

Characterization of Pentagons Determined by Two X-rays

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Abstract

This paper contains some results of pentagons which can be determined by two X-rays. The results reveal this problem is more complicated.

1 Introduction

In computerized tomography, the structure of a planar convex body can be determined by certain sets of four X-rays [2]. But no accurate image, in general, can be determined by two X-rays. This "unique problem" can be back to the work of Lorentz [7]. A lot of works continued after that [1] [3] [4] [6] [5], etc. The problem arises that in what conditions or what is the character, the planar convex body can be determined by two X-rays. Giering [5] gave partial results about triangles and quadrilaterals. This paper gives some results of pentagons. In the following, we give the triangle as an example to demonstrate the issue. (For details about the Steiner symmetral of the triangle, see [5]) Figure 1 shows a Steiner symmetral of the triangle $A_2B_1C_3$ in the horizontal direction; also it shows another Steiner symmetral of the triangle in the vertical direction.

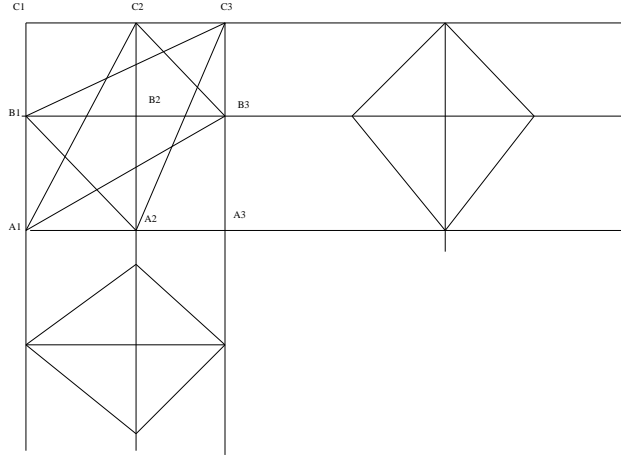


Figure 1: An Example of Steiner Symmetrals of Triangles

In general, the triangle $A_2B_1C_3$ and the triangle $A_1B_3C_2$ should have different Steiner symmetrals. But in some cases, such as, if the chord of the triangle $A_2B_1C_3$ in the direction $B_1B_2B_3$ is equal to the chord of the triangle $A_1B_3C_2$ in the direction $B_1B_2B_3$ and the chord of the triangle $A_2B_1C_3$ in the direction $C_2B_2A_2$ is equal to the chord of the triangle $A_1B_3C_2$ in the direction $C_2B_2A_2$ ("equal chord" condition), then the two different triangles $A_2B_1C_3$ and $A_1B_3C_2$ will have the same Steiner symmetrals; in other words, in this case we cannot determine the two triangles by the Steiner symmetrals.

2 The Case of Pentagons

Details on the classification of pentagon cases are very complicated; it is even more complicated if we mix pentagons with quadrilaterals. Here we only investigate a case of two pentagons, which still reveals very complicated conditions. Figure 2 shows two pentagons: $A_1D_2E_3C_5B_4$ and $B_1C_2E_4D_5A_3$, $A_1A_5E_5E_1$ is a unity square.

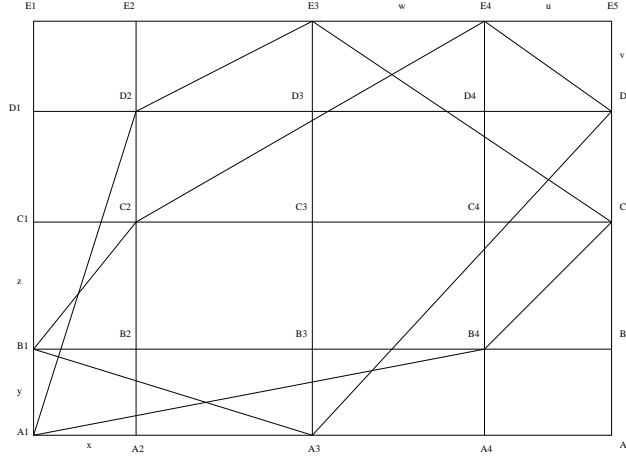


Figure 2: An Example of Two Pentagons

From above "equal chord" condition, we can get the following related equations: from the direction B_1B_5 , we have

$$w - uv - xy - uy - wy = 0,$$

$$1 - v \neq 0$$

from the direction C_1C_5 , we have

$$u + w - x = 0,$$

$$1 - v \neq 0,$$

$$1 - v - y - z \neq 0$$

from the direction D_1D_5 , we have

$$v - 2uv - xv + 2u - 2yu - 2zu - wv + 1 + y + z + x - xy - xz + w - yw - zw = 0,$$

