

Flow-optimized Cooperative Transmission for the Relay Channel

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Abstract

This paper describes an approach for half-duplex cooperative transmission to achieve cooperative diversity in a classical three-node relay channel. Assuming availability of channel state information at the source and relay, the approach makes use of this information to optimize distinct flows through the direct link from the source to the destination and the path via the relay, respectively. It is shown that such a design can effectively obtain diversity advantage of the relay channel in both high-rate and low-rate scenarios. When the rate requirement is low, the proposed design gives a second-order outage diversity performance approaching that of full-duplex relaying. When the rate requirement becomes asymptotically large, the design still gives a close-to-second-order outage diversity performance. The design also achieves the best diversity-multiplexing tradeoff possible for the relay channel.

I. INTRODUCTION

It is well known that the performance of a wireless network can be significantly improved by cooperative transmission among nodes in the network. Many cooperative transmission designs aim to exploit cooperative diversity that is inherently present in the network. Such designs have been suggested in [1], [2] for cellular networks. Recently there has been much interest in achieving cooperative diversity in a classical three-node relay channel [3], which represents the simplest wireless network that can derive advantages from cooperative transmission.

The relay channel has been thoroughly studied in [3]. Bounds on the capacity have been given for the general relay channel, and the capacity has been calculated for the special case of degraded relay channels. To achieve capacity, the coding technique suggested in [3] requires the relay to operate in a full-duplex manner; i.e., it can transmit and receive at the same time. It is commonly argued that full-duplex operation is not practical for most existing wireless transceivers. Thus the restriction of half-duplex operation at the relay is usually considered in cooperative transmission designs.

Since the relay cannot transmit and receive simultaneously, a time-division approach is employed in half-duplex relaying [4]. The source first transmits to the destination, and the relay listens and “captures” [5] the transmission from the source at the same time. Then the relay aids the transmission by sending processed source information to the destination. Note that the source may still send data to the destination when the relay transmits. Several techniques to process and forward the received data by the relay have been suggested. These techniques include the decode-and-forward (DF) and amplify-and-forward (AF) approaches [4]. In the DF approach, the relay decodes the received signal from the source and then forwards a re-encoded signal to the destination. In the AF approach, the relay simply amplifies and forwards the signal received from the source to the destination.

The performance of the DF approach is limited by the capability of the relay to correctly decode the signal received from the source. This in turn depends on the quality of the link from the source to the relay. On the other hand, the AF approach performs poorly in low signal-to-noise ratio (SNR) situations in which the relay forwards mainly noise to the destination. In addition, the time-division approach leads to rate losses which are significant when the relay channel is to support high rates. Some enhanced versions of the AF and DF approaches have been proposed to solve the rate loss problem. A distributed space-time-coding protocol is developed in [6]. An incremental AF technique which requires feedback from the destination to the source is developed in [4]. The non-orthogonal AF and dynamic DF techniques suggested in [7] allow the duration of the relay listening to the transmission from the source to adapt

to the condition of the link from the source to the relay. In particular, the dynamic DF technique is shown to be superior to all the cooperative diversity techniques (except perhaps the incremental relaying techniques) mentioned above. A bursty AF technique is also suggested in [8] to solve the noise forwarding problem of the AF approach when the SNR is low. It is shown that the bursty AF technique achieves the best outage performance at the asymptotically low SNR regime. We note that all these cooperative diversity techniques mentioned so far are designed with the constraint that channel state information is not available at the source and the relay. Some practical code designs for the DF and space-time-coding approaches have been suggested in [9] and [10], respectively.

When the links in the relay channel suffer from slow fading, it is conceivable that the channel state information (or at least the channel quality information) can be estimated and passed to the nodes. The source and relay may then use this information to optimize the cooperative protocol to achieve better performance. Such a design has been considered in [5] which adjusts the transmit power of the source as well as the duration of the relay listening interval of the DF and compress-forward approaches.

In this paper, we assume that the channel state information is available and work on a time-division cooperative diversity design that performs well in both high-rate and low-rate scenarios. The main distinguishing feature of the proposed approach, compared with the cooperative designs mentioned above, is that we do not employ the approach of relay “capturing” the transmission from the source to the destination. Instead, we divide the information to be sent to the destination into three flows. The source employs cooperative broadcasting [11], [12] to intentionally send two *distinct* flows of data to the relay and destination, respectively, in the first time slot. The relay helps to forward, in the DF manner, the data that it receives to the destination in the second time slot, during which the source also concurrently sends the remaining flow of data to the destination. The transmission power and durations of the time slots are optimized according to the link conditions and the rate requirement.

We will show that the proposed approach can efficiently achieves cooperative diversity in both high-rate and low-rate scenarios. In particular, when the rate requirement is asymptotically small, the outage performance of the proposed approach approaches that of full-duplex relaying, giving a second-order diversity performance. On the other hand, when the rate requirement is asymptotically large, the proposed approach still gives a close-to-second-order diversity performance. Because of this, the proposed approach gives the best diversity-multiplexing tradeoff [13] possible for the relay channel.

We note that the two basic building blocks for the proposed approach are cooperative broadcasting (CB) in the first time slot and multiple access (MA) in the second time slot. The combination of CB and MA is crucial to allow distinct flows of data be sent through the relay and through the direct link from

the source to the destination, respectively. This results in minimizing the rate loss due to the time-division operation and is particularly important in the high-rate scenario. We note that distinct flows of data cannot be sent in the other cooperative diversity designs mentioned before. Another advantage of the proposed design is that the basic building blocks are the well known CB and MA approaches. Practical MA coding designs have been well studied, e.g. see [14], [15], while practical CB coding designs are currently under intense investigation [16]–[18].

II. RELAY CHANNEL: FULL-DUPLEX BOUND

Consider a classical three-node relay network, which consists of a source node 1, a relay node 2, and a destination node 3 as shown in Fig. 1. We assume that each link in the figure is a bandpass Gaussian channel with bandwidth W and one-sided noise spectral density N_0 . Let Z_{ij} denote the power gain of the link from node i to node j . The link power gains are assumed to be independent and identically distributed (i.i.d.) exponential random variables with unit mean. This corresponds to the case of independent Rayleigh fading channels with unit average power gains. The results in the sequel can be easily generalized to include the case of non-uniform average power gains.

In this section, we consider the case in which the relay node is capable of supporting full-duplex operation. Our goal is to support an information rate¹ of K nats/s/Hz from the source to the destination. We assume that the link power gains change slowly so that they can be estimated, and hence the power gain information is available at the source and relay nodes. The source and relay nodes can make use of this information to optimally assign transmit power to the source and relay nodes so that the total transmit power P_t is minimized. Because of the non-ergodic scenario considered, we will later evaluate performance based on the probability of the outage, where outage is defined as the event that the required rate K is not supported [19]. In particular, we will express the results in terms of the rate-normalized overall signal-to-noise ratio (RNSNR) of the network:

$$S \triangleq \frac{P_t}{N_0 W} \frac{1}{e^K - 1}.$$

The RNSNR can be interpreted as the additional SNR, in dB, needed to support a certain outage probability at the required rate of K nats/s/Hz, in excess of the SNR required to support the same rate in a simple Gaussian channel with unit gain. This normalization is convenient as we will consider asymptotic cases when K approaches zero and infinity.

¹Strictly speaking, the word “rate” here should be replaced by “spectral efficiency”, since the unit involved is nats/s/Hz. Nevertheless we will use the terminology “rate” throughout this paper for convenience.

We note that the use of the RNSNR to characterize our results has two important implications. First, since the total transmit power of the source and relay is used in defining the RNSNR, no individual power limits are put on the source and relay. The results in this paper can be viewed as bounds for those with individual power limits imposed. Our choice of focusing on the total power comes from the viewpoint that the relay channel considered forms a small component of a larger wireless network. In this sense, it is fairer to compare the total transmit power incurred in sending information from the source to the destination employing cooperative diversity to that incurred in direct transmission. Second, the normalization by the factor $e^K - 1$ implies that the additional power in dB to combat fading can only be a *constant* over the power required to achieve the target rate in a Gaussian channel, regardless of the rate requirement. That is, we restrict the power to increase at the same rate as in a Gaussian channel to cope with increases in the transmission rate through the relay channel. In a sense, this restriction enforces efficiency of power usage.

Employing well known capacity bounds on the relay channel [5], [3], [20], we can obtain the following bounds on the RNSNR to support required spectral efficiency of K nats/s/Hz.

Theorem 2.1: For any positive link power gains Z_{12} , Z_{13} , and Z_{23} , define the bound

$$B_{\text{fd}}^u \triangleq \begin{cases} \frac{Z_{12} + Z_{23}}{Z_{12}(Z_{13} + Z_{23})} & \text{if } Z_{12} > Z_{13} \\ \frac{1}{Z_{13}} & \text{otherwise} \end{cases}$$

and

$$B_{\text{fd}}^l \triangleq \begin{cases} \frac{Z_{12} + Z_{13} + Z_{23}}{(Z_{12} + Z_{13})(Z_{13} + Z_{23})} & \text{if } Z_{12} > Z_{13} \\ \frac{1}{Z_{13}} & \text{otherwise.} \end{cases}$$

Then $S > B_{\text{fd}}^u$ is a sufficient condition in order to support the rate of K nats/s/Hz from the source to destination. Also $S > B_{\text{fd}}^l$ is a necessary condition in order to support the rate of K nats/s/Hz from the source to destination.

Proof: See Appendix A. ■

To achieve the lower bound B_{fd}^u , we need to apply random coding techniques for CB and MA, and perform block Markov encoding at the source and relay [3], [20]. We also need to optimally allocate transmit power between the source and relay nodes. In addition, the availability of channel state information (both magnitudes and phases of the fading coefficients of all three links) as well as symbol timing and carrier phase synchronization at all the three nodes are implicitly assumed in the proof in Appendix A.

III. HALF-DUPLEX PROTOCOLS BASED ON FLOW CONTROL

As mentioned before, most existing wireless transceivers do not support full-duplex operation using a single frequency band. In addition, no practical coding designs currently exist to get close to the bound predicted by Theorem 2.1. Thus the result in the theorem mainly serves as a lower bound on the required RNSNR for practical relay networks.

In this section, we will consider the more practical scenario in which the relay node operates in the following half-duplex fashion. Without loss of generality, we consider a slotted communication system with unit-duration time slots. We partition each unit time slot into two sub-slots with respective durations t_1 and t_2 , where $t_1 + t_2 = 1$. In the first time slot, the source transmits while the relay and destination receive. In the second time slot, the source and relay transmit and the destination receives. Based on this half-duplex mode of operation, we will describe two cooperative communication protocols that make use of the two basic components of cooperative broadcasting (CB) from the source to the relay in the first time slot and multiple access (MA) from the source and relay in the second time slot. The first protocol does not require phase synchronization among the three nodes, while the second protocol does so.

A. Half-Duplex Protocol 1

In this protocol, the information from the source to the destination is divided into three flows of data x_1 , x_2 , and x_3 , where $x_1 + x_2 + x_3 = K$. In the first time slot, the source sends, via CB, two flows of rates x_1/t_1 and x_2/t_1 to the destination and relay, respectively. In the second time slot, the relay and source send, via MA, two flows of rates x_2/t_2 and x_3/t_2 to the destination, respectively. The information flow of rate x_2/t_2 sent by the relay in the second time slot is from the flow of rate x_2/t_1 that it receives and decodes in the first time slot. We choose t_1 , t_2 , x_1 , x_2 , and x_3 to minimize the total power transmitted by the source and relay to support the rate K nats/s/Hz from the source to the destination.

To determine the minimum RNSNR that can support the required rate employing this protocol, we start with the following lemma.

Lemma 3.1: 1) For $0 < t_1 \leq 1$, the source can broadcast at rates x_1/t_1 and x_2/t_1 to the destination and relay, respectively, in the first time slot if and only if the SNR is strictly larger than

$$S_{\text{CB}} = \begin{cases} \frac{1}{Z_{12}}(e^{x_2/t_1} - 1) + \frac{1}{Z_{13}}e^{x_2/t_1}(e^{x_1/t_1} - 1) & \text{if } Z_{13} \geq Z_{12}, \\ \frac{1}{Z_{13}}(e^{x_1/t_1} - 1) + \frac{1}{Z_{12}}e^{x_1/t_1}(e^{x_2/t_1} - 1) & \text{otherwise.} \end{cases}$$

For $t_1 = 0$, $S_{\text{CB}} = 0$.

- 2) For $0 < t_2 \leq 1$, the source and relay can simultaneously transmit at rates x_3/t_2 and x_2/t_2 , respectively, to the destination in the second time slot if and only if the SNR is strictly larger than

$$S_{\text{MA}} = \begin{cases} \frac{1}{Z_{23}}(e^{x_2/t_2} - 1) + \frac{1}{Z_{13}}e^{x_2/t_2}(e^{x_3/t_2} - 1) & \text{if } Z_{13} \geq Z_{23}, \\ \frac{1}{Z_{13}}(e^{x_3/t_2} - 1) + \frac{1}{Z_{23}}e^{x_3/t_2}(e^{x_2/t_2} - 1) & \text{otherwise.} \end{cases}$$

For $t_2 = 0$, $S_{\text{MA}} = 0$.

