

# Ergodic Capacity of Frequency-Selective Rayleigh Fading Channels with Correlated Scattering

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**Abstract**— In this paper, we study the ergodic capacity of a frequency-selective Rayleigh fading channel with correlated scattering. Under an average power constraint, we consider a single-user, single-antenna transmission. Coherent reception is assumed with full CSI at the receiver and no CSI at the transmitter. We distinct between a continuous- and a discrete-time channel, modeled either as random process or random vector with generic covariance. As a practically relevant example, we examine an exponentially attenuated Ornstein-Uhlenbeck process in detail. Finally, we give numerical results and discuss the relation between the continuous- and the discrete time channel model.

## I. INTRODUCTION

Due to the increasing role of wireless communication it is important to determine the maximum achievable information rates over multipath fading channels. Assuming an ergodic fading process and sufficiently relaxed decoding constraints, such that fluctuations of the channel strength can be averaged out, then the *ergodic capacity* is a suitable performance measure. It basically represents the average over all instantaneous channel capacities [1], [2].

In this paper, we examine the ergodic capacity of a single-user, single-antenna channel with full channel state information (CSI) at the receiver. The assumption of coherent reception is reasonable if the fading is slow in the sense that the receiver is able to track the channel variations. The transmitter has no CSI but knows the statistical properties of the fading. Further, we imply an average power constraint on the channel input and Rayleigh-distributed fading.

Under the above constraints, the ergodic capacity was investigated for the case of flat Rayleigh fading e.g. in [3], [4], [5]. The recent interest in ultra-wideband (UWB) technologies makes it important to examine the capacity also of frequency-selective fading channels. As related information-theoretic work, consider for instance [6], [7], where in [6] a model with two scattered components was studied. In [7] a multi-antenna system was considered, which might be adapted to serve as model for a single-antenna, frequency-selective fading channel. However, in

either case uncorrelated scattering is assumed, which does not necessarily apply to UWB channels [8], [9], [10].

In this paper, we study the ergodic capacity of frequency-selective fading channels with correlated scattering. To the best of our knowledge this has not previously been examined in the literature. We consider models appropriate to characterize small-scale fading effects, i.e. fading due to constructive and destructive interference of multiple signal paths. We assume the small-scale fading to be Rayleigh distributed, which is not common in UWB channel modeling [9]. However, results in [10] support this assumption.

We distinct between a continuous- and a discrete-time channel, where the channel impulse response (CIR) is either modeled as random process or random vector with generic covariance. The former is more general and better suited to analysis whereas the latter is more adequate for computer simulations and parameter estimation from measured data. Note, with the discrete approach we model equidistant samples of the CIR (mathematical path) rather than variable-distant physical path as in [11]. Modeling the sampled impulse response better describes the effective channel and is considered more robust since only aggregate physical effects need to be reflected [10]. This is particularly relevant where frequency-selective propagation phenomena occur.

For the continuous-time Rayleigh fading channel, we examine a detailed example with special covariance function. We utilize an exponentially attenuated Ornstein-Uhlenbeck process being mathematically tractable and capturing the common assumption of exponential power decay [9], [11]. Additionally, it incorporates exponentially correlated scattering as measured in [8].

This paper is organized as follows: Section II specifies the channel models and in Section III we derive respective expressions for the ergodic capacity. In Section IV we analyze the example of an exponentially attenuated Ornstein-Uhlenbeck process, give numerical results, and discuss the relations between continuous- and discrete-time model. Section V finally concludes the paper.

The following notation is used. The operator  $E$  denotes expectation. We define the sets  $\mathbb{W} := (-\frac{W}{2}, \frac{W}{2})$ ,  $W > 0$ , and  $\mathbb{Z}_K := \{0, \dots, K-1\}$ ,  $K \in \mathbb{N}$ . Further, we say a random process is stationary, if it is wide-sense stationary.

