

# POLYGON CONVEXITY: ANOTHER $O(n)$ TEST

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ABSTRACT. An  $O(n)$  test for polygon convexity is stated and proved.

## 1. DEFINITIONS AND RESULTS

A *polygon* is defined in this paper as any finite sequence of points (or, interchangeably, vectors) on the Euclidean plane  $\mathbb{R}^2$ ; the same definition was used in [5, 6]. Let here  $\mathcal{P} := (V_0, \dots, V_{n-1})$  be a polygon, which is sequence of  $n$  points; such a polygon is also called an  $n$ -gon. The points  $V_0, \dots, V_{n-1}$  are called the *vertices* of  $\mathcal{P}$ . The smallest value that one may allow for the integer  $n$  is 0, corresponding to a polygon with no vertices, that is, to the sequence  $()$  of length 0. The segments, or closed intervals,

$$[V_i, V_{i+1}] := \text{conv}\{V_i, V_{i+1}\} \quad \text{for } i \in \{0, \dots, n-1\}$$

are called the *edges* of polygon  $\mathcal{P}$ , where

$$V_n := V_0.$$

The symbol  $\text{conv}$  denotes, as usual, the convex hull [9, page 12]. Note that, if  $V_i = V_{i+1}$ , then the edge  $[V_i, V_{i+1}]$  is a singleton set.

In general, our terminology corresponds to that in [9]. Here and in the sequel, we also use the notation

$$\overline{k, m} := \{i \in \mathbb{Z} : k \leq i \leq m\},$$

where  $\mathbb{Z}$  is the set of all integers; in particular,  $\overline{k, m}$  is empty if  $m < k$ .

Let us define the convex hull and dimension of polygon  $\mathcal{P}$  as, respectively, the convex hull and dimension of the set of its vertices:  $\text{conv } \mathcal{P} := \text{conv}\{V_0, \dots, V_{n-1}\}$  and  $\dim \mathcal{P} := \dim\{V_0, \dots, V_{n-1}\} = \dim \text{conv } \mathcal{P}$ .

Given the above notion of the polygon, a *convex polygon* can be defined as a polygon  $\mathcal{P}$  such that the union of the edges of  $\mathcal{P}$  coincides with the boundary  $\partial \text{conv } \mathcal{P}$  of the convex hull  $\text{conv } \mathcal{P}$  of  $\mathcal{P}$ ; cf. e.g. [11, page 5]. Thus, one has

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**Definition 1.1.** A polygon  $\mathcal{P} = (V_0, \dots, V_{n-1})$  is convex if

$$\bigcup_{i \in \overline{0, n-1}} [V_i, V_{i+1}] = \partial \operatorname{conv} \mathcal{P}.$$

Let us emphasize that a polygon in this paper is a sequence and therefore ordered. In particular, even if all the vertices  $V_0, \dots, V_{n-1}$  of a polygon  $\mathcal{P} = (V_0, \dots, V_{n-1})$  are the extreme points of the convex hull of  $\mathcal{P}$ , it does not necessarily follow that  $\mathcal{P}$  is convex. For example, consider the points  $V_0 = (0, 0)$ ,  $V_1 = (1, 0)$ ,  $V_2 = (1, 1)$ , and  $V_3 = (0, 1)$ . Then polygon  $(V_0, V_1, V_2, V_3)$  is convex, while polygon  $(V_0, V_2, V_1, V_3)$  is not.

**Definition 1.2.** Let us say that a polygon  $\mathcal{P} = (V_0, \dots, V_{n-1})$  is

- quasi-ordinary – if for any  $i$  in the set  $\overline{0, n-1}$ , the vertices  $V_i$  and  $V_{i+1}$  are distinct;
- ordinary – if for any two distinct  $i$  and  $j$  in  $\overline{0, n-1}$ , the vertices  $V_i$  and  $V_j$  are distinct;
- strict – if for any three distinct  $i, j$ , and  $k$  in  $\overline{0, n-1}$ , the vertices  $V_i, V_j$ , and  $V_k$  are non-collinear;
- quasi-ordinarily convex – if  $\mathcal{P}$  is quasi-ordinary and convex;

similarly can be defined ordinarily convex and strictly convex polygons.