Proof: See Appendix B. ■

With the help of Lemma 3.1, we can now formulate the optimization of the parameters in Protocol 1 as follows:

$$\begin{aligned} & \min t_1 S_{\text{CB}} + t_2 S_{\text{MA}} \\ & \text{subject to} \quad \text{i. total data requirement:} \quad x_1 + x_2 + x_3 = K \\ & \quad \quad \quad \text{ii. total time requirement:} \quad t_1 + t_2 = 1 \\ & \quad \quad \quad \text{iii. non-negativity requirements:} \quad x_1, x_2, x_3, t_1, t_2 \geq 0 \end{aligned} \tag{1}$$

where S_{CB} and S_{MA} are of the forms in Lemma 3.1. It is not hard to see that (1) is a convex optimization problem and its solution provides the tightest lower bound for the SNR required to support the rate of K nats/s/Hz:

Theorem 3.1: Let $B_1(K)$ be the minimum value achieved in the optimization problem (1), normalized by the factor $e^K - 1$. Then the rate of K nats/s/Hz can be supported from the source to the destination by Protocol 1 if and only if the RNSNR satisfies $S > B_1(K)$.

1) *Description of $B_1(K)$:* To describe the form of the RNSNR bound $B_1(K)$, we need to consider the following few cases. This solution is established by applying the Karush-Kuhn-Tucker (KKT) condition [21] to the convex optimization problem (1) as detailed in Appendix C. For notational convenience, we write

$$M_H(x, y) = \frac{1}{\frac{1}{x} + \frac{1}{y}}$$

as the harmonic mean² of two real numbers x and y .

a) $Z_{13} \geq M_H(Z_{12}, Z_{23})$: The solution is given by

$$\begin{aligned} x_1 &= Kt_1, \\ x_2 &= 0, \\ x_3 &= Kt_2, \end{aligned}$$

²The definition here actually gives one half of the harmonic mean usually defined in the literature. For convenience, we will slightly abuse the common terminology and call $M_H(x, y)$ the harmonic mean.

where t_1 and t_2 can be arbitrarily chosen as long as they satisfy the non-negativity and total-time requirements. This corresponds to directly transmitting all data through the link from the source to destination, without utilizing the relay. The resulting value of $B_1(K)$ is

$$B_1(K) = \frac{1}{Z_{13}}.$$

b) $Z_{13} < M_H(Z_{12}, Z_{23})$: Define

$$\begin{aligned} A_1 &= Z_{23} \left(\frac{1}{Z_{13}} - \frac{1}{Z_{12}} \right), \\ A_2 &= Z_{12} \left(\frac{1}{Z_{13}} - \frac{1}{Z_{23}} \right). \end{aligned}$$

Notice that $A_1 \geq 1$ and $A_2 \geq 1$. Consider two sub-cases:

i. $K > M_H(\log A_1, \log A_2)$:

In this case,

$$B_1(K) = \frac{\min\{\tilde{S}_1(K), \tilde{S}_2(K), \tilde{S}_3(K)\}}{e^K - 1}, \quad (2)$$

where the three SNR terms $\tilde{S}_1(K)$, $\tilde{S}_2(K)$, and $\tilde{S}_3(K)$ are respectively defined in (3), (4), and (5) below.

Define

$$t^* = \frac{\log \left(\frac{Z_{23} \log A_2}{Z_{12} \log A_1} \right) + \log A_2}{\log A_1 + \log A_2}.$$

Employing the well known inequality $\log x \leq x - 1$, it can be shown that $0 \leq t^* \leq 1$. The first SNR term is given by

$$\tilde{S}_1(K) = \begin{cases} \frac{1}{Z_{12}} e^{K+(1-t^*) \log A_2} + \frac{1}{Z_{23}} e^{K+t^* \log A_1} - \frac{1}{Z_{13}} & \text{if } 1 - \frac{K}{\log A_1} \leq t^* \leq \frac{K}{\log A_2} \\ \infty & \text{otherwise.} \end{cases} \quad (3)$$

When $\tilde{S}_1(K)$ is the minimum among the three terms inside the min operator in (2), the corresponding solution to the optimization problem (1) is given by

$$\begin{aligned} x_1 &= Kt^* - t^*(1-t^*) \log A_1, \\ x_2 &= t^*(1-t^*) \log(A_1 A_2), \\ x_3 &= K(1-t^*) - t^*(1-t^*) \log A_2, \end{aligned}$$

with $t_1 = t^*$ and $t_2 = 1 - t^*$.

The second SNR term is given by

$$\tilde{S}_2(K) = \begin{cases} \min_{\frac{K}{\log A_2} \leq t_1 \leq 1} \left\{ \frac{t_1}{Z_{12}} e^{K/t_1} + \frac{1}{Z_{23}} e^{K+t_1 \log A_1} - \frac{t_1}{Z_{13}} - \frac{1-t_1}{Z_{23}} \right\} & \text{if } K \leq \log A_2 \\ \infty & \text{if } K > \log A_2. \end{cases} \quad (4)$$

Write the minimizing value of t_1 in the expression above as t^{**} . When $\tilde{S}_2(K)$ is the minimum among the three terms inside the min operator in (2), the corresponding solution to the optimization problem (1) is given by

$$\begin{aligned} x_1 &= Kt^{**} - t^{**}(1 - t_1) \log A_1, \\ x_2 &= K(1 - t^{**}) + t^{**}(1 - t^{**}) \log A_1, \\ x_3 &= 0, \end{aligned}$$

with $t_1 = t^{**}$ and $t_2 = 1 - t^{**}$.

The third SNR term is given by

$$\tilde{S}_3(K) = \begin{cases} \min_{0 \leq t_1 \leq 1 - \frac{K}{\log A_1}} \left\{ \frac{1 - t_1}{Z_{23}} e^{K/(1-t_1)} + \frac{1}{Z_{12}} e^{K+(1-t_1) \log A_2} - \frac{1 - t_1}{Z_{13}} - \frac{t_1}{Z_{12}} \right\} & \text{if } K \leq \log A_1 \\ \infty & \text{if } K > \log A_1. \end{cases} \quad (5)$$

Write the minimizing value of t_1 in the expression above as t^{***} . When $\tilde{S}_3(K)$ is the minimum among the three terms inside the min operator in (2), the corresponding solution to the optimization problem (1) is given by

$$\begin{aligned} x_1 &= 0, \\ x_2 &= Kt^{***} + t^{***}(1 - t^{***}) \log A_2, \\ x_3 &= K(1 - t^{***}) - t^{***}(1 - t^{***}) \log A_2, \end{aligned}$$

with $t_1 = t^{***}$ and $t_2 = 1 - t^{***}$.

ii. $K \leq M_H(\log A_1, \log A_2)$:

In this case,

$$B_1(K) = \frac{\min\{\hat{S}_1(K), \hat{S}_2(K), \hat{S}_3(K)\}}{e^K - 1}, \quad (6)$$

where the three SNR terms $\hat{S}_1(K)$, $\hat{S}_2(K)$, and $\hat{S}_3(K)$ are respectively defined in (7), (8), and (9) below.

The first SNR term is given by

$$\hat{S}_1(K) = \min_{\frac{K}{\log A_2} \leq t_1 \leq 1 - \frac{K}{\log A_1}} \left\{ \frac{t_1}{Z_{12}} \left[e^{K/t_1} - 1 \right] + \frac{1 - t_1}{Z_{23}} \left[e^{K/(1-t_1)} - 1 \right] \right\}. \quad (7)$$

Write the minimizing value of t_1 in the expression above as t_* . When $\hat{S}_1(K)$ is the minimum among the three terms inside the min operator in (2), the corresponding solution to the optimization problem (1) is given by

$$\begin{aligned} x_1 &= 0, \\ x_2 &= K, \\ x_3 &= 0, \end{aligned}$$

with $t_1 = t_*$ and $t_2 = 1 - t_*$.

The second SNR term is given by

$$\hat{S}_2(K) = \min_{1 - \frac{K}{\log A_1} \leq t_1 \leq 1} \left\{ \frac{t_1}{Z_{12}} e^{K/t_1} + \frac{1}{Z_{23}} e^{K+t_1 \log A_1} - \frac{t_1}{Z_{13}} - \frac{1-t_1}{Z_{23}} \right\}. \quad (8)$$

Write the minimizing value of t_1 in the expression above as t_{**} . When $\hat{S}_2(K)$ is the minimum among the three terms inside the min operator in (2), the corresponding solution to the optimization problem (1) is given by

$$\begin{aligned} x_1 &= K t_{**} - t_{**}(1 - t_{**}) \log A_1, \\ x_2 &= K(1 - t_{**}) + t_{**}(1 - t_{**}) \log(A_1), \\ x_3 &= 0, \end{aligned}$$

with $t_1 = t_{**}$ and $t_2 = 1 - t_{**}$.

The third SNR term is given by

$$\hat{S}_3(K) = \min_{0 \leq t_1 \leq \frac{K}{\log A_2}} \left\{ \frac{1-t_1}{Z_{23}} e^{K/(1-t_1)} + \frac{1}{Z_{12}} e^{K+(1-t_1) \log A_2} - \frac{1-t_1}{Z_{13}} - \frac{t_1}{Z_{12}} \right\}. \quad (9)$$

Write the minimizing value of t_1 in the expression above as t_{***} . When $\hat{S}_3(K)$ is the minimum among the three terms inside the min operator in (2), the corresponding solution to the optimization problem (1) is given by

$$\begin{aligned} x_1 &= 0, \\ x_2 &= K t_{***} + t_{***}(1 - t_{***}) \log A_2, \\ x_3 &= K(1 - t_{***}) - t_{***}(1 - t_{***}) \log A_2, \end{aligned}$$

with $t_1 = t_{***}$ and $t_2 = 1 - t_{***}$.

2) *Asymptotic-rate scenarios:* We are interested in characterizing the required RNSNR in the asymptotic scenarios as the required rate K approaches zero and infinity, respectively. The following corollary of Theorem 3.1 and the description of $B_1(K)$ above provides such characterization:

Corollary 3.1: 1) $B_1(K)$ is an increasing function for all $K > 0$.

$$2) \lim_{K \rightarrow 0} B_1(K) = \begin{cases} \frac{1}{Z_{23}} + \frac{1}{Z_{12}} & \text{if } Z_{13} < M_H(Z_{12}, Z_{23}) \\ \frac{1}{Z_{13}} & \text{if } Z_{13} \geq M_H(Z_{12}, Z_{23}). \end{cases}$$

$$3) \lim_{K \rightarrow \infty} B_1(K) = \begin{cases} \frac{A_1^{t_*}}{Z_{23}} + \frac{A_2^{1-t_*}}{Z_{12}} & \text{if } Z_{13} < M_H(Z_{12}, Z_{23}) \\ \frac{1}{Z_{13}} & \text{if } Z_{13} \geq M_H(Z_{12}, Z_{23}). \end{cases}$$

Proof: See Appendix D. ■

From the solution of the optimization problem described in Section III-A.1 (see the form of solution under (7)), we observe that for a sufficiently low rate requirement, the most power-efficient transmission strategy is to select between the direct link from the source to the destination and the relay path from the source to the relay and then to the destination. The choice of which path to take is determined by comparing the power gains of the two paths. We note that the power gain of the relay path is specified by the harmonic mean of the power gains of the links from the source to the relay and from the relay to the destination, respectively. The form of $\lim_{K \rightarrow 0} B_1(K)$ in part 2) of Corollary 3.1 also suggests this strategy.

When the rate requirement is sufficiently high, the optimal strategy (see the form of solution under (3)) is again to compare the path gains of the direct and relay paths. If the direct path is stronger, all information is still sent through this path. Different from the low-rate case, if the relay path is stronger, most of the information is still sent through the direct path. Only a fixed amount (depends on the link power gains, but not on the rate regardless of how high it is) of information is sent through the relay path. The reduction of this fixed amount of data through the direct path has the equivalent effect of improving the fading margin of the direct path and hence provides diversity advantage. Unlike the low-rate case, this strategy is not readily revealed by the form of $\lim_{K \rightarrow \infty} B_1(K)$ in part 3) of Corollary 3.1. We note that this strategy is possible only because of the use of CB in the first time slot. With the relay capture approach in the other cooperative diversity designs mentioned in the Introduction, the above strategy is not possible because distinct information is not sent across the relay and direct paths.

