

A POLYNOMIAL TIME ALGORITHM FOR 3-SAT

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ABSTRACT. Article describes a class of efficient algorithms for 3SAT and their generalizations on SAT.

INTRODUCTION

3SAT is a decision problem whether a given conjunctive normal form (CNF) with clauses of length three or less is satisfiable. Cook's famous theorem states that 3SAT is a NP-complete problem [1]. One might say that 3SAT is the most beautiful NP-complete problem.

This article describes a matrix encoding for 3SAT instances. The encoding allows the efficient algorithms for 3SAT. That works for conjunctive forms (CF) with clauses of length three or less. In other words, the algorithms detect in polynomial time whether the following system of Boolean equations has a solution:

$$(0.1) \quad c_i = \text{true}, \quad i = 1, 2, \dots, m$$

- where c_i are disjunctions¹ of three or less literals on n Boolean variables,

$$c_i \in \{\alpha \vee \beta \vee \gamma, \alpha \vee \beta, \alpha \mid \alpha, \beta, \gamma \in \{b_j, \neg b_j \mid j = 1, 2, \dots, n\}\},$$

- where b_j are Boolean variables. The algorithms test whether there exists such a true-assignment

$$b_j = \tau_j \in \{\text{true}, \text{false}\}$$

which could satisfy Boolean formula

$$(0.2) \quad f = c_1 \wedge c_2 \wedge \dots \wedge c_m.$$

Regarding this formula, disjunctions c_i are clauses.

Also, we discuss applicability of our method to SAT (clauses may have length bigger than three).

Author would like to express his gratitude to readers and authors [2, and others] for their attention to the method. Author hopes that this version of his article will better explain the compatibility matrix method and the related algorithms.

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¹ c_i can be other Boolean formulas, as well.

1. COMPATIBILITY MATRIX

Let T_i be the truth-table for clause c_i , $i = 1, 2, \dots, m$. Let's arbitrary enumerate the true-assignments for the arguments of c_i and let's write T_i in the following form:

T_i	True-assignments	c_i
1	1-st true-assignment	Value of c_i on the 1-st true-assignment
2	2-nd true-assignment	Value of c_i on the 2-nd true-assignment
\vdots	\vdots	\vdots

Let $S_{i\mu}$ be μ -th string from truth-table T_i . By definition, any two strings $S_{i\mu}$ and $S_{j\nu}$ are compatible if the following two conditions are satisfied:

Condition 1: The μ -th value of c_i is *true* and the ν -th value of c_j is *true*;

Condition 2: The μ -th true-assignment in table T_i and the ν -th true-assignment in table T_j do not contradict each other. In other words, if clauses c_i and c_j share a variable, then this variable has to have the same true-assignment in both strings $S_{i\mu}$ and $S_{j\nu}$.

When at least one of these conditions is not satisfied, then the strings are incompatible. Let us emphasize, i and j may be equal in this definition, i.e. strings $S_{i\mu}$ and $S_{j\nu}$ may be from the same truth-table.

Now, let's build a compatibility box for each of the clause couples (c_i, c_j) , $i, j = 1, 2, \dots, m$. Let k_i and k_j be the lengths of clauses c_i and c_j , appropriately. Then, the compatibility box for clauses c_i and c_j is a Boolean matrix² $C_{ij} = (x_{\mu\nu})_{2^{k_i} \times 2^{k_j}}$ with the following elements:

$$x_{\mu\nu} = \begin{cases} \text{true}, & \text{Strings } S_{i\mu} \text{ and } S_{j\nu} \text{ are compatible} \\ \text{false}, & \text{Otherwise} \end{cases}$$

Having all m^2 compatibility boxes built, we can aggregate them in a box matrix C :

$$C = (C_{ij})_{i,j=1,2,\dots,m}.$$

Matrix C is a Boolean box matrix. For 3SAT, size of C is

$$\left(\sum_{i=1}^m 2^{k_i}\right) \times \left(\sum_{i=1}^m 2^{k_i}\right) \leq (8m) \times (8m) = O(m \times m),$$

- where k_i is length of clause c_i . We call matrix C a compatibility matrix for system 0.1 or formula 0.2.

