

# On Delayed Sequential Coding of Correlated Sources<sup>1</sup>

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## Abstract

Motivated by video coding applications, the problem of sequential coding of correlated sources with encoding and/or decoding frame-delays is studied. The fundamental tradeoffs between individual frame rates, individual frame distortions, and encoding/decoding frame-delays are derived in terms of a single-letter information-theoretic characterization of the rate-distortion region for general inter-frame source correlations and certain types of potentially frame specific and coupled single-letter fidelity criteria. The sum-rate-distortion region is characterized in terms of generalized directed information measures highlighting their role in delayed sequential source coding problems. For video sources which are spatially stationary memoryless and temporally Gauss–Markov, MSE frame distortions, and a sum-rate constraint, our results expose the optimality of idealized differential predictive coding among all causal sequential coders, when the encoder uses a positive rate to describe each frame. Somewhat surprisingly, causal sequential encoding with one-frame-delayed noncausal sequential decoding can *exactly match* the sum-rate-MSE performance of *joint coding* for all nontrivial MSE-tuples satisfying certain positive semi-definiteness conditions. Thus, even a single frame-delay holds potential for yielding significant performance improvements. Generalizations to higher order Markov sources are also presented and discussed. A rate-distortion performance equivalence between, causal sequential encoding with delayed noncausal sequential decoding, and, delayed noncausal sequential encoding with causal sequential decoding, is also established.

## Index Terms

Differential predictive coded modulation, directed information, Gauss–Markov sources, mean squared error, rate-distortion theory, sequential coding, source coding, successive refinement coding, sum-rate, vector quantization, video coding.

## I. INTRODUCTION

Differential predictive coded modulation (DPCM) is a popular and well-established sequential predictive source compression method with a long history of development (see [1]–[8] and the references therein). DPCM has had wide impact on the evolution of compression standards for speech, image, audio, and video coding. The classical DPCM system consists of a causal sequential predictive encoder and a causal sequential decoder. This is aligned with applications having low delay tolerance at both encoder and decoder. However, there are many interesting scenarios where these constraints can be relaxed. There are three additional sequential source coding systems possible when

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limited delays are allowed at the encoder and/or the decoder: (i) causal (C) encoder and noncausal (NC) decoder; (ii) NC-encoder and C-decoder; and (iii) NC-encoder and NC-decoder. Application examples of these include, respectively, non-real-time display of live video for C–NC, zero-delay display of non-real-time encoded video for NC–C, and non-real-time display of non-real-time video for NC–NC (see Figs. 1, 2, 3 and 7). Of special interest, for performance comparison, is joint coding (JC) which may be interpreted as an extreme special case of the C–NC, NC–C, and the NC–NC systems where all frames are jointly processed and jointly reconstructed (Fig. 3(c)).

The goal of this work is to provide a computable (single-letter) characterization of the fundamental information-theoretic rate-distortion performance limits for the different scenarios and to quantify and compare the potential value of systems with limited encoding and decoding delays in different rate-distortion regimes. The primary motivational application of our study is video coding (see Section II-B) with encoding and decoding *frame* delays.<sup>2</sup>

To characterize the fundamental tradeoffs between individual frame-rates, individual expected frame-distortions, encoding and decoding frame-delays, and source inter-frame correlation, we build upon the information-theoretic framework of sequential coding of correlated sources. This mathematical framework was first introduced in [9] (and independently studied in [10], [11] under a stochastic control framework involving dynamic programming) within the context of the *purely C–C*<sup>3</sup> (i.e., without frame-delays). sequential source coding system. As noted in [9], the results for the well-known successive-refinement source coding problem (see [12]–[14]) can be derived from those for the C–C sequential source coding problem by setting all sources to be identically equal to the same source. The complete (single-letter) rate-distortion region for two sources (with a remark regarding generalization to multiple sources) and certain types of perceptually-motivated coupled single-letter distortion criteria were derived in [9]. Our results cover not only the two-frame C–C problem studied in [9] but also the C–NC, the NC–C, the NC–NC, and the JC cases for arbitrary number of sources and for general coupled single-letter distortion criteria. We have also been able to simplify some of the key derivations in [9] (the C–C case).

The benefits of decoding delay on the rate versus MSE performance was investigated in [5], where the video was modeled as a Gaussian process which is spatially independent and temporally first-order-autoregressive. An idealized DPCM structure was imposed on both the encoder and the decoder. In contrast to conventional rate-distortion studies of *scalar* DPCM systems based on *scalar quantization* and *high-rate* asymptotics (see [1]–[3] and references therein), [5] studied DPCM systems with vector-valued sources and large spatial (as opposed to high rate) asymptotics similar in spirit to [9]–[11] but with decoding frame-delays. The main findings of [5] were that (i) NC-decoders offer a significant *relative* improvement in the MSE at medium to low rates for video sources with strong temporal correlation, (ii) most of this improvement can be attained with a modest decoding frame-delay, and (iii) the gains vanish at very high and very low rates.

In contrast to the insistence on DPCM encoders and decoders in [5], here we consider arbitrary rate-constrained coding structures. When specialized to spatially stationary memoryless, temporally Gauss–Markov video sources, with MSE as the fidelity metric and a sum-rate constraint, our results reveal the information-theoretic optimality of idealized DPCM encoders and decoders for the C–C sequential coding system (Corollary 1.3). A second, somewhat surprising, finding is that for  $k$ -th order Gauss–Markov video sources with a sum-rate constraint, a C-encoder with a  $k$ -frame-delayed NC-decoder can *exactly match* the sum-rate-MSE performance of the *joint coding system* which

<sup>2</sup>Accordingly, terms like frame-delay and “causal” and “noncausal” encoding and/or decoding should be interpreted within this application context.

<sup>3</sup>The terminology is ours.

can wait to collect *all* frames of the video segment before jointly processing and jointly reconstructing them<sup>4</sup> (Corollary 5.2). Interestingly, this performance equivalence does not hold for all MSE-tuples. It holds for a *non-trivial* subset which satisfies certain positive semi-definiteness conditions. The performance-matching region expands with increasing frame-delays allowed at the decoder until it completely coincides with the set of all reachable tuples of the JC system. A similar phenomenon holds for Bernoulli-Markov sources with a Hamming distortion metric. Thus, the benefit of even a single frame-delay can be significant. These two specific architectural results constitute the main contributions of this work.

For clarity of exposition, the proofs of achievability and converse coding theorems in this paper are limited to discrete, memoryless, (*spatially*) stationary (DMS) correlated sources taking values in finite alphabets and bounded (but coupled) single-letter fidelity criteria. Analogous results can be established for continuous alphabets (e.g., Gaussian sources) and unbounded distortion criteria (e.g., MSE) using the techniques in [15] but are not discussed here.

The rest of this paper is organized as follows. Delayed sequential coding systems and their associated operational rate-distortion regions are formulated in Section II. To preserve the underlying intuition and flow of ideas, we first focus on 3-stage coding systems and then present natural extensions to general  $T$ -stage coding systems. Coding theorems and associated implications for the C–C, JC, C–NC, and NC–C systems are presented in Sections III, IV, V and VI respectively. Results for  $T$ -stage C–NC and NC–NC systems are presented in Sections VII and VIII. A detailed proof of achievability and converse coding theorems is presented only for the C–NC system with  $T = 3$  frames. The achievability and converse results for other delayed coding systems are similar but lengthy, repetitive, and cumbersome, and are therefore omitted. We conclude in Section IX.

**Notation:** The nonnegative cone of real numbers is denoted by  $\mathbb{R}^+$  and ‘iid’ denotes independent and identically distributed. Vectors are denoted in boldface (e.g.,  $\mathbf{x}$ ,  $\mathbf{X}$ ). The dimension of the vector will be clear from the context. With the exception of  $T$  denoting the size of a group of pictures (GOP) in a video segment and  $R$  denoting a rate, random quantities are denoted in upper case (e.g.,  $X$ ,  $\mathbf{X}$ ), and their specific instantiations in lower case (e.g.,  $X = x$ ,  $\mathbf{X} = \mathbf{x}$ ). When  $A$  denotes a random variable,  $A^n$  denotes the ordered tuple  $(A_1, \dots, A_n)$ ,  $A_m^n$  denotes  $(A_m, \dots, A_n)$ , and  $A(i-)$  denotes  $(A(1), \dots, A(i-1))$ . However, for a set  $\mathcal{A}$ ,  $\mathcal{A}^n$  denotes the  $n$ -fold Cartesian product  $\mathcal{A} \times \dots \times \mathcal{A}$ . For a function  $g(a)$ ,  $g^n(a(1), \dots, a(n))$  denotes the samplewise function  $(g(a(1)), \dots, g(a(n)))$ .

