any individually rational mechanism \mathcal{M} .

However, some of the positive results proven in [14] for $\text{NTUmin}(\mathbf{c})$ go through for $\text{TUmin}(\mathbf{c})$ as well. In particular, one can show that if the set system is a matroid, then for VCG $\phi_{\text{TUmin}} = 1$.

To show that $\phi_{\text{TUmin}}(\text{VCG})$ is at most 1, one must prove that the VCG payment is at most $\text{TUmin}(\mathbf{c})$. This is shown for $\text{NTUmin}(\mathbf{c})$ in the first paragraph of the proof of Theorem 5 in [14]. Their argument does not use condition (1) at all, so it also applies to $\text{TUmin}(\mathbf{c})$. On the other hand, $\phi_{\text{TUmin}}(\text{VCG}) \geq 1$ since $\phi_{\text{TUmin}}(\text{VCG}) \geq \phi_{\text{NTUmin}}(\text{VCG})$ and $\phi_{\text{NTUmin}}(\text{VCG}) \geq 1$ by Proposition 7 of [14] (and also by Claim 26).

6.2 Comparison with VCG payments

Another measure of suitability for payment bounds is whether they can result in frugality ratios that are less than 1 for well-known truthful mechanisms. If this is indeed the case, the payment bound may be too weak, as it becomes too easy to design mechanisms that perform well with respect to it. In particular, a reasonable requirement is that a payment bound should not exceed the total payment by the classical VCG mechanism (see Section 2 for the definition).

The following claim shows that for $\text{NTUmax}(\mathbf{c})$ (and therefore also for $\text{NTUmin}(\mathbf{c})$ and $\text{TUmin}(\mathbf{c})$) the payment bound does not exceed the VCG payment. The proof essentially follows the argument of Proposition 7 of [14].

Claim 26. $\phi_{\text{NTUmax}}(\text{VCG}) \geq 1$.

Proof. Suppose $e \in S$. The VCG payment p_e is $\min_{T: e \notin T} \{c_e + c(T) - c(S)\}$. Let T_e be the feasible set T which achieves the minimum so $p_e = c(T_e) - c(S - e)$. But constraint (2) gives $b(S - T) \leq c(T - S)$ for all T so since $e \notin T_e$, $b_e + b(S - T_e - e) \leq c(T_e - S)$ so

$$b_e \leq c(T_e - S) - b(S - T_e - e). \quad (1)$$

Now by (1), $b(S - T_e - e) \geq c(S - T_e - e)$, so (1) gives

$$b_e \leq c(T_e - S) + c(T_e \cap S) - c(S - T_e - e) - c(T_e \cap S).$$

Since $e \notin T_e \cap S$, we get $b_e \leq c(T_e) - c(S - e) = p_e$. \square

However, the payment bound $\text{TUmax}(\mathbf{c})$ can exceed the total payment of VCG. In particular, for the instance in Claim 13 the VCG payment is less than $\text{TUmax}(\mathbf{c})$ by a factor of $n - 2$. We have already seen that $\text{TUmax}(\mathbf{c}) \geq n - 2$. On the other hand, under VCG, the threshold bid of any X_i , $i = 1, \dots, n - 2$, is 0: if any such vertex bids above 0, it is deleted from the winning set together with X_0 and replaced with X_{n-1} . Similarly, the threshold bid of X_0 is 1, because if X_0 bids above 1, it can be replaced with X_{n-1} .

This result is not surprising: the definition of $\text{TUmax}(\mathbf{c})$ implicitly assumes a fairly high degree of co-operation between the agents, while in our computation of VCG payments we did not allow any interaction between them. Indeed, by working together, the agents can extract much higher payments under VCG, i.e., VCG is not group-strategyproof. This suggests that as a payment bound, $\text{TUmax}(\mathbf{c})$ may be too liberal, at least in the context where there is little or no co-operation between agents. Perhaps, it can be a good benchmark for measuring the performance of mechanisms designed for agents that can form coalitions or make side payments to each other, in particular, group-strategyproof mechanisms.