Obviously, any ordinary polygon is quasi-ordinary. Any  $n$ -gon with  $n \leq 1$  is ordinary. Any  $n$ -gon with  $n \leq 2$  is strictly convex. Any 3-gon is convex, and so, a 3-gon is strictly convex if and only if it is strict. Any strictly convex  $n$ -gon with  $n \geq 3$  is ordinary.

For a polygon  $\mathcal{P} = (V_0, \dots, V_{n-1})$ , let  $x_i$  and  $y_i$  denote the coordinates of its vertices  $V_i$ , so that

$$V_i = (x_i, y_i) \quad \text{for } i \in \overline{0, n-1}.$$

Introduce the determinants

$$(1) \quad \Delta_{\alpha, i, j} := \begin{vmatrix} 1 & x_\alpha & y_\alpha \\ 1 & x_i & y_i \\ 1 & x_j & y_j \end{vmatrix}$$

for  $\alpha, i$ , and  $j$  in the set  $\overline{0, n-1}$ . Let then

$$\begin{aligned} a_i &:= \operatorname{sign} \Delta_{i+1, i-1, i} = \operatorname{sign} \Delta_{i-1, i, i+1}; \\ b_i &:= \operatorname{sign} \Delta_{0, i-1, i}; \\ c_i &:= \operatorname{sign} \Delta_{i, 0, 1} = \operatorname{sign} \Delta_{0, 1, i}. \end{aligned}$$

The following theorem is the main result of [7], which provides an  $O(n)$  test of the strict convexity of a polygon.

**Theorem 1.3.** [7] *An  $n$ -gon  $\mathcal{P} = (V_0, \dots, V_{n-1})$  with  $n \geq 4$  is strictly convex if and only if conditions*

$$(2) \quad \begin{aligned} a_i b_i &> 0, \\ a_i b_{i+1} &> 0, \\ c_i c_{i+1} &> 0 \end{aligned}$$

*hold for all*

$$i \in \overline{2, n-2}.$$

**Proposition 1.4.** [7] *None of the  $3(n-3)$  conditions in Theorem 1.3 can be omitted without (the “if” part of) Theorem 1.3 ceasing to hold.*

Thus, the test given by Theorem 1.3 is exactly minimal.

**Remark 1.5.** [7] *Adding to the  $3(n-3)$  conditions (2) in Theorem 1.3 the equality  $b_2 = c_2$ , which trivially holds for any polygon (convex or not), one can rewrite (2) as the following system of  $3(n-3) + 1$  equations and one inequality:*

$$\begin{aligned} a_2 &= \dots = a_{n-2} \\ = b_2 &= \dots = b_{n-2} = b_{n-1} \\ = c_2 &= \dots = c_{n-2} = c_{n-1} \neq 0. \end{aligned}$$

These results were used in [8].

For any vector  $\vec{v} = (x, y) \in \mathbb{R}^2$  with  $r := |\vec{v}| := \sqrt{x^2 + y^2} \neq 0$ , define the (angle) argument of  $\vec{v}$  as usual, by the formula

$$\arg \vec{v} = \theta \iff (0 \leq \theta < 2\pi \text{ \& } x = r \cos \theta \text{ \& } y = r \sin \theta),$$

so that, for each nonzero vector  $\vec{v} \in \mathbb{R}^2$ , the “angle”  $\arg \vec{v}$  is a uniquely defined number in the interval  $[0, 2\pi)$ . Moreover,

$$(3) \quad \arg \vec{v} = \begin{cases} \arccos \frac{x}{r} & \text{if } \vec{v} \in H_-, \\ 2\pi - \arccos \frac{x}{r} & \text{if } \vec{v} \in H_+, \end{cases}$$

where  $\arccos$  is the branch of the inverse function  $\cos^{-1}$  with values in the interval  $[0, \pi]$  and

$$H_- := \{(x, y) \in \mathbb{R}^2 : y > 0 \text{ or } (y = 0 \text{ \& } x > 0)\},$$

$$H_+ := \{(x, y) \in \mathbb{R}^2 : y < 0 \text{ or } (y = 0 \text{ \& } x < 0)\};$$

note that  $H_- \cup H_+ = \mathbb{R}^2 \setminus \{\vec{0}\}$  and  $H_- \cap H_+ = \emptyset$ .