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## II. CHANNEL MODELS

As general fading multipath channel model we consider a linear, time-varying system with equivalent lowpass impulse response  $(H_{\tau,t})$  being a complex random process in the time-variable  $t \in \mathbb{R}$  and the delay-parameter  $\tau \in \mathbb{R}$ . Then a realization  $h(\tau, t) := H_{\tau,t}(\omega)$  is the channel response at time  $t$  due to an impulse at time  $t - \tau$  [1], [12]. Next, for fixed  $\tau$  we assume the channel to be invariant within coherence intervals of fixed length. Hence, we may consider the random process  $(H_{\tau,n})$ ,  $\tau \in \mathbb{R}$ , in the discrete time-variable  $n \in \mathbb{Z}$ . Further, we imply  $(H_{\tau,n})$  to be stationary and independent for fixed  $\tau$ , which corresponds to the *block fading* model. As a consequence of this major simplification we are able to drop the time index  $n$  and model the channel as random process  $(H_\tau)$  in the delay-variable  $\tau \in \mathbb{R}$ . Another widely-used assumption of uncorrelated scattering, i.e.  $E[H_\tau \bar{H}_{\tau'}] = 0$  for all  $\tau, \tau' \in \mathbb{R}$  with  $\tau \neq \tau'$ , does not necessarily hold [8], [9], [10]. Therefore, we assume correlated scattering which is a main difference to previous work.

In addition to fading we assume additive white Gaussian noise (AWGN) at the receiver. Below, we intend to distinct between a continuous- and a discrete-time channel. Thus the noise is either modeled as complex white Gaussian process  $(Z_t)$ ,  $t \in \mathbb{R}$ , with i.i.d. real and imaginary part, each of zero mean and power spectral density  $\frac{N_0}{2}$  or as complex white Gaussian process  $(Z_n)$ ,  $n \in \mathbb{Z}$ , with i.i.d. real and imaginary part, each of zero mean and variance  $\frac{N_0}{2}$ .

Next, we will specify stochastic properties of  $(H_\tau)$  to obtain a Rayleigh fading channel model for continuous and discrete time.

### A. Continuous-time Rayleigh Fading Channel Model

Let  $(\tilde{X}_\tau)$ ,  $(\tilde{Y}_\tau)$ ,  $\tau \in \mathbb{R}$ , be real, stationary i.i.d. Gaussian processes with zero mean and covariance function  $\tilde{R}$ . Let  $g$  be an integrable function, i.e.  $g \in L_1(\mathbb{R})$ , defined by

$$g(\tau) := u(\tau)I_{[0,\infty)}(\tau), \forall \tau \in \mathbb{R}, \quad (1)$$

with  $I$  the indicator function and  $u$  some suitable function modeling power decay over  $\tau$ . Then  $(X_\tau) := (\tilde{X}_\tau g(\tau))$ ,  $(Y_\tau) := (\tilde{Y}_\tau g(\tau))$ ,  $\tau \in \mathbb{R}$ , are real, i.i.d. Gaussian processes with zero mean and covariance function

$$R(\tau, \tau') = \tilde{R}(\tau - \tau')g(\tau)g(\tau'), \forall \tau, \tau' \in \mathbb{R}. \quad (2)$$

Thus, the mean energy of  $(X_\tau)$  and  $(Y_\tau)$  is finite and we may set  $c := 2 \int_{-\infty}^{\infty} R(s, s)ds$  for normalization. In fact,  $(X_\tau)$ ,  $(Y_\tau)$  are attenuated, non-stationary versions of  $(\tilde{X}_\tau)$ ,  $(\tilde{Y}_\tau)$  with  $X_\tau = Y_\tau = 0$  for all  $\tau < 0$ . Finally, the continuous-time Rayleigh fading channel model is defined as  $(H_\tau) := (X_\tau + jY_\tau)$ ,  $\tau \in \mathbb{R}$ , with  $j$  the imaginary unit.

Note, by now nothing is said about band- or time-limitation of the channel. Further, any real Gaussian process and its Hilbert transform are i.i.d. processes [13]. This motivates the above equivalent lowpass channel model. Using the composition of a stationary process with a decaying function allows us to take advantage of the theory of

stationary processes while capturing the decaying nature of measured channel impulse responses. The condition  $g \in L_1(\mathbb{R})$  we will motivate later.

