

# Graph Operations on Clique-Width Bounded Graphs

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## Abstract

In this paper we survey the behavior of various graph operations on the graph parameters clique-width and NLC-width. We give upper and lower bounds for the clique-width and NLC-width of the modified graphs in terms of the clique-width and NLC-width of the involved graphs. Therefore we consider the binary graph operations join, disjoint union, union, products, corona, substitution, and 1-sum, and the unary graph operations subgraph, edge complement, bipartite edge complement, power of graphs, line graphs, local complementation, switching, edge addition, edge deletion, edge subdivision, vertex identification, and vertex addition.

**Keywords:** clique-width, NLC-width, graph operations

## 1 Introduction

The clique-width of a graph is defined by Courcelle and Olariu in [CO00] through a composition mechanism for vertex-labeled graphs. The operations are the vertex-disjoint union, the addition of edges between vertices controlled by a label pair, and the relabeling of vertices. The clique-width of a graph  $G$  is the minimum number of labels needed to define it. The NLC-width of a graph is defined by Wanke in [Wan94] through a composition mechanism similar to that for clique-width. Every graph of clique-width at most  $k$  has NLC-width at most  $k$  and every graph of NLC-width at most  $k$  has clique-width at most  $2k$  [Joh98]. The only essential difference between the composition mechanisms of clique-width bounded graphs and NLC-width bounded graphs is the addition of edges. In an NLC-width composition the addition of edges is combined with the union operation. This union operation applied to two graphs  $G$  and  $J$  is controlled by a set  $S$  of label pairs such that for every pair  $(a, b) \in S$  all vertices of  $G$  labeled by  $a$  will be connected with all vertices of  $J$  labeled by  $b$ . Both concepts are useful, because it is sometimes much more convenient to use NLC-width expressions instead of clique-width expressions and vice versa, respectively.

Clique-width and NLC-width bounded graphs are particularly interesting from an algorithmic point of view. Several NP-complete graph problems can be solved in polynomial time for graphs of bounded clique-width. For example, all graph properties which are expressible in monadic second order logic with quantifications over vertices and vertex sets (MSO<sub>1</sub>-logic) are

decidable in linear time on clique-width bounded graphs [CMR00]. Furthermore, there are also a lot of NP-complete graph problems which are not expressible in  $\text{MSO}_1$ -logic like Hamiltonicity, partition problems, and bounded degree subgraph problems but which can also be solved in polynomial time on clique-width bounded graphs [Wan94, EGW01, KR03, ST07, GW06].

Distance-hereditary graphs have clique-width at most 3 [GR00]. The set of all graphs of clique-width at most 2 or NLC-width 1 is the set of all co-graphs, i.e.  $P_4$ -free graphs. Brandstädt et al. have analyzed the clique-width of graphs defined by forbidden one-vertex extensions of  $P_4$  [BDLM05]. The clique-width and NLC-width of permutation graphs, interval graphs, grids and planar graphs is not bounded [GR00]. An arbitrary graph with  $n$  vertices has clique-width at most  $n - r$ , if  $2^r < n - r$ , and NLC-width at most  $\lceil n/2 \rceil$  [Joh98]. Every graph of tree-width<sup>1</sup> at most  $k$  has clique-width at most  $3 \cdot 2^{k-1}$  [CR05]. In [GW00], it is shown that every graph of clique-width or NLC-width  $k$  which does not contain the complete bipartite graph  $K_{n,n}$  for some  $n > 1$  as a subgraph has tree-width at most  $3k(n - 1) - 1$ . The recognition problem for graphs of clique-width or NLC-width at most  $k$  is still open for  $k \geq 4$  and  $k \geq 3$ , respectively. The problem whether a graph has clique-width at most 3 is decidable in polynomial time [CHL<sup>+</sup>12] and the problem whether a graph has NLC-width at most 2 is also decidable in polynomial time [Joh00]. By the characterization in terms of co-graphs, it can be decided in linear time whether a graph has clique-width at most 2 or NLC-width 1 [CPS85]. Computing NLC-width and computing clique-width is NP-hard [GW07, FRRS09]. The clique-width of tree-width bounded graphs is computable in linear time [EGW03].

A graph operation  $f$  is an operation which creates a new graph  $f(G_1, \dots, G_n)$  from a number of  $n \geq 1$  input graphs  $G_1, \dots, G_n$ . The graph theory books by Bondy and Murty [BM76] and by Harary [Har69] include a large number of operations on graphs. The behaviour of graph operations on several graph parameters is well studied, e.g. for band-width in [CO86], for tree-width in [Bod98], for clique-width briefly in [HOSG08] and [Cou14], and for rank-width in [HOSG08]. In this paper we examine the behavior of various graph operations on the graph parameters clique-width and NLC-width.

Graph operation can be used to characterize sets of graphs by forbidden graphs. The tree-width, which was defined in the 1980s by Robertson and Seymour in [RS86], is less powerful than clique-width, since the clique-width of a graph can be bounded in its tree-width but not vice-versa. Furthermore the property that a graph has tree-width at most  $k$  is preserved under the operation taking minors, which is used to show that the set of graphs of tree-width at most  $k$  can be characterized by a finite set of forbidden minors [RS85]. Oum and Seymour introduced in [OS06] the rank-width of graphs, which is defined independently of vertex labels, but which is shown to be as powerful as clique-width. In [Oum05b] it is shown that the property that a graph has rank-width at most  $k$  is closed under the operation taking local complementation, which leads to a characterization of graphs of rank-width at most  $k$  by finitely many forbidden vertex-minors (i.e. taking induced subgraphs and local complementations). It is still open if there exists a graph operation that does not increase the NLC-width or clique-width and which can be used to characterize graphs of NLC-width at most  $k$  or clique-width at most  $k$  by a set of finitely many forbidden subgraphs. This is one reason why we want to study the behavior of graph operations on the NLC-width and clique-width of graphs more precisely.

In [OS06] it is shown that for every fixed  $k$  for every given graph  $G$  we can compute in polynomial time a clique-width  $f(k)$ -expression or assert that the clique-width of  $G$  is greater

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<sup>1</sup>See [Bod98] for definition and an overview on tree-width.

than  $k$ . Since  $f(k)$  can be exponential in the clique-width of the given graph and nearly all algorithms for hard graph problems on clique-width bounded graphs have a running time exponential in the number of used labels, it is important to find expressions using few labels. In order to deal with this problem, our survey shows for various graph operations  $f$  how to construct an expression for graph  $f(G)$  from an expression for some graph  $G$ .

This paper is organized as follows. In Section 2, we recall the definitions of clique-width and NLC-width. In Section 3, we give an overview on the behavior of the binary operations join, disjoint union, union, products, corona, substitution, and 1-sum on the clique-width and NLC-width of a given graph. In Section 4, we consider the latter problem for the unary graph operations quotient, subgraph, edge complement, bipartite edge complement, power of graphs, line graphs, switching, local complementation, edge addition, edge subdivision, vertex identification, and vertex addition. In Section 5, we show how these results can be used to give upper bounds on the clique-width and NLC-width of graph classes. In Section 6, we give some conclusions and we compare the shown bounds within two tables.

## 2 Preliminaries

**Graphs** We work with finite undirected *graphs*  $G = (V_G, E_G)$ , where  $V_G$  is a finite set of *vertices* and  $E_G \subseteq \{\{u, v\} \mid u, v \in V_G, u \neq v\}$  is a finite set of *edges*. For a vertex  $v \in V_G$  we denote by  $N_G(v)$  the set of all vertices which are adjacent to  $v$  in  $G$ , i.e.  $N_G(v) = \{w \in V_G \mid \{v, w\} \in E_G\}$ . Vertex set  $N_G(v)$  is called the set of all *neighbors* of  $v$  in  $G$  or *neighborhood* of  $v$  in  $G$ . Please note that  $v$  does not belong to  $N_G(v)$ . The *degree* of a vertex  $v \in V_G$ , denoted by  $\deg_G(v)$ , is the number of neighbors of vertex  $v$  in  $G$ , i.e.  $\deg_G(v) = |N_G(v)|$ . We are discussing graphs only up to isomorphism. This allows us to define the path on  $n$  vertices  $P_n = (\{v_1, \dots, v_n\}, \{\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}\})$ .

**Labeled Graphs** Let  $[k] := \{1, \dots, k\}$  be the set of all integers, which are greater than or to equal to 1 and less than or equal to  $k$ . We also work with finite undirected labeled *graphs*  $G = (V_G, E_G, \text{lab}_G)$ , where  $V_G$  is a finite set of *vertices* labeled by some mapping  $\text{lab}_G : V_G \rightarrow [k]$  and  $E_G \subseteq \{\{u, v\} \mid u, v \in V_G, u \neq v\}$  is a finite set of *edges*. The labeled graph consisting of a single vertex labeled by  $a \in [k]$  is denoted by  $\bullet_a$ . Most of the definitions for unlabeled graphs can be applied to labeled graphs. Thus, we just want to mention subgraphs and isomorphism for labeled graphs.

A labeled graph  $J = (V_J, E_J, \text{lab}_J)$  is a *subgraph* of  $G$  if  $V_J \subseteq V_G$ ,  $E_J \subseteq E_G$  and  $\text{lab}_J(u) = \text{lab}_G(u)$  for all  $u \in V_J$ .  $J$  is an *induced subgraph* of  $G$  if additionally  $E_J = \{\{u, v\} \in E_G \mid u, v \in V_J\}$ .

