A POLYNOMIAL TIME ALGORITHM FOR 3-SAT

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ABSTRACT. Article describes a class of efficient algorithms for 3SAT.

Introduction

This paper presents the compatibility matrix method for 3SAT. The compatibility matrix is an encoding of 3SAT instances using the contradictions between their clauses. This encoding is the first major result of this work.

The compatibility matrix encoding allows parallel testing of all guesses. This paper describes several algorithms which perform the parallel testing of all guesses in polynomial time indeed. The algorithms transform the compatibility matrix into a similar encoding of the satisfying true-assignments. These efficient algorithms for 3SAT are the second major result of this work.

Let us describe 3SAT as follows. Suppose, f is a given conjunctive normal form (CNF) with clauses of length three or less. Let

$$(0.1) f = c_1 \wedge c_2 \wedge \ldots \wedge c_m$$

In formula f, clauses c_i are disjunctions of three or less non-complimentary literals (different Boolean variables or their negations):

$$c_i \in \{\alpha \vee \beta \vee \gamma, \ \alpha \vee \beta, \ \alpha \mid \alpha, \beta, \gamma \in \{b_j, \neg b_j \mid j = 1, 2, \dots, n\}\},\$$

- where b_j are Boolean variables. Suppose, we have to decide whether there does exist such a true-assignment which would satisfy formula f:

$$true = f(b_1, b_2, \dots, b_n)|_{b_i = \tau_i \in \{false, true\}}$$

Such a decision problem is a 3SAT instance.

Cook proved that 3SAT is a NP-complete problem [1]. So, the whole history of the NP-completeness theory is the history of 3SAT. As a matter of fact, 3SAT was one of the first four NP-complete problems discovered [1]. One might say that 3SAT is the most beautiful NP-complete problem discovered.

1. Compatibility matrix

Let T_i be the truth-table for clause c_i , i = 1, 2, ..., m. Let's arbitrarily enumerate the true-assignments for the arguments of c_i and let's write T_i in the following form:

T_i	True-assignments	c_i
1	1-st true-assignment	Value of c_i on the 1-st true-assignment
2	2-nd true-assignment	Value of c_i on the 2-nd true-assignment
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²⁰⁰⁰ Mathematics Subject Classification. Primary 68Q15, 68R10, 90C57.

Key words and phrases. Computational complexity, Algorithms, Satisfiability, SAT, 3SAT.

Let $S_{i\mu}$ be μ -th string from truth-table T_i . By definition, any two strings $S_{i\mu}$ and $S_{j\nu}$ are compatible if the following two conditions are satisfied:

Condition 1: The μ -th value of c_i is true and the ν -th value of c_j is true; Condition 2: The μ -th true-assignment in table T_i and the ν -th true-assignment in table T_j do not contradict each other. In other words, if clauses c_i and c_j share a variable, then this variable has to have the same true-assignment in both strings $S_{i\mu}$ and $S_{j\nu}$.

When at least one of these conditions is not satisfied, then the strings are incompatible. Let us emphasize, i and j may be equal in this definition, i.e. strings $S_{i\mu}$ and $S_{j\nu}$ may be from the same truth-table.

Let's build a compatibility box for each clause couple (c_i, c_j) , i, j = 1, 2, ..., m. Let k_i and k_j be the lengths of clauses c_i and c_j , appropriately. Then, the compatibility box for clauses c_i and c_j is a Boolean matrix $C_{ij} = (x_{\mu\nu})_{2^{k_i} \times 2^{k_j}}$ with the following elements:

$$x_{\mu\nu} = \begin{cases} true, & \text{Strings } S_{i\mu} \text{ and } S_{j\nu} \text{ are compatible} \\ false, & \text{Otherwise} \end{cases}$$

Having all m^2 compatibility boxes built, we can aggregate them in a box matrix C:

$$C = (C_{ij})_{m \times m}$$

- where the size is given in "boxes". Matrix C is a Boolean box matrix. For 3SAT, size of C is

$$\left(\sum_{i=1}^{m} 2^{k_i}\right) \times \left(\sum_{i=1}^{m} 2^{k_i}\right) \le (8m) \times (8m) = O(m \times m),$$

- where k_i is length of clause c_i . We call matrix C a compatibility matrix for formula 0.1.

