

Precision Arithmetic: A New Floating-Point Arithmetic

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1 Abstract

A new floating-point arithmetic called precision arithmetic is developed to track precision for arithmetic calculations. It uses a novel rounding scheme to avoid excessive rounding error propagation of conventional floating-point arithmetic. Unlike interval arithmetic, its uncertainty tracking is based on statistics and its bounding range is much tighter. Its stable rounding error distribution is approximated by truncated normal distribution. Generic standards and systematic methods for validating uncertainty-bearing arithmetics are discussed. The precision arithmetic is found to be better than interval arithmetic in uncertainty-tracking for linear algorithms.

The code of precision arithmetic is available publicly at <http://precisionarithm.sourceforge.net>.

Keywords: Computer arithmetic, Error analysis, Interval arithmetic, Multi-precision arithmetic, Numerical algorithms.

2 Introduction

2.1 Measurement Precision

Except simplest counting, scientific and engineering measurement never gives completely precise results [1][2]. The precisions of measured values range from order-of-magnitude estimation of astronomical measurements, to $10^{-2} \sim 10^{-4}$ of common measurements, to 10^{-14} of state-of-art measurements of basic physics constants [3].

In scientific and engineering measurement the uncertainty of a measurement x is usually characterized by sample deviation δx [1][2][4], though in certain cases such as raw reading from an analog-to-digital converter (ADC) the uncertainty of a measurement x is given as bounding

range Δx [5]¹. If $(x-\Delta x, x+\Delta x)$ crosses 0, x is neither positive nor negative for certainty due to following two possibilities:

- Either Δx is too large to give a precise measurement of x ,
- Or x itself is a measurement of zero.

To distinguish which case it is, additional information is required so that the measurement $x \pm \Delta x$ itself is not significant if $(x-\Delta x, x+\Delta x)$ crosses 0. An insignificant value also has conceptual difficulty to participate in many mathematical operations, such as square root or acting as divisor.

$P \equiv \delta x/|x|$ is defined here as the (relative) precision of the measurement, and its inverse is commonly known as the significance ϕ [1][2]. Precision represents reliable information content of a measurement. Finer precision means higher reliability thus better reproducibility of the measurement [1][2]. Precision in this paper does not mean the maximal significant bit count as in term “variable precision arithmetic”. It takes the traditional definition in measurement [1][2].

2.2 Problem of the Conventional Floating-point Arithmetic

The conventional floating-point arithmetic [6][7][8] assumes a constant and best-possible precision for each value all the time, and constantly generates artificial information during calculation. For example, the following calculation is carried out precisely in integer format:

$$\begin{aligned} \text{Equation 2-1: } & (64919121 \times 205117922 - 159018721 \times 83739041) \\ & = 13316075197586562 - 13316075197586561 = 1; \end{aligned}$$

If calculation in Equation 2-1 is carried out using conventional floating-point arithmetic:

$$\begin{aligned} \text{Equation 2-2: } & (64919121 \times 205117922 - 159018721 \times 83739041) \\ & = 13316075197586562 - 13316075197586560 = 2.0000000000000000; \end{aligned}$$

¹ x is normally an integer at output of ADC. In ideal case, Δx equals half bit of ADC. Δx can be larger if the settle time is not long enough or if the ADC is not ideal.

1. The multiplication results exceed the maximal significance of the 64-bit IEEE floating-point representation, so they are rounded off, generating rounding errors;
2. The normalization of the subtraction result amplifies the rounding error to most significant bit (MSB) by padding zeros.

Equation 2-2 is a show case for the problem of conventional floating-point arithmetic. Because normalization happens after each arithmetic operation [6][7][8], such generation of rounding errors happens very frequently for addition and multiplication, and such amplification of rounding errors happens very frequently for subtraction and division. The accumulation of rounding errors is an intrinsic problem of conventional floating-point arithmetic [9]. For example, even most input data have only $10^{-2} \sim 10^{-4}$ of precision, except for simplest calculations, the 32-bit IEEE floating-point format [6][7][8] which has constant 10^{-7} precision, is usually not fine enough, because the rounding error from lower digits quickly propagates to higher digits.

Self-censored rules are developed to avoid such rounding error propagation [9][10], such as avoiding subtracting results of large multiplication, as in Equation 2-1. However, these rules are not enforceable, and difficult to follow in many cases, e.g., even most carefully crafted algorithm can result in numerical instability. Because the propagation speed of rounding error depends on nature of calculation itself, e.g., faster in nonlinear algorithms than linear algorithms [11], propagation of rounding error in conventional floating-point arithmetic is very difficult to quantify generically [12] so it is difficult to tell if a calculation is improper or becomes excessive for a required result precision. In common practice, reasoning on individual theoretical base is used to estimate the error and validity of calculation results, such from the estimated transfer functions of the algorithms used in the calculation [9][12][14], but such analysis is rare, and generally very difficult to carry out in practice.

Today, most experimental data are collected by ADC [5]. The result from ADC is an integer with a fixed uncertainty, thus a smaller signal value has a coarser precision. When a waveform

containing raw digitalized signals from ADC is converted into conventional floating-point representation, the information content of the digitalized waveform is distorted to favor small signals since all converted data now have same and best-possible precision. The effects of such distortion in signal processing are generally not clear.

So what needed is a floating-point arithmetic that tracks precision automatically. When the calculation is improper or becomes excessive, the results become insignificant. All existing uncertainty-bearing arithmetic is reviewed here.

2.3 Interval Arithmetic

Interval arithmetic [10][15][16] is currently a standard method to track calculation uncertainty. It assumes that a value x is absolutely bounded within its bounding range $[x\downarrow, x\uparrow]$, using the following arithmetic equations [10]:

$$\text{Equation 2-3: } [x_1\downarrow, x_1\uparrow] + [x_2\downarrow, x_2\uparrow] = [x_1\downarrow+x_2\downarrow, x_1\uparrow+x_2\uparrow];$$

$$\text{Equation 2-4: } [x_1\downarrow, x_1\uparrow] - [x_2\downarrow, x_2\uparrow] = [x_1\downarrow-x_2\uparrow, x_1\uparrow-x_2\downarrow];$$

$$\text{Equation 2-5: } [x_1\downarrow, x_1\uparrow] * [x_2\downarrow, x_2\uparrow] \equiv [x\downarrow, x\uparrow];$$

$$x\downarrow = \min(x_1\downarrow * x_2\downarrow, x_1\uparrow * x_2\downarrow, x_1\downarrow * x_2\uparrow, x_1\uparrow * x_2\uparrow);$$

$$x\uparrow = \max(x_1\downarrow * x_2\downarrow, x_1\uparrow * x_2\downarrow, x_1\downarrow * x_2\uparrow, x_1\uparrow * x_2\uparrow);$$

$$\text{Equation 2-6: } [x_1\downarrow, x_1\uparrow] / [x_2\downarrow, x_2\uparrow] \equiv [x\downarrow, x\uparrow];$$

$$x\downarrow = \min(x_1\downarrow / x_2\downarrow, x_1\uparrow / x_2\downarrow, x_1\downarrow / x_2\uparrow, x_1\uparrow / x_2\uparrow);$$

$$x\uparrow = \max(x_1\downarrow / x_2\downarrow, x_1\uparrow / x_2\downarrow, x_1\downarrow / x_2\uparrow, x_1\uparrow / x_2\uparrow);$$

When multiplying x with itself, the result $x\downarrow$ is floored at 0 so that $x\downarrow \geq 0$.

If it is implemented using a floating-point representation with limited resolution, the result bounding range is widened further [16].

A basic problem is that the bounding range used by interval arithmetic is not compatible with usual scientific and engineering measurements, which instead use statistical mean and

deviations to characterize uncertainty. Most measured values are well approximated by normal distribution [1][2][4], which has no limited bounding range. Let bounding leakage be defined as the possibility of the true value to be outside a bounding range. If a bounding range is defined using a statistical rule on bounding leakage, such as the 6σ - 10^{-9} rule for normal distribution [4] (which says the bounding leakage is about 10^{-9} for a bounding range of mean ± 6 -fold of standard deviations), there is no guarantee that the calculation result will also obey the 6σ - 10^{-9} rule using interval arithmetic, since interval arithmetic has no statistical ground.

Another problem is that interval arithmetic only gives worst case of uncertainty propagation and thus tends to over-estimate uncertainty in reality. For example, in addition and subtraction, it gives the result when the two operands are +1 and -1 correlated respectively. However, if the two operands are -1 and +1 correlated respectively instead, the actually bounding range after addition and subtraction reduces. The vast overestimation of bounding ranges in these two cases prompts the developments of affine arithmetic [17][18], which traces error sources using a first-order model. Because of its expense in execution, and dependence on approximate modeling for operations even as basic as multiplication and division, affine arithmetic has not been widely used.

A third problem is that the result of interval arithmetic may depend strongly on the actual expression of an analytic function $f(x)$. For an example, Equation 2-7, Equation 2-8 and Equation 2-9 are different expressions of a same $f(x)$, but only through Equation 2-7 the result is correct, and uncertainty may be exaggerated in the other two forms, e.g., by 67-fold at input range [0.44, 0.56] using Equation 2-8. This is called the dependence problem of interval arithmetic [19]. A practical implementation of interval arithmetic [19] needs to be free from the dependence problem thus much more complicated than using Equation 2-3, Equation 2-4, Equation 2-5 and Equation 2-6 directly.

Equation 2-7: $f(x) = (x - 1/2)^2 - 1/4;$

Equation 2-8: $f(x) = x^2 - x;$

Equation 2-9: $f(x) = (x - 1) x;$

Interval arithmetic has very coarse and algorithm-specific precision but constant zero bounding leakage. It represents the other extreme from conventional floating-point arithmetic. To meet with practice needs, a better uncertainty-bearing arithmetic should be based on statistical propagation of rounding error and it should allow reasonable bounding leakage for normal usages.

2.4 Statistical Propagation of Uncertainty

If the statistical correlation between two operands is known, the result uncertainty is given by statistical propagation of uncertainty [20], with the following arithmetic equations, in which σ is the deviation of a measured value x , P is its precision, and γ is correlation between the two operands x_1 and x_2 :

Equation 2-10: $(x_1 \pm \sigma_1) + (x_2 \pm \sigma_2) = (x_1 + x_2) \pm \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\gamma};$

Equation 2-11: $(x_1 \pm \sigma_1) - (x_2 \pm \sigma_2) = (x_1 - x_2) \pm \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\gamma};$

Equation 2-12: $(x_1 \pm \sigma_1) * (x_2 \pm \sigma_2) = (x_1 * x_2) \pm |x_1 * x_2| \sqrt{P_1^2 + P_2^2 + 2P_1P_2\gamma};$

Equation 2-13: $\frac{x_1 \pm \sigma_1}{x_2 \pm \sigma_2} = \frac{x_1}{x_2} \pm \left| \frac{x_1}{x_2} \right| \sqrt{P_1^2 + P_2^2 - 2P_1P_2\gamma};$

Tracking uncertainty propagation statistically seems an ideal solution. However in practice the correlation between two operands is generally not precisely known, so direct use of statistical propagation of uncertainty is very limited. In this paper, as a proxy for statistical propagation of uncertainty, an independence arithmetic always assumes no correlation between any two operands, whose arithmetic equations are Equation 2-10, Equation 2-11, Equation 2-12 and Equation 2-13 but with $\gamma=0$. Independence arithmetic is actually de facto arithmetic in

engineering data processing [1][2], such as in the common believe that uncertainty after averaging reduces by the square root of number of measurements.

2.5 Significance Arithmetic

Significance arithmetic [21] tries to track reliable bits in an imprecise value during calculation. Except early attempts [22][23], significance arithmetic has not yet been implemented digitally. In these attempts, the implementations of significance arithmetic are based on simple operating rules upon reliable bit counts, rather than on formal statistical approaches. They both treat the reliable bit counts as integers when applying their rules, while in reality a reliable bit count could be a fractional number, so they both can cause artificially quantum reduction of significance. Significance arithmetic is not widely practiced in scientific and engineering calculations [21].

Also can be categorized as significance arithmetic, a stochastic arithmetic [12][24] randomizes least significant bits (LSB) of each of input floating-point values, repeats a same calculation multiple times, and then uses statistics to seek invariant digits among the calculation results as significant digits. This approach may require too much calculation since the number of necessary repeats for each input is specific to each algorithm, especially when the algorithm contains branches. It is based on modeling of rounding errors in conventional floating-point arithmetic, which is quite complicated. A better approach may be to define arithmetic rules that make error tracking by probability easier.

2.6 An Overview of This Paper

In this paper, a new floating-point arithmetic called precision arithmetic [25] is developed to track uncertainty during floating-point calculations, as described in Section 3. Generic standards and systematic methods for validating uncertainty-bearing arithmetics are discussed in Section 4. The precision arithmetic is compared with other uncertainty-bearing arithmetics in Section 5 using FFT algorithms. A brief discussion is provided in Section 6.

3 The Precision Arithmetic

3.1 Basic Assumptions for Precision Arithmetic

Precision arithmetic tracks uncertainty distribution during calculation using specially designed arithmetic rules. It has the independent uncertainty assumption as its basic assumption, which assumes that the uncertainties of any two different identities can be regarded as independent of each other. This assumption can be turned into a realistic statistical requirement on input data to precision arithmetic.

Precision arithmetic has another assumption, the uniform distribution assumption, which assumes that all uncertainty can be bounded by a truncated normal distribution symmetrically around an expected value which is the value given by mathematics when there is no uncertainty. Precision arithmetic may occasionally extend uncertainty range at one side to keep this assumption valid.

In addition, precision arithmetic heavily uses the scaling principle, which says that the result precision should not change when an imprecise value is either multiplied or divided with a non-zero and precise constant. It is concluded from Equation 2-12 and Equation 2-13 for statistical propagation of uncertainty.

3.2 The Independent Uncertainty Assumption

Assume X , Y , and Z are three mutually independent random variables with variance $\sigma^2(X)$, $\sigma^2(Y)$ and $\sigma^2(Z)$ respectively. Let α be a constant, $\text{Cov}()$ be covariance and γ be correlation:

$$\text{Equation 3-1: } \gamma = \frac{\text{Cov}(X+Y, \alpha X+Z)}{\sqrt{\sigma^2(X+Y) \cdot \sigma^2(\alpha X+Z)}} = \frac{\alpha \sigma^2(X)}{\sqrt{\sigma^2(X) + \sigma^2(Y)} \sqrt{\alpha^2 \sigma^2(X) + \sigma^2(Z)}};$$

Assume α is positive. Let $\eta_1^2 \equiv \frac{\sigma^2(Y)}{\sigma^2(X)}$ and $\eta_2^2 \equiv \frac{\sigma^2(Z)}{\sigma^2(\alpha X)} = \frac{\sigma^2(Z)}{\alpha^2 \sigma^2(X)}$:

Equation 3-2:
$$\gamma = \frac{\alpha / |\alpha|}{\sqrt{1+\eta_1^2} \sqrt{1+\eta_2^2}} \equiv \frac{1}{1+\eta^2} ;$$

Equation 3-2 gives the correlation γ between two identities, each of which contains a completely uncorrelated part and a completely correlated part, with η being the average ratio between these two parts.

One special application of Equation 3-2 is the correlation between a measured signal and its true signal, in which noise is the uncorrelated part between the two. Figure 1 shows the effect of noise on most significant two bits of a 4-bit measured signal when $\eta=1/4$. Its top chart shows a triangular waveform between 0 and 16 in black line, and a white noise between -2 and +2 using grey area. The measured signal is the sum of the triangle waveform and the noise. The middle chart of Figure 1 shows the values of the 3rd digit of the true signal in black line, and the mean values of the 3rd bit of the measurement in grey line. The 3rd bit is affected by the noise during its transition between 0 and 1. For example, when the signal is slightly below 8, only a small positive noise can turn the 3rd digit from 0 to 1. The bottom chart of Figure 1 shows the values of the 2nd digit of the signal and the measurement in black line and grey line respectively. Figure 1 clearly shows that the correlation between the measurement and the true signal is less at the 2nd digit than at the 3rd digit. Quantitatively, according to Equation 3-2:

- The 3rd digit of the measurement is 94% correlated to the signal with $\eta=1/4$;
- The 2nd digit of the measurement is 80% correlated to the signal with $\eta=1/2$;
- The 1st digit of the measurement is 50% correlated to the signal with $\eta=1$;
- The 0th digit of the measurement is 20% correlated to the signal with $\eta=2$;

The above conclusion agrees with the common experiences that below noise level of measured signals, noises rather than true signals dominate each digit.

Similarly, while the correlated portion between two identities has exactly same value at each bit of the two identities, the ratio η of uncorrelated portion to the correlated portion increases by

2-fold for each bit down from MSB of the two identities. If two significant identities are overall correlated by γ , at level of uncertainty the correlation between the two identities decreases to γ_ϕ :

$$\text{Equation 3-3: } \left(\frac{1}{\gamma_\phi} - 1\right) = \left(\frac{1}{\gamma} - 1\right)\phi^2, \quad \phi \geq 1;$$

In Equation 3-3, ϕ is the smaller significance of the two identities. Figure 2 plots the relation of γ vs. ϕ for each given γ_ϕ in Equation 3-3. If when γ_ϕ is less than a maximal threshold (e.g., 2%, 5% or 10%), the two identities can be deemed virtually independent of each other at the level of uncertainty, then there is a maximal allowed correlation between any two identities for each ϕ , as shown in Figure 2. If two identities are independent of each other at their uncertainty levels, their uncertainties are independent of each other. For a given γ_ϕ requirement, below the maximal allowed correlation threshold, the independent uncertainty assumption is true for the two identities. Figure 2 shows that for two precisely measured identities, their correlation γ is allowed to be quite high. While to be acceptable in precision arithmetic, each of low-resolution identities should contain enough true noise in its uncertainty, so that they have not much correction through systematic error [1][2]. Thus, the independence uncertainty assumption of precision arithmetic is much weaker than requiring the two identities to be independent of each other, which is the assumption for independence arithmetic.

