

POLYGON CONVEXITY: ANOTHER $O(n)$ TEST

IOSIF PINELIS

ABSTRACT. An n -gon is defined as a sequence $\mathcal{P} = (V_0, \dots, V_{n-1})$ of n points on the plane. An n -gon \mathcal{P} is said to be convex if the boundary of the convex hull of the set $\{V_0, \dots, V_{n-1}\}$ of the vertices of \mathcal{P} coincides with the union of the edges $[V_0, V_1], \dots, [V_{n-1}, V_0]$; if at that no three vertices of \mathcal{P} are collinear then \mathcal{P} is called strictly convex. We prove that an n -gon \mathcal{P} with $n \geq 3$ is strictly convex if and only if a cyclic shift of the sequence $(\alpha_0, \dots, \alpha_{n-1}) \in [0, 2\pi)^n$ of the angles between the x -axis and the vectors $\overrightarrow{V_0 V_1}, \dots, \overrightarrow{V_{n-1} V_0}$ is strictly monotone.

A “non-strict” version of this result is also proved.

1. DEFINITIONS AND RESULTS

A *polygon* is defined in this paper as any finite sequence of points (or, interchangeably, vectors) on the Euclidean plane \mathbb{R}^2 ; the same definition was used in [5–8]. Let here $\mathcal{P} := (V_0, \dots, V_{n-1})$ be a polygon, which is sequence of n points; such a polygon is also called an n -gon. The points V_0, \dots, V_{n-1} are called the *vertices* of \mathcal{P} . The smallest value that one may allow for the integer n is 0, corresponding to a polygon with no vertices, that is, to the sequence $()$ of length 0. The segments, or closed intervals,

$$[V_i, V_{i+1}] := \text{conv}\{V_i, V_{i+1}\} \quad \text{for } i \in \{0, \dots, n-1\}$$

are called the *edges* of polygon \mathcal{P} , where

$$V_n := V_0.$$

The symbol conv denotes, as usual, the convex hull [9, page 12]. Note that, if $V_i = V_{i+1}$, then the edge $[V_i, V_{i+1}]$ is a singleton set. For any two points U and V in \mathbb{R}^2 , let $[U, V] := \text{conv}\{U, V\}$, $[U, V) := [U, V] \setminus \{V\}$, and $(U, V) := [U, V] \setminus \{U, V\}$, so that $(U, V) = \text{ri}[U, V]$, the relative interior of $[U, V]$. Let us define the convex hull and dimension of polygon \mathcal{P} as, respectively, the convex hull and dimension of the set of its vertices: $\text{conv } \mathcal{P} := \text{conv}\{V_0, \dots, V_{n-1}\}$ and $\dim \mathcal{P} :=$

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$\dim\{V_0, \dots, V_{n-1}\} = \dim \operatorname{conv} \mathcal{P}$. In general, our terminology corresponds to that in [9].

In the sequel, we also use the notation

$$\overline{k, m} := \{i \in \mathbb{Z} : k \leq i \leq m\},$$

where \mathbb{Z} is the set of all integers; in particular, $\overline{k, m}$ is empty if $m < k$.

Given the above notion of the polygon, a *convex polygon* can be defined as a polygon \mathcal{P} such that the union of the edges of \mathcal{P} coincides with the boundary $\partial \operatorname{conv} \mathcal{P}$ of the convex hull $\operatorname{conv} \mathcal{P}$ of \mathcal{P} ; cf. e.g. [11, page 5]. Thus, one has

Definition 1.1. A polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is convex if

$$\bigcup_{i \in \overline{0, n-1}} [V_i, V_{i+1}] = \partial \operatorname{conv} \mathcal{P}.$$

Let us emphasize that a polygon in this paper is a sequence and therefore ordered. In particular, even if all the vertices V_0, \dots, V_{n-1} of a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ are the extreme points of the convex hull of \mathcal{P} , it does not necessarily follow that \mathcal{P} is convex. For example, consider the points $V_0 = (0, 0)$, $V_1 = (1, 0)$, $V_2 = (1, 1)$, and $V_3 = (0, 1)$. Then polygon (V_0, V_1, V_2, V_3) is convex, while polygon (V_0, V_2, V_1, V_3) is not.

Definition 1.2. Let us say that a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is

- locally-ordinary – if for any i in the set $\overline{0, n-1}$ the vertices V_i and V_{i+1} are distinct;
- ordinary – if for any two distinct i and j in $\overline{0, n-1}$ the vertices V_i and V_j are distinct;
- locally-strict – if for any i in $\overline{0, n-1}$ the vertices V_{i-1} , V_i , and V_{i+1} are non-collinear, where $V_{-1} := V_{n-1}$;
- quasi-strict – if any two adjacent vertices of \mathcal{P} are not collinear with any other vertex of \mathcal{P} or, more formally, if for any $i \in \overline{0, n-1}$ and any $j \in \overline{0, n-1} \setminus \{i, i \oplus 1\}$ the points V_i , $V_{i \oplus 1}$, and V_j are non-collinear, where

$$i \oplus 1 := \begin{cases} i + 1 & \text{if } i \in \overline{0, n-2}, \\ 0 & \text{if } i = n - 1; \end{cases}$$

- strict – if for any three distinct i , j , and k in $\overline{0, n-1}$ the vertices V_i , V_j , and V_k are non-collinear;
- locally-simple – if for any i in $\overline{0, n-1}$ one has $[V_i, V_{i+1}] \cap [V_{i+1}, V_{i+2}] = \emptyset$, where $V_{n+1} := V_1$;
- simple – if for any two distinct i and j in $\overline{0, n-1}$ one has $[V_i, V_{i+1}] \cap [V_j, V_{j+1}] = \emptyset$;
- locally-ordinarily convex – if \mathcal{P} is locally-ordinary and convex; similarly can be defined ordinarily convex, ..., simply convex polygons.

Remark 1.3. Any (locally-)simple polygon is (locally-)ordinary, since $\{V_i\} \cap \{V_j\} \subseteq [V_i, V_{i+1}) \cap [V_j, V_{j+1})$.

