Solving Sparse, Symmetric, Diagonally-Dominant Linear Systems in Time $O(m^{1.31})$

Daniel A. Spielman [*]
Department of Mathematics
Massachusetts Institute of Technology

Shang-Hua Teng [†]
Department of Computer Science
Boston University and
Akamai Technologies Inc.

February 1, 2008

Second post-FOCS revision.

Abstract

We present a linear-system solver that, given an n-by-n symmetric positive semi-definite, diagonally dominant matrix A with m non-zero entries and an n-vector b, produces a vector $\tilde{x}$ within relative distance ϵ of the solution to $Ax = b$ in time $O(m^{1.31} \log(n\kappa_f(A)/\epsilon)^{O(1)})$, where $\kappa_f(A)$ is the log of the ratio of the largest to smallest non-zero eigenvalue of A. In particular, $\log(\kappa_f(A)) = O(b \log n)$, where b is the logarithm of the ratio of the largest to smallest non-zero entry of A. If the graph of A has genus $m^{2\theta}$ or does not have a K_{m^θ} minor, then the exponent of m can be improved to the minimum of $1 + 5\theta$ and $(9/8)(1 + \theta)$. The key contribution of our work is an extension of Vaidya's techniques for constructing and analyzing combinatorial preconditioners.

1 Introduction

Sparse linear systems are ubiquitous in scientific computing and optimization. In this work, we develop fast algorithms for solving some of the best-behaved linear systems: those specified by symmetric, diagonally dominant matrices with positive diagonals. We call such matrices PSDDD as they are positive semi-definite and diagonally dominant. Such systems arise in the solution of certain elliptic differential equations via the finite element method, the modeling of resistive networks, and in the solution of certain network optimization problems [SF73, McC87, HY81, Var62, You71].

While one is often taught to solve a linear system $Ax = b$ by computing A^{-1} and then multiplying A^{-1} by b, this approach is quite inefficient for sparse linear systems—the best known bound on the time required to compute A^{-1} is $O(n^{2.376})$ [CW82] and the representation of A^{-1} typically requires $\Omega(n^2)$ space. In contrast, if A is symmetric and has m non-zero entries, then one can use the Conjugate Gradient method, as a direct method, to solve for $A^{-1}b$ in $O(nm)$ time and $O(n)$ space! Until Vaidya's revolutionary introduction of combinatorial preconditioners [Vai90], this was the best complexity bound for the solution of general PSDDD systems.

1

The two most popular families of methods for solving linear systems are the direct methods and the iterative methods. Direct methods, such as Gaussian elimination, perform arithmetic operations that produce x treating the entries of A and b symbolically. As discussed in Section 1.4, direct methods can be used to quickly compute x if the matrix A has special topological structure.

Iterative methods, which are discussed in Section 1.5, compute successively better approximations to x. The Chebyshev and Conjugate Gradient methods take time proportional to $m\sqrt{\kappa_f(A)}\log(\kappa_f(A)/\epsilon)$ to produce approximations to x with relative error ϵ, where $\kappa_f(A)$ is the ratio of the largest to the smallest non-zero eigenvalue of A. These algorithms are improved by preconditioning—essentially solving $B^{-1}Ax = B^{-1}b$ for a *preconditioner* B that is carefully chosen so that $\kappa_f(A, B)$ is small and so that it is easy to solve linear systems in B. These systems in B may be solved using direct methods, or by again applying iterative methods.

Vaidya [Vai90] discovered that for PSDDD matrices A one could use combinatorial techniques to construct matrices B that provably satisfy both criteria. In his seminal work, Vaidya shows that when B corresponds to a subgraph of the graph of A, one can bound $\kappa_f(A, B)$ by bounding the dilation and congestion of the best embedding of the graph of A into the graph of B. By using preconditioners derived by adding a few edges to maximum spanning trees, Vaidya's algorithm finds ϵ-approximate solutions to PSDDD linear systems of maximum valence d in time $O((dn)^{1.75}\log(\kappa_f(A)/\epsilon))$. [1] When these systems have special structure, such as having a sparsity graph of bounded genus or avoiding certain minors, he obtains even faster algorithms. For example, his algorithm solves planar linear systems in time $O((dn)^{1.2}\log(\kappa_f(A)/\epsilon))$. This paper follows the outline established by Vaidya: our contributions are improvements in the techniques for bounding $\kappa_f(A, B)$, a construction of better preconditioners, a construction that depends upon average degree rather than maximum degree, and an analysis of the recursive application of our algorithm.

As Vaidya's paper was never published[2], and his manuscript lacked many proofs, the task of formally working out his results fell to others. Much of its content appears in the thesis of his student, Anil Joshi [Jos97]. Gremban, Miller and Zagha[Gre96, GMZ95] explain parts of Vaidya's paper as well as extend Vaidya's techniques. Among other results, they found ways of constructing preconditioners by *adding* vertices to the graphs and using separator trees.

Much of the theory behind the application of Vaidya's techniques to matrices with non-positive off-diagonals is developed in [BGH$^+$]. The machinery needed to apply Vaidya's techniques directly to matrices with positive off-diagonal elements is developed in [BCHT]. The present work builds upon an algebraic extension of the tools used to prove bounds on $\kappa_f(A, B)$ by Boman and Hendrickson [BH]. Boman and Hendrickson [BH01] have pointed out that by applying one of their bounds on support to the tree constructed by Alon, Karp, Peleg, and West [AKPW95] for the k-server problem, one obtains a spanning tree preconditioner B with $\kappa_f(A, B) = m2^{O\left(\sqrt{\log n \log\log n}\right)}$. They thereby obtain a solver for PSDDD systems that produces ϵ-approximate solutions in time $m^{1.5+o(1)}\log(\kappa_f(A)/\epsilon)$. In their manuscript, they asked whether one could possibly augment this tree to obtain a better preconditioner. We answer this question in the affirmative. An algorithm running in time $O(mn^{1/2}\log^2(n))$ has also recently been obtained by Maggs, *et. al.* [MMP$^+$02].

The present paper is the first to push past the $O(n^{1.5})$ barrier. It is interesting to observe that this is exactly the point at which one obtains sub-cubic time algorithms for solving dense PSDDD linear systems.

Reif [Rei98] proved that by applying Vaidya's techniques recursively, one can solve bounded-degree planar positive definite diagonally dominant linear systems to relative accuracy ϵ in time $O(m^{1+o(1)}\log(\kappa(A)/\epsilon))$. We extend this result to general planar PSDDD linear systems.

Due to space limitations in the FOCS proceedings, some proofs have been omitted. These are being gradually included in the on-line version of the paper.

---

[1] For the reader unaccustomed to condition numbers, we note that for an PSDDD matrix A in which each entry is specified using b bits of precision, $\log(\kappa_f(A)) = O(b\log n)$.

[2] Vaidya founded the company Computational Applications and System Integration (http://www.casicorp.com) to market his linear system solvers.

1.1 Background and Notation

A symmetric matrix A is semi-positive definite if $x^T A x \geq 0$ for all vectors x. This is equivalent to having all eigenvalues of A non-negative.

In most of the paper, we will focus on Laplacian matrices: symmetric matrices with non-negative diagonals and non-positive off-diagonals such that for all i, $\sum_j A_{i,j} = 0$. However, our results will apply to the more general family of positive semidefinite, diagonally dominant (PSDDD) matrices, where a matrix is diagonally dominant if $|A_{i,i}| \geq \sum_{j=1}^n |A_{i,j}|$ for all i. We remark that a symmetric matrix is PSDDD if and only if it is diagonally dominant and all of its diagonals are non-negative.

In this paper, we will restrict our attention to the solution of linear systems of the form $A\boldsymbol{x} = \boldsymbol{b}$ where A is a PSDDD matrix. When A is non-singular, that is when A^{-1} exists, there exists a unique solution $x = A^{-1}\boldsymbol{b}$ to the linear system. When A is singular and symmetric, for every $\boldsymbol{b} \in \mathbf{Span}(A)$ there exists a unique $\boldsymbol{x} \in \mathbf{Span}(A)$ such that $A\boldsymbol{x} = \boldsymbol{b}$. If A is the Laplacian of a connected graph, then the null space of A is spanned by $\mathbf{1}$.

There are two natural ways to formulate the problem of finding an approximate solution to a system $A\boldsymbol{x} = \boldsymbol{b}$. A vector $\tilde{\boldsymbol{x}}$ has *relative residual error* ϵ if $\|A\tilde{\boldsymbol{x}} - \boldsymbol{b}\| \leq \epsilon \|\boldsymbol{b}\|$. We say that a solution $\tilde{\boldsymbol{x}}$ is an ϵ-approximate solution if it is at relative distance at most ϵ from the actual solution—that is, if $\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\| \leq \epsilon \|\boldsymbol{x}\|$. One can relate these two notions of approximation by observing that relative distance of $\boldsymbol{x}$ to the solution and the relative residual error differ by a multiplicative factor of at most $\kappa_f(A)$. We will focus our attention on the problem of finding ϵ-approximate solutions.