Let us notice that the compatibility matrix is a symmetric matrix:

$$(1.1) C = C^T, C_{ij} = C_{ji}^T.$$

Also, let us notice that, due to Condition 2 for the string compatibility, diagonal boxes C_{ii} are diagonal matrices, i.e. all non-diagonal elements in these boxes are false:

(1.2)
$$C_{ii} = \begin{pmatrix} t_1 & false & \dots \\ false & t_2 & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}, t_{\mu} \in \{true, false\}$$

Because clauses in formula 0.1 are disjunctions, there is the only one value false among values t_{μ} in matrix 1.2.

As usual, we can use a (0,1)-version of the compatibility matrix. In the version of matrix C, values true are replaced with 1, and values false are replaced with 0.

 $^{^{1}}$ Boolean matrix is a matrix whose elements are true or false.

2. Solution grid and depletion

Traditionally, guessing in 3SAT is the checking whether a true-assignment satisfies formula 0.1. But, any true-assignment for the variables in formula 0.1 contains/consists of the true-assignments for clauses c_i . So, any true-assignment produces in the compatibility matrix a grid of elements, one element per compatibility box:

$$\gamma = \{x_{\mu\nu ij} \mid \mu = \mu(i), \ \nu = \nu(j), \ i, j = 1, 2, \dots, m\},\$$

- where $x_{\mu\nu ij}$ is the (μ, ν) -element of compatibility box C_{ij} , and γ is a grid of such elements in compatibility matrix C, one element per compatibility box.

Based on the grid presentation, we can state that any true-assignment satisfying formula 0.1 is presented in compatibility matrix C with a grid of elements, one element per compatibility box, which all are true. And visa versa, if there is such a grid of elements in the compatibility matrix, one element per compatibility box, whose all elements are true, then formula 0.1 is satisfiable. We call such a grid which consists of all true-elements², one element per compatibility box, a solution grid.

Based on property 1.2 of the compatibility matrix, we could even remove the "one-element-per-box" requirement from the solution grid definition.

Theorem 2.1. Formula 0.1 is satisfiable iff compatibility matrix for that formula contains a solution grid.

Solution grid is a pattern of satisfiability in the compatibility matrix encoding of formula 0.1. And theorem 2.1 reduces the guessing to the searching the matrix for the pattern. In the search, any heuristic is welcome.

Our method is a detection of those elements of the compatibility matrix which do not belong to any solution grid and the assigning to them value false or 0 depending on the presentation of the compatibility matrix. We call the replacement of a true-element with value false a depletion of the compatibility matrix.

Theorem 2.2. Suppose, there is a filter which finds in the compatibility matrix for formula 0.1 all such "true"-elements which do not belong to any solution grid. Suppose, all these elements were replaced with value "false". Then, formula 0.1 is satisfiable iff there is not any compatibility box filled with "false" entirely.

In fact, for unsatisfiable formula 0.1, the matrix emerging from the depletion described in theorem 2.2 will be filled with false entirely.

A compatibility box entirely filled with false we call a pattern of unsatisfiability. One might say that our method is a search of the compatibility matrix for the pattern. For the method, it does not matter what particular filter is used for the depletion. For the filter, the only objective is to preserve at least one solution grid when the grids exist. Then, no information will be lost in the sense of computational complexity.

3. Basic algorithm

The following algorithm is a depletion filter. We call it a basic algorithm:

 $^{^2}True$ -element is an element which is equal true, and false-element is an element which is equal false.

Init: Build the compatibility matrix for formula 0.1.

Let s = 1. Let C_s be the compatibility matrix. Let C_{sij} be the compatibility boxes, where i, j = 1, 2, ..., m.