Two labeled graphs  $G$  and  $J$  are *isomorphic* if there is a bijection  $f : V_G \rightarrow V_J$ , such that for all  $u, v \in V_G$

$$\{u, v\} \in E_G \iff \{f(u), f(v)\} \in E_J$$

and

$$\text{lab}_G(u) = \text{lab}_G(v) \iff \text{lab}_J(u) = \text{lab}_J(v)$$

**Clique-width** The notion of clique-width<sup>2</sup> for labeled graphs is defined by Courcelle and Olariu in [CO00] as follows. Let  $k$  be some positive integer. The class  $\text{CW}_k$  of labeled graphs

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<sup>2</sup>The operations in the definition of clique-width were first considered by Courcelle, Engelfriet, and Rozenberg in [CER93].

is recursively defined as follows.

1. The single vertex graph  $\bullet_a$  for some  $a \in [k]$  is in  $\text{CW}_k$ .
2. Let  $G = (V_G, E_G, \text{lab}_G) \in \text{CW}_k$  and  $J = (V_J, E_J, \text{lab}_J) \in \text{CW}_k$  be two vertex-disjoint labeled graphs, then

$$G \oplus J := (V', E', \text{lab}')$$

defined by  $V' := V_G \cup V_J$ ,  $E' := E_G \cup E_J$ , and for every  $u \in V'$

$$\text{lab}'(u) := \begin{cases} \text{lab}_G(u) & \text{if } u \in V_G \\ \text{lab}_J(u) & \text{if } u \in V_J \end{cases}$$

is in  $\text{CW}_k$ .

3. Let  $a, b \in [k]$  be two distinct integers and  $G = (V_G, E_G, \text{lab}_G) \in \text{CW}_k$  be a labeled graph, then

(a)  $\rho_{a \rightarrow b}(G) := (V_G, E_G, \text{lab}')$  defined by for every  $u \in V_G$

$$\text{lab}'(u) := \begin{cases} \text{lab}_G(u) & \text{if } \text{lab}_G(u) \neq a \\ b & \text{if } \text{lab}_G(u) = a \end{cases}$$

is in  $\text{CW}_k$  and

(b)  $\eta_{a,b}(G) := (V_G, E', \text{lab}_G)$  defined by

$$E' := E_G \cup \{\{u, v\} \mid u, v \in V_G, u \neq v, \text{lab}(u) = a, \text{lab}(v) = b\}$$

is in  $\text{CW}_k$ .

The *clique-width* of a labeled graph  $G$  is the least integer  $k$  such that  $G \in \text{CW}_k$ .

An expression  $X$  built with the operations  $\bullet_a, \oplus, \rho_{a \rightarrow b}, \eta_{a,b}$  for integers  $a, b \in [k]$  is called a *clique-width  $k$ -expression*. The graph defined by expression  $X$  is denoted by  $\text{val}(X)$ . The *clique-width* of an unlabeled graph  $G = (V, E)$  is the smallest integer  $k$ , for which there is some mapping  $\text{lab} : V \rightarrow [k]$  such that the labeled graph  $(V, E, \text{lab})$  has clique-width at most  $k$ .

**NLC-width** The notion of NLC-width<sup>3</sup> of labeled graphs is defined by Wanke in [Wan94] as follows. Let  $k$  be some positive integer. The class  $\text{NLC}_k$  of labeled graphs is recursively defined as follows.

1. The single vertex graph  $\bullet_a$  for some  $a \in [k]$  is in  $\text{NLC}_k$ .
2. Let  $G = (V_G, E_G, \text{lab}_G) \in \text{NLC}_k$  and  $J = (V_J, E_J, \text{lab}_J) \in \text{NLC}_k$  be two vertex-disjoint labeled graphs and  $S \subseteq [k]^2$  be a relation, then

$$G \times_S J := (V', E', \text{lab}')$$

defined by  $V' := V_G \cup V_J$ ,

$$E' := E_G \cup E_J \cup \{\{u, v\} \mid u \in V_G, v \in V_J, (\text{lab}_G(u), \text{lab}_J(v)) \in S\},$$

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<sup>3</sup>The abbreviation NLC results from the *node label controlled* embedding mechanism originally defined for graph grammars.

and for every  $u \in V'$

$$\text{lab}'(u) := \begin{cases} \text{lab}_G(u) & \text{if } u \in V_G \\ \text{lab}_J(u) & \text{if } u \in V_J \end{cases}$$

is in  $\text{NLC}_k$ .

3. Let  $G = (V_G, E_G, \text{lab}_G) \in \text{NLC}_k$  and  $R : [k] \rightarrow [k]$  be a function, then

$$\circ_R(G) := (V_G, E_G, \text{lab}')$$

defined by for every  $u \in V_G$

$$\text{lab}'(u) := R(\text{lab}_G(u))$$

is in  $\text{NLC}_k$ .

The *NLC-width* of a labeled graph  $G$  is the least integer  $k$  such that  $G \in \text{NLC}_k$ .

An expression  $X$  built with the operations  $\bullet_a, \times_S, \circ_R$  for  $a \in [k]$ ,  $S \subseteq [k]^2$ , and  $R : [k] \rightarrow [k]$  is called an *NLC-width  $k$ -expression*. The graph defined by expression  $X$  is denoted by  $\text{val}(X)$ . The *NLC-width* of an unlabeled graph  $G = (V, E)$  is the smallest integer  $k$ , for which there is some mapping  $\text{lab} : V \rightarrow [k]$  such that the labeled graph  $(V, E, \text{lab})$  has NLC-width at most  $k$ .

**Expression Trees** Every NLC-width  $k$ -expression  $X$  has by its recursive definition a tree structure that is called the *NLC-width  $k$ -expression-tree*  $T$  for  $X$ . Tree  $T$  is an ordered rooted tree whose leaves correspond to the vertices of graph  $\text{val}(X)$  and the inner nodes<sup>4</sup> correspond to the operations of  $X$ , see [GW00]. In the same way we define the clique-width  $k$ -expression-tree for every clique-width  $k$ -expression, see [EGW03]. If integer  $k$  is known from the context or irrelevant for the discussion, then we sometimes use the simplified notion *expression-tree* for the notion  *$k$ -expression-tree*. For some node  $u$  of expression-tree  $T$ , let  $T(u)$  be the subtree of  $T$  rooted at  $u$ . Note that tree  $T(u)$  is always an expression-tree. The expression  $X(u)$  defined by  $T(u)$  can simply be determined by traversing the tree  $T(u)$  starting from the root, where the left children are visited first.  $X(u)$  defines a (possibly) relabeled induced subgraph  $G(u)$  of  $G$ . For an inner node  $v$  of some expression-tree  $T$  and a leaf  $u$  of  $T(v)$  we define by  $\text{lab}(u, G(v))$  the label of that vertex of graph  $G(v)$  that corresponds to  $u$ . A node  $u$  of  $T$  is called a *predecessor* of a node  $u'$  of  $T$  if  $u'$  is on a path from  $u$  to a leaf. A node  $u$  of  $T$  is called the *least common predecessor* of two nodes  $u_1$  and  $u_2$  if  $u$  is a predecessor of both nodes  $u_1, u_2$ , and no child of  $u$  is a predecessor of  $u_1, u_2$ .

**Graph Parameters and Relations** There is a very close relation between the clique-width and the NLC-width of a graph.

**Theorem 2.1 ([Joh98])** *Every clique-width  $k$ -expression can be transformed into an equivalent NLC-width  $k$ -expression and every NLC-width  $k$ -expression can be transformed into an equivalent clique-width  $2k$ -expression. Thus, for every graph  $G$  it holds  $\text{NLC-width}(G) \leq \text{clique-width}(G) \leq 2 \cdot \text{NLC-width}(G)$ .*

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<sup>4</sup>To distinguish between the vertices of (non-tree) graphs and trees, we simply call the vertices of trees *nodes*.

In this paper we also refer to notion of tree-width<sup>5</sup> which was defined in the 1980s by Robertson and Seymour in [RS86] by the existence of a tree-decomposition and to the notion of rank-width which was introduced by Oum and Seymour in [OS06].

### 3 Binary Operations

Let  $G_1 = (V_{G_1}, E_{G_1})$  and  $G_2 = (V_{G_2}, E_{G_2})$  be two graphs and let  $f$  be some binary graph operation which creates a new graph  $f(G_1, G_2)$  from  $G_1$  and  $G_2$ . In this section we consider the NLC-width and clique-width of graph  $f(G_1, G_2)$  with respect to the NLC-width or clique-width of  $G_1$  and  $G_2$ .

We start with preliminary results whose proofs are very easy.

#### 3.1 Disjoint Union

The *disjoint union*  $G_1 \oplus G_2$  of two vertex-disjoint graphs  $G_1$  and  $G_2$  is the graph with vertex set  $V_{G_1} \cup V_{G_2}$  and edge set  $E_{G_1} \cup E_{G_2}$ . Since NLC-width and clique-width operations both contain the disjoint union it is easy to see that

$$\text{NLC-width}(G_1 \oplus G_2) = \max(\text{NLC-width}(G_1), \text{NLC-width}(G_2))$$

and

$$\text{clique-width}(G_1 \oplus G_2) = \max(\text{clique-width}(G_1), \text{clique-width}(G_2)).$$

These bounds imply that the NLC-width and clique-width of a graph can be computed by the maximum NLC-width or clique-width of its connected components.

#### 3.2 Join

The *join*  $G_1 \otimes G_2$  of two vertex-disjoint graphs  $G_1$  and  $G_2$  is the graph with vertex set  $V_{G_1} \cup V_{G_2}$  and edge set  $E_{G_1} \cup E_{G_2} \cup \{\{v_1, v_2\} \mid v_1 \in V_{G_1}, v_2 \in V_{G_2}\}$ . It is obviously that

$$\text{NLC-width}(G_1 \otimes G_2) = \max(\text{NLC-width}(G_1), \text{NLC-width}(G_2))$$

and

$$\text{clique-width}(G_1 \otimes G_2) = \max(\text{clique-width}(G_1), \text{clique-width}(G_2), 2).$$

Since the NLC-width does not change when building the edge complement graph (cf. Section 4.8) we conclude that the NLC-width of a graph also can be computed by the maximum NLC-width of its co-connected components.

