

## **Abstract**

The algorithm checks the propositional formulas for patterns of unsatisfiability.

# A Polynomial Time Algorithm for 3-SAT

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## 1 Introduction

3-SAT (or 3SAT) [1, 2, 3, 4] is a problem to determine whether a given logical formula, written in the conjunctive normal form, is satisfiable:

$$f = c_1 \wedge c_2 \wedge \dots \wedge c_m, \quad (1)$$

- where clauses  $c_k$ ,  $k = 1, 2, \dots, m$ , are disjunctions of three or less literals on set of  $n$  Boolean variables

$$B = \{b_1, b_2, \dots, b_n\}.$$

In other words, given formula (1), it is required to determine whether there exists a truth assignment

$$\tau : B \rightarrow \{false, true\},$$

- which satisfies the formula:

$$f(\tau(B)) = true.$$

3-SAT was among four first NP-complete problems identified [1].

Using reductions [3, 4, or other], an effective solution of the problem can be deduced from the solutions of DHC and TCP described in [5]. But algorithm described in this article seems to be simple. Its computational complexity is  $O(m^3)$ , where  $m$  is the number of clauses. The algorithm uses the self-reducibility property of 3-SAT but in reverse. Instead of “bushing” the tree of possibilities, it “trims” the tree.

Each clause in 3-SAT depends on three or less variables. So, it is fair

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simple to build the truth-tables for each of the  $m$  clauses. The algorithm iterates the tables, filtering them against strings' compatibility with strings in the table for the first clause, the second clause, and so on up to the last clause. Each iteration reduces number of possibilities (does not increase it, at least) in the clauses left. The given 3-SAT instance is satisfiable iff there is a compatible combination of strings in the clauses' truth-tables.

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## 2 Solution

Let's calculate truth table  $T_i$  for each clause  $c_i$ ,  $i = 1, 2, \dots, m$ . Let's enumerate strings in each of the tables. Let  $t_{ij}$  be  $j$ -th string in truth-table  $T_i$ . Two strings  $t_{i_1 j_1}$  and  $t_{i_2 j_2}$  from truth-tables  $T_{i_1}$  and  $T_{i_2}$  appropriately will be called compatible iff

- 1). Values of variables in strings  $t_{i_1 j_1}$  and  $t_{i_2 j_2}$  are compatible. This means that if there are common variables in clauses  $c_{i_1}$  and  $c_{i_2}$ , then the values of these variables in strings  $t_{i_1 j_1}$  and  $t_{i_2 j_2}$  must be the same.
- 2). Both clauses  $c_{i_1}$  and  $c_{i_2}$  are *true* in strings  $t_{i_1 j_1}$  and  $t_{i_2 j_2}$ .

In an obvious way, the notion of compatibility of truth-tables' strings can be extended on 3, 4, ...,  $m$  strings.

Let's rewrite formula (1) in the following form:

$$\begin{aligned}
 (c_1 \wedge c_2) \quad \wedge \quad (c_1 \wedge c_3) \quad \wedge \quad \dots \quad \wedge \quad (c_1 \wedge c_m) \\
 \wedge \quad (c_2 \wedge c_3) \quad \wedge \quad \dots \quad \wedge \quad (c_2 \wedge c_m) \\
 \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\
 \wedge \quad (c_{m-2} \wedge c_{m-1}) \quad \wedge \quad (c_{m-2} \wedge c_m) \\
 \qquad \qquad \qquad \wedge \quad (c_{m-1} \wedge c_m)
 \end{aligned} \tag{2}$$

Let's present each conjunction  $c_i \wedge c_j$ ,  $j > i$ ,  $i = 1, 2, \dots, m - 1$  in (2) with a Boolean matrix  $C_{ij}$  calculated in the following way:

- 1). Size of matrix  $C_{ij}$  is  $l_i \times l_j$ , where  $l_i$  and  $l_j$  are numbers of strings in truth-tables  $T_i$  and  $T_j$  appropriately.
- 2). Elements of matrix  $C_{ij}$  are Boolean *true* or *false*.
- 3). Element  $e_{\alpha\beta}$  of matrix  $C_{ij}$  is truth iff strings  $t_{i\alpha}$  and  $t_{j\beta}$  are compatible.