We shall make use of the following result given in [6]. If $\mathcal{P} = (V_0, \dots, V_{n-1})$ is a polygon, let us refer to any subsequence $(V_{i_0}, \dots, V_{i_{m-1}})$ of \mathcal{P} , with $0 \leq i_0 < \dots < i_{m-1} \leq n-1$, as a *sub-polygon* or, more specifically, as a *sub- m -gon* of \mathcal{P} .

Theorem 1.4. [6, Corollary 1.17] *If $\mathcal{P} = (V_0, \dots, V_{n-1})$ is an ordinarily convex polygon, then any sub-polygon of \mathcal{P} is so; in particular, then the sub- $(n-1)$ -gons $\mathcal{P}^{(i)} := (V_0, \dots, V_{i-1}, V_{i+1}, \dots, V_{n-1})$ of \mathcal{P} are ordinarily convex, for all $i \in \overline{0, n-1}$.*

For a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$, let x_i and y_i denote the coordinates of its vertices V_i , so that

$$(1) \quad V_i = (x_i, y_i) \quad \text{for } i \in \overline{0, n-1}.$$

Introduce the determinants

$$(2) \quad \Delta_{\alpha, i, j} := \begin{vmatrix} 1 & x_\alpha & y_\alpha \\ 1 & x_i & y_i \\ 1 & x_j & y_j \end{vmatrix}$$

for α, i , and j in the set $\overline{0, n-1}$. Let then

$$\begin{aligned} a_i &:= \text{sign } \Delta_{i+1, i-1, i} = \text{sign } \Delta_{i-1, i, i+1}; \\ b_i &:= \text{sign } \Delta_{0, i-1, i}; \\ c_i &:= \text{sign } \Delta_{i, 0, 1} = \text{sign } \Delta_{0, 1, i}. \end{aligned}$$

The following theorem is the main result of [7], which provides an $O(n)$ test of the strict convexity of a polygon.

Theorem 1.5. [7] *An n -gon $\mathcal{P} = (V_0, \dots, V_{n-1})$ with $n \geq 4$ is strictly convex if and only if conditions*

$$(3) \quad \begin{aligned} a_i b_i &> 0, \\ a_i b_{i+1} &> 0, \\ c_i c_{i+1} &> 0 \end{aligned}$$

hold for all

$$i \in \overline{2, n-2}.$$

Proposition 1.6. [7] *None of the $3(n-3)$ conditions in Theorem 1.5 can be omitted without (the “if” part of) Theorem 1.5 ceasing to hold.*

Thus, the test given by Theorem 1.5 is exactly minimal.

Remark 1.7. [7] Adding to the $3(n-3)$ conditions (3) in Theorem 1.5 the equality $b_2 = c_2$, which trivially holds for any polygon (convex or not), one can rewrite (3) as the following system of $3(n-3) + 1$ equations and one inequality:

$$\begin{aligned} a_2 &= \cdots = a_{n-2} \\ &= b_2 = \cdots = b_{n-2} = b_{n-1} \\ &= c_2 = \cdots = c_{n-2} = c_{n-1} \neq 0. \end{aligned}$$

These results were used in [8].

For any vector $\vec{v} = (x, y) \in \mathbb{R}^2$ with $r := |\vec{v}| := \sqrt{x^2 + y^2} \neq 0$, define the (angle) argument of \vec{v} as usual, by the formula

$$(4) \quad \arg \vec{v} = \psi \iff (0 \leq \psi < 2\pi \text{ \& } x = r \cos \psi \text{ \& } y = r \sin \psi),$$

so that, for each nonzero vector $\vec{v} \in \mathbb{R}^2$, the “angle” $\arg \vec{v}$ is a uniquely defined number in the interval $[0, 2\pi)$. Moreover,

$$(5) \quad \arg \vec{v} = \begin{cases} \arccos \frac{x}{r} & \text{if } \vec{v} \in H_-, \\ 2\pi - \arccos \frac{x}{r} & \text{if } \vec{v} \in H_+, \end{cases}$$

where \arccos is the branch of the inverse function \cos^{-1} with values in the interval $[0, \pi]$ and

$$\begin{aligned} H_- &:= \{(x, y) \in \mathbb{R}^2 : y > 0 \text{ or } (y = 0 \text{ \& } x > 0)\}, \\ H_+ &:= \{(x, y) \in \mathbb{R}^2 : y < 0 \text{ or } (y = 0 \text{ \& } x < 0)\}; \end{aligned}$$

note that $H_- \cup H_+ = \mathbb{R}^2 \setminus \{\vec{0}\}$ and $H_- \cap H_+ = \emptyset$.

For any locally-ordinary polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$, introduce also the sequence of the angle arguments the edge-vectors $\overrightarrow{V_0V_1}, \dots, \overrightarrow{V_{n-1}V_n}$ of \mathcal{P} , by the formula

$$\arg \mathcal{P} := (\arg \overrightarrow{V_0V_1}, \dots, \arg \overrightarrow{V_{n-1}V_n}).$$

For any two nonzero vectors \vec{u} and \vec{v} in \mathbb{R}^2 , let us write

$$\vec{u} < \vec{v} \quad \text{iff} \quad \arg \vec{u} < \arg \vec{v};$$

similarly defined is the relation $>$ on $\mathbb{R}^2 \setminus \{\vec{0}\}$.