Let us notice that the compatibility matrix is a symmetric matrix:

$$(1.1) \quad C = C^T, \quad C_{ij} = C_{ji}^T.$$

Also, let us notice that, due to Condition 2 for the string compatibility, diagonal boxes C_{ii} are diagonal matrices, i.e. all non-diagonal elements in these boxes are *false*:

$$(1.2) \quad C_{ii} = \begin{pmatrix} t_1 & \text{false} & \dots \\ \text{false} & t_2 & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}, \quad t_\mu \in \{\text{true}, \text{false}\}$$

As usual, we can use a $(0, 1)$ -version of the compatibility matrix. In the version,

²Boolean matrix is a matrix whose elements are *true* or *false*.

values *true* in matrix C are replaced with 1, and values *false* are replaced with 0.

One might say that matrix C encodes all information contained in system 0.1 or formula 0.2. Then, our method will spend this information in order to obtain just one bit - decision “YES” or “NO” about satisfiability of formula 0.2.

2. SOLUTION GRID AND DEPLETION

Any true-assignment for the arguments of system 0.1 or variables in formula 0.2 contains/consists of the true-assignments for clauses c_i . So, any true-assignment produces in the compatibility matrix a grid of elements, one element per compatibility box:

$$\gamma = \{x_{\mu\nu ij} \mid \mu = \mu(i), \nu = \nu(j), i, j = 1, 2, \dots, m\},$$

- where $x_{\mu\nu ij}$ is the $(\mu\nu)$ -element of compatibility box C_{ij} , and γ is a grid of such elements in compatibility matrix C , one element per compatibility box.

Based on the grid presentation, we can state that any true-assignment satisfying formula 0.2 is presented in compatibility matrix C with a grid of elements, one element per compatibility box, which all are *true*. And visa versa, if there is such a grid of elements in the compatibility matrix, one element per compatibility box, whose all elements are *true*, then formula 0.2 is satisfiable. We call such a grid which consists of all *true*-elements³, one element per compatibility box, a solution grid.

Based on property 1.2 of the compatibility matrix, we even could remove that requirement “one element per box” from the solution grid definition.

Theorem 2.1. *Formula 0.2 is satisfiable iff compatibility matrix for that formula contains a solution grid.*

Our method is a detection of those elements of the compatibility matrix which do not belong to any solution grid and the assigning to them value *false* or 0 depending on the presentation of the compatibility matrix. We call the assignment operation a depletion of the compatibility matrix.

Theorem 2.2. *Suppose, there is a filter which finds in the compatibility matrix for formula 0.2 all such “true”-elements which do not belong to any solution grid. Suppose, all these elements were replaced with value “false”. Then, formula 0.2 is satisfiable iff there is not any compatibility box filled with “false” entirely.*

In fact, for unsatisfiable formula 0.2 the resulting compatibility matrix emerging from the depletion will be filled with *false* entirely.

A compatibility box entirely filled with *false* we call a pattern of unsatisfiability. One might say that our method is a search of the compatibility matrix for the pattern. For the method, it does not matter what particular filter is used for the depletion. For the filter, the only objective is to preserve at least one solution grid when the grids exist. Then, no information will be lost in the sense of computational complexity.

The P vs NP problem itself may be seen as a problem of the existence of an efficient filter for any 3SAT instance. And there is no need for those filters to be universal. Some 3SAT instances can be solved with one filter, and others - with another. That corresponds to a polymodal algorithm. Any universal filter will

³*True*-element is an element which is equal *true*, and *false*-element is an element which is equal *false*.

create a unimodal algorithm.

All possible depletion filters can be classified as external or internal. Loosely, the external filters use along with the compatibility matrix some additional information. The internal filters are restricted to that information encoded in the compatibility matrix alone. Let's see some examples.

An external filter: In the $(0, 1)$ -version of the compatibility matrix, deplete that matrix by comparing its powers with the powers of matrix $(1)_{m \times m}$. Actually, that will be sufficient to iteratively compare only the squares of those matrices. But, that is a topic for a special discussion. The algorithm is based on the following theorem.

Theorem 2.3. *Let C be the $(0, 1)$ -version of the compatibility matrix for formula 0.2. Let G be a digraph defined with adjacency matrix C . Then, formula 0.2 is satisfiable iff G has a subgraph with adjacency matrix $(1)_{m \times m}$.*

An internal filter: Our filter described in the previous version of this article is an internal filter. A good explanation of that filter can be found in [2]. This filter preserves all solution grids and has computational complexity $O(m^3)$. There is a simple counter example on the filter in [2]. As described in [2], the filter uses compatibility boxes C_{ij} and C_{ik} to deplete compatibility box C_{jk} . In the same way, boxes C_{ik} and C_{jk} can be used to deplete box C_{ij} , and boxes C_{jk} and C_{ij} - to deplete box C_{ik} . Empowered in such a way, the filter successfully resolved all counter examples known to author. The know-how missed in [2] will be formalized at the end of this article.