The ratio $\kappa_f(A)$ is the finite condition number of A. The l_2 norm of a matrix, $\|A\|$, is the maximum of $\|Ax\| / \|x\|$, and equals the largest eigenvalue of A if A is symmetric. For non-symmetric matrices, $\lambda_{max}(A)$ and $\|A\|$ are typically different. We let $|A|$ denote the number of non-zero entries in A, and $\min(A)$ and $\max(A)$ denote the smallest and largest non-zero elements of A in absolute value, respectively.

The condition number plays a prominent role in the analysis of iterative linear system solvers. When A is PSD, it is known that, after $\sqrt{\kappa_f(A)} \log(1/\epsilon)$ iterations, the Chebyshev iterative method and the Conjugate Gradient method produce solutions with relative residual error at most ϵ. To obtain an ϵ-approximate solution, one need merely run $\log(\kappa_f(A))$ times as many iterations. If A has m non-zero entries, each of these iterations takes time $O(m)$. When applying the preconditioned versions of these algorithms to solve systems of the form $B^{-1}A\boldsymbol{x} = B^{-1}\boldsymbol{b}$, the number of iterations required by these algorithms to produce an ϵ-accurate solution is bounded by $\sqrt{\kappa_f(A,B)} \log(\kappa_f(A)/\epsilon)$ where

$$\kappa_f(A,B) = \left(\max_{x:Ax \neq 0} \frac{\boldsymbol{x}^T A \boldsymbol{x}}{\boldsymbol{x}^T B \boldsymbol{x}} \right) \left(\max_{x:Ax \neq 0} \frac{\boldsymbol{x}^T B \boldsymbol{x}}{\boldsymbol{x}^T A \boldsymbol{x}} \right),$$

for symmetric A and B with $\mathbf{Span}(A) = \mathbf{Span}(B)$. However, each iteration of these methods takes time $O(m)$ plus the time required to solve linear systems in B. In our initial algorithm, we will use direct methods to solve these systems, and so will not have to worry about approximate solutions. For the recursive application of our algorithms, we will use our algorithm again to solve these systems, and so will have to determine how well we need to approximate the solution. For this reason, we will analyze the Chebyshev iteration instead of the Conjugate Gradient, as it is easier to analyze the impact of approximation in the Chebyshev iterations. However, we expect that similar results could be obtained for the preconditioned Conjugate Gradient. For more information on these methods, we refer the reader to [GV89] or [Bru95].

1.2 Laplacians and Weighted Graphs

All weighted graphs in this paper have positive weights. There is a natural isomorphism between weighted graphs and Laplacian matrices: given a weighted graph $G = (V, E, w)$, we can form the Laplacian matrix in

which $A_{i,j} = -w(i,j)$ for $(i,j) \in E$, and with diagonals determined by the condition $A\mathbf{1} = \mathbf{0}$. Conversely, a weighted graph is naturally associated to each Laplacian matrix. Each vertex of the graph corresponds to both a row and column of the matrix, and we will often abuse notation by identifying this row/column pair with the associated vertex.

We note that if G_1 and G_2 are weighted graphs on the same vertex set with disjoint sets of edges, then the Laplacian of the union of G_1 and G_2 is the sum of their Laplacians.

1.3 Reductions

In most of this paper we just consider Laplacian matrices of connected graphs. This simplification is enabled by two reductions.

First, we note that it suffices to construct preconditioners for matrices satisfying $A_{i,i} = \sum_j |A_{i,j}|$, for all i. This follows from the observation in [BGH$^+$] that if $\tilde{A} = A + D$, where A satisfies the above condition, then $\kappa_f(\tilde{A}, B + D) \leq \kappa_f(A, B)$. So, it suffices to find a preconditioner after subtracting off the maximal diagonal matrix that maintains positive diagonal dominance.

We then use an idea of Gremban [Gre96] for handling positive off-diagonal entries. If A is a symmetric matrix such that for all i, $A_{i,i} \geq \sum_j |A_{i,j}|$, then Gremban decomposes A into $D + A_n + A_p$, where D is the diagonal of A, A_n is the matrix containing all negative off-diagonal entires of A, and A_p contains all the positive off-diagonals. Gremban then considers the linear system

$$\left[\begin{array}{cc} D + A_n & -A_p \\ -A_p & D + A_n \end{array} \right] \left[\begin{array}{c} \boldsymbol{x} \\ \boldsymbol{x}' \end{array} \right] = \left[\begin{array}{c} \boldsymbol{b} \\ -\boldsymbol{b} \end{array} \right],$$

and observes that its solution will have $\boldsymbol{x}' = -\boldsymbol{x}$ and that $\boldsymbol{x}$ will be the solution to $A\boldsymbol{x} = \boldsymbol{b}$. Thus, by making this transformation, we can convert any $PSDDD$ linear system into one with non-negative off diagonals. One can understand this transformation as making two copies of every vertex in the graph, and two copies of every edge. The edges corresponding to negative off-diagonals connect nodes in the same copy of the graph, while the others cross copies. To capture the resulting family of graphs, we define a weighted graph G to be a *Gremban cover* if it has $2n$ vertices and

- for $i, j \leq n$, $(i, j) \in E$ if and only if $(i + n, j + n) \in E$, and $w(i,j) = w(i+n, j+n)$,
- for $i, j \leq n$, $(i, j + n) \in E$ if and only if $(i + n, j) \in E$, and $w(i, j+n) = w(i+n, j)$, and
- the graph contains no edge of the form $(i, i + n)$.

When necessary, we will explain how to modify our arguments to handle Laplacians that are Gremban covers.

Finally, if A is the Laplacian of an unconnected graph, then the blocks corresponding to the connected components may be solved independently.

1.4 Direct Methods

The standard direct method for solving symmetric linear systems is Cholesky factorization. Those unfamiliar with Cholesky factorization should think of it as Gaussian elimination in which one simultaneously eliminates on rows and columns so as to preserve symmetry. Given a permutation matrix P, Cholesky factorization produces a lower-triangular matrix L such that $LL^T = PAP^T$. Because one can use forward and back substitution to multiply vectors by L^{-1} and L^{-T} in time proportional to the number of non-zero entries in L, one can use the Cholesky factorization of A to solve the system $A\boldsymbol{x} = \boldsymbol{b}$ in time $O(|L|)$.

Each pivot in the factorization comes from the diagonal of A, and one should understand the permutation P as providing the order in which these pivots are chosen. Many heuristics exist for producing permutations P for which the number of non-zeros in L is small. If the graph of A is a tree, then a permutation P that orders the vertices of A from the leaves up will result in an L with at most $2n - 1$ non-zero entries. In this work, we will use results concerning matrices whose sparsity graphs resemble trees with a few additional edges and whose graphs have small separators, which we now review.

If B is the Laplacian matrix of a weighted graph (V, E, w), and one eliminates a vertex a of degree 1, then the remaining matrix has the form

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & A_{1,} \end{array} \right]$$

where A_1 is the Laplacian of the graph in which a and its attached edge have been removed. Similarly, if a vertex a of degree 2 is eliminated, then the remaining matrix is the Laplacian of the graph in which the vertex a and its adjacent edges have been removed, and an edge with weight $1/(1/w_1 + 1/w_2)$ is added between the two neighbors of a, where w_1 and w_2 are the weights of the edges connecting a to its neighbors.

Given a graph G with edge set $E = R \cup S$, where the edges in R form a tree, we will perform a partial Cholesky factorization of G in which we successively eliminate all the degree 1 and 2 vertices that are not endpoint of edges in S. We introduce the algorithm `trim` to define the order in which the vertices should be eliminated, and we call the *trim order* the order in which `trim` deletes vertices.

Algorithm: `trim`(V, R, S)

1. While G contains a vertex of degree one that is not an endpoint of an edge in S, remove that vertex and its adjacent edge.

2. While G contains a vertex of degree two that is not an endpoint of an edge in S, remove that vertex and its adjacent edges, and add an edge between its two neighbors.

