Tromino Tilings of Domino Deficient Rectangles

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Abstract

We consider tromino tilings of $m \times n$ domino-deficient rectangles, where 3|(mn-2) and $m,n \geq 0$, and characterize all cases of domino removal that admit such tilings, thereby settling the open problem posed by J. M. Ash and S. Golomb in [6]. Based on this characterization, we design a procedure for constructing such a tiling if one exists. We also consider the problem of counting such tilings and derive the exact formula for the number of tilings for $2 \times (3t+1)$ rectangles, the exact generating function for $4 \times (3t+2)$ rectangles, where $t \geq 0$, and an upper bound on the number of tromino tilings for $m \times n$ domino-deficient rectangles. We also consider general 2-deficiency in $n \times 4$ rectangles, where $n \geq 8$, and characterize all pairs of squares which do not permit a tromino tiling.

1 Introduction

Tiling the plane is an interesting field of recreational mathematics. In a 1953 talk at the Harvard Math Club, Solomon Golomb defined a class of geometric figures called *polyominoes*, namely, connected figures formed of congruent squares placed so each square shares one side with at least one other square. *Dominoes*, which use two squares, and *Tetrominoes* (the *Tetris* pieces), which use four squares, are well known to game players. Golomb first published a paper about polyominoes in *The American Mathematical Monthly* [11]. Polyominoes were later popularized by Martin Gardner in his *Scientific American* columns called "Mathematical Games" (see, for example, [4], [5]). A region is *tiled* with a given tile if it is completely covered by its copies without any overlap. Several results about tiling regular shapes with polyominoes are mentioned by Stanley and Ardila [10], and Do [7]. Many of the initial questions asked about polyominoes concern the number of *nominoes* (those formed from *n* squares), and what shapes can be tiled using just one of the polyominoes, possibly leaving one or two squares uncovered.

In this paper we consider tilings of rectangles using 3-ominoes or trominoes of which there are two basic shapes, namely a 1×3 rectangle and an L-shaped figure (more commonly known as the right tromino). We restrict ourselves to tilings only with the right tromino. From now on, we will simply say tromino to mean the right tromino. In the past few years, tromino tilings of rectangles have also been studied quite extensively. Chu and Johnsonbaugh [1], first characterized all $m \times n$ rectangles that permit a tromino tiling.

Theorem 1 (Chu-Johnsonbaugh Theorem [1]) An $m \times n$ rectangle can always be tiled by trominoes if 3|mn, $2 \le m \le n$, except for $3 \times (2k+1)$ rectangles where $k \ge 1$.

A rectangle from which one square has been removed is called a deficient rectangle. Golomb [11] proved that deficient squares whose side length is a power of two can be tiled by trominoes. Chu and Johnsonbaugh extended Golomb's work to the general cases of deficient squares [2]. Ash and Golomb [6] considered the problem of tiling a deficient rectangle. They characterized all positions of a square whose removal still permits a tromino tiling. Their result is as follows:

Theorem 2 (Deficient Rectangle Theorem [6]) An $m \times n$ deficient rectangle, $2 \le m \le n$, $3 \mid (mn-1)$, has a tiling, regardless of the position of the missing square, if and only if (a) neither side has length 2 unless both of them do, and (b) $m \ne 5$.

A slightly weaker version of this theorem was proved by Chu and Johnsonbaugh in [1]. Now consider the problem of tiling a rectangle from which two squares have been removed. This problem was posed as an open problem in [6]. A rectangle from which two squares are missing is called a 2-deficient rectangle. In this paper, we consider tromino tilings of a special class of 2-deficient rectangles, namely, the domino-deficient rectangles.

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These are rectangles from which a domino has been removed. We characterize all cases of domino removal in such rectangles which do not permit a tromino tiling. It turns out that there are only 16 cases of domino removal which prevent a tiling for any $m \times n$ rectangle, where $m, n \geq 7$ and 3 | (mn - 2). These positions are $\{(1,2),(2,2)\},\{(2,1),(2,2)\},\{(2,3),(2,4)\},\{(3,2),(4,2)\}$ and their symmetric counterparts (viewed w.r.t. the other three corners). Based on this characterization, we design a procedure for constructing a tiling of a given domino-deficient rectangle if one exists. We also consider the problem of counting such tilings and derive the exact formula for the number of tilings for $2 \times (3t+1)$ rectangles and the exact generating function for $4 \times (3t+2)$ rectangles, where $t \geq 0$. We derive an upper bound on the number of tromino tilings of arbitrary $m \times n$ domino-deficient rectangles. We also consider general 2-deficiency in $n \times 4$ rectangles, where $n \geq 8$, and characterize all pairs of squares which do not permit a tromino tiling.

1.1 Definitions and notation

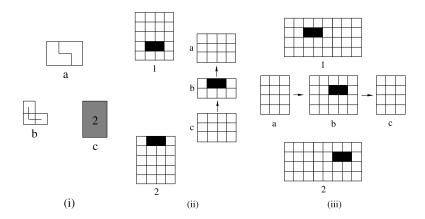


Figure 1: (i) Notations for a tromino tiling of R(i,j). (ii) A (3,4)-vquad shift. (iii) A (4,3)-hquad shift.

Firstly, the reader should note that a 2×3 rectangle can be tiled with trominoes (as shown in Figure 1(i)(a)). We denote a rectangle with i rows and j columns by R(i,j). We will indicate decompositions into non-overlapping subrectangles by means of an additive notation. For example, a $3i \times 2j$ rectangle can be decomposed into ij subrectangles of dimension 3×2 and we write this fact as $R(3i, 2j) = \sum_{a=1}^{i} \sum_{b=1}^{j} R(3, 2) = \sum_{a=1}^{j} R(3, 2)$ ijR(3,2). It follows from this and Figure 1(i)(a) that any $2i \times 3j$ or $3i \times 2j$ rectangle can be tiled with trominoes. For such rectangles, a very "trivial" tiling rule would be to place 2×3 or 3×2 rectangles lengthwise. From now on, any rectangle decomposed into a combination of $3i \times 2j$ subrectangles, $2i \times 3j$ subrectangles and trominoes will be considered as successfully tiled by trominoes. The square lying in row i and column j is denoted as (i,j). To make the notation simple, trominoes are depicted, in the rest of the paper, as a composition of two lines forming an L-shape across an actual tromino (as shown in Figure 1(i)(b). Domino-deficient rectangles with i rows and j columns are denoted in general by $R(i,j)^{--}$. All R(3,2) and R(2,3) rectangles are shown as gray-shaded rectangles labeled by the number 2, since they can be tiled in two ways (as shown in Figure 1(i)(c)). Any pair of missing squares that does not allow a tiling of the resultant structure is referred to as a bad pair. We will always specify bad pairs with respect to the top left corner of the given $m \times n$ rectangle. The reader should note that the bad pairs for $R(m,n)^{--}$ are the same as those for $R(n,m)^{--}$ (the only difference is that their corresponding coordinates change w.r.t. the top left corner). For a given $m \times n$ rectangle, we define a (m,k)-hquad shift to be a process of detaching the leftmost (rightmost) k columns and attaching an $m \times k$ rectangle from the right (left). For rectangles with no deficiency, such an operation has no meaning. However, for domino-deficienct rectangles, such an operation changes the position of the missing domino w.r.t. the top left corner of the rectangle, although size of the given rectangle remains the same. Figure 1(iii) shows a (4,3)-hquad shift for $R(4,8)^{--}$. Similarly, for an $m \times n$ rectangle, we define a (k,n)-vquad shift to be a process of detaching the uppermost (lowermost) $k \times n$ rectangle and attaching a $k \times n$ rectangle from the bottom (top). Figure 1(ii) shows a (3,4)-vquad shift for $R(5,4)^{--}$. We will use a vquad (hquad) shift to change an untileable configuration of a rectangle into a tileable one. The reader should note that we do not specify explicitly the direction associated with such shifts, however, the direction will be clear from the context, since we will only apply such shifts to rectangles where shifting is possible in exactly one direction.

1.2 Organization of the Paper

We present some elementary results for rectangles in Section 2, thereby giving an informal proof of the Chu-Johnsonbaugh Theorem. In Section 2.1, we present a special case of domino-deficiency, where a domino is removed from a corner of the given rectangle. We characterize all rectangles that admit such tilings. We then move on to consider general domino-deficient rectangles, where the position of the missing domino is no more restricted to a corner. For an $m \times n$ domino-deficient rectangle to be tileable by trominoes, the resultant area (mn-2) must be divisible by 3, i.e., $mn \equiv 2 \pmod{3}$. Taking into consideration a rotation by one right angle we assume without loss of generality, that $m \equiv 1 \pmod{3}$ and $n \equiv 2 \pmod{3}$. So, m can assume the values 1, 4, 7, ...(1+3t), where $t \ge 0$, and n can assume the values 2, 5, 8, ...(2+3t), where $t \ge 0$. Since no tromino can fit in one row or column, $m \geq 4$. We present the entire analysis of characterizing bad cases of domino removal in $m \times n$ rectangles in four exhaustive subcases as follows. In Section 3, we consider the simplest of these four cases, where we consider tiling a $R(2,3t+4)^{--}$ rectangle, where $t\geq 0$. We also address the problem of counting the number of tilings for such rectangles in Section 3.1, and derive the exact closed-form formula. In Section 4, we consider tiling $R(4,3t+8)^{--}$ rectangles. We also consider counting tilings for such rectangles in Section 4.1, and derive the exact generating function. In Section 5, we characterize domino-deficiency in $R(5,3t+4)^{--}$ rectangles and then proceed to consider tilings in arbitrary $m \times n$ domino-deficient rectangles in Section 6. We prove that there are only 16 bad cases of domino removal in any $m \times n$ domino-deficient rectangle, where $m, n \geq 7$; we also design a procedure for constructing such a tiling if one exists. In Section 7, we derive an upper bound on the number of tromino tilings of arbitrary $m \times n$ domino-deficient rectangles. Finally, in Section 8, we present an approach to study general 2-deficiency in rectangles and characterize all bad pairs for $n \times 4$ rectangles, where $n \geq 8$.