Below, we describe another efficient internal filter which is ideologically simpler. We call it a basic algorithm.

3. A BASIC ALGORITHM

For system 0.1 or formula 0.2, the basic algorithm consists of the following steps:

Init: Build the compatibility matrix for formula 0.2.

Let $s = 1$. Let C_s be the compatibility matrix. Let C_{sij} be the compatibility boxes, where $i, j = 1, 2, \dots, m$.

Depletion: Let $s = s + 1$. Calculate compatibility matrix C_s as follows⁴:

$$(3.1) \quad C_{sij} = \bigwedge_{k=1}^m C_{s-1,ik} C_{s-1,kj},$$

- where C_{sij} is the (i, j) -th compatibility box of C_s , $i, j = 1, 2, \dots, m$.

Let $x_{s\mu\nu ij}$ be the $(\mu\nu)$ -th element of compatibility box C_{sij} . Then, due to formula 3.1, the element is

$$x_{s\mu\nu ij} = \bigwedge_{k=1}^m \left(\bigvee_{1 \leq \alpha \leq 2^3} x_{s-1, \mu\alpha ik} \wedge x_{s-1, \alpha\nu kj} \right).$$

⁴Here, product of two Boolean matrices $A = (a_{\mu\nu})$ and $B = (b_{\mu\nu})$ of the appropriate sizes (the number of columns in A has to be equal to the number of rows in B) is the following Boolean matrix:

$$AB = \left(\bigvee_k a_{ik} \wedge b_{kj} \right).$$

Conjunction of Boolean matrices of the same size is the matrix of conjunctions of the appropriate elements of the matrices.

Due to property 1.1 of the compatibility matrix, the last expression can be rewritten in any of the following ways:

$$(3.2) \quad x_{s\mu\nu ij} = \bigwedge_k \left(\bigvee_{\alpha} x_{s-1, \alpha\mu ki} \wedge x_{s-1, \alpha\nu kj} \right) = \bigwedge_k \left(\bigvee_{\alpha} x_{s-1, \mu\alpha ik} \wedge x_{s-1, \nu\alpha jk} \right).$$

Iterations: If $C_s \neq C_{s-1}$, then go to the previous step. Otherwise, continue.

Decision: If C_s is entirely filled with *false*, then formula 0.2 is unsatisfiable - decision “NO”. Otherwise, formula 0.2 is satisfiable - decision “YES”.

Obviously, when during an iteration the pattern of unsatisfiability arises (value *false* fills a compatibility box entirely), then the algorithm can be stopped with decision “NO”. Because, formula 3.1 will propagate this value *false* all over the compatibility matrix during the next two iterations, at most.

For 3SAT, the computational complexity of the algorithm can be estimated as $O(m^5)$: there is $O(m^2)$ iterations (each iteration eliminates at least one element from the compatibility matrix); there is m^2 depletions on each iteration; each depletion takes time $O(m)$. But, let us notice the room for improvement of this algorithm’s performance.

The considerations underlying this basic algorithm are clear, but they are really irrelevant. All we need is to check whether the algorithm preserves solution grids in the case of satisfiable formula 0.2, and whether the algorithm produces the pattern of unsatisfiability in the case of unsatisfiable formula 0.2.

Suppose, formula 0.2 is satisfiable. Then, there is a solution grid in the compatibility matrix of that formula. Let elements $\{x_{\mu\nu ij}\}$ of the matrix constitute the solution grid, where element $x_{\mu\nu ij}$ is the $(\mu\nu)$ -th element from compatibility box C_{ij} . Then, formula 3.2 implies

$$\forall s (x_{s\mu\nu ij} \equiv \text{true}).$$

That can be seen with the mathematical induction over s .

Thus, the algorithm preserves all solution grids. Let us support the second part of the algorithm’s correctness with the following considerations.