Proposition 1.1. *The output of* `trim` *is a graph with at most* $4 |S|$ *vertices and* $5 |S|$ *edges.*

Remark 1.2. *If* (V, R) *and* (V, S) *are Gremban covers, then we can implement* `trim` *so that the output graph is also a Gremban cover. Moreover, the genus and maximum size clique minor of the output graph do not increase.*

After performing partial Cholesky factorization of the vertices in the trim order, one obtains a factorization of the form

$$B = LCL^T, \text{where } C = \left[\begin{array}{cc} I & 0 \\ 0 & A_1 \end{array} \right],$$

L is lower triangular, and the left column and right columns in the above representations correspond to the eliminated and remaining vertices respectively. Moreover, $|L| \leq 2n - 1$, and this Cholesky factorization may be performed in time $O(n + |S|)$.

The following Lemma may be proved by induction.

Lemma 1.3. *Let B be a Laplacian matrix and let L and A_1 be the matrices arising from the partial Cholesky factorization of B according to the trim order. Let U be the set of eliminated vertices, and let W be the set of remaining vertices. For each pair of vertices (a, b) in W joined by a simple path containing only vertices of U, let $B_{(a,b)}$ be the Laplacian of the graph containing just one edge between a and b of weight $1/(\sum_i 1/w_i)$, where the w_i are the weights on the path between a and b. Then,*

(a) *the matrix A_1 is the sum of the Laplacian of the induced graph on W and the sum all the Laplacians $B_{(a,b)}$,*

5

(b) $\|A_1\| \le \|B\|$, $\lambda_2(A_1) \ge \lambda_2(B)$, and so $\kappa_f(A_1) \le \kappa_f(B)$.

Other topological structures may be exploited to produce elimination orderings that result in sparse L. In particular, Lipton, Rose and Tarjan [LRT79] prove that if the sparsity graph is planar, then one can find such an L with at most $O(n \log n)$ non-zero entries in time $O(n^{3/2})$. In general, Lipton, Rose and Tarjan prove that if a graph can be dissected by a family of small separators, then L can be made sparse. The precise definition and theorem follow.

Definition 1.4. *A subset of vertices C of a graph $G = (V, E)$ with n vertices is an $f(n)$-separator if $|C| \le f(n)$, and the vertices of $V - C$ can be partitioned into two sets U and W such that there are no edges from U to W, and $|U|, |W| \le 2n/3$.*

Definition 1.5. *Let $f()$ be a positive function. A graph $G = (V, E)$ with n vertices has a family of $f()$-separators if for every $s \le n$, every subgraph $G' \subseteq G$ with s vertices has a $f(s)$-separator.*

Theorem 1.6 (Nested Dissection: Lipton-Rose-Tarjan). *Let A be an n by n symmetric PSD matrix, $\alpha > 0$ be a constant, and $h(n)$ be a positive function of n. Let $f(x) = h(n)x^\alpha$. If $G(A)$ has a family of $f()$-separator, then the Nested Dissection Algorithm of Lipton, Rose and Tarjan can, in $O\left(n + (h(n)n^\alpha)^3\right)$ time, factor A into $A = LL^T$ so that L has at most $O\left((h(n)n^\alpha)^2 \log n\right)$ non-zeros.*

To apply this theorem, we note that many families of graphs are known to have families of small separators. Gilbert, Hutchinson, and Tarjan [GHT84] show that all graphs of n vertices with genus bounded by g have a family of $O(\sqrt{gn})$-separators, and Plotkin, Rao and Smith [PRS94] show that any graph that excludes K_s as minor has a family of $O(s\sqrt{n \log n})$-separators.

1.5 Iterative Methods

Iterative methods such as Chebyshev iteration and Conjugate Gradient solve systems such as $A\boldsymbol{x} = \boldsymbol{b}$ by successively multiplying vectors by the matrix A, and then taking linear combinations of vectors that have been produced so far. The preconditioned versions of these iterative methods take as input another matrix B, called the *preconditioner*, and also perform the operation of solving linear systems in B. In this paper, we will restrict our attention to the preconditioned Chebyshev method as it is easier to understand the effect of imprecision in the solution of the systems in B on the method's output. In the non-recursive version of our algorithms, we will exploit the standard analysis of Chebyshev iteration (see [Bru95]), adapted to our situation:

Theorem 1.7 (Preconditioned Chebyshev). *Let A and B be Laplacian matrices, let $\boldsymbol{b}$ be a vector, and let $\boldsymbol{x}$ satisfy $A\boldsymbol{x} = \boldsymbol{b}$. At each iteration, the preconditioned Chebyshev method multiplies one vector by A, solves one linear system in B, and performs a constant number of vector additions. At the kth iteration, the algorithm maintains a solution $\tilde{\boldsymbol{x}}$ satisfying*

$$\|(\tilde{\boldsymbol{x}} - \boldsymbol{x})\| \le e^{-k/\sqrt{\kappa_f(A,B)}} \kappa_f(A) \sqrt{\kappa_f(B)} \|\boldsymbol{x}\|.$$

In the non-recursive versions of our algorithms, we will pre-compute the Cholesky factorization of the preconditioners B, and use these to solve the linear systems encountered by preconditioned Chebyshev method. In the recursive versions, we will perform a partial Cholesky factorization of B, into a matrix of the form $L[I, 0; 0, A_1]L^T$, construct a preconditioner for A_1, and again use the preconditioned Chebyshev method to solve - the systems in A_1.

2 Support Theory

The essence of support theory is the realization that one can bound $\lambda_f(A, B)$ by constructing an embedding of A into B. We define a *weighted embedding* of A into B to be a function π that maps each edge e of A into a weighted simple path in B linking the endpoints of A. Formally, $\pi : E_A \times E_B \rightarrow \mathbb{R}^+$ is a weighted embedding if for all $e \in A$, $\{f \in B : \pi(e, f) > 0\}$ is a simple path connecting from one endpoint of e to the other. We let $\textbf{path}_\pi(e)$ denote this set of edges in this path in B. For $e \in A$, we define $\textbf{wd}_\pi(e) = \sum_{f \in \textbf{path}_\pi(e)} \frac{a_e}{b_f \pi(e,f)}$. and the weighted congestion of an edge $f \in B$ under π to be $\textbf{wc}_\pi(f) = \sum_{e:f \in \textbf{path}_\pi(e)} \textbf{wd}_\pi(e) \pi(e, f)$.

Our analysis of our preconditioners relies on the following extension of the support graph theory.

Theorem 2.1 (Support Theorem). *Let A be the Laplacian matrix of a weighted graph G and B be the Laplacian matrix of a subgraph F of G. Let π be a weighted embedding of G into F. Then*

$$\kappa_f(A, B) \leq \max_{f \in F} \textbf{wc}_\pi(f).$$

To understand this statement, the reader should first consider the case in which all the weights a_e, b_f and $\pi(e, f)$ are 1. In this case, the Support Theorem says that $\kappa_f(A, B)$ is at most the maximum over edges f of the sum of the lengths of the paths through f. This improves upon the upper bound on $\kappa_f(A, B)$ stated by Vaidya and proved in Bern *et. al.* of the maximum congestion times the maximum dilation, and it improves upon the bound proved by Boman and Hendrickson which was the sum of the dilations. This statement also extends the previous theories by using fractions of edges in B to route edges in A. That said, our proof of the Support Theorem owes a lot to the machinery developed by Boman and Hendrickson and our π is analogous to their matrix M.

We first recall the definition of the support of A in B, denoted $\sigma(A, B)$:

$$\sigma(A, B) = \min\{\tau : \forall t \geq \tau, \quad tB \succeq A\}.$$

Gremban proved that one can use support to characterize λ_f:

Lemma 2.2. *If $\textbf{Null}(A) = \textbf{Null}(B)$, then*

$$\lambda_f(A, B) = \sigma(A, B)\sigma(B, A).$$

Vaidya observed

Lemma 2.3. *If F is a subgraph of the weighted graph G, A is the Laplacian of G and B is the Laplacian of F, then $\sigma(B, A) \leq 1$.*

Our proof of the Support Theorem will use the Splitting Lemma of Bern *et. al.* and the Rank-One Support Lemma of Boman-Hendrickson:

Lemma 2.4 (Splitting Lemma). *Let $A = A_1 + A_2 + \cdots + A_k$ and let $B = A_1 + B_2 + \cdots + B_k$. Then,*

$$\sigma(A, B) \leq \max_i \sigma(A_i, B_i).$$

For an edge $e \in A$ and a weighted embedding π of A into B, we let A_e denote the Laplacian of the graph containing only the weighted edge e and B_e denote the Laplacian of the graph containing the edges $f \in \textbf{path}_\pi(\epsilon)$ with weights $a_f \pi(e, f)$. We have:

Lemma 2.5 (Weighted Dilation). *For an edge $e \in A$,*

$$\sigma(A_e, B_e) = \textbf{wd}_\pi(e).$$

Proof. Follows from Boman and Hendrickson's Rank-One Support Lemma. □

Proof of Theorem 2.1. Lemma 2.5 implies

$$\sigma(A_e, \mathbf{wd}_\pi(e) B_e) = 1.$$

We then have

$$\sigma(A, \max_{f \in B} \mathbf{wc}_\pi(f) B) \leq \sigma(A, \sum_{f \in B} \mathbf{wc}_\pi(f) A_f)$$

$$= \sigma(A, \sum_{e \in A} \mathbf{wd}_\pi(e) B_e)$$

$$\leq \max_{e \in A} \sigma(A_e, \mathbf{wd}_\pi(e) B_e)$$

$$\leq 1,$$

where the second-to-last inequality follows from the Splitting Lemma. □

3 The Preconditioner

In this section, we construct and analyze our preconditioner.