2 Elementary results for rectangles

First consider some rectangles that cannot be tiled. We show by contradiction that a 3×3 square Q cannot be tiled. There are three possible ways of covering the square (3,1), as shown in Figure 2(i)(a)-(c). Orientation 2(i)(c) is immediately ruled out, since square (1,1) cannot be tiled. In the cases 2(i)(a) and (b), a feasible tiling must tile the leftmost 3×2 subrectangle of Q, so that it is also a tiling of the third column R(3,1) of Q, a contradiction. Similarly, one can show by contradiction that a 3×5 rectangle R cannot be tiled. The above argument shows that a feasible tiling must tile the first two columns of R, and hence, also the rightmost three columns of R, a contradiction since we have just shown a 3×3 square to be untileable. Iterating this procedure, one can show that no R(3, odd) can be tiled. It turns out that there are no other untileable rectangles with area divisible by 3. Consider the following three decompositions:

$$R(3t, 2k) = tk \cdot R(3, 2), t, k \ge 1 \tag{1}$$

$$R(6t, 2k+3) = R(6t, 2k) + R(6t, 3), t, k \ge 1$$
(2)

$$R(9+6t,2k+5) = R(9+6t,2k) + R(9,5) + R(6t,2) + R(6t,3), t, k \ge 0$$
(3)

The reader should verify that any $m \times n$ rectangle can be written in the form R(3k, even), R(6k, odd) or R(9+6k,n), $n \geq 5$ and $k \geq 0$, where $2 \leq m \leq n$, such that if one of m and n is 3 then the other is not odd. Equations (1)-(3) show the corresponding tiling rules in each of these cases, the tiling of R(5,9) (and hence R(9,5)) is shown in Figure 2(i)(d). Note that apart from R(9,5), each of the subrectangles obtained in equations (1)-(3) has dimensions $3i \times 2j$ or $2i \times 3j$, where $i, j \geq 1$, and so is tileable. So we have an informal proof of the Chu-Johnsonbaugh Theorem.

2.1 Dog-Eariness in domino-deficient rectangles

We now consider tromino tilings of a special class of domino-deficient rectangles, namely, the domino-deficient dog-eared rectangles, where a domino is removed from a corner of the given rectangle. In our present discussion, we assume that the domino was removed from the top right corner. We will denote an $m \times n$ domino-deficient dog-eared rectangle by $R(m,n)^{--}$. If this rectangle is rotated by π radians, a similar figure with missing lower left-hand corner is created. If it is reflected about a central vertical (resp., horizontal) axis, a similar figure with missing lower right-hand (resp., upper left-hand) corner is created. The problem of tiling the original figure is clearly equivalent to tiling any of the other six cases (note that the missing domino can be either vertical or horizontal). We have the following result:

Theorem 3 [Domino-Deficient Dog-Eared Rectangle Theorem]

 $R(m,n)^{--}$ can always be tiled by trominoes if a domino is removed from a corner, provided $mn \equiv 2 \pmod{3}$ and $m, n \geq 4$.

Proof: Without loss of generality, assume m = 3j' + 1 and n = 3k' + 2, where $j', k' \ge 1$. Set j' = j + 1, k' = 1k+1, and so m=3j+4 and n=3k+5, where $j,k\geq 0$. In each of the cases shown we will break the given $m \times n$ rectangle into subrectangles, which either satisfy the conditions of the Chu-Johnsonbaugh Theorem [1], and so are tileable by trominoes, or are one among the cases shown in Figures 2(ii)(a)-(d), in which case we apply the corresponding tiling shown in Figure 2(ii). We first consider the cases when m=4,7. When m=4, we have only one case $R(4,3k+5)^{--}$ with $k\geq 0$. When m=7, since 7 has different parity, in order to satisfy the criterions of the Chu-Johnsonbaugh Theorem, we divide this case into two subcases, accordingly as k is even or odd. So, we have to tile either $R(7,6l+5)^{--}$, where k=2l and $l\geq 0$, or $R(7,6l+8)^{--}$, where k = 2l + 1 and $l \ge 0$. There correspond these three decompositions:

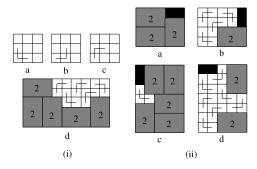


Figure 2: (i) (a)-(c) An impossibility proof. (d) Tiling of R(9,5). (ii) (a) and (b) Tilings of $R(4,5)^{--}$. (c) and (d) Tilings of $R(7,5)^{--}$ (Symmetric cases are also possible).

$$R(4,3k+5)^{--} = R(4,3k) + R(4,5)^{--}$$
(4)

$$R(7,6l+5)^{--} = R(7,6l) + R(7,5)^{--}$$
(5)

$$R(7,6l+8)^{--} = R(7,6l) + R(7,8)^{--}$$

= $R(7,6l) + R(3,8) + R(4,3) + R(4,5)^{--}$ (6)

For the algebraically inclined reader, these decompositions need no further explanation. However, the geometrically inclined reader should draw pictures to visualize them. (All the similar decompositions appearing below have straightforward geometrical interpretations.) Here, in the first two cases, a large rectangle was stripped from the left side of the figure. In the third case, a large rectangle was first stripped from the left side of the figure to obtain $R(7,8)^{--}$, from which the rectangle R(3,8) was removed from the bottom, to get $R(4,8)^{-1}$. R(4,3) was then stripped from its left side to finally get $R(4,5)^{-1}$. All the full rectangles are tileable since they satisfy the conditions of the Chu-Johnsonbaugh Theorem, $R(4,5)^{--}$ and $R(7,5)^{--}$ are tiled as in Figure 2(ii). We now consider the case when m=3j+4, n=3k+5, where $j\geq 2, k\geq 0$. Since m can have different parity, in order to satisfy the conditions of the Chu-Johnsonbaugh Theorem, we divide the above case into two subcases accordingly as k is even or odd. We have the following decompositions:

$$R(3j+4,6l+5)^{--} = R(3j+4,6l) + R(3j+4,5)^{--}$$

$$= R(3j+4,6l) + R(3j,5) + R(4,5)^{--}$$

$$R(3j+4,6l+8)^{--} = R(3j+4,6l) + R(3j+4,8)^{--}$$

$$R(3j+4,6l+8)^{--} = R(3j+4,6l+8)^{--}$$

$$R(3j+4,6l+8) = R(3j+4,6l) + R(3j+4,8)$$

$$= R(3j+4,6l) + R(3j,8) + R(4,3) + R(4,5)^{--}$$
(8)

In these cases too, we strip the $m \times n$ rectangle as above, obtaining $R(4,5)^{--}$ (tileable as in Figure 2(ii)(a)-(b) and full rectangles; R(3j+4,6l), R(3j,5), R(3j,8) and R(4,3), that satisfy the criterions of the Chu-Johnsonbaugh Theorem, and so are tileable by trominoes. The reader should note that R(3j,5)in equation (7) does not violate the conditions of the Chu-Johnsonbaugh Theorem. Since $j \geq 2$, R(3j,5)

can never denote R(3,5), which is untileable. A similar reasoning holds for the case when m=3j'+2 and n=3k'+1, where $j',k'\geq 1$, and so our result follows.

We now consider general domino-deficient rectangles, where the position of the missing domino is no more restricted to a corner. We will denote an $m \times n$ domino-deficient rectangle by $R(m,n)^{--}$. For an $m \times n$ domino-deficient rectangle to be tileable by trominoes, the resultant area (mn-2) must be divisble by 3, i.e., $mn \equiv 2 \pmod{3}$. Taking into consideration a rotation by one right angle we assume without loss of generality, that $m \equiv 1 \pmod{3}$, and $n \equiv 2 \pmod{3}$. So, m can assume the values 1, 4, 7,...(1+3t), where $t \geq 0$, and n can assume the values 2, 5, 8,...(2+3t), where $t \geq 0$. Since no tromino can fit in one row or column, $m \geq 4$. We present the entire analysis of characterizing bad cases of domino removal in $m \times n$ rectangles in four exhaustive subcases as follows. In Section 3, we consider tilings of $R(2,3j+4)^{--}$ rectangles, in Section 4, we consider tilings of $R(4,3j+8)^{--}$ rectangles, in Section 5, we consider tilings of $R(5,3j+4)^{--}$ rectangles, and finally in Section 6, we consider tilings of $R(3j+7,3k+8)^{--}$ rectangles, where $j,k \geq 0$.

3 Tromino tilings of $R(2, 3t+4)^{--}$ rectangles

We start with the simplest of the four cases of domino-deficiency enumerated in the previous section, namely, when one dimension of the given $m \times n$ rectangle (say m) is 2. First consider tromino tilings of $2 \times 3t$ rectangles instead of domino-deficient $2 \times (3t+1)$ rectangles, where $t \geq 1$. Consider the orientations of the tromino covering (1,1) (as shown in Figure 3(a)-(b)). If it covers (2,1) (see Figure 3(a)) then a 2×3 rectangle (R(2,3)) is completed by the tromino covering (2,2), and if it covers (2,2) (see Figure 3(b)) then R(2,3) is completed by the tromino covering (2,1). So, any tiling of R(2,3t) consists of a tiling of R(2,3) on the leftmost side followed by a tiling of R(2,3(t-1)) (see Figure 3(c)). Let T(2,3t) denote the number of tromino tilings of R(2,3t). Taking R(2,3) as the basis case and performing induction on t, using the above argument in the inductive step, we conclude that $T(2,3t) = 2^t$. We state this result in the following lemma:

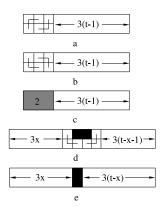


Figure 3: Tilings of R(2,3t) and $R(2,3t+1)^{--}$.

Lemma 1 The number of tromino tilings of a $2 \times 3t$ rectangle, $T(2,3t) = 2^t$.