#### 3.3 Union

The *union*  $G_1 \cup G_2$  of two graphs  $G_1$  and  $G_2$  with  $V_{G_1} = V_{G_2}$  is the graph defined by the edge set which is the union of the edge sets of  $G_1$  and  $G_2$ . Thus two vertices are adjacent in  $G_1 \cup G_2$  if and only if they are adjacent in  $G_1$  or they are adjacent in  $G_2$ .

Let  $G_1$  be the disjoint union of  $m$  paths  $P_n$ , each represented by a row in the adjacency matrix for  $G_1$ , and  $G_2$  be the disjoint union of  $n$  paths  $P_m$ , each represented by a column in the adjacency matrix for  $G_2$ . Then the union  $G_1 \cup G_2$  is an  $n \times m$  grid. Since paths have

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<sup>5</sup>The concept of tree-width already appeared in a work of Halin [Hal76].

clique-width at most 3 and an  $n \times m$ -grid has clique-width at least  $\min(n, m) + 1$  [GR00], it is not possible to bound the clique-width of  $G_1 \cup G_2$  in the clique-width of  $G_1$  and  $G_2$ , even if  $G_1$  and  $G_2$  are of bounded tree-width.

### 3.4 Substitution

Let  $G_1$  and  $G_2$  be two vertex-disjoint graphs and let  $v \in V_{G_1}$  a vertex. The *substitution* of  $v$  by  $G_2$  in  $G_1$ , denoted by  $G_1[v/G_2]$ , is the graph with vertex set  $V_{G_1} \cup V_{G_2} - \{v\}$  and edge set  $E_{G_1} \cup E_{G_2} - \{\{v, w\} \mid w \in N_{G_1}(v)\} \cup \{\{u, w\} \mid u \in V_{G_2}, w \in N_{G_1}(v)\}$ .

**Theorem 3.1** *Let  $G_1$  and  $G_2$  be two graphs and  $v \in V_{G_1}$  a vertex, then it holds*

$$NLC\text{-width}(G_1[v/G_2]) = \max(NLC\text{-width}(G_1), NLC\text{-width}(G_2))$$

and

$$\text{clique-width}(G_1[v/G_2]) = \max(\text{clique-width}(G_1), \text{clique-width}(G_2)).$$

**Proof** Let  $G_1$  be a graph of NLC-width  $k_1$ ,  $v \in V_{G_1}$  a vertex, and  $G_2$  be a graph of NLC-width  $k_2$ . Let  $T_1$  be an NLC-width  $k_1$ -expression-tree for  $G_1$  and  $T_2$  be an NLC-width  $k_2$ -expression-tree for  $G_2$ . Next we construct from  $T_1$  and  $T_2$  an expression-tree  $T$  for  $G_1[v/G_2]$ . We start with a copy  $T$  of  $T_1$ . Let  $x$  be the leaf of  $T$  that corresponds to vertex  $v$ . We relabel  $x$  from  $\bullet_\ell$  into  $\circ_R$ ,  $R(a) = \ell$  for  $a \in [k_2]$ . Then we insert a copy of  $T_2$  in  $T$  and make the root of the copy of  $T_2$  adjacent to leaf  $x$  of  $T$ . The resulting tree is an expression-tree for  $G_1[v/G_2]$  using  $\max(k_1, k_2)$  labels.

The clique-width result can be shown in the same way, see [CO00].  $\square$

Vertex set  $V_{G_2}$  is also called a *module* of graph  $G_1[v/G_2]$ , since all vertices of  $V_{G_2}$  have the same neighbors in the graph  $G_1[v/G_2]$ .

### 3.5 Product

A graph product of two vertex-disjoint graphs  $G_1$  and  $G_2$  is a new graph whose vertex set is  $V_{G_1} \times V_{G_2}$  and for two vertices  $(u_1, u_2)$  and  $(v_1, v_2)$  the adjacency in the product is defined by the adjacency, equality, or non-adjacency of  $u_1$  and  $v_1$  in  $G_1$  and of  $u_2$  and  $v_2$  in  $G_2$ . We consider the following five well known possibilities to define graph products.

Graph Product	Edge set = $\{\{(u_1, u_2), (v_1, v_2)\} \mid$
Cartesian	$(u_1 = v_1 \wedge \{u_2, v_2\} \in E_{G_2}) \vee (u_2 = v_2 \wedge \{u_1, v_1\} \in E_{G_1})\}$
Categorical	$\{u_1, v_1\} \in E_{G_1} \wedge \{u_2, v_2\} \in E_{G_2}\}$
Normal	$(u_1 = v_1 \wedge \{u_2, v_2\} \in E_{G_2}) \vee (\{u_1, v_1\} \in E_{G_1} \wedge u_2 = v_2) \vee$ $(\{u_1, v_1\} \in E_{G_1} \wedge \{u_2, v_2\} \in E_{G_2})\}$
Co-Normal	$\{u_1, v_1\} \in E_{G_1} \vee \{u_2, v_2\} \in E_{G_2}\}$
Lexicographic	$(\{u_1, v_1\} \in E_{G_1}) \vee (u_1 = v_1 \wedge \{u_2, v_2\} \in E_{G_2})\}$

Several results on these graph products and further products can be found in [Har69, IK00, JT94].

The cartesian, categorical, normal, and co-normal graph product applied to two paths  $P_n$  and  $P_m$  leads a graph whose clique-width cannot be bounded independently from  $n$  and  $m$ . Thus it is not possible to bound the clique-width of the cartesian, categorical, normal, or co-normal graph product in the clique-width of the involved graphs.

The lexicographic graph product, which is also known as *graph composition*, of two graphs  $G_1$  and  $G_2$  is denoted by  $G_1[G_2]$ . Let  $G^0 = G_1$  and  $V_{G_1} = \{v_1, \dots, v_n\}$ . Then  $G^i = G^{i-1}[v_i/G_2]$  for  $i = 1, \dots, n$  is a sequence of  $n$  substitutions, such that  $G^n$  defines graph  $G_1[G_2]$ . Thus we can apply Theorem 3.1 to obtain the following bounds.

**Corollary 3.2** *Let  $G_1$  and  $G_2$  be two graphs, then it holds*

$$NLC\text{-width}(G_1[G_2]) = \max(NLC\text{-width}(G_1), NLC\text{-width}(G_2))$$

and

$$clique\text{-width}(G_1[G_2]) = \max(clique\text{-width}(G_1), clique\text{-width}(G_2)).$$

### 3.6 1-Sum

Let  $G_1$  and  $G_2$  be two vertex-disjoint graphs and let  $v \in V_{G_1}$  and  $w \in V_{G_2}$ . The *1-sum*  $G_1 \oplus_{v,w} G_2$  has vertex set  $V_{G_1} \cup V_{G_2} - \{v, w\} \cup \{z\}$  and edge set  $E_{G_1} \cup E_{G_2} - \{\{v, v_1\} \in E_{G_1} \mid v_1 \in V_{G_1}\} - \{\{w, w_1\} \in E_{G_2} \mid w_1 \in V_{G_2}\} \cup \{\{z, z_1\} \mid z_1 \in N_{G_1}(v) \cup N_{G_2}(w)\}$ . That is, by  $G_1 \oplus_{v,w} G_2$  we denote the graph which we obtain by the disjoint union of  $G_1$  and  $G_2$  in which vertices  $v$  and  $w$  are identified.

**Theorem 3.3** *Let  $G_1$  and  $G_2$  be two graphs,  $v \in V_{G_1}$  be a vertex, and  $w \in V_{G_2}$  be a vertex. For  $m_1 = \max(NLC\text{-width}(G_1), NLC\text{-width}(G_2))$  it holds*

$$m_1 \leq NLC\text{-width}(G_1 \oplus_{v,w} G_2) \leq m_1 + 1$$

and for  $m_2 = \max(clique\text{-width}(G_1), clique\text{-width}(G_2))$  it holds

$$m_2 \leq clique\text{-width}(G_1 \oplus_{v,w} G_2) \leq m_2 + 2.$$

**Proof** Let  $G_1$  be a graph of NLC-width  $k_1$ ,  $v \in V_{G_1}$  a vertex,  $G_2$  be a graph of NLC-width  $k_2$  and  $w \in V_{G_2}$  a vertex. Let  $T_1$  be an NLC-width  $k_1$ -expression-tree for  $G_1$  and  $T_2$  be an NLC-width  $k_2$ -expression-tree for  $G_2$ . We now construct an expression-tree  $T$  for graph  $G_1 \oplus_{v,w} G_2$  from  $T_1$  and  $T_2$ , which uses  $m_1 + 1$  labels.

We start with a copy  $T$  of  $T_2$ . Let  $x$  be the leaf of  $T$  that corresponds to vertex  $w$ . We relabel  $x$  to  $\bullet_{m_1+1}$  in order to substitute vertex  $w$  by vertex  $z$ . Now we consider all union nodes  $x_1$  on the path from  $x$  to the root of  $T$  in  $T$ . If  $x$  is a left (right) child of  $x_1$  and union node  $x_1$  is labeled by  $\times_S$  and  $(\text{lab}(x, G(x_1)), \ell) \in S$  ( $(\ell, \text{lab}(x, G(x_1))) \in S$ ) for some  $\ell \in [k_2]$  then we relabel  $x_1$  by  $\times_{S'}$ , where  $S' = S \cup \{(\ell, m_1 + 1) \mid (\ell, \text{lab}(x, G(x_1))) \in S, \ell \in [k_2]\}$  ( $S' = S \cup \{(\ell, m_1 + 1) \mid (\ell, \text{lab}(x, G(x_1))) \in S, \ell \in [k_2]\}$ ). This is done in order to make in  $G_1 \oplus_{v,w} G_2$  all vertices adjacent to  $z$  which are adjacent to  $w$  in  $G_2$ .

