

The Computational Complexity of $3k$ -CLIQUE

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Abstract: In this note, we show that the fastest deterministic and exact algorithm that solves the $3k$ -CLIQUE problem must run in $\Omega(n^{2k})$ time in the worst-case scenario on a classical computer, where n is the number of vertices in the graph.

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The $3k$ -CLIQUE problem is to determine whether or not a clique of size $3k$ exists in a given undirected graph G , where k is a positive integer that is not part of the input of the problem [4]. In this note, we show that the fastest deterministic and exact algorithm that solves $3k$ -CLIQUE must run in $\Omega(n^{2k})$ time in the worst-case scenario on a classical computer, where n is the number of vertices in the graph:

Let G be an undirected graph with n vertices. For every k -clique C in G , create a corresponding vertex $v(C)$ in an auxiliary graph G' . And for every two vertices $v(C_1)$ and $v(C_2)$ in G' , create an edge connecting them in G' if and only if $C_1 \cup C_2$ forms a $2k$ -clique in G . Then G' will have $O(n^k)$ vertices and $O(n^{2k})$ edges. Note that the 3 -CLIQUE problem on G' is equivalent to the $3k$ -CLIQUE problem on G [4]. Let $A = (a_{ij})$ be the adjacency matrix of G' , and let $B = (b_{ij})$ be the matrix in which $B = A^2$. Then G' has no 3 -clique if and only if the Hadamard product of A and B , $A \circ B$, equals zero [2].

By the definition of Hadamard product, in order for an algorithm to determine whether $A \circ B = 0$, it is necessary for the algorithm to compute each entry $a_{ij} \cdot b_{ij}$ of $A \circ B$ until either an entry is found to be nonzero or it is certain that all entries are zero, which takes $\Omega(n^{2k})$ time in the worst-case scenario. Hence, since the problem of determining whether $A \circ B = 0$ is equivalent to the $3k$ -CLIQUE problem on G , any deterministic and exact algorithm that solves the $3k$ -CLIQUE problem on G must also take $\Omega(n^{2k})$ time in the worst-case scenario. And this implies that $P \neq NP$ [1]. \square

This lower bound is confirmed by the fact that the fastest known deterministic and exact algorithm that solves $3k$ -CLIQUE was first published in 1985 and has a running-time of $\Theta(n^{\omega k})$, where $\omega \geq 2$ [1, 3, 4].

References

- [1] F. Eisenbrand and F. Grandoni, "On the complexity of fixed parameter clique and dominating set", *Theoretical Computer Science* 326(1-3): 57-67, 2004.
- [2] A. Itai and M. Rodeh, "Finding a minimum circuit in a graph", *SIAM Journal on Computing* 7(4): 413-423, 1978.
- [3] J. Nešetřil and S. Poljak, "On the complexity of the subgraph problem", *Comment. Math. Univ. Carolin.* 26, 415-419, 1985.
- [4] G. J. Woeginger, "Open problems around exact algorithms", *Discrete Applied Mathematics* 156(3): 397-405, 2008.