

Space-time codes with controllable ML decoding complexity for any number of transmit antennas

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Abstract— We construct a class of linear space-time block codes for any number of transmit antennas that have controllable ML decoding complexity with a maximum rate of 1 symbol per channel use. The decoding complexity for M transmit antennas can be varied from ML decoding of $M/2$ symbols together to single symbol ML decoding. A significant result is obtained that one can construct rate-1, full-diversity space-time codes whose ML decoding involves decoding symbols in pairs for any number of transmit antennas. This result was known thus far for only select number of transmit antennas.

I. INTRODUCTION

Multiple antenna systems have been of great interest in recent times, because of their ability to support higher data rates at same bandwidth and noise conditions; see e.g. [1],[2], [3] and references therein. One of the key aspects of orthogonal designs has been to ensure diversity for *any* symbol constellation. For more than two transmit antennas and complex constellations, these codes offered on the average a rate of less than one symbol per channel use.

The loss of rate has been addressed by the use of quasi-orthogonal codes that make the groups of symbols orthogonal where each group has more than one symbol in general [9], [10], [12], [14]. A fully orthogonal code would have just one symbol per group. Because of this relaxation of constraints, these codes achieve higher code rates that were hitherto not possible with orthogonal codes. It was shown in [11], [13], [15], [16] that performance of above quasi-orthogonal codes can be improved with constellation rotation.

In this paper, we construct that a new class of space-time codes that are inspired from the codes in [14] that have a useful property that the ML decoding is controllable and have maximum rate of one. On one extreme, one can design codes that have single symbol ML decoding and on the other, the ML decoding requires decoding of $M/2$ symbols together, where M are the number of transmit antennas.

It is, however, shown for the constructed codes that for rate one codes with single symbol ML decoding, full-diversity is impossible and for codes that require more than one symbols to be decoded together for ML symbols decoding, it is indeed possible to have full-diversity. Thus one can achieve full-diversity with rate 1 by decoding the symbols in pairs. This is a novel result to the best of our knowledge. Designing codes with higher ML decoding complexity can be done to achieve higher coding gain.

We use the following notation throughout the paper: $*$, T and \dagger denote the conjugate, transpose and conjugate transpose respectively of a matrix or a vector; \mathbf{I}^M and $\mathbf{0}^M$ are $M \times M$ identity and null matrices respectively; $\|A\|_F$, $\det(A)$ and $\text{Tr}(A)$ denote Frobenius norm, determinant and Trace of matrix A respectively; \mathbb{C} denotes the complex number field; $\mathcal{CN}(0, 1)$ denotes a circularly symmetric complex Gaussian variable with zero mean and unit variance.

II. SYSTEM MODEL AND DESIGN CRITERION

Consider a system of M transmit and N receive antennas that we refer to as (M, N) system in this paper. The statistically independent modulated information symbols to be transmitted, are taken P at a time to form a $P \times 1$ vector denoted by $\mathbf{c} = (c_1, \dots, c_P)^T$. This information vector is pre-coded (i.e. multiplied) by a $M \times P$ matrix denoted by \mathcal{R} . Let $\mathbf{s} = (s_1, \dots, s_M)^T$ and

$$\mathbf{s} = \mathcal{R}\mathbf{c} \quad (1)$$

with $E\{|s_i|^2\} = 1$, $i = 1, \dots, M$. As we shall soon see, the choice of \mathcal{R} is central to the construction of codes. \mathbf{s} is the input to a linear space-time block code that outputs a $M \times M$ matrix $G_P[\mathbf{s}]$, where

$$G_M[\mathbf{s}] = \sum_{m=1}^M (C_m s_m + D_m s_m^*), \quad (2)$$

where $C_m, D_m, m = 1, \dots, M$, are $M \times M$ complex matrices, which completely specify the code. This code is transmitted in M channel uses and the average code rate is hence P/M symbols per channel use. For quasi-static fading channel, the received signal is given by

$$X[\mathbf{s}] = \sqrt{\frac{\rho}{M}} G_M[\mathbf{s}] H + V, \quad (3)$$

where X and V are the $M \times N$ received and noise matrices, and H is the $M \times N$ complex channel matrix that is assumed to be constant over M channel uses and varies independently over the next M channel uses and so on. The entries of H and V are assumed to be mutually independent and $\mathcal{CN}(0, 1)$, and ρ is the average SNR per received antenna. We assume that channel is perfectly known at the receiver but is unknown at the transmitter.

It has been shown in [1] by examining the pair-wise probability of error between two distinct information vectors (say $\mathbf{u}, \mathbf{v} \in \mathbb{C}^P$), that for full-diversity, in quasi-static fading channels, $G_M^\dagger[\mathcal{R}(\mathbf{u} - \mathbf{v})]G_M[\mathcal{R}(\mathbf{u} - \mathbf{v})]$ should have a rank of M . For square code matrices, the above criterion could be modified to yield

$$\min_{\mathbf{u}, \mathbf{v}} \det\{G_M[\mathcal{R}(\mathbf{u} - \mathbf{v})]\} \neq 0 \quad (4)$$

III. ITERATIVE CONSTRUCTION OF SPACE-TIME CODES

We first consider the case of M being a power of 2. Case of other M is dealt easily by deleting columns of G_M . The main difference between these codes and those in [14] is the choice of \mathcal{R} that will allow us to vary the ML decoding complexity and construct full-diversity codes with decoding of a pair of symbols.

