

Extractors and the Constructive Dimensions of Weak Truth-Table Degrees

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Abstract

This paper examines the constructive Hausdorff and packing dimensions of weak truth-table degrees. The main result is that every infinite sequence S with constructive Hausdorff dimension $\dim_H(S)$ and constructive packing dimension $\dim_P(S)$ is weak truth-table equivalent to a sequence R with $\dim_H(R) \geq \dim_H(S)/\dim_P(S) - \epsilon$, for arbitrary $\epsilon > 0$. Furthermore, if $\dim_P(S) > 0$, then $\dim_P(R) \geq 1 - \epsilon$. The reduction thus serves as a *randomness extractor* that increases the algorithmic randomness of S , as measured by constructive dimension.

A number of applications of this result shed new light on the constructive dimensions of wtt degrees (and, by extension, Turing degrees). A lower bound of $\dim_H(S)/\dim_P(S)$ is shown to hold for the wtt degree of any sequence S . A new proof is given of a previously-known zero-one law for the constructive packing dimension of wtt degrees. It is also shown that, for any *regular* sequence S (that is, $\dim_H(S) = \dim_P(S)$) such that $\dim_H(S) > 0$, the wtt degree of S has Hausdorff and packing dimension 1.

1 Introduction

Hausdorff [6] initiated the study of dimension as a general framework to define the size of subsets of metric spaces. Recently this framework had been effectivized; Lutz [11] gives an overview of this historical development. Furthermore, Lutz [10, Section 6] reviews early results that anticipated the effectivization of Hausdorff dimension. *Constructive Hausdorff dimension* was defined by Lutz [10] to study effective dimension at the level of computability theory. Intuitively, given an infinite binary sequence S – interpreted as a language or decision problem – the constructive Hausdorff dimension $\dim_H(S)$ of S

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is a real number in the interval $[0,1]$ indicating the density of algorithmic randomness of the sequence. The constructive Hausdorff dimension of a class \mathcal{C} of sequences is the supremum of the dimensions of individual sequences in \mathcal{C} . For many classes \mathcal{C} of interest in computability theory, the problem of determining the constructive Hausdorff dimension of \mathcal{C} remains open.

Reimann [15] investigated in particular whether there are degrees of fractional constructive Hausdorff dimension. Stated in terms of individual sequences, Reimann asked which reducibilities (such as Turing, many-one, weak truth-table, etc.) are capable of increasing the constructive Hausdorff dimension of a sequence. We call such a reduction a *dimension extractor*, since its purpose bears a resemblance to that of the *randomness extractors* of computational complexity [19], which are algorithms that turn a source of weak randomness (a probabilistic source with low entropy) into a source of strong randomness (a source with high entropy). Viewing a sequence with positive, but still fractional, constructive Hausdorff dimension as a weak source of randomness, Reimann essentially asked whether such randomness can be extracted via a reduction to create a sequence with dimension closer to 1. If such extraction is *not* possible for some sequence S , this indicates that the degree of S under the reduction has fractional dimension.

A number of negative results for dimension extractors are known. Reimann and Terwijn [15, Theorem 3.10] proved that there are many-one and bounded truth-table degrees with constructive Hausdorff dimension strictly between 0 and 1. Later Reimann and Slaman [16] extended this result to truth-table degrees. Stephan [21] showed that there is a relativized world in which there exists a wtt degree of constructive Hausdorff dimension between $\frac{1}{4}$ and $\frac{1}{2}$. Furthermore, Nies and Reimann [8] obtained a non-relativized variant of this result and constructed, for each rational α between 0 and 1, a wtt degree of constructive Hausdorff dimension α .

Doty [3] attempted positive results by considering the interaction between constructive Hausdorff dimension and *constructive packing dimension* [1], a dual quantity that is a constructive effectivization of classical packing dimension [23, 22], another widely-studied fractal dimension. The constructive packing dimension $\dim_P(S)$ of a sequence S always obeys $0 \leq \dim_H(S) \leq \dim_P(S) \leq 1$, with each inequality tight in the strong sense that there are sequences S in which $\dim_H(S)$ and $\dim_P(S)$ may take on any values obeying the stated constraint. Doty showed that every sequence S with $\dim_H(S) > 0$ is Turing equivalent to a sequence R with $\dim_P(R) \geq 1 - \epsilon$, for arbitrary $\epsilon > 0$. This implies that the constructive packing dimension of the Turing degree of any sequence S with $\dim_H(S) > 0$ is equal to 1. Unfortunately, since $\dim_H(R) \leq \dim_P(R)$, this Turing reduction constitutes a weaker example of a dimension extractor than that sought by Reimann and it tells us nothing of the constructive dimensions of arbitrary Turing degrees.