## II. PROBLEM FORMULATION

### A. Statistical model for $T$ correlated sources

$T$  correlated DMSs taking values in finite alphabets are defined by

$$(X_1(i), \dots, X_T(i))_{i=1}^n \in (\mathcal{X}_1 \times \dots \times \mathcal{X}_T)^n, \\ |\mathcal{X}_j| < \infty, \quad \forall j = 1, \dots, T.$$

The joint probability distribution of sources is given by

$$\text{for } i = 1, \dots, n, \quad (X_1(i), \dots, X_T(i)) \sim \text{iid } p_{X_1 \dots X_T}(x_1, \dots, x_T).$$

Potentially, the (spatially) iid assumption can be relaxed to spatially stationary ergodic by a general AEP argument, but is not treated in this paper. Of interest are the large- $n$  asymptotics of achievable rate and distortion tuples.

<sup>4</sup>This is similar to the coding of *correlated* parallel vector Gaussian sources but with an *individual* MSE constraint on each source component.

### B. Video coding application context

In Fig. 1,  $\mathbf{X}_1, \dots, \mathbf{X}_T$  represent  $T$  video frames with  $\mathbf{X}_j = (X_j(i))_{i=1}^n, j = 1, \dots, T$ . Here,  $i$  denotes discrete index of the spatial location of a picture element (pixel) relative to a certain spatial scan order (e.g., zig-zag or raster scan), and  $X_j(i)$  denotes discrete pixel intensity level at spatial location  $i$  in frame number  $j$ . Instead of being available simultaneously for encoding, initially, only  $(X_1(i))_{i=1}^n$  is available, then  $(X_2(i))_{i=1}^n$  “arrives”, followed by  $(X_3(i))_{i=1}^n$ , and so on. This temporal structure captures the order in which the frames are processed. The statistical structure assumed in Section II.A above implies that the sources are *spatially independent but temporally dependent*.

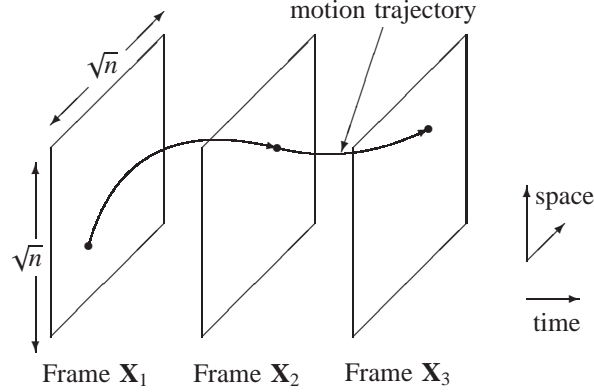


Fig. 1. Illustrating motion-compensated video coding for  $T = 3$  frames.

While this is rarely an accurate statistical model for the *unprocessed* frames of a video segment in a scene (usually corresponding to the GOP in video coding standards), it is a reasonable approximation for the evolution of the video *innovations process* along optical-flow motion trajectories for groups of adjacent pixels (see [5] and references therein). This model assumes arbitrary temporal correlation but iid spatial correlation. The statistical law  $p_{X_1 \dots X_T}$  is assumed to be known here. In practice, this may be learnt from pre-operational training using clips from video databases used by video-codec standardization groups such as H.26x and MPEG-x which is quite similar in spirit to the offline optimization of quantizer tables in commercial video codecs. Single-letter information-theoretic coding results need asymptotics along some problem dimension to exploit some version of the law of large numbers. Here, the asymptotics are in the *spatial dimension* and is matched to video coding applications where it is quite typical to have frames of size  $n = 352 \times 288$  pixels at 30 frames per second (full CIF<sup>5</sup>). It is also fairly common to code video in groups of  $T = 15$  pictures.

### C. Delayed sequential coding systems

For clarity of exposition, we start the discussion with the exemplary case of three frame systems. Systems with an arbitrary number of frames are studied in sections VII and VIII.

- *C–C systems*: The causal (zero-delay) sequential encoding with (zero-delay) causal sequential decoding system is illustrated in Fig. 2. In the first stage, the video encoder can only access  $\mathbf{X}_1$  and encodes it at rate  $R_1$  so that the video decoder is able to reconstruct  $\mathbf{X}_1$  as  $\widehat{\mathbf{X}}_1$  immediately. In the second stage (after one frame-delay), the

<sup>5</sup>CIF stands for Common Intermediate Format. Progressively scanned HDTV is typically  $n = 1280 \times 720 \approx$  one million pixels at 60 frames per second.

encoder has access to both  $\mathbf{X}_1$  and  $\mathbf{X}_2$  and encodes them at rate  $R_2$  so that the decoder can produce  $\widehat{\mathbf{X}}_2$  with help from the encoder's message in the first stage. In the final stage, the encoder has access to all the three sources and encodes them at rate  $R_3$  and the decoder produces  $\widehat{\mathbf{X}}_3$  with help from the encoder's messages from all the previous stages. Note that the processing of information by the video encoder and video decoder in different stages can be conceptually regarded as distinct source encoders and source decoders respectively. Also note that it is assumed that both the encoder and the decoder have enough memory to store all previous frames and messages.

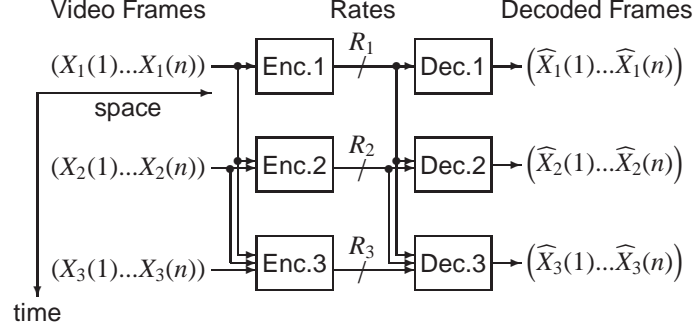


Fig. 2. C-C: Causal (zero-delay) sequential encoding with causal sequential decoding. Sum-rate =  $R_{sum}^{C-C} = R_1 + R_2 + R_3$ .

- *C-NC systems*: The causal sequential encoding with one-stage delayed noncausal sequential decoding system is illustrated in Fig. 3(a). In the figure, all the encoders have access to the same sets of sources as in the C-C system shown in Fig. 2. However, the decoders are delayed (moved downwards) by one stage with respect to Fig. 2. Specifically, the first decoder observes the messages from the first two encoders to produce  $\widehat{\mathbf{X}}_1$ . The second decoder produces  $\widehat{\mathbf{X}}_2$  based on all the three messages from the three encoders. The third decoder also produces  $\widehat{\mathbf{X}}_3$  using all the messages.

- *NC-C systems*: The one-stage delayed noncausal sequential encoding with causal sequential decoding system is

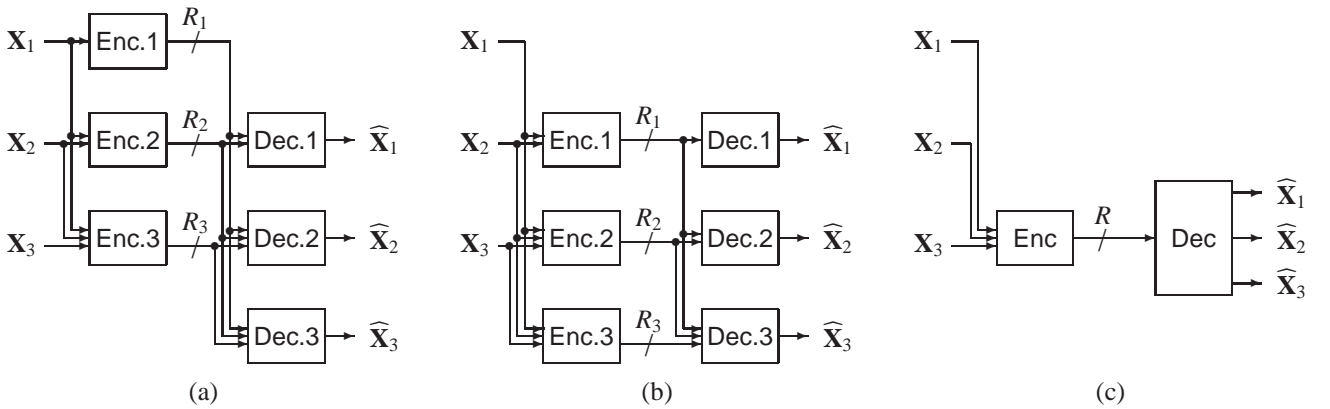


Fig. 3. (a) C-NC: Causal sequential encoding with one-stage delayed noncausal sequential decoding; (b) NC-C: one-stage delayed noncausal sequential encoding with causal sequential decoding; (c) JC:  $(T-1)$ -stage delayed joint (noncausal) encoding with joint (noncausal) decoding.

illustrated in Fig. 3(b). Compared with the C-NC system, the delay is on the encoding side. Specifically, the first

encoder has access to both  $\mathbf{X}_1$  and  $\mathbf{X}_2$ . Both the second and the third encoder have access to all three sources. The decoders have access to the same sets of messages sent by the encoders as in the C–C system.