For any quasi-ordinary polygon  $\mathcal{P} = (V_0, \dots, V_{n-1})$ , introduce also the sequence of the angle arguments the edge-vectors  $\overrightarrow{V_0 V_1}, \dots, \overrightarrow{V_{n-1} V_n}$  of  $\mathcal{P}$ , by the formula

$$\arg \mathcal{P} := (\arg \overrightarrow{V_0 V_1}, \dots, \arg \overrightarrow{V_{n-1} V_n}).$$

For any two nonzero vectors  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^2$ , let us write

$$\vec{u} < \vec{v} \quad \text{iff} \quad \arg \vec{u} < \arg \vec{v};$$

similarly defined is the relation  $>$  on  $\mathbb{R}^2 \setminus \{\vec{0}\}$ .

**Remark 1.6.** Let  $\vec{u} = (s, t)$  and  $\vec{v} = (x, y)$  be any two vectors in  $\mathbb{R}^2 \setminus \{\vec{0}\}$ . Then, using (3), it is elementary but somewhat tedious to check that

$$(4) \quad \vec{u} < \vec{v} \iff \begin{cases} \vec{u} \in H_- \ \& \ \vec{v} \in H_+ & \text{or} \\ \vec{u} \in H_- \ \& \ \vec{v} \in H_- \ \& \ \Delta > 0 & \text{or} \\ \vec{u} \in H_+ \ \& \ \vec{v} \in H_+ \ \& \ \Delta > 0, \end{cases}$$

where

$$\Delta := \begin{vmatrix} 1 & 0 & 0 \\ 1 & s & t \\ 1 & x & y \end{vmatrix} = \begin{vmatrix} s & t \\ x & y \end{vmatrix} = sy - tx.$$

Under the additional condition that  $\vec{u}$  and  $\vec{v}$  are non-collinear, it follows that either  $\vec{u} < \vec{v}$  or  $\vec{v} < \vec{u}$ :

$$\Delta \neq 0 \implies (\vec{u} < \vec{v} \text{ or } \vec{v} < \vec{u}).$$

Note also that

$$(5) \quad \vec{u} < \vec{v} \implies (y \leq 0 \leq t \text{ or } \Delta > 0).$$

**Definition 1.7.** Let us say that a quasi-ordinary polygon  $\mathcal{P} = (V_0, \dots, V_{n-1})$  with  $\arg \mathcal{P} =: (\alpha_0, \dots, \alpha_{n-1})$  is

- increasing – if the sequence  $\arg \mathcal{P}$  is increasing:  $\alpha_0 < \dots < \alpha_{n-1}$ ;
- decreasing – if  $\alpha_0 > \dots > \alpha_{n-1}$ ;
- cyclically increasing or, briefly, c-increasing – if

$$\alpha_k < \dots < \alpha_{n-1} < \alpha_0 < \dots < \alpha_{k-1},$$

for some  $k \in \overline{0, n-1}$ ; (if  $k = 0$  then this chain of inequalities is supposed to read simply as  $\alpha_0 < \dots < \alpha_{n-1}$ , in which case polygon  $\mathcal{P}$  will be increasing);

- cyclically decreasing or, briefly, c-decreasing – similarly, if

$$\alpha_k > \dots > \alpha_{n-1} > \alpha_0 > \dots > \alpha_{k-1},$$

for some  $k \in \overline{0, n-1}$ ;

- cyclically strictly monotone or, briefly, c-strictly monotone – if  $\mathcal{P}$  is either c-increasing or c-decreasing.