We obtain in the current paper stronger positive results for constructive dimension extractors. Our main result is that, given any infinite sequence S and $\epsilon > 0$, there exists $R \equiv_{\text{wtt}} S$ such that $\dim_H(R) \geq \frac{\dim_H(S)}{\dim_P(S)} - \epsilon$ and, if $\dim_P(S) > 0$, then $\dim_P(R) \geq 1 - \epsilon$. This has immediate consequences for the dimensions of weak truth-table degrees:

- Given any sequence S , $\dim_H(\deg_{\text{wtt}}(S)) \geq \frac{\dim_H(S)}{\dim_P(S)}$.
- If $\dim_P(S) > 0$, then $\dim_P(\deg_{\text{wtt}}(S)) = 1$, implying that *every* wtt degree has constructive packing dimension 0 or 1.
- Given any *regular* sequence S such that $\dim_H(S) > 0$, $\dim_H(\deg_{\text{wtt}}(S)) = 1$, where a sequence S is regular if it satisfies $\dim_H(S) = \dim_P(S)$.

Before going into the details of the results, we introduce the concepts and notations formally.

Notation. We refer the reader to the textbooks of Li and Vitányi [7] for an introduction to Kolmogorov complexity and algorithmic information theory and of Odifreddi [14] and Soare [20] for an introduction to computability theory. Although we follow mainly the notation in these books, we nevertheless want to remind the reader on the following definitions, either for the readers' convenience or because we had to choose between several common ways of denoting the corresponding mathematical objects.

All logarithms are base 2. \mathbb{N} denotes the set $\{0, 1, 2, 3, \dots\}$ of the natural numbers including 0. $\{0, 1\}^*$ denotes the set of all finite, binary *strings*. For all $x \in \{0, 1\}^*$, $|x|$ denotes the *length* of x . λ denotes the empty string. $\mathbf{C} = \{0, 1\}^\infty$ denotes the *Cantor space*, the set of all infinite, binary *sequences*. For $x \in \{0, 1\}^*$ and $y \in \{0, 1\}^* \cup \mathbf{C}$, xy denotes the concatenation of x and y , $x \sqsubseteq y$ denotes that x is a *prefix* of y ; that is, there exists $u \in \{0, 1\}^* \cup \mathbf{C}$ such that $xu = y$ and $x \sqsubset y$ denotes that $x \sqsubseteq y$ and $x \neq y$. For $S \in \{0, 1\}^* \cup \mathbf{C}$ and $i, j \in \mathbb{N}$, $S[i]$ denotes the i^{th} bit of S , with $S[0]$ being the leftmost bit, $S[i..j]$ denotes the substring consisting of the i^{th} through j^{th} bits of S (inclusive), with $S[i..j] = \lambda$ if $i > j$.

Reductions and Compression. Let M be a Turing machine and $S \in \mathbf{C}$. We say M *computes* S if M , on input $n \in \mathbb{N}$, outputs the string $S[0..n-1]$. We define an *oracle Turing machine* to be a Turing machine M that can make constant-time queries to an oracle sequence and we let OTM denote the set of all oracle Turing machines. For $R \in \mathbf{C}$, we say M *operates with oracle* R if, whenever M makes a query to index $n \in \mathbb{N}$, the bit $R[n]$ is returned. We write M^R to denote the oracle Turing machine M with oracle R .

Let $S, R \in \mathbf{C}$ and $M \in \text{OTM}$. We say S *is Turing reducible to* R *via* M and we write $S \leq_T R$ *via* M , if M^R computes S . We say S *is Turing reducible to* R and we write $S \leq_T R$, if there exists $M \in \text{OTM}$ such that $S \leq_T R$ via M . We say S *is Turing equivalent to* R and we write $S \equiv_T R$, if $S \leq_T R$ and $R \leq_T S$. The *Turing lower span* of S is $\text{span}_T(S) = \{ R \in \mathbf{C} \mid R \leq_T S \}$ and the *Turing degree* of S is $\deg_T(S) = \{ R \in \mathbf{C} \mid R \equiv_T S \}$.