Theorem 3.1. *Let A be a Laplacian and $G = (V, E, w)$ its corresponding weighted graph. Let G have n vertices and m edges. For any positive integer $t \leq n$, the algorithm* `precondition`, *described below, runs in $O(m \log m)$ time and outputs a spanning tree $R \subseteq E$ of G and a set of edges $S \subseteq E$ such that*

(1) if B is the Laplacian corresponding to $R \cup S$, then $\sigma_f(A, B) \leq \frac{m}{t} 2^{O\left(\sqrt{\log n \log \log n}\right)}$, and

(2) $|S| \leq O\left(t^2 \log n / \log \log n\right)$.

Moreover, if G has genus s^2 or has no K_s minor, then

(2') $|S| \leq O\left(ts \log s \log n / \log \log n\right)$,

and if G is the Gremban cover of such a graph, then the same bound holds and we can ensure that S is a Gremban cover as well.

Proof. Everything except the statement concerning Gremban covers follows immediately from Theorem 2.1 and Lemmas 3.7, 3.8, and 3.13.

In the case that G is Gremban cover, we apply the algorithm `precondition` to the graph that it covers, but keeping all weights positive. We then set R and S to be both images of each edge output by the algorithm. Thus, the size of the set S is at most twice what it would otherwise be.

For our purposes, the critical difference between these two graphs is that a cycle in the covered graph corresponds in the Gremban cover to either two disjoint cycles or a double-traversal of that cycle. Altering the arguments to compensate for this change increases the bound of Lemma 3.10 by at most a factor of 3, and the bound of Lemma 3.13 by at most 9. □

The spanning tree R is built using an algorithm of Alon, Karp, Peleg, and West [AKPW95]. The edges in the set S are constructed by using other information generated by this algorithm. In particular, the AKPW

algorithm builds its spanning tree by first building a spanning forest, then building a spanning forest over that forest, and so on. Our algorithm works by decomposing the trees in these forests, and then adding a representative edge between each set of vertices in the decomposed trees.

Throughout this section, we assume without loss of generality that the maximum weight of an edge is 1.

3.1 The Alon-Karp-Peleg-West Tree

We build our preconditioners by adding edges to the spanning trees constructed by Alon, Karp, Peleg and West [AKPW95]. In this subsection, we review their algorithm, state the properties we require of the trees it produces, and introduce the notation we need to define and analyze our preconditioner.

The AKPW algorithm is run with the parameters $x = 2^{\sqrt{\log n \log \log n}}$ and $\rho = \lceil \frac{3 \log n}{\log x} \rceil$, and the parameters $\mu = 9\rho \log n$ and $y = x\mu$ are used in its analysis.

We assume, without loss of generality, that the maximum weight edge in E has weight 1. The AKPW algorithm begins by partitioning the edge set E by weight as follows:

$$E_i = \left\{ e \in E : 1/y^i < w(e) \leq 1/y^{i-1} \right\}.$$

For each edge $e \in E$, let class (e) be the index such that $e \in E_{\text{class}(e)}$.

The AKPW algorithm iteratively applies a modification of an algorithm of Awerbuch [Awe85], which we call `cluster`, whose relevant properties are summarized in the following lemma.

Lemma 3.2 (Colored Awerbuch). *There exists an algorithm with template*

$$F = \texttt{cluster}(G, x, E_1, \ldots, E_k),$$

where $G = (V, E)$ is a graph, x is a number, $E_1, \ldots, E_k$ are disjoint subsets of E, and F is a spanning forest of V, such that

(1) each forest of F has depth at most $3xk \log n$,

(2) for each $1 \leq i \leq k$, the number of edges in class E_i between vertices in the same tree of F is at least x times the number of edges in class E_i between vertices in distinct trees of F, and

(3) `cluster` runs in time $O(\sum_i |E_i|)$.

Proof. Properties (1) and (2) are established in the proof of Lemma 5.5 in [AKPW95]. To justify the running time bound, we review the algorithm. We first recall that it only pays attention to edges in $\cup_i E_i$. The algorithm proceeds by growing a BFS tree level-by-level from a vertex that is not included in the current forest. It grows this tree until a level is reached at which condition (2) is satisfied. Once condition (2) is satisfied, it adds this tree to the forest, and begins to grow again from a vertex not currently in the forest. □

The other part of the AKPW algorithm is a subroutine with template

$$G' = \texttt{contract}(G, F),$$

that takes as input a graph G and a spanning forest F of G, and outputs the multigraph G' obtained by contracting the vertices of each tree in F to a single vertex. This contraction removes all resulting self-loops (which result from edges between vertices in the same tree), but keeps an image of each edge between distinct trees of F. The classes, weights, and names of the edges are preserved, so that each edge in G' can be mapped back to a unique pre-image in G.

We can now state the AKPW algorithm:

Algorithm: $R = \texttt{AKPW}(G)$

1. Set $j = 1$ and $G^{(j)} = G$.

2. While $G^{(j)}$ has more than one vertex

   (a) Set $R^j = \texttt{cluster}(G^{(j)}, x, E_{j-\rho+1}, \ldots, E_j)$.

   (b) Set $G^{(j+1)} = \texttt{contract}(G^{(j)}, R^j)$

   (c) Set $j = j + 1$.

3. Set $R = \cup_j R^j$

The tree output by the AKPW algorithm is the union of the pre-images of the edges in forests R^j. Our preconditioner will include these edges, and another set of edges S constructed using the forests F^j.

To facilitate the description and analysis of our algorithm, we define

F^j to be the forest on V formed from the union of the pre-images of edges in $R^1 \cup \cdots \cup R^{j-1}$,

T_v^j to be the tree of F^j containing vertex v.

$E_i^j = \left\{ (u, v) \in E_i : T_u^j \neq T_v^j \right\}$,

$H_i^j = E_i^j - E_i^{j+1}$, and $H^j = \cup_i H_i^j$.

We observe that F^{j+1} is comprised of edges from $E_1, \ldots, E_j$, and that each edge in H^j has both endpoints in the same tree of F^{j+1}.

Alon, *et. al.* prove:

Lemma 3.3 (AKPW Lemma 5.4). *The algorithm* AKPW *terminates. Moreover, for every* $i \leq j$, $\left| E_i^j \right| \leq \left| E_i^{(j-1)} \right| / x \leq |E_i| / x^{j-i}$.

We remark that $x^\rho > |E|$, so for $i \leq j - \rho$, $E_i^j = \emptyset$. The following lemma follows from the proof of Lemma 5.5 of [AKPW95] and the observation that $y^\rho \geq |E|$.

Lemma 3.4. *For each simple path P in F^{j+1} and for each l,* $|P \cap E_l| \leq \min(y^{j-l+1}, y^\rho)$.

3.2   Tree Decomposition

Our preconditioner will construct the edge set S by decomposing the trees in the forests produced by the AKPW algorithm, and adding edges between the resulting sub-trees. In this section, we define the properties the decomposition must satisfy and describe the decomposition algorithm.

Definition 3.5. *For a tree T and a set of edges H between the vertices of T, we define an H-decomposition of T to be a pair $(\mathcal{W}, \sigma)$ where $\mathcal{W}$ is a collection of subsets of the vertices of T and σ is a map from H into sets or pairs of sets in $\mathcal{W}$ satisfying*

1. *for each set $W \in \mathcal{W}$, the graph induced by T on W is connected,*

2. *for each edge in T there is exactly one set $W \in \mathcal{W}$ containing that edge, and*

10

3. *for each edge in $e \in H$, if $|\sigma(e)| = 1$, then both endpoints of e lie in $\sigma(e)$; otherwise, one endpoint of e lies in one set in $\sigma(e)$, and the other endpoint lies in the other.*

We note that there can be sets $W \in \mathcal{W}$ containing just one vertex of T.