Another setting in which bounding ϕ_{TUmax} is still of some interest is when for the underlying problem the optimal allocation and VCG payments are NP-hard to compute. In this case, finding a *polynomial-time computable* mechanism with good frugality ratio with respect to $\text{TUmax}(\mathbf{c})$ is a non-trivial task, while bounding the frugality ratio with respect to more challenging payment bounds could be too difficult. To illustrate this point, compare the proofs of Claim 20 and Claim 21: both require some effort, but the latter is much more difficult than the former.

6.3 Negative results for $\text{NTUmin}(\mathbf{c})$ and $\text{TUmin}(\mathbf{c})$

In [14], the authors are able to prove several interesting results about frugality ratios of various mechanisms using $\text{NTUmin}(\mathbf{c})$ as their payment bound. Indeed, it can be argued that ϕ_{NTUmin} is the “best” definition of frugality ratio, because among all reasonable payment bounds (i.e., ones that are at least as large as the cost of the cheapest feasible set), it is most demanding of the algorithm. However, $\text{NTUmin}(\mathbf{c})$ is not always the easiest or the most natural payment bound to work with. In this subsection, we discuss several disadvantages of $\text{NTUmin}(\mathbf{c})$ (and also $\text{TUmin}(\mathbf{c})$) as compared to $\text{NTUmax}(\mathbf{c})$ and $\text{TUmax}(\mathbf{c})$.

6.3.1 Nonmonotonicity

The first problem with $\text{NTUmin}(\mathbf{c})$ is that it is not monotone with respect to \mathcal{F} , i.e., it may increase when one adds a feasible set to \mathcal{F} (It is, however, monotone in the sense that if an agent’s cost increases, it is less likely to be chosen.) Intuitively, a good payment bound should

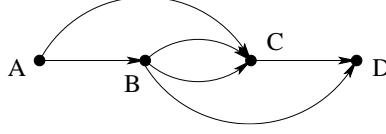


Figure 4: Nonmonotonicity of $NTU\text{Min}$ for path auctions

satisfy this monotonicity requirement, as adding a feasible set increases the competition, so it should drive the prices down. Note that this indeed the case for $NTU\text{max}(\mathbf{c})$ and $TU\text{max}(\mathbf{c})$ since a new feasible set corresponds to a new constraint in (2), thus limiting the solution space for the respective linear program.

Claim 27. *Adding a feasible set to \mathcal{F} can increase $NTU\text{min}(\mathbf{c})$ by a factor of $\Omega(n)$.*

Proof. Let $\mathcal{E} = \{x, a, y_1, \dots, y_n, z_1, \dots, z_n\}$. Set $Y = \{y_1, \dots, y_n\}$, $S = Y \cup \{x\}$, $T_i = Y \setminus \{y_i\} \cup \{z_i\}$, $i = 1, \dots, n$, and suppose that $\mathcal{F} = \{S, T_1, \dots, T_n\}$. The costs are $c_x = 0$, $c_{y_i} = 0$, $c_{z_i} = 1$ for $i = 1, \dots, n$. Note that S is the cheapest feasible set. Let $\mathcal{F}' = \mathcal{F} \cup \{T_0\}$, where $T_0 = Y \cup \{a\}$. For \mathcal{F} , the bid vector $b_{y_1} = \dots = b_{y_n} = 0$, $b_x = 1$ satisfies (1), (2), and (3), so $NTU\text{min}(\mathbf{c}) \leq 1$. For \mathcal{F}' , any optimal solution has $b_x = 0$ (by constraint in (2) with T_0). Condition (3) for y_i implies $b_x + b_{y_i} = c_{z_i} = 1$, so $b_{y_i} = 1$ and $NTU\text{min}(\mathbf{c}) = n$. \square

For path auctions, it has been shown [16] that $NTU\text{min}(\mathbf{c})$ is non-monotone in a slightly different sense, i.e., with respect to adding a new edge (agent) rather than a new feasible set (a team of existing agents). We present this example here for completeness.