We now consider tromino tilings of $R(2, 3t+1)^{--}$ rectangles, where $t \ge 1$. Consider the situation when the removed domino is vertical (see Figure 3(e)). This domino divides the given $2 \times (3t+1)$ rectangle into two subrectangles. These two smaller subrectangles must be completely tileable by trominoes, since each tromino occupies an area of 3, so the area of both these rectangles must be divisible by 3. So, we conclude that a vertical domino can be removed only from the columns x = 3k + 1, where $k \ge 0$. Now consider the case when a horizontal domino is removed from R(2, 3t+1). Without loss of generality, assume that it is present in the 1st row and occupies (1, r) and (1, r+1). The reader can easily see that the only way of covering (2, r) and (2, r+1) is as shown in Figure 3(d). Again, we get two smaller rectangles which must be tileable by trominoes. We conclude that there areas must also be divisible by 3. So, the only possible values of r = 3x + 2, where x > 0. We summarize these results in the following theorem:

Theorem 4 [Domino-Deficient Direc Theorem] In case of a vertical domino removal from R(2, 3t + 4), the remaining area permits a tromino tiling if and only if the missing domino occupies the position $\{(1, 3k + 4), (1, 3k$

1), (2,3k+1)}, where $k \ge 0$. In case of a horizontal domino removal from R(2,3t+4), the remaining area permits a tiling if and only if the missing domino occupies either the position $\{(1,3x+2),(1,3x+3)\}$ or the position $\{(2,3x+2),(2,3x+3)\}$, where $x \ge 0$.

3.1 Counting tilings of $R(2, 3t+4)^{--}$ rectangles

Apart from proving existence of tilings for $R(2,3t+1)^{--}$ rectangles, $t \geq 1$, the Domino-Deficient Direc Theorem is also constructive in nature. This fact can also be used for counting the number of tromino tilings of R(2,3t+1) with 2t trominoes and one domino. Let $T(2,3t+1)^{--}$ denote this number. First consider the case when the removed domino is vertical and let $T_V(2,3t+1)^{--}$ be the number of tilings in this case. From the Domino-Deficient Direc Theorem, the domino can only occupy the columns x = 3k + 1, where $k \geq 0$. So the two smaller rectangles on either side are R(2,3k) and R(2,3(t-k)). Using this fact and Lemma 1, we immediately arrive at the recurrence:

$$T_{V}(2,3t+1)^{--} = \sum_{k=0}^{t} T(2,3k) \times T(2,3(t-k))$$

$$= \sum_{k=0}^{t} 2^{k} \times 2^{t-k}$$

$$= (t+1) \cdot 2^{t}$$
(9)

Now consider the case when the removed domino is horizontal and let $T_H(2, 3t+1)^{--}$ denote the number of tilings in this case. The Domino-Deficient Direc Theorem states that the removed domino can only occupy the pair of columns (3x+2, 3x+3), where $x \ge 0$. So the smaller rectangles on either side are R(2, 3x) and R(2, 3(t-x-1)). The removed domino can be either in the 1st or the 2nd row, introducing an additional factor of 2. From the above conditions and Lemma 1, we get the following recurrence:

$$T_{H}(2,3t+1)^{--} = 2\sum_{x=0}^{t-1} T(2,3x) \times T(2,3(t-x-1))$$

$$= 2\sum_{x=0}^{t-1} 2^{x} \times 2^{t-x-1}$$

$$= t \cdot 2^{t}$$
(10)

The removed domino can be either horizontal or vertical. So, from (9) and (10), we conclude that $T(2,3t+1)^{--} = T_H(2,3t+1)^{--} + T_V(2,3t+1)^{--} = (t+1).2^t + t.2^t = (2t+1).2^t$. We summarize this result in the following theorem:

Theorem 5 The number of tromino tilings of $R(2, 3t+1)^{--}$, where $t \ge 1$, is

$$T(2,3t+1)^{--} = (2t+1) \cdot 2^t \tag{11}$$

The reader may think that this result is surprising and contrary to his expectations. Just by removing a domino from the $2 \times n$ rectangle, the complexity of the number of tromino tilings changes from $\mathcal{O}(2^t)$ to $\mathcal{O}(t.2^t)$, although our intuition says that adding deficiency should be pretty restrictive on the orientations of some trominoes, and so the number of tilings should decrease! The catch lies in observing that the size of the rectangle being considered also increases (from R(2,3t) to R(2,3t+1)), so the number of trominoes remains the same, i.e., 2t. Moreover, we also take into account all permissible positions of the missing domino.

4 Tromino tilings of $R(4, 3t + 8)^{--}$ rectangles

Consider the second case when one dimension of the given $m \times n$ rectangle (say m) is 4. The bad pairs for the case n = 5 will be enumerated separately in another section, so for now we will assume that $n \ge 8$. See Figure 4. The bad pairs for $R(4,8)^{--}$ have been indicated by dark squares. It turns out that these are the

only pairs that do not permit a tromino tiling. As the reader can see, the pairs of squares $\{(1,2),(2,2)\}$, $\{(2,1),(2,2)\}$, $\{(3,1),(3,2)\}$, $\{(3,2),(4,2)\}$, $\{(2,7),(2,8)\}$, $\{(1,7),(2,7)\}$, $\{(3,7),(3,8)\}$, $\{(3,7),(4,7)\}$ make the cornermost squares (1,1), (1,8), (4,1), (4,8) inaccessible, and so, do not permit a tromino tiling. Now consider the four pairs $\{(2,3),(2,4)\}$, $\{(2,5),(2,6)\}$, $\{(3,3),(3,4)\}$, $\{(3,5),(3,6)\}$. In the pair $\{(2,3),(2,4)\}$ the tromino covering (1,3) makes the square (1,1) inaccessible. Similarly, in the case of the pairs $\{(2,5),(2,6)\}$, $\{(3,3),(3,4)\}$, $\{(3,5),(3,6)\}$ the trominoes covering (1,6), (4,3), and (4,6) make the squares (1,8), (4,1) and (4,8) inaccessible. So, we conclude that these pairs are also bad (refer Figure 5(ii)). Finally, consider the pairs $\{(2,3),(3,3)\}$ and $\{(2,6),(3,6)\}$. Since the two pairs are symmetric (reflections of each other in the vertical axis), we need only consider the badness of $\{(2,3),(3,3)\}$. If the tromino covering (1,3) covers (1,2), then the square (1,1) becomes inaccessible. So this tromino must cover (1,4) (see Figure 5(i)). Similarly, the tromino covering (4,3) must cover (4,4). But now we have isolated a 4×2 rectangle which does not satisfy the conditions of the Chu-Johnsonbaugh Theorem, and so is untileable. We conclude that the pairs $\{(2,3),(3,3)\}$ and $\{(2,6),(3,6)\}$ are also bad. We are now ready to prove the following theorem:

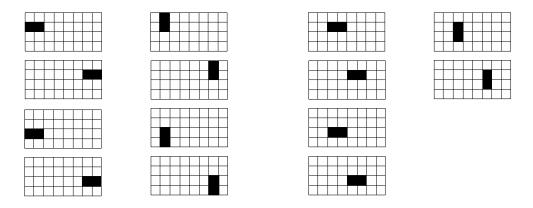


Figure 4: Bad squares for $R(4,8)^{--}$.

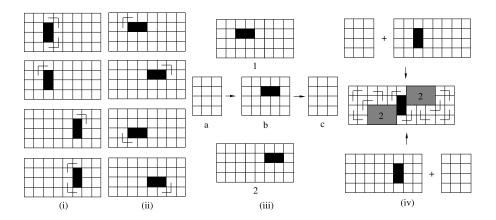


Figure 5: (i) and (ii) Untileability for some bad pairs for $R(4,8)^{--}$. (iii) A (4,3)-hquad shift. (iv) Alternate measure when a (4,3)-hquad shift does not help.

Theorem 6 [Domino-Deficient Quadrec Theorem] The only bad pairs for $R(4,3t+8)^{--}$ are $\{(2,1),(2,2)\}$, $\{(1,2),(2,2)\}$, $\{(2,3t+7),(2,3t+8)\}$, $\{(1,3t+7),(2,3t+7)\}$, $\{(3,1),(3,2)\}$, $\{(3,2),(4,2)\}$, $\{(3,3t+7),(3,3t+8)\}$, $\{(3,3t+7),(4,3t+7)\}$, $\{(2,3),(3,3)\}$, $\{(2,3t+6),(3,3t+6)\}$, $\{(2,3),(2,4)\}$, $\{(2,3t+5),(2,3t+6)\}$, $\{(3,3),(3,4)\}$ and $\{(3,3t+5),(3,3t+6)\}$.

Proof: The proof for the badness of the cases of domino-removal enumerated in the theorem statement is similar to that for $R(4,8)^{--}$, and so is left for the reader as an exercise. We will prove the existence of a tromino tiling in all other cases. Consider tilings of $R(4,3t+8)^{--}$, where $t \ge 1$. This big domino-deficient

rectangle can be viewed as t subrectangles of dimension 4×3 and one domino-deficient $R(4,8)^{--}$ rectangle joined together. Following our additive notation, we write this as:

$$R(4,3t+8)^{--} = t \cdot R(4,3) + R(4,8)^{--}$$
(12)

If the missing domino in the rectangle $R(4,8)^{--}$ does not form a bad pair, then we can tile it. A 4×3 rectangles satisfies the conditions of the Chu-Johnsonbaugh Theorem, and so is tileable. So, in this case, we achieve a tiling of $R(4,3t+8)^{--}$. Consider the situation when the missing domino in $R(4,8)^{--}$ (as above) forms a bad pair. In this case, we perform a (4,3)-hquad shift. The reader should note that we are only considering pairs of squares which are not enumerated in the theorem statement, so such a shift is always possible. If such a shift removes the "badness" of $R(4,8)^{--}$, then we are done. The only case in which a (4,3)-hquad shift will not work, is when $R(4,8)^{--}$ has $\{(2,3),(3,3)\}$ or $\{(2,6),(3,6)\}$ as its bad pair. These pairs are symmetric with respect to a shift by three columns, and so just get interchanged. In this case, we join a removed 4×3 rectangle from the left (or right) as shown in Figure 5(iv). The new rectangle $R(4,11)^{--}$ is tileable (as is evident from the figure). Each of the (t-1) subrectangles of dimension 4×3 is also tileable, so we get a tiling of $R(4,3t+8)^{--}$.