- *JC systems:* Of special interest is the *joint* (noncausal) encoding and decoding system illustrated in Fig. 3(c). All the sources are collected by a single encoder and encoded jointly. The single decoder reconstructs all the frames simultaneously. Note that here the encoding frame delay is  $(T - 1)$ .

$T$ -stage sequential coding systems with  $k$ -stage frame-delays (see Fig. 6 and 7) are natural generalizations of the 3-stage systems discussed so far. The general cases will be discussed in detail in Sections VII and VIII.

The C–C blocklength- $n$  encoders and decoders are formally defined by the maps

$$\begin{aligned} (\text{Enc.}j) \quad f_j^{(n)} &: \mathcal{X}_1^n \times \dots \times \mathcal{X}_j^n \rightarrow \{1, \dots, M_j\}, \\ (\text{Dec.}j) \quad g_j^{(n)} &: \{1, \dots, M_1\} \times \dots \times \{1, \dots, M_j\} \rightarrow \widehat{\mathcal{X}}_j^n \end{aligned}$$

for  $j = 1, \dots, T$ , where  $(\log_2 M_j)/n$  is the  $j$ -th frame coding rate in bits per pixel (bpp) and  $\widehat{\mathcal{X}}_j$  is the  $j$ -th (finite cardinality) reproduction alphabet.

The formal definitions of C–NC encoders are identical to that for the C–C encoders. However, the C–NC decoders with a  $k$ -stage frame-delay are formally defined by the maps

$$(\text{Dec.}j) \quad g_j^{(n)} : \{1, \dots, M_1\} \times \dots \times \{1, \dots, M_{\min\{j+k, T\}}\} \rightarrow \widehat{\mathcal{X}}_j^n,$$

for  $j = 1, \dots, T$ . Similarly, the NC–C decoder definitions are identical to those for the C–C decoders and the NC–C encoders with a  $k$ -stage frame-delay are formally defined by the maps

$$(\text{Enc.}j) \quad f_j^{(n)} : \mathcal{X}_1^n \times \dots \times \mathcal{X}_{\min\{j+k, T\}}^n \rightarrow \{1, \dots, M_j\},$$

for  $j = 1, \dots, T$ . Finally the JC encoder and decoder are defined by the maps

$$\begin{aligned} (\text{Enc.}) \quad f^{(n)} &: \mathcal{X}_1^n \times \dots \times \mathcal{X}_T^n \rightarrow \{1, \dots, M\}, \\ (\text{Dec.}) \quad g^{(n)} &: \{1, \dots, M\} \rightarrow \widehat{\mathcal{X}}_1^n \times \dots \times \widehat{\mathcal{X}}_T^n. \end{aligned}$$

For a frame-delay  $k$ , there are boundary effects associated with the decoders (resp. encoders) of the last  $(k + 1)$  frames for the C–NC (resp. NC–C) systems. For example, the last two decoders in Fig. 3(a) are operationally equivalent to a single decoder since both use the same set of encoded messages. Although redundant, we retain the distinction of the boundary encoders/decoders for clarity and to aid comparison (see Theorem 4 in Section VI and Corollary 6.1 in Section VIII).

#### D. Operational rate-distortion regions

For each  $j = 1, \dots, T$ , the pixel reproduction quality is measured by a single-letter distortion criterion. We allow coupled distortion criteria where the distortion for the current frame can depend on the reproductions in previous frames:

$$d_j : \mathcal{X}_j \times \widehat{\mathcal{X}}_1 \times \dots \times \widehat{\mathcal{X}}_j \rightarrow \mathbb{R}^+.$$

The distortion criteria are assumed to be bounded, i.e.,

$$d_{j,\max} := \max_{x_j, \hat{x}_1, \dots, \hat{x}_j} d_j(x_j, \hat{x}_1, \dots, \hat{x}_j) < \infty.$$

The frame reproduction quality is in terms of the average pixel distortion

$$d_j^{(n)}(\mathbf{x}_j, \hat{\mathbf{x}}_1, \dots, \hat{\mathbf{x}}_j) = \frac{1}{n} \sum_{i=1}^n d_j(x_j(i), \hat{x}_1(i), \dots, \hat{x}_j(i)).$$

Of interest are the expected frame distortions  $E[d_j^{(n)}(\mathbf{X}_j, \widehat{\mathbf{X}}^j)]$ . It is important to notice that these are *frame-specific* distortions as opposed to an average distortion across all frames. This makes the JC problem different from a standard parallel vector source coding problem. Also notice that these fidelity criteria reflect dependencies on previous frame reproductions. For example, the second distortion criterion is given by  $d_2 : X_2 \times \widehat{X}_1 \times \widehat{X}_2 \rightarrow \mathbb{R}^+$ , as opposed to a criterion like  $\tilde{d}_2 : X_2 \times \widehat{X}_2 \rightarrow \mathbb{R}^+$  which is independent of previous reproductions. This model is motivated by the temporal perceptual characteristics of the human visual system where the visibility threshold at a given pixel location depends on the luminance intensity of the same pixel in the previous frames [9].

A rate-distortion-tuple  $(\mathbf{R}, \mathbf{D}) = (R_1, \dots, R_T, D_1, \dots, D_T)$  is said to be admissible for a given delayed sequential coding system if, for every  $\epsilon > 0$ , and all sufficiently large  $n$ , there exist block encoders and decoders satisfying

$$\frac{1}{n} \log M_j \leq R_j + \epsilon, \quad (2.1)$$

$$E[d_j^{(n)}(\mathbf{X}_j, \widehat{\mathbf{X}}^j)] \leq D_j + \epsilon, \quad (2.2)$$

simultaneously for all  $j = 1, \dots, T$ . For system  $A \in \{\text{C-C}, \text{JC}\}$ , the operational rate-distortion region  $\mathcal{R}^A$  is the set of all admissible rate-distortion-tuples. For system  $A \in \{\text{C-NC}, \text{NC-C}\}$  with  $k$ -stage frame-delay, the operational rate-distortion region, denoted by  $\mathcal{R}_k^A$ , is the set of all admissible rate-distortion-tuples. We will abbreviate  $\mathcal{R}_k^A$  to  $\mathcal{R}^A$  when  $k = 1$ . The sum-rate region denoted by  $\mathcal{R}_{sum}^A(\mathbf{D})$  (or  $\mathcal{R}_{k,sum}^A(\mathbf{D})$ ) is the set of all the admissible sum-rates  $\sum_{j=1}^T R_j$  at the distortion tuple  $\mathbf{D}$ .

Note that for any given distortion-tuple the minimum rate of the JC system is also the minimum sum-rate of a C-NC or NC-C system with frame-delay  $(T - 1)$  for the same distortion tuple. For example, in a  $(T - 1)$ -delayed C-NC system, all the decoders become joint decoders and the rate-tuple  $(R_1 = 0, \dots, R_{T-1} = 0, R_T = R_{JC}(\mathbf{D}), \mathbf{D})$  is admissible. Hence  $\mathcal{R}_{(T-1),sum}^{C-NC}(\mathbf{D}) = \mathcal{R}^{JC}(\mathbf{D})$ . Therefore C-NC and NC-C systems for  $T = 2$  are less interesting. The first non-trivial delayed sequential coding system arises for  $T = 3$  (also see the paragraph after Corollary 5.1). This is the reason for commencing the discussion with 3-stage systems.

### III. RESULTS FOR THE 3-STAGE C-C SYSTEM

#### A. Rate-distortion region

The C-C rate-distortion region can be formulated as a single-letter mutual information optimization problem subject to distortion constraints and natural Markov chains involving auxiliary and reproduction random variables and deterministic functions. This characterization is provided by Theorem 1.