Depletion: Let s = s + 1. Calculate compatibility matrix C_s as follows³:

(3.1)
$$C_{sij} = \bigwedge_{k=1}^{m} C_{s-1,ik} C_{s-1,kj},$$

- where C_{sij} is the (i, j)-th compatibility box of C_s , i, j = 1, 2, ..., m. Let $x_{s\mu\nu ij}$ be the $(\mu\nu)$ -th element of compatibility box C_{sij} . Then, due to formula 3.1, the element is

(3.2)
$$x_{s\mu\nu ij} = \bigwedge_{k=1}^{m} \left(\bigvee_{1 \le \alpha \le 2^3} x_{s-1,\mu\alpha ik} \wedge x_{s-1,\alpha\nu kj} \right).$$

Due to property 1.1 of the compatibility matrix, the last expression can be rewritten in any of the following ways:

$$(3.3) x_{s\mu\nu ij} = \bigwedge_{k} (\bigvee_{\alpha} x_{s-1,\alpha\mu ki} \wedge x_{s-1,\alpha\nu kj}) = \bigwedge_{k} (\bigvee_{\alpha} x_{s-1,\mu\alpha ik} \wedge x_{s-1,\nu\alpha jk}).$$

Iterations: If $C_s \neq C_{s-1}$, then go to the previous step. Otherwise, continue. **Decision:** If C_s is entirely filled with false, then formula 0.1 is unsatisfiable - decision "NO". Otherwise, formula 0.1 is satisfiable - decision "YES".

Obviously, when during an iteration the pattern of unsatisfiability arises (value false fills a compatibility box entirely), then the algorithm can be stopped with decision "NO". Because, formula 3.1 will propagate this value false all over the compatibility matrix during the next two iterations, at most.

For 3SAT, the computational complexity of the algorithm can be estimated as $O(m^5)$: there is $O(m^2)$ iterations (each iteration depletes at least one element from the compatibility matrix); there is m^2 depletions on each iteration; and each depletion takes time O(m).

Theorem 3.1. Any "true"-element in the matrix emerging after running of the basic algorithm belongs to a solution grid. Formula 0.1 is unsatisfiable iff the basic algorithm produces the pattern of unsatisfiability for this formula.

Proof. Clauses in formula 0.1 can be expressed in their disjunctive forms (DF). Depending on their length, the clauses are

- where α , β , and γ are literals.

Let's call all conjunctions in DF 3.4 the triads (value false in the formulas is

$$AB = (\bigvee_{k} a_{ik} \wedge b_{kj}).$$

Conjunction of Boolean matrices of the same size is the matrix of conjunctions of the appropriate elements of the matrices.

³Here, product of two Boolean matrices $A=(a_{\mu\nu})$ and $B=(b_{\mu\nu})$ of the appropriate sizes (the number of columns in A has to be equal to the number of rows in B) is the following Boolean matrix:

a triad, too). For each clause, there is a natural one-to-one relation between the clause's triads and true-assignments: that true-assignment which makes a clause equal false relates to triad false; the rest of the true-assignments are in the relation with those triads to which they deliver value true. Let's enumerate the triads in DF 3.4 using that one-to-one relations: triad $\tau_{\mu i}$ is that triad which relates to μ -th true-assignment for clause c_i .

Let's replace clauses in formula 0.1 by their DF 3.4, and let's open all parentheses in the formula. The result of that is the following DF:

$$f = \bigvee_{\mu_1, \mu_2, \dots, \mu_m} \bigwedge_{i=1}^m \tau_{\mu_i i}$$

- where the disjunction is taken over all appropriate combinations of indexes μ_i . Then, formula 0.1 is satisfiable iff there is at least one satisfiable conjunction

(3.5)
$$\Gamma(\mu_1, \mu_2, \dots, \mu_m) = \bigwedge_{i=1}^m \tau_{\mu_i i}$$

In other words, formula 0.1 is satisfiable iff there is at least one conjunction 3.5 without complimentary literals.