4.1 Counting tilings of $R(4, 3t + 8)^{--}$ rectangles

We now proceed to enumerate all tilings of R(4, 3t + 8) rectangles, where $t \ge 0$ with one domino and 4t + 10 trominoes. Let $T(4, 3t + 8)^{--}$ denote this number. Let the number of such tilings be $T_V(4, 3t + 8)^{--}$ ($T_H(4, 3t + 8)^{--}$) when the missing domino is vertical (horizontal). Consider the three interfaces as shown in Figure 6(i). These three interfaces are called *straight*, *deep jog*, and *shallow jog*. For these three interfaces, we define N(t), $N_1(t)$ and $N_2(t)$ respectively as the number of tilings when there are n = 3t columns to the left of the dotted line. Obviously $N_1(t)$ and $N_2(t)$ remain the same if we count tilings of their vertical reflections instead. To express these, we write them as generating functions

$$G(z) = \sum_{t} N(t)z^{t}$$

and similarly for $G_1(z)$ and $G_2(z)$. Finally, N(0) = 1, since there is exactly one way to tile a 4×0 rectangle. Now consider the case when the missing domino is vertical. We set our coordinate system so that the domino occupies the 4th column. Assume the domino occupies the position $\{(1,4),(2,4)\}$. Several cases arise depending on whether a single tromino covers the squares (3,4) and (4,4), or two separate trominoes cover them. Consider the former case first. If a tromino covers (3,4), (4,4), (3,3), then a 2×3 rectangle is completed by the tromino covering (4,3) (see Figure 6(ii)(d)). If this tromino covers (4,3), then the tromino covering (3,3) has three permissible orientations. If it covers (3,2) and (4,2) then a 2×3 rectangle is again completed. If it covers (3,2) and (2,2) (resp. (3,2) and (2,3)), then a 3×2 rectangle is completed by the tromino covering (2,3) (resp. (1,3)) (see Figure 6(ii)(c)). Now consider the case when two different trominoes cover (3,4) and (4,4). If the tromino covering (3,4) covers (3,5) and (4,5), then a 2×3 rectangle is completed by the tromino covering (4,4) (see Figure 6(ii)(b)). If a tromino covers (4,4), (4,5) and (3,5), then there are two possible orientations for the tromino covering (3,4). If it covers (3,3) and (4,3), then a 2×3 rectangle is again completed. If it covers (3,3) and (2,3), then we get the case shown in Figure 6(ii)(a). (The reader should note that these arguments will also hold when the vertical domino is present in the last two rows.) Assume now that the domino occupies the position $\{(2,4),(3,4)\}$. The reader can easily see that only two cases are possible in this situation (see Figures 6(ii)(e) and (f)). Based on the above case analysis, we get equation (13). Suppose $G_V(z) = \sum_{t=1}^{\infty} T_V(4,3t+2)^{--}z^t$. From (13), it can be seen that $G_V(z)$ is basically a sum of convolutions of G(z), $G_1(z)$ and $G_2(z)$. We express this fact in equation (14).

$$T_{V}(4,3t+2)^{--} = 8 \times \{\sum_{x=0}^{t-1} N_{2}(x) \cdot N(t-x-1) + \sum_{x=1}^{t} N_{1}(x) \cdot N(t-x)\}$$

$$+ 4 \times \{\sum_{x=0}^{t-1} N(x) \cdot N_{2}(t-x-1) + \sum_{x=1}^{t-2} N_{2}(x) \cdot N_{2}(t-x-1)\}$$

$$+ 2 \times \{\sum_{x=0}^{t} N(x) \cdot N(t-x) + \sum_{x=1}^{t-2} N_{2}(x) \cdot N_{2}(t-x-1)\}$$

$$(13)$$

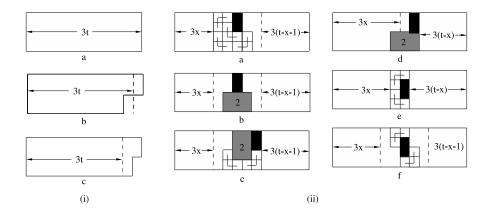


Figure 6: (i) Three kinds of interfaces on rectangles of width 4, (a) straight, (b) deep jog, (c) shallow jog. (ii) Various cases when the domino is vertical.

$$G_V(z) = 12zG_2(z)G(z) + 8G_1(z)G(z) + 6zG_2^2(z) + 2G^2(z)$$
(14)

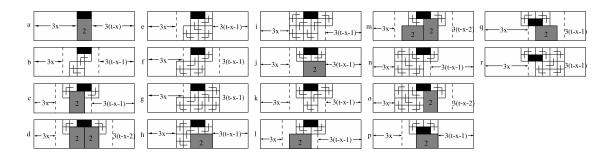


Figure 7: Various cases when the domino is horizontal.

We now consider the case when the missing domino is horizontal. Setting our coordinate system such that this domino is present in columns 4-5. Assume first that it is present in the first row, i.e., it occupies the position $\{(1,4),(1,5)\}$. Several cases arise depending upon the orientation of the tromino covering (2,4), which we call the *crucial tromino* for our present discussion. We shall refer to reflections in the Y-axis as vertical reflections and those in the X-axis as horizontal reflections. If the crucial tromino covers (2,5) and (3,4), then there are two possible orientations of the tromino covering (4,4), as shown in Figure 7(a) and (b). If the crucial tromino covers (3,4) and (3,5), then we get the case shown in Figure 7(e). The case when the crucial tromino covers (2,5) and (3,5) is symmetric to the case when it covers (2,5) and (3,4). Now assume that the crucial tromino covers (2,3) and (3,4). If the tromino covering (4,4) covers (4,3) and (3,3), then there are two cases possible, as shown in Figure 7(c) and (d). If the tromino covering (4,4) covers (4,5) and (3,5), then two cases are possible, which are the vertical reflections of Figure 7(i) and (k). If the crucial tromino covers (3,4) and (3,3), then we get the cases shown in Figure 7(f) and (g). Three cases arise when the crucial tromino covers (2,3) and (3,3). These are shown in Figures 7(h), 7(i), and the vertical reflection of Figure 7(g). Finally, consider the case when the crucial tromino covers (2,3) and (1,3). If the tromino covering (3,4) covers (2,5) and (3,5), then we get the vertical reflection of Figure 7(e). If the tromino covering (3,4) covers (3,5) and (4,5), then there are two possible cases, which are the vertical reflections of Figures 7(h) and (j), and if it covers (4,4) and (3,5), then we get the case in Figure 7(j). If the tromino covering (3,4) covers (4,4) and (4,5), then three cases arise, which are those in Figures 7(j), (k) and the vertical reflection of Figure 7(f). Now assume that the tromino covering (3,4) covers (4,4) and (3,3), then the two cases shown in Figures 7(1) and (m) arise depending on how (2,5) is covered. If the tromino covering (3, 4) covers (4, 4) and (4, 3), then we get the cases shown in Figure 7(1), (n) and (o), and if it covers (3,3) and (4,3), then we get the vertical reflection of Figure 7(j). So, Figures 7(a) -(o) (and their horizontal reflections), show all the various cases which arise when the missing domino is present in the first or fourth row. We now consider the case when the missing domino is present in the second row. Again setting our

coordinate system such that this domino occupies the position $\{(2,4),(2,5)\}$. The reader can easily see that the tromino covering (1,4) ((1,5)) must cover (1,3) and (2,3) ((1,6) and (2,6)). Depending on how (3,5) is covered, three cases essentially arise (or their vertical reflections), as shown in Figure 7(p)-(r). Thus, Figure 7 (and its reflections) shows all the possible cases which can arise when the missing domino is horizontal. Based on the above case analysis, and assuming $G_H(z) = \sum_{t=1}^{\infty} T_H(4,3t+2)^{-z} z^t$, we get the following equation:

$$G_H(z) = 4(1+2z)G^2(z) + 4z(1+6z)G_2^2(z) + 32zG_1(z)G_2(z) + 28zG(z)G_2(z) + 16zG(z)G_1(z)$$
(15)

The reader should note that the missing domino can be either horizontal or vertical, so we have $T(4, 3t + 2)^{--} = T_H(4, 3t + 2)^{--} + T_V(4, 3t + 2)^{--}$. Assuming $F(z) = \sum_{t=1}^{\infty} T(4, 3t + 2)^{--} z^t$, $F(z) = G_V(z) + G_H(z)$. So from equations (11) and (12), we have,

$$F(z) = 6G^{2}(z) + 10zG_{2}^{2}(z) + 32zG_{1}(z)G_{2}(z) + 24z^{2}G_{2}^{2}(z) + 40zG(z)G_{2}(z) + 16zG(z)G_{1}(z) + 8zG^{2}(z) + 8G(z)G_{1}(z)$$
(16)

In [8] Moore had derived the generating functions G(z), $G_1(z)$ and $G_2(z)$ as:

$$G(z) = \frac{1 - 6z}{1 - 10z + 22z^2 + 4z^3} \tag{17}$$

$$G_1(z) = \frac{z(1-2z)}{1-10z+22z^2+4z^3}$$
 (18)

$$G_2(z) = \frac{2z}{1 - 10z + 22z^2 + 4z^3} \tag{19}$$

Putting these values in (16), we get,

$$F(z) = \frac{6 - 56z + 152z^2 - 120z^3 + 160z^4}{(1 - 10z + 22z^2 + 4z^3)^2}$$
 (20)

The asymptotic growth of N(t) (recall that N(t) is the number of tilings for the *straight* interface, and the value of $G(z) = \sum_t N(t)z^t$ is as in (17)) is the reciprocal of the radius of convergence of G's Taylor series. Thus $N(t) \propto \lambda^t$ where λ is the largest positive root of

$$\lambda^3 - 10\lambda^2 + 22\lambda + 4 = 0$$

Numerically, we have

$$\lambda = 6.54560770847481152029.....$$

Now F(z) is basically a convolution of G(z), $G_1(z)$ and $G_2(z)$. It is easy to see that both $N_1(t)$, $N_2(t) \propto \lambda^t$ also. So, we have,

$$T(4,3t+2)^{--} \propto \sum_{r=0}^{t} \lambda^r \times \lambda^{t-r}$$

 $\propto (t+1)\lambda^t$

The reader should once again note that, similar to our observation in the previous section, in this case also, by introducing a small deficiency of 2 in the $4 \times n$ rectangle, the complexity of the number of tromino tilings changes from $\mathcal{O}(\lambda^t)$ to $\mathcal{O}(t \cdot \lambda^t)$. This leads us to proposing the following conjecture:

Conjecture 1 The complexity of the number of tromino tilings of an $m \times n$ rectangle, changes from $N_T(m, n)$ to $\mathcal{O}((m+n) \cdot N_T(m, n))$ in tiling an extended domino-deficient rectangle with minimal dimensions $m \times n'$, where n' > n.