Let us define two disjoint partition vectors that are function of the vector \mathbf{s} (whose length will be clear from the context) denoted by $\mathcal{A}_{M,1}(\mathbf{s})$ and $\mathcal{A}_{M,2}(\mathbf{s})$. These partition vectors have same length as \mathbf{s} and have the same symbols as \mathbf{s} in indices they possess and zeros in other indices. If we denote the first and last M elements of a $2M \times 1$ vector \mathbf{s} by $\mathbf{s}_{M,1}$ and $\mathbf{s}_{M,2}$ respectively, then these partitions are iteratively constructed as

$$\mathcal{A}_{2M,1}(\mathbf{s}) = \mathcal{A}_{M,1}(\mathbf{s}_{M,1}) + \mathcal{A}_{M,2}(\mathbf{s}_{M,2}) \quad (5)$$

$$\mathcal{A}_{2M,2}(\mathbf{s}) = \mathcal{A}_{M,2}(\mathbf{s}_{M,1}) + \mathcal{A}_{M,1}(\mathbf{s}_{M,2}), \quad (6)$$

and the code is iteratively constructed for the i th partition as

$$G_{2M}[\mathcal{A}_{2M,i}(\mathbf{s})] = \begin{bmatrix} G_M[\mathcal{A}_{M,i}(\mathbf{s}_{M,1})] & G_M[\mathcal{A}_{M,\bar{i}}(\mathbf{s}_{M,2})] \\ -G_M[\mathcal{A}_{M,\bar{i}}(\mathbf{s}_{M,2})] & G_M[\mathcal{A}_{M,i}(\mathbf{s}_{M,1})] \end{bmatrix}, \quad (7)$$

where $\bar{i} = 2$, if $i = 1$ and is 1 otherwise, and hence by using linearity, we have

$$G_{2M}[\mathbf{s}] = \begin{bmatrix} G_M[\mathbf{s}_{M,1}] & G_M[\mathbf{s}_{M,2}] \\ -G_M[\mathbf{s}_{M,2}^*] & G_M[\mathbf{s}_{M,1}^*] \end{bmatrix}, \quad (8)$$

where $G_1[\mathbf{s}] \triangleq s_1 \forall \mathbf{s} \in \mathbb{C}^1$, $\mathcal{A}_{1,1} = s_1$, and $\mathcal{A}_{2,1}$ is a null set.

A. Receiver Processing

We give a practical decoding algorithm to have a low complexity ML decoding done over a single partition. We note from (8) that any row of the constructed code either contains the symbols (s_i 's) or its conjugates (with a possible sign change). For any $\mathbf{h} \in \mathbb{C}^{M \times 1}$, define a transformation denoted by \mathcal{T} that takes conjugates of those elements of $M \times 1$ vector $G_M[\mathbf{s}]\mathbf{h}$ that contain conjugates of elements of \mathbf{s} , and we can write

$$\mathcal{T}\{G_M[\mathcal{A}_{M,i}(\mathbf{s})]\mathbf{h}\} = \mathcal{E}_{M,i}(\mathbf{h})v_{M,i}(\mathbf{s}), \quad (9)$$

where $\mathcal{E}_{M,i}$'s are $M \times (M/2)$ matrices dependent only on \mathbf{h} , $v_{M,i}$'s are $(M/2) \times 1$ vectors that contain symbols from partition i , with $i = 1, 2$. We need a few results from [14] that we state here without proof.

Proposition 1: For any $\mathbf{h}, \mathbf{s} \in \mathbb{C}^{M \times 1}$,

$$G_M^\dagger[\mathcal{A}_{M,1}(\mathbf{s})]G_M[\mathcal{A}_{M,2}(\mathbf{s})] + G_M^\dagger[\mathcal{A}_{M,2}(\mathbf{s})]G_M[\mathcal{A}_{M,1}(\mathbf{s})] = \mathbf{0}^M, \quad (10)$$

$$\mathcal{E}_{M,1}^\dagger(\mathbf{h})\mathcal{E}_{M,2}(\mathbf{h}) = \mathbf{0}^{M/2}, \quad (11)$$

$$\det\{G_{2M}[\mathcal{A}_{2M,1}(\mathbf{s})]\} = \det\{G_M[\mathcal{A}_{M,1}(\mathbf{s}_{M,1} - \hat{\mathbf{s}}_{M,2})]\} \times \det\{G_M[\mathcal{A}_{M,1}(\mathbf{s}_{M,1} + \hat{\mathbf{s}}_{M,2})]\}, \quad (12)$$

where for any $2M \times 1$ vector \mathbf{z} , we define a transformation denoted by $\hat{\mathbf{z}}$ that interchanges the two halves of \mathbf{z} with a sign change for the second half, i.e. $\hat{\mathbf{z}} = [-z_{M+1}, \dots, -z_{2M}, z_1, \dots, z_M]$.