Claim 28. *For path auctions, adding an edge to the graph can increase $NTU\text{min}(\mathbf{c})$ by a factor of 2.*

Proof. Consider the graph of Figure 1 with the edge costs $c_{AB} = c_{BC} = c_{CD} = 0$, $c_{AC} = c_{BD} = 1$. In this graph, $ABCD$ is the shortest path, and it is easy to see that $NTU\text{min}(\mathbf{c}) = 1$ with the bid vector $b_{AB} = b_{CD} = 0$, $b_{BC} = 1$. Now suppose that we add a new edge \widehat{BC} of cost 0 between B and C , obtaining the graph of Figure 4. We can assume that the original shortest path $ABCD$ is the lexicographically first shortest path in the new graph, so it gets selected. However, now we have a new constraint in (2), namely, $b_{BC} \leq c_{\widehat{BC}} = 0$, so we have $NTU\text{min}(\mathbf{c}) = 2$ with the bid vector $b_{AB} = b_{CD} = 1$, $b_{BC} = 0$. \square

Remark 29. *It is not hard to modify the example of Claim 28 so that the underlying graph has no multiple edges.*

Remark 30. *We can also show that $NTU\text{min}(\mathbf{c})$ is non-monotone for vertex cover. In this case, adding a new feasible set corresponds to deleting edges from the graph. It turns out that deleting a single edge can increase $NTU\text{min}(\mathbf{c})$ by a factor of $n - 2$; the construction is similar to that of Claim 13.*

6.3.2 NP-Hardness

Another problem with $NTU\text{min}(\mathbf{c})$ is that it is NP-hard to compute even if the number of feasible sets is polynomial in n . Again, this puts it at a disadvantage compared to $NTU\text{max}(\mathbf{c})$ and $TU\text{max}(\mathbf{c})$ (see Remark 1).

Theorem 31. *Computing $\text{NTUmin}(\mathbf{c})$ is NP-hard.*

Proof. We reduce EXACT COVER BY 3-SETS (X3C) to our problem. An instance of X3C is given by a universe $G = \{g_1, \dots, g_n\}$ and a collection of subsets C_1, \dots, C_m , $C_i \subset G$, $|C_i| = 3$, where the goal is to decide whether one can cover G by $n/3$ of these sets. Observe that if this is indeed the case, each element of G is contained in exactly one set of the cover.

Lemma 32. *Consider a minimization problem P of the following form:*

Minimize $\sum_{i=1, \dots, n} b_i$ under conditions

- (1) $b_i \geq 0$ for all $i = 1, \dots, n$
- (2) for any $j = 1, \dots, k$ we have $\sum_{b_i \in S_j} b_i \leq a_j$, where $S_j \subseteq \{b_1, \dots, b_n\}$
- (3) for each b_j , one of the constraints in (2) involving it is tight.

For any such P , one can construct a set system S and a vector of costs c such that $\text{NTUmin}(\mathbf{c})$ is the optimal solution to P .

Proof. The construction is straightforward: there is an element of cost 0 for each b_i , an element of cost a_j for each a_j , the feasible solutions are $\{b_1, \dots, b_n\}$, or any set obtained from $\{b_1, \dots, b_n\}$ by replacing the elements in S_j by a_j . \square

By this lemma, all we have to do is to construct a minimization problem of the form given in the lemma. We do this as follows. For each C_i , we introduce 4 variables x_i , \bar{x}_i , a_i , and b_i . Also, for each element g_j of G there is a variable d_j . We use the following set of constraints:

- In (1), we have constraints $x_i \geq 0$, $\bar{x}_i \geq 0$, $a_i \geq 0$, $b_i \geq 0$, $d_j \geq 0$ for all $i = 1, \dots, m$, $j = 1, \dots, n$.
- In (2), for all $i = 1, \dots, m$, we have the following 5 constraints: $x_i + \bar{x}_i \leq 1$
 $x_i + a_i \leq 1$
 $\bar{x}_i + a_i \leq 1$
 $x_i + b_i \leq 1$
 $\bar{x}_i + b_i \leq 1$.

Also, for all $j = 1, \dots, n$ we have a constraint of the form $x_{i_1} + \dots + x_{i_k} + d_j \leq 1$, where C_{i_1}, \dots, C_{i_k} are the sets that contain g_j .