It is tempting to add true noise to otherwise unqualified identities to make their uncertainties independent of each other. As an extreme case of this approach, if two identities are constructed by adding true noise to a same signal, they are at least 50% correlated at uncertainty level and hence will not satisfy the independent uncertainty assumption. The 50% curve in Figure 2 thus defines maximal possible correlations between any two measured signals. For a given γ_ϕ , there is an upper limit of correlation γ for each ϕ , beyond which adding true noises to identities will not make their correlation less than the γ_ϕ threshold.

3.3 Precision Representation and Precision Round up Rule

Let the content of a floating-point number be denoted as $S@E$, in which S is the significand and E is the exponent of 2 of the floating-point number. In addition, precision representation contains a carry \sim to indicate its rounding error, which can be:

- $+$: The rounding error is positive;
- $-$: The rounding error is negative;
- $?$: The sign of rounding error is unknown;
- $\#$: The precision value contains an error code. Each error code is generated due to a specific illegal arithmetic operation such as dividing by zero. An operand error code is directly transferred to the operation result. In this way, illegal operations can be traced back to the source.

A round up happens when E is incremented by 1 according to the following round up rule:

- A value of $2S\sim@E$ is rounded up to $S\sim@E+1$.
- A value of $(2S+1)+@E$ is rounded up to $(S+1)-@E+1$.
- A value of $(2S+1)-@E$ is rounded up to $S+@E+1$.
- A value of $(2S+1)?@E$ is rounded up randomly to either $(S+1)-@E+1$ or $S+@E+1$ with equal probability.

After each round up, the original rounding error is reduced by half for the new significand. If the original significand is odd, the round up generates a new rounding error of $1/2$, which is added to the existing rounding error. Since the newly generated rounding error always cancels the existing rounding error, the rounding error range is limited to half bit of significand for the precision round up rule.

If initial \sim is wrong, it will be reduced and corrected quickly during rounding up. So the round up process is stable.

The precision arithmetic also tracks rounding error bounding range R , so that the precision representation becomes $S \sim R @ E$.

3.4 Precise Value

When R is zero, the precision value is precise, otherwise it is imprecise. A precise value has no rounding error. After a precise value with odd significand is rounded up once, it becomes an imprecise value of $S ? 1/2 @ E$.

An imprecise value can be decomposed as a precise value plus an imprecise zero:

$$\text{Equation 3-4: } S \sim R @ E = S @ E + 0 \sim R @ E;$$

According to uniform distribution assumption, $S @ E$ carries mathematically expected value while $0 \sim R @ E$ carries uncertainty.

3.5 Probability Distribution of Rounding Errors

An ideal floating-point calculation is carried out to infinitesimal precision before it is rounded up to representation precision [8][10][12]. Thus, rounding up should be an independent process from any calculation, and be evaluated separately. To find out rounding error distribution within its bounding range $[-1/2, +1/2]$, a large count of positive random integers are converted into precision values and then rounded up until each of them has significand larger than a minimal significand threshold. Each rounded up precision value is compared with the original value for rounding error. Figure 3 shows the result histogram of rounding errors for the minimal significand thresholds 0, 1, 4 and 16 respectively. When each bit of significand has equal chance to be either 0 or 1, the result distribution is expected to be uniform within range $(-1/2, +1/2)$. However, the precision round up rule changes such equal chance for few lowest digits of significand. So when the minimal significand threshold is smaller, the bias in rounding error distribution is larger, as shown in Figure 3, and the result distribution is close to uniform only when the minimal significand threshold is 4 and above.

3.6 Result Bounding For Addition and Subtraction

To a floating-point arithmetic, rounding errors are uncertainties [8][10][12]. The precision round up rule incorporates all randomness of an imprecise value into its carry and bounding range, so that it preserves the independent uncertainty assumption between any two identities. The independent uncertainty assumption suggests that the result error distribution of addition is the convolution of the two operand error distributions, while the result error distribution of subtraction is the convolution of the first operand error distribution and the mirror image of the second operand error distribution [4]. Thus, when the exponents of two operands are equal, the results of addition and subtraction are:

$$\text{Equation 3-5: } S_1 \sim_1 R_1 @ E \pm S_2 \sim_2 R_2 @ E = (S_1 \pm S_2) \sim (R_1 + R_2) @ E;$$

Table 1 shows the result \sim for addition while Table 2 shows the result \sim for subtraction. However, since \sim_1 and \sim_2 are only signs of rounding errors, \sim is only a best guess. It will be shown that the \sim immediately after a calculation is actually not important in precision arithmetic.

Assume the rounding errors are uniformly distributed between $[-1/2, +1/2]$ with a density function $P_{1/2}(x)$:

$$\text{Equation 3-6: } P_{1/2}(x) \equiv 1, \quad -1/2 \leq x \leq +1/2;$$

Then the density function $P_{n/2}(x)$ for rounding error distribution of bounding range $R \equiv n/2$ is obtained by convolution:

$$\text{Equation 3-7: } P_{n/2}(x) \equiv \int_{-\infty}^{+\infty} P_{1/2}(y) P_{(n-1)/2}(x-y) dy = \int_{-1/2}^{+1/2} P_{(n-1)/2}(x-y) dy, \quad n=2, 3, 4, \dots;$$

It is easy to prove that the deviation σ of $P_{n/2}(x)$ is determined by its bounding range:

$$\text{Equation 3-8: } \sigma^2 = R/6;$$

Also, a same bounding range can be reached in any combination:

$$\text{Equation 3-9: } P_{(m+n)/2}(x) = \int_{-\infty}^{+\infty} P_{m/2}(y) P_{n/2}(x-y) dy;$$

In reality, $P_{1/2}(x)$ is not strictly uniformly distributed in its bounding range $[-1/2, +1/2]$. As the worst case, let $P_{1/2}(x)$ be the error distribution with a minimal significand threshold of 0 in Figure 3, Figure 4 shows the rounding error distribution after addition and subtraction once and twice, in which:

- $R=1/2$: “1” for no addition and subtraction.
- $R=2/2$: “1+1” for addition once, and “1-1” for subtraction once.
- $R=3/2$: “1+1+1” for addition twice, “1-1-1” for subtraction twice, “1+1-1” for addition once then subtraction once, and “1-1+1” for subtraction once then addition once.

Figure 4 shows that rounding error distributions for same bounding range largely repeat each other, confirming Equation 3-9. Addition and subtraction have slightly different result distribution due to uneven $P_{1/2}(x)$. Figure 5 shows that for all distributions in Figure 3, the deviation of the rounding error distributions increases with the bounding range according to a square root relation to a very high degree with a factor coefficient near $1/\sqrt{6}$, confirming Equation 3-8. Any mean of the distributions is always 10-fold smaller than its deviation thus can be ignored.

The probability density function $D_Y(y)$ after linear transformation ($y = \alpha x + \beta$) of a generic probability density function $D_X(x)$ is [4]:

$$\text{Equation 3-10: } D_Y(y) = D_X((y - \beta)/\alpha) / \alpha ;$$

Let $N(x)$ be standard normal distribution. Since $P_{n/2}(x)$ is sum of $P_{1/2}(x)$ by n -times, according to central limit theorem [4], Equation 3-8 and Equation 3-10:

$$\text{Equation 3-11: } P_{n/2}(y) \xrightarrow{d} N(y/\sigma) / \sigma , \quad y \in (-R, +R) ;$$

Equation 3-11 shows that $P_{n/2}(x)$ approaches a normal distribution of same deviation with increased n . The uniform distribution assumption is based on such convergence to normal distribution in precision arithmetic.

3.7 Result Bounding for Rounding Up and Rounding Down

When an imprecise value is rounded up once, both its original bounding range R and its original deviation σ are reduced to half for the new significand. According to Equation 3-8, when R is reduced to 1/2-fold, σ is reduced to $1/\sqrt{2}$ -fold, while when σ is reduced to 1/2-fold, R is reduced to 1/4-fold. The former is called round up by range, while the latter is called round up by deviation. Figure 6 compares these two ways of rounding up when the original error range is $R=8$, in which $R=4$ is rounded up by range, while $R=2$ is rounded up by deviation. It clearly shows that round up by deviation results in a much similar rounding error distribution. After rounded up, the stable distribution in Equation 3-11 becomes $N(y/\frac{\sigma}{2})/\frac{\sigma}{2}, y \in [-\frac{R}{2}, +\frac{R}{2}]$, which is better approximated as $N(y/\frac{\sigma}{2})/\frac{\sigma}{2}, y \in [-\frac{R}{4}, +\frac{R}{4}]$ when R is sufficiently large. Round up by deviation is also required by the scaling principle.

Round up by deviation also introduces bounding leakage called round-up leakage. In Figure 6, the 8/2 distribution of rounding error outside the range $[-2, +2]$ contributes to a round-up leakage of 0.05%. Figure 7 shows that the round-up leakage decreases exponentially with increased bounding range R . Smaller round-up leakage means that the actual rounding error distribution is more similar to the rounding error distribution chosen by rounding up by deviation.

When R is above a threshold R_{\max} , round-up leakage is small enough, so that rounding up by deviation can be applied repeatedly. This is the normalization process in precision arithmetic. An imprecise value $S \sim R @ E$ is in normalized format if its R is in the range of $[R_{\max}/4, R_{\max})$. Otherwise it is called a nearly precise value. According to Figure 7, when $R_{\max}=16$, the maximal normalization leakage is 10^{-6} , which is small enough for most applications. One function of normalization is to enforce the correctness of carry.

Because of normalization, $P_{n/2}(x)$ is extended to $P_R(x)$ for 2's fractional $R \in [1/2, R_{\max})$, so that the deviation δx and bounding range Δx of $S \sim R @ E$ is:

$$\text{Equation 3-12: } \delta x = \sigma 2^E, \quad \sigma < \sqrt{R_{\max}/6};$$

$$\text{Equation 3-13: } \Delta x = R 2^E, \quad R < R_{\max};$$

$$\text{Equation 3-14: } \Delta x / \delta x = R / \sigma = \sqrt{6R};$$

The stable probability density function in Equation 3-11 becomes probability density function $\rho(y)$ in Equation 3-15.

$$\text{Equation 3-15: } \rho(y) = N(y/\delta x) / \delta x, \quad y \in (-\Delta x, \Delta x);$$

All characteristics of a distribution are decided by its moments [4]. A normal distribution decreases very fast at its two tails so that its tails do not contribute significantly to moment calculation [4][26]. Thus, according to Equation 3-14, when R is sufficiently large, $\rho(y)$ has identical moments of the corresponding normal distribution, and it is truly the normal distribution but with truncated range. For example, in Figure 7, bounding leakage actually indicates the difference of 0th moment from 1 with increased R . It is assumed that when $R \geq R_{\max}/4$, R is sufficiently large. Equation 3-4 can be reinterpreted as Equation 3-16, in which x is the mathematically expected value and y is a ρ -distributed random variable.

$$\text{Equation 3-16: } x \pm \delta x = x + y, \quad y \in \rho(y);$$

The round down rule is defined using the scaling principle. After round down once, $S \sim R @ E$ becomes $2S \sim 4R @ E-1$. Round-down reduces bounding leakage.

To add or subtract two operands with different exponents, the operand with larger exponent is first round down to the other exponent, and the result of addition or subtraction using Equation 3-5 is normalized afterward. If the operand with larger exponent is normalized, the result exponent is at least as large as the larger operand exponent.

Adding an operand to itself is same as multiplying the operand by 2, and the result bounding range is $\sqrt{2}$ -fold as adding two similar operands according to Equation 3-18. Subtracting an operand from itself results in precise 0.

3.8 Bounding Initiation

An integer S is initialized as a precise value $S@0$.

A conventional 64-bit floating-point value $S@E$ is usually initialized as a nearly precise value $S?1/2@E$, because the IEEE floating-point standard [8] guarantees accuracy to half bit of significand.

A mean-deviation pair $(x \pm \delta x)$ of 64-bit conventional floating-point values is initialized as an imprecise value $S \sim R@E$ by (1) rounding up δx until Equation 3-12 is satisfied; (2) obtaining R and E from final δx ; (3) rounding up x to E ; (4) obtaining S and \sim from final x . Similarly, a median-range pair $(x \pm \Delta x)$ is initialized as an imprecise value $S \sim R@E$ using Equation 3-13. Immediately after initialization, \sim is accurate due to rounding up. If the precision of the measured value is worse than 10^{-16} , the result value is normalized. Practically all measured values are normalized.

Precision arithmetic has limited calculation inside uncertainty, and the resolution of its significand is determined by its precision. In addition to other characteristics, Table 3 shows bits calculation inside uncertainty for different R_{\max} using precision arithmetic. Although a larger R_{\max} has smaller normalization leakage, it achieves this by having larger bounding range, thus it greatly increase the chance for a value to be insignificant. According to Table 3, only when $R_{\max}=16$ the precision arithmetic has a comparable bounding range as de facto $6\sigma \cdot 10^{-9}$ rule. Thus, $R_{\max}=16$ is used in this paper. While calculating many bits inside uncertainty does not seem meaningful, not calculating at all inside uncertainty may not be an optimal approach either. Thus, Equation 3-12 is modified as Equation 3-17, in which χ is a small constant positive integer, to introduce χ -bit calculation inside uncertainty by reinterpreting the precision for $S \sim R@E$.

Equation 3-17: $\delta x = \sqrt{R/6} \cdot 2^{-\chi} 2^{E+\chi};$

Table 4 shows examples of precision arithmetic with different χ for $R_{\max}=16$. χ will be found empirically to be 2 later in this paper.

Limited resolution of significands does not necessarily mean that precision arithmetic has larger calculation error. The following example shows that symmetry of precision representation cancels out rounding errors:

- $1/3 \pm 0.001 + 2/3 \pm 0.001 = 341 + 6.29 @ -10 + 683 - 6.29 @ -10 = 1024 \pm 12.6 @ -12 = 1 \pm 0.0014;$

3.9 Result Bounding For Multiplication

After $S \sim R @ E$ is multiplied by 2, both its range and deviation increase by 2-fold. If the scaling principle is applied, the result is $2S \sim 4R @ E$. When $S \sim R @ E$ is normalized, the result is then normalized as $S \sim R @ E+1$, and there is neither bound widening nor bounding leakage. Generally, the direct result of multiplying $S_1 \sim_1 R_1 @ E_1$ by $S_2 @ E_2$ is:

$$\text{Equation 3-18: } S_1 \sim_1 R_1 @ E_1 \times S_2 @ E_2 = S_1 S_2 \sim_1 S_2^2 R_1 @ (E_1 + E_2);$$

Equation 3-18 obeys scaling principle, and the result uncertainty is still ρ -distributed.

According to independent uncertainty assumption, the product bounding range of multiplying $0 \sim_1 R_1 @ E_1$ by $0 \sim_2 R_2 @ E_2$ is $R_1 R_2 @ (E_1 + E_2)$. Thus:

$$\begin{aligned} \text{Equation 3-19: } S_1 \sim_1 R_1 @ E_1 \times S_2 \sim_2 R_2 @ E_2 = & (S_1 S_2) @ (E_1 + E_2) + \\ & 0 \sim_1 R_1 S_2^2 @ (E_1 + E_2) + 0 \sim_2 R_2 S_1^2 @ (E_1 + E_2) + 0 (\sim_1 \sim_2) R_1 R_2 @ (E_1 + E_2); \end{aligned}$$

In Equation 3-19, $0 \sim_1 R_1 S_2^2 @ (E_1 + E_2)$, $0 \sim_2 R_2 S_1^2 @ (E_1 + E_2)$ and $0 (\sim_1 \sim_2) R_1 R_2 @ (E_1 + E_2)$ are still ρ -distributed, and the addition further converges the result uncertainty toward ρ -distributed.

The result precision P of multiplication is:

$$\text{Equation 3-20: } P^2 = \frac{1}{6} \frac{(S_2^2 R_1 + S_1^2 R_2 + R_1 R_2)}{(S_1 S_2)^2} = P_1^2 + P_2^2 + \frac{1}{6} P_1^2 P_2^2;$$

Equation 3-20 shows that precision cannot be improved during multiplication. It is identical to Equation 2-12 except their cross term, representing difference in their statistical requirements.

A special case of multiplication is between an operand and itself. If x is $N(x)$ distributed, x^2 is χ^2 -distributed with freedom 1 [4], which has mean 1 and variance of 2. $N(x)$ and $\chi^2(x)$ have quite different characteristics, e.g., $\chi^2(x)$ only roughly resembles half of $N(x)$. According to

Equation 3-10, the square of $0 \pm \delta x$ has mean of δx and variance of $2 (\delta x)^4$. The uniform distribution assumption extends x^2 distribution to the other side of the mathematically expected value, and absorb mean δx into final variance:

$$\text{Equation 3-21: } (S \sim R @ E)^2 = S^2 @ 2E + 0 \sim 4RS^2 @ 2E + 0 + 3R^2 @ 2E;$$

3.10 Result Bounding For Division

The reverse of Equation 3-18 defines:

$$\text{Equation 3-22: } S_1 \sim R_1 @ E_1 / S_2 @ E_2 \equiv (S_1 / S_2) \sim_1 (R_1 / S_2^2) @ (E_1 - E_2);$$

In Equation 3-22, the rounding error decreases by S_2 -fold, but the bounding range decreases by S_2^2 -fold, so there is bounding leakage. To limit the bounding leakage to acceptable level, the result is rounded down until normalized. In this case, rounding down the direct result is equivalent to round down the dividend $S_1 \sim R_1 @ E_1$ by the same times before the division.

The scaling principle is used to define the result of dividing $S_1 @ E_1$ by $S_2 \pm R_2 @ E_2$:

$$\text{Equation 3-23: } S_1 @ E_1 / S_2 \sim_2 R_2 @ E_2 \equiv (S_1 / S_2) (-\sim_2) (R_2 S_1^2 / S_2^4) @ (E_1 - E_2);$$

The result also satisfies the reciprocal relation—when $S_1 @ E_1$ is divided by it, the result is exactly $S_2 \sim_2 R_2 @ E_2$. To limit the bounding leakage to acceptable level, the result is rounded down until normalized.

