

Low SNR Capacity of Fading Channels – MIMO and multipath

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Abstract—Discrete-time noncoherent Rayleigh fading multiple-input multiple-output (MIMO) channels are considered, with no channel state information at either the transmitter or the receiver. The fading is assumed to be correlated in time and independent from antenna to antenna. Peak and average transmit power constraints are imposed, either on the sum over antennas, or on each individual antenna. In both cases, an upper bound and an asymptotic lower bound, as the signal-to-noise ratio approaches zero, on the channel capacity are presented. The capacity asymptote is identified under the sum power constraints. For the individual power constraints, bounds on the capacity asymptote are presented and shown to be tight under certain conditions. These results are applied to a SISO channel with multipath fading. For such a channel, the multiple transmission paths play the role of multiple transmit antennas. Upper and lower bounds on the low SNR capacity asymptote are presented and shown to coincide under certain conditions.

Index Terms

Low SNR, channel capacity, correlated fading, multipath fading, MIMO, Gauss Markov fading

I. INTRODUCTION

Discrete-time noncoherent Rayleigh fading multiple-input multiple-output (MIMO) channels are considered, with no side information at either the transmitter or the receiver. The fading is assumed to be correlated in time and independent for distinct (input,output) antenna pairs. The main focus of this paper is the low signal to noise ratio (SNR) behavior of the channel capacity. In the low SNR regime, it is known that capacity achieving input signals are bursty [1,2]. To limit this behavior, a hard peak constraint, in addition to the average power constraint, is imposed in this paper. Two cases are considered: either the peak and average power constraints are imposed on the sum over the antennas, or they are imposed

on each antenna. In each case, an upper bound on the capacity of MIMO channels is derived. In the sum peak constraint case, the capacity asymptote at low SNR is identified and the upper bound is found to be tight. In the individual peak constraint case, asymptotic tightness of the upper bound is established under certain conditions. Insight about optimal signaling strategies is derived in each of the above cases. Also, comments on the benefits of having multiple antennas in the low SNR regime are presented. This work extends previous work of the authors [3,4] for SISO channels, to MIMO and multipath fading channels.

For similar work in the high SNR regime, see [5] and references therein. The capacity of this channel in the low SNR regime has recently been of heightened research interest [6–8].

Next, a single-input single-output (SISO) channel with multipath (i.e. frequency selective) fading is considered. The fading is assumed to be modeled by a finite number of taps. The fading processes are assumed to be independent across taps, and allowed, within each tap, to be correlated in time. Here, bounds on the low SNR capacity asymptote are presented and shown to coincide under some conditions.

II. MODEL AND NOTATION

Consider a single-user discrete-time MIMO channel with no channel state information at either the transmitter or receiver. The channel includes additive noise and multiplicative noise (Rayleigh flat fading). Let N_T be the number of transmit antennas and N_R be the number of receive antennas. Let the input at time $n \in \mathbb{N}$ be denoted by $\sqrt{\rho} \underline{Z}_{N_T \times 1}(n)$: so the input on antenna k at time n is $\sqrt{\rho} Z_k(n)$. Here, the signal to noise ratio is represented by $\rho > 0$. Let the corresponding output be denoted by $\underline{Y}_{N_R \times 1}(n)$. Then,

$$Y_l(n) = \sqrt{\rho} \sum_{k=0}^{N_T-1} H_{k,l}(n) Z_k(n) + W_l(n) \quad (1)$$

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where $l \in [0, N_R - 1]$ is the index of the receive antennas. The above sum is over the N_T transmit antennas. The fading H and the additive noise W processes are mutually independent and together independent of the input. The additive noise on each antenna is modeled by an independent and identically distributed (iid) proper complex normal (PCN) process with zero mean and unit variance.