By taking conjugates appropriately, we can derive a modified signal model from (3) for receive antenna n , ($n = 1, \dots, N$), as

$$\hat{X}_n(\mathbf{s}) = \sqrt{\frac{\rho}{M}} [\mathcal{E}_{M,1}(H_n)v_{M,1}(\mathbf{s}) + \mathcal{E}_{M,2}(H_n)v_{M,2}(\mathbf{s})] + \hat{V}_n, \quad (13)$$

where H_n is the n th column of H and \hat{X}_n and \hat{V}_n are derived from n th column of X and V respectively by taking the conjugates of some or all their elements. Let the singular value decomposition (SVD) of $\mathcal{E}_{M,i}(H_n)$ be given by

$$\mathcal{E}_{M,i}(H_n) = U_{M,i}S_{M,i}W_{M,i}^\dagger, \quad (14)$$

where $U_{M,i}$ and $W_{M,i}$ are unitary and $S_{M,i}$ is a $M \times (M/2)$ diagonal matrix. Let $\hat{S}_{M,i}$ be a $M \times (M/2)$ diagonal matrix whose diagonal elements are inverse of diagonal elements of $S_{M,i}$ and hence

$$\hat{S}_{M,i}S_{M,i}^\dagger = \begin{bmatrix} \mathbf{I}^{M/2} & \mathbf{0}^{M/2} \\ \mathbf{0}^{M/2} & \mathbf{0}^{M/2} \end{bmatrix} \quad (15)$$

and $\hat{S}_{M,i}S_{M,i}^\dagger S_{M,i} = S_{M,i}$. Multiplying both sides of (13) by $U_{M,i}\hat{S}_{M,i}W_{M,i}^\dagger\mathcal{E}_{M,i}^\dagger(H_n) = U_{M,i}\hat{S}_{M,i}S_{M,i}^\dagger U_{M,i}^\dagger$, we get

$$U_{M,i}\hat{S}_{M,i}S_{M,i}^\dagger U_{M,i}^\dagger \hat{X}_n(\mathbf{s}) = \sqrt{\frac{\rho}{M}} \mathcal{E}_{M,i}(H_n)v_{M,i}(\mathbf{s}) + U_{M,i}\hat{S}_{M,i}S_{M,i}^\dagger U_{M,i}^\dagger V_n, \quad (16)$$

where we have used $\mathcal{E}_{M,1}^\dagger(H_n)\mathcal{E}_{M,2}(H_n) = 0$ to cancel the contribution of other partition. Note that using (15), it follows that $\hat{V}_n = U_{M,i}\hat{S}_{M,i}S_{M,i}^\dagger U_{M,i}^\dagger V_n$ has the same the statistics as V_n . We can rewrite Eq. (16) as

$$\hat{X}_n(\mathbf{s}) = \sqrt{\frac{\rho}{M}} S_{M,i}W_{M,i}^\dagger v_{M,i}(\mathbf{s}) + \hat{V}_n \quad (17)$$

Using (7), one can iteratively generate the equivalent channels for each partitions with $\mathbf{h}_{M,1} = [h_1, \dots, h_M]$ and $\mathbf{h}_{M,2} = [h_{M+1}, \dots, h_{2M}]$, as

$$\mathcal{E}_{2M,1}(\mathbf{h}) = \begin{bmatrix} \mathcal{E}_{M,1}(\mathbf{h}_{M,1}) & \mathcal{E}_{M,2}(\mathbf{h}_{M,2}) \\ \mathcal{E}_{M,1}^*(\mathbf{h}_{M,2}) & -\mathcal{E}_{M,2}^*(\mathbf{h}_{M,1}) \end{bmatrix}, \quad (18)$$

$$\mathcal{E}_{2M,2}(\mathbf{h}) = \begin{bmatrix} -\mathcal{E}_{M,2}(\mathbf{h}_{M,1}) & -\mathcal{E}_{M,1}(\mathbf{h}_{M,2}) \\ -\mathcal{E}_{M,2}^*(\mathbf{h}_{M,2}) & \mathcal{E}_{M,1}^*(\mathbf{h}_{M,1}) \end{bmatrix} \quad (19)$$

B. Codes with controllable decoding complexity

Before we get to the code design, we first prove some properties that are given in the following propositions.

Proposition 2: The matrices

$$\mathbf{T}_{2M,1}(\mathbf{h}_{2M}) = \mathcal{E}_{2M,1}^\dagger(\mathbf{h}_{2M})\mathcal{E}_{2M,1}(\mathbf{h}_{2M}), \quad (20)$$

$$\mathbf{T}_{2M,2}(\mathbf{h}_{2M}) = \mathcal{E}_{2M,2}^\dagger(\mathbf{h}_{2M})\mathcal{E}_{2M,2}(\mathbf{h}_{2M}), \quad (21)$$

$$\begin{aligned} \mathbf{K}_M(\mathbf{h}_{M,1}, \mathbf{h}_{M,2}) &= \mathcal{E}_{M,1}^\dagger(\mathbf{h}_{M,1})\mathcal{E}_{M,2}(\mathbf{h}_{M,2}) - \\ &\quad \mathcal{E}_{M,1}^T(\mathbf{h}_{M,2})\mathcal{E}_{M,2}^*(\mathbf{h}_{M,1}), \end{aligned} \quad (22)$$