For any transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and any polygon  $\mathcal{P} = (V_0, \dots, V_{n-1})$ , define the corresponding transformation of  $\mathcal{P}$  as the polygon  $T\mathcal{P} := (TV_0, \dots, TV_{n-1})$ ; if  $T$  is an orthogonal or homothetical transformation of the plane, let us say that the polygon  $T\mathcal{P}$  is, respectively, an orthogonal or homothetical transformation of polygon  $\mathcal{P}$ . (A homothetical transformation is understood here as one of the form  $\mathbb{R}^2 \ni \vec{v} \rightarrow \lambda \vec{v}$  for some  $\lambda > 0$ .) A rotation is any orthogonal (and hence linear) transformation with determinant 1; any rotation can be represented as the linear transformation  $R_\alpha$  with matrix  $\begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$  for some real number  $\alpha$ , so

that  $R_\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix}$ . The reflection is denoted here by  $R$  and defined by the formula  $\mathbb{R}^2 \ni (x, y) \mapsto R(x, y) := (x, -y)$ . Any orthogonal transformation can be represented as  $R_\alpha R$  (as well as  $RR_\beta$ ) for appropriate  $\alpha$  and  $\beta$ .

**Proposition 1.8.** *The properties of being convex, quasi-ordinary, ordinary, and strict are each preserved for any polygon under any nonsingular affine transformation. The properties of being c-increasing and c-decreasing are each preserved under any rotation or homothetical transformation or a parallel translation. The properties of being c-increasing and c-decreasing are interchanged under the reflection.*

*Proof of Proposition 1.8.* Suppose that an  $n$ -gon  $\mathcal{P}$  with  $(\alpha_0, \dots, \alpha_{n-1}) := \arg(\mathcal{P})$  is c-increasing, that is,

$$(\text{Incr}_k) \quad \alpha_k < \dots < \alpha_{n-1} < \alpha_0 < \dots < \alpha_{k-1},$$

for some  $k \in \overline{0, n-1}$ . Let  $(\beta_0, \dots, \beta_{n-1}) := \arg(R\mathcal{P})$ , the argument sequence of the reflected polygon  $R\mathcal{P}$ . Then  $\beta_i = 2\pi - \alpha_i$  for all  $i \neq k$ , while  $\beta_k = 2\pi - \alpha_k$  if  $\alpha_k \neq 0$  and  $\beta_k = 0$  if  $\alpha_k = 0$ . It follows that

$$(\text{Decr}_k) \quad \beta_k > \dots > \beta_{n-1} > \beta_0 > \dots > \beta_{k-1}$$

if  $\alpha_k \neq 0$ , and the  $\beta_i$ 's satisfy  $(\text{Decr}_{k \oplus 1})$  if  $\alpha_k = 0$ , where  $k \oplus 1 := k+1$  if  $k \in \overline{0, n-2}$  and  $k \oplus 1 := 0$  if  $k = n-1$ .

Similarly, if  $\arg \mathcal{P} = (\beta_0, \dots, \beta_{n-1})$  satisfies condition  $(\text{Decr}_k)$  for some  $k \in \overline{0, n-1}$ , then  $(\alpha_0, \dots, \alpha_{n-1}) := \arg(R\mathcal{P})$  satisfies condition  $(\text{Incr}_k)$  if  $\alpha_{k-1} \neq 0$ , and the  $\alpha_i$ 's satisfy  $(\text{Incr}_{k \ominus 1})$  if  $\alpha_{k-1} = 0$ , where  $k \ominus 1 := k-1$  if  $k \in \overline{1, n-1}$  and  $k \ominus 1 := n-1$  if  $k = 0$ .

Thus, reflection  $R$  interchanges the properties of being c-increasing and c-decreasing.