The channel fading processes are assumed to be stationary and ergodic, and jointly PCN. Also, they are assumed to be spatially independent; i.e., if $(k, l) \neq (k', l')$, the fading processes $H_{k,l}$ and $H_{k',l'}$ are mutually independent. Further, for each transmit and receive antenna pair (k, l) , the fading process $H_{k,l}$ is allowed to be correlated in time, with autocorrelation function $R_{k,l}(n)$, defined by $R_{k,l}(n) = E[H_{k,l}(n)H_{k,l}^*(0)]$, and spectral density function $(S_{k,l}(\omega))$.

Also, let $\phi_{k,l}$ be defined by

$$\phi_{k,l} = \sum_{n=1}^{\infty} |R_{k,l}(n)|^2 \quad (2)$$

It is assumed that $\phi_{k,l}$ is finite for all valid k and l . We call a fading process $H_{k,l}$ *ephemeral* if $2\phi_{k,l} \leq R_{k,l}^2(0)$ and *nonephemeral* otherwise.

For a fixed value of $\rho > 0$, let a set of noisy observations of the past be denoted by

$$\mathcal{V}_{k,l}(\rho) = \{\sqrt{\rho}H_{k,l}(m) + W_l(m)\}_{m=-\infty}^{-1} \quad (3)$$

Let $\sigma_{k,l}^2(\rho)$ denote the minimum mean square error (MMSE) for estimation of $H_{k,l}(0)$ based on $\mathcal{V}_{k,l}(\rho)$. The estimation error can be expressed in terms of the spectral density function of the fading process [9, Chapter XII.4, Theorem 4.3] (also see [10]), as follows.

$$\log(\rho\sigma_{k,l}^2(\rho) + 1) = \int_{-\pi}^{\pi} \log(1 + \rho S_{k,l}(\omega)) \frac{d\omega}{2\pi} \quad (4)$$

At low SNR ($\rho \ll 1$), $\sigma_{k,l}^2(\rho)$ can be approximated as

$$\sigma_{k,l}^2(\rho) = R_{k,l}(0) - \rho\phi_{k,l} + o(\rho) \quad (5)$$

A few subclasses of the set of MIMO channels we have described are now defined. A channel is *transmit iid* if the fading statistics are identical across transmit antennas for every receive antenna; i.e., $R_{k,l}(n) = R_l(n)$ for all k and n . In this case, we write ϕ_l instead of $\phi_{k,l}$. A channel is *spatially iid* if the fading statistics are identical across all transmit receive antenna pairs: i.e., $R_{k,l}(n) = R(n)$ for all k, l, n . In this case we write ϕ instead of $\phi_{k,l}$. A channel is *spatially separable* if $R_{k,l}(n) = \alpha_k \beta_l R(n)$ for all k, l , and n , for some autocorrelation function $\{R(n) : n \in \mathbb{N}\}$, and nonnegative constants $\{\alpha_k\}_{k=0}^{N_T-1}$ and $\{\beta_l\}_{l=0}^{N_R-1}$. In this case, $\phi_{k,l} = \alpha_k \beta_l^2 \phi$, where $\phi = \sum_{n=1}^{\infty} R^2(n)$. Finally, a MIMO channel is said to be *nonephemeral* if all of the constituent fading processes $\{H_{k,l}\}$ are nonephemeral.

III. MIMO CHANNELS WITH SUM CONSTRAINTS

This section concerns the case that both peak and average power constraints are placed on sums over antennas. The

instantaneous power transmitted on antenna k at time n is $|Z_k(n)|^2$, and the sum of instantaneous powers is $\|\underline{Z}(n)\|_2^2 = \sum_{k=0}^{N_T-1} |Z_k(n)|^2$. The *sum peak power constraint* is

$$\|\underline{Z}\|_2^2 \leq 1 \quad \forall n \quad (6)$$

and the *sum average power constraint* is

$$E[\|\underline{Z}\|_2^2] \leq \frac{1}{\beta} \quad \forall n, \quad (7)$$

where β is a parameter with $\beta \geq 1$. The parameter β is the ratio of the peak power constraint to the average power constraint. If an input were to meet both the peak and average power constraints with equality, then β would be the peak to average power ratio. Taking $\beta = 1$ reduces to the case that only a peak power constraint is imposed.