**Theorem 1** (*C-C rate-distortion region*) The single-letter rate-distortion region for a  $T = 3$  frame C-C system is



given by

$$\begin{aligned}
\mathcal{R}^{C-C} &= \{(\mathbf{R}, \mathbf{D}) \mid \exists U^2, \widehat{X}^3, g_1(\cdot), g_2(\cdot, \cdot), s.t. \\
&R_1 \geq I(X_1; U_1), \\
&R_2 \geq I(X^2; U_2|U_1), \\
&R_3 \geq I(X^3; \widehat{X}_3|U^2), \\
&D_j \geq E[d_j(X_j, \widehat{X}^j)], \quad j = 1, 2, 3, \\
&\widehat{X}_1 = g_1(U_1), \quad \widehat{X}_2 = g_2(U_1, U_2), \\
&U_1 - X_1 - X_2^3, \quad U_2 - (X^2, U_1) - X_3\} \tag{3.3}
\end{aligned}$$

where  $\{U_1, U_2, \widehat{X}_1, \widehat{X}_2, \widehat{X}_3\}$  are auxiliary and reproduction random variables taking values in alphabets  $\{\mathcal{U}_1, \mathcal{U}_2, \widehat{\mathcal{X}}_1, \widehat{\mathcal{X}}_2, \widehat{\mathcal{X}}_3\}$  satisfying the cardinality bounds

$$\begin{aligned}
|\mathcal{U}_1| &\leq |X_1| + 6, \\
|\mathcal{U}_2| &\leq |X_1|^2 |X_2| + 6|X_1| |X_2| + 4,
\end{aligned}$$

and  $\{g_1(\cdot), g_2(\cdot, \cdot)\}$  are deterministic functions.

The rate-distortion region in [9] is for the 2-stage C–C problem, whereas the above region is for the 3-stage C–C problem. The above region differs from what one might expect to get from a natural extension of the 2-stage C–C rate-distortion region in [9]. This is because the characterization in Theorem 1 has different rate inequalities and fewer Markov chain conditions than what one might expect from the extension. One of the advantages of the characterization of the rate-distortion region in (3.3) is that it is more intuitive (as explained below) and this intuition carries over with little effort to the case of multiple frames (see Section VII and VIII). Another advantage of the characterization of the rate-distortion in (3.3) is that it is convex and closed as defined. The convexity can be shown along the lines of the time-sharing argument in Appendix C.II which is part of the converse proof of the coding theorem for C–NC systems. The closedness can be shown along the lines of the convergence argument in Appendix C.IV. Therefore, unlike the characterization provided in [9], there is no need to take the convex hull and closure in (3.3).

The proof of achievability can be carried out using standard random coding and random binning arguments and will be similar in spirit to the derivation for the  $T = 2$  frame case in [9], but with a different intuitive interpretation. Hence we will only present the intuition and informally sketch the steps leading to the proof of Theorem 1 in the following paragraph. As remarked in the introduction, a detailed proof of achievability and converse results will be presented only for the C–NC system with  $T = 3$  frames (Appendices II and III). The proofs of achievability and converse results for other systems can be carried out in a similar manner but the derivations become lengthy, repetitive, and cumbersome, and are therefore omitted.

The region in Theorem 1 has the following natural interpretation. First,  $\mathbf{X}_1$  is quantized to  $\mathbf{U}_1$  using a random codebook-1 for encoder-1 without access to  $\mathbf{X}_2^3$ . Decoder-1 recovers  $\mathbf{U}_1$  and reproduces  $\mathbf{X}_1$  as  $\widehat{\mathbf{X}}_1 = g_1^r(\mathbf{U}_1)$ . Next, the tuple  $\{\mathbf{X}^2, \mathbf{U}_1\}$  is (jointly) quantized to  $\mathbf{U}_2$  without access to  $\mathbf{X}_3$  using a random codebook-2 for encoder-2. The codewords are further randomly distributed into bins and the bin index of  $\mathbf{U}_2$  is sent to the decoder. Decoder-2 identifies  $\mathbf{U}_2$  from the bin with the help of  $\mathbf{U}_1$  as side-information (available from decoder-1) and reproduces  $\mathbf{X}_2$  as



$\widehat{\mathbf{X}}_2 = g_2^n(\mathbf{U}_1, \mathbf{U}_2)$ . Finally, encoder-3 (jointly) quantizes  $\{\mathbf{X}^3, \mathbf{U}^2\}$  into  $\widehat{\mathbf{X}}_3$  using encoder-3's random codebook, bins the codewords and sends the bin index of  $\widehat{\mathbf{X}}_3$  such that decoder-3 can identify  $\widehat{\mathbf{X}}_3$  with the help of  $\mathbf{U}^2$  as side-information available from decoders 1 and 2. The constraints on the rates and Markov chains ensure that with high probability (for all large enough  $n$ ) both encoding (quantization) and decoding (recovery) succeed and the recovered words are jointly strongly typical with the source words to meet the target distortions. Notice that the conditioning random variables that appear in the conditional mutual information expressions at each stage correspond to quantities that are known to both the encoding and decoding sides at that stage due to the previous stages. Using this observation, one can intuitively write down an achievable rate-distortion region for general delayed sequential coding systems by inspection.

### B. Sum-rate region

The sum-rate region can be obtained from the rate-distortion region  $\mathcal{R}^{C-C}$  as shown in the following corollary. The main simplification is the *absence* of the auxiliary random variables  $U^2$ .

**Corollary 1.1** (*C-C Sum-rate region*) The sum-rate region for the C-C system is  $\mathcal{R}_{sum}^{C-C}(\mathbf{D}) = [R_{sum}^{C-C}(\mathbf{D}), \infty)$  where the minimum sum-rate is

$$R_{sum}^{C-C}(\mathbf{D}) = \min_{\substack{E[d_j(X_j, \widehat{X}^j)] \leq D_j, j=1,2,3, \\ \widehat{X}_1 - X_1 - X_2^3, \widehat{X}_2 - (X^2, \widehat{X}_1) - X_3}} I(X^3; \widehat{X}^3). \quad (3.4)$$

*Proof:* For any point  $(\mathbf{R}, \mathbf{D}) \in \mathcal{R}^{C-C}$ , there exist auxiliary random variables and functions satisfying all the constraints in (3.3). Since the Markov chains  $U_1 - X_1 - X_2^3$  and  $U_2 - (X^2, U_1) - X_3$  hold, and  $\widehat{X}^2$  is a function of  $U^2$ , we have

$$\begin{aligned} R_1 + R_2 + R_3 &\geq I(X_1; U_1) + I(X^2; U_2|U_1) + I(X^3; \widehat{X}_3|U^2) \\ &= I(X^3; U_1) + I(X^3; U_2|U_1) + I(X^3; \widehat{X}_3|U^2) \\ &= I(X^3; U^2, \widehat{X}_3) \\ &= I(X^3; U^2, \widehat{X}^3) \\ &\geq I(X^3; \widehat{X}^3). \end{aligned}$$

It can be verified that Markov chains  $\widehat{X}_1 - X_1 - X_2^3$  and  $\widehat{X}_2 - (X^2, \widehat{X}_1) - X_3$  hold. Therefore the right hand side of (3.4) is not greater than the minimum sum rate.

On the other hand, because  $\{U_1 = \widehat{X}_1, U_2 = \widehat{X}_2\}$  is a possible choice of  $\{U_1, U_2\}$ ,

$$R_{sum}^{C-C}(\mathbf{D}) = \min I(X^3; U^2, \widehat{X}_3) \leq \min I(X^3; \widehat{X}^3),$$

where the first minimization is subject to the constraints in (3.3), and the second minimization is subject to the constraints in (3.4). Therefore (3.4) holds.  $\blacksquare$

As will become clear in the sequel, the minimum sum-rate for any type of delayed sequential coding system is given by the minimization of the mutual information between the source random variables  $X^T$  and the reproduction random variables  $\widehat{X}^T$  subject to several expected distortion and Markov-chain constraints involving these random variables of a form similar to (3.4).

### C. Sum-rate region for Gaussian source and MSE

In the case of Gaussian sources and MSE distortion criteria, the minimum sum-rate of any delayed sequential coding system (see Corollaries 1.1, 3.1, 5.1 and Theorem 2) can be achieved by reproduction random variables which are jointly Gaussian with the source random variables. This is contained in the following lemma.

