

## **Abstract**

The algorithm checks the propositional formulas for patterns of non-satisfiability.

# A Polynomial Time Algorithm for 3-SAT

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## 1 Introduction

3-SAT (or 3SAT) [1, 2, 3, 4] is a problem to determine whether a given logical formula, written in conjunctive normal form, is satisfiable:

$$d_1 \wedge d_2 \wedge \dots \wedge d_m, \quad (1)$$

- where clauses  $d_k$ ,  $k = 1, 2, \dots, m$ , are disjunctions of at most [1] three Boolean variables (or their negations) from a given set of  $n$  Boolean variables

$$B = \{b_1, b_2, \dots, b_n\}. \quad (2)$$

In other words, given the set of Boolean variables (2), the set of literals

$$L = \{b_1, \bar{b}_1, b_2, \bar{b}_2, \dots, b_n, \bar{b}_n\}, \quad (3)$$

and  $m$  normalized clauses  $d_1, d_2, \dots, d_m$  from the set of all 3-disjunctions of literals

$$D = \{\eta \vee \xi \vee \zeta \mid \eta, \xi, \zeta \in L\},$$

it is required to determine whether there is a truth assignment making (1) *true*.

The 3-SAT problem was one of the first four NP-complete problems identified [1]. This article describes an algorithm solving the problem in polynomial time.

Using reductions [3, 4, or other], a polynomial time algorithms for the problem can be derived from the algorithms for DHC and TSP described in [5]. But the algorithm described in this article is ideologically and technically simpler. Although, the base idea of all these algorithms is the same:

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to search the string encoding a problem's instance for a YES/NO pattern. Really, how else a universal Turing machine can “decide” to stop? And any NP-complete problem is a mathematical model of a Turing machine universal for all NP-problems. Such approach directs the quest for an efficient solution of a NP-complete problem to the search for such mathematical model of the problem, which would produce a polynomial number of YES/NO patterns. Then, a polynomial time algorithm could be realized as the brute-force search through all those patterns. Author hopes that the next section will prove that 3-SAT is one of such models.

## 2 Solution of 3-SAT

**Theorem 1.** (*Patterns of non-satisfiability*).

Let (1) - (2) be the given instance of 3-SAT. Let  $C$  be the set of clauses of the instance:

$$C = \{d_1, d_2, \dots, d_m\}.$$

The instance is non-satisfiable iff at least one of the following is true:

**Pattern 1.** There is  $\alpha \in B$ :

$$\{\alpha, \bar{\alpha}\} \subseteq C;$$

**Pattern 2.** There are different  $\alpha, \beta \in B$ :

$$\{\alpha \vee \beta, \alpha \vee \bar{\beta}, \bar{\alpha} \vee \beta, \bar{\alpha} \vee \bar{\beta}\} \subseteq C;$$

**Pattern 3.** There are different  $\alpha, \beta, \gamma \in B$ :

$$\{\alpha \vee \beta \vee \gamma, \alpha \vee \beta \vee \bar{\gamma}, \alpha \vee \bar{\beta} \vee \gamma, \alpha \vee \bar{\beta} \vee \bar{\gamma}, \\ \bar{\alpha} \vee \beta \vee \gamma, \bar{\alpha} \vee \beta \vee \bar{\gamma}, \bar{\alpha} \vee \bar{\beta} \vee \gamma, \bar{\alpha} \vee \bar{\beta} \vee \bar{\gamma}\} \subseteq C.$$

*Proof.* Let's use distributive laws and rewrite (1) in disjunctive normal form:

$$c_1 \vee c_2 \vee \dots \vee c_k, \tag{4}$$

- where all  $c_i$ ,  $i = 1, 2, \dots, k$ , are conjunctions of appropriate literals from (3). Formula (1) is non-satisfiable iff formula (4) is non-satisfiable. And the latter is non-satisfiable iff each of the conjunctions  $c_i$ ,  $i = 1, 2, \dots, k$ , contains at least one couple of complimentary literals:

$$c_i = \dots \wedge \delta_i \wedge \dots \wedge \bar{\delta}_i \wedge \dots$$

Thus, formula (1) is non-satisfiable iff it contains the conjunctive normal form of the following formula:

$$\delta_1 \wedge \bar{\delta}_1 \vee \delta_2 \wedge \bar{\delta}_2 \vee \dots \vee \delta_k \wedge \bar{\delta}_k. \quad (5)$$

Since the length of clauses in (1) is limited by 3 (it is 3-SAT), formula (5) contains three different variables at most.  $\square$

A 3-SAT instance with  $n$  variables produces  $C_n^1 + C_n^2 + C_n^3 = O(n^3)$  different patterns of non-satisfiability. Thus, the following algorithm solves 3-SAT in  $O(n^3)$ -time:

**Algorithm 1.** In a sequential order, generate patterns of non-satisfiability and check them on inclusion in the set of clauses of the given instance. If a pattern enters in the set, then stop - the 3-SAT instance is non-satisfiable. If all patterns were checked, then the instance is satisfiable.

The following  $O(n^4)$ -time algorithm finds a truth assignment satisfying a satisfiable 3-SAT instance:

**Algorithm 2.** In a sequential order, for each variable assign value *true* or *false* and check the resulting instance of 3-SAT for satisfiability. Fix that value, which produces a satisfiable instance.

## References

- [1] Stephen Cook. The complexity of theorem-proving procedures. In Conference Record of Third Annual ACM Symposium on Theory of Computing. p.151-158, 1971
- [2] Stephen Cook. The P versus NP problem. [http://www.claymath.org/millennium/P\\_vs\\_NP/pvsnp.pdf](http://www.claymath.org/millennium/P_vs_NP/pvsnp.pdf)
- [3] Richard M. Karp. Reducibility Among Combinatorial Problems. In Complexity of Computer Computations, Proc. Sympos. IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y. New York: Plenum, p.85-103, 1972.
- [4] M.R. Garey and D.S. Johnson. Computers and Intractability, a Guide to the Theory of NP-Completeness. W.H. Freeman and Co. San Francisco, 1979.
- [5] Sergey Gubin. A Polynomial Time Algorithm for The Traveling Salesman Problem. [http://arxiv.org/PS\\_cache/cs/pdf/0610/0610042.pdf](http://arxiv.org/PS_cache/cs/pdf/0610/0610042.pdf)