### B. Discrete-time Rayleigh Fading Channel Model

For the discrete-time model we assume the channel to be band-limited to  $W$ . Then  $(H_\tau)$  can be sampled at  $\tau = \frac{l}{W}$ ,  $l \in \mathbb{Z}$ , to obtain the complex random process  $(H_l)$  in the discrete delay-variable  $l \in \mathbb{Z}$ . Note, we have infinite expansion in delay due to band-limitation and  $H_l = 0$  for all  $l < 0$  due to  $H_\tau = 0$  for all  $\tau < 0$ . In the following we will refer to  $H_l$  as the  $l$ -th channel tap. Next, we approximate  $(H_l)$ ,  $l \in \mathbb{Z}$ , by a random vector  $H := (H_0, \dots, H_{L-1})$  of size  $L := \lfloor WT_d \rfloor + 1$ . There,  $L$  models the number of significant channel taps with  $T_d$  the channel delay spread. This practically feasible approximation is well-founded since CIRs are sufficiently close to zero for delays  $\tau > T_d$  which is mathematically captured by  $u$  in (1). Note, for flat fading we have  $L = 1$ . Finally, we can denote the  $L$ -dimensional complex random vector  $H$  as  $H = X + jY$ , where  $X, Y$  are real, i.i.d. Gaussian vectors with zero mean and covariance matrix

$$\Gamma := (\gamma_{ik}), \gamma_{ik} := \varrho_{i,k} \sigma_i \sigma_k, i, k \in \mathbb{Z}_L. \quad (3)$$

Therein,  $\sigma_l^2 := E[H_l^2]$  is the mean power of the  $l$ -th channel tap  $H_l$  and  $(\sigma_0^2, \dots, \sigma_{L-1}^2)$  is the mean power delay profile related to  $u$  in (1). The coefficients  $\varrho_{i,k}$  represent the normalized correlation between tap  $H_i$  and  $H_k$ . For normalization we set the mean power of  $H$  to 1, i.e.  $\sum_{l=0}^{L-1} \sigma_l^2 = \frac{1}{2}$ . Subsequently, we refer to the above defined model as the discrete-time Rayleigh fading channel.

## III. CALCULATION OF ERGODIC CAPACITY

We now calculate the ergodic capacity for the defined continuous- and discrete-time Rayleigh fading channel under the general conditions specified in Section I. Unless stated otherwise, capacity expressions are given in [bits/s] for the continuous model and in [bits/s/Hz] for the discrete model.

### A. Capacity formulae

In the continuous case the ergodic capacity within the frequency band  $W$  is calculated by [6], [1]

$$C = E \int_{-W/2}^{W/2} \log_2 (1 + \alpha |\hat{H}_f|^2) df, \quad (4)$$

where  $(\hat{H}_f)$ ,  $f \in \mathbb{R}$ , is the Fourier transform of the process  $(H_\tau)$  and  $\alpha := \frac{P}{N_0 W}$  defines the signal-to-noise ratio (SNR). This expression is valid assuming an information carrying complex envelope input signal band-limited to  $W$  with constant power spectral density  $\frac{P}{W}$  and mean power constraint  $P$ .

For the discrete-time channel model (4) is also applicable if the discrete-time Fourier transform (DTFT), i.e.  $\hat{H}_f := \sum_{l=0}^{L-1} H_l e^{-j2\pi f l/W}$ ,  $f \in W$ , is used to calculate the

spectrum of the channel vector  $H$ . However, with the discrete approach we aim at an numerically easy-to-compute model, preferably discrete in the frequency domain as well. Therefore, we approximate the DTFT by an  $N$ -point DFT, i.e. evaluating the spectrum at  $N$  points. Thus we calculate  $\hat{H}_n := \hat{H}_f|_{f=nW/N}, n \in \mathbb{Z}_N$  and obtain the complex random vector  $\hat{H} := (\hat{H}_0, \dots, \hat{H}_{N-1})$ . This actually means we are dividing the spectrum into  $N$  flat, parallel sub-channels which corresponds to an OFDM-based system approach with  $N$  sub-carriers [2, chap. 5.3.3/5.4.7]. The ergodic capacity is then given by