We insert a new root  $r$  labeled by  $\circ_R$  and an edge from  $r$  to the old root of  $T$  into  $T$ . The relabeling  $R$  maps every label from  $[m_1]$  to  $m_1 + 1$  and label  $m_1 + 1$  to  $\ell$ , if the leaf  $y$  in  $T_1$  which corresponds to vertex  $v$  is labeled by  $\bullet_\ell$ . Formally we have  $R : [m_1 + 1] \rightarrow [m_1 + 1]$  and  $R(a) = m_1 + 1$  if  $1 \leq a \leq m_1$  and  $R(a) = \ell$  if  $a = m_1 + 1$ .

Further we insert a copy of  $T_1$  in  $T$  and replace leaf  $y$  by the root  $r$ . The labeling  $\ell$  for vertex  $z$  ensures that all which are adjacent to  $v$  in  $G_1$  become adjacent to  $z$  in  $G_1 \oplus_{v,w} G_2$ . The new root of  $T$  is the root of  $T_1$ . Now  $T$  defines graph  $G_1 \oplus_{v,w} G_2$ .

Since  $G_1$  and  $G_2$  are induced subgraphs of  $G_1 \oplus_{v,w} G_2$ , the NLC-width of  $G_1 \oplus_{v,w} G_2$  is at least the maximum of  $NLC\text{-width}(G_1)$  and  $NLC\text{-width}(G_2)$ .



In the same way we can show the clique-width result. The only difference is that we need one more label in order to realize the relabeling operation.  $\square$

The shown NLC-width bounds are tight for  $m_1 = 1$  and  $m_1 = 2$ , which can be verified by the 1-sums  $P_2 \oplus_{v,w} P_3$  and  $P_5 \oplus_{v,w} P_6$ , where  $v$  and  $w$  are vertices of degree one in the combined paths.

Vertex  $z$  is also called an *articulation vertex* of graph  $G_1 \oplus_{v,w} G_2$ , since  $G_1 \oplus_{v,w} G_2$  without  $z$  has more connected components than  $G_1 \oplus_{v,w} G_2$ . The maximal connected subgraphs of some graph  $G$  without any articulation vertex are called *blocks* of  $G$ . The bounds of Theorem 3.3 imply that the NLC-width and clique-width of a graph can be estimated by the maximum NLC-width or clique-width of its blocks and its number of blocks. It is also known from [BL02, LR04b] that every graph of clique-width  $k$  contains a block whose clique-width is at least  $k - 2$ .

### 3.7 Corona

The corona of graphs was introduced by Frucht and Harary in [FH70] as follows. The *corona*  $G_1 \wedge G_2$  of two vertex-disjoint graphs  $G_1$  and  $G_2$  consists of the disjoint union of one copy of  $G_1$  and  $|V_{G_1}|$  copies of  $G_2$  and each vertex of the copy of  $G_1$  is connected to all vertices of one copy of  $G_2$ , i.e.  $|V_{G_1}| \cdot |V_{G_2}|$  edges are inserted in the disjoint union of the  $|V_{G_1}| + 1$  graphs.

The corona of  $G_1$  and  $G_2$  can be obtained by applying 1-sum operations as follows. Let  $V_1 = \{v_1, \dots, v_n\}$  be the vertex set of  $G_1$ . For  $i = 1 \dots, n$  we take a copy of  $G_2$  and insert a dominating vertex  $w_i$  to that copy and obtain a graph  $G_{2,i}$ . Then by the sequence of 1-sums  $(\dots((G_1 \oplus_{v_1, w_1} G_{2,1}) \oplus_{v_2, w_2} G_{2,2}) \dots) \oplus_{v_n, w_n} G_{2,n}$  we obtain the corona  $G_1 \wedge G_2$ .

By this observation, we can bound the NLC-width and clique-width of  $G_1 \wedge G_2$  in the NLC-width or clique-width of its combined graphs by applying the idea of the proof of Theorem 3.3 on every leaf in an expression-tree for  $G_1$ .

**Corollary 3.4** *Let  $G_1$  and  $G_2$  be two graphs. For  $m_1 = \max(\text{NLC-width}(G_1), \text{NLC-width}(G_2))$  it holds*

$$m_1 \leq \text{NLC-width}(G_1 \wedge G_2) \leq m_1 + 1$$

*and for  $m_2 = \max(\text{clique-width}(G_1), \text{clique-width}(G_2))$  it holds*

$$m_2 \leq \text{clique-width}(G_1 \wedge G_2) \leq m_2 + 2.$$

## 4 Unary Operations

Let  $G = (V_G, E_G)$  be a graph and  $f$  be some unary graph operation which creates a new graph  $f(G)$  from  $G$ . In this section we consider the NLC-width and clique-width of graph  $f(G)$  with respect to the NLC-width or clique-width of  $G$ .

### 4.1 Vertex Deletion and Vertex Addition

**Vertex Deletion** Let  $G$  be a graph and  $v \in V_G$ . By  $G - v$  we denote the graph which we obtain from  $G$  by removing vertex  $v$  and all edges incident to  $v$ . That is,

$$G - v = (V_G - \{v\}, E_G - \{\{v, v'\} \mid v' \in N(v)\}).$$

An expression for graph  $G - v$  can be obtained by an expression  $X$  from graph  $G$  by restricting  $X$  to the vertices of  $G - v$ .

**Vertex Addition** Let  $G$  be a graph and  $v \notin V_G$ . By  $G + v$  we denote the graph which we obtain from  $G$  by inserting vertex  $v$  with an arbitrary neighborhood  $N(v) \subseteq V_G$ . That is,

$$G + v = (V_G \cup \{v\}, E_G \cup \{\{v, v'\} \mid v' \in N(v)\}).$$

In the special case where  $N(v) = \{v'\}$  for some  $v' \in V_G$  we denote  $v$  as a *pendant vertex* and where  $N(v) = V_G$  we denote  $v$  as a *dominating vertex*.

We consider the NLC-width and clique-width of graph  $G + v$ .

**Theorem 4.1** *Let  $G$  be a graph and  $x \notin V_G$ , then it holds*

$$NLC\text{-width}(G) \leq NLC\text{-width}(G + v) \leq 2 \cdot NLC\text{-width}(G)$$

and

$$clique\text{-width}(G) \leq clique\text{-width}(G + v) \leq 2 \cdot clique\text{-width}(G).$$

**Proof** Let  $G$  be a graph of NLC-width  $k$ ,  $v \notin V_G$  be a vertex, and  $T$  be an NLC-width  $k$ -expression-tree that defines graph  $G$  and We now define an NLC-width  $2k$ -expression-tree that defines graph  $G + v$ . We start with a copy  $T'$  of  $T$ .

First we want to separate the neighborhood of  $v$  from the non-neighborhood by introducing  $k$  further labels  $k + 1, \dots, 2k$ . Every leaf of  $T'$  that corresponds to a vertex from  $G$  which is not from  $N(v)$  will be relabeled from label  $\bullet_a$ ,  $a \in [k]$ , into  $\bullet_{a+k}$ .

Then we consider all nodes  $x$  on the paths from these relabeled leaves to the root of the so defined tree. If node  $x$  is a union node labeled by some  $\times_S$ ,  $S \subseteq [k]^2$ , then we relabel  $x$  by  $\times_{S'}$  where  $S' = \{(a, b), (a, b + k), (a + k, b), (a + k, b + k) \mid (a, b) \in S\}$ . If node  $x$  is a relabeling node labeled by some  $\circ_R$ ,  $R : [k] \rightarrow [k]$ , then we relabel  $x$  by  $\circ_{R'}$ , where  $R' : [2k] \rightarrow [2k]$  and  $R'(a) = R(a)$ , if  $i \leq k$  and  $R'(a) = R(a) + k$ , if  $k + 1 \leq a \leq 2k$ . The resulting tree is denoted by  $T''$ .

In a last step we insert two additional nodes  $t_v$  and  $t_r$  labeled by  $\bullet_1$  and  $\times_{\{(1, a) \mid a \in [k]\}}$ , respectively and two additional arcs from  $t_r$  to  $t_v$  and from  $t_r$  to the root of  $T''$  in  $T''$ , such that  $t_v$  is the left child of  $t_r$ .

The resulting tree is denoted by  $T'''$ . Tree  $T'''$  is an NLC-width  $2k$ -expression-tree and  $T'''$  defines the graph  $G + v$ .

Since  $G$  is an induced subgraph of  $G + v$ , the NLC-width of  $G + v$  cannot be less than the NLC-width of  $G$ .

To prove the corresponding clique-width bound, we have to find a label for vertex  $v$ , which is not used in the graph defined by  $G(T'')$ , since clique-width does not allow edge insertions between equal labeled vertices. This can be done by relabeling all vertices  $G(T'')$  labeled by  $k + 1, \dots, 2k$  by e.g.  $k + 1$  and then we can take, for  $k \geq 2$ , one of the free labels e.g. label  $2k$  to label the inserted vertex  $v$ . In the case  $k = 1$ ,  $G$  consists of isolated vertices and  $G + v$  is the disjoint union of one  $K_{1,p}$ , for some  $p$ , and isolated vertices. Thus also in this case  $G + v$  has clique-width  $2k = 2$ .  $\square$

The shown NLC-width bounds are tight for graphs of width one and two. If we insert a vertex in a path of length 2 to get a path of length 3, we insert a vertex in a graph NLC-width 1 and obtain a graph of NLC-width 2. If we insert vertex  $v$  in the graph  $H - v$  of Figure 1, we insert a vertex in a graph NLC-width 2 and obtain a graph of NLC-width 4.