B. Half-Duplex Protocol 2

In this protocol, the information from the source to the destination is again divided into three flows of data x_1 , x_2 , and x_3 , where $x_1 + x_2 + x_3 = K$. In the first time slot, the source sends, via CB, two flows of rates x_1/t_1 and x_2/t_1 to the destination and relay, respectively, as before. In the second time slot, the relay sends the information that it receives in the first time slot to the destination with a flow of rates x_2/t_2 . The source, on the other hand, simultaneously sends two flows of information to the destination in the second time slot. The first flow is the exact same flow of rate x_2/t_2 sent by the relay. The other flow has rate x_3/t_2 containing new information. Like before, we choose t_1 , t_2 , x_1 , x_2 , and x_3 to minimize the total power transmitted by the source and relay to support the rate K nats/s/Hz from the source to the destination.

To send the same flow of data (with rate x_2/t_2) in the second time slot, the source and relay use the same codebook. The codeword symbols from the source and relay are sent in such a way that the

corresponding received symbols arrive at the destination in phase and hence add up coherently. In order to do so, the source and relay need to be phase synchronized and to have perfect channel state information of the links. We note that these two assumptions are also needed in the full-duplex approach described in Section II. In addition, the codebooks used by the source to send the two different flows in the second time slot are independently selected so that the transmit power of the source is the sum of the power of the two codewords sent.

Since the transmission procedure is the same as that of Protocol 1 in the first time slot, Lemma 3.1 part 1) gives the minimum SNR that can support the required CB transmission in the first time slot. The minimum SNR required in the second time slot is given by the following lemma:

Lemma 3.2: For $0 < t_2 \leq 1$, suppose that the source transmits a flow of data at rate x_3/t_2 to the destination in the second time slot. Then the source and relay can jointly send another in-phase flow of data at rate x_2/t_2 to the destination in the second time slot if and only if the SNR is strictly larger than

$$\hat{S}_{\text{MA}} = \frac{1}{Z_{13}}(e^{x_3/t_2} - 1) + \frac{1}{Z_{13} + Z_{23}}e^{x_3/t_2}(e^{x_2/t_2} - 1).$$

For $t_2 = 0$, $\hat{S}_{\text{MA}} = 0$.

Proof: See Appendix E. ■

Let us define $\tilde{Z}_{23} = Z_{13} + Z_{23}$. Then we note that the expression of \hat{S}_{MA} above can be obtained by putting \tilde{Z}_{23} in place of Z_{23} in the expression of S_{MA} in Lemma 3.1. This means that as far as minimum SNR is concerned, Protocol 2 is equivalent to Protocol 1 with the power gain of the link from the relay to the destination specified by \tilde{Z}_{23} instead. Using this equivalence, we obtain the following counterparts of Theorem 3.1 and Corollary 3.1 for Protocol 2:

Theorem 3.2: The rate of K nats/s/Hz can be supported from the source to the destination by Protocol 2 if and only if the RNSNR satisfies $S > B_2(K)$, where $B_2(K)$ is obtained by replacing Z_{23} with \tilde{Z}_{23} in the description of $B_1(K)$ given in Section III-A.1.

We note that $B_2(K) \leq B_1(K)$ since Protocol 1 can be seen as an unoptimized version of Protocol 2 with the zero power assigned to the transmission of the flow of rate x_2/t_2 from the source to the destination during the second time slot.

Corollary 3.2: 1) $B_2(K)$ is an increasing function for all $K > 0$.
 2) $\lim_{K \rightarrow 0} B_2(K) = \begin{cases} \frac{1}{\tilde{Z}_{23}} + \frac{1}{Z_{12}} & \text{if } Z_{13} < M_H(Z_{12}, \tilde{Z}_{23}) \\ \frac{1}{Z_{13}} & \text{if } Z_{13} \geq M_H(Z_{12}, \tilde{Z}_{23}). \end{cases}$

$$3) \lim_{K \rightarrow \infty} B_2(K) = \begin{cases} \frac{\tilde{A}_1^{\tilde{t}^*}}{\tilde{Z}_{23}} + \frac{\tilde{A}_2^{1-\tilde{t}^*}}{\tilde{Z}_{12}} & \text{if } Z_{13} < M_H(Z_{12}, \tilde{Z}_{23}) \\ \frac{1}{\tilde{Z}_{13}} & \text{if } Z_{13} \geq M_H(Z_{12}, \tilde{Z}_{23}). \end{cases}$$

In parts 2) and 3), \tilde{A}_1 , \tilde{A}_2 , and \tilde{t}^* are the same as A_1 , A_2 , and t^* , respectively, with Z_{23} replaced by \tilde{Z}_{23} .

IV. OUTAGE ANALYSIS

As mentioned previously, we model the link power gains Z_{13} , Z_{12} , and Z_{23} as i.i.d. exponential random variables with unit mean and consider the outage probability of which the rate K cannot be supported at the RNSNR S . Let us denote the outage probabilities of full-duplex relaying, half-duplex relaying using Protocol 1, and half-duplex relaying using Protocol 2 by $P_{\text{fd}}(K, S)$, $P_1(K, S)$, and $P_2(K, S)$, respectively. Then by Theorems 2.1, 3.1, and 3.2, we have

$$\begin{aligned} \Pr(S \leq B_{\text{fd}}^l) &\leq P_{\text{fd}}(K, S) \leq \Pr(S \leq B_{\text{fd}}^u) \\ P_1(K, S) &= \Pr(S \leq B_1(K)) \\ P_2(K, S) &= \Pr(S \leq B_2(K)). \end{aligned}$$

In this section, we will investigate the asymptotic behavior of the outage probabilities when the RNSNR is large.

To this end, we obtain the following bounds on the outage probabilities. Let $f(x)$ and $g(x)$ be real-valued functions and a be a constant. We say that the function $f(x)$ is of order $ag(x)$ asymptotically, denoted by $f(x) \sim \mathcal{O}(ag(x))$, if $\lim_{x \rightarrow \infty} f(x)/g(x) = a$. Moreover, we denote the ν th-order modified Bessel function of the second kind by $K_\nu(x)$.

Theorem 4.1: 1) For all $K > 0$,

$$P_{\text{fd}}(K, S) \geq 1 - e^{-\frac{1}{S}} - 2e^{-\frac{1.5}{S}} + 2e^{-\frac{2}{S}} \sim \mathcal{O}\left(\frac{1.25}{S^2}\right).$$

2) For all $K > 0$,

$$\begin{aligned} P_1(K, S) &\leq 1 - e^{-\frac{1}{S}} - \int_0^{\frac{1}{S}} 2zK_1(2z)e^{-3z}dz \\ &\quad + \int_0^{\frac{\sqrt{2}}{S}} \left[2zK_1(2z)e^{-2z} - \frac{4}{S^2z}K_1\left(\frac{4}{S^2z}\right)e^{-\frac{4}{S^2z}} \right] e^{-z}dz \sim \mathcal{O}\left(\frac{4 \log S}{S^2}\right). \end{aligned}$$

3) For all $K > 0$,

$$P_1(K, S) \geq \left[1 - \frac{2}{S}K_1\left(\frac{2}{S}\right)e^{-\frac{2}{S}} \right] \cdot \left[1 - e^{-\frac{1}{S}} \right] \sim \mathcal{O}\left(\frac{2}{S^2}\right).$$

Equality above is achieved when K approaches 0.

4) For all $K > 0$,

$$P_2(K, S) \leq 1 - e^{-\frac{1}{S}} - \int_0^{\frac{1}{S}} 2zK_1(2z)e^{-2z} dz + \int_0^{\frac{\sqrt{2}}{S}} \left[2zK_1(2z)e^{-2z} - \frac{4}{S^2 z} K_1\left(\frac{4}{S^2 z}\right) e^{-\frac{4}{S^2 z}} \right] dz \sim \mathcal{O}\left(\frac{4 \log S}{S^2}\right).$$

5) For all $K > 0$,

$$P_2(K, S) \geq 1 - e^{-\frac{1}{S}} - \frac{2}{S^2} K_1\left(\frac{2}{S}\right) e^{-\frac{2}{S}} \sim \mathcal{O}\left(\frac{1.5}{S^2}\right).$$

Equality above is achieved when K approaches 0.

Proof: See Appendix F. ■

The various bounds in this theorem are illustrated in Fig. 2.

For comparison purpose, it is easy to verify that the outage probability for direct transmission from source to destination is $P_{\text{dt}}(K, S) = \Pr(S > 1/Z_{13}) = 1 - e^{-\frac{1}{S}} \sim \mathcal{O}\left(\frac{1}{S}\right)$. From parts 1) and 5) of the theorem, we see that $\mathcal{O}\left(\frac{1.25}{S^2}\right) \leq P_{\text{fd}}(K, S) \leq \mathcal{O}\left(\frac{1.5}{S^2}\right)$. Hence full-duplex relaying provides a second-order diversity outage performance as expected. In addition, when the rate requirement is small and the RNSNR is large, the loss in outage performance due to the restriction of half-duplex relaying is at most 0.4dB by using Protocol 2. If phase synchronization between the source and relay is impractical, then employing Protocol 1 results in an additional loss of about 0.6dB. All these observations are readily illustrated in Fig. 2.

When the rate requirement increases, the loss of half-duplex relaying starts to increase. In Figs. 3 and 4, we plot the outage probabilities achieved using Protocols 1 and 2, respectively. In each of the figures, we include the outage probabilities when the rate requirement approaches 0, 1, 3, 6, and ∞ bits/s/Hz. For comparison, we also plot the lower bound of outage probability for full-duplex relaying based on the lower bound $B_{\text{fd}}^{l,3}$ in Theorem 2.1 and the outage probability for direction transmission in the figures. All the results in the figures are obtained using Monte Carlos simulations. From Fig. 3, for Protocol 1, we see that the loss, with respect to full-duplex relaying at the outage probability of 10^{-4} , is roughly 0.9dB at 1 bits/s/Hz. The loss increases to about 2.1dB and 3.6dB when the rate increases to 3 and 6 bits/s/Hz, respectively. A similar trend is observed in Fig. 4 for Protocol 2. There is minimal loss (~ 0.1 dB) at asymptotically small rates. At 1 bits/s/Hz, the loss is about 0.3dB. The loss increases to 1.6dB and 3.4dB when the rate increases to 3 and 6 bits/s/Hz, respectively. Moreover, at all these values of K considered

³It can be shown that the outage probability corresponding to the full-duplex rate inner bound B_{fd}^u in Theorem 2.1 is of $\mathcal{O}\left(\frac{1.5}{S^2}\right)$. Thus Protocol 2 achieves the same outage performance as the full-duplex achievable outage performance at asymptotically small rates.

for both Protocols 1 and 2, the simulation results seem to indicate that the outage probability is of the order of $\mathcal{O}\left(\frac{a}{S^2}\right)$ for some constant a , whose value is different for the different cases.

When the rate requirement becomes asymptotically large, Theorem 4.1 parts 2) and 4) state that the outage probabilities for Protocols 1 and 2 are at most of order $\mathcal{O}\left(\frac{4\log S}{S^2}\right)$. This implies that both Protocols 1 and 2 give close-to-second-order diversity performance at asymptotically high rates. From the simulation results shown in Figs. 3 and 4, it appears that the outage probabilities for both Protocols 1 and 2 do in fact have the order of $\mathcal{O}\left(\frac{a\log S}{S^2}\right)$, where a is about 2.85. This corresponds to a performance loss of about 5dB at the outage probability of 10^{-4} , and the bound in parts 2) and 4) is about 0.8dB loose (cf. Fig. 2). We also note that Protocol 2 does not improve the outage performance, compared to Protocol 1, at asymptotically large rates. This is contrary to the finite rate cases in which Protocol 2 does provide performance advantage over Protocol 1, although the amount of advantage decreases as the rate requirement increases. In summary, Protocol 1 seems to be of higher practical utility than Protocol 2 since the former does not require phase synchronization between the source and relay, while it only suffers from a performance loss of about 0.6dB.

A. Diversity-multiplexing tradeoff

It is also interesting to investigate the diversity-multiplexing tradeoff of [13] for Protocols 1 and 2. To this end, we need to follow [4] to change the parameterization of the outage probabilities from (K, S) to (\tilde{K}, \tilde{S}) , where \tilde{S} is the SNR and \tilde{K} is the multiplexing gain ($0 < \tilde{K} < 1$) defined by

$$\tilde{K} = \frac{K}{\log(1 + \tilde{S})}.$$

With the parameterization (\tilde{K}, \tilde{S}) , the diversity orders [13], [4] achieved by Protocols 1 and 2 are defined as

$$\Delta_i(\tilde{K}) = \lim_{\tilde{S} \rightarrow \infty} \frac{-\log P_i(\tilde{K}, \tilde{S})}{\log \tilde{S}},$$

for $i = 1$ and 2 , respectively. Then the diversity orders can be readily obtained in the following corollary of Theorem 4.1.

Corollary 4.1: For $i = 1$ and 2 , $\Delta_i(\tilde{K}) = 2(1 - \tilde{K})$.