**Lemma** If  $(X_1, \dots, X_T)$  are jointly Gaussian, the minimum value of  $I(X^T; \widehat{X}^T)$  subject to MSE constraints  $E[(X_j - \widehat{X}_j)^2] \leq D_j, j = 1, \dots, T$  and Markov chain constraints involving  $X^T$  and  $\widehat{X}^T$  is achieved by reproduction random variables  $\widehat{X}^T$  which are jointly Gaussian with  $X^T$ .

*Proof:* Given any reproduction random vector  $\widehat{\mathbf{X}} = (\widehat{X}_1, \dots, \widehat{X}_T)$  satisfying the MSE and Markov chain constraints, we can construct a new random vector  $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \dots, \widetilde{X}_T)$  which is *jointly Gaussian* with  $\mathbf{X} = (X_1, \dots, X_T)$  with the same second-order statistics. Specifically,  $\text{cov}(\widehat{\mathbf{X}}) = \text{cov}(\widetilde{\mathbf{X}})$  and  $\text{cov}(\mathbf{X}, \widehat{\mathbf{X}}) = \text{cov}(\mathbf{X}, \widetilde{\mathbf{X}})$ . Since MSEs are fully determined from second-order statistics,  $\widetilde{\mathbf{X}}$  automatically satisfies the same MSE constraints as  $\widehat{\mathbf{X}}$ . The Markov chain constraints for  $\widehat{\mathbf{X}}$  imply corresponding conditional uncorrelatedness constraints for  $\widetilde{\mathbf{X}}$ , which will also hold for  $\widetilde{\mathbf{X}}$ . Since  $\widetilde{\mathbf{X}}$  is jointly Gaussian, conditional uncorrelatedness is equivalent to conditional independence. Therefore  $\widetilde{\mathbf{X}}$  will also satisfy the corresponding Markov chain constraints.

Let the linear MMSE estimate of  $\mathbf{X}$  based on  $\widetilde{\mathbf{X}}$  be given by  $A\widetilde{\mathbf{X}}$  where  $A$  is a matrix. Note that by the orthogonality principle and the joint Gaussianity of  $\mathbf{X}$  and  $\widetilde{\mathbf{X}}$  we have  $(\mathbf{X} - A\widetilde{\mathbf{X}}) \perp \widetilde{\mathbf{X}}$ , and further  $(\mathbf{X} - A\widetilde{\mathbf{X}}) \perp\!\!\!\perp \widetilde{\mathbf{X}}$ . Therefore,

$$\begin{aligned} I(\mathbf{X}; \widehat{\mathbf{X}}) &= h(\mathbf{X}) - h(\mathbf{X} - A\widehat{\mathbf{X}}|\widehat{\mathbf{X}}) \\ &\geq h(\mathbf{X}) - h(\mathbf{X} - A\widetilde{\mathbf{X}}) \\ &\stackrel{(b)}{\geq} h(\mathbf{X}) - h(\mathbf{X} - A\widetilde{\mathbf{X}}) \\ &\stackrel{(c)}{=} h(\mathbf{X}) - h(\mathbf{X} - A\widetilde{\mathbf{X}}|\widetilde{\mathbf{X}}) \\ &= I(\mathbf{X}; \widetilde{\mathbf{X}}). \end{aligned}$$

Step (b) is because  $(\mathbf{X} - A\widetilde{\mathbf{X}})$  has the same second-order statistics as  $(\mathbf{X} - A\widehat{\mathbf{X}})$  and it is a *jointly Gaussian* random vector. Step (c) is because  $(\mathbf{X} - A\widetilde{\mathbf{X}})$  is independent of  $\widetilde{\mathbf{X}}$ .

In conclusion, given an *arbitrary* reproduction vector, we can construct a *Gaussian* random vector  $\widetilde{\mathbf{X}}$  satisfying the same MSE and Markov chain constraints as  $\widehat{\mathbf{X}}$  and  $I(\mathbf{X}; \widehat{\mathbf{X}}) \geq I(\mathbf{X}; \widetilde{\mathbf{X}})$ . Hence the minimum value of  $I(X^T; \widehat{X}^T)$  subject to MSE and Markov chain constraints will be achieved by a reproduction random vector which is jointly Gaussian with  $\mathbf{X}$ . ■

Since Gaussian vectors are characterized by means and covariance matrices, the minimum sum-rate computation reduces to a determinant optimization problem involving Markov chain and second-order moment constraints.

For Gauss–Markov sources,  $p_{X_1 X_2 X_3} = \mathcal{N}(\mathbf{0}, \Sigma_X)(x_1, x_2, x_3)$  where the covariance matrix  $\Sigma_X$  has the following structure

$$\Sigma_X = \begin{pmatrix} \sigma_1^2 & \rho_1 \sigma_1 \sigma_2 & \rho_1 \rho_2 \sigma_1 \sigma_3 \\ \rho_1 \sigma_1 \sigma_2 & \sigma_2^2 & \rho_2 \sigma_2 \sigma_3 \\ \rho_1 \rho_2 \sigma_1 \sigma_3 & \rho_2 \sigma_2 \sigma_3 & \sigma_3^2 \end{pmatrix},$$

which is consistent with the Markov chain relation  $X_1 - X_2 - X_3$  associated with the Gauss–Markov assumption.

Define a distortion region  $\mathcal{D}^{C-C} := \{\mathbf{D} \mid D_1 \leq \sigma_1^2, D_2 \leq \sigma_{W_2}^2, D_3 \leq \sigma_{W_3}^2\}$  where

$$\sigma_{W_j}^2 = \rho_{j-1}^2 \frac{\sigma_j^2}{\sigma_{j-1}^2} D_{j-1} + (1 - \rho_{j-1}^2) \sigma_j^2, \quad j = 2, 3 \quad (3.5)$$

whose significance will be discussed below. The C-C minimum sum-rate evaluated for any MSE tuple  $\mathbf{D}$  in this region is given by the following corollary.

**Corollary 1.2** (*C-C minimum sum-rate for Gauss-Markov sources and MSE*) In the distortion region  $\mathcal{D}^{C-C}$ , the C-C minimum sum-rate for Gauss-Markov sources and MSE is

$$R_{sum}^{C-CGM}(\mathbf{D}) = \frac{1}{2} \log \left( \frac{\sigma_1^2}{D_1} \right) + \frac{1}{2} \log \left( \frac{\sigma_{W_2}^2}{D_2} \right) + \frac{1}{2} \log \left( \frac{\sigma_{W_3}^2}{D_3} \right). \quad (3.6)$$

The proof of Corollary 1.2 is given in Appendix I. The form of (3.6) suggests the following idealized (achievable) coding scheme which is explained with reference to Fig. 4 and the upper bound argument in the proof of Corollary 1.2 in Appendix I. Encoder-1 initially quantizes  $\mathbf{X}_1$  into  $\widehat{\mathbf{X}}_1$  to meet the target MSE  $D_1$  using an ideal Gaussian rate-distortion quantizer and decoder-1 recovers  $\widehat{\mathbf{X}}_1$ . Since the quantizer is ideal, the joint distribution of  $(\mathbf{X}_1, \widehat{\mathbf{X}}_1)$  will follow the test-channel distribution of the rate-distortion function for a memoryless Gaussian source [16, p. 345, 370]. This idealization holds in the limit as the blocklength  $n$  tends to infinity. Let  $\mathbf{W}_1 := \mathbf{X}_1$  and  $\widehat{\mathbf{W}}_1 := \widehat{\mathbf{X}}_1$ . Next, encoder-2 makes the causal minimum mean squared error (MMSE) prediction of  $\mathbf{X}_2$  based on  $\widehat{\mathbf{X}}_1$  and quantizes the prediction error  $\mathbf{W}_2$  into  $\widehat{\mathbf{W}}_2$  using an ideal Gaussian rate-distortion quantizer so that decoder-2 can form  $\widehat{\mathbf{X}}_2$  to meet the target MSE  $D_2$  with help from  $\widehat{\mathbf{W}}_1$ . The asymptotic per-component variance of  $\mathbf{W}_2$  will be consistent with (3.5) because the rate-distortion quantizer is ideal. Specifically, decoder-2 recovers  $\widehat{\mathbf{W}}_2$  and creates the reproduction  $\widehat{\mathbf{X}}_2$  as the causal MMSE estimate of  $\mathbf{X}_2$  based on  $\widehat{\mathbf{W}}^2$ . Finally, encoder-3 makes the causal MMSE prediction of  $\mathbf{X}_3$  based on  $\widehat{\mathbf{W}}^2$  and quantizes the prediction error  $\mathbf{W}_3$  into  $\widehat{\mathbf{W}}_3$  using an ideal Gaussian rate-distortion quantizer so that decoder-3 can form  $\widehat{\mathbf{X}}_3$  to meet the target MSE  $D_3$  with help from  $\widehat{\mathbf{W}}^3$ . Decoder-3 recovers  $\widehat{\mathbf{W}}_3$  and makes the reproduction  $\widehat{\mathbf{X}}_3$  as the MMSE estimate of  $\mathbf{X}_3$  based on  $\widehat{\mathbf{W}}^3$ . The C-C coding scheme just described is an idealized version of DPCM (see [1]–[3], [5], [6] and references therein) because the rate-distortion quantizer is idealized. The above arguments lead to the following corollary.