For a weighted set of edges H and an H-decomposition $(\mathcal{W}, \sigma)$, we define the H-weight of a set $W \in \mathcal{W}$ by $w_H(W) \stackrel{\text{def}}{=} \sum_{e \in H : W \in \sigma(e)} w(e)$.

We also define $w_{tot}(H) \stackrel{\text{def}}{=} \sum_{e \in H} w(e)$.

Our preconditioner will use an algorithm for computing small H-decompositions in which each set $W \in \mathcal{W}$ with $|W| > 1$ has bounded H-weight.

Lemma 3.6 (Tree Decomposition). *There exists an algorithm with template*

$$(\mathcal{W}, \sigma) = \mathtt{decompose}(T, H, \phi)$$

that runs in time $O(|H| + |T|)$ and outputs an H-decomposition $(\mathcal{W}, \sigma)$ satisfying

1. *for all $W \in \mathcal{W}$ such that $|W| > 1$, $w_H(W) \leq \phi$, and*

2. *$|\mathcal{W}| \leq 4 w_{tot}(H)/\phi$.*

Proof. We let $T(v)$ denote the set of vertices in the subtree rooted at v, and for a set of vertices W, let $H(W) = \{e \in H : e \cap H \neq \emptyset\}$. We then define $\bar{w}(v) \stackrel{\text{def}}{=} H(T(v))$. Let v_0 denote the root of the tree. Our algorithm will proceed as if it were computing $\bar{w}(v_0)$ via a depth-first traversal of the tree, except that whenever it encounters a subtree of weight more than $\phi/2$, it will place nodes from that subtree into a set in $\mathcal{W}$ and remove them from the tree. There are three different cases which determine how the nodes are placed into the set and how σ is constructed.

If, when processing a node v, the algorithm has traversed a subset of the children of v, $\{v_1, \ldots, v_k\}$ such that $\bar{w}(v_1) + \cdots + \bar{w}(v_k) \geq \phi/2$, then a set W is created, all the nodes in $\{v\} \cup \bigcup_{i=1}^{k} T(v_i)$ are placed in W, and those nodes in $\bigcup_{i=1}^{k} T(v_i)$ are deleted from the tree. If a node v is encountered such that $\phi/2 \leq H(T(v)) \leq \phi$, then a set W is created, the nodes in $T(v)$ are placed in W, and those nodes in W are deleted from the tree. In either case, for each node $e \in H(W)$ we set $\sigma(e) = \sigma(e) \cup \{W\}$.

If a node v is encountered which is not handled by either of the preceeding cases and for which $\bar{w}(v) > \phi$, then two sets $W_1 = T(v)$ and $W_2 = \{v\}$ are created, and those nodes in $T(v)$ are deleted from the tree. For each edge $e \in H(v)$, W_2 is added to $\sigma(e)$ and for each edge $e \in H(T(v) - \{v\})$, W_1 is added to $\sigma(e)$.

When the algorithm finally returns from examining the root, all the remaining nodes are placed in a final set, and this set is added to $\sigma(e)$ for each edge $e \in H$ with endpoints in this set. The algorithm maintains the invariant that whenever it returns from examining a node v, it has either deleted v, or removed enough vertices below v so that $\bar{w}(v) < \phi/2$. To see that the algorithm produces at most $4 w_{tot}/\phi$ sets, we note that each edge in H can contribute its weight to at most two sets, and that every time the algorithm forms sets, it either forms one set with weight at least $\phi/2$ or two sets with total weight at least ϕ. $\qquad \square$

3.3 Constructing the Preconditioner

We can now describe our algorithm for constructing the preconditioner. We will defer a discussion of how to efficiently implement the algorithm to Lemma 3.8.

The algorithm will make use of the parameter

$$\theta^{(j)} \stackrel{\text{def}}{=} \begin{cases} x^{j-1} & \text{if } j \leq \rho \\ x^{\rho} y^{j - \rho - 1} & \text{otherwise} \end{cases}$$

Algorithm: $(R, S) =$ `Precondition`(G)

1. Run $R = $ `AKPW`(G). Set h to the number of iterations taken by `AKPW`, and record $R^1, \ldots, R^h$ and $H^1, \ldots, H^h$.

2. For $j = 1$ to h

   (a) let $\{T_1, \ldots, T_k\}$ be the set of trees in F^{j+1}.

   (b) for $i = 1$ to k

      i. let H be the subset of edges in H_j with endpoints in T_i

      ii. Set $(\{W_1, \ldots, W_l\}, \sigma)$ to
         `decompose`$(T_i, H, |E| / t\theta^{(j)})$

      iii. for each $\mu \leq \nu \leq l$, let $a_{\mu,\nu}$ be the maximum weight edge in H between W_μ and W_ν, and add $a_{\mu,\nu}$ to S.

Lemma 3.7. *Let S be the set of edges produced by* `Precondition`. *Then,*

$$|S| \leq 8\rho^2 t^2 = O\left(t^2 \log n / \log \log n\right).$$

Moreover, if G has no K_s minor, then $|S| = O\left(ts \log s \log n / \log \log n\right).$

Proof. Let c_j be the total number of sets produced by applying `decompose` to the trees in F^{j+1}. We first bound $\sum_j c_j$. We have

$$\sum_j c_j \leq \sum_j 4t\theta^{(j)} w_{tot}(H^j) / |E| = (4t / |E|) \sum_j \theta^{(j)} \sum_i w_{tot}(H_i^j).$$

To bound this sum, we set

$$h_i^j = \begin{cases} 0 & \text{if } j < i, \\ \sum_{l \leq i} |H_i^l| & \text{if } i = j, \text{ and} \\ \left|H_i^j\right| & \text{if } j > i. \end{cases}$$

We observe that Lemma 3.3 implies $h_i^j \leq |E_i| / x^{j-i}$, and $h_i^j = 0$ for $j \geq i + \rho$. As $\theta^{(j)}$ is increasing, we have

$$\sum_j \theta^{(j)} \sum_i w_{tot}(H_i^j) \leq \sum_j \theta^{(j)} \sum_{i=j-\rho+1}^{j} h_i^j / y^{i-1}$$

$$= \sum_i \sum_{j=i}^{i+\rho-1} \theta^{(j)} h_i^j / y^{i-1}$$

$$\leq \sum_i |E_i| \sum_{j=i}^{i+\rho-1} \theta^{(j)} / \left(x^{j-i} y^{i-1}\right)$$

$$\leq \sum_i |E_i| \rho$$

$$\leq |E| \rho,$$

as $\theta^{(j)} \leq x^{j-i} y^{i-1}$ for $j \leq i + \rho - 1$. Thus, $\sum_j c_j \leq 4\rho t$, and, because we add at most one edge between each pair of these sets, we have $|S| \leq 8\rho^2 t^2$.

As observed by Vaidya, a result of Mader [Bol78] implies that if a graph does not have a complete graph on s vertices as a minor, then the average degree of every minor of G is $O(s \log s)$. Hence, the number of edges added to S at iteration j is at most $c_j s \log s$, and so

$$|S| \leq \sum_j c_j s \log s \leq 8\rho t s \log s.$$

Finally, a graph of genus s^2 does not have a $K_{\Theta(s)}$ minor. $\qquad\square$

Using the dynamic trees data structure of Sleator and Tarjan [ST83], we prove:

Lemma 3.8. *If G is a graph with n vertices and m edges, then the output of* `precondition` *can be produced in $O(m \log m)$ time.*

Proof. We first observe that `AKPW` can be implemented to run in time $O(m \log m)$, as each edge appears in at most $\rho = O(\log m)$ calls to `coloredAwerbuch`, and the contractions can be implemented using standard techniques to have amortized complexity $O(\log m)$ per node.

As j could be large, it could be impractical for the preconditioning algorithm to actually examine the entire forest F^{j+1} for each j. To overcome this obstacle, we observe that the determination of which edges $a_{\mu,\nu}$ to include in S only depends upon the projection of the sets in the decompositions onto vertices at endpoints of edges in H^j. That is, rather than passing (T^j, H) to `decompose`, it suffices to pass the topological tree induced by restricting T^j to vertices with endpoints in H (*i.e.,* with non-essential degree 2 nodes removed). As this tree has size at most $O(|H|)$, we can implement the algorithm in linear time plus the time required to produce these trees. There are many data structures that allow one to dynamically add edges to a tree and, for any set of vertices in the tree, to produce the induced tree on all least common ancestors of those vertices. For example, one can do this if one can determine (i) the nearest common ancestor of any pair of vertices, and (ii) which of a pair of vertices comes first in an in-order. The dymanic trees of Sleator and Tarjan [ST83] enable edge additions and nearest common ancestor queries at an amortized cost of $O(\log n)$ each, and any algorithm that balances search trees using tree rotations, such as red-black trees, enables one to determine relative order of nodes in an in-order at a cost of $O(\log n)$ per addition and querry. $\qquad\square$

3.4 Analyzing the Preconditioner

We will use weighted embeddings of edges into paths in $R \cup S$ to bound the quality of our preconditioners. The weights will be determined by a function $\tau(j,l)$, which we now define to be

$$\tau(j,l) = \begin{cases} 1 & j - l < \rho \\ \frac{(j-l-\rho+1)^2}{y^{j-l-\rho+1}} & \text{Otherwise.} \end{cases}$$

For each edge $e \in H^j$ and each edge $f \in \mathbf{path}_\pi(e)$, we will set $\pi(e,f) = \tau(j, \text{class}(f))$. We will construct π so as to guarantee class $(e) < $ class $(f) + \rho$.