Let us now verify the preservation of the c-increasing property under any rotation  $R_\alpha$ . W.l.o.g.,  $0 \leq \alpha < 2\pi$ . Suppose again that an  $n$ -gon  $\mathcal{P}$  with  $(\alpha_0, \dots, \alpha_{n-1}) := \arg(\mathcal{P})$  satisfies condition  $(\text{Incr}_k)$ . Then the cyclic permutation  $\mathcal{Q} := \mathcal{P}\theta^k := (V_k, \dots, V_{n-1}, V_0, \dots, V_{k-1})$  of polygon  $\mathcal{P}$  with  $(\beta_0, \dots, \beta_{n-1}) := \arg(\mathcal{Q}) = (\alpha_k, \dots, \alpha_{n-1}, \alpha_0, \dots, \alpha_{k-1})$  is an increasing  $n$ -gon. Let  $(\psi_0, \dots, \psi_{n-1}) := \arg(R_\alpha \mathcal{Q})$ . Let  $J := \{i \in \overline{0, n-1} : \beta_i + \alpha \geq 2\pi\}$ , and let  $j := \min J$  if  $J \neq \emptyset$  and  $j := n$  if  $J = \emptyset$ . Then  $\psi_i := \beta_i + \alpha$  for  $i \in \overline{0, j-1}$  and  $\psi_i := \beta_i + \alpha - 2\pi$  for  $i \in \overline{j, n-1}$ . Hence, the sequence  $\arg(R_\alpha \mathcal{Q}\theta^j) = (\varphi_0, \dots, \varphi_{n-1})$  is increasing, where  $\varphi_i := \beta_{i+j} + \alpha - 2\pi$  for  $i \in \overline{0, n-j-1}$ , and  $\varphi_i := \beta_{i+j-n} + \alpha$  for  $i \in \overline{n-j, n-1}$ . Thus, the cyclic permutation  $R_\alpha \mathcal{P}\theta^m = R_\alpha \mathcal{P}\theta^{k+j} = R_\alpha \mathcal{Q}\theta^j$  of polygon  $R_\alpha \mathcal{P}$  is increasing, where  $m := k+j$  if  $k+j < n$  and  $m := k+j-n$  if  $k+j \geq n$ . Thus,  $R_\alpha \mathcal{P}$  is c-increasing.

The preservation of the c-decreasing property under any rotation is verified quite similarly.

The other claims stated in Proposition 1.8 are only easier to check. □

The following theorem is the main result of this paper.

**Theorem 1.9.** *An  $n$ -gon with  $n \geq 3$  is strictly convex iff it is c-strictly monotone.*

**Remark 1.10.** Any  $n$ -gon with  $n \leq 1$  is, trivially, both strictly convex and  $c$ -strictly monotone. Any 2-gon is, trivially, strictly convex; however, a 2-gon is  $c$ -strictly monotone only if it is quasi-ordinary (and hence ordinary). It is easy to see that any strict 3-gon is  $c$ -strictly monotone, so that Theorem 1.9 is trivial for  $n = 3$ . Note also that an  $n$ -gon is both  $c$ -increasing and  $c$ -decreasing iff it is quasi-ordinary and  $n = 2$ .

A suggestion to use  $c$ -strict monotonicity to test for polygon convexity was given in [3], without a proof. A result, similar to Theorem 1.9, with a non-strict version of  $c$ -monotonicity, was presented in [2, Lemma 5 in Section 10.3], with a very brief, heuristic proof.

*Proof of Theorem 1.9.* Let  $\mathcal{P} = (V_0, \dots, V_{n-1})$  be an  $n$ -gon with  $n \geq 3$ , vertices  $V_i = (x_i, y_i)$ , and argument  $(\alpha_0, \dots, \alpha_{n-1}) := \arg \mathcal{P}$ . In view of Proposition 1.8, the rotation  $R_{2\pi-\alpha_0}$  and any homothetical transformation will preserve both the convexity and  $c$ -monotonicity properties of  $\mathcal{P}$ . Therefore, assume without loss of generality (w.l.o.g.) that  $\alpha_0 = 0$  and, moreover,  $V_0 = (0, 0)$  and  $V_1 = (1, 0)$ .

**“If”** When proving this part, assume w.l.o.g. that  $\mathcal{P}$  is  $c$ -increasing, that is,  $\alpha_k < \dots < \alpha_{n-1} < \alpha_0 < \dots < \alpha_{k-1}$ . (Indeed, in view of Proposition 1.8, the reflection transformation  $R$  will preserve the convexity property of  $\mathcal{P}$  and interchange the property of  $\mathcal{P}$  being  $c$ -increasing with it being  $c$ -decreasing; also,  $R$  will preserve the property  $\alpha_0 = 0$ .) Then the conditions  $\alpha_0 = 0$  and  $\alpha_i \in [0, 2\pi) \forall i$  imply that  $k = 0$  and  $\alpha_0 = 0 < \dots < \alpha_{n-1}$ ; that is, the sequence  $\arg \mathcal{P}$  is increasing.