The direct result of division between two operands $S_1 \sim_1 R_1 @ E_1$ and $S_2 \sim_2 R_2 @ E_2$ is calculated as $(S_1 \sim_1 R_1 @ E_1) \times 1 / (S_2 \sim_2 R_2 @ E_2)$:

$$\text{Equation 3-24: } S_1 \sim_1 R_1 @ E_1 / S_2 \sim_2 R_2 @ E_2 = (S_1 / S_2) @ (E_1 - E_2) + \\ 0 \sim_1 R_1 / S_2^2 @ (E_1 - E_2) - 0 \sim_2 R_2 S_1^2 / S_2^4 @ (E_1 - E_2) - 0 (\sim_1 \sim_2) R_1 R_2 / S_2^4 @ (E_1 - E_2);$$

The result precision P of division is:

$$\text{Equation 3-25: } P^2 = \frac{1}{6} \frac{(R_1 / S_2^2 + R_2 S_1^2 / S_2^4 + R_1 R_2 / S_2^4)}{(S_1 / S_2)^2} = P_1^2 + P_2^2 + \frac{1}{6} P_1^2 P_2^2;$$

Equation 3-25 is identical to Equation 3-20, because $1 / S_2 \sim_2 R_2 @ E_2$ has same precision as $S_2 \sim_2 R_2 @ E_2$.

Dividing an operand by itself results in precise 1.

3.11 Function Evaluation

The uncertainty of function $f(x)$ at $x \pm \delta x$ is evaluated by set $\{f(x+y)-f(x)\}$, in which y is a random variable defined by Equation 3-15. Because the uniform distribution assumption of precision arithmetic centers on mathematically expected value, the variance $(\delta f)^2$ of the set is calculated as:

$$\text{Equation 3-26: } (\delta f)^2 \equiv \int (f(x+y) - f(x))^2 \rho(y) dy ;$$

An analytical expression of $(\delta f)^2$ solely in term of x^2 and $(\delta x)^2$ may be turned into an arithmetic operation executed directly in precision representation. Otherwise, the result can be obtained from the $(f \pm \delta f)$ pair.

If function $f(x)$ is Taylor expandable at x , $f(x+y)-f(x)$ is calculated according to Equation 3-27, in which $f^{(n)}(x)$ is n th deviation of $f(x)$ at x . Using Equation 3-27, the definition of uncertainty of $f(x)$ at x can be extended to the cases when range $[x-\Delta x, x+\Delta x]$ is partially outside the definition range of $f(x)$. Denote $M(n)$ as n th moment of standard Normal distribution. $(\delta f)^2$ is given by Equation 3-28 and Equation 3-29.

$$\text{Equation 3-27: } f(x+y) - f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(x)}{n!} y^n ;$$

$$\text{Equation 3-28: } M(2j+1) = 0 ;$$

$$M(2j+2) = \prod_{k=0}^j (2k+1) ;$$

$$\text{Equation 3-29: } (\delta f)^2 = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{f^{(n)}(x)}{n!} \frac{f^{(j)}(x)}{j!} (\delta x)^{n+j} M(n+j) ;$$

According to Equation 3-29:

- $f(x) = \alpha x$: $(\delta f)^2 = \alpha^2 (\delta x)^2$, confirming Equation 3-18.

- $f(x) = x^2$: $(\delta f)^2 = 3 (\delta x)^4 + 4 x^2 (\delta x)^2$, confirming Equation 3-21.
- $f(x) = \sqrt{x}$: When $x \pm \delta x$ is significant, $|x| > R$, $|x| \gg \delta x$, $(\delta f)^2 \approx (\delta x)^2 / (4|x|)$, which is comparable to the result for $f(x) = x^2$ when $|x| \gg \delta x$.
- $f(x) = 1/x$: When $x \pm \delta x$ is significant, $(\delta f)^2 \approx (\delta x)^2 / x^4$, confirming Equation 3-23.

Taylor expansion can also be used to find the result $(\delta f)^2$ of function $f(x_1, x_2)$, in which $f^{(m,n)}(x_1, x_2)$ is the m th and n th partial deviation of x_1 and x_2 respectively, and the independent uncertainty assumption between x_1 and x_2 leads to independence between y_1 and y_2 in Equation 3-30:

$$\text{Equation 3-30: } f(x_1 + y_1, x_2 + y_2) - f(x_1, x_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{f^{(m,n)}(x_1, x_2)}{m!n!} y_1^m y_2^n ;$$

$$\text{Equation 3-31: } (\delta f)^2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{f^{(m,n)}(x_1, x_2)}{m!n!} \frac{f^{(i,j)}(x_1, x_2)}{i!j!} (\delta x_1)^{m+i} (\delta x_2)^{n+j} M(m+i)M(n+j) ;$$

Such approach can be extended to a function of arbitrary number of input variables. According to Equation 3-31:

- $f(x_1, x_2) = x_1 \pm x_2$: $(\delta f)^2 = (\delta x_1)^2 + (\delta x_2)^2$, confirming Equation 3-5.
- $f(x_1, x_2) = x_1 x_2$: $(\delta f)^2 = x_2^2 (\delta x_1)^2 + x_1^2 (\delta x_2)^2 + (\delta x_1)^2 (\delta x_2)^2$, confirming Equation 3-15.

If $f(x)$ is a black-box function, because $N(x)$ is well-known, standard methods exist to divide range $[x - \Delta x, x + \Delta x]$ into equal-probability quantiles [4], and δf can be found numerically by sampling. For example, a κ -point monotonic sampling requires that (1) each numerically monotonic region contains at least κ consecutive sampling points; and (2) the whole range $[x - \Delta x, x + \Delta x]$ has been divided into monotonic regions only. When κ is sufficiently large, the chance of missing peak and valley is small enough, so that the sampling is fair enough. Range $[x - \Delta x, x + \Delta x]$ is first divided into κ equal-probability quantiles, and at each additional step each quantile is further divided into equal number of sub quantiles, until both sampling requirements are met. Then $(\delta f)^2$ is calculated as sample variance plus the square of sample mean.

3.12 Implementation

The conventional 64-bit floating-point standard IEEE-754 [5][7][8] has:

- 11 bits for storing exponent E;
- Including a hidden MSB, 53 bits for storing significand S.
- 1 bit for storing sign;

To be a super set of conventional 64-bit floating-point standard, an 80-bit implementation of precision arithmetic has:

- 11 bits for storing exponent E;
- 53 bits for storing significand S (without using the hidden MSB);
- 1 bit for storing sign;
- 2 bit for storing carry ~;
- 13 bits to store bounding range R as a fixed-point value.

The precision arithmetic is implemented in C++. With heavy additional code to count for statistics and to detect implementation errors, it runs now about 7-fold slower than the implementation of interval arithmetic using Equation 2-3, Equation 2-4, Equation 2-5 and Equation 2-6. It is probably comparable in speed to the implementation of interval arithmetic without dependence problem [19]. With code weight-trimming and optimization, its speed is expected to be improved by at least 3-fold. Unlike conventional floating-point arithmetic, it only calculates limited number of significand bits, e.g., 12 bits instead of all 53 bits when the precision is 10^{-3} . Its slowest but very frequent operation is to find the position of highest non-zero significand digit, which can be found instantly with a decoder [3]. So hardware optimization can also improve the speed of precision arithmetic by another few folds.

Appendix provides more implementation details. The code for precision arithmetic and its validation is available publicly at <http://precisionarithm.sourceforge.net>.

4 Standards and Methods for Validating Uncertainty-Bearing Arithmetic

4.1 Validating Standards and Methods

Algorithms with known analytic result are used to validate an uncertainty-bearing arithmetic. The difference between the arithmetic result and the analytic result is defined as the value error. The question is whether the uncertainty bounding range or uncertainty deviation is enough to cover value error with increased amount of calculation for any input. Corresponding to two different goals for uncertainty-bearing, there are actual two different sets of measurements:

- The ratio of absolute value error to uncertainty deviation is defined as error significand for each output value. An ideal uncertainty-tracking arithmetic should have an average error significands close to 1.
- The ratio of absolute value error to uncertainty bounding range is defined as bounding ratio for each output value. An ideal uncertainty-bounding arithmetic should have a maximal bounding ratio less than but close to 1. If the maximal bounding ratio is more than 1, bounding leakage measures the probability for errors to be outside uncertainty bounding ranges, and it should be a constant.

In both cases, all measurements should be stable for an algorithm—they should not change significantly for different input deviation, input data, or the amount of calculation.

Precision arithmetic tracks both uncertainty bounding range and uncertainty deviation, so it can be evaluated for both goals. Independence arithmetic has no uncertainty bounding range, while interval arithmetic has no uncertainty deviation. To be able to compare all the three arithmetics, $[x-6\delta x, x+6\delta x]$ is used artificially as the bounding range for an average value x with deviation δx , and vice versa.

The statistical assumption of precision arithmetic is weaker than that of independence arithmetic but stronger than that of interval arithmetic, so after executing a same algorithm on a same input data, the output deviation and bounding range of precision arithmetic are expected

to be more than those of independence arithmetic but less than those of interval arithmetic. According to Equation 3-8 and Equation 3-5, when the exponents of the two operands are same, for addition and subtraction using precision arithmetic, the result deviation propagates in the same way as that of independence arithmetic, while the result bounding range propagates in the same way as that of interval arithmetic. According to Equation 3-20 and Equation 3-25, the result precision of multiplication and division by precision arithmetic is always larger than those by independence arithmetic. However, if both operands have precisions much less than 1, the result precision of multiplication and division is very close to that of independence arithmetic, which provides a very good approximation to precision arithmetic in this case. Thus, the result of precision arithmetic should be much closer to that of independence arithmetic.

4.2 Input Data to Use

To test input data of any precision, a precise input value can be cast to any specific input deviation using precision representation. There are actually two ways of implementing such casting:

- A clean signal is obtained by directly casting a perfect signal to a specific precision. Such casting may contain systematic rounding errors. For an example, if a perfect sine signal repeat 2^n times in 2^{n+2} samples, the signal contains only values 0, ± 1 and $\pm 1/\sqrt{2}$, with each value repeated multiple times in the signal. Such value repeat may translate to artificial repeat of rounding errors. On the other hand, the symmetry of an arithmetic may be tested by the output symmetry of clean input signals.
- A noisy signal is obtained by adding Gaussian noise of same deviation as the input deviation to a perfect signal before casting. It represents realistic signal, and it should be used in validating arithmetic on uncertainty propagation.

5 Validating Uncertainty-Bearing Arithmetics Using FFT

5.1 Frequency Response of Discrete Fourier Transformation

Each testing algorithm needs to be scrutinized carefully. One important issue is whether the digital implementation of the algorithm is faithful for the original analytic algorithm. For an example, discrete Fourier transformation is only faithful for continuous Fourier transformation at certain frequencies, and it has different degree of faithfulness for other frequencies. This is called the frequency response of discrete Fourier transformation in this paper.

For each signal sequence $h[k]$, $k = 0, 1 \dots N-1$, in which N is a positive integer, the discrete Fourier transformation $H[n]$, $n = 0, 1 \dots N-1$ is [9]:

$$\text{Equation 5-1: } H[n] = \sum_{k=0}^{N-1} h[k] \cdot e^{i2\pi \cdot n \cdot \frac{k}{N}} ;$$

The corresponding reverse transformation is [9]:

$$\text{Equation 5-2: } h[k] = \frac{1}{N} \sum_{n=0}^{N-1} H[n] \cdot e^{-i2\pi \cdot k \cdot \frac{n}{N}} ;$$

The $H[n]$ of a pure sine signal $h[k] = \sin(2\pi f k/N)$ is calculated as:

$$\text{Equation 5-3: } H[n] = \frac{\sum_{k=0}^{N-1} e^{i2\pi \cdot (n+f) \cdot \frac{k}{N}} - \sum_{k=0}^{N-1} e^{i2\pi \cdot (n-f) \cdot \frac{k}{N}}}{2i} ;$$

When f is an integer F , $0 \leq F < N/2$, so that F is one of the index frequencies of $H[n]$:

$$\text{Equation 5-4: } H[n] = i\delta_{n,F} N/2 ;$$

Otherwise:

$$\text{Equation 5-5: } H[n] = \frac{1}{2} \frac{\sin(2\pi f - 2\pi \frac{f}{N}) + \sin(2\pi \frac{f}{N}) - \sin(2\pi f) \cdot e^{-i2\pi \frac{n}{N}}}{\cos(2\pi \frac{n}{N}) - \cos(2\pi \frac{f}{N})} ;$$

Let F be the closest integer to f , and $\Delta f \equiv f - F$:

Equation 5-6: $|\Delta f| \rightarrow 0: H[n] \rightarrow i\delta_{n,F} N/2;$

$|\Delta f| \rightarrow 1/2: H[n] \rightarrow N/\pi;$

Discrete Fourier transformation $H[n]$ of a signal $h[k]$ is the digital implementation of continuous Fourier transformation $H(s)$ of signal $h(t)$ [9], in which $H(s)=i\delta(s-f)$ for $h[k]=\sin(2\pi f)$. From Equation 5-6, when a signal frequency of the original signal falls between two index frequencies of the transformation, the peak is lower and wider with a wrong phase, depending on the fractional frequency. Thus, a discrete Fourier transformation is only faithful for signal components with exactly one of the index frequencies of the transform, and it suppresses and widens unfaithful signal components, each of which has different phase than its closest faithful representation, with the phase of a sine wave distorted toward that of a cosine wave, and vice versa. Examples of unfaithful representation of fractional frequency by discrete Fourier transformation are shown in Figure 8. The calculation is done on 1024 samples using FFT on a series of perfect sine signals having amplitude of 1 and slightly different frequencies as shown in legends. In the drawing, x axis shows the frequency, y axis shows either intensity or phase (inlet). A faithful representation at 256 is also included for comparison, whose phase is 90° at 256, and undetermined at other frequencies.

Due to its width, an unfaithful transformation may interact with other frequency components of the discrete Fourier spectrum, thus sabotages the whole idea of using Fourier transformation to decompose a signal into independent frequency components. Because the reverse Fourier transformation mathematically restores the original $h[k]$ for any $\{H[n]\}$, it exaggerates and narrows all unfaithful signal components correspondingly. This means that the common method of signal processing in Fourier space [9][12][14] may generate artifacts due to its uniform treatment of faithful and unfaithful signal components, which probably coexist in reality. Unlike aliasing [9][12][14], unfaithful representation of discrete Fourier transformation has equal effect in whole frequency range, so that it cannot be avoided by different samplings.

Unfaithful representation arises from implied assumption of discrete Fourier transformation.

The continuous Fourier transformation has infinitive signal range, so that:

$$\text{Equation 5-7: } h(t) \Leftrightarrow H(s): \quad h(t-\tau) \Leftrightarrow H(s) \exp(i2\pi s \tau);$$

As an analog, the discrete Fourier transformation $G[n]$ of signal $h[k]$, $k = 1 \dots N$ can be calculated mathematically from the discrete Fourier transformation $H[n]$ of $h[k]$, $k = 0 \dots N-1$:

$$\text{Equation 5-8: } G[n] = (H[n] + h[N] - h[0]) \exp(i2\pi n/N);$$

So for $G[n]$ to be equivalent to $H[n]$, $h[N] = h[0]$, or discrete Fourier transformation has an implied assumption that the signal $h[k]$ repeats itself outside the region of $[0, N-1]$. For a unfaithful frequency, $h[N-1]$ and $h[N]$ are discontinuous in regard to signal periodicity, resulting in larger peak width, lower peak height, and wrong phase.

Because of unfaithful representation of discrete Fourier transformation, the discrete Fourier transformation of a generic sine signal is provided by Equation 5-5. The most convenient signals to test uncertainty-bearing arithmetics are sine or cosine signals with index frequencies. A linear signal with slope λ , $h[k] = \lambda k$, provides a generic test, whose Fourier spectrum is:

$$\text{Equation 5-9: } H[n] = -\lambda \frac{N}{2} \left(1 + \frac{i}{\tan(\pi \cdot n / N)} \right);$$

5.2 FFT (Fast Fourier Transformation)

When $N = 2^L$, in which L is a positive integer, the generalized Danielson-Lanczos lemma [9] can be applied in Fourier transformation as FFT (Fast Fourier Transformation) [9]:

$$\text{Equation 5-10: } H\left[n, \frac{k}{2^L}\right] = h[j], \quad k, n = 0, 1 \dots N-1, \quad j \text{ is bit-reverse of } n;$$

$$\text{Equation 5-11: } H\left[n, \frac{k}{2^m}\right] = H\left[n, \frac{k}{2^{m+1}}\right] + H\left[n, \frac{k}{2^{m+1}}\right] \cdot e^{+i2\pi n \frac{1}{2^{L-m}}}, \quad m = 0, 1 \dots L-1$$

$$\text{Equation 5-12: } H[n] = H\left[n, \frac{k}{2^0}\right];$$

Thus, each output value is obtained after applying Equation 5-11 L times. L is called FFT order in this paper.

The calculation of term $e^{i2\pi n \frac{1}{2^{L-m}}}$ in Equation 5-11 can be simplified. Let \ll denotes bit left-shift operation and let $\&$ denote bitwise AND operation:

Equation 5-13: $\varphi[n] \equiv e^{i2\pi n \frac{1}{2^L}} ; \quad e^{i2\pi n \frac{1}{2^{L-m}}} = \varphi[(n \ll m) \& ((1 \ll L) - 1)] ;$

It is important to have an accurate phase factor array $\varphi[n]$ when tracking the FFT calculation error. The accuracy of $\varphi[n]$ can be checked rigidly within itself by trigonometric relations, so that no significant error is introduced from trigonometric functions.

5.3 Evaluating Calculation inside Uncertainty

Figure 9 shows the output deviations and value errors for a noisy sine signal after forward FFT. It shows that the output deviations using precision arithmetic are slightly larger than those using independence arithmetic, but much less than those using interval arithmetic. For a fixed input deviation, the output deviation using independence arithmetic is a constant for each FFT. Because value and uncertainty interact with each other through normalization in precision arithmetic, output deviations of Equation 5-11 are no longer a constant. One interesting consequence is that only in precision arithmetic the output deviations for a noisy input signal are larger than those for a corresponding clean input signal.