Remark 1.8. Let $\vec{u} = (s, t)$ and $\vec{v} = (x, y)$ be any two vectors in $\mathbb{R}^2 \setminus \{\vec{0}\}$. Then, using (5), it is elementary but somewhat tedious to check that

$$(6) \quad \vec{u} < \vec{v} \iff \begin{cases} \vec{u} \in H_- \text{ \& } \vec{v} \in H_+ & \text{or} \\ \vec{u} \in H_- \text{ \& } \vec{v} \in H_- \text{ \& } \Delta > 0 & \text{or} \\ \vec{u} \in H_+ \text{ \& } \vec{v} \in H_+ \text{ \& } \Delta > 0, \end{cases}$$

where

$$\Delta := \begin{vmatrix} 1 & 0 & 0 \\ 1 & s & t \\ 1 & x & y \end{vmatrix} = \begin{vmatrix} s & t \\ x & y \end{vmatrix} = sy - tx.$$

Under the additional condition that \vec{u} and \vec{v} are non-collinear, it follows that either $\vec{u} < \vec{v}$ or $\vec{v} < \vec{u}$:

$$\Delta \neq 0 \implies (\vec{u} < \vec{v} \text{ or } \vec{v} < \vec{u}).$$

Note also that

$$(7) \quad \vec{u} < \vec{v} \implies (y \leq 0 \leq t \text{ or } \Delta > 0).$$

Definition 1.9. Let us say that a locally-ordinary polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ with $\arg \mathcal{P} =: (\alpha_0, \dots, \alpha_{n-1})$ is

- increasing – if the sequence $\arg \mathcal{P}$ is increasing: $\alpha_0 < \dots < \alpha_{n-1}$;
- decreasing – if $\alpha_0 > \dots > \alpha_{n-1}$;
- cyclically increasing or, briefly, c-increasing – if

$$\alpha_k < \dots < \alpha_{n-1} < \alpha_0 < \dots < \alpha_{k-1},$$

for some $k \in \overline{0, n-1}$; (if $k = 0$ then this chain of inequalities is supposed to read simply as $\alpha_0 < \dots < \alpha_{n-1}$, in which case polygon \mathcal{P} will be increasing);

- cyclically decreasing or, briefly, c-decreasing – similarly, if

$$\alpha_k > \dots > \alpha_{n-1} > \alpha_0 > \dots > \alpha_{k-1},$$

for some $k \in \overline{0, n-1}$;

- cyclically strictly monotone or, briefly, c-strictly monotone – if \mathcal{P} is either c-increasing or c-decreasing.

The notions of nondecreasing, nonincreasing, c-nondecreasing, c-nonincreasing, and c-monotone polygons are defined similarly, with signs \leq and \geq replacing $<$ and $>$, respectively.

For any $k \in \overline{0, n-1}$, define the cyclic shift (or, briefly, c-shift) θ^k of the sequence (u_0, \dots, u_{n-1}) of any objects u_0, \dots, u_{n-1} by the formula

$$(u_0, \dots, u_{n-1})\theta^k := (u_k, \dots, u_{n-1}, u_0, \dots, u_{k-1}).$$

Remark 1.10. For any polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ and any $k \in \overline{0, n-1}$, one has $\arg(\mathcal{P}\theta^k) = (\arg \mathcal{P})\theta^k$. It follows that \mathcal{P} is c-increasing iff \mathcal{P} is a c-shift of an increasing polygon iff a c-shift of \mathcal{P} is increasing; similarly for “decreasing” vs. “c-decreasing” and for other such pairs of terms. Also, all c-shifts preserve the polygon convexity and all properties defined in Definition 1.2 as well as all the “cyclic” properties defined in Definition 1.9: being c-increasing, being c-increasing,...

For any transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and any polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$, define the corresponding transformation of \mathcal{P} as the polygon $T\mathcal{P} := (TV_0, \dots, TV_{n-1})$. A rotation is any orthogonal (and hence linear) transformation with determinant 1; any rotation can be represented as the linear transformation R_α with matrix $\begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$ for some real number α , so that $R_\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix}$. The reflection is denoted here by R and defined by the formula $\mathbb{R}^2 \ni (x, y) \mapsto R(x, y) := (x, -y)$. Any orthogonal transformation can be represented as $R_\alpha R$ (as well as RR_β) for appropriate α and β . A homothetical transformation is understood here as one of the form $\mathbb{R}^2 \ni \vec{v} \rightarrow \lambda \vec{v}$ for some $\lambda > 0$.

Proposition 1.11. *The properties of being convex; locally-ordinary; ordinary; locally-strict; quasi-strict; strict; locally-simple; and simple are each preserved for any polygon under any nonsingular affine transformation. The properties of being c-increasing, c-decreasing, c-nondecreasing, and c-nonincreasing are each preserved under any rotation or any homothetical transformation or any parallel translation. The properties of being c-increasing and c-decreasing are interchanged under the reflection.*

All the necessary proofs are deferred to Section 2.

The following theorem is one of the main results of this paper.

Theorem 1.12. *An n -gon with $n \geq 3$ is strictly convex iff it is c-strictly monotone.*

Remark 1.13. *Any n -gon with $n \leq 1$ is, trivially, both strictly convex and c-strictly monotone. Any 2-gon is, trivially, strictly convex; however, a 2-gon is c-strictly monotone only if it is locally-ordinary (and hence ordinary). It is easy to see that any strict 3-gon is c-strictly monotone, so that Theorem 1.12 is trivial for $n = 3$. Note also that an n -gon is both c-increasing and c-decreasing iff it is locally-ordinary and $n = 2$.*

Theorem 1.12 is complemented by the following proposition, which will also be of use in the proof of Theorem 1.12.

Proposition 1.14. *For any n -gon \mathcal{P} with $n \geq 3$ the following statements are equivalent to one another:*

- (I) \mathcal{P} is ordinary and locally-strictly convex;
- (II) \mathcal{P} is quasi-strictly convex;
- (III) \mathcal{P} is strictly convex.

Remark. *The conditions in Proposition 1.14 that $n \geq 3$ and \mathcal{P} is ordinary cannot be dropped. Indeed, all 2-gons are strictly convex, but not all of them are ordinary. On the other hand, the polygon $(V_0, V_1, V_2, V_0, V_1, V_2)$ is locally-strictly convex if the points V_0, V_1, V_2 are non-collinear, but it is not ordinary and not strictly convex.*

The following theorem is a “non-strict” counterpart of Theorem 1.12.

Theorem 1.15. *For any n -gon \mathcal{P} with $n \geq 3$ the following statements are equivalent to one another*

- (I) \mathcal{P} is ordinary and locally-simply convex;
- (II) \mathcal{P} is simply convex;
- (III) \mathcal{P} is c-monotone and $\dim \mathcal{P} = 2$.