Let $S, R \in \mathbf{C}$ and $M \in \text{OTM}$ such that $S \leq_T R$ via M . Define $\#(M^R, S[0..n-1])$ to be the *query usage of* M^R *on* $S[0..n-1]$, the number of bits of R queried by M when

computing the string $S[0..n-1]$.¹ We say S is *weak truth-table (wtt) reducible to R via M* and we write $S \leq_{\text{wtt}} R$ via M , if $S \leq_{\text{T}} R$ via M and there is a computable function $q : \mathbb{N} \rightarrow \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $\#(M^R, S[0..n-1]) \leq q(n)$. Define $S \leq_{\text{wtt}} R$, $S \equiv_{\text{wtt}} R$, $\text{span}_{\text{wtt}}(S)$ and $\text{deg}_{\text{wtt}}(S)$ analogously to their counterparts for Turing reductions. Define

$$\begin{aligned}\rho_M^-(S, R) &= \liminf_{n \rightarrow \infty} \frac{\#(M^R, S[0..n-1])}{n}, \\ \rho_M^+(S, R) &= \limsup_{n \rightarrow \infty} \frac{\#(M^R, S[0..n-1])}{n}.\end{aligned}$$

Viewing R as a compressed version of S , $\rho_M^-(S, R)$ and $\rho_M^+(S, R)$ are respectively the best- and worst-case compression ratios as M decompresses R into S . Note that $0 \leq \rho_M^-(S, R) \leq \rho_M^+(S, R) \leq \infty$.

The following lemma is useful when one wants to compose two reductions:

Lemma 1.1. [2] *Let $S, S', S'' \in \mathbf{C}$, and $M_1, M_2 \in \text{OTM}$ such that $S' \leq_{\text{T}} S$ via M_1 and $S'' \leq_{\text{T}} S'$ via M_2 . There exists $M \in \text{OTM}$ such that $S'' \leq_{\text{T}} S$ via M and:*

$$\begin{aligned}\rho_M^+(S'', S) &\leq \rho_{M_2}^+(S'', S') \rho_{M_1}^+(S', S). \\ \rho_M^-(S'', S) &\leq \rho_{M_2}^-(S'', S') \rho_{M_1}^-(S', S). \\ \rho_M^-(S'', S) &\leq \rho_{M_2}^+(S'', S') \rho_{M_1}^-(S', S).\end{aligned}$$

(the last bound is not explicitly stated in [2], but it holds for the same reason as the second one).

For $S \in \mathbf{C}$, the *lower and upper Turing compression ratios of S* are respectively defined

$$\begin{aligned}\rho^-(S) &= \min_{\substack{R \in \mathbf{C} \\ M \in \text{OTM}}} \{ \rho_M^-(S, R) \mid S \leq_{\text{T}} R \text{ via } M \}, \\ \rho^+(S) &= \min_{\substack{R \in \mathbf{C} \\ M \in \text{OTM}}} \{ \rho_M^+(S, R) \mid S \leq_{\text{T}} R \text{ via } M \}.\end{aligned}$$

Doty [2] showed that the above minima exist. Note that $0 \leq \rho^-(S) \leq \rho^+(S) \leq 1$.

Constructive Dimension. Lutz [10] gives an introduction to the theory of constructive dimension. We use Mayordomo's characterization [12] of the constructive dimensions of sequences. For all $S \in \mathbf{C}$, the *constructive Hausdorff dimension* and the *constructive packing dimension* of S are respectively defined

$$\dim_{\text{H}}(S) = \liminf_{n \rightarrow \infty} \frac{C(S[0..n-1])}{n}, \text{ and } \dim_{\text{P}}(S) = \limsup_{n \rightarrow \infty} \frac{C(S[0..n-1])}{n},$$

¹If we instead define $\#(M^R, S[0..n-1])$ to be the index of the rightmost bit of R queried by M when computing $S[0..n-1]$, all results of the present paper still hold.

where $C(w)$ denotes the *Kolmogorov complexity* of $w \in \{0, 1\}^*$ (see [7]). If $\dim_H(S) = \dim_P(S)$, we say S is a *regular* sequence. Doty [2] showed that, for all $S \in \mathbf{C}$, $\rho^-(S) = \dim_H(S)$ and $\rho^+(S) = \dim_P(S)$.