$$C_N = E \left[ \frac{1}{N} \sum_{n=0}^{N-1} \log_2(1 + \alpha |\hat{H}_n|^2) \right], \quad (5)$$

where  $\alpha$  again means SNR. Considering the parallel channels in frequency, the AWGN process at the receiver translates into a complex zero mean Gaussian random vector with independent real and imaginary parts with  $N$  independent components each having variance  $\frac{N\alpha}{2}$ . The average power constraint of  $P$  on each discrete-time channel input symbol converts to  $NP$  on the set of sub-channels (per OFDM-symbol). Note that  $\frac{WC_N}{N} \rightarrow C$  as  $N \rightarrow \infty$ .

In the following we evaluate (4) and (5).

### B. Continuous-time Rayleigh Fading Channel

*Theorem 1:* The ergodic capacity (4) of the continuous-time Rayleigh fading channel is given by

$$C = \frac{1}{\ln(2)} \int_{-W/2}^{W/2} \exp\left(\frac{1}{2\alpha \hat{\sigma}^2(f)}\right) \text{Ei}_1\left(\frac{1}{2\alpha \hat{\sigma}^2(f)}\right) df, \quad (6)$$

where

$$\hat{\sigma}^2(f) := \int_0^\infty \int_0^\infty R(\tau, \tau') \cos(2\pi(\tau - \tau')f) d\tau d\tau' \quad (7)$$

for all  $f \in \mathbb{R}$  and  $\text{Ei}_m(z) := \int_1^\infty e^{-tz} t^{-m} dt$ . Particularly,  $\text{Ei}_1(z) = -\text{Ei}(-z)$  with Ei the exponential integral [14, 5.1].

*Proof:* Due to limited space, here we just give an outline in terms of main propositions rigorously proofed in [13].

- (i) *Existence of  $(\hat{H}_f)$ :* Let  $(\tilde{H}_\tau) := (\tilde{X}_\tau + j\tilde{Y}_\tau)$  where  $(\tilde{X}_\tau), (\tilde{Y}_\tau)$  are the stationary processes as defined in Section II-A. Then  $(H_\tau) = (\tilde{H}_\tau g(\tau))$  with  $g$  as in (1). If  $g \in L_1(\mathbb{R})$ , then  $(\hat{H}_f) = (\int_{-\infty}^\infty e^{-j2\pi f\tau} H_\tau d\tau)$  exists almost surely. Note, the Fourier transform of the stationary process  $(\tilde{H}_\tau)$  does not exist!
- (ii) *Distribution of  $|\hat{H}_f|^2$ :* Let  $(\hat{X}_f) := (\text{Re}[\hat{H}_f]), (\hat{Y}_f) := (\text{Im}[\hat{H}_f])$ , then  $(\hat{X}_f), (\hat{Y}_f)$  are real, i.i.d. Gaussian processes with zero mean and covariance function  $\hat{R}(f, f') = \int_0^\infty \int_0^\infty R(\tau, \tau') \cos(2\pi(\tau f - \tau' f')) d\tau d\tau'$  for all  $f, f' \in \mathbb{R}$  with  $R$  as in (2). Particularly,  $\hat{X}_f, \hat{Y}_f$  are i.i.d. Gaussian random variables with zero mean and variance  $\hat{\sigma}^2(f) := \hat{R}(f, f)$  for all  $f \in \mathbb{R}$ . (This is already sufficient for (iii).) Thus,  $|\hat{H}_f|^2 = \hat{X}_f^2 + \hat{Y}_f^2$  is an exponentially distributed random variable with parameter  $\frac{1}{2\hat{\sigma}^2(f)}$ .