Remark. None of the following conditions: (i) $n \geq 3$, (ii) \mathcal{P} is ordinary, or (iii) $\dim \mathcal{P} = 2$ in Theorem 1.15 can be dropped. Indeed, (i) all 2-gons are simply convex but not all of them are ordinary; (ii) for any three non-collinear points V_0, V_1, V_2 , the 6-gon $(V_0, V_1, V_2, V_0, V_1, V_2)$ is of dimension 2 and locally-simply convex but not simply convex or c-monotone (or ordinary); (iii) for any three distinct collinear points V_0, V_1, V_2 , the 3-gon (V_0, V_1, V_2) is c-monotone and ordinary but not simply convex or locally-simply convex (or of dimension 2).

A suggestion to use c-strict monotonicity to test for polygon convexity was given in [3], without a proof. A result, similar to Theorem 1.15 was presented in [2, Lemma 5 in Section 10.3], with a very brief, heuristic proof.

2. PROOFS

Proof of Proposition 1.11. Suppose that an n -gon \mathcal{P} with $(\alpha_0, \dots, \alpha_{n-1}) := \arg(\mathcal{P})$ is c-increasing, that is,

$$(\text{Incr}_k) \quad \alpha_k < \dots < \alpha_{n-1} < \alpha_0 < \dots < \alpha_{k-1},$$

for some $k \in \overline{0, n-1}$. Let $(\beta_0, \dots, \beta_{n-1}) := \arg(R\mathcal{P})$, the argument sequence of the reflected polygon $R\mathcal{P}$. Then $\beta_i = 2\pi - \alpha_i$ for all $i \neq k$, while $\beta_k = 2\pi - \alpha_k$ if $\alpha_k \neq 0$ and $\beta_k = 0$ if $\alpha_k = 0$. It follows that

$$(\text{Decr}_k) \quad \beta_k > \dots > \beta_{n-1} > \beta_0 > \dots > \beta_{k-1}$$

if $\alpha_k \neq 0$, and the β_i 's satisfy $(\text{Decr}_{k \oplus 1})$ if $\alpha_k = 0$, where $k \oplus 1 := k+1$ if $k \in \overline{0, n-2}$ and $k \oplus 1 := 0$ if $k = n-1$.

Similarly, if $\arg \mathcal{P} = (\beta_0, \dots, \beta_{n-1})$ satisfies condition (Decr_k) for some $k \in \overline{0, n-1}$, then $(\alpha_0, \dots, \alpha_{n-1}) := \arg(R\mathcal{P})$ satisfies condition (Incr_k) if $\alpha_{k-1} \neq 0$, and the α_i 's satisfy $(\text{Incr}_{k \ominus 1})$ if $\alpha_{k-1} = 0$, where $k \ominus 1 := k-1$ if $k \in \overline{1, n-1}$ and $k \ominus 1 := n-1$ if $k = 0$.

Thus, reflection R interchanges the properties of being c-increasing and c-decreasing.

Let us now verify the preservation of the c-increasing property under any rotation R_α . W.l.o.g., $0 \leq \alpha < 2\pi$. Suppose again that an n -gon \mathcal{P} with $(\alpha_0, \dots, \alpha_{n-1}) := \arg(\mathcal{P})$ satisfies condition (Incr_k) . Then the c-shift $\mathcal{Q} := \mathcal{P}\theta^k := (V_k, \dots, V_{n-1}, V_0, \dots, V_{k-1})$ of polygon \mathcal{P} with $(\beta_0, \dots, \beta_{n-1}) := \arg(\mathcal{Q}) = (\alpha_k, \dots, \alpha_{n-1}, \alpha_0, \dots, \alpha_{k-1})$ is an increasing n -gon. Let $(\psi_0, \dots, \psi_{n-1}) := \arg(R_\alpha \mathcal{Q})$. Let $J := \{i \in \overline{0, n-1} : \beta_i + \alpha \geq 2\pi\}$, and let $j := \min J$ if $J \neq \emptyset$ and $j := n$ if $J = \emptyset$. Then $\psi_i := \beta_i + \alpha$ for $i \in \overline{0, j-1}$ and $\psi_i := \beta_i + \alpha - 2\pi$ for $i \in \overline{j, n-1}$. Hence, the sequence $\arg(R_\alpha \mathcal{Q}\theta^j) = (\varphi_0, \dots, \varphi_{n-1})$ is increasing, where $\varphi_i := \beta_{i+j} + \alpha - 2\pi$ for $i \in \overline{0, n-j-1}$, and $\varphi_i := \beta_{i+j-n} + \alpha$ for $i \in \overline{n-j, n-1}$. Thus, the cyclic permutation $R_\alpha \mathcal{P}\theta^m = R_\alpha \mathcal{P}\theta^{k+j} = R_\alpha \mathcal{Q}\theta^j$ of polygon $R_\alpha \mathcal{P}$ is increasing, where $m := k+j$ if $k+j < n$ and $m := k+j-n$ if $k+j \geq n$. Thus, $R_\alpha \mathcal{P}$ is c-increasing.

The preservation of the c-decreasing property under any rotation is verified quite similarly.

The other claims stated in Proposition 1.11 are only easier to check. \square

Proof of Theorem 1.12. Let $\mathcal{P} = (V_0, \dots, V_{n-1})$ be an n -gon with $n \geq 3$, vertices $V_i = (x_i, y_i)$, and argument $(\alpha_0, \dots, \alpha_{n-1}) := \arg \mathcal{P}$. In view of Proposition 1.11, the rotation $R_{2\pi-\alpha_0}$ and any homothetical transformation will preserve both the convexity and c-monotonicity properties of \mathcal{P} . Therefore, assume without loss of generality (w.l.o.g.) that $\alpha_0 = 0$ and, moreover, $V_0 = (0, 0)$ and $V_1 = (1, 0)$.