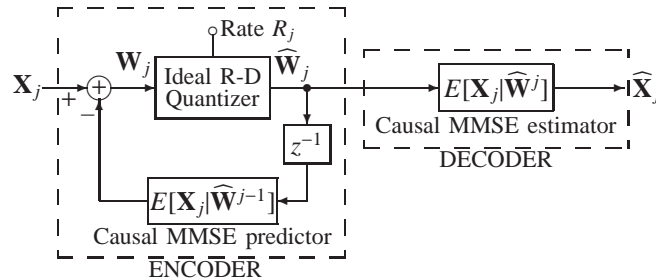


Fig. 4. Illustrating idealized DPCM.

**Corollary 1.3** (*C-C Optimality of idealized DPCM for Gauss-Markov sources and MSE*) The C-C minimum sum-rate-MSE performance for Gauss-Markov sources is achieved by idealized DPCM for all distortion tuples  $\mathbf{D}$  in the

distortion region  $\mathcal{D}^{C-C}$ .

The distortion region  $\mathcal{D}^{C-C}$  is the set of distortion tuples for which the DPCM encoder uses a positive rate for each frame. Note that  $\mathcal{D}^{C-C}$  has a *non-zero volume* for nonsingular sources ( $\sigma_j \neq 0, \rho_j \neq \pm 1$ ). Hence, the assertion that DPCM is optimal for C-C systems is a nontrivial statement.

#### IV. RESULTS FOR THE 3-STAGE JC SYSTEM

**Theorem 2** (*JC rate-distortion function, [17, Problem 14, p.134]*) The single-letter rate-distortion function for the joint coding system is given by

$$R^{JC}(\mathbf{D}) = \min_{E[d_j(X_j, \widehat{X}^j)] \leq D_j, j=1,2,3} I(X^3; \widehat{X}^3). \quad (4.7)$$

Compared to  $R_{sum}^{C-C}(\mathbf{D})$  given by (3.4), the JC rate-distortion function  $R^{JC}(\mathbf{D})$  given by (4.7) having no Markov chain constraints is a lower bound for  $R_{sum}^{C-C}(\mathbf{D})$ . While this follows from a direct comparison of the single-letter rate-distortion functions, from the operational structure of C-C, C-NC, NC-C, and JC systems it is clear that the JC rate-distortion function is in fact a lower bound for the sum-rates for *all* delayed sequential coding systems.

Similar to Corollary 1.2 which is for a C-C system, Gaussian sources, and MSE distortion criteria, we have the following corollary for a JC system.

**Corollary 2.1** (*JC rate-MSE function for Gauss-Markov sources*)

(i) For the distortion region  $\mathcal{D}^{JC} := \{\mathbf{D} \mid (\Sigma_X - \text{diag}(\mathbf{D})) \geq 0\}$ , the JC rate-MSE function for jointly Gaussian sources is given by

$$R^{JCGM}(\mathbf{D}) = \frac{1}{2} \log \left( \frac{|\Sigma_X|}{D_1 D_2 D_3} \right). \quad (4.8)$$

(ii) For the distortion region  $\mathcal{D}^{JC}$ , the JC rate-MSE function for Gauss-Markov sources is given by

$$\begin{aligned} R^{JCGM}(\mathbf{D}) &= \frac{1}{2} \log \left( \frac{\sigma_1^2}{D_1} \right) + \frac{1}{2} \log \left( \frac{\sigma_2^2(1 - \rho_1^2)}{D_2} \right) + \\ &\quad + \frac{1}{2} \log \left( \frac{\sigma_3^2(1 - \rho_2^2)}{D_3} \right). \end{aligned} \quad (4.9)$$

Formula (4.8) is the Shannon lower bound [2], [3] of the JC rate-distortion function. It can be achieved in the distortion region  $\mathcal{D}^{JC}$  by the test channel

$$\widehat{\mathbf{X}} + \mathbf{Z} = \mathbf{X} \quad (4.10)$$

where  $\mathbf{Z} = (Z_1, Z_2, Z_3)$  and  $\widehat{\mathbf{X}} = (\widehat{X}_1, \widehat{X}_2, \widehat{X}_3)$  are independent Gaussian vectors with covariance matrices

$$\Sigma_Z = \text{diag}(\mathbf{D}), \quad \Sigma_{\widehat{X}} = \Sigma_X - \text{diag}(\mathbf{D}),$$

and  $\mathbf{X} = (X_1, X_2, X_3)$ . The existence of this channel is guaranteed by the definition of  $\mathcal{D}^{JC}$ .

Comparing (3.6) and (4.9) for  $\mathbf{D} \in \mathcal{D}^{JC} \cap \mathcal{D}^{C-C}$  which generally has a nonempty interior, we find that in general the C–C sum-rate  $R_{sum}^{C-CGM}(\mathbf{D})$  is *strictly* greater than the JC rate  $R^{JCGM}(\mathbf{D})$ . However, as  $\mathbf{D} \rightarrow \mathbf{0}$ , the two rates are asymptotically equal.

We would like to draw some parallels between C–C sequential coding of correlated sources and Slepian-Wolf distributed coding of correlated sources [16]. In the Slepian-Wolf coding problem we have spatially correlated sources, temporal asymptotics, and a distributed coding constraint. In the C–C sequential coding problem we have temporally correlated sources, spatial asymptotics, and a sequential coding constraint. The roles of time and space are approximately exchanged. In Slepian-Wolf coding, the sources  $\mathbf{X}_1$ ,  $\mathbf{X}_2$ , and  $\mathbf{X}_3$  can be individually encoded at the rates  $H(X_1)$ ,  $H(X_2|X_1)$ , and  $H(X_3|X^2)$  respectively and decoded sequentially by first reconstructing  $\mathbf{X}_1$ , then  $\mathbf{X}_2$ , and finally  $\mathbf{X}_3$  (see Fig. 5). The sum-rate is equal to the joint entropy of the three sources which is the rate required for jointly coding the three sources. The fact that as  $\mathbf{D} \rightarrow \mathbf{0}$  the C–C sum-rate approaches the JC sum-rate is consistent with the fact that in the Slepian-Wolf coding problem, sequential encoding and decoding does not entail a rate-loss with respect to joint coding. As  $\mathbf{D} \rightarrow \mathbf{0}$  we are approaching near-lossless compression.

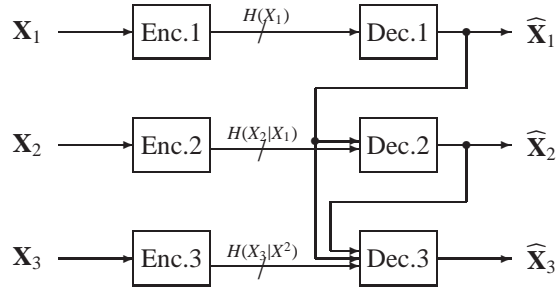


Fig. 5. Slepian-Wolf coding with a sequential decoding

## V. RESULTS FOR THE 3-STAGE C–NC SYSTEM

Similar to Theorem 1 and Corollary 1.1 for the C–C system, the rate-distortion and sum-rate regions for a C–NC system are characterized by Theorem 3 and Corollary 3.1 respectively as follows.