- (iii) *Calculation of  $C$ :* As  $C < \infty$  we can apply Fubini's theorem to (4) and get  $\int_{-W/2}^{W/2} E[\log_2(1 + \alpha |\hat{H}_f|^2)] df$ . Then we get (6) by computing  $E[\log_2(1 + \alpha |\hat{H}_f|^2)]$  using (ii). ■

### C. Discrete-time Rayleigh Fading Channel

*Theorem 2:* The ergodic capacity (5) of the discrete-time Rayleigh fading channel is given by

$$C_N = \frac{1}{N \ln(2)} \sum_{n=0}^{N-1} \exp\left(\frac{1}{2\alpha \hat{\sigma}_n^2}\right) \text{Ei}_1\left(\frac{1}{2\alpha \hat{\sigma}_n^2}\right) \quad (8)$$

where

$$\hat{\sigma}_n^2 := \sum_{i=0}^{L-1} \sum_{k=0}^{L-1} \gamma_{i,k} \cos(2\pi(i-k)n/N), \forall n \in \mathbb{Z}_N. \quad (9)$$

If the covariance (3) satisfies  $\Gamma = \text{diag}(\sigma_0^2, \dots, \sigma_{L-1}^2)$ , we have uncorrelated channel taps (uncor. scattering) and get

$$C_N = \frac{1}{\ln(2)} \exp\left(\frac{1}{\alpha}\right) \text{Ei}_1\left(\frac{1}{\alpha}\right) =: C_{us}, \quad (10)$$

independent of  $L, N$ , and  $(\sigma_0^2, \dots, \sigma_{L-1}^2)$ . (10) is well known and identical to the capacity of the flat Rayleigh fading case.

*Proof:* Theorem 2 is analogously proved as Theorem 1 but with less effort. The existence of  $\hat{H} := (\hat{H}_0, \dots, \hat{H}_{N-1})$  is evident and exchanging summation and expectation follows directly from the linearity of  $E$ . For the distribution of  $|\hat{H}_n|^2$  we have: If  $\hat{X} := \text{Re}[\hat{H}], \hat{Y} := \text{Im}[\hat{H}]$  (componentwise), then  $\hat{X}, \hat{Y}$  are real, i.i.d. Gaussian vectors with zero mean and covariance matrix  $\hat{\Gamma} := (\hat{\gamma}_{nn'})$ ,  $n, n' \in \mathbb{Z}_N$ , where  $\hat{\gamma}_{nn'} = \sum_{i=0}^{L-1} \sum_{k=0}^{L-1} \gamma_{i,k} \cos(2\pi(in - kn')/N)$  with  $\gamma_{i,k}$  as in (3). Particularly,  $\hat{X}_n, \hat{Y}_n$  are i.i.d. Gaussian random variables with zero mean and variance  $\hat{\sigma}_n^2 := \hat{\gamma}_{nn}$  for all  $n \in \mathbb{Z}_N$ . Thus,  $|\hat{H}_n|^2 = \hat{X}_n^2 + \hat{Y}_n^2$  is an exponentially distributed random variable with parameter  $\frac{1}{2\hat{\sigma}_n^2}$ . The rest of part (iii) applies accordingly.

Finally, if we have uncorrelated scattering, i.e.  $\Gamma$  is diagonal, then  $\hat{\sigma}_n^2 = \sum_{l=0}^{L-1} \sigma_l^2 = \frac{1}{2}$  for all  $n \in \mathbb{Z}_N$  and we get (10) from (8). ■

## IV. EXAMPLE: EXPONENTIALLY ATTENUATED ORNSTEIN-UHLENBECK PROCESS

Now, we consider an example for the continuous-time Rayleigh fading channel for special covariance  $R$  (2). Further, we discuss the relation to the discrete-time model.

### A. Definition

A stationary Ornstein-Uhlenbeck process is a real, Gaussian processes with zero mean and covariance function  $\hat{R}(\tau - \tau') := \frac{d}{2a} e^{-a|\tau - \tau'|}$  for all  $\tau, \tau' \in \mathbb{R}$  with parameters  $a, d > 0$  [15, chap. 3.7.2/3.7.3]. Using the notation of Section II-A we set  $(\tilde{X}_\tau), (\tilde{Y}_\tau)$  to be normalized Ornstein-Uhlenbeck processes with  $d := 2a$  and specify the attenuation function  $u$  in (1) as  $u(\tau) := \sqrt{bce^{-b\tau}}$  for all  $\tau \in \mathbb{R}$  with parameter  $b > 0$  and  $c > 0$  as in Section II-A. We obtain the attenuated Ornstein-Uhlenbeck processes

$(X_\tau) = (\tilde{X}_\tau g(\tau)), (Y_\tau) = (\tilde{Y}_\tau g(\tau)), \tau \in \mathbb{R}$ , representing independent real and imaginary part of  $(H_\tau)$ , each with covariance function

$$R(\tau, \tau') = ce^{-a|\tau-\tau'|}be^{-b(\tau+\tau')}I_{[0,\infty)}(\tau)I_{[0,\infty)}(\tau'). \quad (11)$$

for all  $\tau, \tau' \in \mathbb{R}$ .

