

# The Computational Complexity of $3k$ -CLIQUE

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**Abstract:** In this note, we show that the fastest deterministic and exact algorithm that solves the  $3k$ -CLIQUE problem must run in  $\Omega(n^{2k})$  time in the worst-case scenario on a classical computer, where  $n$  is the number of vertices in the graph.

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The  $3k$ -CLIQUE problem is to determine whether or not a clique of size  $3k$  exists in a given undirected graph  $G$ , where  $k$  is a positive integer that is not part of the input of the problem [4]. In this note, we show that the fastest deterministic and exact algorithm that solves  $3k$ -CLIQUE must run in  $\Omega(n^{2k})$  time in the worst-case scenario on a classical computer, where  $n$  is the number of vertices in the graph:

Let  $G$  be an undirected graph with  $n$  vertices. For every  $k$ -clique  $C$  in  $G$ , create a corresponding vertex  $v(C)$  in an auxiliary graph  $G'$ . And for every two vertices  $v(C_1)$  and  $v(C_2)$  in  $G'$ , create an edge connecting them in  $G'$  if and only if  $C_1 \cup C_2$  forms a  $2k$ -clique in  $G$ . Then  $G'$  will have  $O(n^k)$  vertices and  $O(n^{2k})$  edges. Note that the  $3$ -CLIQUE problem on  $G'$  is equivalent to the  $3k$ -CLIQUE problem on  $G$  [4].

Let  $A$  be the adjacency matrix of  $G'$ . Then  $G'$  has no  $3$ -clique if and only if  $A \circ A^2 = 0$ , where  $A \circ A^2$  is the Hadamard product of  $A$  and  $A^2$  [2]. Because the equation  $A \circ A^2 = 0$  could consist of  $\Theta(n^{2k})$  algebraic equations in which all but a constant number are true, the fastest algorithm that determines whether  $A \circ A^2 = 0$  must run in  $\Omega(n^{2k})$  time in the worst-case scenario. Then since determining whether  $A \circ A^2$  is nonzero is equivalent to the  $3k$ -CLIQUE problem on  $G$ , it must also take  $\Omega(n^{2k})$  time in the worst-case scenario for any deterministic and exact algorithm to solve the  $3k$ -CLIQUE problem on  $G$ . And this implies that  $P \neq NP$  [1].  $\square$

This lower bound is confirmed by the fact that the fastest known deterministic and exact algorithm that solves  $3k$ -CLIQUE was first published in 1985 and has a running-time of  $\Theta(n^{\omega k})$ , where  $\omega \geq 2$  [1, 3, 4].

## References

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