Proof: First notice that since $\tilde{S} = S(e^K - 1)$, $S = \frac{\tilde{S}}{(1 + \tilde{S})^{\tilde{K}} - 1}$. Hence $S \sim \mathcal{O}(\tilde{S}^{1-\tilde{K}})$. As a result, applying Theorem 4.1 with the parameterization (\tilde{K}, \tilde{S}) , for sufficiently large \tilde{S} and $i = 1, 2$,

$$\mathcal{O}\left(\frac{a_i}{\tilde{S}^{2(1-\tilde{K})}}\right) \leq P_i(\tilde{K}, \tilde{S}) \leq \mathcal{O}\left(\frac{4(1 - \tilde{K}) \log \tilde{S}}{\tilde{S}^{2(1-\tilde{K})}}\right),$$

where $a_1 = 2$ and $a_2 = 1.5$. Applying $-\log$, dividing the result by $\log \tilde{S}$, and finally taking limit as $\tilde{S} \rightarrow \infty$ on each item in the inequality equation above give the desired result. ■

This result indicates that the proposed half-duplex relaying approach with Protocols 1 and 2 achieves the best diversity-multiplexing tradeoff possible for the relay channel.

V. CONCLUSIONS

With channel state information available at the source and relay, we have shown that a half-duplex cooperative transmission design, based on optimizing distinct flows through the direct link from the source to the destination and the path via the relay, can effectively obtain diversity advantage of the relay channel in both high-rate and low-rate scenarios. Specifically, the proposed design gives outage performance approaching that of full-duplex relaying at asymptotically low rates. When the rate requirement becomes asymptotically large, the design still gives a close-to-second-order outage diversity performance. In addition, as the transmission rate increases, the transmit power required to achieve this performance needs only to grow at the same rate as that required in a simple Gaussian channel. The design also gives the best diversity-multiplexing tradeoff possible for the relay channel.

In addition to the good performance, a perhaps more important advantage of the proposed relaying design is that only flow-level design is needed to optimize the use of the rather standard components of cooperative broadcasting and multiple access. This advantage makes generalizations of the design to more complicated relay networks manageable. For a general relay network, it may be possible to characterize the design by a convex optimization problem on a graph that describes the network topology and the information flows. We will explore this general design in a sequel to this paper.

APPENDIX

A. Proof of Theorem 2.1

To prove Theorem 2.1, we consider two different cases:

1) $Z_{12} > Z_{13}$: Let $0 \leq \alpha \leq 1$ be the fraction of the total power, P_t , allocated to the source node. Then the transmit power of the relay node is $(1 - \alpha)P_t$. First consider the sufficient condition, by [3, Theorem 1] (also see [5]), the following rate is achievable by the relay channel when the relay decodes and re-encodes its received signal:

$$R_{\text{fd}}^u(\alpha) = \max_{0 \leq \beta \leq 1} \min \left\{ C \left(\frac{P_t}{N_0 W} \left[\alpha Z_{13} + (1 - \alpha) Z_{23} + 2\sqrt{(1 - \beta)\alpha(1 - \alpha)Z_{13}Z_{23}} \right] \right), \right. \\ \left. C \left(\frac{P_t}{N_0 W} \alpha \beta Z_{12} \right) \right\}, \quad (10)$$

where $C(x) = \log(1+x)$. We can further maximize the capacity by optimally allocating transmit power between the source and relay node, i.e.,

$$\begin{aligned} R_{\text{fd}}^u &= \max_{0 \leq \alpha \leq 1} C_{\text{fd}}^u(\alpha) \\ &= C \left(\frac{P_t}{N_0 W} \max_{0 \leq \alpha, \beta \leq 1} \min \left\{ \alpha Z_{13} + (1-\alpha) Z_{23} + 2\sqrt{(1-\beta)\alpha(1-\alpha)Z_{13}Z_{23}}, \alpha\beta Z_{12} \right\} \right), \end{aligned}$$

where the second equality results from the fact that $C(x)$ is an increasing function. Hence, the requirement of $R_{\text{fd}}^u \geq K$ implies that the RNSNR should satisfy $S > 1/Z_{\text{fd}}^u$, where

$$Z_{\text{fd}}^u = \max_{0 \leq \alpha, \beta \leq 1} \min \left\{ \alpha Z_{13} + (1-\alpha) Z_{23} + 2\sqrt{(1-\beta)\alpha(1-\alpha)Z_{13}Z_{23}}, \alpha\beta Z_{12} \right\}. \quad (11)$$

Thus it reduces to solving the optimization problem in (11).

To solve (11), we write $Z_{\text{fd}}^u(\alpha) = \max_{0 \leq \beta \leq 1} \min \left\{ \alpha Z_{13} + (1-\alpha) Z_{23} + 2\sqrt{(1-\beta)\alpha(1-\alpha)Z_{13}Z_{23}}, \alpha\beta Z_{12} \right\}$. and consider two cases:

a) $0 \leq \alpha \leq \frac{Z_{23}}{Z_{23} + Z_{12} - Z_{13}}$: Under this case, the second term inside the min operator is smaller than the first term for all $0 \leq \beta \leq 1$. Hence $Z_{\text{fd}}^u(\alpha) = \max_{0 \leq \beta \leq 1} \alpha\beta Z_{12} = \alpha Z_{12}$.

b) $\frac{Z_{23}}{Z_{23} + Z_{12} - Z_{13}} < \alpha \leq 1$: Under this case, notice that the first and second terms inside the min operator are strictly decreasing and increasing in β , respectively. Moreover the two terms equalize at some $0 \leq \beta_* \leq 1$. Hence $Z_{\text{fd}}^u(\alpha) = \alpha\beta_* Z_{12}$. Solving for the equalizing β_* , we get

$$Z_{\text{fd}}^u(\alpha) = \alpha Z_{13} + (1-\alpha) Z_{23} \left(1 - 2\frac{Z_{13}}{Z_{12}} \right) + 2\sqrt{(1-\alpha)\frac{Z_{13}}{Z_{12}} \left(1 - \frac{Z_{13}}{Z_{12}} \right) Z_{23} [\alpha Z_{12} - (1-\alpha) Z_{23}]}.$$

Now we maximize $Z_{\text{fd}}^u(\alpha)$ over $0 \leq \alpha \leq 1$. For case a), $\max_{\alpha} Z_{\text{fd}}^u(\alpha) = \frac{Z_{12}Z_{23}}{Z_{23} + Z_{12} - Z_{13}}$. For case b), a direct but tedious calculation shows that $\max_{\alpha} Z_{\text{fd}}^u(\alpha) = \frac{Z_{12}(Z_{13} + Z_{23})}{Z_{12} + Z_{23}}$. It is not hard to verify that the maximum value in case b) is larger than the maximum value in case a). Hence $Z_{\text{fd}}^u = \frac{Z_{12}(Z_{13} + Z_{23})}{Z_{12} + Z_{23}}$ and the sufficient condition is $S > \frac{Z_{12} + Z_{23}}{Z_{12}(Z_{13} + Z_{23})}$.

For the necessary condition, we employ the max-flow-min-cut bound of [20, Theorem 14.10.1] to obtain an upper bound, $R_{\text{fd}}^l(\alpha)$, on the rate of the relay channel. It turns out [5] that the expression for $R_{\text{fd}}^l(\alpha)$ is obtained simply by replacing every occurrence of Z_{12} by $Z_{12} + Z_{13}$ in (10) above. In addition, the power optimization procedure above carries through directly for this case with every occurrence of Z_{12} replaced by $Z_{12} + Z_{13}$. Thus we obtain the necessary condition as $S > \frac{Z_{12} + Z_{13} + Z_{23}}{(Z_{12} + Z_{13})(Z_{13} + Z_{23})}$.

2) $Z_{12} \leq Z_{13}$: First note that the capacity of the relay channel is upper bounded by the maximum sum rate of the CB channel from the source to the relay and destination. This CB channel is a degraded

Gaussian broadcast channel and the individual rates R_{13} and R_{12} from the source to the destination and relay, respectively, satisfy [20, Ch. 14]

$$\begin{aligned} R_{13} &< C\left(\frac{\alpha Z_{13} P_t}{N_0 W}\right) \\ R_{12} &< C\left(\frac{(1-\alpha) Z_{12} P_t}{\alpha Z_{12} P_t + N_0 W}\right), \end{aligned} \quad (12)$$

for any $0 \leq \alpha \leq 1$. To have $R_{13} + R_{12} \geq K$, we need

$$\begin{aligned} K &< \max_{0 \leq \alpha \leq 1} \left\{ C\left(\frac{\alpha Z_{13} P_t}{N_0 W}\right) + C\left(\frac{(1-\alpha) Z_{12} P_t}{\alpha Z_{12} P_t + N_0 W}\right) \right\} \\ &= \log \left[\left(1 + Z_{12} \frac{P_t}{N_0 W}\right) \cdot \max_{0 \leq \alpha \leq 1} \left(\frac{1 + \alpha Z_{13} \frac{P_t}{N_0 W}}{1 + \alpha Z_{12} \frac{P_t}{N_0 W}} \right) \right] \\ &= C\left(Z_{13} \frac{P_t}{N_0 W}\right), \end{aligned}$$

where the last equality is obtained by choosing $\alpha = 1$, due to the condition that $Z_{13} \geq Z_{12}$. Hence $S > 1/Z_{13}$. This lower bound is achievable by sending all information directly from the source to the destination without using the relay.

B. Proof of Lemma 3.1

- 1) The case of $t_1 = 0$ trivially requires $x_1 = x_2 = 0$, and hence $S_{CB} = 0$. So we consider $0 < t_1 \leq 1$. If $Z_{13} > Z_{12}$, we have a degraded broadcast channel during this time slot. Thus rate constraints in (12) must be satisfied with $R_{13} = x_1/t_1$ and $R_{12} = x_2/t_1$. Combining the two inequalities to remove α , it is easy to obtain the stated lower bound S_{CB} of the SNR $P_t/N_0 W$. We note that this lower bound corresponds to the optimal choice $\alpha = \frac{(e^{x_1/t_1} - 1)/Z_{13}}{(e^{x_2/t_1} - 1)/Z_{12} + e^{x_2/t_1}(e^{x_1/t_1} - 1)/Z_{13}}$. Interchanging the roles of Z_{13} and Z_{12} , we get the stated SNR lower bound for the case of $Z_{13} \leq Z_{12}$.
- 2) The case of $t_2 = 0$ trivially requires $x_2 = x_3 = 0$, and hence $S_{MA} = 0$. So we consider $0 < t_2 \leq 1$. The capacity region of this Gaussian MA channel is specified by [20, Ch. 14]:

$$\begin{aligned} \frac{x_3}{t_2} &< C\left(\frac{\alpha Z_{13} P_t}{N_0 W}\right), \\ \frac{x_2}{t_2} &< C\left(\frac{(1-\alpha) Z_{23} P_t}{N_0 W}\right), \\ \frac{x_2 + x_3}{t_2} &< C\left(\frac{[\alpha Z_{13} + (1-\alpha) Z_{23}] P_t}{N_0 W}\right), \end{aligned} \quad (13)$$

for any $0 \leq \alpha \leq 1$, where α and $1 - \alpha$ are the fractions of the transmit power assigned to the source and relay, respectively. We want to optimally choose α so that the SNR $P_t/N_0 W$ required

to satisfy (13) is minimized. First, suppose that $Z_{13} > Z_{23}$. Then rearranging the second and third inequalities in (13) gives

$$\begin{aligned}\alpha &< 1 - \frac{1}{Z_{23}} \left(e^{x_2/t_2} - 1 \right) \frac{N_0 W}{P_t}, \\ \alpha &> \frac{1}{Z_{13} - Z_{23}} \cdot \left\{ \left[e^{(x_2+x_3)/t_2} - 1 \right] \frac{N_0 W}{P_t} - Z_{23} \right\},\end{aligned}$$

respectively. Combining these two inequalities to remove α , we obtain the stated lower bound S_{MA} . We note that the corresponding optimal choice $\alpha = \frac{(e^{x_3/t_2} - 1)/Z_{13}}{(e^{x_2/t_2} - 1)/Z_{23} + e^{x_2/t_2}(e^{x_3/t_2} - 1)/Z_{13}}$. Interchanging the roles of Z_{13} and Z_{23} , we get the stated SNR lower bound for the case of $Z_{13} < Z_{23}$. When $Z_{13} = Z_{23}$, the third inequality in (13) gives the stated lower bound S_{MA} . The choice of optimal α in this case exactly the same as the one in the case of $Z_{13} > Z_{23}$ (or $Z_{13} < Z_{23}$).

C. Solution to optimization problem (1)

Suppose that t_1 and t_2 are fixed, satisfying both the non-negativity and total time requirements. Then we can view the optimization problem (1) as a convex optimization problem in x_1, x_2 and x_3 . Rewriting it in standard form [21]:

$$\begin{aligned}\min_{x_1, x_2, x_3} \quad & \tilde{S}(t_1, t_2) = t_1 S_{\text{CB}} + t_2 S_{\text{MA}} \\ \text{Subject to} \quad & \text{i. total data requirement:} \quad x_1 + x_2 + x_3 = K \\ & \text{ii. non-negativity requirements:} \quad -x_1, -x_2, -x_3 \leq 0\end{aligned}\tag{14}$$

Since this optimization problem is convex, the rate tuple (x_1, x_2, x_3) is a solution if it satisfies the following KKT conditions:

- K1. $\nabla \tilde{S} + \sum_{i=1}^3 \lambda_i \nabla(-x_i) + \mu \nabla(x_1 + x_2 + x_3 - K) = 0$
- K2. $\lambda_i(-x_i) = 0$ for $i = 1, 2, 3$
- K3. $\lambda_i \geq 0$ for $i = 1, 2, 3$
- K4. $-x_i \leq 0$ for $i = 1, 2, 3$
- K5. $x_1 + x_2 + x_3 = K$.