For all $X \subseteq \mathbf{C}$, the *constructive Hausdorff dimension* and the *constructive packing dimension* of X are respectively defined

$$\dim_H(X) = \sup_{S \in X} \dim_H(S), \text{ and } \dim_P(X) = \sup_{S \in X} \dim_P(S).$$

2 Constructive Dimension Extractors

Ryabko [17, 18] discovered the next theorem.

Theorem 2.1 (Ryabko [17, 18]). *For all $S \in \mathbf{C}$ and $\delta > 0$, there exists $R \in \mathbf{C}$ and $N_d \in \text{OTM}$ such that*

1. $S \leq_T R$ via N_d and $R \leq_T S$.
2. $\rho_{N_d}^-(S, P) \leq \dim_H(S) + \delta$.

The following theorem was shown in [2].

Theorem 2.2 (Doty [2]). *There is an oracle Turing machine M_d such that, for all $S \in \mathbf{C}$, there exists $R \in \mathbf{C}$ such that*

1. $S \leq_{\text{wtt}} R$ via M_d .
2. $\rho_{M_d}^-(S, R) = \dim_H(S)$.
3. $\rho_{M_d}^+(S, R) = \dim_P(S)$.

The following theorem, which is similar to Ryabko's Theorem 2.1, shows that the decoding machine M_d of Theorem 2.2 can also be reversed if the compression requirements are weakened.

Theorem 2.3. *Let M_d be the oracle Turing machine from Theorem 2.2. For all $\delta > 0$, there is an oracle Turing machine M_e such that, for all $S \in \mathbf{C}$, there is a sequence $R' \in \mathbf{C}$ such that*

1. $S \leq_{\text{wtt}} R'$ via M_d and $R' \leq_{\text{wtt}} S$ via M_e .
2. $\rho_{M_d}^+(S, R') \leq \dim_P(S) + \delta$.

Proof. Let $S \in \mathbf{C}$ and choose R for S as in Theorem 2.2. Let $\delta > 0$ and let $D \in (\dim_P(S), \dim_P(S) + \delta)$ be rational. By Theorem 2.2, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $\#(M_d^R, S[0..n-1]) < Dn$.

M_e will make use of the oracle Turing machine M_d . The proof of Theorem 2.2 in [2] shows that M_d has the following useful properties. First, write $S = s_1 s_2 s_3 \dots$ and $R = r_1 r_2 r_3 \dots$, where each $s_i, r_i \in \{0, 1\}^*$ are blocks such that $|s_i| = i$ and $|r_i| \leq |s_i| + o(|s_i|)$.

- M_d computes S from R in stages, where it outputs the block s_i on the i^{th} stage.
- Assuming that M_d has already computed $s_1 \dots s_i$, M_d uses only the block r_{i+1} and the prefix $s_1 \dots s_i$ to compute s_{i+1} .

Because of these properties, we can use M_d to search for a sequence R' that satisfies requirements 1 and 2 in the statement of Theorem 2.3. By Theorem 2.2, R satisfies these requirements, so such an R' will exist. By the above two properties of M_d , if we find a string $r' = r'_1 \dots r'_i$ that satisfies requirements 1 and 2 (in the sense described below), we will always be able to find an extension $r'' = r'_{i+1} \dots r'_j$ (for some $j > i$) such that $r'r''$ continues to satisfy the requirements. It will not matter if $r' \not\sqsubseteq R$, since M_d does not use the portion of R coming before block r_{i+1} to compute s_{i+1} . In other words, to reverse the computation of $M_d^{R'}$ and compute R' from S , we don't need to find the R from Theorem 2.2; we need only to find an R' that is “close enough”.

Define the oracle Turing machine M_e with oracle $S \in \mathbf{C}$ as follows. Let $i \in \mathbb{N}$ and assume inductively that the prefix $r' = r'_1 \dots r'_i \sqsubset R'$ has been computed, so that, letting $|s_1 \dots s_i| = n$,

- (a) $M_d^{r'}(n)$ outputs $S[0 \dots n-1]$,
- (b) for all $n_0 \leq k \leq n$, $\#(M_d^{r'}, S[0 \dots k-1]) \leq Dk$.