Figure 9 also shows that the value errors calculated using precision arithmetic are comparable to those using conventional floating-point arithmetic, and they are both comparable to the output deviations using either precision arithmetic or independence arithmetic. In another word, the result of calculating 2-bit or 53-bit into uncertainty are quite comparable, so that limited calculation inside uncertainty is reasonable.

Figure 10 compares the output value errors of precision arithmetics calculating different bit inside uncertainty. With no calculation inside uncertainty, the output value errors are only on

four levels. Such quantum distribution is reduced noticeable by the 2-bit calculation inside uncertainty, while the 3-bit calculation inside uncertainty has only marginal improvement in addition. Precision arithmetic with 2-bit calculation inside uncertainty is used for further tests.

5.4 Evaluating Uncertainty Distribution

As indicated by the uniform distribution assumption of precision arithmetic, which further embodies as Equation 3-26, the goal of uncertainty tracking in precision arithmetic is not to obtain an actual uncertainty distribution, but to provide a truncated normal distribution which bounds the actual uncertainty distribution according to the independent uncertainty assumption. Thus, the actual distribution could be narrower than the bounding distribution. FFT provides a good test for such probability bounding. Its forward and reverse algorithms are identical except a constant, so that they result in exactly the same bounding probability distributions. On the other hand, forward FFT condenses a sine signal into only two non-zero imaginary values by mutual cancellation of signal components, while reverse FFT spreads only two non-zero imaginary values to construct a sine signal. Thus forward FFT is more sensitive to calculation errors than reverse FFT, and should have a broader actual uncertainty distribution.

The occurrence frequencies of output value error significand are counted for different bounding ranges and normalized as measured probability densities. Because value errors are only observable as errors of result significand, the theoretical rounding error distributions obtained from Equation 3-15 are integrated around each integer value of rounding error, to result in theoretical probability densities of value error significand for different bounding ranges. Figure 11 and Figure 12 shows the theoretical and empirical distribution of significand errors for different bounding ranges on noisy sine signals for forward and reverse FFT respectively. They shows that the uncertainty distribution of forward FFT is very close to the bounding distribution, while the uncertainty distribution of reverse FFT is indeed narrower. Reverse FFT has slightly smaller average error significands than those of forward FFT.

5.5 Evaluating Uncertainty-Tracking

Figure 13 shows that for a same input deviation, the output deviations of forward FFT increases exponentially with FFT order using all three arithmetics. Figure 14 shows that for a same FFT order, the output deviations of forward FFT increases linearly with input deviation using all three arithmetics. The output deviation does not change with input frequency, so that all data of a same input deviation and FFT order but different input frequencies can be pooled together during analysis. The trends in Figure 13 and Figure 14 is modeled by Equation 5-14, in which L is FFT order, δx is input deviation, δy is output average deviation, α and β are empirical fitting constants:

$$\text{Equation 5-14: } \delta y = \alpha \beta^L \delta x;$$

β measures propagation speed of deviation with increased amount of calculation in Equation 5-14. It is called propagation base rate. Unless β is close to 1, β dominates α in fitting thus determines characteristics of Equation 5-14.

It turns out that Equation 5-14 is very good fits for both average output deviations and value errors for all three arithmetics, such as demonstrated in Figure 15. Because uncertainty-tracking is a competition between error propagation and uncertainty propagation, the average output error significand for forward FFT is expected to fit Equation 5-15 and Equation 5-16, in which z is average output error significand, L is FFT order, $(\alpha_{\text{dev}}, \beta_{\text{dev}})$ and $(\alpha_{\text{err}}, \beta_{\text{err}})$ are fitting parameters of Equation 5-14 for average output deviations and value errors respectively:

$$\text{Equation 5-15: } z = \alpha \beta^L;$$

$$\text{Equation 5-16: } \alpha = \alpha_{\text{err}}/\alpha_{\text{dev}}; \quad \beta = \beta_{\text{err}}/\beta_{\text{dev}};$$

The estimated average output error significand can then be compared with the measured ones to evaluate the predictability of uncertainty-tracking mechanism. One example of measured average output error significands is shown in Figure 16, which shows that the average output error significands using precision arithmetic are a constant despite that both average output

uncertainty deviations and value errors increases linearly with input deviation and exponentially with FFT order. Equation 5-14 and Equation 5-15 are found empirically to be good fit for any FFT algorithm with any input signal using any arithmetic.

Reverse FFT algorithm is identical to Forward FFT algorithm, except:

- Reverse FFT algorithm uses constant $(-i)$ instead $(+i)$ in Equation 5-11.
- Reverse FFT algorithm divides the result further by 2^L .

Thus, the output average deviations and value errors of reverse FFT algorithm are expected to obey Equation 5-14 and Equation 5-17, in which $(\alpha_{for}, \beta_{for})$ are corresponding fitting parameters of Equation 5-14 for forward FFT, while the output average error significands is expected to obey Equation 5-15 with same α and β as those of forward FFT.

Equation 5-17: $\alpha = \alpha_{for}; \quad \beta = \beta_{for}/2;$

Roundtrip FFT is forward FFT followed by reverse FFT, with the output of forward FFT as input to reverse FFT. Thus, both its average output deviations and value errors are expected to fit Equation 5-14 and Equation 5-18, in which $(\alpha_{for}, \beta_{for})$ and $(\alpha_{rev}, \beta_{rev})$ are corresponding fitting parameters of Equation 5-14 for forward FFT and reverse FFT respectively. And its error significands are expected to fit Equation 5-15 and Equation 5-18, in which $(\alpha_{for}, \beta_{for})$ and $(\alpha_{rev}, \beta_{rev})$ are corresponding fitting parameters of Equation 5-15 for forward FFT and reverse FFT respectively.

Equation 5-18: $\alpha = \alpha_{for} \alpha_{rev}; \quad \beta = \beta_{for} \beta_{rev};$

Figure 17, Figure 18 and Figure 19 show the fitting of β for independent, precision and interval arithmetic for all the three algorithms respectively. These three figures show that all measured β makes no distinction between input signals for any algorithms using any arithmetic, e.g., no difference between real and imaginary part for a sine signal. The estimated β for average error significands is obtained from Equation 5-16. The estimated β for average uncertainty deviations and value errors for Reverse and Roundtrip algorithms are obtained from

Equation 5-17 and Equation 5-18 respectively. The measured β and the estimated β agree well with each other in all cases, confirming the notion that uncertainty-tracking is a simple competition between error propagation and uncertainty propagation:

- Equation 5-11 sums up two mutually independent operands, so the error propagation in a FFT algorithm is precisely tracked by independence arithmetic. Figure 17 confirms that independence arithmetic is ideal for uncertainty-tracking for FFT algorithms: (1) β for error significands is a constant 1; and (2) β for both the average output deviations and value errors are both 1 for roundtrip FFT because the result signal after roundtrip FFT should be restored as the original signal.
- Precision arithmetic has β for average output deviations slightly larger than those of value errors, resulting in β for average output error significands to be a constant slightly less than 1. Its β for average output deviations is slightly larger than the corresponding β of independence arithmetic, so its average output deviations propagate slightly faster with increased FFT order than those of independent arithmetic. Such slightly faster increase with the amount calculation arithmetic is anticipated by the difference between Equation 3-20 and Equation 2-12 with $\gamma=0$.
- Interval arithmetic has β for average output deviations always much larger than β for average output value errors, resulting in β for average output error significands of about 0.62 for forward and reverse FFT, and about 0.39 for roundtrip FFT. So its average output deviations propagate much faster with the amount of calculations than its value error does. Because Equation 5-11 always sums up two mutually independent operands, the dependence problem of interval arithmetic should not present in this test, and the fast propagation of uncertainty ranges is intrinsic to interval arithmetic due to its worst-case assumption.

Figure 20 shows that for forward FFT the measured average output error significands using either precision arithmetic or independence arithmetic are both approximately constant of 0.8 regardless of FFT order. In contrast, it shows that using interval arithmetic the measured average output error significands decrease exponentially with FFT order L . Such trends of average error significands hold for all three FFT algorithms and all input signals. Thus, in this case, the direct uncertainty tracking provided by precision arithmetic is better than the indirect uncertainty tracking provided by interval arithmetic.

5.6 Evaluating Uncertainty-Bounding

While uncertainty-tracking is the result of propagation competition between average output deviations and average values errors with increased amount of calculations, uncertainty-bounding is the result of propagation competition between maximal output bounding ranges and maximal value errors, both of which still fits Equation 5-14 well experimentally using any arithmetic. Equation 5-15 and Equation 5-16 can be used to estimate maximal bounding ratio as well. For example, Figure 21 shows that the maximal output bounding ratios using precision arithmetic fits Equation 5-15 well. Unlike average output error significands in Figure 16, using precision arithmetic, the maximal output bounding ratios increase slowly with FFT order.

To characterize trend of maximal output bounding ratios with FFT order, β for measured average output deviation vs. measured maximal output range, β for measured average vs. maximal output value errors, and β for measured average error significand vs. measured maximal bounding ratios are plotted side-by-side respectively in an arithmetic characterization plot for a particular input signal. Figure 22 shows the characterization plot for independence arithmetic while Figure 23 shows the characterization plot for precision arithmetic. In both characterization plots, β for maximal value error exceeds β for maximal uncertainty bounding ranges, suggesting that maximal bound ratio should increase with FFT order, which results in β for maximal bounding ratio slightly larger than 1. Figure 25 shows that the output maximal

bounding ratios indeed increase with FFT order of forward FFT for both arithmetics. Using either arithmetic, β for maximal bounding ratio is nearly constant for all these FFT algorithms, which is defined as maximal bounding ratio β for the type of signal. Figure 26 shows that β for different types of signal using either arithmetic is a constant so that each characterization plot is generic for the arithmetic. The bounding leakage for independent arithmetic is about 10^{-9} by statistical estimation, while precision arithmetic has empirical bounding leakage about 10^{-4} , which seems independent of input precision or amount of calculations. As shown in Figure 24 and Figure 25, like its average error significands, interval arithmetic has its maximal bounding ratios decreasing exponentially with increased FFT order for all algorithms while keeps its bounding leakages at constant 0. Like independence arithmetic and precision arithmetic, its characterization plot is stable among different types of input signals. Whether precision arithmetic is better than interval arithmetic in uncertainty bounding depends on the statistical requirements for the uncertainty-bounding:

- In situation when absolute bounding is required, interval arithmetic is the only choice.
- In range estimation [1] involving low-resolution measurements whose sources of uncertainty are unclear, interval arithmetic is the only choice because the independence uncertainty assumption of precision arithmetic may not be satisfied.
- Otherwise if precision arithmetic can also be used, precision arithmetic should be more suitable for normal usages because most input data already contain bounding leakage.

5.7 Evaluating Deterministic Precision Round-up Rule

The result of current precision round-up rule is not deterministic for a value of $(2S+1)?@E$. If deterministic is a requirement, rounding $(2S+1)?@E$ always to $(S+1)-@E+1$ provides slightly higher significance than rounding it always to $S+@E+1$. Compared with the random precision round-up rule, both flavors of precision arithmetic have identical result in uncertainty tracking

and very similar but slightly worse result in uncertainty bounding, although both are not conceptually as good.

6 Conclusion and Discussion

6.1 Summary

Equation 5-14 is expected to describe general value error propagation for linear algorithms in which L is the amount of calculations [11]. For all three arithmetics, the average and maximal output deviation and value errors are shown to obey Equation 5-14, while the average output error significands and maximal bounding ratios are shown to obey Equation 5-15 and Equation 5-16. FFT algorithms provide a good linear platform to test any uncertainty-bearing arithmetic, with (1) a clearly defined value measuring the amount of calculation, (2) a known error propagation mechanism, (3) no condition execution in the algorithm, and (4) usage of basic arithmetic operations only. For FFT algorithms, precision arithmetic seems more suitable for uncertainty-tracking as well as uncertainty-bounding than interval arithmetic in normal usages. Precision arithmetic with 2-bit calculation inside uncertainty gives almost ideal performance for both uncertainty tracking and uncertainty bounding.

6.2 Dependence Problem

Equation 3-29 accurately accounts for all correlation within $f(x)$, and it provides a clean solution for δf . It is tempting to applying basic arithmetic operations to $f(x)$ directly. However, such approach may apply the independent uncertainty assumption wrongly between an identity and its constant mathematical expression, such as between x and x^2 , so that it may result in dependence problem similar to that of interval arithmetic [19]. For example, same as using interval arithmetic, only Equation 2-7 gives correct result for precision arithmetic.

Conventional arithmetic has no uncertainty and it can decompose an analytic expression $f(x_1, x_2, \dots)$ into many simpler form $e(g(x_1, x_2, \dots), h(x_1, x_2, \dots))$, until each simpler form contains a

constant mathematical expression of one identity. Equation 2-7, Equation 2-8 and Equation 2-9 are three examples of different decompositions of a same function $f(x)$. The flexibility of such decomposition makes conventional arithmetic very powerful. Using precision arithmetic, the uncertainty independence between x_1 and x_2 does not necessarily mean the uncertainty independence between $g(x_1, x_2)$ and $h(x_1, x_2)$. Thus, if $g(x_1, x_2, \dots)$ and $h(x_1, x_2, \dots)$ has any identity x_j in common, Equation 3-30 and Equation 3-31 can only be applied on final expression $f(x_1, x_2, \dots)$, but not through intermediate expressions $g(x_1, x_2, \dots)$, $h(x_1, x_2, \dots)$ and $e(g(x_1, x_2, \dots), h(x_1, x_2, \dots))$. For example, Equation 3-30 cannot be applied separately to x^2 and x then to Equation 2-8. Similar problem also plagues interval arithmetic [19] and independence arithmetic. This rigid requirement may limit both capability and convenience of use for the uncertainty-bearing arithmetics. It may even be a generic requirement because an uncertainty-bearing arithmetic may need to work on the final expression to avoid counting uncertainty excessively, e.g., an uncertainty-bearing arithmetic no longer has freedom to add mutually canceling terms to an expression. To obtain the final expression $f(x_1, x_2, \dots)$, symbolic calculation may be required, so that implementation of algorithms using an uncertainty-bearing arithmetic may be more complicated than its counterpart using conventional floating-point arithmetic.

6.3 Condition Execution

Condition execution based on the comparison relation between two values is frequently used in practical algorithms. When each value has associated uncertainty, the comparison relation between two values becomes quite different [15]. This is particularly true for interval arithmetic, in which a value can be anywhere inside its bounding range. On the other hand, in precision arithmetic, each value has a mathematically expected value plus a well-defined bounding distribution for uncertainty. The relation between two imprecise values in precision representation can either be defined by their mathematically expected values, or by their statistical comparison relations [4]. However, the usage of condition execution in a traditional

algorithm needs to be reevaluated conceptually with uncertainty statistics in mind when upgrading an algorithm to use precision arithmetic.

6.4 Improvements Needed for Precision Arithmetic

The uniform distribution assumption of precision arithmetic has rejected the effect of uncertainty on expected value by incorporating value shift due to uncertainty as increase of variance, such as for $f(x)=x^2$. The primary reason for this approach is that precision arithmetic only provides a bounding probability distribution, not the actual probability distribution. An alternative is to absorb the values shift due to uncertainty into expected values. Another alternative is to provide asymmetrical bounding in the uniform itself. Further study may be needed on this topic.

Although Taylor expansion method gives a clean solution to function evaluation in precision arithmetic, in reality Taylor expansion such as in Equation 3-27 can only be carried out to limited terms, and it may converge only in certain conditions, such as requiring x to be significant for $f(x)=1/x$. How to find the convergence conditions and how to determine the upper limit of a Taylor expansion remain unanswered generic questions for precision arithmetic at this moment.

Before applying it generally, precision arithmetic still needs more ground works and testing. However, it seems a worthwhile alternative to interval arithmetic and statistical propagation of uncertainty. It should be tested further in other problems, such as improper integrations and solutions to differential equations. This paper just provides a proof of concept.

6.5 Acknowledge

As an independent researcher, the author of this paper feels indebted to encouragements and valuable discussions with Dr. Zhong Zhong from Brookhaven National Laboratory, the organizers of AMCS 2005 with Prof. Hamid R. Arabnia from University of Georgia in particular, and the organizers of NKS Mathematica Forum 2007 with Dr. Stephen Wolfram in particular.

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8 Tables

\sim_1 VS. \sim_2	$\sim_1 = \sim_2$	$\sim_1 \neq \sim_2$				
		$\sim_1 = ?$	$\sim_2 = ?$	$R_1 > R_2$	$R_1 < R_2$	$R_1 = R_2$
\sim	\sim_1	\sim_2	\sim_1	\sim_1	\sim_2	$?$

Table 1: Result \sim in $S1 \sim 1R1@E + S2 \sim 2R2@E = (S1+S2) \sim (R1+R2)@E$.

\sim_1 VS. \sim	$\sim_1 = \sim_2$				$\sim_1 \neq \sim_2$	
	$\sim_1 = ?$	$R_1 > R_2$	$R_1 < R_2$	$R_1 = R_2$	$\sim_1 = ?$	$\sim_1 \neq ?$
\sim	$?$	\sim_1	\sim_2	$?$	\sim_2	\sim_1

Table 2: Result \sim in $S1 \sim 1R1@E - S2 \sim 2R2@E = (S1-S2) \sim (R1+R2)@E$.