Further it is possible to bound the NLC-width and clique-width of  $G + v$  in the NLC-width and clique-width of  $G$  and the vertex degree  $d$  of  $v$ . The main idea is to label each vertex of

$G$  which should be adjacent to vertex  $v$  by a new label from  $\{k+1, \dots, k+d\}$ . Then, the new vertex can easily be inserted in a last step. If we use clique-width operations we first have to relabel at least one of the used labels from  $\{1, \dots, k\}$  to get a free label in order to insert the new vertex.

**Corollary 4.2** *Let  $G$  be graph and  $G + v$  be obtained from  $G$  by inserting a new vertex  $v$  of degree  $d$ , then it holds*

$$NLC\text{-width}(G) \leq NLC\text{-width}(G + v) \leq NLC\text{-width}(G) + d$$

and

$$\text{clique-width}(G) \leq \text{clique-width}(G + v) \leq \text{clique-width}(G) + d.$$

The addition of a vertex of high degree  $d' = |V| - d$  can be done more efficiently by adding a vertex of degree  $d$  in the edge complement and building the edge complement of the result. By the bounds of Section 4.8 and Theorem 2.1 we get the following results.

**Corollary 4.3** *Let  $G$  be graph and  $G + v$  be obtained from  $G$  by inserting a new vertex  $v$  of degree  $|V| - d$ , then it holds*

$$NLC\text{-width}(G) \leq NLC\text{-width}(G + v) \leq NLC\text{-width}(G) + d$$

and

$$\text{clique-width}(G) \leq \text{clique-width}(G + v) \leq 2 \cdot \text{clique-width}(G) + 2d.$$

## 4.2 Edge Addition and Edge Deletion

Let  $G$  be a graph and  $v, w \in V_G$ . For  $\{v, w\} \notin E_G$  we define by  $G + \{v, w\}$  the graph we obtain from  $G$  by adding edge  $\{v, w\}$ . That is,

$$G + \{v, w\} = (V_G, E_G \cup \{\{v, w\}\}).$$

For  $\{v, w\} \in E_G$  we define by  $G - \{v, w\}$  the graph we obtain from  $G$  by deleting edge  $\{v, w\}$ . That is,

$$G - \{v, w\} = (V_G, E_G - \{\{v, w\}\}).$$

Our next theorem shows that we can insert or delete an edge in a graph using at most 2 more labels.

**Theorem 4.4** *Let  $G$  be a graph and  $v, w \in V_G$  be two different vertices, then it holds*

$$NLC\text{-width}(G) - 2 \leq NLC\text{-width}(G \pm \{v, w\}) \leq NLC\text{-width}(G) + 2$$

and

$$\text{clique-width}(G) - 2 \leq \text{clique-width}(G \pm \{v, w\}) \leq \text{clique-width}(G) + 2.$$

**Proof** In order to show the upper bound on the NLC-width, let  $G$  be a graph of NLC-width  $k$  and let  $v$  and  $w$  be two non-adjacent vertices of  $G$ . Further, let  $T$  be an NLC-width  $k$ -expression-tree that defines  $G$ . We now define a new NLC-width  $(k+2)$ -expression-tree that defines  $G + \{v, w\}$ . We start with a copy  $T'$  of  $T$ . Let  $x$  and  $y$  be the leaves of  $T'$  that

correspond to vertices  $v$  and  $w$ , respectively, of graph  $G$ . First, we relabel leaf  $x$  and  $y$  in  $T'$  by  $\bullet_{k+1}$  and  $\bullet_{k+2}$ , respectively.

Now we consider all union nodes  $x_1$  on the path from  $x$  to the root of  $T'$  in  $T'$ . If  $x$  is a left (right) child of  $x_1$  and union node  $x_1$  is labeled by  $\times_S$  and  $(\text{lab}(x, G(x_1)), \ell) \in S$  ( $(\ell, \text{lab}(x, G(x_1))) \in S$ ) for some  $\ell \in [k]$  then we relabel  $x_1$  by  $\times_{S'}$ , where  $S' = S \cup \{(k+1, \ell) \mid (\text{lab}(x, G(x_1)), \ell) \in S, \ell \in [k]\}$  ( $S' = S \cup \{(\ell, k+1) \mid (\ell, \text{lab}(x, G(x_1))) \in S, \ell \in [k]\}$ ). This is done in order to make all vertices which are adjacent to  $v$  in  $G$  also adjacent to  $v$  in  $G + \{v, w\}$ . In the same way we have to modify all union nodes on the path from  $y$  to the root of  $T'$  in order to make all vertices of  $G + \{v, w\}$  adjacent to  $w$ , if they are in  $G$ .

Last we have to relabel the least common predecessor  $z$  of  $x$  and  $y$  in  $T'$  to create the edge between  $v$  and  $w$ . Since  $z$  is always a union node in  $T'$ ,  $z$  is labeled by  $\times_S$  for some  $S \subseteq [k]^2$ . If  $x$  is the left (right) child and  $y$  is the right (left) child of  $z$  in  $T'$  then we relabel  $z$  by  $\times_{S \cup \{(k+1, k+2)\}} (\times_{S \cup \{(k+2, k+1)\}})$ .

The resulting tree is denoted by  $T''$ . Tree  $T''$  is an NLC-width  $(k+2)$ -expression-tree and  $T''$  defines graph  $G + \{v, w\}$ .

The proof for edge deletion runs similar, we just have to leave out the above described relabeling of the the least common predecessor  $z$  of  $x$  and  $y$  in  $T'$  to create the edge between  $v$  and  $w$ .

For the lower bounds assume that  $H$  is obtained from  $G$  by inserting (deleting) an edge  $e$  and it holds  $\text{NLC-width}(H) < \text{NLC-width}(G) - 2$ . Then by deleting (inserting) edge  $e$  from (in)  $H$  we obtain a graph  $G'$  isomorphic to  $G$ . By our shown upper bound we conclude that  $\text{NLC-width}(G') < \text{NLC-width}(G)$ , which leads to a contradiction.

The results for clique-width can be shown by similar argumentations.  $\square$

If we add or delete an edge in a graph of NLC-width 1, i.e. a co-graph, then we always obtain a graph of NLC-width at most 2, since we can label both end vertices of the new edge (both end vertices of the deleted edge) by the same label 2.

The graphs in Figure 1 show that the NLC-width bounds of Theorem 4.4 cannot be improved for  $k = 2$ . Graph  $G$  has NLC-width 2 and graph  $H$ , which we obtain after inserting edge  $e = \{v, w\}$  in  $G$ , has NLC-width 4. Further the graphs of Figure 1 show that the bound for edge deletion is best possible for  $k = 2$ . The edge complement graph  $\overline{G}$  of  $G$  has NLC-width 2 and contains edge  $\{v, w\}$ . If we remove edge  $\{x, y\}$  from  $\overline{G}$  we obtain the graph  $\overline{H}$  which has NLC-width 4.

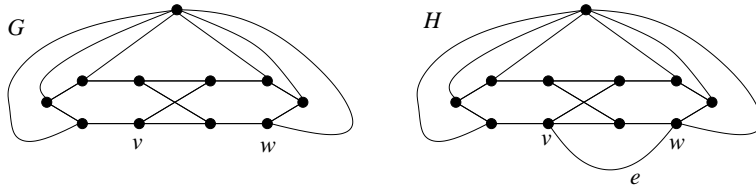


Figure 1: Graph  $G$  has NLC-width 2. Graph  $H$  can be obtained from  $G$  by adding edge  $e$  and  $H$  has NLC-width 4.

Our last theorem gives an answer of Question 6.3 of [CO00]. It remains to verify whether the given clique-width bounds are tight.

**Problem 4.5** *Is there a graph  $G$  and  $v, w \in V_G$ , such that  $|\text{clique-width}(G) - \text{clique-width}(G \pm \{v, w\})| > 1$ ?*

$|\{v, w\}| = 2$  holds?

### 4.3 Edge Subdivision

For some graph  $G$  and an edge  $\{v, w\} \in E_G$  the *subdivision*  $G_{v,w}$  of  $G$  has vertex set  $V_G \cup \{u\}$  and edge set  $E_G - \{\{v, w\}\} \cup \{\{v, u\}, \{w, u\}\}$ . The subdivision operation is also known as *elementary refinement*.

At least after subdividing all edges of a graph the resulting graph is bipartite. After subdividing every edge of a graph  $G$  the resulting graph is called the *incidence graph* of the given graph. Incidence graphs have unbounded clique-width in general, but incidence graphs of graphs of bounded tree-width have bounded clique-width, since subdivisions do not change the tree-width.

**Theorem 4.6** *Let  $G$  be a graph and  $\{v, w\} \in E_G$  an edge, then it holds*

$$\text{NLC-width}(G) - 2 \leq \text{NLC-width}(G_{v,w}) \leq \text{NLC-width}(G) + 2$$

and

$$\text{clique-width}(G) - 2 \leq \text{clique-width}(G_{v,w}) \leq \text{clique-width}(G) + 2.$$

**Proof** First we want to show the upper bound. Let  $G$  be a graph of NLC-width  $k$  and let  $\{v, w\}$  be an edge of  $G$ . Let  $T$  be an NLC-width  $k$ -expression-tree that defines  $G$ . We now define a new NLC-width  $(k + 2)$ -expression-tree that defines  $G_{v,w}$ .

Let  $T'$  be defined for  $T$  as in the proof of Theorem 4.4 for edge removing. In  $T'$  we insert a new root  $r$  labeled by  $\times_{\{(k+1,k+1),(k+2,k+1)\}}$  and a new node  $z$  (defining the vertex  $u$  which subdivides edge  $\{v, w\}$ ) labeled by  $\bullet_{k+1}$  and two edges, one from  $r$  to  $z$  and one from  $r$  to the root of  $T'$  such that  $z$  is the right child of  $r$ .