Our approach to solve the original optimization problem (1) is to first solve the sub-problem (14) for each pair of (t_1, t_2) , and then minimize $\min_{x_1, x_2, x_3} \tilde{S}(t_1, t_2)$ over all allowable pairs. To this end, we consider the following cases and obtain solution to the optimization problem (14) by directly checking the KKT conditions. Note that we assume in below that both t_1 and t_2 are positive. For $t_2 = 0$ ($t_1 = 0$), Lemma 3.1 tells us that the transmission in the first (second) reduces trivially to over the direct link from the source to the destination. Hence $\min_{x_1, x_2, x_3} \tilde{S}(1, 0) = \min_{x_1, x_2, x_3} \tilde{S}(0, 1) = (e^K - 1) / Z_{13}$.

1) $Z_{13} \geq Z_{12}$ and $Z_{13} \geq Z_{23}$: From Lemma 3.1,

$$\tilde{S}(t_1, t_2) = \frac{t_1}{Z_{12}} (e^{x_2/t_1} - 1) + \frac{t_1}{Z_{13}} e^{x_2/t_1} (e^{x_1/t_1} - 1) + \frac{t_2}{Z_{23}} (e^{x_2/t_2} - 1) + \frac{t_2}{Z_{13}} e^{x_2/t_2} (e^{x_3/t_2} - 1).$$

The condition K1 yields

$$\begin{aligned} \frac{1}{Z_{13}} e^{(x_1+x_2)/t_1} - \lambda_1 + \mu &= 0 \\ \frac{1}{Z_{12}} e^{x_2/t_1} + \frac{1}{Z_{13}} e^{x_2/t_1} (e^{x_1/t_1} - 1) + \frac{1}{Z_{23}} e^{x_2/t_2} + \frac{1}{Z_{13}} e^{x_2/t_2} (e^{x_3/t_2} - 1) - \lambda_2 + \mu &= 0 \\ \frac{1}{Z_{13}} e^{(x_2+x_3)/t_2} - \lambda_3 + \mu &= 0. \end{aligned}$$

It is then easy to check that the following solution satisfy the KKT conditions:

$$\begin{aligned} x_1 &= Kt_1, & \lambda_1 &= 0 \\ x_2 &= 0, & \lambda_2 &= \frac{1}{Z_{12}} + \frac{1}{Z_{23}} - \frac{1}{Z_{13}} + \frac{1}{Z_{13}} (e^K - 1) \\ x_3 &= Kt_2, & \lambda_3 &= 0 \\ \mu &= -\frac{1}{Z_{13}} e^K. \end{aligned}$$

2) $Z_{13} \geq Z_{12}$ and $Z_{13} < Z_{23}$: From Lemma 3.1,

$$\tilde{S}(t_1, t_2) = \frac{t_1}{Z_{12}} (e^{x_2/t_1} - 1) + \frac{t_1}{Z_{13}} e^{x_2/t_1} (e^{x_1/t_1} - 1) + \frac{t_2}{Z_{13}} (e^{x_3/t_2} - 1) + \frac{t_2}{Z_{23}} e^{x_3/t_2} (e^{x_2/t_2} - 1).$$

The condition K1 yields

$$\begin{aligned} \frac{1}{Z_{13}} e^{(x_1+x_2)/t_1} - \lambda_1 + \mu &= 0 \\ \frac{1}{Z_{12}} e^{x_2/t_1} + \frac{1}{Z_{13}} e^{x_2/t_1} (e^{x_1/t_1} - 1) + \frac{1}{Z_{23}} e^{(x_2+x_3)/t_2} - \lambda_2 + \mu &= 0 \\ \frac{1}{Z_{13}} e^{x_3/t_2} + \frac{1}{Z_{23}} e^{x_3/t_2} (e^{x_2/t_2} - 1) - \lambda_3 + \mu &= 0. \end{aligned}$$

It is then easy to check that the following solution satisfy the KKT conditions:

$$\begin{aligned} x_1 &= Kt_1, & \lambda_1 &= 0 \\ x_2 &= 0, & \lambda_2 &= \frac{1}{Z_{12}} - \frac{1}{Z_{13}} + \frac{1}{Z_{23}} e^K \\ x_3 &= Kt_2, & \lambda_3 &= 0 \\ \mu &= -\frac{1}{Z_{13}} e^K. \end{aligned}$$

3) $Z_{13} < Z_{12}$ and $Z_{13} \geq Z_{23}$: From Lemma 3.1,

$$\tilde{S}(t_1, t_2) = \frac{t_1}{Z_{13}} (e^{x_1/t_1} - 1) + \frac{t_1}{Z_{12}} e^{x_1/t_1} (e^{x_2/t_1} - 1) + \frac{t_2}{Z_{23}} (e^{x_2/t_2} - 1) + \frac{t_2}{Z_{13}} e^{x_2/t_2} (e^{x_3/t_2} - 1).$$

The condition K1 yields

$$\begin{aligned} \frac{1}{Z_{13}} e^{x_1/t_1} + \frac{1}{Z_{12}} e^{x_1/t_1} (e^{x_2/t_1} - 1) - \lambda_1 + \mu &= 0 \\ \frac{1}{Z_{12}} e^{(x_1+x_2)/t_1} + \frac{1}{Z_{23}} e^{x_2/t_2} + \frac{1}{Z_{13}} e^{x_2/t_2} (e^{x_3/t_2} - 1) - \lambda_2 + \mu &= 0 \\ \frac{1}{Z_{13}} e^{(x_2+x_3)/t_2} - \lambda_3 + \mu &= 0. \end{aligned}$$

It is then easy to check that the following solution satisfy the KKT conditions:

$$\begin{aligned} x_1 &= Kt_1, & \lambda_1 &= 0 \\ x_2 &= 0, & \lambda_2 &= \frac{1}{Z_{12}}e^K + \frac{1}{Z_{23}} - \frac{1}{Z_{13}} \\ x_3 &= Kt_2, & \lambda_3 &= 0 \\ \mu &= -\frac{1}{Z_{13}}e^K. \end{aligned}$$

4) $M_H(Z_{12}, Z_{23}) \leq Z_{13} < \min\{Z_{12}, Z_{23}\}$: From Lemma 3.1,

$$\tilde{S}(t_1, t_2) = \frac{t_1}{Z_{13}} \left(e^{x_1/t_1} - 1 \right) + \frac{t_1}{Z_{12}} e^{x_1/t_1} \left(e^{x_2/t_1} - 1 \right) + \frac{t_2}{Z_{13}} \left(e^{x_3/t_2} - 1 \right) + \frac{t_2}{Z_{23}} e^{x_3/t_2} \left(e^{x_2/t_2} - 1 \right).$$

The condition K1 yields

$$\begin{aligned} \frac{1}{Z_{13}} e^{x_1/t_1} + \frac{1}{Z_{12}} e^{x_1/t_1} (e^{x_2/t_1} - 1) - \lambda_1 + \mu &= 0 \\ \frac{1}{Z_{12}} e^{(x_1+x_2)/t_1} + \frac{1}{Z_{23}} e^{(x_2+x_3)/t_2} - \lambda_2 + \mu &= 0 \\ \frac{1}{Z_{13}} e^{x_3/t_2} + \frac{1}{Z_{23}} e^{x_3/t_2} (e^{x_2/t_2} - 1) - \lambda_3 + \mu &= 0. \end{aligned}$$

It is then easy to check that the following solution satisfy the KKT conditions:

$$\begin{aligned} x_1 &= Kt_1, & \lambda_1 &= 0 \\ x_2 &= 0, & \lambda_2 &= \left(\frac{1}{Z_{12}} + \frac{1}{Z_{23}} - \frac{1}{Z_{13}} \right) e^K \\ x_3 &= Kt_2, & \lambda_3 &= 0 \\ \mu &= -\frac{1}{Z_{13}} e^K. \end{aligned}$$

5) $Z_{13} < M_H(Z_{12}, Z_{23})$: The expression for $\tilde{S}(t_1, t_2)$ in case 4) still holds. However, we need to consider the following two sub-cases in order to express the solution to the optimization problem (14):

a) $K > M_H(\log A_1, \log A_2)$:

i. For $1 - \frac{K}{\log A_1} \leq t_1 \leq \frac{K}{\log A_2}$, the following solution satisfies the KKT conditions:

$$\begin{aligned} x_1 &= Kt_1 - t_1 t_2 \log A_1, & \lambda_1 &= 0 \\ x_2 &= t_1 t_2 \log(A_1 A_2), & \lambda_2 &= 0 \\ x_3 &= Kt_2 - t_1 t_2 \log A_2, & \lambda_3 &= 0 \\ \mu &= -\frac{1}{Z_{12}} e^{K+t_2 \log A_2} - \frac{1}{Z_{23}} e^{K+t_1 \log A_1}. \end{aligned}$$

ii. For $\frac{K}{\log A_2} < t_1 < 1$, the following solution satisfies the KKT conditions:

$$\begin{aligned} x_1 &= Kt_1 - t_1 t_2 \log A_1, & \lambda_1 &= 0 \\ x_2 &= Kt_2 + t_1 t_2 \log A_1, & \lambda_2 &= 0 \\ x_3 &= 0, & \lambda_3 &= \frac{1}{Z_{13}} - \frac{1}{Z_{23}} - \frac{1}{Z_{12}} e^{K/t_1} \\ \mu &= -\frac{1}{Z_{23}} e^{K+t_1 \log A_1} - \frac{1}{Z_{12}} e^{K/t_1}. \end{aligned}$$

iii. For $0 < t_1 < 1 - \frac{K}{\log A_1}$, the following solution satisfies the KKT conditions:

$$\begin{aligned} x_1 &= 0, & \lambda_1 &= \frac{1}{Z_{13}} - \frac{1}{Z_{12}} - \frac{1}{Z_{23}} e^{K/t_2} \\ x_2 &= Kt_1 + t_1 t_2 \log A_2, & \lambda_2 &= 0 \\ x_3 &= Kt_2 - t_1 t_2 \log A_2, & \lambda_3 &= 0 \\ \mu &= -\frac{1}{Z_{12}} e^{K+t_2 \log A_2} - \frac{1}{Z_{23}} e^{K/t_2}. \end{aligned}$$

b) $K \leq M_H(\log A_1, \log A_2)$:

i. For $\frac{K}{\log A_2} \leq t_1 \leq 1 - \frac{K}{\log A_1}$, the following solution satisfies the KKT conditions:

$$\begin{aligned} x_1 &= 0, & \lambda_1 &= \frac{1}{Z_{13}} - \frac{1}{Z_{12}} - \frac{1}{Z_{23}} e^{K/t_2} \\ x_2 &= K, & \lambda_2 &= 0 \\ x_3 &= 0, & \lambda_3 &= \frac{1}{Z_{13}} - \frac{1}{Z_{23}} - \frac{1}{Z_{12}} e^{K/t_1} \\ \mu &= -\frac{1}{Z_{12}} e^{K/t_1} - \frac{1}{Z_{23}} e^{K/t_2}. \end{aligned}$$

ii. For $1 - \frac{K}{\log A_1} < t_1 < 1$, the following solution satisfies the KKT conditions:

$$\begin{aligned} x_1 &= Kt_1 - t_1 t_2 \log A_1, & \lambda_1 &= 0 \\ x_2 &= Kt_2 + t_1 t_2 \log A_1, & \lambda_2 &= 0 \\ x_3 &= 0, & \lambda_3 &= \frac{1}{Z_{13}} - \frac{1}{Z_{23}} - \frac{1}{Z_{12}} e^{K/t_1} \\ \mu &= -\frac{1}{Z_{23}} e^{K+t_1 \log A_1} - \frac{1}{Z_{12}} e^{K/t_1}. \end{aligned}$$

iii. For $0 < t_1 < \frac{K}{\log A_2}$, the following solution satisfies the KKT conditions:

$$\begin{aligned} x_1 &= 0, & \lambda_1 &= \frac{1}{Z_{13}} - \frac{1}{Z_{12}} - \frac{1}{Z_{23}} e^{K/t_2} \\ x_2 &= Kt_1 + t_1 t_2 \log A_2, & \lambda_2 &= 0 \\ x_3 &= Kt_2 - t_1 t_2 \log A_2, & \lambda_3 &= 0 \\ \mu &= -\frac{1}{Z_{12}} e^{K+t_2 \log A_2} - \frac{1}{Z_{23}} e^{K/t_2}. \end{aligned}$$

For cases 1)–4), direction substitution of the solution yields $\min_{x_1, x_2, x_3} \tilde{S}(t_1, t_2) = (e^K - 1) / Z_{13}$. Since this solution is independent of the choice of (t_1, t_2) , the solution to the optimization problem (1) in these four cases is simply $(e^K - 1) / Z_{13}$. Also note that these four cases can be collectively specified by the condition $Z_{13} \geq M_H(Z_{12}, Z_{23})$.