R_{\max}	16	64	256
Normalized bounding range	$[4.9\sigma, 9.8\sigma)$	$[9.8\sigma, 19.6\sigma)$	$[19.6\sigma, 39.2\sigma)$
Maximal normalization leakage	10^{-6}	10^{-22}	10^{-85}
Bit calculation inside uncertainty	0	1	2
$0.5 \pm 0.001 =$	$512?6.29@-10$	$1024?25.16@-11$	$2048?100.64@-12$
$1 \pm 0.001 =$	$1024?6.29@-10$	$2048?25.16@-11$	$4096?100.64@-12$
$1 \pm 0.002 =$	$512?6.29@-9$	$1024?25.16@-10$	$2048?100.64@-11$

Table 3: Characterization of precision arithmetic with different R_{\max}

χ	0	1	2
$0.5 \pm 0.001 =$	512?6.29@-10	1024?12.58@-11	2048?25.16@-12
$1 \pm 0.001 =$	1024?6.29@-10	2048?12.58@-11	4096?25.16@-12
$1 \pm 0.002 =$	512?6.29@-9	1024?12.58@-10	2048?25.16@-11

Table 4: Examples of precision arithmetic with different χ for $R_{\max}=16$

9 Figures

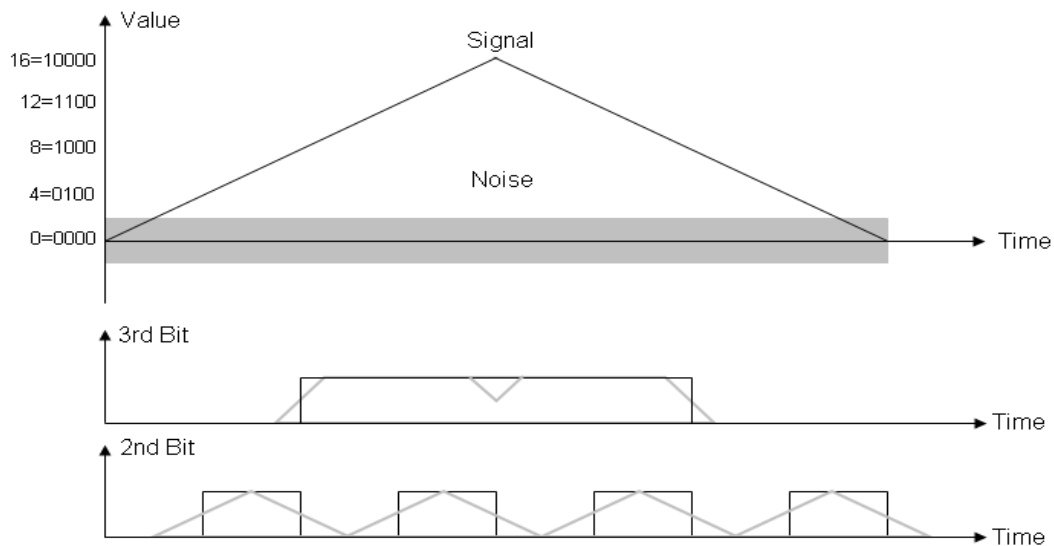


Figure 1: Effect of noise on bit values of a measured value.

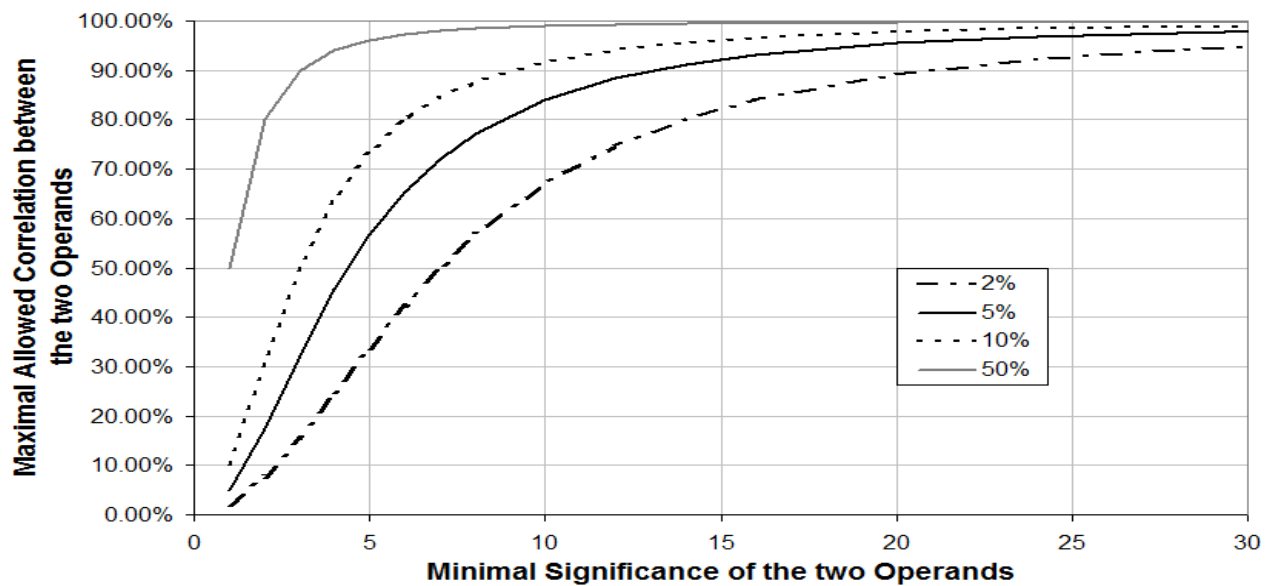


Figure 2: Allowed maximal correlation between two operands for different significance for precision arithmetic

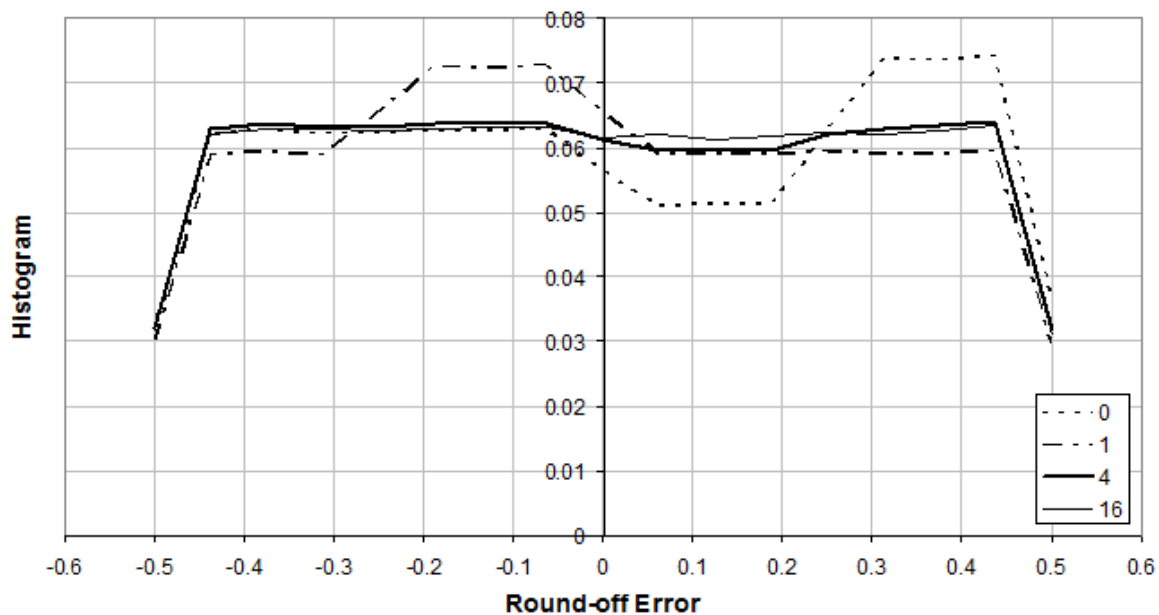


Figure 3: Measured probability distribution of rounding errors of precision round-up rule for the minimal significant thresholds 0, 1, 4, and 16 respectively.

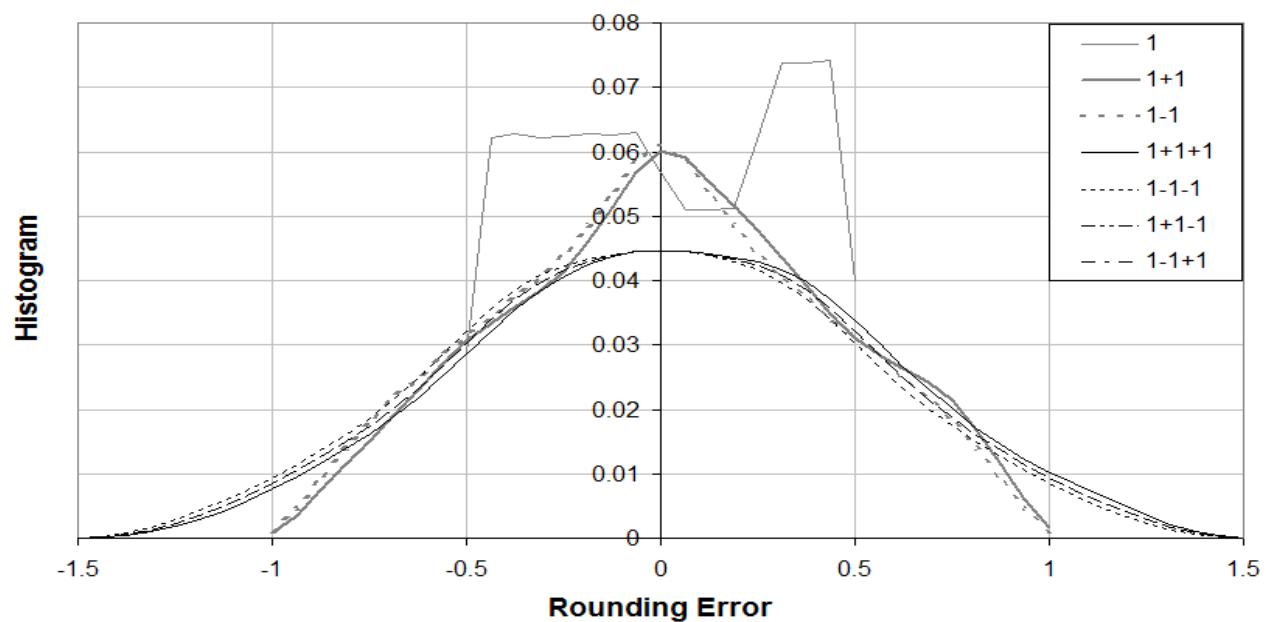


Figure 4: Measured probability distribution of rounding error after addition and subtraction.

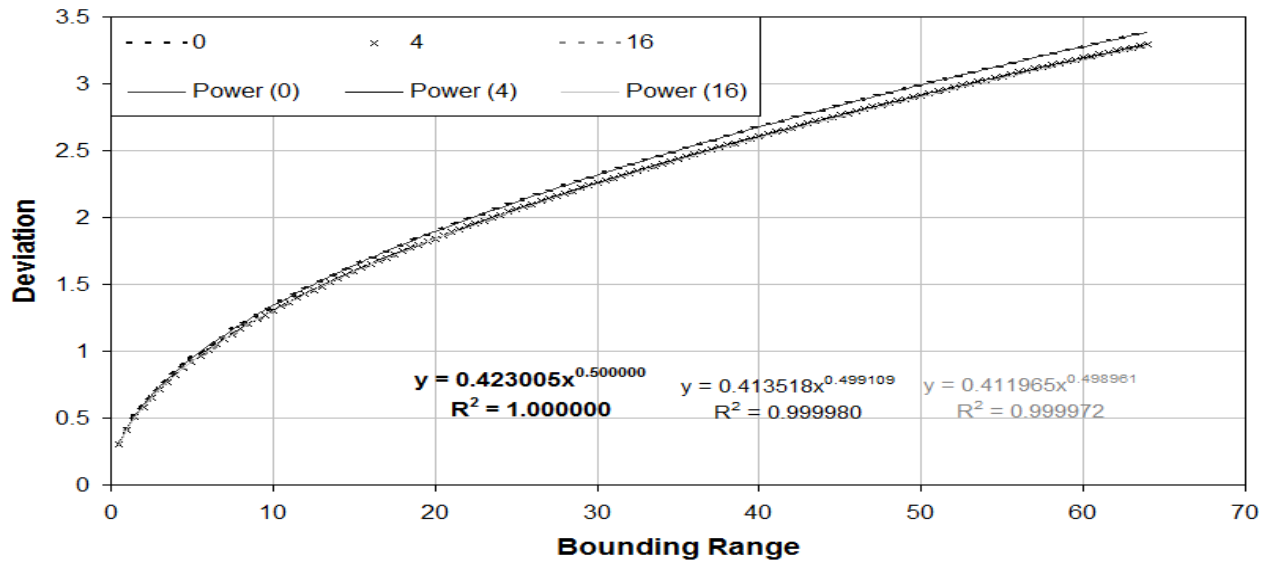


Figure 5: Deviations vs. bounding ranges of measured rounding error distributions of different minimal significant thresholds.

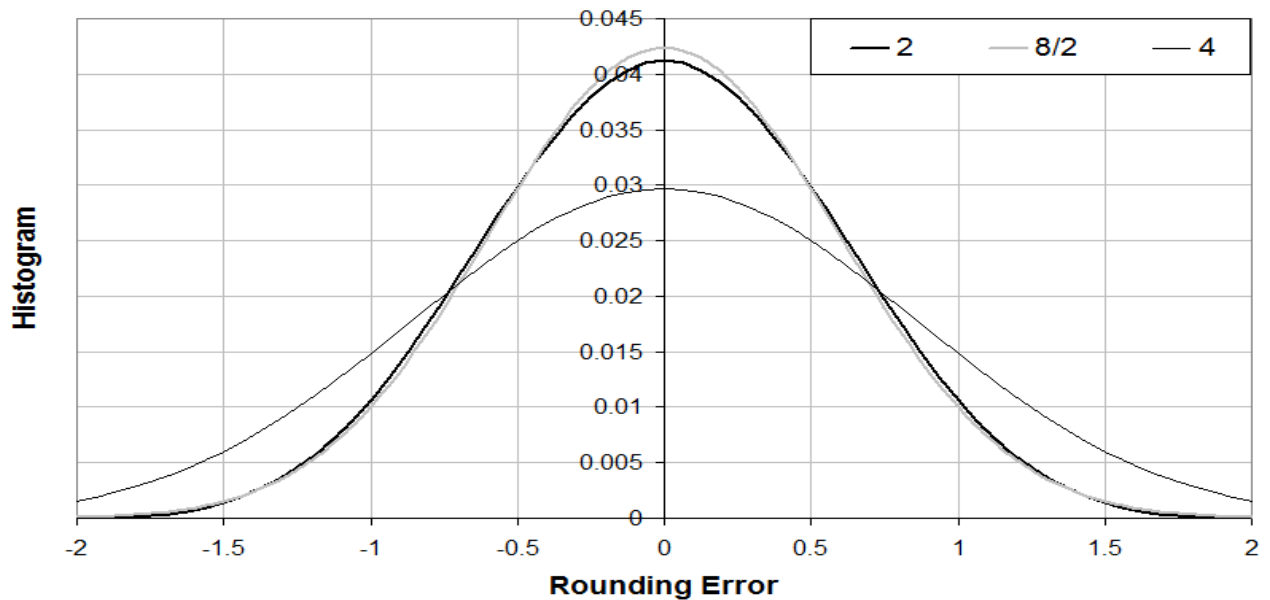


Figure 6: The result rounding error distribution $R=8/2$ after the original error distribution $R=8$ is rounded up once. The $R=8/2$ distribution is compared with the $R=4$ distribution and the $R=2$ distribution, which have same bounding range and deviation respectively.

Precision Arithmetic: A New Floating-Point Arithmetic

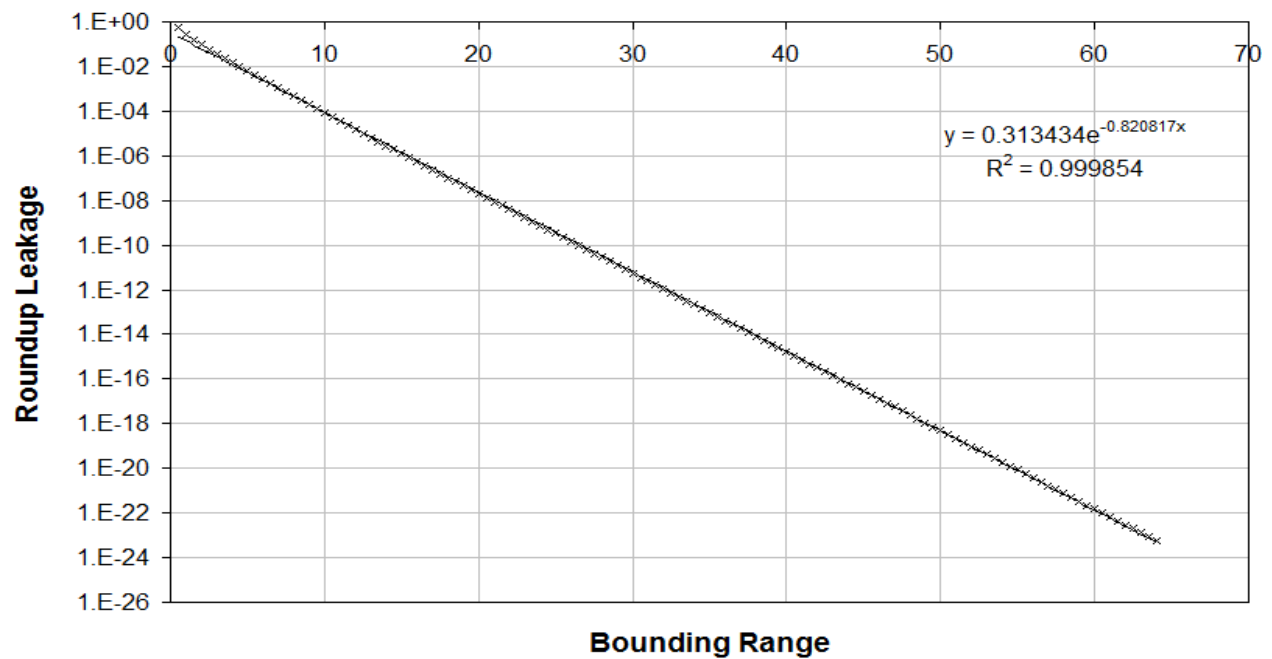


Figure 7: Roundup leakage vs. bounding ranges after rounding up by deviation once.