Let $C_{mimo-s}(\rho, \beta)$ denote the information theoretic capacity of the MIMO channel under the sum power constraints (6) and (7). Let

$$\mathcal{A}(\beta) = \left\{ (a_0, \dots, a_{N_T-1}) : a_k \geq 0 \quad \forall k, \quad \sum_{k=0}^{N_T-1} a_k \leq \frac{1}{\beta} \right\},$$

and

$$U_{mimo-s}(\rho, \beta) = \max_{\underline{a} \in \mathcal{A}(\beta)} \sum_{l=0}^{N_R-1} \left\{ \log(1 + \rho \sum_{k=0}^{N_T-1} R_{k,l}(0) a_k) - \sum_{k=0}^{N_T-1} a_k \log(1 + \rho \sigma_{k,l}^2(\rho)) \right\}.$$

Proposition 3.1: $C_{mimo-s}(\rho, \beta) \leq U_{mimo-s}(\rho, \beta)$.

Proposition 3.2: For $\beta \geq 1$ fixed,

$$\begin{aligned} \lim_{\rho \rightarrow 0} \frac{C_{mimo-s}(\rho, \beta)}{\rho^2} &= \lim_{\rho \rightarrow 0} \frac{U_{mimo-s}(\rho, \beta)}{\rho^2} \\ &= \frac{1}{2} \max_{\underline{a} \in \mathcal{A}(\beta)} \sum_{l=0}^{N_R-1} \left\{ \sum_{k=0}^{N_T-1} a_k (R_{k,l}^2(0) + 2\phi_{k,l}) - \left(\sum_{k=0}^{N_T-1} a_k R_{k,l}(0) \right)^2 \right\}. \end{aligned}$$

If the channel is transmit iid (so $R_{k,l}(0) = R_l$, $\sigma_{k,l}^2(\rho) = \sigma_l^2(\rho)$, and $\phi_{k,l} = \phi_l$), the expression for $U_{mimo-s}(\rho, \beta)$ simplifies to:

$$\max_{0 \leq a \leq \frac{1}{\beta}} \sum_{l=0}^{N_R-1} \{ \log(1 + \rho a R_l) - a \log(1 + \rho \sigma_l^2(\rho)) \}$$

and the limit in Proposition 3.2 simplifies to

$$\frac{1}{2} \max_{0 \leq a \leq \frac{1}{\beta}} \sum_{l=0}^{N_R-1} \{ a(R_l^2(0) + 2\phi_l) - a^2 R_l(0)^2 \}. \quad (8)$$

IV. MIMO CHANNELS WITH INDIVIDUAL CONSTRAINTS

This section concerns the case that both peak and average power constraints are placed on the individual antennas. The *individual peak power constraints* are

$$|Z_k(n)|^2 \leq 1 \quad \forall k \in [0, N_T - 1] \quad n \in \mathbb{Z}, \quad (9)$$

and the *individual average power constraints* are

$$E[|Z_k(n)|^2] \leq \frac{1}{\beta} \quad \forall k \in [0, N_T - 1] \quad n \in \mathbb{Z}, \quad (10)$$

where β is a parameter with $\beta \geq 1$. Such constraints are seen in practice when each transmit antenna is powered by its own analog driver, so, at any time instant, the available instantaneous power for each antenna is not immediately constrained by the instantaneous powers of the other antennas. Let $C_{mimo-i}(\rho, \beta)$ denote the information theoretic capacity of the channel under the individual power constraints (9) and (10).