$$\begin{aligned} \mathbf{Y}_M(\mathbf{h}_{M,1}, \mathbf{h}_{M,2}) &= \mathcal{E}_{M,1}^\dagger(\mathbf{h}_{M,1})\mathcal{E}_{M,1}(\mathbf{h}_{M,2}) + \\ &\quad \mathcal{E}_{M,1}^\dagger(\mathbf{h}_{M,2})\mathcal{E}_{M,1}(\mathbf{h}_{M,1}), \end{aligned} \quad (23)$$

$$\begin{aligned} \mathbf{Z}_M(\mathbf{h}_{M,1}, \mathbf{h}_{M,2}) &= \mathcal{E}_{M,2}^\dagger(\mathbf{h}_{M,1})\mathcal{E}_{M,2}(\mathbf{h}_{M,2}) + \\ &\quad \mathcal{E}_{M,2}^\dagger(\mathbf{h}_{M,2})\mathcal{E}_{M,2}(\mathbf{h}_{M,1}) \end{aligned} \quad (24)$$

are real $\forall \mathbf{h}_{2M} \in \mathbb{C}^{2M \times 1}$, $\mathbf{h}_{M,1} \in \mathbb{C}^{M \times 1}$, $\mathbf{h}_{M,2} \in \mathbb{C}^{M \times 1}$.

Proof: Omitted. ■

Proposition 3: For any $\mathbf{h}_{M,1}, \mathbf{h}_{M,2} \in \mathbb{C}^{M \times 1}$, if $\mathbf{Y}_M(\mathbf{h}_{M,1}, \mathbf{h}_{M,2})$ and $\mathbf{T}_{M,1}(\mathbf{h}_{M,1})$ have the same eigen-vectors and $\mathbf{Z}_M(\mathbf{h}_{M,1}, \mathbf{h}_{M,2})$ and $\mathbf{T}_{M,2}(\mathbf{h}_{M,1})$ have the same eigen-vectors, then for any $\mathbf{h}_{2M}, \mathbf{g}_{2M} \in \mathbb{C}^{2M \times 1}$, eigen-vectors of $\mathbf{Y}_{2M}(\mathbf{h}_{2M}, \mathbf{g}_{2M})$, $\mathbf{T}_{2M,1}(\mathbf{h}_{2M})$ are the same, and similarly, the eigen-vectors of $\mathbf{Z}_{2M}(\mathbf{h}_{2M}, \mathbf{g}_{2M})$, $\mathbf{T}_{2M,2}(\mathbf{h}_{2M})$ are also the same.

Proof: Omitted. ■

Proposition 4: If for any $\mathbf{h}_{4M} \in \mathbb{C}^{4M \times 1}$

$$\begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \end{bmatrix} \quad (25)$$

is an eigen-vector for $\mathbf{T}_{4M,1}(\mathbf{h}_{4M})$ with λ_{4M} as the associated eigen-value, then the eigen-vector of $\mathbf{T}_{4M,2}(\mathbf{h}_{4M})$ is

$$\begin{bmatrix} \mathbf{b}_{4M} \\ -\mathbf{a}_{4M} \end{bmatrix} \quad (26)$$

with the same eigen-value λ_{4M} . Furthermore,

$$\mathbf{K}_{2M}\mathbf{b}_{4M} = \lambda_{4M}^k \mathbf{a}_{4M}, \quad (27)$$

$$\mathbf{K}_{2M}^\dagger \mathbf{a}_{4M} = \lambda_{4M}^k \mathbf{b}_{4M}, \quad (28)$$

where the dependence of \mathbf{K} on the channel realization is dropped for convenience.

Proof: Omitted. ■

Proposition 5: If

$$\begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \end{bmatrix} \quad (29)$$

is an eigen-vector of $\mathbf{T}_{4M,1}(\mathbf{h}_{4M})$, then the eigen-vectors of $\mathbf{T}_{8M,1}(\mathbf{h}_{8M})$ are

$$\frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \\ \mathbf{b}_{4M} \\ -\mathbf{a}_{4M} \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \\ -\mathbf{b}_{4M} \\ \mathbf{a}_{4M} \end{bmatrix} \quad (30)$$

Proof: We prove this by induction. It is easy to check it for $M = 8$. Let us assume that this is true for $\mathbf{T}_{k,1}(\mathbf{h}_k) \forall k \leq 4M$ i.e. if $\begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \end{bmatrix}$ is an eigen-vector of $\mathbf{T}_{4M,1}(\mathbf{h}_{4M})$, then $\mathbf{a}_{4M} = \begin{bmatrix} \mathbf{a}_{2M} \\ \mathbf{b}_{2M} \end{bmatrix}$ is an eigen-vector of $\mathbf{T}_{2M,1}(\mathbf{h}_{2M})$. By using Proposition 2, $\mathbf{T}_{8M,1}(\mathbf{h}_{8M})$ can be written as into smaller parts as

$$\mathbf{T}_{8M,1}(\mathbf{h}_{8M}) = \begin{bmatrix} \mathbf{T}_{4M,1}(\mathbf{h}_{4M,1}) + \mathbf{T}_{4M,1}(\mathbf{h}_{4M,2}) & \mathbf{K}_{4M}(\mathbf{h}_{4M,1}, \mathbf{h}_{4M,2}) \\ \mathbf{K}_{4M}^T(\mathbf{h}_{4M,1}, \mathbf{h}_{4M,2}) & \mathbf{T}_{4M,2}(\mathbf{h}_{4M,2}) + \mathbf{T}_{4M,2}(\mathbf{h}_{4M,1}) \end{bmatrix} \quad (31)$$