For case 5a), the three functions $\tilde{S}_1(K)$, $\tilde{S}_2(K)$, and $\tilde{S}_3(K)$ respectively described in (3), (4), and (5) can be obtained by direct substitution of the solutions in the 3 sub-cases (i., ii., and iii, respectively), and then minimizing the corresponding $\min_{x_1, x_2, x_3} \tilde{S}(t_1, t_2)$ over the range of t_1 specified in each sub-cases.

Hence the final solution of the optimization problem (1) is obtained by finding the minimum among the these three functions. For case 5b), a similar procedure yields the fact that the solution to the optimization problem (1) is the minimum among the three functions $\hat{S}_1(K)$, $\hat{S}_2(K)$, and $\hat{S}_3(K)$ respectively described in (7), (8), and (9).

D. Proof of Corollary 3.1

The proof of the results in this corollary is based on the fact that $B_1(K)$ is the (normalized) solution to the optimization problem (1) and the form of $B_1(K)$ described in Section III-A.1.

- 1) From the description of $B_1(K)$ in Section III-A.1, to show $B_1(K)$ is increasing, it suffices to establish that under the condition $Z_{13} < M_H(Z_{12}, Z_{23})$, the functions $\frac{\tilde{S}_1(K)}{e^K - 1}$, $\frac{\tilde{S}_2(K)}{e^K - 1}$, $\frac{\tilde{S}_3(K)}{e^K - 1}$, $\frac{\hat{S}_1(K)}{e^K - 1}$, $\frac{\hat{S}_2(K)}{e^K - 1}$, and $\frac{\hat{S}_3(K)}{e^K - 1}$ are all increasing in the corresponding ranges of K that the functions are finite. To this end, we will repeatedly employ the following form of Young's inequality:

$$x^t y^{1-t} \leq tx + (1-t)y,$$

for nonnegative x, y , and $0 \leq t \leq 1$.

Now suppose that $Z_{13} < M_H(Z_{12}, Z_{23})$. First let us consider $\tilde{S}_1(K)$. Suppose $K > M_H(\log A_1, \log A_2)$ and t^* falls within the range described in (3) for which $\tilde{S}_1(K)$ is finite. Then

$$\frac{d}{dK} \frac{\tilde{S}_1(K)}{e^K - 1} = \left(\frac{1}{Z_{13}} - \frac{A_1^{t^*}}{Z_{23}} - \frac{A_2^{1-t^*}}{Z_{12}} \right) \frac{e^K}{(e^K - 1)^2}.$$

But by Young's inequality,

$$\begin{aligned} \frac{1}{Z_{13}} - \frac{A_1^{t^*}}{Z_{23}} - \frac{A_2^{1-t^*}}{Z_{12}} &= \frac{1}{Z_{13}} - \left(\frac{1}{Z_{23}} \right)^{1-t^*} \left(\frac{1}{Z_{13}} - \frac{1}{Z_{12}} \right)^{t^*} - \left(\frac{1}{Z_{12}} \right)^{t^*} \left(\frac{1}{Z_{13}} - \frac{1}{Z_{23}} \right)^{1-t^*} \\ &\geq \frac{1}{Z_{13}} - \frac{1-t^*}{Z_{23}} - \frac{t^*}{Z_{13}} + \frac{t^*}{Z_{12}} - \frac{t^*}{Z_{12}} - \frac{1-t^*}{Z_{13}} + \frac{1-t^*}{Z_{23}} \\ &= 0. \end{aligned}$$

Thus $\frac{\tilde{S}_1(K)}{e^K - 1}$ is increasing.

Next consider $\tilde{S}_2(K)$. Suppose $K \leq \log A_2$ so that $\tilde{S}_2(K)$ is finite. For any $\frac{K}{\log A_2} \leq t \leq 1$,

$$\begin{aligned} \frac{d}{dK} \left(\frac{\frac{t}{Z_{12}} e^{K/t} + \frac{1}{Z_{23}} e^{K+t \log A_1} - \frac{t}{Z_{13}} - \frac{1-t}{Z_{23}}}{e^K - 1} \right) \\ = \left\{ \frac{1}{Z_{12}} \left[(1-t)e^{K/t} + t - e^{K(1-t)/t} \right] + \frac{1-t-A_1^t}{Z_{23}} - \frac{t}{Z_{12}} + \frac{t}{Z_{13}} \right\} \cdot \frac{e^K}{(e^K - 1)^2}. \end{aligned}$$

Again by Young's inequality,

$$(1-t)e^{K/t} + t - e^{K(1-t)/t} \geq e^{K(1-t)/t} \cdot 1^t - e^{K(1-t)/t} = 0$$

and

$$\begin{aligned}
\frac{1-t-A_1^t}{Z_{23}} - \frac{t}{Z_{12}} + \frac{t}{Z_{13}} &= \frac{1-t}{Z_{23}} - \frac{t}{Z_{12}} + \frac{t}{Z_{13}} - \left(\frac{1}{Z_{23}}\right)^{1-t} \left(\frac{1}{Z_{13}} - \frac{1}{Z_{12}}\right)^t \\
&\geq \frac{1-t}{Z_{23}} - \frac{t}{Z_{12}} + \frac{t}{Z_{13}} - \frac{1-t}{Z_{23}} - \frac{t}{Z_{13}} + \frac{t}{Z_{12}} \\
&= 0.
\end{aligned}$$

Hence $\frac{\tilde{S}_2(K)}{e^K-1}$ is increasing. Finally, we note that the increasing nature of the functions $\frac{\tilde{S}_3(K)}{e^K-1}$, $\frac{\hat{S}_1(K)}{e^K-1}$, $\frac{\hat{S}_2(K)}{e^K-1}$, and $\frac{\hat{S}_3(K)}{e^K-1}$ can be proven in the same way.

- 2) When K is sufficiently small, $B_1(K) = \frac{\hat{S}_1(K)}{e^K-1}$. Then a simple application of L'Hospital's rule gives the desired result.
- 3) When K is sufficiently large, $B_1(K) = \frac{\tilde{S}_1(K)}{e^K-1}$. Then simply taking limit gives the desired result.

E. Proof of Lemma 3.2

The case of $t_2 = 0$ trivially requires $x_2 = x_3 = 0$, and hence $\hat{S}_{\text{MA}} = 0$. So we consider $0 < t_2 \leq 1$. Suppose that the transmit power of the relay is P_2 and the transmit power of the source is $P_1 + P_3$, where P_1 is the power employed to transmit the flow of rate x_2/t_2 while P_3 is the power of the flow of rate x_3/t_2 . Then the transmission procedure in the second time slot of Protocol 2 (cf. Section III-B) describes the transmission over an equivalent two-user Gaussian MA channel in which one user of rate x_2/t_2 has power $(\sqrt{Z_{13}P_1} + \sqrt{Z_{23}P_2})^2$ and another user of rate x_3/t_2 has power $Z_{13}P_3$. From the capacity region of this Gaussian MA channel specified by [20, Ch. 14], P_1 , P_2 and P_3 must satisfy:

$$\begin{aligned}
\frac{x_3}{t_2} &< C \left(\frac{Z_{13}P_3}{N_0W} \right) \\
\frac{x_2}{t_2} &< C \left(\frac{(\sqrt{Z_{13}P_1} + \sqrt{Z_{23}P_2})^2}{N_0W} \right) \\
\frac{x_2 + x_3}{t_2} &< C \left(\frac{(\sqrt{Z_{13}P_1} + \sqrt{Z_{23}P_2})^2 + Z_{13}P_3}{N_0W} \right).
\end{aligned}$$

To minimize the total power given the rates of transmission, we consider the following optimization problem:

$$\begin{aligned}
& \min P_1 + P_2 + P_3 \\
& \text{subject to } f_1(\mathbf{P}) = a - Z_{13}P_3 \leq 0 \\
& f_2(\mathbf{P}) = b - (\sqrt{Z_{13}P_1} + \sqrt{Z_{23}P_2})^2 \leq 0 \\
& f_3(\mathbf{P}) = c - (\sqrt{Z_{13}P_1} + \sqrt{Z_{23}P_2})^2 - Z_{13}P_3 \leq 0 \\
& f_4(\mathbf{P}) = -P_1 \leq 0 \\
& f_5(\mathbf{P}) = -P_2 \leq 0 \\
& f_6(\mathbf{P}) = -P_3 \leq 0
\end{aligned}$$

where $\mathbf{P} = (P_1, P_2, P_3)$, $a = N_0W(e^{x_3/t_2} - 1)$, $b = N_0W(e^{x_2/t_2} - 1)$, and $c = N_0W(e^{(x_2+x_3)/t_2} - 1)$.

Notice that $c \geq a + b$. It can be shown that this is a convex optimization problem. The power tuple (P_1, P_2, P_3) is a solution if it satisfies the following KKT conditions:

$$\begin{aligned}
\text{K1. } & \nabla(P_1 + P_2 + P_3) + \sum_{i=1}^6 \lambda_i \nabla f_i(\mathbf{P}) = 0 \\
\text{K2. } & \lambda_i f_i(\mathbf{P}) = 0 \text{ for } i = 1, 2, \dots, 6 \\
\text{K3. } & \lambda_i \geq 0 \text{ for } i = 1, 2, \dots, 6 \\
\text{K4. } & f_i(\mathbf{P}) \leq 0 \text{ for } i = 1, 2, \dots, 6.
\end{aligned}$$

The condition K1 yields

$$\begin{aligned}
1 - (\lambda_2 + \lambda_3)(\sqrt{Z_{13}P_1} + \sqrt{Z_{23}P_2})\sqrt{\frac{Z_{13}}{P_1}} - \lambda_4 &= 0 \\
1 - (\lambda_2 + \lambda_3)(\sqrt{Z_{13}P_1} + \sqrt{Z_{23}P_2})\sqrt{\frac{Z_{23}}{P_2}} - \lambda_5 &= 0 \\
1 - (\lambda_1 + \lambda_3)Z_{13} - \lambda_6 &= 0.
\end{aligned}$$

It is then easy to check that the following solution satisfies the KKT conditions:

$$\begin{aligned}
P_1 &= \frac{(c-a)Z_{13}}{(Z_{13} + Z_{23})^2}, \quad \lambda_1 = \frac{1}{Z_{13}} - \frac{1}{Z_{13} + Z_{23}}, \\
P_2 &= \frac{(c-a)Z_{23}}{(Z_{13} + Z_{23})^2}, \quad \lambda_3 = \frac{1}{Z_{13} + Z_{23}}, \\
P_3 &= \frac{a}{Z_{13}}, \quad \lambda_2 = \lambda_4 = \lambda_5 = \lambda_6 = 0.
\end{aligned}$$

Then normalizing the sum of this choice of P_1 , P_2 , and P_3 by N_0W gives the stated expression of \hat{S}_{MA} in Lemma 3.2.

F. Proof of Theorem 4.1

To prove the theorem, we need to use the following result:

Claim 1: For any $x \geq z \geq 0$,

$$\Pr\left(\frac{1}{Z_{12}} + \frac{1}{Z_{23} + z} \geq \frac{1}{x}\right) = 1 - 2xK_1(2x)e^{-2x+z}.$$

Proof:

$$\begin{aligned} \Pr\left(\frac{1}{Z_{12}} + \frac{1}{Z_{23} + z} \geq \frac{1}{x}\right) &= \int_0^\infty \Pr\left(\frac{1}{Z_{12}} \geq \frac{1}{x} - \frac{1}{y+z}\right) e^{-y} dy \\ &= \int_0^{x-z} e^{-y} dy + \int_{x-z}^\infty \left(1 - e^{-\frac{1}{\frac{1}{x} - \frac{1}{y+z}}}\right) e^{-y} dy \\ &= 1 - \int_{x-z}^\infty e^{-\left(\frac{1}{\frac{1}{x} - \frac{1}{y+z}} + y\right)} dy \\ &= 1 - e^{-2x+z} \int_0^\infty e^{-\left(y + \frac{x^2}{y}\right)} dy, \end{aligned}$$

where the integral in the last line is an integral representation of the function $2xK_1(2x)$ [22, pp. 53]. ■

We note that the same result is obtained for the special case of $z = 0$ in [23] using moment generating functions of exponential random variables.