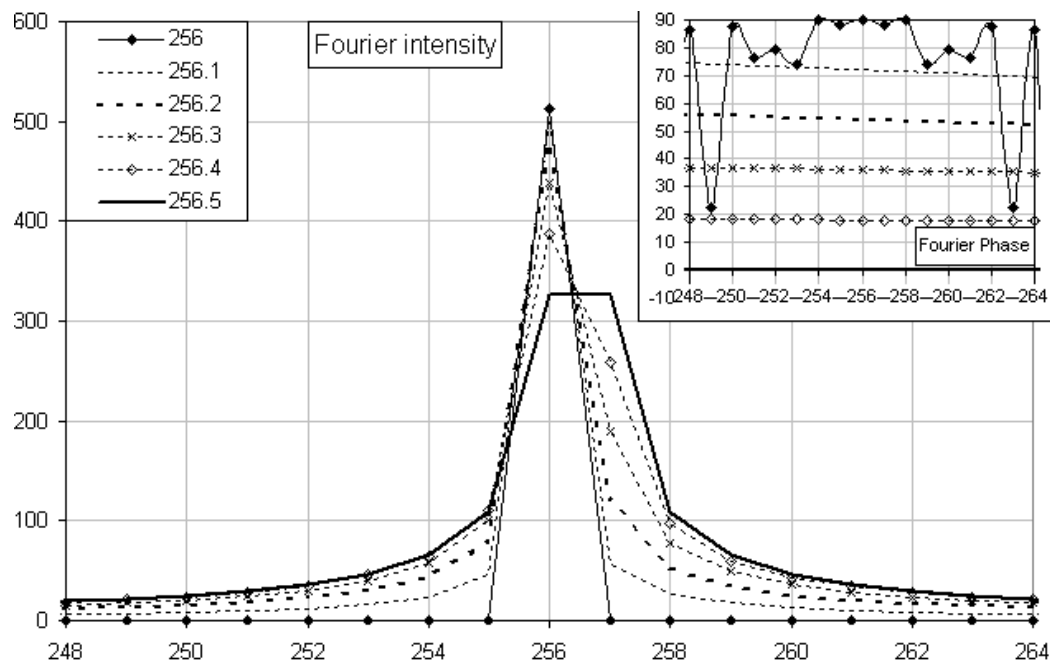


Figure 8: Unfaithful representations of perfect sine signals in discrete Fourier transformation.

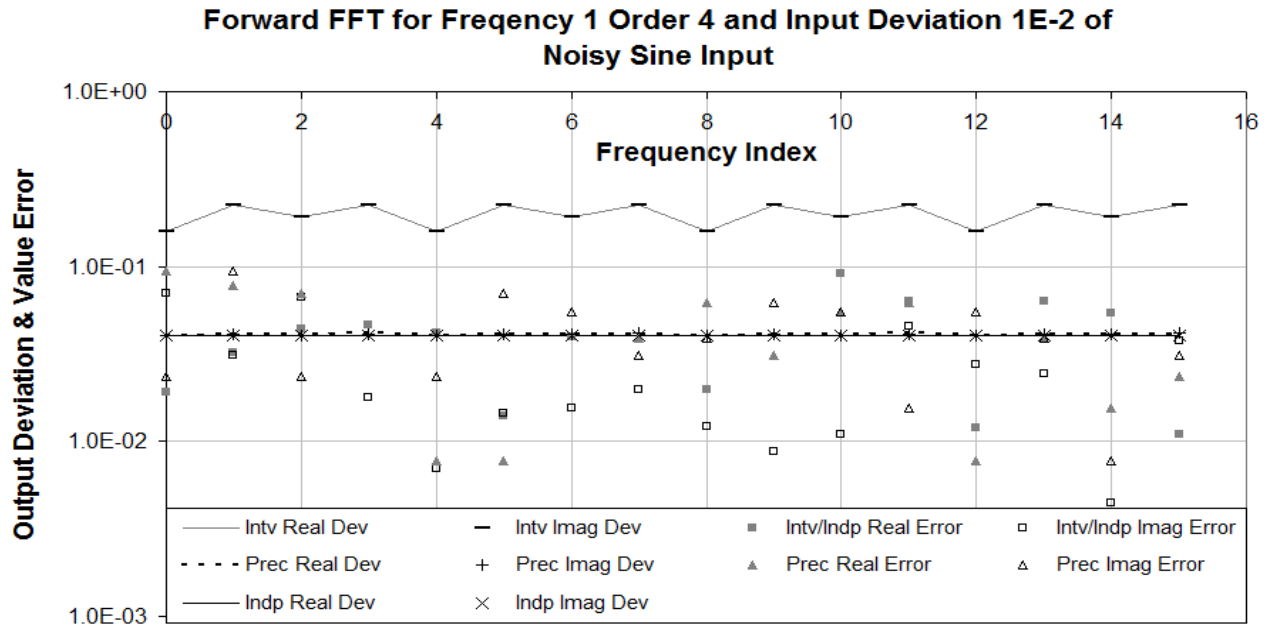


Figure 9: The output deviations and value errors of forward FFT on a noisy signal of index frequency 1 and input deviation 10^{-2} .

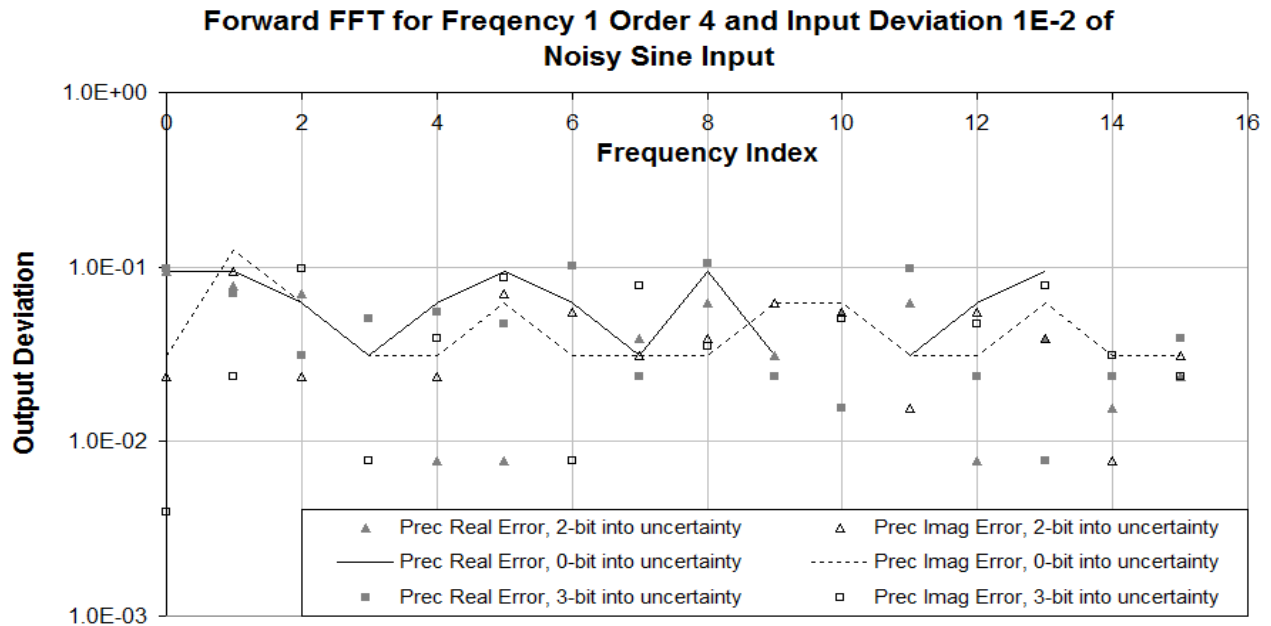


Figure 10: The output value errors of forward FFT on a noisy signal of index frequency 1 and input deviation 10^{-2} using precision arithmetic with different bit inside uncertainty

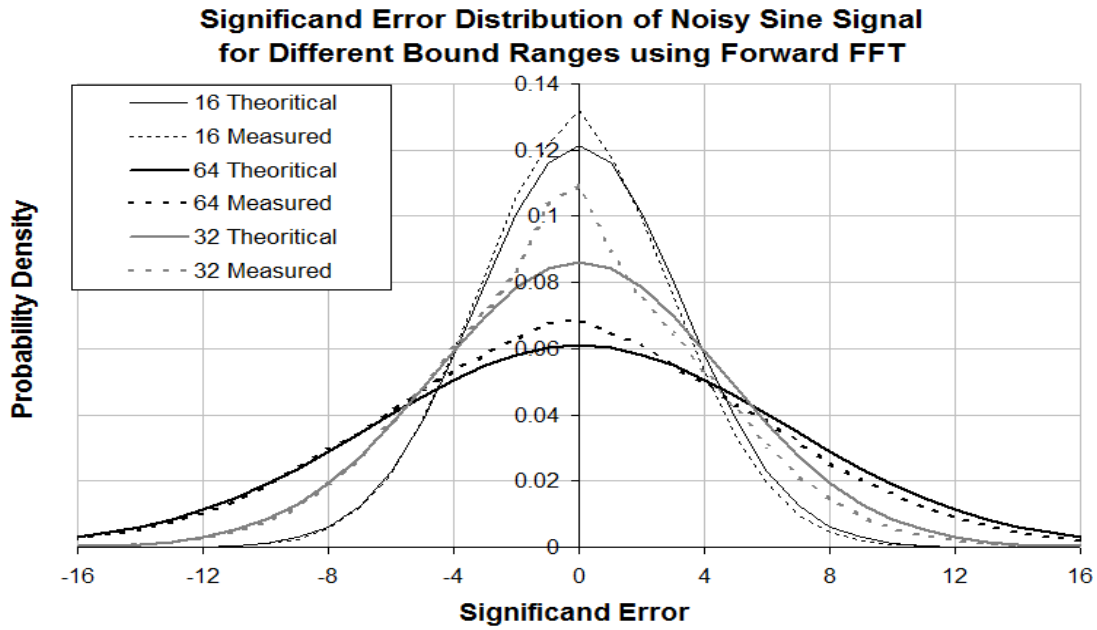


Figure 11: The theoretical and empirical distribution of significand errors for different bounding ranges using precision arithmetic for forward FFT on noisy sine signals.

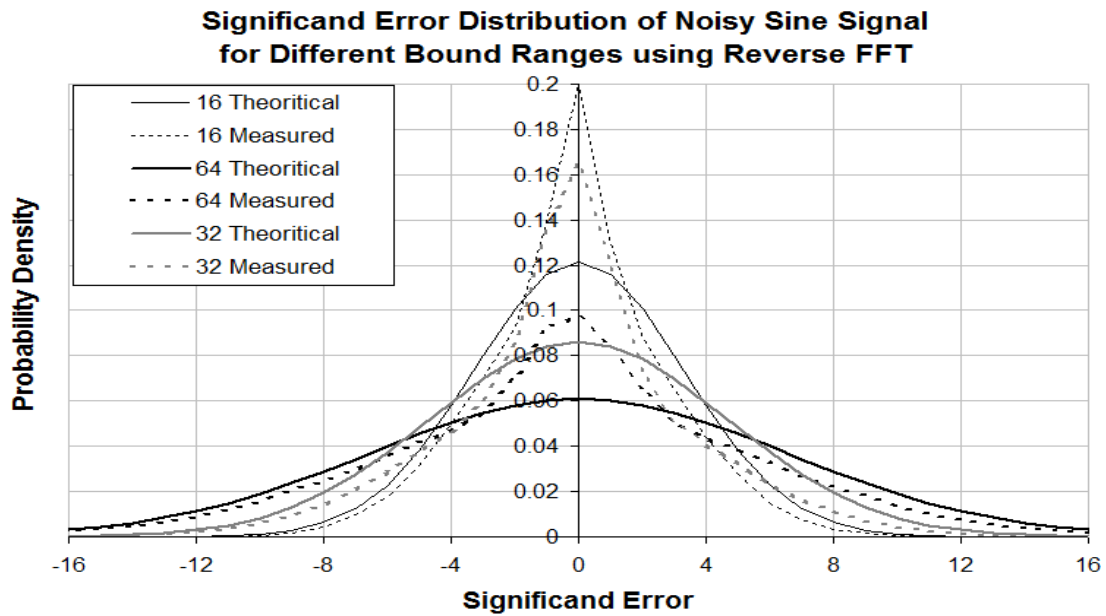


Figure 12: The theoretical and empirical distribution of significand errors for different bounding ranges using precision arithmetic for reverse FFT on noisy sine signals.

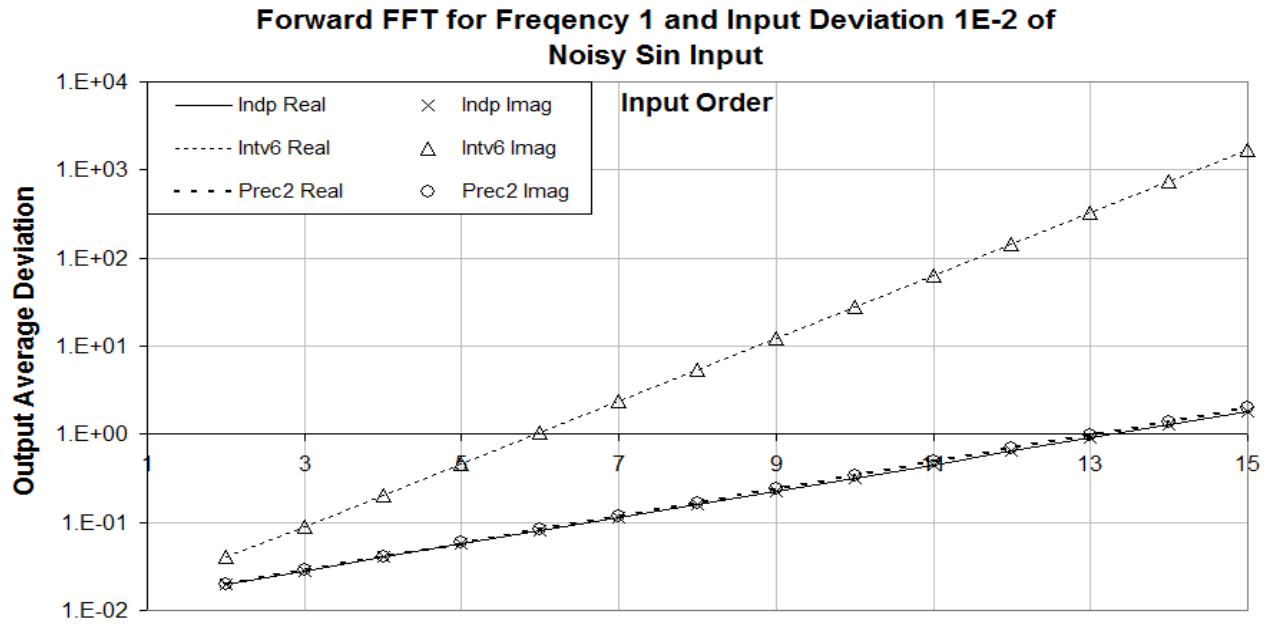


Figure 13: For a same input deviation, the average output deviations of forward FFT increase exponentially with FFT order.

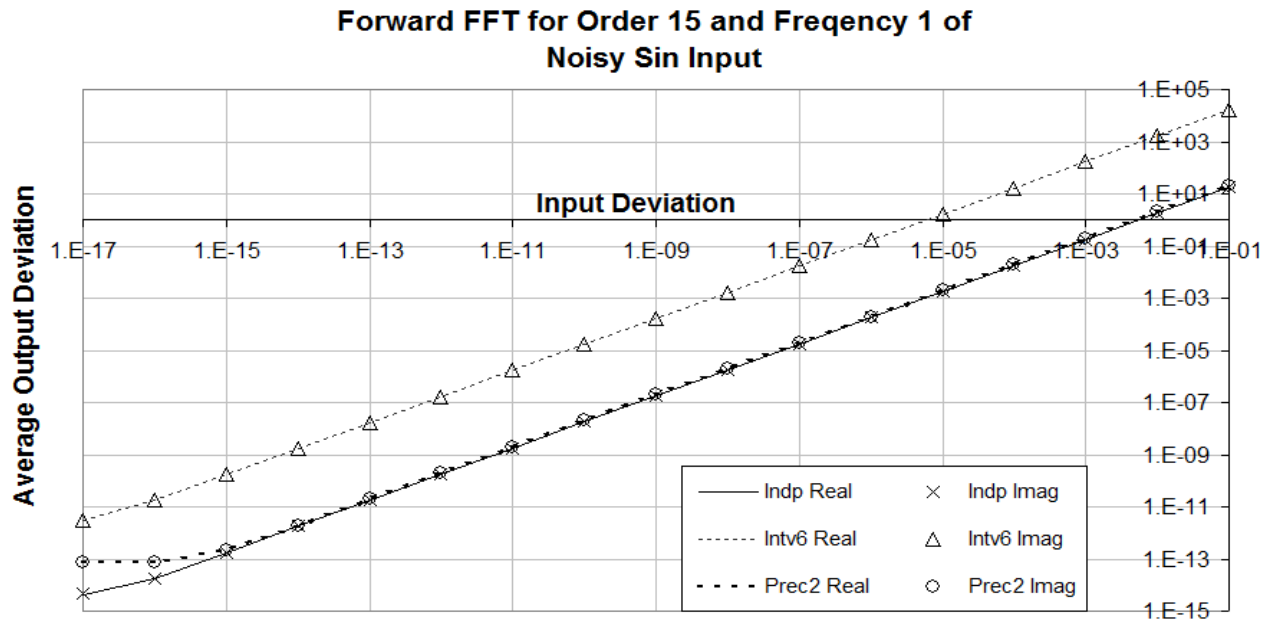


Figure 14: For a same order of FFT calculation, the average output deviations of forward FFT increases linearly with input deviation.