It remains to define the paths over which edges are embedded. For an edge $e = (u,v)$ in H^j, if $e \in R \cup S$ then we set $\mathbf{path}_\pi(e) = e$ and $\pi(e,e) = 1$. Otherwise, we let T be the tree in F^{j+1} containing the endpoints of e and let σ be the function output by `decompose` on input T. If $|\sigma(e)| = 1$, then we let $\mathbf{path}_\pi(e)$ be the simple path in T connecting the endpoints of e. Otherwise, we let $\{W_\nu, W_\mu\} = \sigma(e)$ and let $a_{\nu,\mu}$ be the edge added between W_ν and W_μ. We then let $\mathbf{path}_\pi(e)$ be the concatenation of the simple path in T from u to $a_{\nu,\mu}$, the edge $a_{\nu,\mu}$ and the simple path in T from $a_{\nu,\mu}$ to v.

The two properties that we require of τ are encapsulated in the following lemma.

Lemma 3.9. (a) *For all $j \geq 1$, $\sum_{l=1}^{j} \frac{y^l \min(y^{j-l+1}, y^\rho)}{\tau(j,l)} \leq y^{j+1}(\rho + 2)$, and*

(b) *For all $l \geq 1$, $\sum_{j \geq l} \tau(j,l) \leq (\rho+1)$.*

Proof. The first property follows from

$$\sum_{l=1}^{j} \frac{y^l \min\left(y^{j-l+1}, y^\rho\right)}{\tau(j,l)} = \sum_{l=1}^{j-\rho} \frac{y^l y^\rho y^{j-l-\rho+1}}{(j-l+\rho+1)^2} + \sum_{l=j-\rho+1}^{j} y^l y^{j-l+1}$$

$$= \sum_{l=1}^{j-\rho} \frac{y^{j+1}}{(j-l+\rho+1)^2} + \sum_{l=j-\rho+1}^{j} y^{j+1}$$

$$\leq y^{j+1}(\rho+2),$$

as $\sum_{l=1}^{j-l}(j-l+\rho+1)^{-2} \leq 2$.

The second property follows from $\sum_{i \geq 1} i^2 y^{-i} \leq 1$, which holds because y is greater than the real root of $y^3 - 4y^2 + 2y - 1$, which is about 3.51155. $\square$

We now derive the upper bound we need on the maximum weighted congestion of the embedding π.

Lemma 3.10. *For each j and each simple path P in F^{j+1},*

$$\sum_{f \in P} \frac{1}{w(f)\tau(j, class\,(f))} \leq (\rho+2)y^{j+1}.$$

Proof.

$$\sum_{f \in P} \frac{1}{w(f)\tau(j, class\,(f))} \leq \sum_{l=1}^{j} \sum_{f \in P \cap E_l} \frac{1}{w(f)\tau(j,l)}$$

$$\leq \sum_{l=1}^{j} \frac{\min\left(y^{j-l+1}, y^\rho\right)}{w(f)\tau(j,l)}$$

$$\leq \sum_{l=1}^{j} \frac{y^l \min\left(y^{j-l+1}, y^\rho\right)}{\tau(j,l)}$$

$$\leq y^{j+1}(\rho+2)$$

where the third-to-last inequality follows from Lemma 3.4, the second-to-last inequality follows from $f \in E_l$, and the last inequality follows from Lemma 3.9 (a). $\square$

Lemma 3.11. *For each edge $e \in E$,*

$$\boldsymbol{wd}_\pi\,(e) \leq (2\rho+5)y^{j+1}w(e).$$

Proof. Let $e \in H_i^j$, let T be the forest in F^{j+1} containing the endpoints of e, and let $(\mathcal{W}, \sigma)$ be the output of decompose on input T. If $|\sigma(e)| = 1$, the e is routed over the simple path in T connecting its endpoints, so we can apply Lemma 3.10 to show

$$\mathbf{wd}_\pi\,(e) \leq (\rho+2)y^{j+1}w(e).$$

Otherwise, let $\sigma(e) = \{W_\nu, W_\mu\}$, and observe that $\mathbf{path}_\pi\,(e)$ contains two simple paths in T and the edge $a_{\nu,\mu}$. Applying Lemma 3.10 to each of these paths and recalling class $(a_{\nu,\mu}) \leq j$, which implies $w(a_{\nu,\mu}) \geq 1/y^j$, we obtain

$$\mathbf{wd}_\pi\,(e) \leq 2(\rho+2)y^{j+1}w(e) + y^j w(e) \leq (2\rho+5)y^{j+1}w(e).$$

$\square$

Lemma 3.12. *For each $f \in R \cup S$ and for each j*

$$\sum_{e \in H^j : f \in \boldsymbol{path}_\pi(e)} \boldsymbol{wd}_\pi(e) \leq (2\rho + 5)\,\mu^\rho y^2\,|E|\,/t.$$

Proof. Let T be the tree in F^{j+1} containing the endpoints of f, and let $(\mathcal{W}, \sigma)$ be the output of `decompose` on input T. There are two cases two consider: f can either be an edge of T, or f can be one of the edges $a_{\nu,\mu}$. If f is an edge of T, let W be the set in $\mathcal{W}$ containing its endpoints. Otherwise, if f is one of the edges $a_{\nu,\mu}$, let W be the larger of the sets W_ν or W_μ. If $|W_\nu| = |W_\mu| = 1$, then the only edge having f in its path is f itself, in which case the lemma is trivial. So, we may assume $|W| > 1$. In either case, each edge e for which $f \in \mathbf{path}_\pi(e)$ must have $W \in \sigma(e)$. Thus,

$$\begin{aligned}
\sum_{e \in H^j : f \in \mathbf{path}_\pi(e)} \mathbf{wd}_\pi(e) &\leq \sum_{e \in H^j : W \in \sigma(e)} \mathbf{wd}_\pi(e) \\
&\leq \sum_{e \in H^j : W \in \sigma(e)} w(e)\,(2\rho + 5)\,y^{j+1} \\
&= w_{H^j}(W)\,(2\rho + 5)\,y^{j+1} \\
&\leq (2\rho + 5)\,y^{j+1}\,|E|\,/t\theta^{(j)} \\
&\leq (2\rho + 5)\,y^2 \mu^\rho\,|E|\,/t.
\end{aligned}$$ $\qquad\square$

Lemma 3.13. *Let R, S and π be constructed as above. Then,*

$$\max_{f \in R \cup S} \boldsymbol{wc}_\pi(f) = \frac{m}{t} 2^{O\left(\sqrt{\log n \log \log n}\right)}.$$

Proof. For any edge $f \in R \cup S$, we let $l = \mathrm{class}(f)$ and compute

$$\begin{aligned}
\mathbf{wc}_\pi(f) &= \sum_{e \in E : f \in \mathbf{path}_\pi(e)} \mathbf{wd}_\pi(e)\,\pi(e, f) \\
&= \sum_j \sum_{e \in H^j : f \in \mathbf{path}_\pi(e)} \mathbf{wd}_\pi(e)\,\tau(j, l) \\
&\leq \sum_j \tau(j, l)\,(2\rho + 5)\,\mu^\rho y^2\,|E|\,/t \\
&\leq (\rho + 1)\,(2\rho + 5)\,\mu^\rho y^2\,|E|\,/t, \\
&= 2^{O\left(\sqrt{\log n \log \log n}\right)}\,|E|\,/t.
\end{aligned}$$

where the second-to-last inequality follows from Lemma 3.12, the last inequality follows from Lemma 3.9 (b), and the last equality follows from $\mu^\rho = 2^{O\left(\sqrt{\log n \log \log n}\right)}$. $\qquad\square$

4 One-Shot Algorithms

Our first algorithm constructs a preconditioner B for the matrix A, performs a partial Cholesky factorization of B by eliminating the vertices in trim order to obtain $B = L[I, 0; 0, A_1]L^T$, performs a further Cholesky factorization of A_1 into $L_1 L_1^T$, and applies the preconditioned Chebyshev algorithm. In each iteration of the preconditioned Chebyshev algorithm, we solve the linear systems in B by back-substitution through the Cholesky factorizations.