“If” When proving this part, assume w.l.o.g. that \mathcal{P} is c-increasing, that is, $\alpha_k < \dots < \alpha_{n-1} < \alpha_0 < \dots < \alpha_{k-1}$. (Indeed, in view of Proposition 1.11, the reflection transformation R will preserve the convexity property of \mathcal{P} and interchange the property of \mathcal{P} being c-increasing with it being c-decreasing; also, R will preserve the property $\alpha_0 = 0$.) Then the conditions $\alpha_0 = 0$ and $\alpha_i \in [0, 2\pi) \forall i$ imply that $k = 0$ and $\alpha_0 = 0 < \dots < \alpha_{n-1}$; that is, the sequence $\arg \mathcal{P}$ is increasing.

Hence, inequality $\alpha_1 \geq \pi$ would imply $\alpha_i \in (\pi, 2\pi)$ for all $i \in \overline{2, n-1}$. Hence and because $n \geq 3$, one would have $0 = y_1 \geq y_2 > y_3 > \dots > y_n = y_0 = 0$, and at least one inequality here is strict (since $n \geq 3$), which is a contradiction.

The case $\alpha_1 < \pi$ is similar. In this case, $y_2 > 0$. To obtain a contradiction, suppose that the set $L := \{i \in \overline{2, n-1} : y_i \leq 0\}$ is non-empty and then let $\ell := \min L$, so that $\ell \in \overline{3, n-1}$, $y_{\ell-1} > 0$, and $y_\ell \leq 0$. Then $\alpha_{\ell-1} \in (\pi, 2\pi)$. Hence and because the sequence $\arg \mathcal{P}$ is increasing, one has $\alpha_i \in (\pi, 2\pi)$ for all $i \in \overline{\ell-1, n-1}$. Therefore, $0 \geq y_\ell > \dots > y_n = y_0 = 0$, which is a contradiction. This contradiction means that $L = \emptyset$, so that $y_i > 0$ for all $i \in \overline{2, n-1}$; that is, according to [7, Definition 2.4], the polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is strictly to one side of its edge $[V_0, V_1]$.

Similarly it is proved that \mathcal{P} is strictly to one side of any other one of its edges; that is, \mathcal{P} is strictly to-one-side. To complete the proof of the “if” part of Theorem 1.12, it remains to refer to [7, Lemmas 2.6 and 2.11].

“Only if” Here is assumed that polygon \mathcal{P} is strictly convex. Again w.l.o.g. one has $\alpha_0 = 0$. Also, by Remark 1.13, w.l.o.g. $n \geq 4$. Again by the “reflection” part of Proposition 1.11, w.l.o.g. $y_2 \geq 0$. Moreover, because of the strictness of \mathcal{P} and the assumptions $V_0 = (0, 0)$ and $V_1 = (1, 0)$, one actually has $y_2 > 0$, so that $\alpha_0 = 0 < \alpha_1 < \pi$ and $\Delta_{0,1,2} = y_2 > 0$. So, the strict convexity of \mathcal{P} and Remark 1.7 yield $\Delta_{0,1,i} = y_i > 0$ for all $i \in \overline{2, n-1}$. The strictness of \mathcal{P} also implies that all the values $\alpha_0, \dots, \alpha_{n-1}$ are distinct from one another.

Thus, it suffices to show that $\alpha_i \leq \alpha_{i+1}$ for all $i \in \overline{1, n-2}$. Suppose the contrary, that $\alpha_i > \alpha_{i+1}$ for some $i \in \overline{1, n-2}$. Consider separately the following three cases.

Case 1: $i = 1$. Then $\alpha_1 > \alpha_2$. By (7), this implies that $\Delta_{1,2,3} \leq 0$ or $y_2 - y_1 \leq 0 \leq y_3 - y_2$; but $y_2 - y_1 = y_2 > 0$, so that one must have $\Delta_{1,2,3} \leq 0$; now inequalities $\Delta_{1,2,3} \leq 0$ and $\Delta_{0,1,2} > 0$ contradict Remark 1.7.

Case 2: $i = n - 2$. Then $\alpha_{n-2} > \alpha_{n-1}$. This case is quite similar to Case 1. Indeed, by (7), here one has $\Delta_{0,n-2,n-1} = \Delta_{n-2,n-1,0} \leq 0$ or $0 \leq y_0 - y_{n-1}$; but $y_0 - y_{n-1} = -y_{n-1} < 0$, so that $\Delta_{0,n-2,n-1} \leq 0$; now inequalities $\Delta_{0,n-2,n-1} \leq 0$ and $\Delta_{0,1,2} > 0$ contradict Remark 1.7.

Case 3: $i \in \overline{2, n-3}$ and $\alpha_i > \alpha_{i+1}$. Then the 5-gon $\mathcal{Q} := (V_0, V_1, V_i, V_{i+1}, V_{i+2})$ is a sub-polygon of \mathcal{P} , so that \mathcal{Q} is strictly convex, by [6, Corollary 1.17]. On the other hand, $\arg \mathcal{Q} = (\alpha_0, \beta, \alpha_i, \alpha_{i+1}, \gamma)$, for some real numbers β and γ . Thus, w.l.o.g. $\mathcal{P} = \mathcal{Q}$, $n = 5$, and so, one has all of the following: $\mathcal{P} = (V_0, V_1, V_2, V_3, V_4)$; $i = 2$; $\alpha_2 > \alpha_3$; and $\Delta_{0,1,i} = y_i > 0$ for all $i \in \overline{2, 4}$. By Remark 1.7, one now also sees that the determinants $\Delta_{2,3,4}$, $\Delta_{0,2,3}$, and $\Delta_{0,3,4}$ are all strictly positive as well. Therefore, the condition $\alpha_2 > \alpha_3$ and implication (7) yield $y_3 - y_2 \leq 0 \leq y_4 - y_3$. One can verify the identity

$$(y_4 - y_3) \Delta_{0,2,3} + (y_2 - y_3) \Delta_{0,3,4} + \Delta_{2,3,4} \Delta_{0,1,3} = 0,$$

whose left-hand side is strictly positive, since all the determinants $\Delta_{\cdot,\cdot,\cdot}$ in this identity are strictly positive and because of the condition $y_3 - y_2 \leq 0 \leq y_4 - y_3$. Thus, one obtains a contradiction.