First, some notation is introduced and an upper bound on $C_{mimo-i}(\rho, \beta)$ is presented. Let \mathcal{B} denote the unit cube in N_T dimensions; i.e., $\mathcal{B} = \{(b_0, \dots, b_{N_T-1}) : 0 \leq b_k \leq 1 \forall k\}$ and let $\hat{\mathcal{B}}$ denote the set of 2^{N_T} corner points of \mathcal{B} , with elements $\underline{e}_m = (e_{m,0}, \dots, e_{m,N_T-1})$ for $0 \leq m \leq 2^{N_T} - 1$. In addition, let $\mathcal{P}(\beta)$ denote the set of probability distributions \underline{p} on $\hat{\mathcal{B}}$ such that $\sum_m p_m e_{m,k} \leq \frac{1}{\beta}$ for all k , and let Γ be defined by:

$$\Gamma = \{\underline{\gamma} = (\gamma_0 \dots \gamma_{N_T-1})^T : \gamma_k \geq 0 \forall k, \sum_k \gamma_k \leq 1\}.$$

For any $\underline{\gamma} \in \Gamma$, let

$$U_{mimo-i}(\rho, \beta, \underline{\gamma}) \quad (11)$$

$$= \max_{\underline{p} \in \mathcal{P}(\beta)} \sum_{l=0}^{N_R-1} \left\{ \log \left(1 + \rho \sum_{k=0}^{N_T-1} \left(R_{k,l}(0) \sum_{m=0}^{2^{N_T}-1} p_m e_{m,k} \right) \right) - \sum_{m=0}^{2^{N_T}-1} p_m \log \left(1 + \rho \sum_{k=0}^{N_T-1} e_{m,k} \hat{\sigma}_{k,l}^2(\rho) \right) \right\} \quad (13)$$

where

$$\hat{\sigma}_{k,l}^2(\rho) = MMSE \left[H_{k,l}(0) \left| \sqrt{\frac{\rho}{\gamma_k}} H_{k,l}(n) + W_{k,l}''(n) \right\}_{n=-\infty}^{-1} \right] \text{ and}$$

Note that $\hat{\sigma}_{k,l}^2(\rho) = \sigma_{k,l}^2(\rho/\gamma_k)$.

Proposition 4.1: For any $\beta \geq 1$ and $\rho > 0$,

$$C_{mimo-i}(\rho, \beta) \leq \inf_{\underline{\gamma} \in \Gamma} U_{mimo-i}(\rho, \beta, \underline{\gamma}).$$

If the channel is transmit iid the expression for $U_{mimo-i}(\rho, \beta, \underline{\gamma})$ simplifies to an expression involving a maximum over a probability distribution π on $\{0, \dots, N_T\}$, representing the probability distribution of the number of

transmit antennas that are ON, rather than over a probability distribution $p \in \mathcal{P}(\beta)$:

$$U_{mimo-i}(\rho, \beta, \underline{\gamma}) = \max_{\pi : \sum_{i=0}^{N_T} i \pi_i \leq \frac{N_T}{\beta}} \sum_{l=0}^{N_R-1} \left\{ \log \left(1 + \rho R_l \sum_{i=0}^{N_T} i \pi_i \right) - \sum_{i=0}^{N_T} \pi_i \log \left(1 + \rho \sigma_l^2(\rho/\gamma_k) i \right) \right\} \quad (14)$$

In the full version of this paper, an expression is given for $\lim_{\rho \rightarrow 0} U_{mimo-i}/\rho^2$, and a lower bound is given on $\liminf_{\rho \rightarrow 0} C_{mimo-i}/\rho^2$. These expressions are not given here due to space limitation. The upper and lower bounds are not, in general, equal. Nevertheless, we here list several cases in which there is no gap, and the asymptote of the capacity is thus determined.