The goal is to minimize $z = \sum_i (x_i + \bar{x}_i + a_i + b_i) + \sum_j d_j$.

Observe that for each j , there is only one constraint involving d_j , so by condition (3) it must be tight.

Consider the two constraints involving a_i . One of them must be tight, and therefore $x_i + \bar{x}_i + a_i + b_i \geq x_i + \bar{x}_i + a_i \geq 1$. Hence, for any feasible solution to (1)–(3) we have $z \geq m$.

Now, suppose that there is an exact set cover. Set $d_j = 0$ for $j = 1, \dots, n$. Also, if C_i is included in this cover, set $x_i = 1$, $\bar{x}_i = a_i = b_i = 0$, otherwise set $\bar{x}_i = 1$, $x_i = a_i = b_i = 0$. Clearly, all inequalities in (2) are satisfied (we use the fact that each element is covered exactly once), and for each variable, one of the constraints involving it is tight. This assignment results in $z = m$.

Conversely, suppose there is a feasible solution with $z = m$. As each addend of the form $x_i + \bar{x}_i + a_i + b_i$ contributes at least 1, we have $x_i + \bar{x}_i + a_i + b_i = 1$ for all i , $d_j = 0$ for all j .

We will now show that for each i , either $x_i = 1$ and $\bar{x}_i = 0$, or $x_i = 0$ and $\bar{x}_i = 1$. For the sake of contradiction, suppose that $x_i = \delta < 1$, $\bar{x}_i = \delta' < 1$. As one of the constraints involving a_i must be tight, we have $a_i \geq \min\{1 - \delta, 1 - \delta'\}$. Similarly, $b_i \geq \min\{1 - \delta, 1 - \delta'\}$. Hence, $x_i + \bar{x}_i + a_i + b_i = 1 = \delta + \delta' + 2 \min\{1 - \delta, 1 - \delta'\} > 1$. To finish the proof, note that for each $j = 1, \dots, m$ we have $x_{i_1} + \dots + x_{i_k} + d_j = 1$ and $d_j = 0$, so the subsets that correspond to $x_i = 1$ constitute a set cover. \square

Remark 33. Observe that both in the non-monotonicity proof and in the NP-hardness proof all constraints in (1) are of the form $b_e \geq 0$. Hence, the same constructions can be used to show that $\text{TUmin}(\mathbf{c})$ is non-monotone with respect to \mathcal{F} as well as NP-hard to compute.

Remark 34. For shortest-path auctions, the size of \mathcal{F} can be superpolynomial. However, there is a polynomial-time separation oracle for constraints in (2) (it is easy to construct one based on any algorithm for finding shortest paths), so one can compute $\text{NTUmax}(\mathbf{c})$ and $\text{TUmax}(\mathbf{c})$ for this problem in polynomial time. We do not know if computing $\text{NTUmin}(\mathbf{c})$ for shortest-path auctions is NP-hard; this was suggested as an open problem in [14].

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A Nash equilibria and frugality ratios

Karlin et al. [14], argue that the payment bound ν can be viewed as the total payment in a Nash equilibrium of a certain game. In this section, we build on this intuition to justify the four payment bounds introduced above. We consider two variants of a game that differ in how profit is shared between the winning players. We will call these variants the TU game and the NTU game (standing for “transferable utility” and “non-transferable utility” respectively). We then show that $\text{NTUmax}(\mathbf{c})$ and $\text{NTUmin}(\mathbf{c})$ correspond to the worst and the best Nash equilibrium of the NTU game, and $\text{TUmax}(\mathbf{c})$ and $\text{TUmin}(\mathbf{c})$ correspond to the worst and the best Nash equilibrium of the TU game. $\text{NTUmin}(\mathbf{c})$ corresponds to the payment bound ν of [14].

In both versions, the players are the elements of the ground set \mathcal{E} . Each player has an associated cost that is known to all parties. The game starts by the centre selecting a cheapest feasible set $S \in \mathcal{F}$ (with respect to the true costs), resolving ties lexicographically. Then the elements of S are allowed to make bids, and the centre decides whether or not to accept them. Intuitively, S ought to be able to win the auction, and we seek bids from S that are

low enough to win, and high enough that no member of S has an incentive to raise his bid (because that would cause him to lose).