The resulting tree is denoted by  $T''$ . Then  $T''$  is an NLC-width  $(k + 2)$ -expression-tree and it is easy to show that  $T''$  defines graph  $G_{v,w}$ .

For the lower bounds assume that graph  $G_{v,w}$  is obtained from  $G$  by subdividing an edge  $\{v, w\}$  and  $\text{NLC-width}(G_{v,w}) < \text{NLC-width}(G) - 2$ . Then we obtain by removing the inserted vertex  $u$  and inserting  $\{v, w\}$  in  $G_{v,w}$  a graph  $G'$  isomorphic to  $G$  with  $\text{NLC-width}(G') < \text{NLC-width}(G)$ , by our upper bound in Theorem 4.4, and thus a contradiction.

Since the clique-width operations do not allow edge insertions between equal labeled vertices, we have to do one additional relabeling  $\rho_{k+1 \rightarrow k+2}$  in order to label vertices  $v$  and  $w$  in the proof of Theorem 4.4 by  $k + 2$  before inserting the new vertex in  $T$ .  $\square$

The upper bound for  $\text{NLC-width}(G_{v,w})$  of Theorem 4.6 cannot be improved, since first subdividing an edge and deleting the new vertex corresponds to edge deletion, which needs two additional labels in general, see Figure 1.

In the appendix of [CO00] it is shown that in a graph  $G$  of clique-width at least 4 every path of length at least 5, consisting of vertices which all have degree 2 in  $G$  and one end vertex of degree 1 in  $G$ , can be extended by subdivisions without increasing the clique-width of  $G$ .

There are several examples where a subdivision increases the NLC-width and clique-width, e.g. a  $P_3$ , and several examples where a subdivision does not change the NLC-width and clique-width, e.g. a  $P_4$ . It remains open, whether a subdivision can decrease the NLC-width and clique-width of graphs.

**Problem 4.7** *Is there some graph  $G$  and some  $\{v, w\} \in E_G$ , such that  $NLC\text{-width}(G_{v,w}) < NLC\text{-width}(G)$  or  $\text{clique-width}(G_{v,w}) < \text{clique-width}(G)$ ?*

#### 4.4 Vertex Identification and Edge Contraction

For some graph  $G$  and two different vertices  $v, w \in V_G$  the *identification*  $G^{v,w}$  has vertex set  $V_G - \{v, w\} \cup \{u\}$  and edge set

$$(E_G - \{\{v', v''\} \mid v' \in V_G, v'' \in \{v, w\}\}) \cup \{\{v', u\} \mid v' \notin \{v, w\} \text{ and } \{v', v\} \in E_G \text{ or } \{v', w\} \in E_G\}.$$

Next we analyze the identification of two vertices in a graph with respect to the NLC-width and clique-width of the involved graphs.

**Theorem 4.8** *Let  $G$  be a graph and  $v, w \in V_G$ , then it holds*

$$1/4 \cdot NLC\text{-width}(G) \leq NLC\text{-width}(G^{v,w}) \leq 2 \cdot NLC\text{-width}(G)$$

and

$$1/4 \cdot \text{clique-width}(G) \leq \text{clique-width}(G^{v,w}) \leq 2 \cdot \text{clique-width}(G).$$

**Proof** For the upper bound we can delete  $v, w$  and insert  $u$ , see Theorem 4.1, with neighborhood  $N(v) \cup N(w)$ . The lower bound holds since we can obtain  $G$  from  $G^{v,w}$  by removing  $u$  and inserting the vertices  $v$  and  $w$ , each with a factor of 2.  $\square$

If the two vertices  $v$  and  $w$  of an identification are adjacent, i.e.  $\{v, w\} \in E_G$ , we call the corresponding operation *edge contraction*, which is a well known minor operation. Courcelle has recently shown in [Cou14] that there is a graph of clique-width 3, which yields a graph of clique-width greater than 3 by the contraction of a single edge. This disproves Conjecture 4.4 in [LPRW12] on the closure of graphs of bounded clique-width under edge contractions.

In the appendix of [CO00] it is shown that in a graph  $G$  of clique-width at least 4 every path of length at least 2, consisting of vertices which all have degree 2 in  $G$  and one end vertex of degree 1 in  $G$ , can be decreased by edge contractions without increasing the clique-width of  $G$ .

#### 4.5 Subgraph

**Subgraph** For the subgraph operation  $f_s$  applied on a graph  $G$ , and thus also for the minor operation, the clique-width of  $f_s(G)$  and NLC-width of  $f_s(G)$  cannot be bounded in the clique-width or NLC-width of  $G$ . This can easily be shown by the example of complete graphs, which all have NLC-width 1 and clique-width 2, while their subgraphs may have arbitrary large NLC-width and clique-width. By taking the number of removed edges into account, the bounds of Section 4.2 can be used to estimate the NLC-width and clique-width of subgraphs.

**Induced Subgraph** Since the induced subgraph operation  $f_i$  applied on a graph  $G$  can be realized by vertex deletions, by Section 4.1 it holds

$$NLC\text{-width}(f_i(G)) \leq NLC\text{-width}(G)$$

and

$$\text{clique-width}(f_i(G)) \leq \text{clique-width}(G).$$

Although taking induced subgraphs does not increase the NLC-width and clique-width of a graph, characterizations for the classes  $\text{NLC}_k$ ,  $k \geq 2$ , and  $\text{CW}_k$ ,  $k \geq 3$ , by sets of forbidden subgraphs are unknown until now.

**Quotient** If we remove all but one vertices of a module  $V' \subseteq V_G$  from graph  $G$ , we denote the obtained graph as a *quotient graph* of  $G$ . Since every quotient graph of  $G$  is an induced subgraph of  $G$ , the quotient operation does not increase the NLC-width or clique-width.

The substitution operation (see Section 3.4) and quotient operation are used in [Joh98] and [CMR00] to show that the NLC-width and clique-width of a graph can be obtained by the maximum NLC-width or clique-width of its prime subgraphs appearing as quotient graphs in a modular decomposition.

## 4.6 Power of a Graph

The  $d$ -th power  $G^d$  of a graph  $G$  is a graph with the same set of vertices as  $G$  and an edge between two vertices if and only if there is a path of length at most  $d$  between them. Suchan and Todinca have shown in [ST07] the following bound.

$$\text{NLC-width}(G^d) \leq 2 \cdot (d + 1)^{\text{NLC-width}(G)}$$

## 4.7 Line Graph

The *line graph*  $L(G)$  of a graph  $G$  has a vertex for every edge of  $G$  and an edge between two vertices if the corresponding edges of  $G$  are adjacent [Whi32]. For some line graph  $L(G)$ , graph  $G$  is called the *root graph* of  $L(G)$ . Even for complete graphs  $K_n$ , the line graph operation generates graphs whose NLC-width cannot be bounded in the NLC-width of their root graphs [GW07]. But it is possible to bound the NLC-width and clique-width of line graphs in the tree-width of their root graphs, and even vice versa by the following bounds, which have been shown in [GW07].

$$\begin{aligned} 1/4 \cdot (\text{tree-width}(G) + 1) &\leq \text{NLC-width}(L(G)) \leq \text{tree-width}(G) + 2 \\ 1/4 \cdot (\text{tree-width}(G) + 1) &\leq \text{clique-width}(L(G)) \leq 2 \cdot \text{tree-width}(G) + 2 \end{aligned}$$

## 4.8 Edge Complement

The *edge complement graph*  $\overline{G}$  of graph  $G$  has the same vertex set as  $G$  and two vertices in  $\overline{G}$  are adjacent if and only if they are not adjacent in  $G$ , i.e.  $\overline{G} = (V_G, \{\{u, v\} \mid u, v \in V_G, u \neq v, \{u, v\} \notin E_G\})$ . The following bounds are known from [Wan94] and [CO00].

$$\begin{aligned} \text{NLC-width}(\overline{G}) &= \text{NLC-width}(G) \\ 1/2 \cdot \text{clique-width}(G) &\leq \text{clique-width}(\overline{G}) \leq 2 \cdot \text{clique-width}(G) \end{aligned}$$

## 4.9 Bipartite Complement

Let  $G$  be a bipartite graph with vertex partition  $V_G = V_1 \cup V_2$ , such that there are no edges between two vertices of  $V_1$  and no edges between two vertices of  $V_2$ . The *bipartite complement*

$\overline{G}^{\text{bip}}$  of  $G$  has vertex set  $V_G$  and edge set  $\{\{u, v\} \mid \{u, v\} \notin E_G, u \in V_1, v \in V_2\}$ . In [LR04a] it is shown that for every bipartite graph  $G$  the following bounds hold.

$$1/4 \cdot \text{clique-width}(G) \leq \text{clique-width}(\overline{G}^{\text{bip}}) \leq 4 \cdot \text{clique-width}(G)$$

In the same way one can show that

$$1/2 \cdot \text{NLC-width}(G) \leq \text{NLC-width}(\overline{G}^{\text{bip}}) \leq 2 \cdot \text{NLC-width}(G).$$

#### 4.10 Local Complementation

For some graph  $G$  and a vertex  $v \in V_G$  the *local complementation*  $LC(G, v)$  is defined by Bouchet in [Bou94] as follows. Graph  $LC(G, v)$  is obtained from graph  $G$  by replacing the subgraph of  $G$  defined by  $N(v)$  by its edge complement, i.e.  $LC(G, v)$  has vertex set  $V_G$  and edge set

$$(E_G - \{\{u, w\} \mid u, w \in N_G(v), \{u, w\} \in E_G\}) \cup \{\{u, w\} \mid u, w \in N_G(v), u \neq w, \{u, w\} \notin E_G\}.$$

**Theorem 4.9** *Let  $G$  be a graph and  $v \in V_G$ , then*

$$1/2 \cdot \text{NLC-width}(G) \leq \text{NLC-width}(LC(G, v)) \leq 2 \cdot \text{NLC-width}(G)$$

and

$$1/3 \cdot \text{clique-width}(G) \leq \text{clique-width}(LC(G, v)) \leq 3 \cdot \text{clique-width}(G).$$

**Proof** Let  $T$  be an NLC-width  $k$ -expression-tree that defines graph  $G$ . We now define a new NLC-width  $2k$ -expression-tree that defines graph  $LC(G, v)$ . We start with a copy  $T'$  of  $T$ . The main idea is to separate the vertices in  $N(v)$  from the vertices in  $V - N(v)$ . Let  $n' = |N(v)|$  and  $x_1, \dots, x_{n'}$  be the leaves of  $T'$  that corresponds to vertices in  $N(v)$  of  $G$ .