1) By Theorem 2.1,

$$\begin{aligned} P_{\text{fd}}(K, S) &\geq \Pr(S \leq B_{\text{fd}}^l) \\ &= \underbrace{\Pr\left(S \leq \frac{1}{Z_{13}} \mid Z_{13} \geq Z_{12}\right) \cdot \Pr(Z_{13} \geq Z_{12})}_a \\ &\quad + \underbrace{\Pr\left(S \leq \frac{Z_{12} + Z_{13} + Z_{23}}{(Z_{12} + Z_{13})(Z_{13} + Z_{23})} \mid Z_{13} < Z_{12}\right) \cdot \Pr(Z_{13} < Z_{12})}_b. \end{aligned}$$

A simple calculation shows that $a = \frac{1}{2} \left(1 + e^{-\frac{2}{s}}\right) - e^{-\frac{1}{s}}$. Conditioned on the event $\{Z_{13} < Z_{12}\}$, $\left\{S \leq \frac{1}{Z_{12} + Z_{13}}\right\} \cup \left\{S \leq \frac{1}{Z_{13} + Z_{23}}\right\} \subseteq \left\{S \leq \frac{Z_{12} + Z_{13} + Z_{23}}{(Z_{12} + Z_{13})(Z_{13} + Z_{23})}\right\}$. Hence

$$\begin{aligned} b &\geq \Pr\left(\left\{S \leq \frac{1}{Z_{12} + Z_{13}}\right\} \cup \left\{S \leq \frac{1}{Z_{13} + Z_{23}}\right\} \mid Z_{13} < Z_{12}\right) \cdot \Pr(Z_{13} < Z_{12}) \\ &= \Pr(Z_{13} < Z_{12}) - \Pr\left(\left\{S > \frac{1}{Z_{12} + Z_{13}}\right\} \cap \left\{S > \frac{1}{Z_{13} + Z_{23}}\right\} \cap \{Z_{13} < Z_{12}\}\right) \\ &= \int_0^\infty \left[\Pr(Z_{12} > z) - \Pr\left(Z_{12} > \max\left\{z, \frac{1}{S} - z\right\}\right) \cdot \Pr\left(Z_{23} > \frac{1}{S} - z\right)\right] e^{-z} dz \\ &= \int_0^{\frac{1}{2S}} \left(e^{-z} - e^{-\frac{1}{S}+z} \cdot e^{-\frac{1}{S}+z}\right) e^{-z} dz + \int_{\frac{1}{2S}}^{\frac{1}{S}} \left(e^{-z} - e^{-z} \cdot e^{-\frac{1}{S}+z}\right) e^{-z} dz \\ &\quad + \int_{\frac{1}{S}}^\infty (e^{-z} - e^{-z} \cdot 1) e^{-z} dz \\ &= \frac{1}{2} - 2e^{-\frac{1.5}{S}} + \frac{3}{2}e^{-\frac{2}{S}}. \end{aligned}$$

It is also easy to see that $\lim_{S \rightarrow \infty} \frac{a+b}{1/S^2} = 1.25$.

- 2) By Corollary 3.1, $B_1(K) \leq \lim_{K' \rightarrow \infty} B_1(K')$ for all $K > 0$. Note that when K is sufficiently large, $B_1(K) = \frac{\tilde{S}_1(K)}{e^K - 1}$. Now instead of choosing the optimal t^* in (3), we choose $t_1 = 1/2$ and normalize the suboptimal solution by the factor $e^K - 1$. Taking limit as $K \rightarrow \infty$, we get

$$\tilde{B}_1 = \begin{cases} \sqrt{\frac{1}{Z_{23}} \left(\frac{1}{Z_{13}} - \frac{1}{Z_{12}} \right)} + \sqrt{\frac{1}{Z_{12}} \left(\frac{1}{Z_{13}} - \frac{1}{Z_{23}} \right)} & \text{if } Z_{13} < M_H(Z_{12}, Z_{23}) \\ \frac{1}{Z_{13}} & \text{if } Z_{13} \geq M_H(Z_{12}, Z_{23}). \end{cases}$$

Obviously, $\lim_{K' \rightarrow \infty} B_1(K') \leq \tilde{B}_1$ because of the suboptimality of the choice $t_1 = 1/2$. Thus, $P_1(K, S) \leq \Pr(S \leq \tilde{B}_1)$. Moreover, when $Z_{13} < M_H(Z_{12}, Z_{23})$,

$$\begin{aligned} \sqrt{\frac{1}{Z_{23}} \left(\frac{1}{Z_{13}} - \frac{1}{Z_{12}} \right)} + \sqrt{\frac{1}{Z_{12}} \left(\frac{1}{Z_{13}} - \frac{1}{Z_{23}} \right)} &= \frac{2 \left(\frac{1}{2} \sqrt{\frac{Z_{12}}{Z_{13}} - 1} + \frac{1}{2} \sqrt{\frac{Z_{23}}{Z_{13}} - 1} \right)}{\sqrt{Z_{12} Z_{23}}} \\ &\leq \frac{2 \sqrt{\frac{Z_{12} + Z_{23}}{2 Z_{13}} - 1}}{\sqrt{Z_{12} Z_{23}}} \\ &< \sqrt{\frac{2}{Z_{13}}} \cdot \sqrt{\frac{1}{Z_{12}} + \frac{1}{Z_{23}}}, \end{aligned}$$

where the second line is due to the concavity of the square-root function. Hence

$$\begin{aligned} P_1(K, S) &\leq \underbrace{\Pr \left(S \leq \frac{1}{Z_{13}} \mid \frac{1}{Z_{12}} + \frac{1}{Z_{23}} \geq \frac{1}{Z_{13}} \right)}_a \cdot \Pr \left(\frac{1}{Z_{12}} + \frac{1}{Z_{23}} \geq \frac{1}{Z_{13}} \right) \\ &\quad + \underbrace{\Pr \left(S < \sqrt{\frac{2}{Z_{13}}} \cdot \sqrt{\frac{1}{Z_{12}} + \frac{1}{Z_{23}}} \right)}_b \cdot \Pr \left(\frac{1}{Z_{12}} + \frac{1}{Z_{23}} < \frac{1}{Z_{13}} \right). \end{aligned}$$

By Claim 1,

$$\begin{aligned} a &= \int_0^{\frac{1}{S}} \Pr \left(\frac{1}{Z_{12}} + \frac{1}{Z_{23}} \geq \frac{1}{z} \right) e^{-z} dz \\ &= \int_0^{\frac{1}{S}} [1 - 2z K_1(2z) e^{-2z}] e^{-z} dz \\ &= 1 - e^{\frac{1}{S}} - \int_0^{\frac{1}{S}} 2z K_1(2z) e^{-3z} dz, \end{aligned}$$

and

$$\begin{aligned} b &= \int_0^{\frac{\sqrt{2}}{S}} \Pr \left(\frac{S^2 z}{2} < \frac{1}{Z_{12}} + \frac{1}{Z_{23}} < \frac{1}{z} \right) e^{-z} dz \\ &= \int_0^{\frac{\sqrt{2}}{S}} \left[2z K_1(2z) e^{-2z} - \frac{4}{S^2 z} K_1 \left(\frac{4}{S^2 z} \right) e^{-\frac{4}{S^2 z}} \right] e^{-z} dz. \end{aligned}$$

Now by repeated uses of L'Hospital's rule, we have

$$\begin{aligned}
\lim_{S \rightarrow \infty} \frac{a}{1/S^2} &= \lim_{u \rightarrow 0} \frac{1 - e^u - \int_0^u 2z K_1(2z) e^{-3z} dz}{u^2} \\
&= \lim_{u \rightarrow 0} e^{-u} \cdot \frac{1 - 2u K_1(2u) e^{-2u}}{2u} \\
&= \lim_{u \rightarrow 0} 2u [K_1(2u) + K_0(2u)] e^{-2u} \\
&= 1,
\end{aligned}$$

where the third equality is due to the fact that the derivative of $-u K_1(u) e^{-u}$ is $u [K_1(u) + K_0(u)] e^{-u}$ [23], and the last equality is due to the facts that $\lim_{u \rightarrow 0} u K_1(u) = 1$ and $\lim_{u \rightarrow 0} u K_0(u) = 0$ [22]. To find the asymptotic order of b , let us write

$$c = \int_0^{\frac{\sqrt{2}}{S}} \left[2z K_1(2z) e^{-2z} - \frac{4}{S^2 z} K_1 \left(\frac{4}{S^2 z} \right) e^{-\frac{4}{S^2 z}} \right] dz.$$

First, we note that

$$e^{-\frac{\sqrt{2}}{S}} c \leq b \leq c.$$

Then again by repeated applications of L'Hospital's rule, we have

$$\begin{aligned}
\lim_{S \rightarrow \infty} \frac{c}{\log(S)/S^2} &= \lim_{u \rightarrow 0} \frac{\int_0^{\sqrt{2u}} \left[2z K_1(2z) e^{-2z} - \frac{4u}{z} K_1 \left(\frac{4u}{z} \right) e^{-\frac{4u}{z}} \right] dz}{-\frac{1}{2} u \log u} \\
&= 8 \lim_{u \rightarrow 0} \frac{\int_{2\sqrt{2u}}^{\infty} [K_1(x) + K_0(x)] e^{-x} dx}{-\log u - 1} \\
&= 4 \lim_{u \rightarrow 0} 2\sqrt{2u} [K_1(2\sqrt{2u}) + K_0(2\sqrt{2u})] e^{-2\sqrt{2u}} \\
&= 4,
\end{aligned}$$

where the second equality is obtained by a change of integration variable after the use of L'Hospital's rule. As a consequence, $\lim_{S \rightarrow \infty} \frac{b}{\log(S)/S^2} = 4$.

- 3) By Corollary 3.1, $B_1(K) \geq \lim_{K' \rightarrow 0} B_1(K') = \min \left\{ \frac{1}{Z_{13}}, \frac{1}{Z_{23}} + \frac{1}{Z_{12}} \right\}$ for all $K > 0$. In addition, the bound is achieved as K approaches zero. Thus

$$\begin{aligned}
P_1(K, S) &\geq \Pr \left(S \leq \min \left\{ \frac{1}{Z_{13}}, \frac{1}{Z_{23}} + \frac{1}{Z_{12}} \right\} \right) \\
&= \Pr \left(S \leq \frac{1}{Z_{13}} \right) \cdot \Pr \left(S \leq \frac{1}{Z_{23}} + \frac{1}{Z_{12}} \right) \\
&= \left[1 - e^{-\frac{1}{S}} \right] \left[1 - \frac{2}{S} K_1 \left(\frac{2}{S} \right) e^{-\frac{2}{S}} \right],
\end{aligned}$$

where the last line is due to Claim 1 and the bound is achieved as $K \rightarrow 0$ by monotone convergence.

Moreover,

$$\begin{aligned} \lim_{S \rightarrow \infty} \frac{\left[1 - \frac{2}{S} K_1\left(\frac{2}{S}\right) e^{-\frac{2}{S}}\right] \left[1 - e^{-\frac{1}{S}}\right]}{1/S^2} &= \lim_{u \rightarrow 0} \frac{1 - 2u K_1(2u) e^{-2u}}{u} \cdot \lim_{u \rightarrow 0} \frac{1 - e^{-u}}{u} \\ &= 2 \cdot 1. \end{aligned}$$

4) By Corollary 3.2 and similar to part 2), we have

$$\begin{aligned} P_2(K, S) &\leq \underbrace{\Pr\left(S \leq \frac{1}{Z_{13}} \mid \frac{1}{Z_{12}} + \frac{1}{Z_{13} + Z_{23}} \geq \frac{1}{Z_{13}}\right) \cdot \Pr\left(\frac{1}{Z_{12}} + \frac{1}{Z_{13} + Z_{23}} \geq \frac{1}{Z_{13}}\right)}_a \\ &\quad + \underbrace{\Pr\left(S < \sqrt{\frac{2}{Z_{13}}} \cdot \sqrt{\frac{1}{Z_{12}} + \frac{1}{Z_{13} + Z_{23}}}\right) \cdot \Pr\left(\frac{1}{Z_{12}} + \frac{1}{Z_{13} + Z_{23}} < \frac{1}{Z_{13}}\right)}_b. \end{aligned}$$

By Claim 1,

$$\begin{aligned} a &= \int_0^{\frac{1}{S}} \Pr\left(\frac{1}{Z_{12}} + \frac{1}{Z_{23} + z} \geq \frac{1}{z}\right) e^{-z} dz \\ &= 1 - e^{-\frac{1}{S}} - \int_0^{\frac{1}{S}} 2z K_1(2z) e^{-2z} dz, \end{aligned}$$

and

$$\begin{aligned} b &= \int_0^{\frac{\sqrt{2}}{S}} \Pr\left(\frac{S^2 z}{2} < \frac{1}{Z_{12}} + \frac{1}{Z_{23} + z} < \frac{1}{z}\right) e^{-z} dz \\ &= \int_0^{\frac{\sqrt{2}}{S}} \left[2z K_1(2z) e^{-2z} - \frac{4}{S^2 z} K_1\left(\frac{4}{S^2 z}\right) e^{-\frac{4}{S^2 z}}\right] dz. \end{aligned}$$

Now by repeated uses of L'Hospital's rule, we have

$$\begin{aligned} \lim_{S \rightarrow \infty} \frac{a}{1/S^2} &= \lim_{u \rightarrow 0} \frac{1 - e^{-u} - \int_0^u 2z K_1(2z) e^{-2z} dz}{u^2} \\ &= \lim_{u \rightarrow 0} \frac{e^{-u} - 2u K_1(2u) e^{-2u}}{2u} \\ &= 1/2. \end{aligned}$$

As derived in part 2), $\lim_{S \rightarrow \infty} \frac{b}{\log(S)/S^2} = 4$.