Theorem 4.1 (One-Shot). *Let A be an n-by-n PSDDD matrix with m non-zero entries. Using a single application of our preconditioner, one can solve the system $A\boldsymbol{x} = \boldsymbol{b}$ to relative accuracy ϵ in time $O\left(m^{18/13+o(1)}\log(\kappa(A)/\epsilon)\right)$. Moreover, if if the sparsity graph of A does not contain a minor isomorphic to the complete graph on m^θ vertices, or if it has genus at most $m^{2\theta}$, for $\theta < 1/3$, then the exponent of m can be reduced to $1.125(1 + \theta) + o(1)$.*

Proof. The time taken by the algorithm is the sum of the time required to compute the preconditioner, perform the partial Cholesky factorization of B, pre-process A_1 (either performing Cholesky factorization or inverting it), and the product of the number of iterations and the time required per iteration. In each case, we will set $t = m^\gamma$ for some constant γ, and note that the number of iterations will be $m^{(1-\gamma)/2+o(1)}$, and that the matrix A_1 will depend on m^γ.

If we do not assume that A has special topological structure, then A_1 is a matrix on $m^{2\gamma+o(1)}$ vertices. If we solve systems in A_1 by Cholesky factorization, then it will take time $O\left(m + m^{6\gamma+o(1)}\right)$ to perform the factorization and time $O\left(m + m^{4\gamma+o(1)}\right)$ to solve each system. So, the total time will be $m^{(1-\gamma)/2+4\gamma+o(1)} + m^{6\gamma+o(1)}$. Setting $\gamma = 3/13$, we obtain the first result.

If the graph has genus θ^2 or does not have a K_{m^θ} minor, or is the Gremban cover of such a graph, then can apply part $(2')$ of Theorem 3.1. Thus, A_1 is a matrix on $m^{\gamma+\theta+o(1)}$ vertices. In the Gremban cover case, the preconditioner is a Gremban cover, and so the partial Cholesky factorization can ensure that A_1 is a Gremban cover as well. As the Gremban cover of a graph has a similar family of separators to the graph it covers, in either case we can apply the algorithm of Lipton, Rose and Tarjan to obtain the Choleksy factorization of A_1. By Theorem 1.6, with $\alpha = 1/2 + \theta$, the time required to perform the factorization will be $O\left(m + m^{\gamma(3\theta+3/2)+o(1)}\right)$, and the time required to solve the system will be

$$O\left(m^{(1-\gamma)/2}(m + m^{\gamma(2\theta+1)+o(1)})\right) = O\left(m^{(1-\gamma)/2+1+o(1)}\right),$$

provided $\gamma(2\theta + 1) \leq 1$. We will obtain the desired result by setting $\gamma = (3 - 9\theta)/4$. □

5 Recursive Algorithms

We now show how to apply our algorithm recursively to improve upon the running time of the algorithm presented in Theorem 4.1.

For numerical reasons, we will use partial LDL^T-factorization in this section instead of partial Cholesky factorizations. We remind the reader that the LDL^T-factorization of a matrix B is comprised of a lower-triangular matrix L with ones on the diagonal, and a diagonal matrix D. The partial LDL^T factorization of a matrix B_1 has the form

$$B_1 = L \left(\begin{array}{cc} D & 0 \\ 0 & A_1 \end{array} \right) L^T,$$

where D is diagonal L has the form

$$L = \left(\begin{array}{cc} L_{1,1} & 0 \\ L_{2,1} & I \end{array} \right)$$

and $L_{1,1}$ has 1s on the diagonal.

The recursive algorithm is quite straightforward: it first constructs the top-level preconditioner B_1 for matrix $A_0 = A$. It then eliminates to vertices of B_1 in the trim order to obtain the partial LDL^T-factorization $B_1 = L_1 C_1 L_1^T$, where $C_1 = [D_1, 0; 0, A_1]$. When an iteration of the preconditioned Chebyshev algorithm needs to solve a linear system in B_1, we use forward- and backward-substitution to solve the systems in L_1 and L_1^T, but recursively apply our algorithm to solve the linear system in A_1.

We will use a recursion of depth r, a constant to be determined later. We let $A_0 = A$ denote the initial matrix. We let B_{i+1} denote the preconditioner for A_i, $L_i C_i L_i^T$ be the partial LDL^T factorization of B_i in trim order, and $C_i = [D_i, 0; 0, A_i]$. To analyze the algorithm, we must determine the relative error ϵ_i to which we will solve the systems in A_i. The bound we apply is derived from the following lemma, which we derive from a result of Golub and Overton [GO88].

Lemma 5.1 (Preconditioned Inexact Chebyshev Method). *Let A and B be Laplacian matrices satisfying $\sigma(B, A) \geq 1$. Let $\boldsymbol{x}$ be the solution to $A\boldsymbol{x} = \boldsymbol{b}$. If, in each iteration of the preconditioned Chebyshev Method, a vector $\boldsymbol{z}_k$ is returned satisfying*

$$B\boldsymbol{z}_k = \boldsymbol{r}_k + \boldsymbol{q}_k, \text{ where } \|\boldsymbol{q}_k\| \leq \delta \|\boldsymbol{r}_k\|,$$

where $\delta \leq \left(128\sqrt{\kappa_f(B)}\sigma(A, B)\right)^{-1}$, then the k-th iterate, $\boldsymbol{x}_k$, output by the algorithm will satisfy

$$\|\boldsymbol{x} - \boldsymbol{x}_k\| \leq 6 \cdot 2^{-k/\sqrt{\kappa_f(A,B)}} \kappa_f(A)\sqrt{\kappa_f(B)} \|\boldsymbol{x}\|.$$

Our main theorem is:

Theorem 5.2 (Recursive). *Let A be an n-by-n PSDDD matrix with m non-zero entries. Using the recursive algorithm, one can solve the system $A\boldsymbol{x} = \boldsymbol{b}$ to relative accuracy ϵ in time*

$$O\left(m^{1.31+o(1)}(\log(\epsilon^{-1})\log(n\kappa(A)))^{O(1)}\right).$$

Moreover, if the graph of A does not contain a minor isomorphic to the complete graph on m^θ vertices, or has genus at most $m^{2\theta}$, or is the Gremban cover of such a graph, then the exponent of m can be reduced to $1 + 5\theta + o(1)$.

We note that if $G(A)$ is planar, then the algorithm take time nearly linear in m.

The following two lemmas allow us to bound the accuracy of the solutions to systems in B_i in terms of the accuracy of the solutions to the corresponding systems in A_i.

Lemma 5.3. *Let LCL^T be a partial LDL^T-decomposition of a symmetric diagonally dominant matrix. Then,*

$$\kappa(L) \leq 2n^{3/2}.$$

Proof. As L is column diagonally-dominant and has 1s on its diagonal, $\|L\|_1 \leq 2$; so, $\|L\| \leq 2\sqrt{n}$. By a result of Malyshev [Mal00, Lemma 1], $\|L^{-1}\| \leq n$ (also see Peña [Peñ98]). $\square$

Lemma 5.4. *Let B be a Laplacian matrix, let LCL^T be the partial LDL^T-factorization obtained by eliminating vertices of B in the trim order. Then, $\kappa(C) \leq \kappa(B)$.*

Proof. We recall that C has form

$$\begin{pmatrix} D & 0 \\ 0 & A_1 \end{pmatrix}.$$

The factor A_1 is identical to that obtained from partial Cholesky factorization, so $\kappa(A_1) \leq \kappa(B)$ follows from Lemma 1.3. To now bound $\kappa(C)$, we need merely show that each entry of D lies between the smallest and largest non-zero eigenvalues of B. This follows from the facts that the ith diagonal of D equals the value of the diagonal of the corresponding vertex in the lower factor right before it is eliminated, this value lies between the smallest and largest non-zero elements of the corresponding factor, and by Lemma 1.3, these lie between the largest and smallest non-zero eigenvalues of B. $\square$