### B. Analytical Calculations

The expressions given in this subsection are derived in the Appendix in condensed form. For details and alternative representations of (13) please refer to [13].

Calculating (7) using (11) we get

$$\hat{\sigma}^2(f) = \frac{c(a+b)}{(a+b)^2 + (2\pi f)^2} \quad (12)$$

by elementary integration. One representation of the closed form solution of (6) is given by

$$C = \frac{W}{\ln(2)} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n \frac{1}{k+2} L_k^{\frac{1}{2}}(\beta_1 W^2/4) L_{n-k}^{-\frac{1}{2}}(\beta_1 \beta_2), \quad (13)$$

where  $L_k^\mu$  is the generalized Laguerre polynomial of order  $k$  [16, 8.970],  $\beta_1 := 2\pi^2/(\alpha c(a+b))$ , and  $\beta_2 := (a+b)^2/(4\pi^2)$ .

Upper and lower bounds for (13) are given by

$$C_\theta = W \log_2(1 + 2e^{-\theta} \alpha \hat{\sigma}^2(W/2)) - \frac{4\sqrt{\beta_2}}{\ln(2)} \arctan\left(\frac{W/2}{\sqrt{\beta_2}}\right) + \frac{4\sqrt{\beta_2 + e^{-\theta}/\beta_1}}{\ln(2)} \arctan\left(\frac{W/2}{\sqrt{\beta_2 + e^{-\theta}/\beta_1}}\right), \quad (14)$$

where we have an upper bound for  $\theta = 0$  and a lower bound for  $\theta = \gamma$  with  $\gamma$  the Euler constant.

In case the effect of the channel outside the band  $W$  is negligible, i.e.  $W$  is sufficiently large depending on the parameters  $a, b, c, \alpha$ , then (13) can closely be approximated by integrating (6) with (12) over entire  $\mathbb{R}$  yielding

$$C_\approx = \frac{\pi}{\ln(2)\sqrt{\beta_1}} \exp(\beta_1 \beta_2) \Gamma\left(\frac{1}{2}, \beta_1 \beta_2\right), \quad (15)$$

where  $\Gamma(\mu, z) := \int_z^\infty e^{-t} t^{\mu-1} dt$  is the incomplete gamma function [16, 8.350.2].

Note, (15) is just an approximate expression for proper parameter constellations but not the capacity for infinite bandwidth, i.e. not  $\lim_{W \rightarrow \infty} C$ , since  $\alpha$  and thus  $\beta_1$  depend on  $W$  as well.

### C. Numerical Results and Relation to the Discrete-time Rayleigh Fading Channel

Here, we give numerical examples for the previously derived expressions and show the connection to the discrete-time channel. To do so, we first define two constants for the continuous-time channel. Let  $\hat{\varepsilon} \in (0, 1)$  be the portion of the channel within the frequency band  $W$  in terms of mean energy, i.e.  $\hat{\varepsilon} := \frac{2}{c} \int_{-W/2}^{W/2} \hat{\sigma}^2(f) df$ . If we fix  $\hat{\varepsilon}$ , then we get  $W = \frac{a+b}{\pi} \tan(\frac{\pi}{2}\hat{\varepsilon})$  by simple integration. Further let  $\varepsilon \in (0, 1)$  be the portion of the channel within the time interval  $[0, T_d]$  in terms of mean energy, i.e.  $\varepsilon := \frac{2}{c} \int_0^{T_d} R(s, s) ds$ . If we fix  $\varepsilon$ , then we get  $T_d = \frac{-1}{2b} \ln(1 - \varepsilon)$  by simple integration.