Let N be the smallest integer greater than 2^n such that $S[0 \dots N-1] = s_1 \dots s_{i'}$, for some $i' \in \mathbb{N}$. M_e^S searches all strings $r'' \in \{0, 1\}^N$ until it finds one that satisfies

- (a) $M_d^{r'r''}(N)$ outputs $S[0 \dots N-1]$,
- (b) for all $n_0 \leq k \leq N$, $\#(M_d^{r'r''}, S[0 \dots k-1]) \leq Dk$.

M_e^S then outputs r'' and saves it for the computation of the next extension of R' . By the existence of R from Theorem 2.2 and a simple induction on the stages of computation that M_e performs and the fact that N is asymptotically larger than n , M_e^S will always be able to find such an r'' . Therefore, in the output sequence R' , for all but finitely many N , requirement (b) will be satisfied. Therefore the sequence R' will satisfy the two requirements of Theorem 2.3.

Finally, for any n , $M_e(n)$ makes no more than 2^{2n} queries to S and therefore M_e computes a wtt reduction. \square

The following theorem is the main result of this paper. It states that constructive packing dimension can be almost optimally extracted from a sequence of positive packing dimension, while at the same time, constructive Hausdorff dimension is *partially* extracted this sequence, if it has positive Hausdorff dimension and packing dimension less than 1. The machine M_e from Theorem 2.3 serves as the extractor. Intuitively, this works because M_e compresses the sequence S into the sequence R . Since R is a compressed representation of S , R must itself be more incompressible than S . However, because dimension measures the compressibility of a sequence, this means that the constructive dimensions R are greater than those of S .

Theorem 2.4. *For all $\epsilon > 0$ and $S \in \mathbf{C}$ such that $\dim_{\mathbf{P}}(S) > 0$, there exists $R \equiv_{\text{wtt}} S$ such that $\dim_{\mathbf{P}}(R) \geq 1 - \epsilon$ and $\dim_{\mathbf{H}}(R) \geq \frac{\dim_{\mathbf{H}}(S)}{\dim_{\mathbf{P}}(S)} - \epsilon$.*

Proof. Let $\epsilon > 0$ and $S \in \mathbf{C}$ such that $\dim_{\mathbf{P}}(S) > 0$. Let $\delta > 0$ and R', M_d be as in Theorem 2.3. Let $R'' \in \mathbf{C}$ and $M \in \text{OTM}$ such that $R' \leq_{\mathbf{T}} R''$ via M , $\rho_M^-(R', R'') = \dim_{\mathbf{H}}(R')$ and $\rho_M^+(R', R'') = \dim_{\mathbf{P}}(R')$ (the existence of M and R'' is asserted by Theorem 2.2). By Lemma 1.1, we have

$$\rho^+(S) \leq \rho_{M_d}^+(S, R') \rho_M^+(R', R'')$$

which, by construction of R' and R'' implies

$$\rho^+(S) \leq (\dim_{\mathbf{P}}(S) + \delta) \dim_{\mathbf{P}}(R')$$

and since $\rho^+(S) = \dim_{\mathbf{P}}(S)$,

$$\dim_{\mathbf{P}}(R') \geq \frac{\dim_{\mathbf{P}}(S)}{\dim_{\mathbf{P}}(S) + \delta}.$$

Moreover (by Lemma 1.1 again):

$$\rho^-(S) \leq \rho_{M_d}^+(S, R') \rho_M^-(R', R'')$$

which, by construction of R' and R'' , implies

$$\rho^-(S) \leq (\dim_{\mathbf{P}}(S) + \delta) \dim_{\mathbf{H}}(R')$$

and since $\rho^-(S) = \dim_{\mathbf{H}}(S)$,

$$\dim_{\mathbf{H}}(R') \geq \frac{\dim_{\mathbf{H}}(S)}{\dim_{\mathbf{P}}(S) + \delta}.$$

Taking δ small enough, we get by the above inequalities: $\dim_{\mathbf{P}}(R) \geq 1 - \epsilon$ and $\dim_{\mathbf{H}}(R) \geq \frac{\dim_{\mathbf{H}}(S)}{\dim_{\mathbf{P}}(S)} - \epsilon$. \square

Theorem 2.4 has a number of applications, stated in the following corollaries, which shed light on the constructive dimensions of sequences, spans and degrees.