Corollary 4.1: If the channel is transmit iid, then

$$\lim_{\rho \rightarrow 0} \frac{C_{mimo-i}(\rho, \beta)}{\rho^2} = \frac{N_T^2}{2} \max_{0 \leq a \leq \frac{1}{\beta}} \sum_{l=0}^{N_R-1} \{a(2\phi_l + R_l^2(0)) - a^2 R_l^2(0)\}.$$

When the fading is transmit iid, the input strategy used to obtain the asymptotic lower bound is as follows. All the N_T transmit antennas are used together. With probability a^* , the input signal is nonzero; a complex scalar with amplitude set to the maximum value allowed by the peak constraint is generated, and the same input is transmitted on all the selected antennas. With probability $1 - a^*$, the input signal is zero. Information is transmitted both in amplitude (in the position of the ON and OFF intervals) and in the input phase (when the signal is non-zero).

Corollary 4.2: If the channel is spatially separable,

$$\lim_{\rho \rightarrow 0} \frac{C_{mimo-i}(\rho, \beta)}{\rho^2} = \frac{1}{2} \left(\sum_{k=0}^{N_T-1} \alpha_k \right)^2 \left(\sum_{l=0}^{N_R-1} \beta_l^2 \right) \cdot \max_{0 \leq p \leq \frac{1}{\beta}} \{p(R^2(0) + 2\phi) - p^2 R^2(0)\}$$

Corollary 4.3: If the channel is nonfading and if no average power constraint is imposed (i.e. $\beta = 1$) then, with $\underline{\gamma} = (\frac{1}{N_T}, \dots, \frac{1}{N_T})$,

$$\limsup_{\rho \rightarrow 0} \frac{C_{mimo-i}(\rho)}{\rho^2} \leq \lim_{\rho \rightarrow 0} \frac{U_{mimo-i}(\rho, \underline{\gamma})}{\rho^2} = N_T \sum_{k,l} \phi_{k,l}$$

$$\liminf_{\rho \rightarrow 0} \frac{C_{mimo-i}(\rho)}{\rho^2} \geq \sum_{l=0}^{N_R-1} \sum_{n=1}^{\infty} \left| \sum_{k=0}^{N_T-1} R_{k,l}(n) \right|^2.$$

If the channel is nonfading, the input strategy used to obtain the asymptotic lower bound is to use all the transmit antennas constantly at the peak power, and the same input signal is carried by all the transmit antennas. So all the information is in the input phases.

The final corollary is a special case of either Corollary 4.1 or 4.3:

Corollary 4.4: If the channel is transmit iid and nonfading, and if no average power constraint is imposed (i.e. $\beta = 1$), then the bounds in Corollary 4.3 coincide and

$$\lim_{\rho \rightarrow 0} \frac{C_{mimo-i}(\rho)}{\rho^2} = N_T^2 \sum_{l=0}^{N_R-1} \phi_l.$$

V. DISCUSSION OF MIMO RESULTS

A. On the benefits of more antennas

Consider a nonfading spatially iid channel with N_T transmit and N_R receive antennas in the presence of a peak power constraint, but no average power constraint (i.e. $\beta = 1$). Under the assumptions on the channel, $a = 1$ gives the maximum in (8). Therefore, under a sum peak constraint, the low SNR capacity asymptote is $N_R \phi$. Here, the capacity is independent of the number of transmit antennas, so adding more transmit antennas without an increase in the peak sum power constraint is useless with respect to capacity. The intuitive reason is that any benefit due to diversity brought by multiple transmit antennas is nulled by the cost of tracking the additional fading processes.

If, for the same channel, individual peak constraints are imposed, Corollary 4.4 yields that the capacity asymptote is $N_T^2 N_R \phi$. This additional factor of N_T^2 , as compared to the sum peak constraint case, is explained by the two facts: (i) the average received power is a factor N_T higher under the individual peak constraints (9) than under the sum peak constraint (6), and (ii) the capacity asymptote, obtained by dividing by ρ^2 , scales quadratically with an effective factor change in the SNR, ρ . So, for the peak power constraint adjusted to yield a given maximum *received* power, it makes no difference for this channel whether the peak constraint is applied as a sum peak constraint or as individual peak constraints per antenna.