An interesting question would be to analyze the complexity of the number of tromino tilings $N_T(m,n)$ of an $m \times n$ rectangle on introducing a deficiency of k dominoes. How does $N_T(m,n)$ vary when the number of trominoes remains the same, but k increases? What happens if the size of the rectangle remains the same? We state the following problem for the interested reader:

Open question 1 How does the number of tromino tilings $N_T(m, n)$ of an $m \times n$ rectangle, where $m, n, k \ge 1$ and $3 \mid (mn-2k)$, vary with k, the number of dominoes which are removed, when (a) The number of trominoes remains the same, and (b) The size of the rectangle remains the same?

5 Tromino Tilings of $R(5, 3t + 4)^{--}$ Rectangles

Consider now the case when one dimension of the given rectangle is 5. It turns out that the bad pairs for this case are different from rest of the cases considered so far. First consider the simplest rectangle of this case, a 4×5 rectangle. Figure 8(1) shows all the bad pairs for $R(4,5)^{--}$. The bad pairs $\{(2,2),(2,3)\},\{(2,3),(2,4)\},$ $\{(3,2),(3,3)\}$ and $\{(3,3),(3,4)\}$ are symmetric, and the reader can see that the tromino covering (1,3) (in the first two cases) and (4,3) (in the last two cases) make the squares (1,5) (resp., (1,1), (4,5), and (4,1)) inaccessible. See Figure 8(1)(a)-(d). So we conclude that these pairs are bad. Now consider the bad pairs $\{(1,3),(2,3)\}$ and $\{(3,3),(4,3)\}$. Since these pairs are symmetric, we will only prove the badness of the first pair. There are two possible orientations for the tromino covering (1,1) and (1,2). If it covers (2,2), then a 3×2 rectangle is completed by the tromino covering (2,1). It can be easily seen that the square (4,1) becomes inaccessible in this case. So in order to permit a tiling, the tromino covering (1,1) and (1,2)must not complete a 3×2 rectangle. So the only possible case is that shown in Figure 8(1)(f). The reader can see that (4,5) becomes inaccessible in this case. So we conclude that these two pairs are also bad. Finally, consider the pair $\{(2,3),(3,3)\}$. In this case at least two corner squares are made inaccessible by the trominoes covering (1,3) and (4,3). So this pair is also bad. We now present a very interesting observation regarding bad pairs for a 5×7 rectangle. The reader should note that the bad pairs for $R(m,n)^{--}$ are same as those for $R(n,m)^{--}$, so we will be alternately considering either of the above two cases.

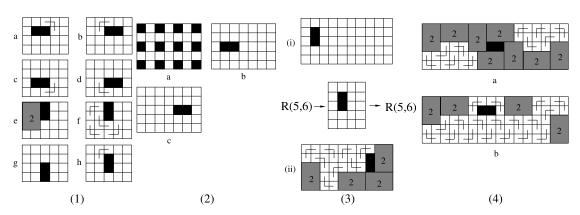


Figure 8: (1) Bad pairs for $R(4,5)^{--}$. (2) Bad pairs for $R(5,7)^{--}$. (3) A (5,6)-hquad shift. (4) Alternate approach when a (5,6)-hquad shift fails.

Lemma 2 [Deficient 5×7 Lemma] If both the x or y coordinates of the position of the domino removed from R(5,7) are even, then the resulting shape is not tileable.

Proof: We form a kind of checkerboard by marking each of the 12 squares as shown in Figure 8(2)(a). If both the x or y coordinates of the removed domino are even, then any tiling of $R(5,7)^{--}$ must contain one tromino for each of the 12 marked squares, so that the tiling must have area at least $12 \cdot 3 = 36$, which is absurd since the area of $R(5,7)^{--}$ is 33. Thus, all pairs which satisfy the above criterion are bad.

Apart from the pairs proved bad by the *Deficient* 5×7 *Lemma*, we have two more bad pairs for $R(5,7)^{--}$, namely $\{(3,2),(3,3)\}$ and $\{(3,5),(3,6)\}$ (as shown in Figure 8(2)(b)-(c)). Due to symmetry, we need consider the badness of the first pair only. Consider the tromino covering (3,1) in the former case. The reader can easily see that either (1,1) or (5,1) becomes inaccessible. We now move on to larger rectangles. For $R(5,10)^{--}$, the only bad pairs are $\{(2,1),(2,2)\}$, $\{(2,9),(2,10)\}$, $\{(4,1),(4,2)\}$,

 $\{(4,9),(4,10)\},\ \{(1,2),(2,2)\},\ \{(1,9),(2,9)\},\ \{(4,2),(5,2)\},\ \{(4,9),(5,9)\},\ \{(2,3),(2,4)\},\ \{(2,7),(2,8)\},\ \{(4,3),(4,4)\},\ \{(4,7),(4,8)\},\ \{(2,2),(3,2)\},\ \{(3,2),(4,2)\},\ \{(2,9),(3,9)\},\ \{(3,9),(4,9)\},\ \{(3,2),(3,3)\}\$ and $\{(3,8),(3,9)\}.$ We leave it as an exercise for the reader to prove the badness of the pairs mentioned above. It turns out that for $R(5,13)^{--}$, the only bad pairs are the corresponding analogues for those of $R(5,10)^{--}$ (as mentioned above). We now move on to prove something stronger, that the bad pairs for any $R(5,3t+10)^{--}$, where $t \geq 0$, are the corresponding analogues for the pairs mentioned above.

Theorem 7 [Domino-Deficient Pentrec Theorem] The only bad pairs for $R(5, 3t+10)^{--}$, where $t \ge 0$, are $\{(2,1),(2,2)\}$, $\{(2,3t+9),(2,3t+10)\}$, $\{(4,1),(4,2)\}$, $\{(4,3t+9),(4,3t+10)\}$, $\{(1,2),(2,2)\}$, $\{(1,3t+9),(2,3t+9)\}$, $\{(4,2),(5,2)\}$, $\{(4,3t+9),(5,3t+9)\}$, $\{(2,3),(2,4)\}$, $\{(2,3t+7),(2,3t+8)\}$, $\{(4,3),(4,4)\}$, $\{(4,3t+7),(4,3t+8)\}$, $\{(2,2),(3,2)\}$, $\{(3,2),(4,2)\}$, $\{(2,3t+9),(3,3t+9)\}$, $\{(3,3t+9),(4,3t+9)\}$, $\{(3,2),(3,3)\}$ and $\{(3,3t+8),(3,3t+9)\}$.

Proof: The proof for the badness of the above cases of domino-removal is similar to that for $R(5,10)^{--}$ and $R(5,13)^{--}$, and so is left for the reader as an exercise. We will only prove the existence of a tiling in all other cases. Since R(5,3) does not admit a tromino tiling, in order to satisfy the criterions of the Chu-Johnsonbaugh Theorem, we divide the case of tiling $R(5,3t+10)^{--}$ into two subcases, accordingly as t is even or odd. First consider the case when t is even, i.e., t=2l. Any tiling of $R(5,6l+10)^{--}$ can be viewed as a tiling of l subrectangles of dimension 5×6 and one domino-deficient $R(5,10)^{--}$ rectangle. This can be written as:

$$R(5,6l+10)^{--} = l \cdot R(5,6) + R(5,10)^{--}$$
(21)

A 5×6 rectangle satisfies the conditions of the Chu-Johnsonbaugh Theorem and so is tileable. If the missing domino in $R(5,10)^{--}$ does not form a bad pair, then we can tile it, thereby achieving a tiling of $R(5,6l+10)^{--}$. So consider the situation when this missing domino forms a bad pair. In this case, we perform a (5,6)-hquad shift (see Figure 8(3)). If by using this technique, the given $R(5,10)^{--}$ rectangle becomes tileable, then we are done. However, if such a shift fails to remove the "badness" of $R(5,10)^{--}$, then the bad pairs must be one of $\{(2,1),(2,2)\}$ (resp., $\{(2,9),(2,10)\}$, $\{(4,1),(4,2)\}$, $\{(4,9),(4,10)\}$), $\{(2,3),(2,4)\}$ (resp., $\{(2,7),(2,8)\}$, $\{(4,3),(4,4)\}$, $\{(4,7),(4,8)\}$), or $\{(3,2),(3,3)\}$ (resp., $\{(3,8),(3,9)\}$). The bad pair $\{(2,1),(2,2)\}$ changes to $\{(2,7),(2,8)\}$ on a (5,6)-hquad shift, while the pair $\{(3,2),(3,3)\}$ changes to $\{(3,8),(3,9)\}$ (the other bad pairs change analogously). In this case, we join a removed 5×6 rectangle from the left (or right) as shown in Figure 8(4) (symmetric cases are also possible). Note that such a join is always possible since the missing domino does not occupy the positions enumerated in the theorem statement. So, we get a tileable $R(5,16)^{--}$ and (l-1) subrectangles of dimension 5×6 , thereby achieving a tiling of $R(5,6l+10)^{--}$.