We have to show that if $\begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \end{bmatrix}$ is an eigen-vector of $\mathbf{T}_{4M,1}(\mathbf{h}_{4M})$, then

$$\mathbf{T}_{8M,1}(\mathbf{h}_{8M}) \begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \\ \mathbf{b}_{4M} \\ -\mathbf{a}_{4M} \end{bmatrix} = \lambda_{8M} \begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \\ \mathbf{b}_{4M} \\ -\mathbf{a}_{4M} \end{bmatrix} \quad (32)$$

From the induction assumption, $\begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \end{bmatrix}$ is an eigen-vector of $\mathbf{T}_{4M,1}(\mathbf{h}_{4M,1})$ and $\mathbf{T}_{4M,1}(\mathbf{h}_{4M,2})$ with eigen-values λ_{4M} , λ_{4M}^a respectively and using Proposition 4, these are also the eigen-vectors of $\mathbf{T}_{4M,2}(\mathbf{h}_{4M,1})$ and $\mathbf{T}_{4M,2}(\mathbf{h}_{4M,2})$. Substituting in Eqs. (31) and (32), we have to show that

$$\mathbf{K}_{4M}(\mathbf{h}_{4M,1}, \mathbf{h}_{4M,2}) \begin{bmatrix} \mathbf{b}_{4M} \\ -\mathbf{a}_{4M} \end{bmatrix} = \lambda_{8M}^a \begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \end{bmatrix} \quad (33)$$

$$\mathbf{K}_{4M}^\dagger(\mathbf{h}_{4M,1}, \mathbf{h}_{4M,2}) \begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \end{bmatrix} = \lambda_{8M}^a \begin{bmatrix} \mathbf{b}_{4M} \\ -\mathbf{a}_{4M} \end{bmatrix} \quad (34)$$

If $\begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \end{bmatrix}$ is an eigen-vector of $\mathbf{T}_{4M,1}(\mathbf{h}_{4M})$, then it follows from the induction assumption that \mathbf{a}_{4M} is an eigen-vector of $\mathbf{T}_{2M,1}$ (and \mathbf{Y}_{2M}) and \mathbf{b}_{4M} is an eigen-vector of $\mathbf{T}_{2M,2}$ (and \mathbf{Z}_{2M}), where the dependence of \mathbf{T} on the channel realization is dropped for convenience. Using Propositions 3 and 4, we have

$$\mathbf{K}_{2M}(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,2})\mathbf{b}_{4M} = \lambda_{4M}^k(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,2})\mathbf{a}_{4M}, \quad (35)$$

$$\mathbf{K}_{2M}^\dagger(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,2})\mathbf{a}_{4M} = \lambda_{4M}^k(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,2})\mathbf{b}_{4M}, \quad (36)$$

$$\mathbf{Y}_{2M}(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,2})\mathbf{a}_{4M} = \lambda_{4M}^c(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,2})\mathbf{a}_{4M}, \quad (37)$$

$$\mathbf{Z}_{2M}(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,2})\mathbf{a}_{4M} = \lambda_{4M}^c(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,2})\mathbf{b}_{4M} \quad (38)$$

Note that $\mathbf{K}_{4M}(\mathbf{h}_{4M,1}, \mathbf{h}_{4M,2})$

$$= \begin{bmatrix} -\mathbf{K}_{2M}(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,3}) + \mathbf{K}_{2M}(\mathbf{h}_{2M,4}, \mathbf{h}_{2M,2}) \\ -\mathbf{Y}_{2M}(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,4}) + \mathbf{Y}_{2M}^*(\mathbf{h}_{2M,2}, \mathbf{h}_{2M,3}) \\ -\mathbf{Z}_{2M}(\mathbf{h}_{2M,2}, \mathbf{h}_{2M,3}) + \mathbf{Z}_{2M}^*(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,4}) \\ \mathbf{K}_{2M}^\dagger(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,3}) - \mathbf{K}_{2M}^\dagger(\mathbf{h}_{2M,4}, \mathbf{h}_{2M,2}) \end{bmatrix}, \text{ hence}$$

using Eqs. (35), (36), (37), (38), we have

$$\mathbf{K}_{4M}(\mathbf{h}_{4M,1}, \mathbf{h}_{4M,2}) \begin{bmatrix} \mathbf{b}_{4M} \\ -\mathbf{a}_{4M} \end{bmatrix} = \lambda_{8M}^a \begin{bmatrix} \mathbf{b}_{4M} \\ -\mathbf{a}_{4M} \end{bmatrix}, \quad (39)$$