Hence, inequality  $\alpha_1 \geq \pi$  would imply  $\alpha_i \in (\pi, 2\pi)$  for all  $i \in \overline{2, n-1}$ . Hence and because  $n \geq 3$ , one would have  $0 = y_1 \geq y_2 > y_3 > \dots > y_n = y_0 = 0$ , and at least one inequality here is strict (since  $n \geq 3$ ), which is a contradiction.

The case  $\alpha_1 < \pi$  is similar. In this case,  $y_2 > 0$ . To obtain a contradiction, suppose that the set  $L := \{i \in \overline{2, n-1} : y_i \leq 0\}$  is non-empty and then let  $\ell := \min L$ , so that  $\ell \in \overline{3, n-1}$ ,  $y_{\ell-1} > 0$ , and  $y_\ell \leq 0$ . Then  $\alpha_{\ell-1} \in (\pi, 2\pi)$ . Hence and because the sequence  $\arg \mathcal{P}$  is increasing, one has  $\alpha_i \in (\pi, 2\pi)$  for all  $i \in \overline{\ell-1, n-1}$ . Therefore,  $0 \geq y_\ell > \dots > y_n = y_0 = 0$ , which is a contradiction. This contradiction means that  $L = \emptyset$ , so that  $y_i > 0$  for all  $i \in \overline{2, n-1}$ ; that is, according to [7, Definition 2.4], the polygon  $\mathcal{P} = (V_0, \dots, V_{n-1})$  is strictly to one side of its edge  $[V_0, V_1]$ .

Similarly it is proved that  $\mathcal{P}$  is strictly to one side of any other one of its edges; that is,  $\mathcal{P}$  is strictly to-one-side. To complete the proof of the “if” part of Theorem 1.9, it remains to refer to [7, Lemmas 2.6 and 2.11].

**“Only if”** Here is assumed that polygon  $\mathcal{P}$  is strictly convex. Again w.l.o.g. one has  $\alpha_0 = 0$ . Also, by Remark 1.10, w.l.o.g.  $n \geq 4$ . Again by the “reflection” part of Proposition 1.8, w.l.o.g.  $y_2 \geq 0$ . Moreover, because of the strictness of  $\mathcal{P}$  and the assumptions  $V_0 = (0, 0)$  and  $V_1 = (1, 0)$ , one actually has  $y_2 > 0$ , so that  $\alpha_0 = 0 < \alpha_1 < \pi$  and  $\Delta_{0,1,2} = y_2 > 0$ . So, the strict convexity of  $\mathcal{P}$  and Remark 1.5

yield  $\Delta_{0,1,i} = y_i > 0$  for all  $i \in \overline{2, n-1}$ . The strictness of  $\mathcal{P}$  also implies that all the values  $\alpha_0, \dots, \alpha_{n-1}$  are distinct from one another.

Thus, it suffices to show that  $\alpha_i \leq \alpha_{i+1}$  for all  $i \in \overline{1, n-2}$ . Suppose the contrary, that  $\alpha_i > \alpha_{i+1}$  for some  $i \in \overline{1, n-2}$ . Consider separately the following three cases.

*Case 1:  $i = 1$ .* Then  $\alpha_1 > \alpha_2$ . By (5), this implies that  $\Delta_{1,2,3} \leq 0$  or  $y_2 - y_1 \leq 0 \leq y_3 - y_2$ ; but  $y_2 - y_1 = y_2 > 0$ , so that one must have  $\Delta_{1,2,3} \leq 0$ ; now inequalities  $\Delta_{1,2,3} \leq 0$  and  $\Delta_{0,1,2} > 0$  contradict Remark 1.5.

*Case 2:  $i = n-2$ .* Then  $\alpha_{n-2} > \alpha_{n-1}$ . This case is quite similar to Case 1. Indeed, by (5), here one has  $\Delta_{0,n-2,n-1} = \Delta_{n-2,n-1,0} \leq 0$  or  $0 \leq y_0 - y_{n-1}$ ; but  $y_0 - y_{n-1} = -y_{n-1} < 0$ , so that  $\Delta_{0,n-2,n-1} \leq 0$ ; now inequalities  $\Delta_{0,n-2,n-1} \leq 0$  and  $\Delta_{0,1,2} > 0$  contradict Remark 1.5.