Now consider the case when t is odd. Assuming t = 2l + 1, where $l \ge 0$, we have to tile $R(5, 6l + 13)^{--}$. Similar to the above case, we can view a tiling of $R(5, 6l + 13)^{--}$ as a tiling of l subrectangles of dimension 5×6 and one domino-deficient $R(5, 13)^{--}$ rectangle. Following our additive notation,

$$R(5,6l+13)^{--} = l \cdot R(5,6) + R(5,13)^{--}$$
(22)

If this $R(5,13)^{--}$ rectangle is tileable, then we are done. Otherwise, the reader should note that none of the bad pairs (enumerated in the theorem statement) changes to another bad pair on applying a (5,6)-hquad shift. So, we can always change an untileable configuration of $R(5,13)^{--}$ (as above) into a tileable one. \Box

6 Tilings of Arbitrary Domino-Deficient Rectangles

We now move on to proving our major result. We characterize bad pairs for arbitrary $m \times n$ domino-deficient rectangles, where $m, n \geq 7$ and 3 | (mn - 2). We first note that the only bad pairs for $R(7,8)^{-}$ are $\{(2,1),(2,2)\}, \{(6,1),(6,2)\}, \{(2,7),(2,8)\}, \{(6,7),(6,8)\}, \{(1,2),(2,2)\}, \{(6,2),(7,2)\}, \{(1,7),(2,7)\}, \{(6,7),(7,7)\}, \{(2,3),(2,4)\}, \{(2,5),(2,6)\}, \{(6,3),(6,4)\}, \{(6,5),(6,6)\}, \{(3,2),(4,2)\}, \{(4,2),(5,2)\}, \{(3,7),(4,7)\},$ and $\{(4,7),(5,7)\}$. It turns out that the corresponding analogues for $R(7,11)^{-}$ and $R(10,8)^{-}$ are the only bad pairs for these two rectangles also. Based on this observation, we have the following lemma:

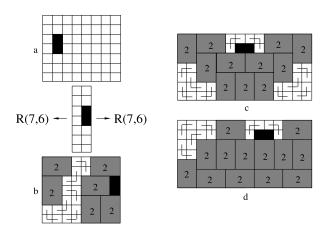


Figure 9: (a) and (b) A (10,3)-hquad shift. (c) and (d) Tilings of $R(7,14)^{--}$ obtained by joining R(7,6) to $R(7,8)^{--}$ having the missing domino at $\{(2,1),(2,2)\}$ or $\{(2,3),(2,4)\}$.

Lemma 3 The only bad pairs for $R(7,3t+8)^{--}$, where $t \ge 0$, are $\{(2,1),(2,2)\}$, $\{(6,1),(6,2)\}$, $\{(2,3t+7),(2,3t+8)\}$, $\{(6,3t+7),(6,3t+8)\}$, $\{(1,2),(2,2)\}$, $\{(6,2),(7,2)\}$, $\{(1,3t+7),(2,3t+7)\}$, $\{(2,3),(2,4)\}$, $\{(2,3t+5),(2,3t+6)\}$, $\{(6,3),(6,4)\}$, $\{(6,3t+5),(6,3t+6)\}$, $\{(3,2),(4,2)\}$, $\{(4,2),(5,2)\}$, $\{(3,3t+7),(4,3t+7)\}$, and $\{(4,3t+7),(5,3t+7)\}$. Furthermore, the corresponding analogues are the only bad pairs for $R(10,3t+8)^{--}$.

Proof: The proof for the badness of the above cases of domino-removal is similar to that given in previous sections and so is left for the reader as an exercise. Since R(7,3) does not admit a tromino tiling, in order to satisfy the criterions of the Chu-Johnsonbaugh Theorem, we divide the case of tiling $R(7,3t+8)^{--}$ rectangles into two subcases, accordingly as t is even or odd. We first consider tiling $R(7,6l+11)^{--}$ and $R(10,3t+8)^{--}$ rectangles, where $t,l \geq 0$. Following our additive decomposition notation, we have,

$$R(7,6l+11)^{--} = l \cdot R(7,6) + R(7,11)^{--}$$
(23)

$$R(10,3t+8)^{--} = t \cdot R(10,3) + R(10,8)^{--}$$
(24)

It turns out that none of the bad pairs (enumerated in the lemma statement above) converts to another bad pair on applying a (7,6)-hquad shift on $R(7,11)^{--}$ and a (10,3)-hquad shift on $R(10,8)^{--}$. Also, a 7×6 rectangle satisfies the criterions of the Chu-Johnsonbaugh Theorem, and so is tileable. Using these facts and equations (23) and (24), we conclude that $R(7,6l+11)^{--}$ and $R(10,3t+8)^{--}$ always permit a tromino tiling when the missing domino does not occupy the pairs enumerated above. So we need only consider tiling $R(7,6l+8)^{--}$ rectangles, where $l \ge 0$. We first note that,

$$R(7,6l+8)^{--} = l \cdot R(7,6) + R(7,8)^{--}$$
(25)

If the missing domino in $R(7,8)^{--}$ does not form a bad pair, then we can tile it, thereby achieving a tiling of $R(7,6l+8)^{--}$. If the missing domino forms a bad pair, and is removed after performing a (7,6)-hquad shift (see Figure 9(a)-(b)), then we do the same. However, if such a shift fails to make $R(7,8)^{--}$ tileable, then the reader can verify that the bad pair must be either $\{(2,1),(2,2)\}$ (resp., $\{(6,1),(6,2)\}$, $\{(2,3t+7),(2,3t+8)\}$, $\{(6,3t+7),(6,3t+8)\}$) or $\{(2,3),(2,4)\}$ (resp., $\{(2,3t+5),(2,3t+6)\}$, $\{(6,3),(6,4)\}$, $\{(6,3t+5),(6,3t+6)\}$). The bad pair $\{(2,1),(2,2)\}$ changes to $\{(2,3),(2,4)\}$ upon such a shift, and the other pairs change correspondingly. In this case, we join a removed 7×6 rectangle from the left (right) to form $R(7,14)^{--}$, and apply the tiling shown in Figure 9(c)-(d) (symmetric cases are also possible). We note that such a join is always possible since the bad pair is not one among the pairs enumerated in the lemma statement. So we get a tileable $R(7,14)^{--}$ and (l-1) subrectangles of dimension 7×6 , which tile $R(7,6l+8)^{--}$ completely.

We are now ready to prove our major result, thereby settling the open problem posed by Ash and Golomb in [6], for tiling an $m \times n$ rectangle when a domino has been removed from it, where $m, n \geq 7$ and $3 \mid (mn-2)$. We have the following theorem:

Theorem 8 [Domino-Deficient Rectangle Theorem] $An\ m \times n$ rectangle, where $m, n \geq 7$ and $3 \mid (mn-2)$, from which a domino has been removed, can always be tiled with trominoes provided the domino does not occupy the positions $\{(2,1),(2,2)\}$, $\{(2,n-1),(2,n)\}$, $\{(m-1,1),(m-1,2)\}$, $\{(m-1,n-1),(m-1,n)\}$, $\{(1,2),(2,2)\}$, $\{(1,n-1),(2,n-1)\}$, $\{(m-1,2),(m,2)\}$, $\{(m-1,n-1),(m,n-1)\}$, $\{(2,3),(2,4)\}$, $\{(2,n-3),(2,n-2)\}$, $\{(m-1,3),(m-1,4)\}$, $\{(m-1,n-3),(m-1,n-2)\}$, $\{(3,2),(4,2)\}$, $\{(3,n-1),(4,n-1)\}$, $\{(m-3,2),(m-2,2)\}$, and $\{(m-3,n-1),(m-2,n-1)\}$.

Proof: Without loss of generality, we assume $m \equiv 1 \pmod{3}$, and $n \equiv 2 \pmod{3}$. We treat the cases m = 7,10 individually. Note that if $m \geq 13$, then $m - 6 \geq 6$, and so the missing domino is not present in either the top or bottom 6 rows. So we successively slice a full rectangle of height 6 off the top (bottom) of $R(m,n)^{--}$, that is, $R(m,n)^{--} = R(m-6,n)^{--} + R(6,n)$, until the dimension $m \leq 13$ is reached. In the above equation, the last term R(6,n) satisfies the criterions of the Chu-Johnsonbaugh Theorem, and so is tileable. Following the above procedure, we eventually land up with either $R(7,3t+8)^{--}$ or $R(10,3t+8)^{--}$. If the missing domino is not one among the bad pairs for either of these two subrectangles (as enumerated in Lemma 3), then this subrectangle can be tiled by trominoes (see Lemma 3). Consider the case when the missing domino is one among the bad pairs enumerated in Lemma 3. In this case, we join a removed $6 \times n$ rectangle from above or below to this subrectangle, to obtain $R(13,3t+8)^{--}$ or $R(16,3t+8)^{--}$. We divide this subrectangle by one of the following decompositions:

$$R(13,6l+8)^{--} = l \cdot R(13,6) + R(13,8)^{--}$$
(26)

$$R(13,6l+11)^{--} = l \cdot R(13,6) + R(13,11)^{--}$$
(27)

$$R(16,3t+8)^{--} = t \cdot R(16,3) + R(16,8)^{--}$$
(28)

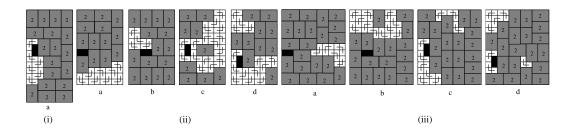


Figure 10: (i) Tilings of $R(16,8)^{--}$. (ii) Tilings of $R(13,8)^{--}$. (iii) Tilings of $R(13,11)^{--}$.