Let’s assume that the algorithm works correct for all 3SAT instances with the number of clauses less than m . Now, suppose that there is the following unsatisfiable 3SAT instance with m clauses:

$$\psi = c_1 \wedge c_2 \wedge c_3 \wedge \dots \wedge c_{m-1} \wedge c_m.$$

Suppose, the algorithm did not produce the pattern of unsatisfiability for the instance. Then, due to our assumption and formula 3.1, the following three 3SAT instances are satisfiable:

$$\begin{aligned} \psi_1 &= c_1 \wedge c_2 \wedge c_3 \wedge \dots \wedge c_{m-1} \\ \psi_2 &= c_2 \wedge c_3 \wedge \dots \wedge c_{m-1} \\ \psi_3 &= c_2 \wedge c_3 \wedge \dots \wedge c_{m-1} \wedge c_m \end{aligned}$$

Let us notice that the compatibility matrices for ψ_1 and ψ_3 overlap over the compatibility matrix for ψ_2 . Then, due to our assumption that the algorithm eliminates all *true*-elements which do not belong to any solution grid, all true-assignments satisfying ψ_1 and all true-assignments satisfying ψ_3 have to produce such solution grids in the compatibility matrix for ψ , which coincide on the compatibility matrix for ψ_2 . That creates in the compatibility matrix for ψ a grid of *true*-elements except two boxes are not involved - compatibility boxes $C_{1,m}$ and $C_{m,1}$, $C_{m,1} = C_{1,m}^T$. Let’s continue that grid in box $C_{1,m}$. If there is an intersection of the lines of that grid

in box $C_{1,m}$, then there is a solution grid for ψ - contradiction with our assumption that ψ is unsatisfiable. But, if there is not any intersections of the lines in box $C_{1,m}$, then compatibility box C_{1m} is filled with *false* entirely - contradiction with our assumption that the algorithm did not produce the pattern of unsatisfiability.

Thus, to prove the second part of our algorithm's correctness we could select any m and test whether the algorithm finds all unsatisfiable 3SAT instances with the number of clauses less than m . The smallest unsatisfiable 3SAT instance known to author (the real 3SAT) is the counter example from [2]. That is the following formula:

$$\omega_3 = (P \vee Q \vee R) \wedge \bar{P} \wedge \bar{Q} \wedge \bar{R}.$$

Let's see how our algorithm works for this 3SAT instance.

Clauses:

$$c_1 = P \vee Q \vee R, \quad c_2 = \bar{P}, \quad c_3 = \bar{Q}, \quad c_4 = \bar{R}.$$

Init. Truth tables for the clauses:

T_1	P	Q	R	c_1		T_2	P	c_2		T_3	Q	c_3		T_4	R	c_4
1	0	0	0	0												
2	0	0	1	1												
3	0	1	0	1		1	0	1		1	0	1		1	0	1
4	1	0	0	1		2	1	0		2	1	0		2	1	0
5	0	1	1	1												
6	1	0	1	1												
7	1	1	0	1												
8	1	1	1	1												

Compatibility matrix:

$$C = \left(\begin{array}{cccccccc|cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right),$$

- where the matrix is divided on the compatibility boxes $C_{1,1}$, $C_{1,2}$, $C_{1,3}$, $C_{1,4}$, $C_{2,1}$, $C_{2,2}$, $C_{2,3}$, $C_{2,4}$, $C_{3,1}$, $C_{3,2}$, $C_{3,3}$, $C_{3,4}$, $C_{4,1}$, $C_{4,2}$, $C_{4,3}$, and $C_{4,4}$, appropriately. The first depletion:

$$C_{1,1} = C_{1,1}C_{1,1} \wedge C_{1,2}C_{2,1} \wedge C_{1,3}C_{3,1} \wedge C_{1,4}C_{4,1}.$$

Let's calculate:

$$C_{1,1}C_{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C_{1,2}C_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_{1,3}C_{3,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_{1,4}C_{4,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The new value of C_{11} is the conjunction of these four Boolean matrices:

$$C_{1,1} = (0)_{8 \times 8}.$$

Thus, our algorithm produced the pattern of unsatisfiability. Thus, our algorithm correctly detected the unsatisfiability of formula ω_3 .

Before go further, let us notice the following. Due to Condition 1 of the strings compatibility and the basic algorithm, the true-assignments, which deliver to their clauses value *false*, can be missed when building the compatibility matrix. Because, they will produce in their compatibility boxes the rows and columns entirely filled with *false*. So, the compatibility matrix can be built solely on the contradictions between the satisfying true-assignments for different clauses.

4. LEXICOGRAPHICAL VERSION OF THE METHOD

Let's transform formula 0.2 in a disjunctive form (DF) by the opening of its parentheses. Obviously, formula 0.2 is satisfiable iff there is a multiplicand in the resulting DF. That gives an idea to build the formula's compatibility matrix based on the complementarity of literals from different clauses. We call that a lexicographical version of the compatibility matrix method. The above theorems 2.1 and 2.2 will work for such built compatibility matrix, as well.