where $\lambda_{8M}^a = -\lambda_{4M}^k(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,3}) + \lambda_{4M}^k(\mathbf{h}_{2M,4}, \mathbf{h}_{2M,2}) + \lambda_{4M}^c(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,4}) - \lambda_{4M}^c(\mathbf{h}_{2M,2}, \mathbf{h}_{2M,3})$. This proves Eq. (33). Hence

$$\begin{bmatrix} -\mathbf{K}_{2M}(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,3}) + \mathbf{K}_{2M}(\mathbf{h}_{2M,4}, \mathbf{h}_{2M,2}) \\ -\mathbf{Y}_{2M}(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,4}) + \mathbf{Y}_{2M}^*(\mathbf{h}_{2M,2}, \mathbf{h}_{2M,3}) \\ -\mathbf{Z}_{2M}(\mathbf{h}_{2M,2}, \mathbf{h}_{2M,3}) + \mathbf{Z}_{2M}^*(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,4}) \\ \mathbf{K}_{2M}^\dagger(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,3}) - \mathbf{K}_{2M}^\dagger(\mathbf{h}_{2M,4}, \mathbf{h}_{2M,2}) \end{bmatrix} \begin{bmatrix} \mathbf{b}_{4M} \\ -\mathbf{a}_{4M} \end{bmatrix} = \lambda_{8M}^a \begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \end{bmatrix}$$

By interchanging the first half of the rows with the second half, then interchanging the first half of the columns with the second half, then multiplying the first half of columns and the second half of rows with -1 , and using the fact that $\mathbf{Y}_M, \mathbf{Z}_M$ are real, Hermitian matrices, we can write the above equations

$$\begin{bmatrix} -\mathbf{K}_{2M}^\dagger(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,3}) + \mathbf{K}_{2M}^\dagger(\mathbf{h}_{2M,4}, \mathbf{h}_{2M,2}) \\ -\mathbf{Z}_{2M}^\dagger(\mathbf{h}_{2M,2}, \mathbf{h}_{2M,3}) + \mathbf{Z}_{2M}^{\dagger*}(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,4}) \\ -\mathbf{Y}_{2M}^\dagger(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,4}) + \mathbf{Y}_{2M}^{\dagger*}(\mathbf{h}_{2M,2}, \mathbf{h}_{2M,3}) \\ \mathbf{K}_{2M}(\mathbf{h}_{2M,1}, \mathbf{h}_{2M,3}) - \mathbf{K}_{2M}(\mathbf{h}_{2M,4}, \mathbf{h}_{2M,2}) \end{bmatrix} \begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \end{bmatrix} = \lambda_{8M}^a \begin{bmatrix} \mathbf{b}_{4M} \\ -\mathbf{a}_{4M} \end{bmatrix}$$

or

$$\mathbf{K}_{4M}^\dagger(\mathbf{h}_{4M,1}, \mathbf{h}_{4M,2}) \begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \end{bmatrix} = \lambda_{8M}^a \begin{bmatrix} \mathbf{b}_{4M} \\ -\mathbf{a}_{4M} \end{bmatrix} \quad (40)$$

Hence if $\begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \end{bmatrix}$ is an eigen-vector of $\mathbf{T}_{4M,1}(\mathbf{h}_{4M})$, $\begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \\ \mathbf{b}_{4M} \\ -\mathbf{a}_{4M} \end{bmatrix}$ is an eigen-vector of $\mathbf{T}_{8M,1}(\mathbf{h}_{8M})$. Similarly, by

using Eqs. (33) and (34), one can show that $\begin{bmatrix} \mathbf{a}_{4M} \\ \mathbf{b}_{4M} \\ -\mathbf{b}_{4M} \\ \mathbf{a}_{4M} \end{bmatrix}$ is also an eigen-vector for $\mathbf{T}_{8M,1}(\mathbf{h}_{8M})$. Q.E.D. ■

Example: For $M = 4$, the eigen-vector matrix for $T_{4,1}$ (or $\bar{W}_{M,1}$ in Eq. (14)) is computed as

$$W_{4,1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (41)$$

and using Proposition 5, the eigen-vector matrix for $T_{8,1}$ is given by

$$W_{8,1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \quad (42)$$

An important aspect of the eigen-vector matrices of $T_{M,1}$ and $T_{M,2}$ is that they are independent of the channel realization. This property is quite useful in constructing codes with controllable ML decoding complexity.

Proposition 6: The determinant for the first partition of the code matrix is given by

$$\det \{G_M[\mathcal{A}_{M,1}(\mathbf{s})]\} = f \left(Q_{M,1}^\dagger v_{M,1}(\mathbf{s}) \right), \quad (43)$$

where $Q_{M,1} = \sqrt{M/2} W_{M,1}$ and for any n length vector \mathbf{q} , $f(\mathbf{q}) = \prod_{k=1}^n |q_k|^2$.

Proof: Firstly, note that $\sqrt{M/2}$ is a normalizing factor such that the elements of $Q_{M,1}$ are ± 1 .