Both 13×6 and 10×3 rectangles satisfy the conditions of the Chu-Johnsonbaugh Theorem, and so are tileable. So we need only consider tilings of $R(13,8)^{--}$, $R(13,11)^{--}$ and $R(16,8)^{--}$. Note that the bad cases of domino-removal $\{(2,n-1),(2,n)\}$, $\{(m-1,1),(m-1,2)\}$, $\{(m-1,n-1),(m-1,n)\}$, are actually the symmetric counterparts of the case $\{(2,1),(2,2)\}$, the cases $\{(1,n-1),(2,n-1)\}$, $\{(m-1,2),(m,2)\}$, $\{(m-1,n-1),(m,n-1)\}$, are the symmetric counterparts of the case $\{(1,2),(2,2)\}$,.... and so on, when viewed from the other three corners of the given rectangle. So we need only consider the bad pairs for $R(7,8)^{--}$, $R(7,11)^{--}$ and $R(10,8)^{--}$, before joining R(6,8) or R(6,11) from the top (bottom), to be one of the cases $\{(2,1),(2,2)\}$, $\{(1,2),(2,2)\}$, $\{(2,3),(2,4)\}$, and $\{(3,2),(4,2)\}$ (the rest follow from symmetry). For $(13,8)^{--}$ and $R(13,11)^{--}$, the corresponding tilings are shown in Figure 10(ii)-(iii) for all the four cases of domino removal. The situation is slightly different in case of $R(16,8)^{--}$. For the bad cases $\{(2,3),(2,4)\}$ and $\{(2,1),(2,2)\}$ in $R(10,8)^{--}$, the "badness" is removed if we perform a (6,8)-vquad shift. The pairs $\{(1,2),(2,2)\}$ and $\{(3,2),(4,2)\}$ of $R(10,8)^{--}$ actually become symmetrical when we join R(6,8) with $R(10,8)^{--}$ from the top (bottom). So only one tiling is shown in Figure 10(i) which suffices for both these cases.

The above proof is constructive in nature, i.e., it also identifies the tiling rule if there exists one. Based on the above characterization, we suggest a procedure for tiling an $m \times n$ domino-deficient rectangle. This procedure takes as input the dimensions m, n of the given domino-deficient rectangle, and the position of the missing domino, all of which can be represented in $\mathcal{O}(\log m + \log n)$ bits. It outputs a specific tiling for $R(m, n)^{--}$ if there exists one; the five steps above may be viewed as a set of rules for generating such a tiling.

Algorithm 1 The Domino-Deficient Tiling Procedure

```
1: Remove the rectangle R(6 \cdot |i/6|, n) from the top.
2: if 6|(m-7) then
      Remove the rectangle R(m-6 \cdot |i/6|-7, n) from the bottom
4:
      Remove the rectangle R(m-6 \cdot |i/6|-10, n) from the bottom
6: end if
   if The missing domino does not occupy the bad pairs stated in Lemma 3 then
      Identify the corresponding tiling rule for R(7,n)^{--} or R(10,n)^{--}.
      return(Tiling Exists!).
9:
10: else
      Join R(6,n) from the top or bottom, and identify the tiling rule from Figure 10, tiling R(13,n-8)
11:
      (R(13, n-11) \text{ or } R(16, n-8)) \text{ by } R(2,3)'s.
      return(Tiling Exists!).
12:
13: end if
14: return(No tiling exists!)
```

7 Estimating the Number of Domino-Deficient Tilings

We now proceed to estimate the number of tromino tilings of arbitrary $m \times n$ domino-deficient rectangles. A natural question to ask is how does the number of tromino tilings change when the number of trominoes are kept the same, but some deficiencies are introduced in the given rectangle? We address this question for the case when the deficiency introduced is a domino removal. The first comparison between the number of tromino and domino tilings was done by Aanjaneya and Pal in [12]. They showed that if $N_T(m,n)$ and $N_D(m,n)$ represent the number of tromino and domino tilings of an $m \times n$ rectangle, then the following result holds:

Theorem 9 For all rectangles R(m,n), such that 3|mn and m,n>0, the following inequality holds:

$$N_T(m,n) \le 2^{\frac{4mn}{3}} \{ \min[N_D(m,2n), N_D(2m,n)] \}$$
(29)

where the number of domino tilings of R(2m, 2n) is given by the formula,

$$N_D(2m, 2n) = 4^{mn} \prod_{j=1}^m \prod_{k=1}^n \left\{ \cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right\}$$
 (30)

We use techniques similar to those in [12] for producing an upper bound. For the sake of convenience, we review some definitions and notations. A monodic tiling of R(m,n) is a tiling with $\frac{mn}{3}$ dominoes and $\frac{mn}{3}$ monominoes. A domino with an arrow (as shown in Figure 11) is called a directed domino. A monodic tiling with directed dominoes is called a directed monodic tiling. A tromino tiling can be converted to a directed monodic tiling by using the one-one mapping shown in Figure 11(a)-(d) (by converting every tromino into a combination of a directed domino and a monomino). A directed monodic tiling of R(m,n) is valid if it was obtained from a tromino tiling of R(m,n) via the one-one mapping shown in Figure 11. A valid directed monodic tiling can be converted back to a tromino tiling by attaching to every directed domino the monomino to the right of its arrowhead. It can be easily seen that every tromino tiling converts to a unique directed monodic tiling by the above procedure. As depicted in Figure 11(1), a tromino tiling of R(m,n), after being converted to the corresponding directed monodic tiling via the mapping function shown in Figure 11(1)(a)-(d), is first made undirected and coloured, and then stretched either horizontally or vertically to give a corresponding domino tiling. By stretching we double either the length n or the breadth m of the rectangle R(m,n). (Colouring is defined below.) In this process, every monomino gets converted to a domino, and a domino gets converted into either two horizontal dominoes lying side-by-side (in case of a horizontal stretching of R(m,n), or it gets converted to two horizontal dominoes one lying on top of the other (in case of a vertical stretching of R(m,n)). These conversions are shown in Figures 11(1) (i)-(j). So a tromino tiling of R(2,3) (as shown in Figure 11(1)(e) gets converted to a domino tiling of either R(2,6) or R(4,3) (as shown in Figures 11(1)(g) and (h)). The reader must note that not every domino tiling of R(m,2n) is obtained by stretching

a monodic tiling of R(m, n). Note, for example, the domino tiling of R(3, 4) shown in Figure 11(2). The reader can verify that this domino tiling cannot be obtained by stretching (either horizontally or vertically) any monodic tiling of R(3, 2).

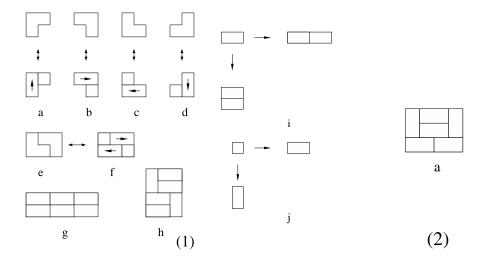


Figure 11: (1) Mapping function from a tromino tiling to a domino tiling. (2) A domino tiling which is not obtained via the given mapping function.

Now let us define our operations in case of an $m \times n$ domino-deficient rectangle. Similar to a monodic tiling, we define a deficient monodic tiling for an $m \times n$ domino-deficient rectangle, as a tiling with $\frac{mn-2}{3}+1$ dominoes and $\frac{mn-2}{3}$ monominoes. While converting a tromino tiling of the given rectangle to a directed monodic tiling, we just don't attach any direction with the missing domino. We call the resulting tiling a deficient directed monodic tiling. The reader can easily see that a deficient directed monodic tiling can be converted back to the original domino-deficient tromino tiling by the same procedure described above (the missing domino is simply left untouched). If we mutiply the total number of deficient monodic tilings of R(m,n) by $2^{\frac{mn-2}{3}}$, we get the total number of deficient directed monodic tilings of R(m,n). First, let us focus only on the case when the stretching done is horizontal. With slight modification, the original streching procedure can also be converted to a one-to-one mapping function which converts every deficient monodic tiling of R(m,n) to a domino tiling of R(m,2n). We call a domino tiling of R(m,2n) valid if it was obtained by stretching a deficient monodic tiling of R(m,n). Any valid domino tiling can be converted back to a deficient monodic tiling of R(m,n) simply by unstretching (i.e., compressing lengthwise) the valid domino tiling of R(m,2n).

Suppose we have a deficient monodic tiling of R(m,n). We colour this tiling in the following manner, every monomino is coloured blue and every domino is coloured red. We now stretch R(m,n) length-wise to produce R(m,2n). The outcome will be a coloured domino tiling of R(m,2n). Note that the dominoe(s) produced by the stretching of a coloured monomino (domino) will be of the same colour as that of the monomino (domino). And similarly, we define the unstretching of a coloured domino tiling. We have the following lemma.

Lemma 4 Distinct coloured deficient monodic tilings of R(m,n) give rise to distinct coloured domino tilings of R(m,2n).

Proof: We prove the above claim by the method of contradiction. We assume on the contrary that the above claim is false. First suppose that two distinct coloured deficient monodic tilings of R(m,n) give rise to the same coloured domino tiling of R(m,2n) via stretching. Since the two monodic tilings of R(m,n) being considered are distinct, we know that the position of at least one monomino is different in these two tilings. This means that the position of a monomino in the first tiling of R(m,n) is covered by a domino in the second tiling of R(m,n). Let this monomino be (i,j). It can be easily seen that after stretching of R(m,n), the squares (i,2j) and (i,2j+1) correspond to the square (i,j) in R(m,n). These two squares in R(m,2n) will be covered by a blue domino via the first monodic tiling and a red domino via the second. But this is impossible because both the coloured deficient monodic tilings of R(m,n) being considered give rise to the same coloured domino tiling of R(m,2n). Thus, we arrive at a contradiction.