The proof of the “only if” part and thus of entire Theorem 1.12 is now complete. \square

Let $U _ V _ W _ \dots$ (respectively, $U, \widehat{V, W}, \dots$) mean that U, V, W, \dots are collinear (respectively, non-collinear) points on the plane.

Lemma 2.1. [Cf. [7, Lemma 2.7].] *For any choice of α , β , i , and j in $\overline{0, n-1}$, points V_α and V_β are to one side of $[V_i, V_j]$ if and only if $\Delta_{\alpha,i,j} \Delta_{\beta,i,j} \geq 0$, where $\Delta_{\alpha,i,j}$ are given by (2); for an exact definition of “to one side”, see [7, Definition 2.4].*

Proof of Lemma 2.1. This proof can be done quite similarly to that of [7, Lemma 2.7]. Alternatively and more simply, Lemma 2.1 can be easily deduced from [7, Lemma 2.7] by observing that $V_\alpha _ V_i _ V_j$ if and only if $\Delta_{\alpha,i,j} = 0$. \square

Proof of Proposition 1.14. W.l.o.g., $n \geq 4$.

(I) \implies (II) Here it is assumed that a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is ordinary and locally-strictly convex. By the c-shift invariance (Remark 1.10), at this point it suffices to show that $\widehat{V_0, V_1, V_i}$ for each $i \in \overline{2, n-1}$ or just only for each $i \in \overline{3, n-1}$, because $\widehat{V_0, V_1, V_2}$ follows from \mathcal{P} being locally-strict.

To obtain a contradiction, assume first that $V_0 _ V_1 _ V_3$. Since \mathcal{P} is locally-strictly convex, one has $\widehat{V_0, V_1, V_2}$, so that, by the affine invariance (Proposition 1.11), w.l.o.g. $V_0 = (0, 0)$, $V_1 = (1, 0)$, and $V_2 = (1, 1)$. Then $V_0 _ V_1 _ V_3$ implies that $V_3 = (x_3, 0)$ for some real x_3 . Since \mathcal{P} is ordinary, one has $x_3 \notin \{0, 1\}$. Hence, there are only the following three cases to consider at this point:

Case 1: $x_3 > 1$. Then $\Delta_{1,2,0} \Delta_{1,2,3} = 1 - x_3 < 0$. By Lemma 2.1, this means that V_0 and V_3 are not on one side of $[V_1, V_2]$; by [6, Lemma 2.3], this contradicts the

convexity of polygon \mathcal{P} .

Case 2: $0 < x_3 < 1$. Then $\Delta_{2,3,0} \Delta_{2,3,1} = x_3(x_3 - 1) < 0$, so that V_0 and V_1 are not on one side of $[V_2, V_3]$, which contradicts the convexity of polygon \mathcal{P} .

Case 3: $x_3 < 0$. Then $V_{n-1} = (x_{n-1}, y_{n-1})$ for some real x_{n-1} and y_{n-1} such that $y_{n-1} \neq 0$ – because \mathcal{P} is locally strict and hence $\widehat{V_{n-1}, V_0, V_1}$. Hence, $\Delta_{n-1,0,1} \Delta_{n-1,0,3} = y_{n-1}^2 x_3 < 0$, so that V_0 and V_3 are not on one side of edge $[V_{n-1}, V_0]$, which contradicts the convexity of polygon \mathcal{P} .

Thus, in all cases the assumption $\widehat{V_0, V_1, V_3}$ leads to a contradiction, so that one has $\widehat{V_0, V_1, V_3}$. Similarly one proves that $\widehat{V_1, V_3, V_4}$ (for $n = 4$ this has been already proved, for then $V_4 = V_0$). That is, the sub-polygon $\mathcal{P}^{(2)} := (V_0, V_1, V_3, \dots, V_{n-1})$ is locally strict. Also, by Theorem 1.4, $\mathcal{P}^{(2)}$ inherits the property of \mathcal{P} of being ordinarily convex. Now it follows by induction that $\mathcal{P}^{(2)}$ is quasi-strict, so that indeed $\widehat{V_0, V_1, V_i}$ for each $i \in \overline{3, n-1}$. This proves (I) \implies (II).

(II) \implies (III) This is immediate from [7, Lemma 2.11].

(III) \implies (I) This is immediate from [7, Lemmas 2.11 and 2.5]. \square

Introduce the “direction” equivalence on $\mathbb{R}^2 \setminus \{\vec{0}\}$ defined by the formula

$$\vec{u} \uparrow\uparrow \vec{v} \stackrel{\text{def}}{\iff} \vec{v} = \lambda \vec{u} \text{ for some } \lambda > 0,$$

for any \vec{u} and \vec{v} in $\mathbb{R}^2 \setminus \{\vec{0}\}$. Note that

$$(8) \quad \vec{u} \uparrow\uparrow \vec{v} \implies (\vec{u} + \vec{v}) \uparrow\uparrow \vec{u}.$$

Lemma 2.2. *For any two vectors \vec{u} and \vec{v} in $\mathbb{R}^2 \setminus \{\vec{0}\}$, one has $\arg \vec{u} = \arg \vec{v}$ iff $\vec{u} \uparrow\uparrow \vec{v}$.*

Proof of Lemma 2.2. This follows immediately from (4). \square

Lemma 2.3. *Suppose that a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is locally-ordinary and for some j and k in $\overline{0, n-1}$ such that $j \leq k$ one has $\alpha_j = \dots = \alpha_k$, where $(\alpha_0, \dots, \alpha_{n-1}) := \arg \mathcal{P}$. Then $\overrightarrow{V_j V_{k+1}} \neq \vec{0}$ and*

- (I) $\arg \overrightarrow{V_j V_{k+1}} = \arg \overrightarrow{V_i V_{i+1}}$ for all i in $\overline{j, k}$;
- (II) $[V_j, V_{k+1}] = \bigcup_{i=j}^k [V_i, V_{i+1}]$;
- (III) $[V_i, V_{i+1}] \cap [V_t, V_{t+1}] = \emptyset$ for all i and t in $\overline{j, k}$ such that $i \neq t$.