Consider the continuous-time channel within the frequency band  $W$ . To represent this channel by the discrete-time model we sample the covariance function  $R$  of (11) over the range  $[0, T_d] \times [0, T_d]$  by  $\frac{1}{W}$ -spacing to get the covariance matrix  $\Gamma$  of (3). We normalize  $\Gamma$  to have channel vectors with unit mean energy as stated in Section II-B. Additionally, we choose  $\varepsilon$  to be close to 1, i.e. truncation in time is negligible. Note, this discrete version of the channel resembles the behavior of the continuous channel in time domain. However, the spectrum is different due to aliasing, since the Ornstein-Uhlenbeck process is not band-limited because of (12). This, in turn, is negligible if  $\hat{\varepsilon}$  is close to 1. To obtain equality in frequency domain, we have to sample a lowpass filtered version of the continuous channel, which of course has different behavior in time. Since, we aim at equivalence in time, we consider the former approach. Finally, for the discrete and the continuous model to be comparable we have to set  $c := W/\hat{\varepsilon}$ .

As a numerical example, we set  $a := b := \frac{1}{2}$ , and  $\varepsilon := 0.998$  resulting in  $T_d = 6.2146$ , when normalized to [s]. In Example 1 (Fig. 1), we set  $\hat{\varepsilon} := 0.998$  leading to  $W = 101.32$ , when normalized to [Hz],  $c = 101.52$ , and  $L = 630$ . In Example 2 (Fig. 2), we set  $\hat{\varepsilon} := 0.800$  leading to  $W = 0.9796$ ,  $c = 1.2245$ , and  $L = 7$ . Capacity expressions are considered as function of  $\alpha$  and are given in [bits/s/Hz], where normalization to the respective bandwidth  $W$  is performed if required. The values of  $C/W$  are obtained by numerically evaluating (6) and  $C_N$  is computed with  $N = 6300$ . As reference curves the AWGN channel capacity  $C_{awgn} = \log_2(1 + \alpha)$  is plotted. Another reference is the capacity  $C_{us}$  for uncorrelated channel taps (10), which is an upper bound for the correlated case due to Jensen's inequality.

We observe differences between  $C_N$  and  $C/W$  which are due to the applied approximations, i.e. truncation in time, aliasing, discretization of spectrum. They are primarily minor, particularly in Example 2, but increase with  $\alpha$ . The distance to  $C_{us}$  is considerable in Fig. 1 but small in Fig. 2. This depends on the concentration of the spectrum within the considered band, which in turn is controlled by the degree of correlation (parameter  $a$ ) and the delay spread (parameter  $b$ ). The tightness of the bounds is parameter-dependent. Especially the lower bound in Fig. 2 is very tight for  $\alpha > 15$  dB. Finally,  $C$  is well approximated by  $C_\approx$  in Example 1 for  $\alpha < 15$  dB but is useless in Example 2, since  $\hat{\varepsilon}$  is not close enough to 1.

## V. CONCLUSION

In this paper, we determined the ergodic capacity of a frequency-selective Rayleigh fading channel with correlated scattering. We considered a continuous- and a discrete-time channel and examined a detailed example incorporating exponential power decay and exponentially correlated scattering. Analytical, approximate, and bounding expressions as well as numerical results were pre-



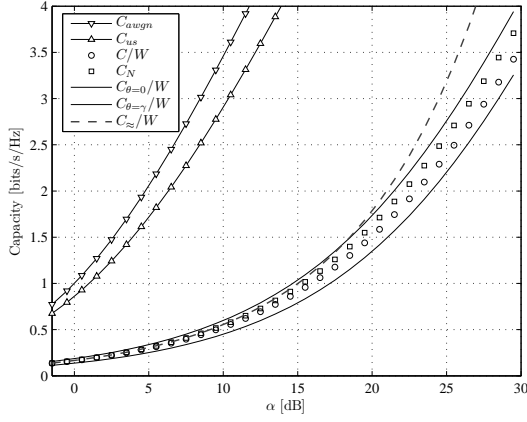


Fig. 1. Ergodic capacity Example 1:  $a = b = \frac{1}{2}$ ,  $\hat{\varepsilon} = 0.998$ ,  $\varepsilon = 0.998$