For every leaf  $x_i$ ,  $i = 1, \dots, n'$ , we modify the nodes  $x$  on the paths from  $x_i$  to the root of  $T'$  in  $T'$  as follows.

1. If  $x$  is a leaf  $x_i$ ,  $i = 1, \dots, n'$ , labeled by  $\bullet_\ell$  in  $T'$ , then we relabel  $x$  by  $\bullet_{\ell+k}$ .
2. If  $x$  is a relabeling node labeled by  $\circ_R$ , then we relabel  $x$  by  $\circ_{R'}$ , such that  $R'(a) = R(a)$ , if  $1 \leq a \leq k$  and  $R'(a) = R(a - k) + k$ , if  $k + 1 \leq a \leq 2k$ .
3. If  $x$  is a union node labeled by  $\times_S$ , then we relabel  $x$  by  $\times_{S'}$ , such that  $S' = S \cup S_1 \cup S_2$ , where  $S_1 = \{(a + k, b + k) \mid (a, b) \notin S\}$  and  $S_2 = \{(a, b + k), (a + k, b) \mid (a, b) \in S\}$ .  $S_1$  creates an edge between two nodes in  $N(v)$ , if and only if these vertices are not adjacent in  $G$ ,  $S_2$  creates an edge between one vertex of  $V_G - N(v)$  and one vertex of  $N(v)$ , if and only if these vertices are adjacent in  $G$ .

This is necessary in order to create the complement graph of the graph induced by  $N(v)$ . The resulting tree is denoted by  $T''$ . Tree  $T''$  is an NLC-width  $2k$ -expression-tree and defines graph  $LC(G, v)$ .

The lower bound follows since by  $L(L(G, v), v)$  we obtain  $G$ .

For the clique-width bounds we need  $k$  additionally labels to distinguish the vertices in  $N(v)$  from those in  $V - N(v)$  and  $k$  further labels to create the complement graph of the subgraph induced by vertex set  $N(v)$ .  $\square$



The example of a paw (3-pan), see Figure 2, shows that the local complementation can increase or decrease the NLC-width and clique-width of a graph. If we apply a local complementation on the paw at one of the vertices of degree 2, we obtain a path on four vertices of NLC-width 2 and clique-width 3.

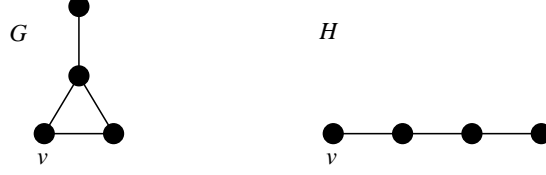


Figure 2: The graph  $G$  on the left side is called *paw* or *3-pan* and has NLC-width 1 (clique-width 2), the graph  $H$  on the right side is called  $P_4$  has NLC-width 2 (clique-width 3).

The proof of Theorem 4.9 implies the following bounds for the NLC-width and clique-width of graph  $LC(G, v)$  using the vertex degree of  $v$  in the graph  $G$ .

**Corollary 4.10** *Let  $G$  be a graph of NLC-width  $k$ , and  $v \in V_G$ , then*

$$\text{NLC-width}(LC(G, v)) \leq k + \min(k, \deg_G(v))$$

*and for every graph  $G$  of clique-width  $k$*

$$\text{clique-width}(LC(G, v)) \leq k + \min(2k, 2 \cdot \deg_G(v)).$$

Two graphs  $G$  and  $G'$  on the same vertex set are called *locally equivalent* if there is a sequence of vertices  $(v_1, \dots, v_\ell)$  such that  $G^0 = G$ ,  $G^i = LC(G^{i-1}, v_i)$  for  $i = 1, \dots, \ell$  and  $G^\ell = G'$ .

**Theorem 4.11** *Let  $G$  be a graph and  $G'$  a graph which is locally equivalent to  $G$ , then it holds*

$$\text{NLC-width}(G') \leq 2^{\text{NLC-width}(G)+1} - 1$$

*and*

$$\text{clique-width}(G') \leq 2^{\text{clique-width}(G)+1} - 1.$$

**Proof** Let  $G$  be a graph of NLC-width  $k$ . By adjusting Proposition 6.3. of [OS06] to the relationship between NLC-width and rank-width, graph  $G$  has rank-width at most  $k$ . Since the rank-width of a graph does not change by applying local complementations [Oum05b], every graph  $G'$  which is obtained by a sequence of local complementations on  $G$  also has rank-width at most  $k$ . Again by [OS06] we know that  $G'$  has NLC-width at most  $2^{k+1} - 1$ .

The clique-width bound follows by Proposition 6.3. of [OS06].  $\square$

#### 4.11 Switching

The *switching* operation is defined by Lint and Seidel in [vLS66] as follows. Let  $G$  be a graph and  $v \in V_G$  be a vertex. Graph  $S(G, v)$  has the same vertex set as  $G$  and its edge set is the

edge set of  $G$  but changing the neighbors of  $v$  to non neighbors and vice versa. That is, graph  $S(G, v)$  has vertex set  $V_G$  and edge set

$$E_G - \{\{v, w\} \mid w \in V_G, \{v, w\} \in E_G\} \cup \{\{v, w\} \mid w \in V_G, v \neq w, \{v, w\} \notin E_G\}.$$

Next we will show that one switching operation in a graph increases or decreases its NLC-width by at most one.

**Theorem 4.12** *Let  $G = (V_G, E_G)$  be a graph and  $v \in V_G$ , then it holds*

$$NLC\text{-width}(G) - 1 \leq NLC\text{-width}(S(G, v)) \leq NLC\text{-width}(G) + 1$$

and

$$\frac{1}{2} \cdot \text{clique-width}(G) \leq \text{clique-width}(S(G, v)) \leq 2 \cdot \text{clique-width}(G).$$

**Proof** Let  $T$  be an NLC-width  $k$ -expression-tree that defines  $G$  and  $v \in V_G$ . We now define a new NLC-width  $(k+1)$ -expression-tree that defines  $S(G, v)$ . We start with a copy  $T'$  of  $T$ . Let  $x$  be the leaf of  $T'$  that corresponds to vertex  $v$  of  $G$ . We relabel leaf  $x$  in  $T'$  by  $\bullet_{k+1}$ .

Now we consider the union nodes  $x_1$  on the path from  $x$  to the root of  $T'$  in  $T'$ . If  $x$  is a left (right) child of  $x_1$  and union node  $x_1$  is labeled by  $\times_S$  then we relabel  $x_1$  by  $\times_{S'}$ , where  $S' = S \cup \{(k+1, \ell) \mid (\text{lab}(x, G(x_1)), \ell) \notin S, \ell \in [k]\}$  ( $S' = S \cup \{(\ell, k+1) \mid (\ell, \text{lab}(x, G(x_1))) \notin S, \ell \in [k]\}$ ). This is necessary in order to make all vertices adjacent to  $v$  which are not adjacent to  $v$  in  $G$ , and vice versa.

The resulting tree is denoted by  $T''$ . Tree  $T''$  is an NLC-width  $(k+1)$ -expression-tree and  $T''$  defines graph  $S(G, v)$ .

The lower bound follows since by  $S(S(G, v), v)$  we obtain  $G$ .

In order to show a bound on the clique-width of graph  $S(G, v)$  we cannot bound the clique-width by a constant independently on  $k$ , because of the different edge insertion operation in the clique-width model. Clearly we can first remove vertex  $v$  from a graph  $G$  and insert a new vertex with neighborhood  $V_G - N(v)$  to obtain the graph  $S(G, v)$  of clique-width at most  $2k$ , see Section 4.1. In this case it is much more convenient to use NLC-width operations.  $\square$

The bounds on the NLC-width given in Theorem 4.12 are best possible. For the upper bound consider the graph  $G$  of NLC-width 1 in Figure 2 (which is called *paw* or *3-pan* [BLS99]). A switching operation on graph  $G$  at one of the vertices of degree 2 creates a graph  $H$  which is isomorphic to a  $P_4$ , which has NLC-width 2. Further by  $S(H, v)$  we obtain the graph  $G$ , thus the lower bound is best possible too.

Two graphs  $G$  and  $G'$  on the same vertex set are called *switching equivalent* if there is a sequence of vertices  $(v_1, \dots, v_\ell)$  such that  $G^0 = G$ ,  $G^i = S(G^{i-1}, v_i)$  for  $i = 1, \dots, \ell$  and  $G^\ell = G'$ . It is shown in [CC80] that deciding if two graphs are switching equivalent is an isomorphism complete problem.