Proof of Lemma 2.3. By Lemma 2.2, the condition $\alpha_j = \dots = \alpha_k$ implies that for each i in $\overline{0, n-2}$ one has $\overrightarrow{V_{i+1} V_{i+2}} \uparrow\uparrow \overrightarrow{V_i V_{i+1}}$, so that, in view of (8), $\overrightarrow{V_j V_{k+1}} \uparrow\uparrow \overrightarrow{V_i V_{i+1}}$, which is equivalent to (I), again by Lemma 2.2.

In particular, these observations imply that the points V_j, \dots, V_{k+1} lie on the same straight line, say ℓ . Let A be any non-singular affine mapping of ℓ onto \mathbb{R} such that $AV_j < AV_{j+1}$ (such a mapping exists because \mathcal{P} is locally-ordinary and hence $V_j \neq V_{j+1}$). Then the condition $\overrightarrow{V_{i+1} V_{i+2}} \uparrow\uparrow \overrightarrow{V_i V_{i+1}}$ implies $V_{i+1} \in (V_i, V_{i+2})$, whence $AV_{i+1} \in (AV_i, AV_{i+2})$, again for each i in $\overline{0, n-2}$, so that $AV_j < \dots <$

AV_{k+1} , $[AV_j, AV_{k+1}] = \bigcup_{i=j}^k [AV_i, AV_{i+1}]$, and $[AV_i, AV_{i+1}] \cap [AV_t, AV_{t+1}] = \emptyset$ for all distinct i and t in $\overline{j, k}$. It remains to note that $[AU, AV] = A[U, V]$ for any points U and V on ℓ and recall that A is non-singular and hence one-to-one. \square

Lemma 2.4. *Any strictly convex n -gon with $n \geq 3$ is simple.*

Proof of Lemma 2.4. Suppose that an n -gon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is strictly convex and take any i and j in $\overline{0, n-1}$ such that $i < j$. We have then to show that $[V_i, V_{i+1}] \cap [V_j, V_{j+1}] = \emptyset$. By the c-shift invariance (Remark 1.10), w.l.o.g. $i = 0$ and then $j \in \overline{1, n-1}$. By Remark 1.3, \mathcal{P} is ordinary, so that $V_1 \neq V_0$. Let $\ell := V_0V_1$, the straight line through V_0 and V_1 . By [7, Lemma 2.6], there is an open half-plane H such that $\partial H = \ell$ and $\{V_2, \dots, V_{n-1}\} \subset H$. Then (with $i = 0$)

$$(9) \quad [V_i, V_{i+1}] \cap [V_j, V_{j+1}] = [V_0, V_1] \cap (\ell \cap [V_j, V_{j+1}]) \subseteq [V_0, V_1] \cap \{V_j\} = \emptyset;$$

the set inclusion in (9) follows because $(V_j, V_{j+1}) = \text{ri}[V_j, V_{j+1}] \subseteq \text{interior}(\ell \cup H) = H \subseteq \mathbb{R}^2 \setminus \ell$; the last equality in (9) is trivial for $j = 1$ and takes place for $j \in \overline{2, n-1}$ because then $[V_0, V_1] \cap \{V_j\} \subseteq \ell \cap H = \emptyset$. \square

Lemma 2.5. *Suppose that a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ is ordinary and locally-simple. Suppose also that for some j and k in $\overline{0, n-1}$ one has $V_j \dots \dots V_{k+1}$. Then $\alpha_j = \dots = \alpha_k$, where $(\alpha_0, \dots, \alpha_{n-1}) := \arg \mathcal{P}$.*

Proof of Lemma 2.5. Take any i in $\overline{j, k-1}$. Then $\overrightarrow{V_{i+1}V_{i+2}} = \mu_i \overrightarrow{V_iV_{i+1}}$ for some $\mu_i \neq 0$, given the conditions that $V_j \dots \dots V_{k+1}$ and \mathcal{P} is ordinary. Then inequality $\mu_i < 0$ would imply $[V_i, V_{i+1}] \cap [V_{i+1}, V_{i+2}] \supseteq (\tilde{V}_i, V_{i+1}) \neq \emptyset$, since \mathcal{P} is ordinary, where $\tilde{V}_i := V_i$ if $\mu_i \leq -1$ and $\tilde{V}_i := V_{i+2}$ if $-1 < \mu_i < 0$; so, this would contradict the condition that \mathcal{P} is locally-simple.

It follows that $\mu_i > 0$ and hence $\overrightarrow{V_{i+1}V_{i+2}} \uparrow \overrightarrow{V_iV_{i+1}}$ for all i in $\overline{j, k-1}$. It remains to refer to Lemma 2.2. \square

Proof of Theorem 1.15.

(I) \implies (III) Here it is assumed that an n -gon $\mathcal{P} = (V_0, \dots, V_{n-1})$, with $n \geq 3$ and $(\alpha_0, \dots, \alpha_{n-1}) := \arg \mathcal{P}$, is ordinary and locally-simply convex. We have to prove that \mathcal{P} is c-monotone. Introduce the set

$$M := \{i \in \overline{0, n} : \widehat{V_{i-1}, V_i, V_{i+1}}\},$$

where $V_{-1} := V_{n-1}$ and $V_{n+1} := V_1$.

Note that $M \neq \emptyset$. Indeed, otherwise one would have $\alpha_0 = \dots = \alpha_{n-1}$, by Lemma 2.5; then Lemma 2.3(II) would imply $[V_0, V_1] \subseteq [V_0, V_n] = \{V_0\}$, which would contradict the condition that \mathcal{P} is ordinary.