Now, let's write the following symbolic box matrix Y:

$$Y = (Y_{ij})_{m \times m},$$

- where Y_{ij} are the boxes. The symbolic boxes in matrix Y are as follows:

$$Y_{ij} = (\tau_{\mu i} \wedge \tau_{\nu j})_{2^{k_i} \times 2^{k_j}},$$

- where k_i and k_j are lengths of clauses c_i and c_j appropriately. Let's calculate powers of matrix Y using formulas 3.1 and 3.2:

$$Y^{1} = Y, Y^{s+1} = (\bigwedge_{k=1}^{m} Y_{s,ik} Y_{s,kj})_{m \times m}, s \ge 1$$

- where boxes $Y_{s,*,*}$ are the appropriate boxes in box matrix Y^s . Due to formula 3.2, elements of matrix $Y^s = (y_{\alpha\beta})_{m \times m}$ are some DF of the following kind:

(3.6)
$$y_{\alpha\beta} = \bigvee_{\sigma_1, \sigma_2, \dots} \tau_{\mu_{\sigma_1}, \sigma_1} \wedge \tau_{\mu_{\sigma_2}, \sigma_2} \wedge \dots$$

- where $(\sigma_1, \sigma_2, \ldots)$ are all appropriate combinations of the triads' indexes.

The most important for us is to notice that, starting from some power $s = O(m^2)$ of matrix Y, each conjunction in DF 3.6 contains at least m different triads. But, any conjunction of more than m different triads contains at least two different triads from the same clause. But, any such conjunction equals false because there are complimentary literals in those different triads from the same clause. Thus, starting from that power, all DF 3.6 get finalized as follows:

(3.7)
$$y_{\alpha\beta} = \bigvee_{\mu_1, \mu_2, \dots, \mu_m} \Gamma(\mu_1, \mu_2, \dots, \mu_m)$$

- where the disjunction is taken over all appropriate combinations of indexes μ_i . Let's rewrite $\Gamma(\mu_1, \mu_2, \dots, \mu_m)$ as follows:

$$\Gamma(\mu_1, \mu_2, \dots, \mu_m) = \bigwedge_{i=1}^m \tau_{\mu_i i} =$$

$$= (\tau_{\mu_1,1} \wedge \tau_{\mu_1,1}) \wedge (\tau_{\mu_1,1} \wedge \tau_{\mu_2,2}) \wedge \dots \wedge (\tau_{\mu_1,1} \wedge \tau_{\mu_m,m}) \wedge (\tau_{\mu_2,2} \wedge \tau_{\mu_1,1}) \wedge (\tau_{\mu_2,2} \wedge \tau_{\mu_2,2}) \wedge \dots \wedge (\tau_{\mu_2,2} \wedge \tau_{\mu_m,m}) \wedge \wedge \dots \wedge (\tau_{\mu_m,m} \wedge \tau_{\mu_1,1}) \wedge (\tau_{\mu_m,m} \wedge \tau_{\mu_2,2}) \wedge \dots \wedge (\tau_{\mu_m,m} \wedge \tau_{\mu_m,m})$$

Each parenthesis in the last presentation is an element of the compatibility matrix for formula 0.1:

$$\tau_{\mu i} \wedge \tau_{\nu j} = x_{\mu \nu i j}$$

- where $x_{\mu\nu ij}$ is the (μ, ν) -the element in compatibility box C_{ij} . Element $x_{\mu\nu ij}$ is true iff triads $\tau_{\mu i}$ and $\tau_{\nu j}$ contain no complimentary literals. Thus, conjunctions 3.5 are associated with the grids of the compatibility matrix's elements, one element per compatibility box:

(3.8)
$$\Gamma(\mu_{1}, \mu_{2}, \dots, \mu_{m}) = \begin{array}{cccc} x_{\mu_{1}, \mu_{2}, 1, 1} \wedge x_{\mu_{1}, \mu_{2}, 1, 2} \wedge \dots \wedge x_{\mu_{1}, \mu_{m}, 1, m} & \wedge \\ \wedge & x_{\mu_{2}, \mu_{1}, 2, 1} \wedge x_{\mu_{2}, \mu_{2}, 2, 2} \wedge \dots \wedge x_{\mu_{2}, \mu_{m}, 2, m} & \wedge \\ \wedge & & & & & \wedge \\ & & & & & \wedge \end{array}$$

$$\wedge & x_{\mu_{m}, \mu_{1}, m, 1} \wedge x_{\mu_{m}, \mu_{2}, m, 2} \wedge \dots \wedge x_{\mu_{m}, \mu_{m}, m, m}$$