**Theorem 4.13** *Let  $G$  be a graph and  $G'$  a graph which is switching equivalent to  $G$  by sequence  $(v_1, \dots, v_\ell)$ , then it holds*

$$NLC\text{-width}(G') \leq 2^{NLC\text{-width}(G)+1} - 1$$

and

$$\text{clique-width}(G') \leq 2^{\text{clique-width}(G)+2} - 1.$$

**Proof** Let  $G$  be a graph,  $v \in V_G$  be a vertex and  $G' = S(G, v)$ . Let  $G_0$  be the graph obtained from  $G$  by adding a dominating vertex  $v_0$  and  $G'_0$  be the graph obtained from  $G_0$  by adding a dominating vertex  $v_0$ . It is easy to check that  $G'_0$  can be obtained from  $G_0$  by applying three local complementations at  $v$  at  $v_0$  and at  $v$ , which implies that the rank-width of  $G'_0$  and  $G_0$  are equal by [Oum05b].

Now, suppose that  $G_1$  and  $G_2$  are two switching equivalent graphs. Then  $G_{1,0}$  and  $G_{2,0}$  (both obtained by adding a dominating vertex  $v_0$ ) are locally equivalent and therefore  $G_{1,0}$  and  $G_{2,0}$  have the same rank-width. By [OS06] and [Oum05b] it holds

$$\begin{aligned} \text{NLC-width}(G_1) &= \text{NLC-width}(G_{1,0}) \\ &\leq 2^{\text{rank-width}(G_{1,0})+1} - 1 \\ &= 2^{\text{rank-width}(G_{2,0})+1} - 1 \\ &\leq 2^{\text{NLC-width}(G_{2,0})+1} - 1 \\ &= 2^{\text{NLC-width}(G_2)+1} - 1 \end{aligned}$$

The clique-width bound can be similar, only for the dominating vertex we need one more label.  $\square$

## 5 Application to Graph Classes of Bounded Clique-width

In this section we want to give two examples how the consideration of graph operations implies results on the clique-width and NLC-width of graph classes.

A class of graphs  $\mathcal{L}$  has *bounded clique-width* if there is some integer  $k$  such that every graph in  $\mathcal{L}$  has clique-width at most  $k$ , i.e. there is some  $k$  such that  $\mathcal{L} \subseteq \text{CW}_k$ . The minimal  $k$ , if exists, is defined as *clique-width* of class  $\mathcal{L}$ . If there is no  $k \in \mathbb{N}$  such that  $\mathcal{L} \subseteq \text{CW}_k$ , we say that  $\mathcal{L}$  has *unbounded clique-width*. The same notations can be defined for NLC-width. By Theorem 2.1 a graph class has bounded clique-width if and only if it has bounded NLC-width. See [KLM09] for a survey on the clique-width of graph classes.

First we consider *tree-cographs* which are defined by Tinhofer in [Tin89] recursively as follows.

- (i) Every tree is a tree-cograph.
- (ii) If  $G$  is a tree-cograph, then the complement graph  $\overline{G}$  is a tree-cograph.
- (iii) If  $G_1, \dots, G_k$ ,  $k \geq 2$  are connected tree-cographs then the disjoint union  $G_1 \oplus \dots \oplus G_k$  is also a tree-cograph.

Since trees have NLC-width at most 3 [Wan94] and edge-complement and disjoint union do not increase the NLC-width by Section 4.8 and Section 3.1 we know that tree-cographs have NLC-width at most 3 and thus by Theorem 2.1 clique-width at most 6. The clique-width bound can be improved by the following observations. For two tree-cographs  $T_1, T_2$  we can compute  $\overline{T_1 \oplus T_2}$  by  $\overline{T_1} \otimes \overline{T_2}$ . Thus we can equivalently define tree-cographs by the following four rules.

- (i) Every tree is a tree-cograph.
- (ii) Every complement of a tree is a tree-cograph.

- (iii) If  $G_1, \dots, G_k$ ,  $k \geq 2$  are connected tree-cographs then the disjoint union  $G_1 \oplus \dots \oplus G_k$  is also a tree-cograph.
- (iv) If  $G_1, \dots, G_k$ ,  $k \geq 2$  are connected tree-cographs then the join  $G_1 \otimes \dots \otimes G_k$  is also a tree-cograph.

Because trees have clique-width at most 3 and complement of trees have clique-width at most 4 we follow from the last characterization and the bounds of Sections 3.1 and 3.2 that tree-cographs have clique-width at most 4.

**Theorem 5.1** *Tree-cographs have NLC-width  $\leq 3$  and clique-width  $\leq 4$ .*

As a second example we introduce the graph class of *cograph-trees*.

- (i) Every tree is a cograph-tree.
- (ii) If we substitute a vertex  $x$  of a cograph-tree  $G$  by a co-graph  $H$ , as defined in Section 3.4, we obtain by  $G[x/H]$  a cograph-tree.

Since trees have NLC-width and clique-width at most 3 and cographs have NLC-width 1 and clique-width at most 2 and substitution does not increase the NLC-width or clique-width, we obtain the following result.

**Theorem 5.2** *Cographs-trees have NLC-width  $\leq 3$  and clique-width at most  $\leq 3$ .*

More general, we consider every graph class  $\mathcal{G}$  which can recursively be defined as follows.

- (i) A class of (basic) graphs of bounded clique-width is in  $\mathcal{G}$ .
- (ii) If we apply some of the operations disjoint union, join, composition, substitution, and edge complement on graphs of  $\mathcal{G}$ , we obtain a graph of  $\mathcal{G}$ .

By our results it follows that every such class  $\mathcal{G}$  has bounded clique-width.

## 6 Conclusions and Overview

In Section 3, we considered a number of binary graph operations  $f$  which create a new graph  $f(G_1, G_2)$  from two graphs  $G_1$  and  $G_2$ . In all cases in which it is possible to bound the NLC-width and clique-width of the combined graph  $f(G_1, G_2)$  in the NLC-width and clique-width of graphs  $G_1$  and  $G_2$  we show how to compute the corresponding expression in linear time in the size of the corresponding expression for  $G$ . Thus our results are constructive. In Table 1 we compare our results concerning binary graph operations.

In Section 4, we give an overview how the NLC-width and clique-width of a given graph changes if we apply certain unary graph operations  $f$  on this graph. In all cases in which it is possible to bound the NLC-width and clique-width of the resulting graph  $f(G)$  we also show how to compute the corresponding expression in linear time in the size of the corresponding expression for  $G$ . Thus our results are constructive. Although clique-width is the more famous concept, we obtain in all cases closer bounds for  $\text{NLC-width}(f(G))$  for local operations  $f$ . Therefore in our results the results for NLC-width are mentioned at first. In Table 2 we compare our results concerning unary graph operations.

operation $f$	NLC-width( $f(G_1, G_2)$ )	clique-width( $f(G_1, G_2)$ )
disjoint union	$\max(k_1, k_2)$	$\max(k_1, k_2)$
join	$\max(k_1, k_2)$	$\max(k_1, k_2, 2)$
composition	$\max(k_1, k_2)$	$\max(k_1, k_2)$
corona	$\max(k_1, k_2) + 1$	$\max(k_1, k_2) + 2$
substitution	$\max(k_1, k_2)$	$\max(k_1, k_2)$
1-sum	$\max(k_1, k_2) + 1$	$\max(k_1, k_2) + 2$

Table 1: Let  $G_1$  and  $G_2$  be two graphs of NLC-width  $k_1$  and  $k_2$ , respectively, and  $f$  be a binary graph operation of the first column. The second column of the table shows the upper bound of the NLC-width of graph  $f(G_1, G_2)$ . The third column gives the results for clique-width.

operation $f$	NLC-width( $f(G)$ )	clique-width( $f(G)$ )
induced subgraph	$k$	$k$
edge complement	$k$	$2k$
bipartite complement	$2k$	$4k$
local complementation	$2k$	$3k$
switching	$k + 1$	$2k$
edge insertion	$k + 2$	$k + 2$
edge subdivision	$k + 2$	$k + 2$
vertex insertion	$2k$	$2k + 1$
edge deletion	$k + 2$	$k + 2$
edge contraction	$2k$	$2k$

Table 2: Let  $G$  be a graph of NLC-width  $k$  and  $f$  be a unary graph operation of the first column. The second column of the table shows the upper bound of the NLC-width of graph  $f(G)$ . The third column gives the results for clique-width.

Since the computation of NLC-width and clique-width is NP-hard [GW07, FRRS09], it seems to be difficult to find an optimal  $k$ -expression for some given graph. Our results may help to find an expression for some given graph  $f(G)$  from a known expression of a similar graph  $G$ . For example, we can construct an NLC-width  $(k + \ell)$ -expression for every graph which is switching equivalent to some graph with known NLC-width  $k$ -expression, where  $\ell$  is the number of necessary switching operations. As well, we can construct an  $(k + 2)$ -expression for every graph which differs only by one edge from a graph with known  $k$ -expression.

In nearly all cases, it remains to show that our bounds are best possible, or to improve them. Especially the clique-width bounds on the complement, bipartite complement, and local complementation seem to be improvable.

Further it remains open if there are graph operations, which do not increase the clique-width or NLC-width of a given graph and make the given graph smaller, in order to define useful reduction rules or a characterization by forbidden graphs for graphs of bounded clique-width or graphs of bounded NLC-width. Among our considered operations only the operation induced subgraph does not increase the clique-width or NLC-width, which implies that there exist characterizations by sets of forbidden graphs for  $\text{NLC}_k$  and  $\text{CW}_k$  for every integer  $k$ . Unfortunately only for  $\text{NLC}_1$  and  $\text{CW}_2$  these sets are known.

It is also an open problem to find graph operations that increase or decrease the NLC-width or clique-width of some graph by a fixed constant or a fixed factor, e.g. an operation such that for every graph  $G$  there is a positive integer  $c$  such that  $\text{NLC-width}(f(G)) = c + \text{NLC-width}(G)$  or  $\text{NLC-width}(f(G)) = c \cdot \text{NLC-width}(G)$ . This would imply a useful means in order to decrease the NLC-width or clique-width in a controlled way. For rank-width such an operation using a factor of  $c = 2$  is known, see Lemma 5.3 in [Oum05a].

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