**Theorem 3** (*C–NC rate-distortion region*) The single-letter rate-distortion region for a C–NC system with one-stage decoding frame-delay is given by

$$\begin{aligned}
 \mathcal{R}^{C-NC} &= \{(\mathbf{R}, \mathbf{D}) \mid \exists U^2, \widehat{X}^3, g_1(\cdot, \cdot), s.t. \\
 R_1 &\geq I(X_1; U_1), \\
 R_2 &\geq I(X^2; U_2|U_1), \\
 R_3 &\geq I(X^3; \widehat{X}_2^3|U^2), \\
 D_j &\geq E[d_j(X_j, \widehat{X}^j)], \quad j = 1, 2, 3, \\
 \widehat{X}_1 &= g_1(U_1, U_2), \\
 U_1 &= X_1 - X_2^3, \quad U_2 = (X^2, U_1) - X_3\}
 \end{aligned} \tag{5.11}$$

where  $g_1(\cdot, \cdot)$  is a deterministic function and  $\{U_1, U_2\}$  are auxiliary random variables satisfying cardinality bounds

$$\begin{aligned} |\mathcal{U}_1| &\leq |X_1| + 6, \\ |\mathcal{U}_2| &\leq |X_1|^2 |X_2| + 6|X_1| |X_2| + 5. \end{aligned}$$

Note that  $\mathcal{R}^{C-C} \subseteq \mathcal{R}^{C-NC}$  because the encoders and decoders of a C-C system can also be used in a C-NC system. As in Theorem 1, the characterization of the rate-distortion region given in Theorem 3 is both convex and closed and there is no need to take the convex hull and closure.

The proof of the forward part of Theorem 3 is given in Appendix II. The region in Theorem 3 has the following natural interpretation. First,  $\mathbf{X}_1$  is quantized to  $\mathbf{U}_1$  using a random codebook-1 for encoder-1 without access to  $\mathbf{X}_2^3$ . Next, the tuple  $\{\mathbf{X}^2, \mathbf{U}_1\}$  is (jointly) quantized to  $\mathbf{U}_2$  without access to  $\mathbf{X}_3$  using a random codebook-2 for encoder-2. The codewords are further randomly distributed into bins and the bin index of  $\mathbf{U}_2$  is sent to the decoder. Decoder-1 recovers  $\mathbf{U}_1$  from the message sent by encoder-1. Then it identifies  $\mathbf{U}_2$  from the bin with the help of  $\mathbf{U}_1$  as side-information and reproduces  $\mathbf{X}_1$  as  $\widehat{\mathbf{X}}_1 = g_1^n(\mathbf{U}_1, \mathbf{U}_2)$ . Finally, encoder-3 (jointly) quantizes  $\{\mathbf{X}^3, \mathbf{U}^2\}$  into  $\widehat{\mathbf{X}}_2^3$  using encoder-3's random codebook, bins the codewords and sends the bin index such that decoder-2 and decoder-3 can identify  $\widehat{\mathbf{X}}_2^3$  with the help of  $\mathbf{U}^2$  as side-information available from decoders 1 and 2. The constraints on the rates and the Markov chains ensure that with high probability (for all large enough  $n$ ) both encoding (quantization) and decoding (recovery) succeed and the recovered words are jointly strongly typical with the source words to meet the target distortions.

The (weak) converse part of Theorem 3 is proved in Appendix III using standard information inequalities by defining auxiliary random variables  $U_j(i) = (S_j, X_j(i-)), j = 1, 2$ , where  $S_j$  denotes the message sent by the  $j$ -th encoder satisfying all the Markov-chain and distortion constraints, and a convexification (time-sharing) argument as in [16, p.397]. The cardinality bounds of the auxiliary random variables are also derived in Appendix C.III using the Carathéodory theorem.

**Corollary 3.1** (*C-NC sum-rate region*) The sum-rate region for the one-stage delayed C-NC system is  $\mathcal{R}_{sum}^{C-NC}(\mathbf{D}) = [R_{sum}^{C-NC}(\mathbf{D}), \infty)$  where the minimum sum-rate is

$$R_{sum}^{C-NC}(\mathbf{D}) = \min_{\substack{E[d_j(X_j, \widehat{X}^j)] \leq D_j, j=1,2,3, \\ \widehat{X}_1 - X^2 - X_3}} I(X^3; \widehat{X}^3). \quad (5.12)$$

*Proof:* The proof is similar to that of Corollary 1.1. The main simplification is the absence of the auxiliary random variables  $U^2$ . For any point  $(\mathbf{R}, \mathbf{D}) \in \mathcal{R}^{C-NC}$ , there exist auxiliary random variables and functions satisfying all the constraints in  $\mathcal{R}^{C-NC}$ . Since the Markov chains  $U_1 - X_1 - X_2^3$  and  $U_2 - (X^2, U_1) - X_3$  hold, and  $\widehat{X}_1$  is a function of  $U^2$ , we have

$$\begin{aligned} R_1 + R_2 + R_3 &\geq I(X_1; U_1) + I(X^2; U_2 | U_1) + I(X^3; \widehat{X}_2^3 | U^2) \\ &= I(X^3; U_1) + I(X^3; U_2 | U_1) + I(X^3; \widehat{X}_2^3 | U^2) \\ &= I(X^3; U^2, \widehat{X}_2^3) \\ &= I(X^3; U^2, \widehat{X}^3) \\ &\geq I(X^3; \widehat{X}^3). \end{aligned}$$

It can be verified that the Markov chain  $\widehat{X}_1 - X^2 - X_3$  holds. Therefore the right hand side of (5.12) is not greater than the minimum sum rate.

On the other hand, because  $\{U_1 = 0, U_2 = \widehat{X}_1\}$  is a possible choice of  $\{U_1, U_2\}$ ,

$$R_{sum}^{C-NC}(\mathbf{D}) = \min I(X^3; U^2, \widehat{X}_2^3) \leq \min I(X^3; \widehat{X}^3),$$

where the first minimization is subject to the constraints in (5.11), and the second minimization is subject to the constraints in (5.12). Therefore (5.12) holds.  $\blacksquare$

As noted earlier, the JC rate-distortion function (4.7) having no Markov chain constraints is a lower bound for  $R_{sum}^{C-NC}(\mathbf{D})$ . Remarkably, for Gauss–Markov sources and certain nontrivial MSE tuples  $\mathbf{D}$  discussed below,  $R_{sum}^{C-NC}(\mathbf{D})$  coincides with the JC rate  $R^{JC}(\mathbf{D})$ .

**Corollary 3.2** (*JC-optimality of a one-stage delayed C–NC system for Gauss–Markov sources and MSE*) For all distortion tuples  $\mathbf{D}$  belonging to the distortion region  $\mathcal{D}^{JC}$  defined in Section IV, Corollary 2.1(i), we have

$$R_{sum}^{C-NCGM}(\mathbf{D}) = R^{JCGM}(\mathbf{D}).$$

*Proof:* The JC rate-distortion function is achieved by the test channel (4.10) in the distortion region  $\mathcal{D}^{JC}$ . We will verify that the Markov chain  $\widehat{X}_1 - X^2 - X_3$  holds for this test channel.

Note that because all the variables are jointly Gaussian, they have the property that  $A \perp\!\!\!\perp B$  and  $A \perp\!\!\!\perp C$  implies  $A \perp\!\!\!\perp \{B, C\}$  for any Gaussian vector  $(A, B, C)$ .

By the Markov chain  $X_1 - X_2 - X_3$ , the MMSE estimate of  $X_3$  based on  $X_1$  and  $X_2$  is

$$X_3 = \rho_2 \frac{\sigma_3}{\sigma_2} X_2 + N \quad (5.13)$$

where  $N$  is Gaussian and independent of  $\{X_1, X_2\}$ .

By the structure of the test channel,  $Z_1 \perp\!\!\!\perp \{Z_2, Z_3, \widehat{X}_2, \widehat{X}_3\}$  implies  $Z_1 \perp\!\!\!\perp \{X_2, X_3\}$ , which further implies  $Z_1 \perp\!\!\!\perp N$ . Moreover, because  $N \perp\!\!\!\perp \{X_1, Z_1\}$ , we have  $N \perp\!\!\!\perp \widehat{X}_1$ . Therefore  $N \perp\!\!\!\perp \{X_1, X_2, \widehat{X}_1\}$ . So the best estimate of  $X_3$  based on  $\{X_1, X_2, \widehat{X}_1\}$  is still formula (5.13). It follows that the Markov chain  $X_3 - X_2 - (X_1, \widehat{X}_1)$  holds which in turn implies that  $\widehat{X}_1 - X^2 - X_3$  holds and completes the proof.  $\blacksquare$

Recall that the JC rate-distortion function is a lower bound for the minimum sum-rate for all delayed sequential coding systems. Corollary 3.2 implies that the JC rate-distortion performance is achievable in terms of sum-rate with only a single frame decoding delay for Gauss–Markov sources and MSE tuples in the region  $\mathcal{D}^{JC}$ . The first-order Markov assumption on sources  $X_1 - X_2 - X_3$  is essential for this optimality. An interpretation is that  $\mathbf{X}_2$  supplies all the help from  $\mathbf{X}_3$  to generate the optimum  $\widehat{\mathbf{X}}_1$ . More generally (for  $T > 3$ ), as shown in Section VII, C–NC encoders need access to only the present and past frames together with *one* future frame to match the rate-distortion function of the JC system in which *all* future frames are simultaneously available for encoding. Thus, the neighboring future frame supplies all the help from the entire future through the Markovian property of sources. The benefit of one frame-delay is so significant that it is equivalent to arbitrary frame-delay for Gauss–Markov sources and MSE criteria when  $\mathbf{D} \in \mathcal{D}^{JC}$ .