Now suppose that two distinct coloured domino tilings of R(m, 2n) give rise to the same coloured deficient monodic tiling of R(m, n) after unstretching. Again, since the two coloured domino tilings are distinct, the position of at least one square (i, j) (say) in one domino tiling is covered by a red (blue) domino and which is covered by a blue (red) domino in the other. The reader can see that after unstretching the rectangle R(m, 2n), each square (i, j) in R(m, 2n) maps to (i, [j/2]) in R(m, n) (here [x] represents the integer part of x). When we unstretch the two coloured tilings, this means that the square (i, [j/2]) in R(m, n), corresponding to the square (i, j) in R(m, 2n), will be red (blue) via the first tiling and blue (red) via the second tiling. Thus, we again arrive at a contradiction. So, we conclude that distinct coloured deficient monodic tilings of R(m, n) map to distinct coloured domino tilings of R(m, 2n).

We have already shown that not all domino tilings of R(m,2n) are valid. Thus, via the two mappings just described above, we have defined an injective function from the set of domino-deficient tromino tilings of $R(m,n)^{--}$ to a proper subset of coloured domino tilings of R(m,2n). Since the direction of stretching is unimportant in the above arguments, the reader should convince himself that the above claims also hold for vertical stretching of rectangles. The number of coloured domino tilings of R(m,2n) is simply 2^{mn} times the number of domino tilings of R(m,2n) (since every domino can be coloured either red or blue). Let $N_T(m,n)^{--}$ and $N_D(m,n)$, denote the number of tromino tilings of $R(m,n)^{--}$ and the number of domino tilings of R(m,n). From a given tromino tiling of an $m \times n$ domino-deficient rectangle, we can get at most $2^{\frac{mn-2}{3}}$. k deficient directed monodic tilings, where k is the number of deficient monodic tilings of the given rectangle. From each tiling thus obtained, we can get a unique coloured domino tiling of R(m,2n), of which there are at most $2^{mn}N_D(m,2n)$ of them. We summarize these arguments in the following theorem:

Theorem 10 For all rectangles R(m,n), such that 3|(mn-2) and m,n>0, the following inequality holds:

$$N_T(m,n)^{--} \le 2^{\frac{4mn-2}{3}} \{ \min[N_D(m,2n), N_D(2m,n)] \}$$
(31)

where the number of domino tilings of R(2m, 2n) is given by the formula,

$$N_D(2m, 2n) = 4^{mn} \prod_{j=1}^m \prod_{k=1}^n \left\{ \cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right\}$$
 (32)

The upper bound derived above shows that the number of tilings of an $m \times n$ domino-deficient rectangle is at most exponential in both m and n. So, the number of binary bits required in encoding any tiling of the rectangle is upper bounded by a polynomial in m and n.

8 A step towards general 2-deficiency in rectangles

Now consider general 2-deficiency in rectangles. In this section, we denote 2-deficient $m \times n$ rectangles by $R(m,n)^{--}$. General 2-deficiency as such is a lot more complicated than domino-deficiency, and till date no characterizations exist for any classes of rectangles. In this paper, we restrict ourselves to studying general 2-deficiency in $n \times 4$ rectangles. For a few examples of the bad pairs for the case when one dimension of the given $m \times n$ rectangle, say m = 7 see [9]. Apart from the bad pairs shown in Figure 4 of Section 4, Figure 12 shows all bad pairs for $R(8,4)^{--}$. We encourage the reader to try placing trominoes on $R(8,4)^{--}$ with the 2-deficiencies shown in Figure 12, and verify for himself that the pairs shown are actually bad. It turns out that for $R(3t+8,4)^{--}$, where $t \geq 0$, the corresponding analogues of the pairs shown in Figure 12 are the only bad pairs. We prove this fact in the following theorem:

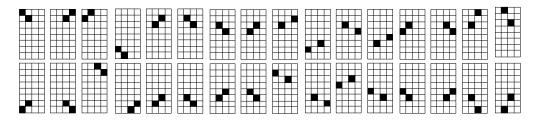


Figure 12: Bad pairs for the general $R(8,4)^{--}$.

Theorem 11 The only bad pairs (apart from those enumerated in section 4) for $R(3t+8,4)^{--}$, where $t \geq 0$, are those corresponding to the pairs in Figure 12.

Proof: We leave the proof of the badness of the pairs shown in Figure 12 for the reader. We prove the existence of a tiling in all other cases. First consider the case when there exist two distinct 4×4 squares such that each square contains exactly one missing square, and the remaining area of R(3t + 8, 4) can be divided into R(3,4) rectangles. Following our additive decomposition notation, we can decompose $R(3t+8,4)^{--}$ as shown in equation (33). A 3×4 rectangles satisfies the conditions of the Chu-Johnsonbaugh Theorem, and so is tileable, while $R(4,4)^-$ is tileable since it satisfies the conditions of the Deficient Rectangle Theorem. It follows from above that $R(3t+8,4)^{--}$ can be tiled in this case.

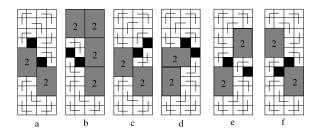


Figure 13: Tiling the general $R(3t + 8, 4)^{--}$

$$R(3t+8,4)^{--} = t \cdot R(3,4) + 2R(4,4)^{-}$$

$$R(3t+8,4)^{--} = t \cdot R(3,4) + R(8,4)^{--}$$
(33)

$$R(3t+8,4)^{--} = t \cdot R(3,4) + R(8,4)^{--}$$
(34)

We now consider the case when the position of the two missing squares does not permit the decomposition in the former case. It is easy to see that in this case, we can decompose $R(3t+8,4)^{--}$ as shown in (34). If the missing squares in $R(8,4)^{--}$ do not form a bad pair, then we are done. Otherwise, the missing squares form one of the bad pairs shown in Figure 12. In this case, we join a removed 3×4 rectangle from the top (bottom), and apply the corresponding tiling rule among the various cases shown in Figure 13. So, in this case, we tile $R(11,4)^{--}$ as shown, and separately tile (t-1) subrectangles of dimension 3×4 , to get a tiling of $R(3t+8,4)^{--}$.

9 Final Remarks

We are currently exploring general 2-deficiency in rectangles and wish to characterize bad pairs for arbitrary $m \times n$ rectangles. The proof of the last theorem is particularly important because it suggests an approach for solving general 2-deficiency in large rectangles. When the missing squares are far apart, the given rectangle can be broken down into two subrectangles, such that each subrectangle contains exactly one missing square. Now the Deficient Rectangle Theorem may be used to find a tiling rule, if one exists. However, one would have to consider certain cases as "basis" cases, before using the above argument, as we did in the proof of the Domino-Deficient Rectangle Theorem in Section 6. We observed that the number of bad pairs for these "basis" cases is extremely large. So, analysis using existing techniques becomes extremely complicated. Moreover, showing non-existence of a tiling by this method seems to be rather hard. We believe that enumeration of bad pairs may be analysed by studying possible transformations that map between classes of bad pairs. The hquad and vquad shifts are examples of such transformations, however, they cannot generate all bad pairs starting from a single one. Furthermore, as a sequel to 2-deficiency problems, one may also consider the question of k-deficiency, for k > 3.

Another research direction would be to establish a lower bound on the number of tromino tilings of $m \times n$ domino-deficient rectangles; it is an interesting open problem to determine whether there exists a lower bound comparable to our upper bound derived in inequality (31), Section 7. No such lower bounds have been established till date. Partitioning schemes such as the one used in the Domino-Deficient Tiling Procedure in Section 6, lead to lower bounds on the number of tilings of the entire rectangle obtained by multiplying lower bounds on the number of tilings for each part. Devising partitioning schemes that lead to good lower bounds is an interesting open problem.

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References

- [1] I.P. Chu and R. Johnsonbaugh, Tiling Boards with Trominoes, J. Rec. Math. 18(1985-86), 188-193.
- [2] I.P. Chu and R. Johnsonbaugh, *Tiling Deficient Boards with Trominoes*, Integre Technical Publishing Co., Mathematics Magazine 59 (1986), 34-40, MR87c:05044.
- [3] S. Golomb, *Polyominoes*, Charles Scribner's Sons, N.Y.
- [4] Gardner Martin, More About Tiling the Plane: The Possibilities of Polyominoes, Polyiamonds and Polyhexes, Scientific American 233, No. 2, August 1975: Mathematical Games Column, pp. 112-115.
- [5] Gardner Martin, More about the Shapes that can be made with Complex Dominoes, Scientific American 203, No. 5, November 1960: The Mathematical Games Column, pp. 186-194.
- [6] J. Marshall Ash and S. Golomb, Tiling Deficient Rectangles with Trominoes, Integre Technical Publishing Co., Mathematics Magazine (2003), 46-55.
- [7] Norman Do, Mathellaneous: The Art of Tiling with Rectangles.
- [8] Cristopher Moore, Some Polyomino Tilings of the Plane; eprint math.CO/9905012, v1, May 3, 1999, available on http://www.arxiv.org/
- [9] Mridul Aanjaneya, Tromino Tilings of Domino Deficient Rectangles, Technical Report No. IIT/CSE/TR/2006/MA/1, Department of Computer Science and Engineering, Indian Institute of Technology, Kharagpur - 721302, India, June 05, 2006.
- [10] F. Ardila and R.P. Stanley, Tilings.
- [11] S. W. Golomb, *Checkerboards and Polyominoes*, American Mathematical Monthly, Vol. 61, No. 10, Dec. 1954, pp. 675-682.
- [12] Mridul Aanjaneya and Sudebkumar Prasant Pal, Faultfree Tromino Tilings of Rectangles; eprint math.CO/0610925, v1, Oct 30, 2006, available on http://www.arxiv.org/
- [13] M. Fisher and H. Temperley, Dimer problem in statistical mechanics an exact result, Philos. Mag. 6 (1961), 1061-1063.
- [14] P. Kasteleyn, The statistics of dimers on a lattice I. The number of dimer arrangements on a quadratic lattice, Phys. 27 (1961), 1209-1225.