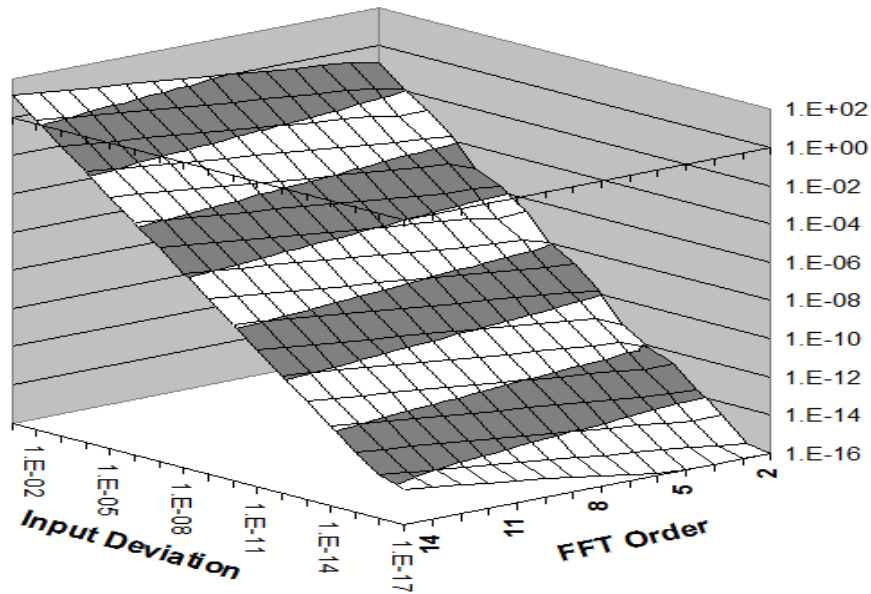


Figure 15: The average output value errors using precision arithmetic increase exponentially with FFT order and linearly with input deviation respectively.

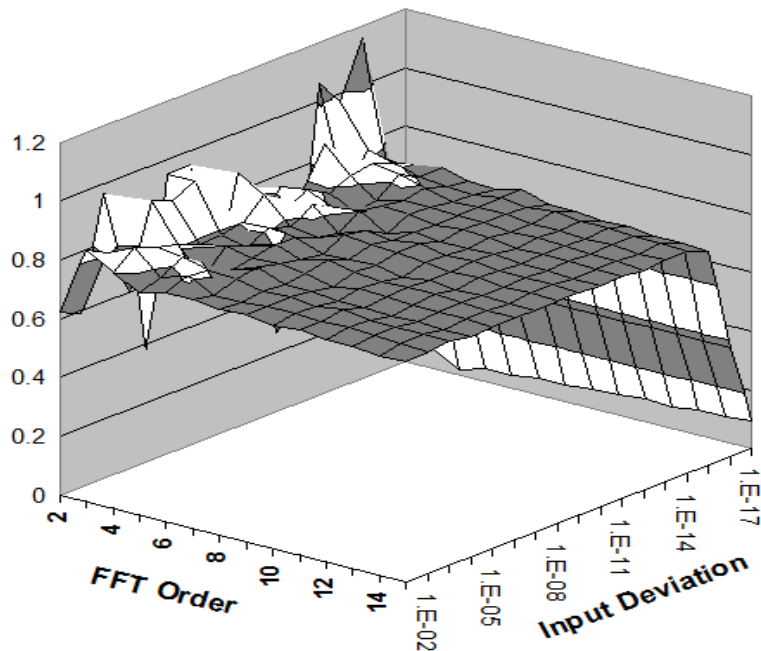


Figure 16: The average output error significands using precision arithmetic is a constant when the input deviation is larger than 10^{-15} and the FFT order is more than 4.

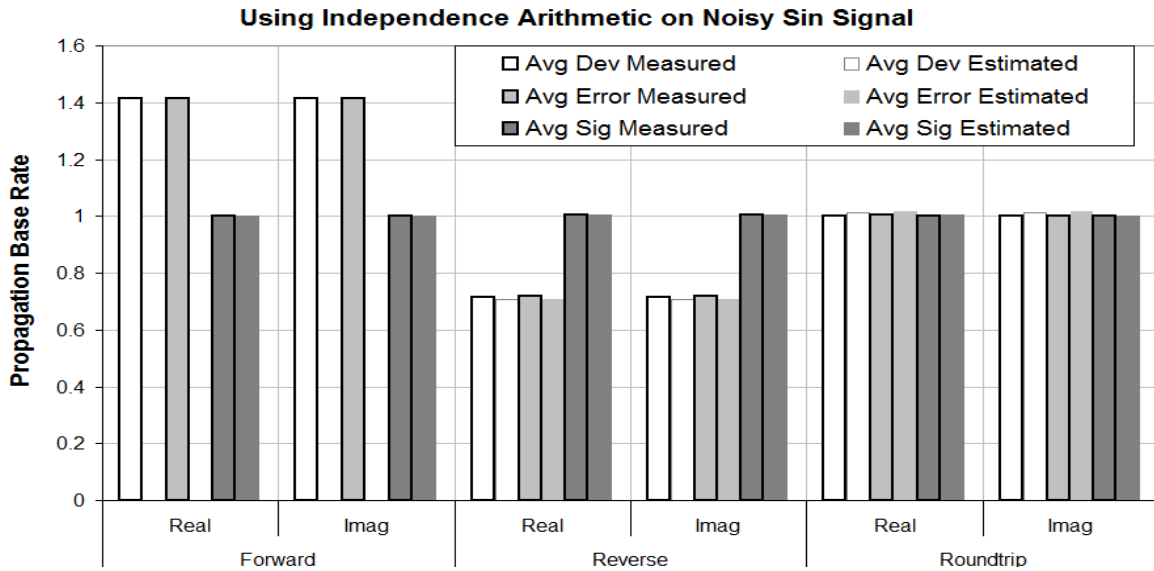


Figure 17: β for fitting average output deviations, value errors and error significands for forward, reverse and roundtrip FFT using independence arithmetic on noisy sine signals.

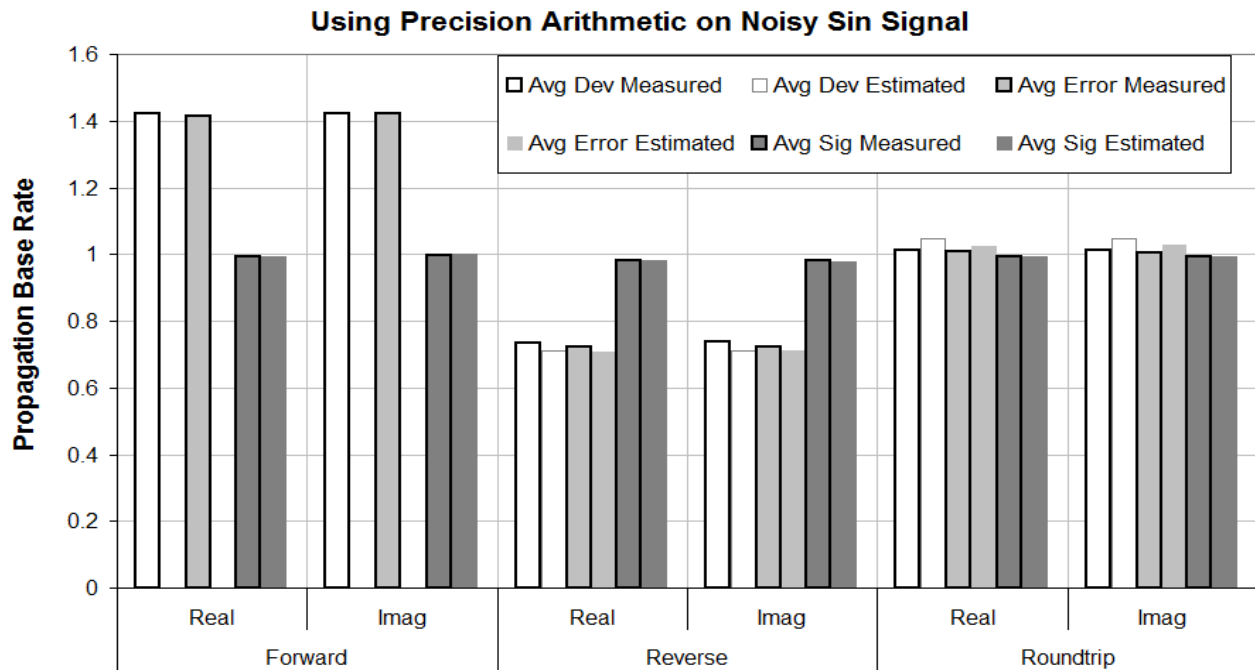


Figure 18: β for fitting average output deviations, value errors and error significands for forward, reverse and roundtrip FFT using precision arithmetic on noisy sine signals.

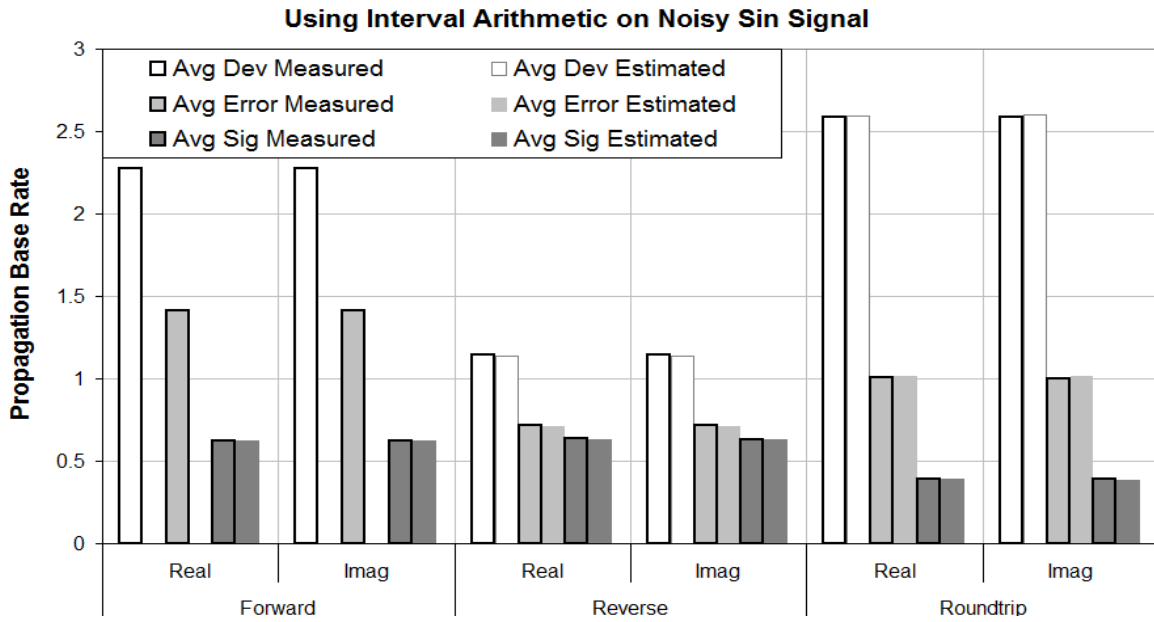


Figure 19: β for fitting average output deviations, value errors and error significands for forward, reverse and roundtrip FFT using interval arithmetic on noisy sine signals.

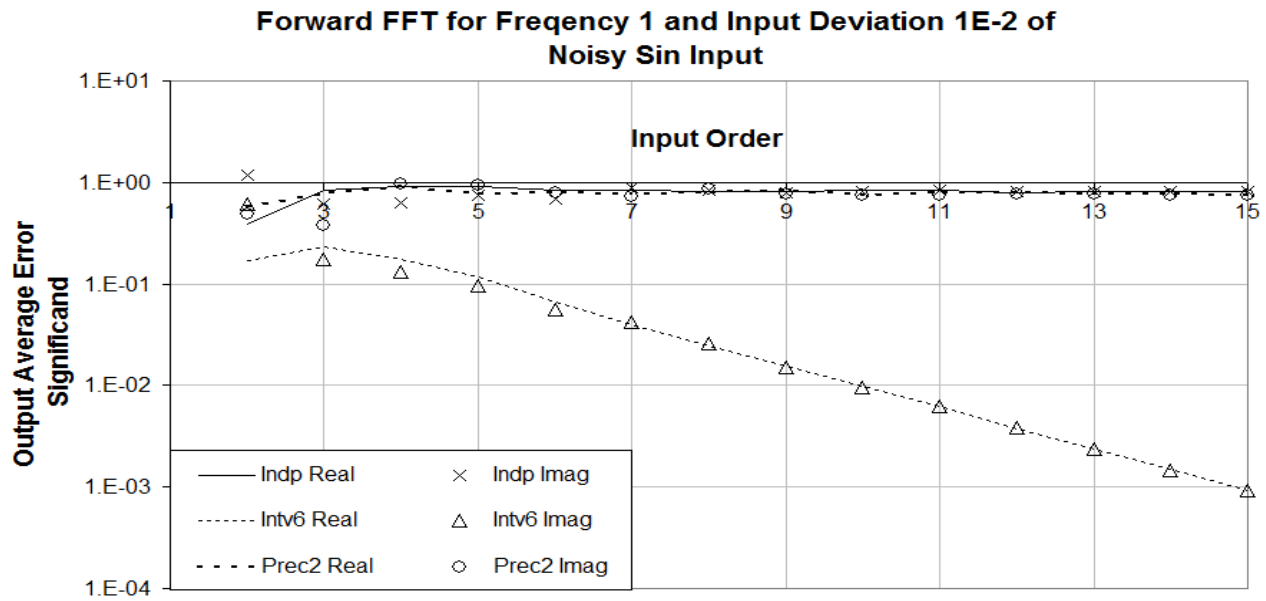


Figure 20: The output average error significands vs. FFT order of forward FFT for all three arithmetics.

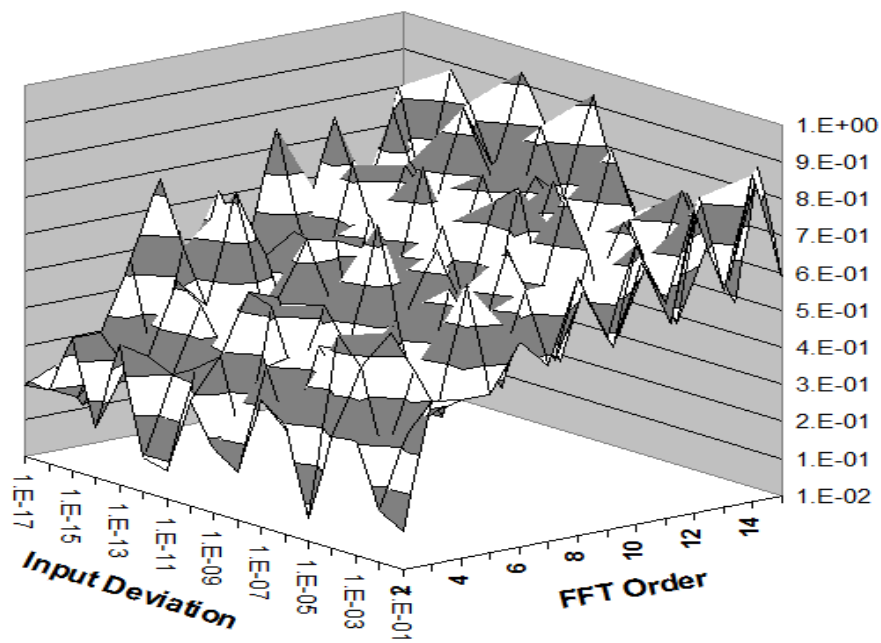


Figure 21: The maximal output bounding ratios using precision arithmetic increases slowly with FFT order.

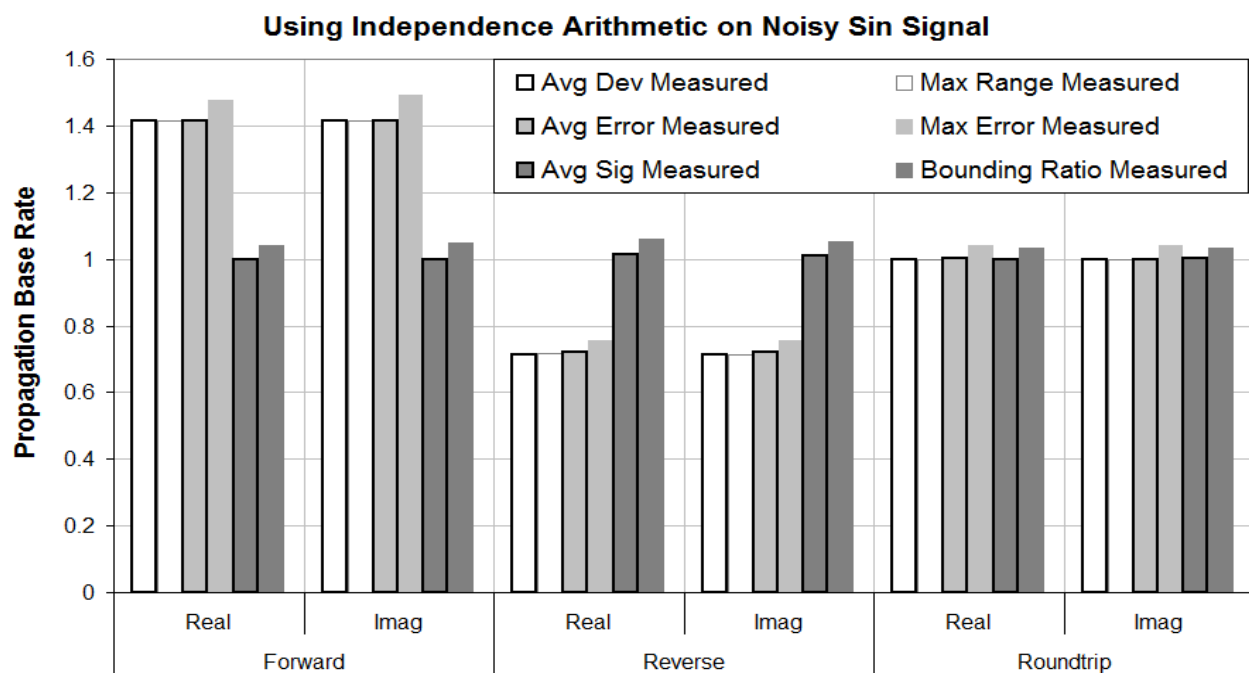


Figure 22: The characterization plot of independence arithmetic on noisy sine signal.