*Case 3:  $i \in \overline{2, n-3}$  and  $\alpha_i > \alpha_{i+1}$ .* Then the 5-gon  $\mathcal{Q} := (V_0, V_1, V_i, V_{i+1}, V_{i+2})$  is a sub-polygon of  $\mathcal{P}$ , so that  $\mathcal{Q}$  is strictly convex, by [6, Corollary 1.17]. On the other hand,  $\arg \mathcal{Q} = (\alpha_0, \beta, \alpha_i, \alpha_{i+1}, \gamma)$ , for some real numbers  $\beta$  and  $\gamma$ . Thus, w.l.o.g.  $\mathcal{P} = \mathcal{Q}$ ,  $n = 5$ , and so, one has all of the following:  $\mathcal{P} = (V_0, V_1, V_2, V_3, V_4)$ ;  $i = 2$ ;  $\alpha_2 > \alpha_3$ ; and  $\Delta_{0,1,i} = y_i > 0$  for all  $i \in \overline{2, 4}$ . By Remark 1.5, one now also sees that the determinants  $\Delta_{2,3,4}$ ,  $\Delta_{0,2,3}$ , and  $\Delta_{0,3,4}$  are all strictly positive as well. Therefore, the condition  $\alpha_2 > \alpha_3$  and implication (5) yield  $y_3 - y_2 \leq 0 \leq y_4 - y_3$ . One can verify the identity

$$(y_4 - y_3) \Delta_{0,2,3} + (y_2 - y_3) \Delta_{0,3,4} + \Delta_{2,3,4} \Delta_{0,1,3} = 0,$$

whose left-hand side is strictly positive, since all the determinants  $\Delta_{\cdot,\cdot,\cdot}$  in this identity are strictly positive and because of the condition  $y_3 - y_2 \leq 0 \leq y_4 - y_3$ . Thus, one obtains a contradiction.

The proof of the “only if” part and thus of entire Theorem 1.9 is now complete.  $\square$

## Literature

- [1] F. S. Hill, Jr. *The pleasures of “perp dot” products*. In Graphics Gems IV, P. S. Heckbert, ed., Academic Press, London, 138–148, 1994.
- [2] K. Mehlhorn and St. Näher, *The LEDA Platform of Combinatorial and Geometric Computing*, Cambridge University Press, 1999. <http://www.mpi-sb.mpg.de/~mehlhorn/LEDAbook.html>.
- [3] Mehlhorn, K. et al., *Checking Geometric Programs or Verification of Geometric Structures*. Computational Geometry, 12(1-2):85–103, 1999.
- [4] B. M. E. Moret and H. D. Shapiro, *Algorithms from N to NP. Volume 1. Design and Efficiency*, The Benjamin/Cummings Publishing Company, 1991.
- [5] I. Pinelis, Cyclic polygons with given edge lengths: existence and uniqueness. *J. Geom.* **82** (2005), no. 1-2, 156–171.
- [6] I. Pinelis, *Convexity of sub-polygons of convex polygons*. Preprint (2006), <http://arxiv.org/abs/math.GM/0609698>.
- [7] Pinelis, I., *Polygon Convexity: A Minimal  $O(n)$  Test*. Preprint (2006), <http://arxiv.org/abs/cs.CG/0609141>.
- [8] Pinelis, I., *An order-refined and generalized version of the Erdős-Szekeres theorem on convex polygons*. Preprint (2006), <http://arxiv.org/abs/math.CO/0611802>.

- [9] R. T. Rockafellar, *Convex Analysis*, Princeton University Press 1970.
- [10] E. W. Weisstein. Convex polygon. From MathWorld—A Wolfram Web Resource.  
<http://mathworld.wolfram.com/ConvexPolygon.html> .
- [11] I. M. Yaglom and V. G. Boltyanskii, *Convex Figures*, Holt, Rinehart and Winston 1961.

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