Among all conjunctions 3.5, only those are equal true whose elements constitute a solution grid in the compatibility matrix:

Compatibility matrix is an encoding of formula 0.1. Theorem 3.1 shows that the basic algorithm transforms this encoding into an encoding of those true-assignments which satisfy the formula. Each true-element in the resulting matrix belongs to a solution grid. The true-assignments appropriate to any resulting true-element are a part of a satisfying true-assignment. These parts can be used to compute the whole satisfying true-assignments by the method of self-reducibility.

Let us notice that, due to formula 3.2, the length of conjunctions in DF 3.6 grows exponentially over power of matrix Y. So, the expected number of iterations in the basic algorithm can be estimated as $O(\log m)$. Thus, the expected running time of the algorithm can be estimated as $O(m^3 \log m)$.

Also, in the case of unsatisfiable formula 0.1, number m in all estimations of time can be replaced by the number of clauses in the minimal unsatisfiable subformula of formula 0.1. That is due to the conjunctions in the formula 3.1: the basic algorithm restricted to those compatibility boxes which are built for a minimal unsatisfiable sub-formula will deplete less true-elements but it will produce the pattern of unsatisfiability.

Before going further, let us notice that, due to Condition 1 of the strings compatibility, the true-assignments, which deliver to their clauses value *false*, can be missed when building the compatibility matrix, because they will produce in their compatibility boxes the rows and columns entirely filled with *false*. So, the compatibility matrix can be built solely on the contradictions between true-assignments satisfying different clauses separately.

4. Asynchronous depletion

Obviously, depletions 3.1 could use the current values of the compatibility boxes which were computed during the current iteration. Such an asynchronous depletion can be expressed with the following back-feeds:

$$(4.1) C_{ij} = C_{ij} \wedge C_{ik}C_{kj}, \ 1 \le i, j, k \le m$$

Really, diagonal boxes C_{ii} in the compatibility matrix are the diagonal Boolean matrices, and⁴

$$C_{ij} = C_{ii}C_{ij} = C_{ij}C_{jj}, C_{ii} \leq C_{ij}C_{ij}^T, C_{jj} \leq C_{ij}^TC_{ij}$$

Thus, depletions 3.1 are conjunctions of depletions 4.1.

Then, the basic algorithm can be reformulated as follows:

Init: Compose the compatibility boxes for formula 0.1 and select a schema to iterate index-triplets

$$(4.2) (i,k,j): 1 \le i,j,k \le m$$

The schemes can be context-sensitive and have other specifics. For example, a schema could "visit" the triplets with different frequencies or even miss some of them. The simplest schemes just cycle trough all index-triplets;

Depletions: Follow the selected iteration schema and deplete the appropriate compatibility boxes using transformations 4.1 until each of these boxes became an invariant under the depletions.

Decision: If the pattern of unsatisfiability emerges during the depletions, then formula 0.1 is unsatisfiable. Otherwise, the formula is satisfiable.

The freedom of selection the iteration schema creates a family of depletion filters: the asynchronous algorithms differ one from another by their iteration schemes. Theorem 3.1 implies

Theorem 4.1. Let an iteration schema traverse all index-triplets 4.2. Then, formula 0.1 is unsatisfiable iff depletions 4.1 produce the pattern of unsatisfiability for this formula.

For 3SAT, computational complexity of the asynchronous algorithms can be estimated as $O(m^3)$: there are m^3 index-triplets 4.2; during iterations 4.1, each of these triplets involves O(1) Boolean operations to deplete the appropriate compatibility boxes of size $2^3 \times 2^3$ or less.

Obviously, depletions 4.1 will detect the pattern of unsatisfiability (if there is one) in time $O(s^3)$, where s is the number of clauses in the shortest unsatisfiable sub-formulas of formula 0.1.