The lexicographical version of compatibility matrix is built as follows. We arbitrarily enumerate literals in each clause, and then we build the compatibility boxes C_{ij} for each couple of clauses (c_i, c_j) , $i, j = 1, 2, \dots, m$; $i \neq j$. Element $x_{\mu\nu ij}$ in compatibility box C_{ij} is *false* when the μ -th literal in clause c_i and the ν -th literal in clause c_j are complementary; otherwise, the element is *true*. Diagonal boxes are the "identity matrices" of the appropriate sizes. The compatibility matrix aggregates the compatibility boxes in accordance with their indexes.

The space savings allow the straightforward generalization of the compatibility matrix method on SAT. For a SAT instance with m clauses of length l or less, the lexicographically built compatibility matrix has size $O(lm \times lm)$. And, for example, the basic algorithm described in the previous section has computational complexity $O(l^5 m^5)$: there will be $O(l^2 m^2)$ iterations (each iteration will eliminate at least one element of the compatibility matrix); each iteration will deplete m^2 compatibility boxes; each depletion will take time $O(l^3 m)$.

Let us illustrate the lexicographical method on the following formula:

$$\omega_l = (P_1 \vee P_2 \vee \dots \vee P_l) \wedge \bar{P}_1 \wedge \bar{P}_2 \wedge \dots \wedge \bar{P}_l.$$

Compatibility boxes:

$$\begin{aligned} C_{1,1} &= \text{diag}(\underbrace{1, 1, \dots, 1}_l), \\ C_{j,1} &= C_{1,j}^T = (\underbrace{1, \dots, 1}_{j-2}, 0, \underbrace{1, \dots, 1}_{l+1-j})_{1 \times l}, \quad j = 2, \dots, l+1 \\ C_{ij} &= (1)_{1 \times 1}, \quad 1 < i, j \leq l+1. \end{aligned}$$

Let's use the basic algorithm from the previous section. The first depletion:

$$C_{1,1} = C_{1,1} C_{1,1} \wedge C_{1,2} C_{2,1} \wedge \dots \wedge C_{1,l+1} C_{l+1,1}.$$

Let's calculate:

$$\begin{aligned} C_{1,1} C_{1,1} &= \text{diag}(\underbrace{1, 1, \dots, 1}_l), \\ \text{diag} C_{1,j} C_{j,1} &= (\underbrace{1, \dots, 1}_{j-2}, 0, \underbrace{1, \dots, 1}_{l+1-j}), \quad j = 2, \dots, l+1 \end{aligned}$$

The new value of compatibility box $C_{1,1}$ is the conjunction of these $l+1$ Boolean matrices. Thus,

$$C_{1,1} = (0)_{l \times l}.$$

Our algorithm produced the pattern of unsatisfiability. Thus, formula ω_l is unsatisfiable.

Our depletion filter discussed in [2] is restricted to those compatibility boxes C_{ij} for which

$$1 < i < j \leq m.$$

These boxes are located in the upper triangular part of the compatibility matrix. The depletion may be expressed with the following formula:

$$C_{jk} = C_{jk} \wedge C_{ij}^T C_{ik}.$$

As mentioned the above, the know-how is to add, for example, the following depletions:

$$C_{ij} = C_{ij} \wedge C_{ik} C_{jk}^T, \quad C_{ik} = C_{ik} \wedge C_{ij} C_{jk}.$$

Let's see how it works for formula ω_l . For example,

$$C_{1,2} = C_{1,2} \wedge C_{1,3} C_{2,3}^T = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \end{pmatrix}_{l \times 1} \wedge \begin{pmatrix} 1 \\ 0 \\ 1 \\ \vdots \end{pmatrix}_{l \times 1} (1)_{1 \times 1} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}_{l \times 1},$$

- etc. So, eventually (in cubic time) the algorithm will produce the pattern of unsatisfiability for formula ω_l .

5. CONCLUSION

In this article, we described on high-level several polynomial time algorithms for 3SAT and SAT. The designs are based on the compatibility matrix method. The base of the method is a matrix encoding of SAT instances - the compatibility matrix. The encoding simplifies the automation of decision making in one way or another. Generally, the efficient 3SAT and SAT solvers described in this article simulate the human activities during solving a jigsaw puzzle.

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