

EVIDENCE THAT $P \neq NP$

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Abstract: The question of whether the class of decision problems that can be solved by deterministic polynomial-time algorithms, P , is equal to the class of decision problems that can be solved by nondeterministic polynomial-time algorithms, NP , has been open since it was first formulated by Cook, Karp, and Levin in 1971. In this paper, we give evidence that they are not equal by examining the SUBSET-SUM problem.

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Let P be the class of decision problems that can be solved by deterministic polynomial-time algorithms and NP be the class of decision problems that can be solved by nondeterministic polynomial-time algorithms. It has been an open question since the early 1970's whether or not $P = NP$. In this note, we give evidence that they are not equal. The proofs in this note are put forward in a purely heuristic spirit and should not be interpreted as rigorous proofs (except for the proof of Lemma 3); however, the author is by no means claiming that a more rigorized version of the argument presented here is impossible.

And the author welcomes and challenges anyone to produce a rigorized version, as he has no plans of even trying, because he is pretty tired of working on this problem and if he had to do it over again would never have even attempted it, not even for the prize of a million dollars offered for solving it - it's just not worth all of the headache... Anyway, let us start out by considering the following commonly known NP problem:

SUBSET-SUM: *Given $n \in \mathcal{N}$, vector $\mathbf{a} \in \mathcal{Z}^n$, and scalar $b \in \mathcal{Z}$ (each represented in binary), determine whether there exists a vector, $\mathbf{x} \in \{0, 1\}^n$, such that $\mathbf{a} \cdot \mathbf{x} = b$.*

Let $S = \{\mathbf{a} \cdot \mathbf{x} : \mathbf{x} \in \{0, 1\}^n\}$, so the SUBSET-SUM problem is to determine whether $b \in S$. And consider the following algorithm, which we shall call algorithm \mathcal{A} , found in a paper by G.J. Woeginger (2003) and brought to the attention of the author by R.B. Lyngsoe, which runs in $O(2^{\frac{n}{2}})$ time (assuming that the algorithm can perform arithmetic in constant-time and that the algorithm can sort in linear-time) and can be described as follows:

Sort sets $b - S^- = b - \{\mathbf{a} \cdot \mathbf{x} \in S : x_{\lfloor \frac{n}{2} \rfloor + 1} = \dots = x_n = 0\}$ and $S^+ = \{\mathbf{a} \cdot \mathbf{x} \in S : x_1 = \dots = x_{\lfloor \frac{n}{2} \rfloor} = 0\}$ in ascending order. Compare the first two elements in each of the lists. If there is a match, then stop and output that there is a solution. If not, then compare the greater element with the next element on the other list. Continue this process until there is a match, in which case there is a solution, or until one of the lists runs out of elements, in which case there is no solution.

We now state and argue two propositions:

Proposition 1: *Algorithm \mathcal{A} has the best running-time (with respect to $n \geq N$ for large N) of all algorithms which solve SUBSET-SUM, assuming that the algorithms can perform arithmetic in constant-time and that the algorithms can sort in linear-time.*

Proof: We use induction on n : We shall leave it to the reader to find a large enough N to verify the basis step. Let us assume true for n and prove true for $n + 1$: When an algorithm solves SUBSET-SUM given input $([a : a_{n+1}], b)$, it is in essence solving two subproblems by determining whether $b - a_{n+1} \in S$ or $b \in S$ (where S is defined as above for problems of size n). Now, if these two subproblems were completely unrelated to one another, then the fastest algorithm that solves SUBSET-SUM for problems of size $n + 1$ would solve both subproblems individually using algorithm \mathcal{A} , by the induction hypothesis; however, the two subproblems both involve set S , so information obtained from solving one subproblem may in fact be used to solve the other subproblem - Notice that if n is odd and solving one of the subproblems takes 6 units of time, 3 units to sort lists S^- and S^+ and 3 units to go through the lists, then it is possible to solve the other subproblem in 3 units of time, since the lists are already sorted from solving the first subproblem. Such a procedure takes a total of 9 units of time, instead of the 12 units of time that it would take to solve both subproblems individually with algorithm \mathcal{A} .

Can we do better? Yes! If n is odd and the algorithm sorts sets $S^- \cup (S^- + a_{n+1})$ and S^+ instead of sorting sets S^- and S^+ , then the algorithm will take 4 units of time to sort the lists and 4 units of time to go through the lists, a total of 8 units of time instead of the 12 units of time that the algorithm would take to solve both subproblems individually with algorithm \mathcal{A} . And when n is even, the running-time of such an improved strategy will not differ from that of the first intelligent strategy that we mentioned, which takes 9 units of time. When this improved strategy is implemented, the algorithm still solves each subproblem in the fastest way possible, by the induction hypothesis, but also computes (what appears to be) the maximum amount of information that can be used to solve both subproblems without slowing down the total running-time. (Lemma 3 may help to clarify this statement.) Therefore, such an algorithm will have the best running-time of all algorithms which solve SUBSET-SUM. Since this procedure is, in fact, descriptive of how algorithm \mathcal{A} works on problems of size $n + 1$, we have evidence that algorithm \mathcal{A} has the best running-time of all algorithms which solve SUBSET-SUM. \square

Proposition 2: *Algorithm \mathcal{A} has the best running-time (with respect to $n \geq N$ for large N) of all algorithms that solve SUBSET-SUM without using any pre-computed information (for example, pre-computed look-up-tables) where the input is of polynomial size, restricted so that each $|a_i| < 2^n$ and $|b| < n \cdot 2^n$.*

Proof: Since S could be of size 2^n , computing the elements of set S by say a dynamic programming strategy (which takes polynomial time only when the magnitudes of each a_i and b are bounded by a polynomial) will not be faster than algorithm \mathcal{A} . Then the same arguments in Proposition 1 which show that \mathcal{A} is the best algorithm to solve the unrestricted SUBSET-SUM problem together with the fact that the algorithm does not use any pre-computed information also apply to the restricted SUBSET-SUM problem where each $|a_i| < 2^n$ and $|b| < n \cdot 2^n$. So since SUBSET-SUM is in NP and \mathcal{A} runs in super-polynomial time with respect to n for input of polynomial size $O(n^2)$, we have evidence that $P \neq NP$. \square

We now present a lemma with a rigorous proof that may help to clarify the preceding arguments.

Lemma 3: The fastest algorithm that solves SUBSET-SUM without using any pre-computed information does so by directly comparing pairs of elements in $S \cup \{b\}$.

Proof: Any strategy for solving SUBSET-SUM without using any pre-computed information must lead to information about specific relationships between elements in S and b . (If anyone does not believe this, they should attempt to prove it by induction on n , as an exercise.) Therefore by generalization, any strategy for solving SUBSET-SUM must lead to information about specific relationships between pairs of elements in $S \cup \{b\}$. The fastest method that is always successful in doing such is to directly compare pairs of elements in $S \cup \{b\}$.

Now, there are other reasonable strategies that do not always lead to information about specific relationships between pairs of elements in $S \cup \{b\}$, for instance, the strategy of determining whether $\gcd(a_1, \dots, a_n)$ divides b and concluding that there is no solution to SUBSET-SUM if not; however, these strategies will increase the worst-case running-time of those algorithms which use them, since whenever they fail to lead to any new information about specific relationships between pairs of elements in $S \cup \{b\}$, they cannot help to determine whether $b \in S$ and the extra computations used in the attempt are wasted. Therefore, algorithms which use such strategies must have a slower running-time than the best algorithm that works by directly comparing pairs of elements in $S \cup \{b\}$. \square

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