Corollary 2.5. *Let $S \in \mathbf{C}$ and assume that $\dim_{\mathbf{H}}(S) > 0$. Then $\dim_{\mathbf{H}}(\deg_{\mathbf{T}}(S))$, $\dim_{\mathbf{H}}(\deg_{\text{wtt}}(S))$, $\dim_{\mathbf{H}}(\text{span}_{\mathbf{T}}(S))$ and $\dim_{\mathbf{H}}(\text{span}_{\text{wtt}}(S))$ are all at least $\frac{\dim_{\mathbf{H}}(S)}{\dim_{\mathbf{P}}(S)}$.*

We obtain a zero-one law for the constructive packing dimension of Turing and weak truth-table lower spans and degrees.

Corollary 2.6. *For all $S \in \mathbf{C}$, the dimensions $\dim_{\mathbf{P}}(\deg_{\mathbf{T}}(S))$, $\dim_{\mathbf{P}}(\deg_{\text{wtt}}(S))$, $\dim_{\mathbf{P}}(\text{span}_{\mathbf{T}}(S))$ and $\dim_{\mathbf{P}}(\text{span}_{\text{wtt}}(S))$ are each either 0 or 1.*

Thus, the theorem of Nies and Reimann [8] establishing the existence of wtt degrees of fractional constructive Hausdorff dimension does not extend to constructive packing dimension. Because of the Nies-Reimann theorem, we must settle for more conditional results for constructive Hausdorff dimension. We therefore focus attention on regular sequences.

Corollary 2.7. *For all $\epsilon > 0$ and all regular $S \in \mathbf{C}$ such that $\dim_{\mathbf{H}}(S) > 0$, there exists $R \equiv_{\text{wtt}} S$ such that $\dim_{\mathbf{H}}(R) \geq 1 - \epsilon$.*

Corollary 2.8. *For all regular $S \in \mathbf{C}$ such that $\dim_{\mathbf{H}}(S) > 0$, the constructive Hausdorff and packing dimension of the wtt and Turing degree and lower span of S are all 1.*

It remains open whether every Turing lower span or degree of positive constructive Hausdorff dimension contains a regular sequence of positive constructive Hausdorff dimension. If so, this would imply a zero-one law for constructive Hausdorff dimension similar to Corollary 2.6.

We note that the zero-one law for the constructive packing dimension of Turing and weak truth-table lower spans and degrees also follows from the following theorem due to Fortnow, Hitchcock, Pavan, Vinodchandran and Wang [5], giving a polynomial-time extractor for constructive packing dimension. For $R, S \in \mathbf{C}$, write $R \leq_{\mathbf{T}}^{\mathbf{P}} S$ if $R \leq_{\mathbf{T}} S$ via an OTM that, on input n , runs in time polynomial in n , and similarly for $\equiv_{\mathbf{T}}^{\mathbf{P}}$.

Theorem 2.9. [5] *For all $\epsilon > 0$ and all $S \in \mathbf{C}$ such that $\dim_{\mathbf{P}}(S) > 0$, there exists $R \leq_{\mathbf{T}}^{\mathbf{P}} S$ such that $\dim_{\mathbf{P}}(R) \geq 1 - \epsilon$.*

In fact, Theorem 2.9 holds for any resource-bounded packing dimension [9] defined by Turing machines allowed at least polynomial space, which includes constructive packing dimension as a special case, thus proving a spectrum of zero-one packing dimension laws for various degrees and lower spans that are at least polynomial-time computable. Furthermore, defining $\tilde{R} \in \mathbf{C}$ as $\tilde{R}[x] = S[y]$ if $x = y^2$ and $\tilde{R}[x] = R[x]$ if x is not a square, one obtains a new language which has the same packing dimension as R and the same polynomial time Turing degree as S . Therefore, the preordering $\leq_{\mathbf{T}}^{\mathbf{P}}$ in Theorem 2.9 can be replaced with the corresponding equivalence relation $\equiv_{\mathbf{T}}^{\mathbf{P}}$.

Acknowledgments. We thank Joe Miller for assistance with the proof of Theorem 2.3, as well as John Hitchcock, Jan Reimann and André Nies for their insightful comments. We also thank the American Institute of Mathematics which generously invited us to the Workshop on Effective Randomness; this paper is a result of a workgroup discussing open questions during this workshop. Besides the American Institute of Mathematics, we would also like to thank the organizers Denis Hirschfeldt and Joe Miller of the workshop as well as the participants who discussed this research topic with us.

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