## Variations on the Fibonacci Universal Code

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**Abstract:** This note presents variations on the Fibonacci universal code, that may also be called the Gopala-Hemachandra code, that can have applications in source coding as well as in cryptography.

### Introduction

For the purposes of Fibonacci coding, the Fibonacci sequence of order two, F(i), where  $i \in \mathbb{N}$ , i > 0 is defined as [1]:

$$F(0) = 1$$
,  $F(1) = 2$ ,  $F(i) = F(i-1) + F(i-2)$ , for  $i > 2$ .

We will consider this familiar Fibonacci sequence  $F(i)=\{1, 2, 3, 5, 8, 13, ...\}$  as well as alternative sequences with different initial values.

The Fibonacci code is a type of universal coding scheme that maps the positive integers, which represent the probability rank of source messages, into variable length codewords. The codeword elements have a binary alphabet set, and are defined according to the following rule for a given positive integer n. Construct a column vector B(n) of Fibonacci numbers such that the ith element of B(n),  $B(n)_i = F(i)$ , i = 0, 1, ..., d, where F(d) is the largest Fibonacci number less than or equal to n. A row vector A(n) of binary digits is then chosen, also with dimension d, such that A(n) \* B(n) = n, and  $A(n)_d = 1$ . The codeword, FB(n), is the row vector with dimension d + 1 where  $FB(n)_k = A(n)_k$  for  $1 \le k \le d$ , and  $FB(n)_{d+1} = 1$ .

For example, suppose we wish to construct a codeword for the integer 10. Our vectors B(n), A(n), and FB(n) can be given as below:

$$d = 4$$
,  $B(n) = [1 \ 2 \ 3 \ 5 \ 8]^{T}$ ,  $A(n) = [0 \ 1 \ 0 \ 0 \ 1]$ ,  $FB(n) = [0 \ 1 \ 0 \ 0 \ 1 \ 1]$ .

Zeckendorf's theorem states that every positive integer has a unique representation as the sum of nonconsecutive Fibonacci numbers [2]. An integer written in such a fashion is said to be in Zeckendorf representation. Therefore while the recursive nature of the Fibonacci numbers allow some integers to have multiple representations using the above scheme, we can always choose A(n) such that there are no two consecutive 1's. For example, the decimal number 10 can be represented as F(1) + F(2) + F(3) by  $A(10) = [0\ 1\ 1\ 1]$  or in Zeckendorf representation as F(1) + F(4) by  $F(1) = [0\ 1\ 0\ 0\ 1]$ . Since  $F(1) = F(1) = F(1) = [0\ 1\ 0\ 0\ 1]$  will be at its termination, thus giving the code the prefix condition.

n	A(n)	$B(n)^{T}$	Fibonacci Codeword, FB(n)	
1	[1]	[1]	11	
2	[0 1]	[1 2]	011	
3	[0 0 1]	[1 2 3]	0011	
4	[1 0 1]	[1 2 3]	1011	
5	[0 0 0 1]	[1 2 3 5]	00011	
6	[1 0 0 1]	[1 2 3 5]	10011	
7	[0 1 0 1]	[1 2 3 5]	01011	
8	[0 0 0 0 1]	[1 2 3 5 8]	000011	
9	[1 0 0 0 1]	[1 2 3 5 8]	100011	
10	[0 1 0 0 1]	[1 2 3 5 8]	010011	
11	[0 0 1 0 1]	[1 2 3 5 8]	001011	
12	[1 0 1 0 1]	[1 2 3 5 8]	101011	
13	[0 0 0 0 0 1]	[1 2 3 5 8 13]	0000011	
14	[1 0 0 0 0 1]	[1 2 3 5 8 13]	1000011	
15	[0 1 0 0 0 1]	[1 2 3 5 8 13]	0100011	

Table 1, The Fibonacci Code, n = 1, ..., 15

# **Gopala-Hemachandra Sequence and Codes**

A variation to the Fibonacci sequences is the more general Gopala-Hemachandra sequence [3]:

$$a, b, a+b, a+2b, 2a+3b, 3a+5b, ...$$

for any pair a, b, which for the case a = 1, b = 2 represents the Fibonacci numbers. For a historical context of these sequences, see the papers [4,5,6].