We can check this result for $M = 2$. Let us assume that it is true for $2M$. Then we know from Proposition 1 that the determinant of the $G_{4M}(\mathcal{A}_{2M,1})$ can be broken down into the product of determinants of $G_{2M}(\mathcal{A}_{M,1})$. Thus to show that

$$f \left(Q_{4M,1}^\dagger v_{4M,1}(\mathbf{s}_{4M}) \right) = f \left(Q_{2M,1}^\dagger v_{2M,1}(\mathbf{s}_{2M,1} - \hat{\mathbf{s}}_{2M,2}) \right) \times f \left(Q_{2M,1}^\dagger v_{2M,1}(\mathbf{s}_{2M,1} + \hat{\mathbf{s}}_{2M,2}) \right)$$

we can write

$$v_{2M,1}(\mathbf{s}_{2M,1} - \hat{\mathbf{s}}_{2M,2}) = \mathcal{D}_1 v_{4M,1}(\mathbf{s}_{4M}) \quad (44)$$

$$v_{2M,1}(\mathbf{s}_{2M,1} + \hat{\mathbf{s}}_{2M,2}) = \mathcal{D}_2 v_{4M,1}(\mathbf{s}_{4M}), \quad (45)$$

where $\mathcal{D}_1, \mathcal{D}_2$ are sparse $M/2 \times M$ matrices with diagonal entries as unity and (k, l) th entry of \mathcal{D}_i is denoted by

$$\mathcal{D}_i^{k,l} = \begin{cases} 1, & k = l \\ (-1)^{i-1}, & k = 2M - l - 1, k \leq M \\ (-1)^i, & k = 2M - l - 1, k > M \end{cases} \quad (46)$$

It follows easily from Proposition 5 that the rows of $Q_{2M,1} \mathcal{D}_i$, $i = 1, 2$ are the rows of $Q_{4M,1}$ though not necessarily in the same order. This proves the result. ■

For the second partition, due to similarity with the above Proposition, we omit it.

We note here from Eq. (17) that it is $W_{M,i}$ that dictates the ML decoding complexity. For example, we could precode the information-carrying symbol vector \mathbf{c} in Eq. (1) such that

$$v_{M,i}(\mathbf{c}) = W_{M,i} v_{M,i}(\mathbf{s}) \quad (47)$$

According to Eq. (17), this code will admit single symbol ML decoding. But this will give the determinant of the error code matrix as

$$\det \{G_M[\mathcal{A}_{M,1}(W_{M,1}(\mathbf{c} - \mathbf{e}))]\} = \|v_{M,i}(\mathbf{c} - \mathbf{e})\|_F^2 \quad (48)$$

Since the elements of \mathbf{c} and \mathbf{e} are drawn from the same constellation, hence

$$\min_{\mathbf{c}, \mathbf{e}} \det \{G_M[\mathcal{A}_{M,1}(W_{M,1}(\mathbf{c} - \mathbf{e}))]\} = 0 \quad (49)$$

Hence one cannot achieve full-diversity with single symbol ML decoding. But we can achieve full-diversity with double

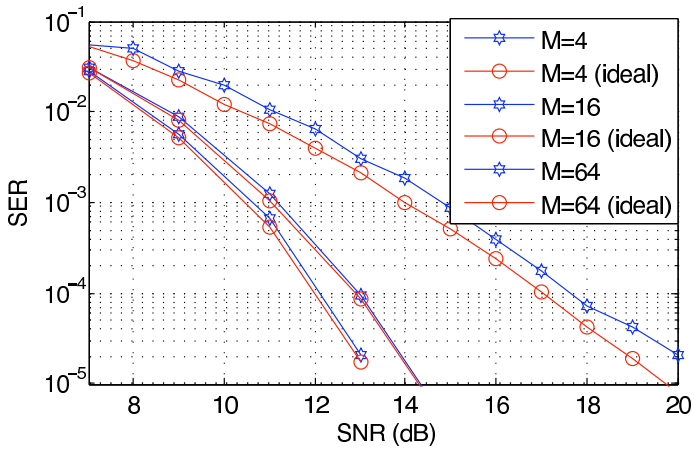


Fig. 1. SER versus SNR for various M and $N = 1$ with QPSK modulation for the rate-1 constructed and the *ideal* codes.

symbol ML decoding and we give a constructive proof.

Proposition 7: By choosing the precoding matrix such that the precoding for the first partition comes out to be

$$\mathcal{R}_{1v_{M,1}}(\mathbf{s}) = W_{M,1}B_{M,1}v_{M,1}(\mathbf{c}), \quad (50)$$

where $B_{M,1}$ is a block diagonal matrix consisting of 2×2 matrices $M/4$ in number along its diagonal each of which is $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, we can ensure that full-diversity is achieved.

Proof: As per Proposition 6, the determinant of the error code matrix will have product of terms of the form $|c_{i_1} + c_{i_2} - e_{i_1} - e_{i_2}|^2 \times |c_{i_1} - c_{i_2} - e_{i_1} + e_{i_2}|^2$. One can employ various methods like constellation rotation [11], [13], [15], [16] for c_{i_2} (and hence e_{i_2}) that will ensure that the determinant will never be zero and hence imply full-diversity. ■

By using this block diagonal structure, we can construct codes with ML decoding of different symbols together and hence the ML decoding complexity can be controlled. Precoding techniques in [4], [5], [7] ensure that full-diversity is obtained.