5) By Corollary 3.2 and similar to part 3), we have

$$\begin{aligned}
P_2(K, S) &\geq \Pr \left(S \leq \min \left\{ \frac{1}{Z_{13}}, \frac{1}{Z_{13} + Z_{23}} + \frac{1}{Z_{12}} \right\} \right) \\
&= \int_0^{\frac{1}{S}} \Pr \left(S \leq \frac{1}{Z_{12}} + \frac{1}{Z_{23} + z} \right) e^{-z} dz \\
&= \int_0^{\frac{1}{S}} \left[e^{-z} - \frac{2}{S} K_1 \left(\frac{2}{S} \right) e^{-\frac{2}{S}} \right] dz \\
&= 1 - e^{-\frac{1}{S}} - \frac{2}{S^2} K_1 \left(\frac{2}{S} \right) e^{-\frac{2}{S}},
\end{aligned}$$

where the equality in the third line is established by Claim 1 and the bound is achieved as $K \rightarrow 0$ by monotone convergence. Moreover,

$$\begin{aligned}
\lim_{S \rightarrow \infty} \frac{1 - e^{-\frac{1}{S}} - \frac{2}{S^2} K_1 \left(\frac{2}{S} \right) e^{-\frac{2}{S}}}{1/S^2} &= \lim_{u \rightarrow 0} \frac{1 - e^{-u} - 2u^2 K_1(2u) e^{-2u}}{u^2} \\
&= \lim_{u \rightarrow 0} \frac{e^{-u} - 2u K_1(2u) e^{-2u}}{2u} + \lim_{u \rightarrow 0} 2u [K_1(2u) + K_0(2u)] e^{-2u} \\
&= \lim_{u \rightarrow 0} \frac{-e^{-u} - 4u [K_1(2u) + K_0(2u)] e^{-2u}}{2} + 1 \\
&= 1/2 + 1 = 3/2.
\end{aligned}$$

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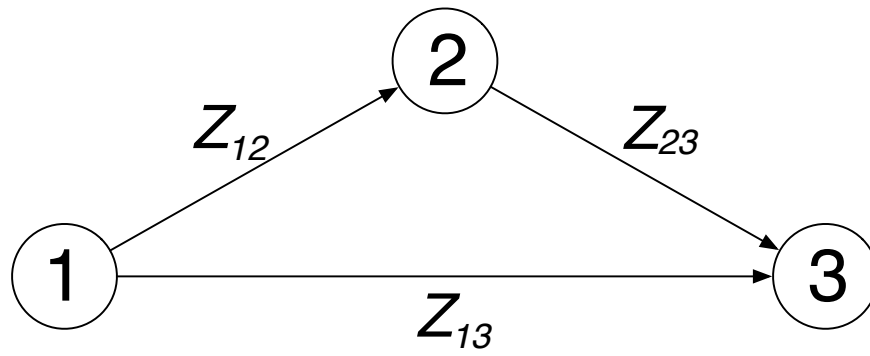


Fig. 1. The classical 3-node relay channel.

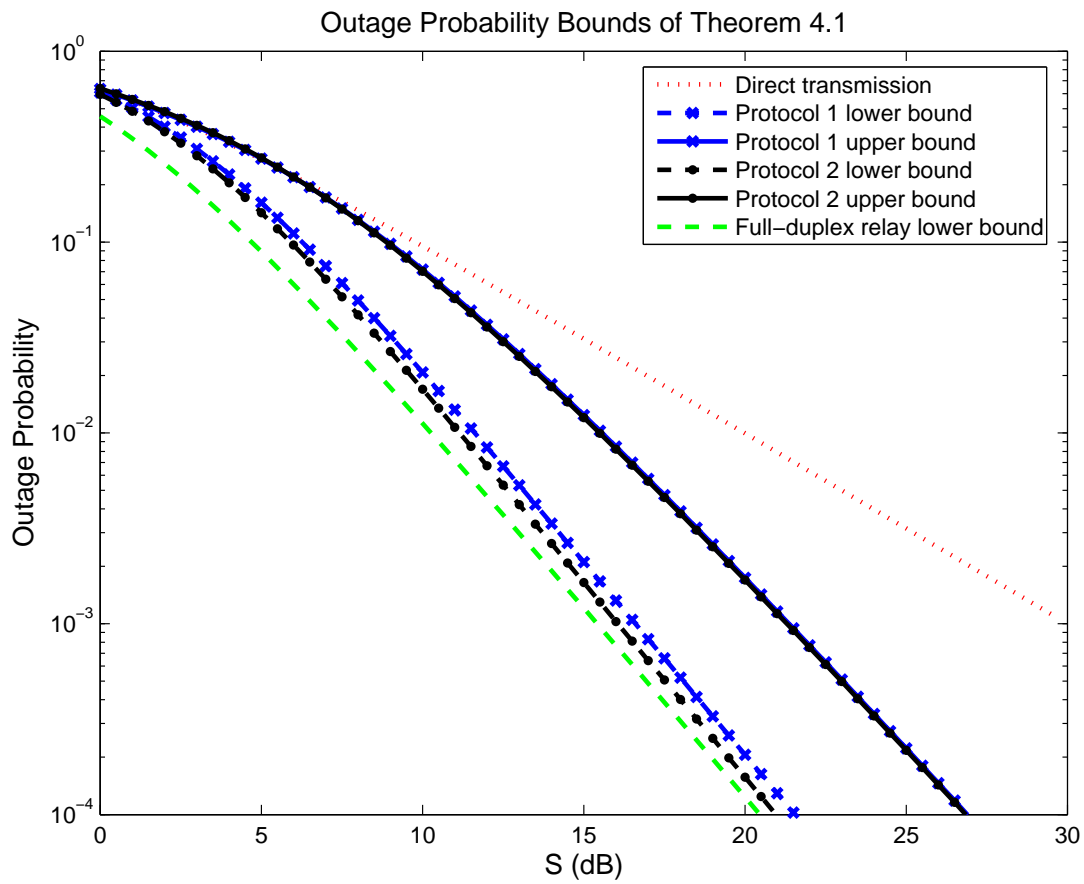


Fig. 2. Outage probability bounds of Theorem 4.1.

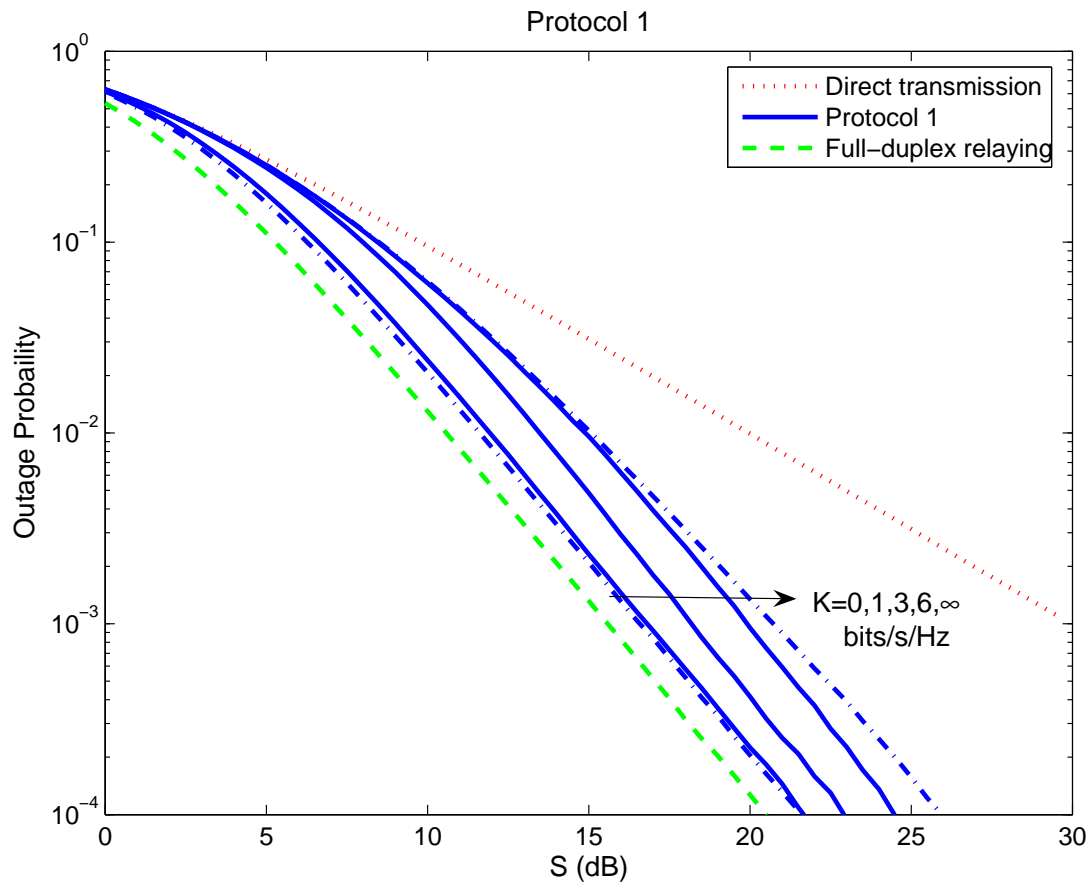


Fig. 3. Outage probabilities for Protocol 1 obtained from Monte Carlos simulations.

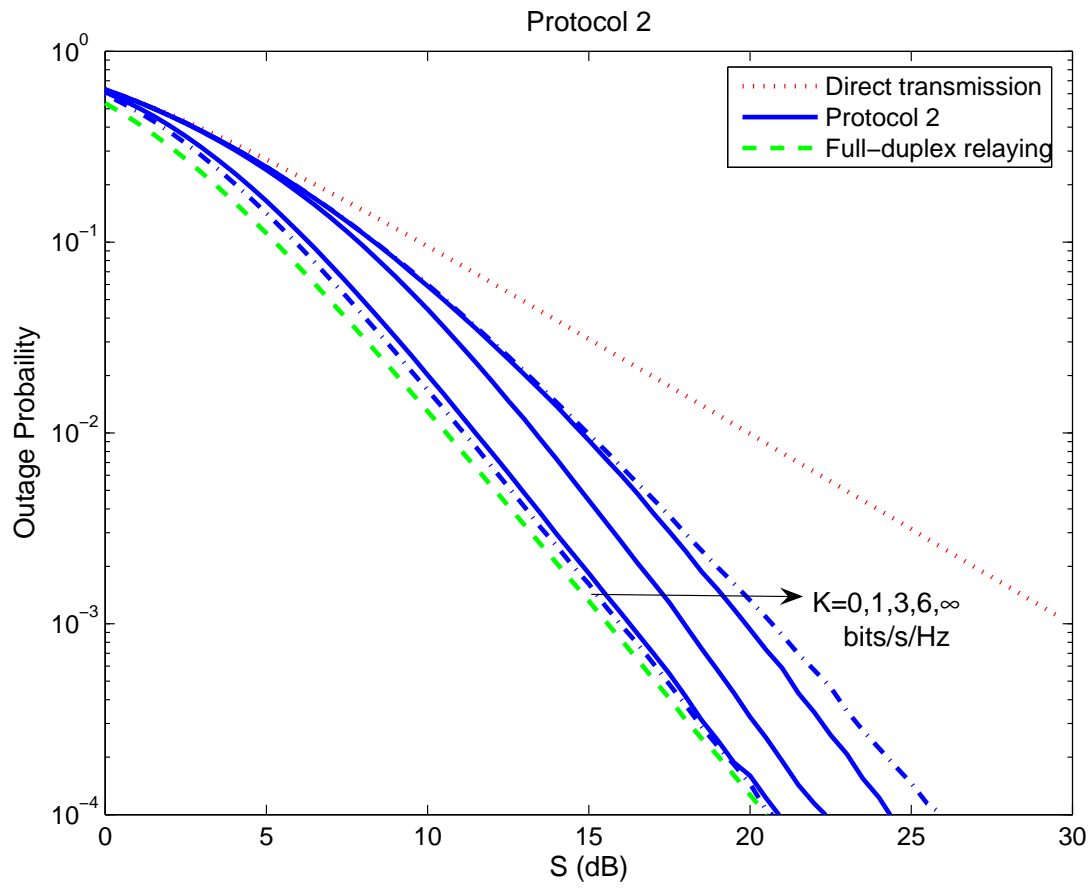


Fig. 4. Outage probabilities for Protocol 2 obtained from Monte Carlos simulations.