We now introduce a variation on the Fibonacci coding scheme by using the Gopala-Hemachandra sequence to construct B(n). Define a second order Variant Fibonacci sequence,  $VF_a(n)$ , as the Gopala-Hemachandra sequence above such that b = 1 - a. That is,  $VF_a(0) = a$ ,  $(a \in \mathbb{Z})$ ,  $VF_a(1) = 1 - a$ , and for n > 1,  $VF_a(n) = VF_a(n-1) + VF_a(n-2)$ . With this definition we obtain (e.g.)  $VF_{-2}(n)$  as  $\{-2, 3, 1, 4, 5, 9, 14, 23, \ldots\}$ . We note in passing that similar Variant Fibonacci sequences have been investigated in the construction of hypercubes [7].

While the term "Zeckendorf representation" is properly used only in reference to the standard Fibonacci sequence, we will use it when discussing similar representations of numbers based on Variant Fibonacci sequences. Daykin proved that only the standard Fibonacci sequence F(n) gives all positive integers a unique Zeckendorf representation [8]. Thus the Variant Fibonacci sequences allow for multiple Zeckendorf representations of the same integer. With these Variant Fibonacci sequences, we can obtain a new universal source code, which we call the Gopala-Hemachandra code (or simply G-H code), using the same rule used to generate the standard Fibonacci code FB(n) above.

We quickly discover that not all values of a will generate a Variant Fibonacci sequence which is suitable for all applications of universal coding. For a = -5 (e.g.), we obtain  $VF_{-5}(n) = \{-5, 6, 1, 7, 8, 15, 23, 38 ...\}$ . It is easily seen that there is no Zeckendorf representation of the integers 5 or 12 using this sequence.

n	GH <sub>-2</sub> (n)	GH <sub>-3</sub> (n)	GH₋₄(n)	<i>GH</i> <sub>-5</sub> ( <i>n</i> )
1	0011	0011	0011	0011
2	10011	10011	10011	10011
3	011 or 100011	100011	100011	100011
4	00011 or 101011	011 or 101011	101011	101011
5	000011	00011	011	N/A
6	001011	000011	00011	011
7	01011 or 1000011	001011	000011	00011
8	010011 or 1010011	1000011	001011	000011
9	0000011	01011 or 1010011	1000011	001011
10	0010011	010011	1010011	1000011
11	1001011	0000011	01011	1010011
12	0100011 or 10000011	0010011	010011	N/A
13	0001011 or 10100011	1001011	0000011	01011
14	0000011	10000011	0010011	010011
15	00100011	0100011 or 10100011	1001011	0000011

Table 2, Some G-H Codes, n = 1, ..., 15

Even though the G-H codes are longer than the standard Fibonacci code, and therefore would less desirable by themselves, the family of G-H codes,  $GH_a(n)$ , which satisfy our condition for encoding all of the desired integers  $(1 \le n \le M)$ , allows us to have many more universal codes at our disposal when transmitting a message. Even those G-H codes such as  $GH_{-5}(n)$ , which lack the ability to encode certain positive integers, could be used on portions of a message signal that contained only those source messages which they are able to encode.

### **Conclusions**

The encoding scheme outlined in this paper offers some highly desirable cryptographic properties. Since the G-H codes are uniquely determined by their initial value a, the codebook could be easily changed multiple times during transmission, making decoding much more difficult. In addition, the presence of multiple representations of the same integer allow for a codebook that appears larger than it actually is. As shown in Table 2, we also see that the codeword lengths are not always increasing. This property seems to be undesirable by itself, but it could also offer cryptographic advantages.

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