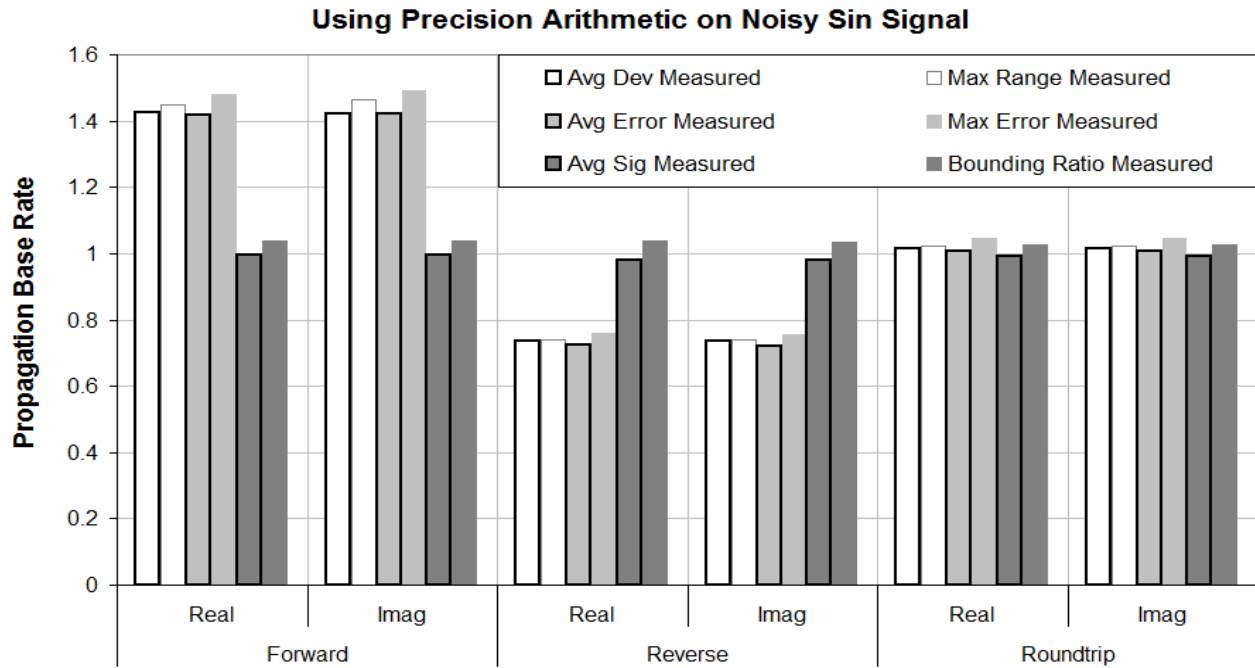


Figure 23: The characterization plot of precision arithmetic on noisy sine signal.

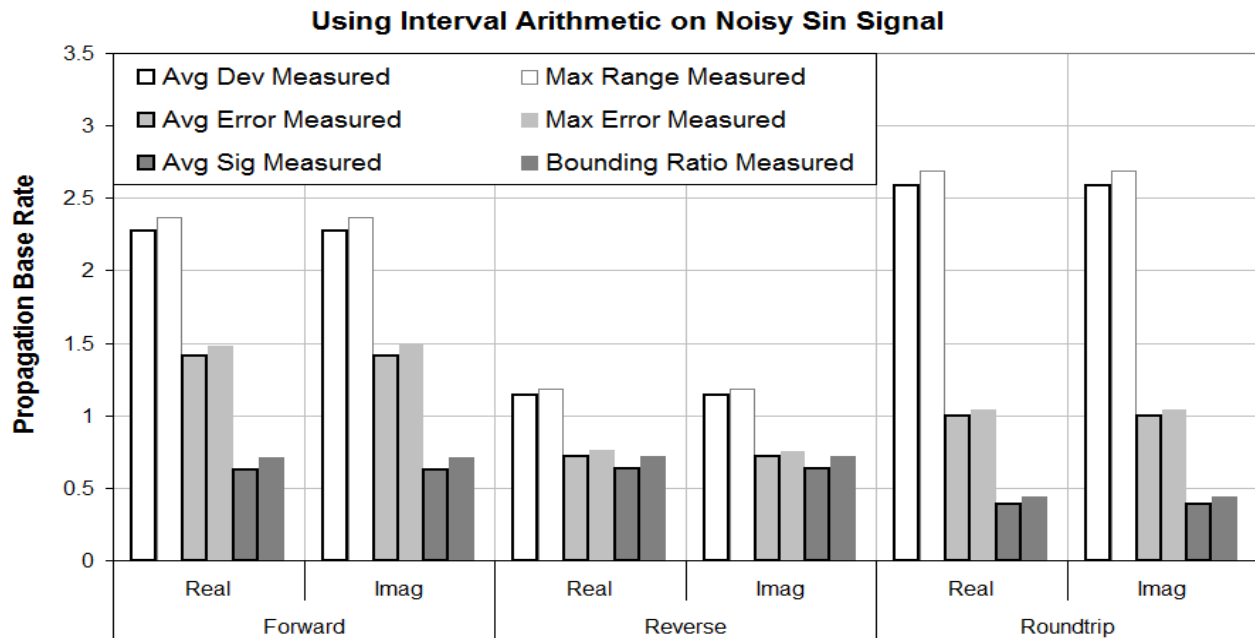


Figure 24: The characterization plot of interval arithmetic on noisy sine signal.

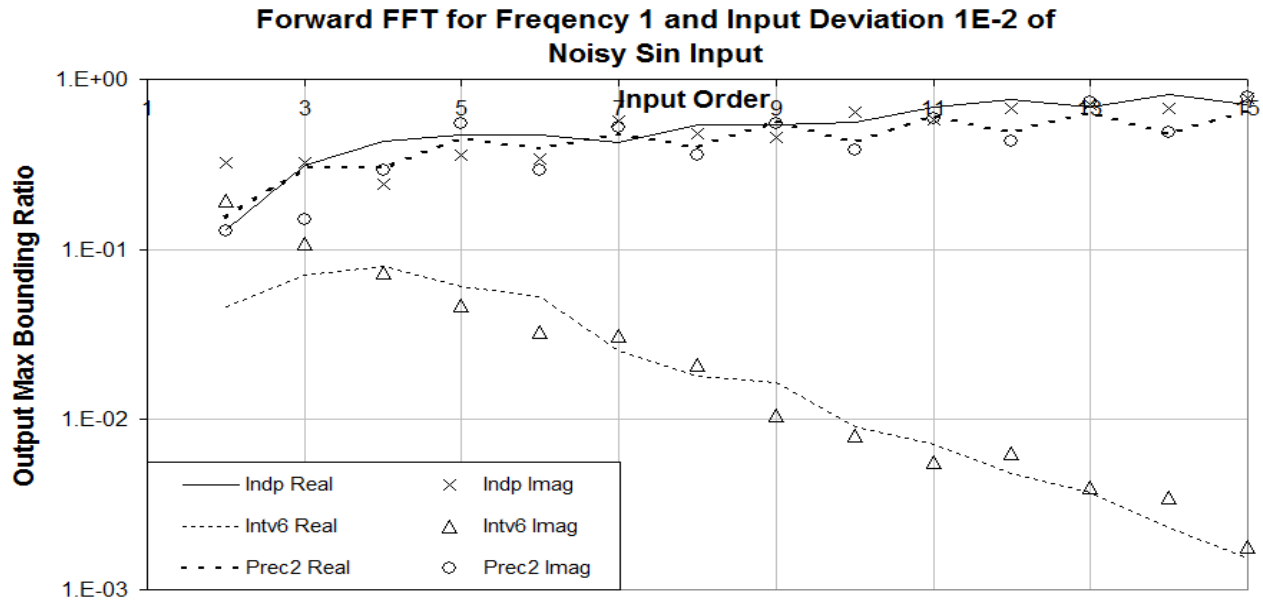


Figure 25: The output maximal bounding ratios vs. FFT order of forward FFT for all three arithmetics.

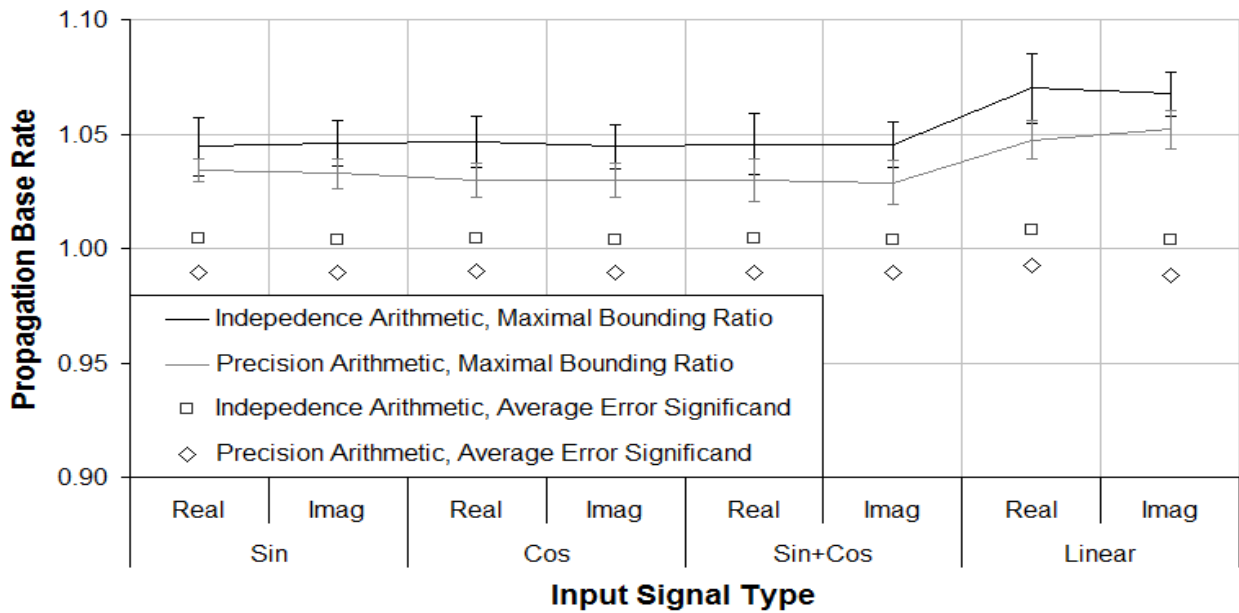


Figure 26: The β_s for output maximal bounding ratios vs. different input signal types for independence arithmetic and precision arithmetic.

10 Appendix: Project Precision Arithmetic in Source Forge

The source codes using visual studio express 2008 are published as project Precision Arithmetic in source forge at <http://www.precisionarithm.sourceforge.com>. The source code can be divided into four parts:

- `RealTypes.lib`: It contains C++ template classes for precision arithmetic, independence arithmetic and interval arithmetic. Implicit interface `IRReal` contains basic requirements for any uncertainty-bearing floating-point arithmetic. Each implementation of arithmetic has a short name. Class `TInterval<unsigned t_Sigma=6>` and class `TIndp<unsigned t_Sigma=6>` provide implementations for independence arithmetic and interval arithmetic respectively, in which `t_Sigma` defines the σ rule of the relation between uncertainty deviation and bounding range of these two arithmetic. Their short names are “Intv<t_Sigma>” and “Indp<t_Sigma>” respectively, such as “Intv6” and “Indp6” for the 6σ - 10^{-9} rule. Class `TPrecision<unsigned t_ExtraSigBits=0, enum ECarry t_CarryWhenUncertain=CARRY_UNKNOWN>` implements precision arithmetic, in which `t_ExtraSigBits` is the extra significand bits to be calculated inside uncertainty, and `t_CarryWhenUncertain` defines the precision round up rule for a value of $(2S+1)?R@E$ as either `CARRY_UNKNOWN` or `CARRY_POSITIVE` or `CARRY_NEGATIVE`. Its short name is “Prec” followed by “<t_ExtraSigBits>” then followed by “+” or “-” when `t_CarryWhenUncertain` equals `CARRY_POSITIVE` or `t_CarryWhenUncertain` respectively. For example, the short name for a precision arithmetic with random precision round up rule and 2 bit calculation inside uncertainty is “Prec2”, while it is “Prec2+” when the precision round up rule is deterministic and positive.
- `Precision.exe`: It tests any floating-point arithmetic in `RealTypes.dll` specified by a short name on an input signal (among “Sin”, “Cos”, “SinCos”, “Linear”) with noise type

("" for none, "GN" for Gaussian noise) for all three FFT algorithms ("For"--Forward, "Rnd"--Round-trip and "Inv"--Reverse). It generates output files named as "<short name>_<algorithm>_<signal>_<noise type>.txt" in folder "../Output/<short name>". It outputs detailed FFT calculation in files named as "<short name>_<signal>_<FFT order>_<frequency>_<input deviation>_<noise type>.txt" in folder "../Output/<short name>/FFT", and measured error significand distribution in files named as "<short name>_<signal>_<input deviation>_<noise type>.txt" in folder "../Output/<short name>/Dist". It also demonstrates basic math using the floating-point arithmetic and outputs the result in file "../Output/<short name>/DemoMath.txt". It is usually executed through a batch file `Precision.bat`, which accepts a short name and tests the arithmetic for all input signals and noise types. File `"log<short name>.txt"` and `"err<short name>.txt"` in folder "../Output/<short name>" capture standard output and error output of the batch process, with the former contains execution times.

- `Parser.exe`: It either selects or normalizes the large amount of output data from `Precision.exe`, and outputs files named as "<short name>_<signal>_<noise type>_<algorithm>.txt" in folder "../Output/<short name>/Parser". It is usually executed through a batch file `Parser.bat` embedded in `Precision.bat`. Thus, only `Precision.bat` needs to be executed for each arithmetic short name.
- `ErrorTracing.exe`: It accepts a minimal significand threshold, use Monte Carlo method to generate the probability distribution for bounding range of 1/2, calculates theoretical rounding error distributions and round up leakage for different bounding ranges, and outputs files in folder "../Output/Tracking/<minimal significand threshold>". Files named as "Range_<bounding range>_<method>.txt" contains theoretical rounding error distributions for the bounding range, in which <method> can be "p" for repeated addition, "m" for repeated subtraction, and "pm" for repeated addition followed by

subtraction. File "All.txt" contains round up leakage for different bounding ranges. File "MonteCarlo" contains precision round up process using random integer as original value.

Each output file contains tab-separated tables with column heading, which can be analyzed directly by any math package such as R statistics software. To cater for wider audiences, the output files are imported into excel files which fit and plot the data in folder "../Doc/<short name>". For example, in "../Doc/Prec0":

- File "Prec Sin.xls", "Prec Cos.xls", "Prec SinCos.xls" and "Prec Linear.xls" displays FFT results for a particular input deviation and signal frequency. Figure 9 is taken directly from "Prec Sin.xls".
- File "Prec Dist 0.xls", "Prec Dist 1e-1.xls", "Prec Dist 1e-2.xls", "Prec Dist 1e-3.xls", "Prec Dist 1e-4.xls", "Prec Dist 1e-5.xls" and "Prec Dist Total.xls" displays error significant distribution for each bounding range, input deviation and FFT algorithm. Figure 12 is taken directly from "Prec Dist 1e-4.xls".
- File "For Charts.xls", "Rnd Charts.xls" and "Inv Charts.xls" compare results for Forward, Roundtrip and Reverse algorithms with the corresponding results of "Indp6" and "Intv6" respectively. Figure 13, Figure 14, Figure 20 and Figure 25 are taken directly from "For Charts.xls".
- File "Prec For.xls", "Prec Rnd.xls" and "Prec Inv.xls" contain fitting results for Forward, Roundtrip and Reverse FFT algorithms respectively. Figure 15, Figure 16 and Figure 21 are taken directly from file "Prec For.xls".
- File "Prec FFT.xls" compares the fitting results from "Prec For.xls", "Prec Rnd.xls" and "Prec Inv.xls". Figure 18 and Figure 22 are taken directly from file "Prec FFT.xls".

Thus, xls files under "../Doc/Prec0" gives a complete picture of the "Prec0" arithmetic. The outputs of `ErrorTracing.exe` are also analyzed by excel. File "../Output/Tracking/Tracking

<minimal significand threshold>.xls” displays rounding error distributions and round up leakage for a particular bounding range. For example, Figure 4 and Figure 6 are taken from file “Tracking 0.xls”. File “../Output/Tracking/Tracking All.xls” compares characteristics of different minimal significand threshold, which contains Figure 3 and Figure 5.

The current implementation of precision arithmetic can be improved in several ways:

- Function `TPrecision::Normalize()` and `TPrecision::RoundUpTo()` apply precision round-up rule repeatedly. Because the rounded up significand is always the closest value to the original significand at each precision, they could calculate the rounded up significand at targeted exponent and then sum up the rounding errors in one step.
- To calculate significand and bounding range strictly, class `UInt` provides an implementation of unsigned integer with variable significant bits. A 64-bit unsigned integer is shown to provide very good approximation but with a 2-fold improvement on speed.
- Function `UInt::Mul()` and `UInt::Div()` can be replaced by few assembly instructions respectively.
- In current approach, an instance of template is first identified from a short name, then the corresponding instantiated template class or function is called. All available instances of template class and associated friend functions need to be instantiated in the static library `RealTypes.lib`, as well as in the clients of the library, `Precision.exe` and `Parser.exe`. This leads to repeated and cumbersome codes. `RealTypes.lib` should contain a type factory which outputs a type from an inputted short name, then use this type to instantiate template classes and functions in run-time. However, unlike C#, C++ does not allow instantiating template from type info in run-time.