Note that we have assumed in Proposition 7 that $P = M$ i.e. unit rate. One could design codes with $P < M$ that may have additional coding gain while sacrificing code rate.

IV. DISCUSSION AND RESULTS

The basic ideas presented in this paper are as follows. The iterative construction ensures that by appropriate conjugation at the receiver, one could write the signal of interest as a symbol vector multiplied by an equivalent channel, whose SVD reveals an interesting aspect that the right unitary matrix is *constant* for any number of transmit antennas. One could hence precode the signal vector and control the number of symbols that will interfere with each other. It is shown that it is impossible to attain full-diversity with single symbol ML decoding but it can indeed be attained if allow more than one symbols to interfere with each other with constellation rotation (ML decoding of symbols in pairs for example).

The symbol error rate (SER) versus the average SNR per receive antenna for the proposed rate-1 code that admits decoding in pairs of symbols is plotted in Fig. 1 with QPSK modulation for $M = 4, 8, 16$ and $N = 1$. Also plotted is the performance of an *ideal* rate-1 orthogonal space-time codes (non-existent for $M > 2$) with equivalent channel as $\|H\|_F$.

V. CONCLUSIONS

We have constructed a class of linear space-time codes that have controllable ML decoding complexity for any number of transmit antennas. For the codes with single symbol ML decoding, it is impossible to have full-diversity. For codes with ML decoding involving more than one symbol, one can construct codes that offer full-diversity by using constellation rotation for example. One can thus construct codes that offer full-diversity whose ML decoding involve the decoding of symbols in pairs. This result was known thus far for only select number of transmit antennas.

REFERENCES

- [1] V. Tarokh, N. Seshadri and A.R. Calderbank, "Space-time codes for high data rate wireless communications : Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, pp. 744-765, March 1998.
- [2] V. Tarokh, H. Jafarkhani and A.R. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1456-1467, July 1999.
- [3] O. Tirkkonen and A. Hottinen, "Square-matrix embeddable space-time block codes for complex signal constellations," *IEEE Trans. Inform. Theory*, vol. 48, pp. 384-395, Feb. 2002.
- [4] V.M. DaSilva and E.S. Sousa, "Fading-resistant modulation using several transmitter antennas," *IEEE Trans. Commun.*, vol.45, pp. 1236-1244, Oct. 1997.
- [5] M.O. Damen, K. Abed-Meraim and J.-C. Belfiore, "Diagonal algebraic space-time block codes," *IEEE Trans. Inform. Theory*, vol. 48, pp. 628-636, March 2002.
- [6] Y. Xin, Z. Wang and G.B. Giannakis, "Space-time diversity systems based on linear constellation pre-coding," *IEEE Trans. Wireless Commun.*, vol. 2, pp. 294-309, March 2003.
- [7] J. Boutros and E. Viterbo, "Signal space diversity: A power and bandwidth efficient diversity technique for the Rayleigh fading channel," *IEEE Trans. Inform. Theory*, vol. 44, pp. 1453-1467, July 1998.
- [8] X. Ma and G.B. Giannakis, "Full-diversity full-rate complex-field space-time coding," *IEEE Trans. Signal Process.*, Nov. 2003.
- [9] H. Jafarkhani, "A quasi-orthogonal space-time block code," *IEEE Trans. Commun.*, vol. 49, pp. 1-4, Jan. 2001.
- [10] O. Tirkkonen, A. Boariu and A. Hottinen, "Minimal non-orthogonality rate 1 space-time block code for 3+ Tx antennas," in *Proc. IEEE ISSSTA*, Parsippany, NJ, Sept. 2000.
- [11] O. Tirkkonen, "Optimizing space-time block codes by constellation rotations," in *Proc. Finnish Wireless Commun. Workshop 2001*, Oct. 2001.
- [12] C. B. Papadakis and G. J. Foschini, "Capacity-approaching space-time codes for systems employing four transmitter antennas," *IEEE Trans. on Inform. Theory*, vol. 49, pp. 726-732, March 2003.
- [13] N. Sharma and C.B. Papadakis, "Improved quasi-orthogonal codes through constellation rotation," *IEEE Trans. Commun.*, vol. 51, pp. 332-335, March 2003.
- [14] N. Sharma and C.B. Papadakis, "Full-rate full-diversity linear quasi-orthogonal space-time codes for any number of transmit antennas," *Proc. Allerton Conf. Commun. Control Computing*, Monticello, IL, Oct. 2003, also in *EURASIP J. Applied Signal Processing (Special Issue on Advances in Smart Antennas)*, vol. 2004, no. 9, pp. 1246-1256, Aug. 2004.
- [15] D. Wang and X. Xia, "Optimal diversity product rotations for quasi-orthogonal STBC with MPSK symbols," *IEEE Commun. Lett.*, vol. 9, pp. 420-422, May 2005.
- [16] L. Xian and H. Liu, "Optimal rotation angles for quasi-orthogonal space-time codes with PSK modulation," *IEEE Commun. Lett.*, vol. 9, pp. 676-678, Aug. 2005.