$$1 - y - z \neq 0$$

from the direction E_2A_2 , we have

$$1 - 2u - v + 2uv - 2xy - z + zw + 2zu + u^2 - u^2v + 2xyu - zwu - zu^2$$

$$-w + wu + wv - uvw + xyw = 0,$$

$$1 - u \neq 0,$$

$$1 - w - u \neq 0$$

from the direction E_3A_3 , we have

$$-y + 2yu - yu^2 + xy - xyu - xyw + w - uw - zw + zwu = 0,$$

$$1 - u \neq 0,$$

$$1 - x - u \neq 0$$

from the direction E_4A_4 , we have

$$zw - yu - vw = 0,$$

$$u + w \neq 0$$

After eliminating the variables z and w , we have

$$(y - 3yu + yu^2 - 3xy + 3xyu + u)v + (-y + 2yu + 3xy - 3xyu - 2x^2y^2 - u) = 0,$$

$$(-4x + 7xu + 4x^2 - 2u^2 + u)v^2 + (-10xu - 2x^2 + 4u^2 - 2u - 2xy + 6xyu - 6x^2y)v \\ + (3xu - 2u^2 + 2x^2y + 2x + 2xy + u + 2x^2 - 6xyu - 4x^2y^2) = 0,$$

$$(4x - 8xu - u + 2u^2 + 4xu^2 - u^3)v^2 + (-6x + 12xu + 2u - 4u^2 - 6xu^2 + 2u^3 \\ + 12x^2y - 12x^2yu - 2xyu + 2xyu^2)v \\ + (2x - 4xu - 8x^2y - u + 2u^2 + 2xu^2 + 8x^2yu - u^3 + 2xyu - 2xyu^2 + 4x^3y^2) = 0$$

When we need to reconstruct or determine a pentagon from Steiner symmetrals, we can draw a pentagon as one in the Figure 2 using Steiner symmetrals (chords of a pentagon to be reconstructed can be obtained by translating chords of Steiner symmetrals in the directions). Then we can get the values of the variables u and v , and then we can use the values of u and v and above three related equations to plot a curve of (x, y) . If the point $B_2(x, y)$ (Figure 2) happens to be on the curve we plotted, it means the pentagon we just reconstructed is not unique; there are a group of pentagons having the same Steiner symmetrals.

We can further eliminate the variables v from above three related equations. After eliminating the variables v , we have

$$(-2y^2)u^6 + (5y^2 - 11xy^2 - 6xy^3)u^5 \\ + (-4y^2 - 4xy - 10x^2y^2 - 8x^2y^3 - 2xy^2 + 26xy^3 - 4x^2y^4)u^4 \\ + (y^2 + 26x^2y^2 + 8xy - 13xy^2 - 32xy^3 + 6x^3y^2 + 2x^2y - 26x^3y^3 + 22x^2y^3 - 12x^3y^4 + 24x^2y^4)u^3 \\ + (-16x^4y^3 + 8xy^2 + 14xy^3 + 24x^3y + 36x^4y^2 + 4x^2 - 34x^2y + 4xy - 2x - 94x^3y^2 + 52x^3y^3 \\ + 26x^2y^2 - 22x^2y^3 + 56x^3y^4 - 20x^4y^4 - 44x^2y^4)u^2 \\ + (36x^5y^3 - 8x^5y^4 - 32x^4y^3 - 2xy^3 - 60x^4y^2 - 24x^3y + 122x^3y^2 + 20x^2y - 4xy \\ + 4xy^2 - 18x^3y^3 - 50x^2y^2 + 16x^2y^3 + 64x^4y^4 - 72x^3y^4 + 24x^2y^4)u \\ + (16x^6y^4 - 36x^5y^3 + 60x^4y^3 + 20x^5y^4 + 36x^4y^2 + 16x^2y^2 - 42x^3y^2 - 2xy^2 \\ + 8x^3y^3 - 36x^4y^3 - 4x^2y^3 + 20x^3y^4 - 36x^4y^4 - 4x^2y^4) = 0, \\ (-y)u^7 + (4y + 2xy - 2xy^2)u^6 + (-6y - 6x^2y - 2x - 6xy + 10xy^2 + 6x^2y^2)u^5 \\ + (4y + 6x + 2xy - 16xy^2 + 28x^2y - 26x^2y^2 + 8x^3y^3)u^4 \\ + (-y - 28x^2y - 6x + 8xy + 10xy^2 + 22x^2y^2 - 24x^4y^2 - 4x^2 - 20x^3y + 8x^4y^3 + 52x^3y^2 - 40x^3y^3)u^3 \\ + (4x^2y + 2x - 8xy - 20x^5y^3 - 2xy^2 + 44x^3y - 128x^3y^2 + 72x^4y^2 + 4x^2 + 10x^2y^2 - 16x^4y^3 + 60x^3y^3)u^2 \\ + (2xy + 40x^5y^3 - 72x^4y^2 - 20x^3y - 16x^2y^2 + 88x^3y^2 - 28x^3y^3 + 8x^4y^3 + 2x^2y)u \\ + (4x^3y^3 + 4x^2y^2 + 24x^4y^2 - 20x^3y^2 - 20x^5y^3) = 0$$

We can further reduce above two equations to be one, which is (x, y) curve, using the method of the resultants, but the result is too complicated to be written here. We just show the first term of the equation as follows:

$$16^7x^{42}y^{34} + \dots$$

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