Given that S is supposed to win, we modify the game to rule out behaviour such as elements of S bidding unnecessarily high and losing. One way to enforce the requirement that S wins is via fines. If S is not the among the cheapest sets with respect to the bids (where the new cost of a set T is the sum of the total cost of $T \setminus S$ and the total bid of $S \setminus T$), the centre rejects the solution and every element of S who bids above its true cost pays a fine of size $C = \max_{e \in \mathcal{E}} c_e$, while other elements pay 0. Otherwise, members of S are paid their bids (which may then be shared amongst members of S). This ensures that in a Nash equilibrium, the resulting bids are never rejected as a result of S not being the cheapest feasible set.

In the NTU game, we assume that players cannot make payments to each other, i.e., the utility of each player in S is exactly the difference between his bid and his true cost. In particular, this means that no agent will bid below his true cost, which is captured by condition (1). In a Nash equilibrium, S is the cheapest set with respect to the bids, which is captured by condition (2). Now, suppose that condition (3) is not satisfied for some bidder e . Then the vector of bids is not a Nash equilibrium: e can benefit from increasing his bids by a small amount. Conversely, any vector of bids that satisfies (1), (2) and (3) is a Nash equilibrium: no player wants to decrease its bid, as it would lower the payment it receives, and no player can increase its bid, as it would violate (2) and will cause this bidder to pay a fine. As $\text{NTUmin}(\mathbf{c})$ minimises $\sum_{e \in S} b_e$ under conditions (1), (2), and (3), and $\text{NTUmax}(\mathbf{c})$ maximises it, these are, respectively, the best and the worst Nash equilibrium, from the centre's point of view.

In the TU game, the players in S redistribute the profits among themselves in equal shares, i.e., each player's utility is the difference between the total payment to S and the total cost of S , divided by the size of S . We noted in Section 6.1 that when S is *required* to be the winning set, this may result in Nash equilibria where members of S make a loss collectively, and not just individually as a result of condition (1) not applying. (Recall that we do assume that agents' bids are non-negative; condition (1*).) $\text{TUmin}(\mathbf{c})$ thus represents a situation in which "winners" are being coerced into accepting a loss-making contract.

$\text{TUmax}(\mathbf{c})$ does not have the above problem, since it is larger than the other payment bounds, so members of S will not make a loss. The meaning of conditions (2) and (3) remains the same: the agents do not want the centre to reject their bid, and no agent can improve the total payoff by raising their bid. Note that we are not allowing coalitions (see remark 35), i.e., coordinated deviations by two or more players: even though the players share the profits, they cannot make joint decisions about their strategies. Similarly to the NTU game, it is easy to see that $\text{TUmax}(\mathbf{c})$ and $\text{TUmin}(\mathbf{c})$ are, respectively, the worst and the best Nash equilibria of this game from the centre's viewpoint.

Remark 35. *Allowing payment redistribution within a set is different from allowing players to form coalitions (as in, e.g., the definition of strong Nash equilibrium): in the latter case, players are allowed to make joint decisions about their bids, but they cannot make payments to each other.*

Remark 36. *Both of our games are different from the one implicitly suggested by [14], as we do not allow bids from the players not in the winning set, i.e., our games are NOT true first-price auctions. The reason for this choice is that $\text{NTUmax}(\mathbf{c})$ and $\text{TUmax}(\mathbf{c})$ do not have a good characterization in terms of first-price auctions, neither when players are allowed*

to form coalitions or redistribute payments, nor when they are not allowed to do so. One of the reasons for that is that an agent not in the cheapest set may bid above his true cost at a Nash equilibrium; while the agent itself does not benefit from it, it has no disincentive to do so, and it increases the total payment. Hence, the worst Nash equilibrium in a first-price auction may be much worse than $\text{NTUmax}(\mathbf{c})$ or $\text{TUmax}(\mathbf{c})$. We can penalize the agents not in the selected set for bidding above their true value, but this restriction does not seem natural in the context of first-price auctions, and essentially reduces them to the game considered above.