The size of the matrix is  $8 \times 8$  at most. Let's call  $C_{ij}$  a matrix of compatibility of clauses  $c_i$  and  $c_j$ .

In accordance with (2), let's build the following box-matrix:

$$\begin{array}{cccccc}
C_{12} & C_{13} & \dots & C_{1m-1} & C_{1m} & \\
& C_{23} & \dots & C_{2m-1} & C_{2m} & \\
& & \ddots & & \vdots & \\
& & & C_{m-2m-1} & C_{m-2m} & \\
& & & & C_{m-1m} & 
\end{array} \quad (3)$$

The box-matrix contains at most  $32m(m-1)$  elements, which are sorted over  $m(m-1)/2$  compatibility matrices for clauses of formula (1).

**Theorem 1.** *(The algorithm)*

*The following  $O(m^3)$ -time algorithm decides whether formula (1) is satisfiable.*

**Start:** *Build box-matrix (3). This step takes time  $O(m^2)$ . If there is any matrix, whose elements all equal false, then stop - formula (1) is unsatisfiable.*

**Step k:** *For each  $k = 2, \dots, m-1$ , deplete all matrices  $C_{ij}$ ,  $i \geq k$ , in (3) ridding them of such elements  $e_{\alpha\beta}$ , that  $\alpha$  column of matrix  $C_{k-1,i}$  and  $\beta$  column of matrix  $C_{k-1,j}$  do not have any true element in the same string. This step takes time  $O(m^2)$ . If in the result there is any matrix, whose elements all equal false, then stop - formula (1) is unsatisfiable.*

**Finish:** *Stop - formula (1) is satisfiable. It took time  $O(m^3)$  to reach this point.*

*Proof.* Correctness: there is an element in matrix (3), which survives all iterations, iff there are  $m$  different strings in the clauses' truth-tables, which are compatible.

Time: each of  $m-2$  steps of the algorithm takes time  $O(m^2)$ .  $\square$

Without any change, the algorithm can be applied to SAT per se. The computational complexity of such method will be  $O(2^{k+l}m^3)$ , where  $k$  and  $l$  are numbers of literals in the two longest clauses. In case of 3-SAT, the numbers are less than or equal 3.

Thus, formula (1) is satisfiable iff the algorithm does not create any matrix filled with *false* completely. Let's call such *false*-matrix a pattern of unsatisfiability.

### 3 Formalization

Let  $k$  be an integer between 1 and  $m$ . Let  $C_{ij,0}$ ,  $i > k$ , be the initial compatibility matrix of clauses  $c_i$  and  $c_j$ . Let  $C_{ij,k}$  be that matrix (or, more precisely, what is left of it) after  $k$ -th step of the algorithm. Then, Step  $k$  of the algorithm can be formalized with the following formula

$$C_{ij,k} = (C_{ki,k-1}^T \times C_{kj,k-1}) \wedge C_{ij,k-1}, \quad (4)$$

- where the operations with Boolean matrices are defined in the following way. Let

$$C_{ki,k-1} = (x_{\alpha\beta}), \quad C_{kj,k-1} = (y_{\alpha\gamma}), \quad C_{ij,k-1} = (z_{\beta\gamma}).$$

Then,

$$\begin{aligned} C_{ki,k-1}^T &= (x_{\beta\alpha}); \\ C_{ki,k-1}^T \times C_{kj,k-1} &= \left( \bigvee_{\alpha} x_{\beta\alpha} \wedge y_{\alpha\gamma} \right) = (w_{\beta\gamma}); \\ C_{ij,k} &= (w_{\beta\gamma}) \wedge (z_{\beta\gamma}) = (z_{\beta\gamma} \wedge w_{\beta\gamma}). \end{aligned}$$

The formula (4) shows how the tree of possibilities does collapse.

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