Lemma 5.5. *Let B be a Laplacian matrix and let LCL^T be the partial LDL^T-factorization obtained by eliminating vertices of B in the trim order. For any $c \in \mathbf{Span}(B)$, let s be the solution to $Cs = L^{-1}c$ and let $\tilde{s}$ satisfy $\|s - \tilde{s}\| \leq \epsilon \|s\|$. Let $\tilde{y}$ be the solution to $L^T \tilde{y} = \tilde{s}$. Then*

$$\|c - B\tilde{y}\| \leq \epsilon \kappa(L)\kappa(C) \|c\|.$$

Proof of Lemma 5.5. First, note that $c - B\tilde{y} = LC(s - \tilde{s})$ and $c = LCs$. Moreover, $L^{-1}c$ must lie in $\mathbf{Span}(C)$. Thus, $\|Cs\| \geq \lambda_2(C) \|s\|$, and so

$$\|C(s - \tilde{s})\| \leq \epsilon \kappa_f(C) \|Cs\|.$$

As L is non-degenerate, we may conclude

$$\|LC(s - \tilde{s})\| \leq \epsilon \kappa(L)\kappa_f(C) \|LCs\|.$$

$\square$

Proof of Theorem 5.2. For $A_0, \ldots, A_r$, $B_1, \ldots, B_r$, $C_1, \ldots, C_r$, and $L_1, \ldots, L_r$ as defined above, we can apply Lemma 5.4 and Theorem 3.1 to show:

- $\kappa_f(A_i) \leq \kappa_f(B_i) \leq m^{i(1+o(1))}\kappa_f(A)$,

- $\kappa_f(B_i) \leq m^{1+o(1)}\kappa_f(A_{i-1}) \leq m^{i(1+o(1))}\kappa_f(A)$

In the recursive algorithm we will solve systems in A_i, for $i \geq 1$, to accuracy

$$\epsilon_i = \left(128 m^{i(1+o(1))}(2n^{3/2}\kappa(A))\right)^{-1}.$$

By Lemma 5.5 and the above bounds, we then obtain solutions to the systems in B_i to sufficient accuracy to apply Lemma 5.1.

Let m_i be the number of edges of A_i. When constructing the preconditioner, we set $t_i = (m_i)^{\gamma}$, for a γ to be chosen later. Thus, by Theorem 3.1 and Proposition 1.1, $m_i \leq m^{(2\gamma)^i}$, and $\kappa_f(A_i, B_{i+1}) = m^{(2\gamma)^i(1-\gamma)+o(1)}$.

We now prove by induction that the running time of the algorithm obtained from a depth r recursion is

$$O\left(m^{\beta_r + o(1)} \left(r \log(n\kappa(A))\right)^r\right), \text{ where}$$

$$\beta_r \stackrel{\text{def}}{=} \left(\frac{1-\gamma}{2}\right) \sum_{i=1}^{r} (2\gamma)^{i-1} + 2(2\gamma)^r,$$

and $\gamma \stackrel{\text{def}}{=} (3 - \sqrt{5})/2$. In the limit, β_r approaches $\beta_\infty \stackrel{\text{def}}{=} (3 + \sqrt{5})/4$ from above. The base case, $r = 1$, follows from Theorem 4.1.

The preprocessing time is negligible as the partial Cholesky factorizations used to produce the C_i take linear time, and the full Cholesky factorization is only performed on A^r.

Thus, the running time is bounded by the iterations. The induction follows by observing that the iteration time is $m^{\frac{1-\gamma}{2}+o(1)}\left(m + m_1^{\beta_{r-1}}\right) \log\left(\kappa(A_r)\kappa(B_r)/\epsilon_r\right)$, which proves the inductive hypothesis because $m_1^{\beta_{r-1}} > m$. As $1.31 > \beta_\infty$, there exists an r for which $\beta_r < 1.31$.

When the graph of A does not contain a K_{m^θ} minor or has genus at most $m^{2\theta}$, we apply a similar analysis. In this case, we have $m_i \leq m_{i-1}m^{\theta+o(1)}$. Otherwise, our proof is similar, except that we set $\gamma = (3 - \theta - \sqrt{1 + 6\theta + \theta^2})/2$, and obtain $\beta_\infty = (3 + \theta + \sqrt{1 + 6\theta + \theta^2})/4$, and note that $\beta_\infty \leq 1 + 5\theta$. $\square$

References

[AKPW95] Noga Alon, Richard M. Karp, David Peleg, and Douglas West. A graph-theoretic game and its application to the k-server problem. *SIAM Journal on Computing*, 24(1):78–100, February 1995.

[Awe85] Baruch Awerbuch. Complexity of network synchronization. *Journal of the ACM*, 32(4):804–823, October 1985.

[BCHT] Erik Boman, Doron Chen, Bruce Hendrickson, and Sivan Toledo. Maximum-weight-basis preconditioners. *to appear in Numerical Linear Algebra and Applications.*

[BGH$^+$] M. Bern, J. Gilbert, B. Hendrickson, N. Nguyen, and S. Toledo. Support-graph preconditioners. *submitted to SIAM J. Matrix Anal. & Appl.*

[BH] Erik Boman and B. Hendrickson. Support theory for preconditioning. *submitted to SIAM J. Matrix Anal. & Appl (Revised 10/02).*

[BH01] Erik Boman and B. Hendrickson. On spanning tree preconditioners. Manuscript, Sandia National Lab., 2001.

[Bol78] B. Bollobas. *Extremal Graph Theory*. Academic Press, London, 1 edition, 1978.

[Bru95] Are Magnus Bruaset. *A Survey of Preconditioned Iteratitve Methods*. Longman Scientific and Technical, 1995.

[CW82] D. Coppersmith and S. Winograd. On the asymptotic complexity of matrix multiplication. *SIAM Journal on Computing*, 11(3):472–492, August 1982.

[GHT84] John R. Gilbert, Joan P. Hutchinson, and Robert Endre Tarjan. A separator theorem for graphs of bounded genus. *Journal of Algorithms*, 5(3):391–407, September 1984.

[GMZ95] Keith Gremban, Gary Miller, and Marco Zagha. Performance evaluation of a new parallel preconditioner. In *9th IPPS*, pages 65–69, 1995.

[GO88] G. H. Golub and M. Overton. The convergence of inexact Chebychev and Richardson iterative methods for solving linear systems. *Numerische Mathematik*, 53:571–594, 1988.

[Gre96] Keith Gremban. *Combinatorial Preconditioners for Sparse, Symmetric, Diagonally Dominant Linear Systems*. PhD thesis, Carnegie Mellon University, CMU-CS-96-123, 1996.

[GV89] G. H. Golub and C. F. Van Loan. *Matrix Computations, 2nd. Edition*. The Johns Hopkins University Press, Baltimore, MD, 1989.

[HY81] Louis A. Hageman and David M. Young. *Applied iterative methods*. Computer Science and Applied Mathematics. Academic Press, New York, NY, USA, 1981.

[Jos97] Anil Joshi. *Topics in Optimization and Sparse Linear Systems*. PhD thesis, UIUC, 1997.

[LRT79] Richard J. Lipton, Donald J. Rose, and Robert Endre Tarjan. Generalized nested dissection. *SIAM Journal on Numerical Analysis*, 16(2):346–358, April 1979.

[Mal00] A. N. Malyshev. A note on the stablity of gauss-jordan elimination for diagonally dominant matrices. *Computing*, 65:281–284, 2000.

[McC87] S. F. McCormick. *Multigrid Methods*, volume 3 of *Frontiers in Applied Mathematics*. SIAM Books, Philadelphia, 1987.

[MMP⁺02] Bruce M. Maggs, Gary L. Miller, Ojas Parekh, R. Ravi, and Shan Leung Maverick Woo. Solving symmetric diagonally-dominant systems by preconditioning. 2002.

[Peñ98] J. M. Peña. Pivoting strategies leading to diagonal dominane by rows. *Numerische Mathematik*, 81:293–304, 1998.

[PRS94] Serge Plotkin, Satish Rao, and Warren D. Smith. Shallow excluded minors and improved graph decompositions. In *5th SODA*, pages 462–470, 1994.

[Rei98] John Reif. Efficient approximate solution of sparse linear systems. *Computers and Mathematics with Applications*, 36(9):37–58, 1998.

[SF73] Gilbert Strang and George J. Fix. *An Analysis of the Finite Element Method*. Prentice-Hall, Englewood Cliffs, NJ 07632, USA, 1973.

[ST83] Daniel D. Sleator and Robert E. Tarjan. A data structure for dynamic trees. *Journal of Computer and System Sciences*, 26(3):362–391, 1983.

[Vai90] Pravin M. Vaidya. Solving linear equations with symmetric diagonally dominant matrices by constructing good preconditioners. Unpublished manuscript UIUC 1990. A talk based on the manuscript was presented at the IMA Workshop on Graph Theory and Sparse Matrix Computation, October 1991, Minneapolis., 1990.

[Var62] Richard S. Varga. *Matrix Iterative Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey, 1962. Second Edition, Springer, Berlin, 2000.

[You71] David M. Young. *Iterative solution of large linear systems*. Academic Press, New York, NY, USA, 1971.