5. Triangular algorithms

Due to the symmetry of the compatibility matrix, any asynchronous depletion can be restricted to those compatibility boxes which are located over/under the major diagonal of the matrix. For example, the compatibility boxes allocated over the major diagonal of the compatibility matrix are described with the following index-triplets:

$$(5.1) 1 \le i < k < j \le m$$

 $^{^{4}}$ false = false, false < true, true = true.

For these index-triplets, depletions 4.1 have to be rewritten as follows:

(5.2)
$$\begin{cases} C_{ij} = C_{ij} \wedge C_{ik}C_{kj} \\ C_{ik} = C_{ik} \wedge C_{ij}C_{kj}^T \\ C_{kj} = C_{kj} \wedge C_{ik}^T C_{ij} \end{cases}$$

Theorem 4.1 implies the following.

Theorem 5.1. Let an iteration schema traverse all index-triplets 5.1. Then, formula 0.1 is unsatisfiable iff depletions 5.2 produce the pattern of unsatisfiability for this formula.

Any such algorithm we call a triangular algorithm.

6. Square algorithms

For the (0,1)-version of the compatibility matrix, depletions 3.3 can be rewritten "sub-tropically":

$$(6.1) x_{\mu\nu ij} = \min\{x_{\mu\nu ij}, \min\{1, \left| \frac{1}{m} \sum_{\beta=1}^{m} \max\{x_{\mu\alpha i\beta} x_{\alpha\mu\beta j} \mid 1 \le \alpha \le 2^{k_{\beta}}\} \right| \}\},$$

- where $x_{\mu\nu ij}$ is the (μ, ν) -th element of compatibility box C_{ij} , k_{β} is the number of literals in clause c_{β} , and |*| is the whole part of a number.

Obviously, all the above algorithms do work with depletions 6.1 and have the same computational complexity. The above theorems imply

Theorem 6.1. Formula 0.1 is unsatisfiable iff a square algorithm produces the pattern of unsatisfiability for this formula.

We call such algorithms the square algorithms because of the following reason. Obviously, the compatibility matrix for formula 0.1 contains a solution grid iff there are m such permutation matrices X_1, X_2, \ldots , and X_m which would satisfy the following system of inequalities⁵:

(6.2)
$$X_i C_{ij} X_j^T \ge \begin{pmatrix} 1 & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}_{2^{k_i} \times 2^{k_j}}, \ i, j = 1, 2, \dots, m$$

- where k_i and k_j are the numbers of literals in clauses c_i and c_j appropriately. This system just reflect that fact that, for a satisfiable formula 0.1, there is such a re-enumeration of true-assignments in truth-tables T_i which would move a solution grid's elements in the upper left corners of their compatibility boxes.

System 6.2 is a system of integral bilinear inequalities. Iterations 6.1 just solve this system. Formula 6.1 iteratively compares the squares of the compatibility matrix with the squares of that matrix on the right side of system 6.2 and depletes the compatibility matrix appropriately.

The result of such depletions is a general solution of system 6.2 in the form of the compatibility matrix encoding of all true-assignments satisfying formula 0.1.

⁵For matrices $A=(a_{\mu\nu})$ and $B=(b_{\mu\nu})$ of the same size, relation $A\geq B$ means that $\forall \mu, \nu \ (a_{\mu\nu}\geq b_{\mu\nu})$.

Conclusion

This paper described on high-level several polynomial time algorithms for 3SAT. The algorithms are based on a original encoding of 3SAT instances - a compatibility matrix. The encoding reflects the contradictions between true-assignments satisfying different clauses separately. Such a "hologram" of 3SAT instances allows parallel testing of all guesses. The algorithms transform the compatibility matrix into a matrix encoding of the satisfying true-assignments.

Basically, the compatibility matrix presents 3SAT instances in the form a jigsaw puzzle, and the algorithms simulate the human activities during the solving of the puzzle.

For some of the algorithms, there are demos at [2]. Readers can use these demos to probe the algorithms with their own 3SAT instances.

References

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- [2] 3SAT Solver, Polynomial Times, http://www.timescube.com