Having multiple receive antennas is tremendously useful at low SNR, for either peak or sum constraints. The amount of information learned about the input by each receive antenna is so small (at low SNR) that each additional receive antenna gathers information that is almost independent of that gathered by the other antennas. Therefore, as can be seen in the asymptotes, the channel capacity is linear in the number of receive antennas at low SNR.

B. Signaling strategies

In Sections III and IV, specific input distributions are used to construct tight asymptotic lower bounds on the MIMO channel capacities. These distributions yield insight into the structure of input signaling schemes that perform well on the noncoherent MIMO channel at low SNR.

First, the structure of the input magnitude is commented on. Suppose there is only a peak power constraint, and no average power constraint ($\beta = 1$). In the sum peak constraint case, pick a transmit antenna (any one is equally good) and use it at a maximum power. In the individual peak constraint case, all the transmit antennas work at maximum power, and the

phase of the input signal is the same across transmit antennas. This in-phase signaling means that the receiver has to track and estimate only N_R processes, namely $\{\sum_{k=0}^{N_T-1} H_{k,l}(n)\}$ instead of all $N_T N_R$ fading processes. In the sum peak power constraint case, the presence of an average power constraint would imply the following variation in the optimal input strategy. Time is divided into ON and OFF intervals with each interval being of a significant duration. During the ON intervals, at most one antenna is selected – if an antenna is selected, it is used for the entire interval at maximum power. In the OFF intervals, nothing is transmitted. The relative frequency of the ON and OFF time slots is chosen such that the average power constraint is not violated. Information is encoded both into the selection of the sequence of ON and OFF intervals (amplitude modulation) and in the phase of the input during ON intervals.

VI. MULTIPATH CHANNELS

The low SNR regime is largely motivated by broadband channels, where the total transmit power is constrained while the available number of degrees of freedom is virtually unlimited. Since broadband channels are typically channels with multipath fading, the capacity of multipath channels is of interest.

Médard and Gallager [1] (also see [2]) consider a broadband channel with WSSUS fading, and show that the mutual information achievable using spread spectrum signaling dies to zero with increasing bandwidth. There, burstiness is constrained through the use of constraints on fourth moments. Telatar and Tse [11] consider the WSSUS channel with specular multipath, and show that the capacity of the channel in the wideband limit is the same as that of a wideband Gaussian channel with the same average received power. In addition, they show that the mutual information achievable using spread spectrum signals is inversely proportional to the number of resolvable paths. A central theme here is that burstiness in input signaling is necessary to achieve capacity in channels with multipath fading. So the motivation for a peak power constraint at the transmitter applies to multipath channels too.

Consider the following discrete time channel with specular multipath fading:

$$Y(n) = \sqrt{\rho} \sum_{k=0}^{K-1} H_k(n) Z(n-k) + W(n) \quad (15)$$

Here, Z is the input (complex scalar) and Y is the output. The fading is assumed to be independent across the K taps, and correlated in time within each tap. The correlation function for tap k is given by $\{R_k(n) : n \in \mathbb{Z}\}$. The additive noise is modeled by W , an iid PCN process with zero mean and unit variance, and is assumed to be independent of the fading and the input.

Let the input be subject to the following peak and average power constraints: for any $n \in \mathbb{Z}$,

$$|Z(n)| \leq 1 \quad (16)$$

$$E[|Z(n)|^2] \leq \frac{1}{\beta} \quad (17)$$

Let the capacity of the multipath channel with the above power constraints be denoted by $C_{multi}(\rho, \beta)$.