Therefore, $\dim \mathcal{P} = 2$. Also, by the c-shift invariance (Remark 1.10), w.l.o.g. $0 \in M$ and hence (by the definition of M) one has $n \in M$, so that

$$M = \{j_0, \dots, j_m\}$$

for some $m \in \overline{0, n-1}$ and integers j_0, \dots, j_m such that $0 = j_0 < \dots < j_m = n$, and, moreover,

$$(10) \quad \begin{array}{ccc} \widehat{V_{n-1}, V_0}, V_1, & V_0 = V_{j_0} \dash\dash\dash V_{j_1}; \\ \widehat{V_{j_1-1}, V_{j_1}, V_{j_1+1}}, & V_{j_1} \dash\dash\dash V_{j_2}; \\ \vdots & \\ \widehat{V_{j_{m-1}-1}, V_{j_{m-1}}, V_{j_{m-1}+1}}, & V_{j_{m-1}} \dash\dash\dash V_{j_m} = V_n = V_0. \end{array}$$

By Theorem 1.4, the sub-polygon

$$(11) \quad \mathcal{Q} := (U_0, \dots, U_{m-1}) := (V_{j_0}, \dots, V_{j_{m-1}})$$

of the ordinarily convex polygon \mathcal{P} is ordinarily convex as well.

Note also that \mathcal{Q} is locally-strict. To check this, in view of the c-shift invariance (Remark 1.10) it suffices to show that $\widehat{U_0, U_1, U_2}$ or, equivalently, $\widehat{V_{j_0}, V_{j_1}, V_{j_2}}$. But this follows by the ordinariness of \mathcal{P} and construction of \mathcal{Q} (whereby $\widehat{V_{j_1-1}, V_{j_1}, V_{j_1+1}}, \widehat{V_{j_0} \dash\dash V_{j_1-1} \dash\dash V_{j_1}},$ and $\widehat{V_{j_1} \dash\dash V_{j_1+1} \dash\dash V_{j_2}}$).

Hence, by Proposition 1.14, \mathcal{Q} is strictly convex. Now, in view of Theorem 1.12, \mathcal{Q} is c-strictly monotone. Applying (if necessary) the reflection transformation R and a cyclic shift θ^k and referring to Proposition 1.11 and Remark 1.10, assume w.l.o.g. that polygon \mathcal{Q} is increasing:

$$(12) \quad \arg \overrightarrow{V_{j_0} V_{j_1}} < \dots < \arg \overrightarrow{V_{j_{m-1}} V_{j_m}}.$$

To complete the proof of implication (I) \implies (III) of Theorem 1.15, it remains to refer to (10), Lemma 2.5, and (12), whereby

$$\begin{aligned} \alpha_0 &= \alpha_{j_0} = \dots = \alpha_{j_1-1} = \arg \overrightarrow{V_{j_0} V_{j_1}} \\ &< \arg \overrightarrow{V_{j_1} V_{j_2}} = \alpha_{j_1} = \dots = \alpha_{j_2-1} \\ &\vdots \\ &< \arg \overrightarrow{V_{j_{m-1}} V_{j_m}} = \alpha_{j_{m-1}} = \dots = \alpha_{j_m-1} = \alpha_{n-1}. \end{aligned}$$

(III) \implies (II) Here it is assumed that a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ with $(\alpha_0, \dots, \alpha_{n-1}) := \arg \mathcal{P}$ is c-monotone. We have to prove that \mathcal{P} is simply convex. Applying the reflection transformation R and a cyclic shift θ^k , w.l.o.g. polygon \mathcal{P} is non-decreasing:

$$\alpha_0 \leq \dots \leq \alpha_{n-1}.$$

That is,

$$(13) \quad \alpha_{j_0} = \dots = \alpha_{j_1-1} < \dots < \alpha_{j_{m-1}} = \dots = \alpha_{j_m-1}$$

for some natural m and integer j_0, \dots, j_m such that

$$(14) \quad 0 = j_0 < \dots < j_m = n.$$

Define polygon \mathcal{Q} again by (11). Then, in view of Lemma 2.3(I), one has $\arg \mathcal{Q} = (\alpha_{j_0}, \dots, \alpha_{j_{m-1}})$, so that \mathcal{Q} is increasing and hence, by Theorem 1.12, strictly convex.

By Lemma 2.3(II), one has

$$[U_p, U_{p+1}] = [V_{j_p}, V_{j_{p+1}}] = \bigcup_{i=j_p}^{j_{p+1}-1} [V_i, V_{i+1}]$$

for all $p \in \overline{0, m-1}$, so that $\text{edg } \mathcal{Q} = \text{edg } \mathcal{P}$. Therefore, moreover,

$$\{V_0, \dots, V_{n-1}\} \subseteq \text{edg } \mathcal{P} = \text{edg } \mathcal{Q} \subseteq \text{conv } \mathcal{Q} \subseteq \text{conv } \mathcal{P},$$

whence $\text{conv } \mathcal{P} = \text{conv } \mathcal{Q}$. Thus, the convexity of \mathcal{Q} implies the convexity of \mathcal{P} and $\dim \mathcal{Q} = \dim \mathcal{P} = 2$, so that $m \geq 3$.

Let us now show that \mathcal{P} simple. That is, let us take any i and j in $\overline{0, n-1}$ such that $i \neq j$ and show that $[V_i, V_{i+1}] \cap [V_j, V_{j+1}] = \emptyset$. By (14), there exist p and q in $\overline{0, m-1}$ such that $i \in J_p$ and $j \in J_q$, where $J_s := \overline{j_s, j_{s+1}-1}$ for all $s \in \overline{0, m-1}$.

If $p = q$ then $[V_i, V_{i+1}] \cap [V_j, V_{j+1}] = \emptyset$ follows from Lemma 2.3(III) and (13).

If $p \neq q$ then, by Lemma 2.3(II),

$$[V_i, V_{i+1}] \cap [V_j, V_{j+1}] \subseteq [U_p, U_{p+1}] \cap [U_q, U_{q+1}] = \emptyset,$$

the latter equality taking place since $m \geq 3$ and the m -gon \mathcal{Q} is strictly convex and hence (by Lemma 2.4) simple.

This completes the proof of implication (III) \implies (II) of Theorem 1.15.

(II) \implies (I) This implication is the easiest to prove. Indeed, if a polygon \mathcal{P} is simple then it is (trivially) locally-simple and (by Remark 1.3) ordinary. This completes the proof of Theorem 1.15. \square

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Department of Mathematical Sciences
Michigan Technological University
Houghton, MI 49931
USA
e-mail: ipinelis@mtu.edu