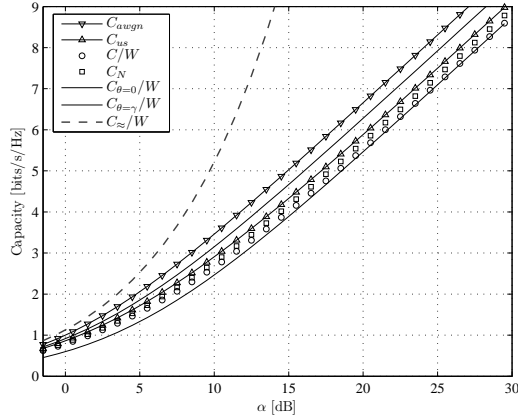


Fig. 2. Ergodic capacity Example 2:  $a = b = \frac{1}{2}$ ,  $\hat{\varepsilon} = 0.998$ ,  $\varepsilon = 0.800$

sented and the relation between continuous and discrete-time models were discussed.

Future work includes for example: considering other stationary processes, assuming non-perfect CSI, further elaborating (13), analyzing the outage capacity, or estimating model parameters from measured data.

## APPENDIX

### A. Derivation of (13):

- (i) Using (12) in (6) and  $\text{Ei}_1(x)e^x = \sum_{n=0}^{\infty} \frac{1}{n+1} L_n(x)$  [17, 5.11.1.4] we get  $C = \frac{1}{\ln(2)} \sum_{n=0}^{\infty} \frac{1}{n+1} I_1$  with  $I_1 := \int_{-W/2}^{W/2} L_n(\beta_1(\beta_2 + f^2)) df$  by Lebesgue's theorem.
- (ii) Using the substitution  $s = \beta_1 f^2$  and  $L_n(x+y) = \sum_{k=0}^n L_k^{-\frac{1}{2}}(x) L_{n-k}^{-\frac{1}{2}}(y)$  [17, 4.4.2.3] then yields  $I_1 = \beta_1^{-\frac{1}{2}} \sum_{k=0}^n L_{n-k}^{-\frac{1}{2}}(\beta_1 \beta_2) I_2$  with the remaining integral  $I_2 := \int_0^{\beta_1(W/2)^2} s^{-\frac{1}{2}} L_k^{-\frac{1}{2}}(s) ds$ .
- (iii) Finally, we evaluate the term  $I_2$  using  $\int_0^t s^\mu L_k^\mu(s) ds = \frac{1}{k+\mu+1} t^{\mu+1} L_k^{\mu+1}(t)$  [17, 1.14.3.4] to get (13).

### B. Derivation of (14):

We simply use the inequalities  $\ln(1 + \frac{e^{-\gamma}}{x}) < \text{Ei}_1(x)e^x$  [18, eq. (13)] and  $\text{Ei}_1(x)e^x < \ln(1 + \frac{1}{x})$  [14, 5.1.20] to replace the integrand. Then elementary integration and the monotony of the integral yields (14).

### C. Derivation of (15):

- (i) Using (12) and infinite integration boundaries in (6) we get  $C \approx \frac{e^{\beta_1 \beta_2}}{\ln(2)\sqrt{\beta_1}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} \text{Ei}_1(s + \beta_1 \beta_2) ds$  by again substituting  $s = \beta_1 f^2$ .
- (ii) With  $\int_0^\infty \frac{e^{-t}}{t^{\mu-\nu}} \text{Ei}_1(t+\nu) dt = \frac{\pi \nu^{(\mu-1)/2} e^{-\nu/2}}{\sin(\mu\pi)} W_{\frac{\mu-1}{2}, \frac{\mu}{2}}(\nu)$  [17, 2.5.3.14] and  $W_{-\frac{1}{4}, \frac{1}{4}}(y) = y^{\frac{1}{4}} e^{\frac{y}{2}} \Gamma(\frac{1}{2}, y)$  [16, 9.236.1, 8.359.3], where  $W_{\kappa, \lambda}$  is the Whittaker's W-function [16, 9.220], we finally get (15).

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