The channel in (15) can be represented in the form of a MISO channel with an additional constraint on the vector input:

$$Y(n) = \sqrt{\rho} \sum_{k=0}^{K-1} H_k(n) Z_k(n) + W(n)$$

Here, the vector input process is constrained to satisfy

$$Z_k(n) := Z(n - k) \quad (18)$$

The power constraints (16) and (17) translate to the following constraints for the MISO channel: for every valid antenna index k and every $n \in \mathbb{Z}$,

$$|Z_k(n)| \leq 1 \quad (19)$$

$$E[|Z_k(n)|^2] \leq \frac{1}{\beta} \quad (20)$$

Clearly, the MISO channel capacity under constraints (18) - (20) is equal to $C_{multi}(\rho, \beta)$. This is no larger than the MISO channel capacity under the power constraints (19) and (20) alone. These power constraints are identical to those discussed in Section IV, for the case of one receive antenna ($N_R = 1$). So the MISO channel capacity under power constraints (19) and (20), denoted by $C_{miso-i}(\rho, \beta)$, is an upper bound on $C_{multi}(\rho, \beta)$. Consequently, for any $\gamma \in \Gamma$, $U_{miso-i}(\rho, \beta, \gamma)$ defined in (14) is an upper bound on $C_{multi}(\rho, \beta)$.

An asymptotic lower bound can be obtained by considering the mutual information rate for an input with the following distribution. Let $a \in [0, \frac{1}{\beta}]$. For $0 \leq n < N$, let

$$Z(n) = I_{\{U \leq a\}} \exp(j\theta(n))$$

Here U is an auxiliary random variable uniformly distributed in $[0, 1]$. The phases $\{\theta(n) : n \in [0, N - 1]\}$ are chosen as follows:

$$\theta(n) = n\vartheta$$

where ϑ is an auxiliary random variable uniformly distributed over the set $\{2\pi \frac{i}{N} : 0 \leq i < N\}$. This input distribution satisfies the power constraints. The signaling scheme can be understood as a form of FSK combined with on-off amplitude modulation. (Alternatively, ϑ could be chosen to be uniformly distributed in $[0, 2\pi]$ – such a choice results in an identical asymptotic lower bound on capacity. Here, the input modulation is similar to a limiting case of FSK, with the number of tones going to infinity.) The asymptotic lower bound corresponding to this choice of input distribution is

$$\liminf_{\rho \rightarrow 0} \frac{C_{multi}(\rho, \beta)}{\rho^2} \geq \max_{0 \leq a \leq \frac{1}{\beta}} L_{multi}(a)$$

where $L_{multi}(a) = \frac{1}{2}[a(R'^2 + 2\phi') - a^2(R'^2)]$,

$$R' = \sum_{k=0}^{K-1} R_k(0), \text{ and } \phi' = \sum_{n=1}^{\infty} \left| \sum_{k=0}^{K-1} R_k(n) \right|^2.$$

The multipath fading channel is called *spatially separable* if $R_k(n) = \alpha_k R(n)$ for every $k \in [0, K - 1]$ and $n \in \mathbb{N}$, for some constants $\alpha_k \geq 0$ and some autocorrelation function $\{R(n) : n \in \mathbb{N}\}$. As usual, the fading is assumed to be independent across taps.

If the mutlipath channel is separable, the corresponding MISO channel is spatially separable. Therefore, Corollary 4.2 implies that $\limsup_{\rho \rightarrow 0} C_{multi}(\rho, \beta)/\rho^2 \leq \xi(\beta)$ where

$$\xi(\beta) = \frac{R^2(0)}{2} \left(\sum_{k=0}^{K-1} \alpha_k \right)^2 \max_{0 \leq p \leq \frac{1}{\beta}} \left\{ \left(1 + \frac{2\phi}{R^2(0)} \right) p - p^2 \right\} \quad (21)$$

Let p'' be the maximizing choice of p in (21). Clearly $p'' \in [0, \frac{1}{\beta}]$. It is easy to verify that $L_{multi}(p'')$ coincides with $\xi(\beta)$ thus proving the following corollary.

Proposition 6.1: For an peak power constraint ρ and peak to average ratio $\beta \geq 1$, the capacity of a spatially separable mutipath fading channel has the following asymptote.

$$\lim_{\rho \rightarrow 0} C_{multi}(\rho, \beta)/\rho^2 = \xi(\beta)$$

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