It is of interest to compare Corollary 3.2 with the real-time source coding problem in [18]. In [18] it is shown that for Markov sources, a C–C encoder may ignore the previous sources and only use the current source and decoder’s memory without loss of performance. This is a purely structural result (no spatial asymptotics and computable single-letter information-theoretic characterizations) exclusively focused on C–C systems. In contrast, Corollary 3.2



is about achieving the JC-system performance with a C–NC system. Additionally, [18] deals with a frame-averaged expected distortion criterion as opposed to frame-specific individual distortion constraints treated here.

The JC-optimality of the one-stage delayed C–NC system is guaranteed to hold within the distortion region  $\mathcal{D}^{JC}$  defined as the set of all distortion tuples  $\mathbf{D}$  satisfying the positive semidefiniteness condition  $(\Sigma_X - \text{diag}(\mathbf{D})) \geq 0$ . For nonsingular sources  $\Sigma_X > 0 \Rightarrow \lambda_{\min}(\Sigma_X) > 0$  where  $\lambda_{\min}(\Sigma_X)$  is the smallest eigenvalue of the positive definite symmetric (covariance) matrix  $\Sigma_X$ . For any point  $\mathbf{D}$  in the closed hypercube  $[0, \lambda_{\min}]^T$ ,

$$\Sigma_X - \text{diag}(\mathbf{D}) = (\Sigma_X - \lambda_{\min}I) + \text{diag}(\lambda_{\min}\mathbf{e} - \mathbf{D})$$

where  $I$  is the identity matrix and  $\mathbf{e} = (1, \dots, 1)$  is the all-one vector. Because both terms are positive semidefinite matrices, the sum is also positive semidefinite. Therefore  $\mathcal{D}^{JC}$  contains this hypercube, which has a strictly positive volume in  $\mathbb{R}^T \Rightarrow \mathcal{D}^{JC}$  has a *non-zero volume*. Hence, the JC-optimality of a C–NC system with one-stage decoding delay discussed here is a nontrivial assertion.  $\mathcal{D}^{JC}$  includes all distortion tuples with components below certain thresholds corresponding to “sufficiently good” reproduction qualities. However, it should be noted that this is *not* a high-rate (vanishing distortion) asymptotic.

On the contrary, the JC-optimality of a C–NC system with one-stage decoding frame-delay does not hold for all distortion tuples as the following counter example shows.

*Counter example:* Consider Gauss–Markov sources  $X^3$  where  $X_1 = X_2$  and MSE tuple  $\mathbf{D}$  where  $D_1 = D_2 = D$ . The JC problem reduces to a *two-stage JC problem* where the encoder jointly quantizes  $(\mathbf{X}_1, \mathbf{X}_3)$  into  $(\widehat{\mathbf{X}}_1, \widehat{\mathbf{X}}_3)$  and the decoder simply sets  $\widehat{\mathbf{X}}_2 = \widehat{\mathbf{X}}_1$ . However, the C–NC problem reduces to a *two-stage C–C problem* with sources  $(\mathbf{X}_1, \mathbf{X}_3)$  because the first two C–NC encoders are operationally equivalent to the first C–C encoder observing  $\mathbf{X}_1$  and the last C–NC encoder is operationally equivalent to the second C–C encoder observing all sources. As mentioned in the last but one paragraph of Section IV, generally speaking, a two-stage C–C system does not match (in sum-rate) the JC-system rate-distortion performance. Therefore the three-stage C–NC system also does not match the JC performance for these specific sources and certain distortion tuples  $\mathbf{D}$ . Note that these sources are actually singular ( $\Sigma_X$  has a zero eigenvalue) and  $\mathcal{D}^{JC}$  only contains trivial points (either  $D = 0$  or  $D_3 = 0$ ). So for the nontrivial distortion tuples  $\mathbf{D}$  described above (which do not belong to  $\mathcal{D}^{JC}$ ), the JC-optimality of a C–NC system with a one-stage decoding delay fails to hold.

To construct a counter example with nonsingular sources, one can slightly perturb  $\Sigma_X$  such that it becomes positive definite. However, the JC rate and C–NC sum-rate only change by limited amounts due to continuity properties of the sum-rate-distortion function with respect to the source distributions (similar to [17, Lemma 2.2, p.124]). Therefore we can find a small enough perturbation such that the rates do not match.

The JC-optimality of the one-stage delayed C–NC system is not a unique property of Gaussian sources and MSE. It also holds for symmetrically correlated binary sources with a Hamming distortion. These sources can be described as follows. Let  $X_1, N_1, N_2$  be mutually independent  $\text{Ber}(1/2)$ ,  $\text{Ber}(p_1)$ ,  $\text{Ber}(p_2)$  random variables respectively.  $X_2 = X_1 \oplus N_1$ ,  $X_3 = X_2 \oplus N_2$ , where  $\oplus$  indicates the Boolean exclusive OR operation. One can verify that the sum-rate-distortion performance of a C–NC system matches the JC rate-distortion performance for these sources and Hamming distortion within a certain distortion region of a nonzero volume. We omit the proof because it is cumbersome.

## VI. RESULTS FOR THE 3-STAGE NC–C SYSTEM

We can derive the rate-distortion region for an NC–C system by mimicking the derivations for the C–NC system discussed till this point. However, due to the operational structural relationship between C–NC and NC–C systems,

it is not necessary to re-derive the results for the NC–C system at certain operating points, in particular, for the sum-rate region:

**Theorem 4** (“Equivalence” of C–NC and NC–C rate-distortion regions)

(i) The rate-distortion region for the one-stage delayed NC–C system is given by

$$\begin{aligned}\mathcal{R}^{NC-C} = \{(\mathbf{R}, \mathbf{D}) \mid & \exists U^2, \widehat{X}^3, g_1(\cdot), g_2(\cdot, \cdot), s.t. \\ & R_1 \geq I(X^2; U_1), \\ & R_2 \geq I(X^3; U_2|U_1), \\ & R_3 \geq I(X^3; \widehat{X}_3|U^2), \\ & D_j \geq E[d_j(X_j, \widehat{X}^j)], \quad j = 1, 2, 3, \\ & \widehat{X}_1 = g_1(U_1), \widehat{X}_2 = g_2(U_1, U_2), \\ & U_1 - X^2 - X_3\}.\end{aligned}$$

with the following cardinality bounds

$$\begin{aligned}|\mathcal{U}_1| & \leq |\mathcal{X}_1| + 6, \\ |\mathcal{U}_2| & \leq |\mathcal{X}_1|^2|\mathcal{X}_2|^2|\mathcal{X}_3| + 6|\mathcal{X}_1||\mathcal{X}_2||\mathcal{X}_3| + 4.\end{aligned}$$

(ii) For an arbitrary distortion tuple  $\mathbf{D}$ , the rate regions  $\mathcal{R}^{NC-C}$  and  $\mathcal{R}^{C-NC}$  are related in the following manner:

$$\begin{aligned}(R_1, R_2, R_3, \mathbf{D}) \in \mathcal{R}^{C-NC} & \Rightarrow (R_1 + R_2, R_3, 0, \mathbf{D}) \in \mathcal{R}^{NC-C}, \\ (R_1, R_2, R_3, \mathbf{D}) \in \mathcal{R}^{NC-C} & \Rightarrow (0, R_1, R_2 + R_3, \mathbf{D}) \in \mathcal{R}^{C-NC}.\end{aligned}$$

(iii) For an arbitrary distortion tuple  $\mathbf{D}$ , the minimum sum-rates of one-stage delayed C–NC and NC–C systems are equal:

$$R_{sum}^{C-NC}(\mathbf{D}) = R_{sum}^{NC-C}(\mathbf{D}).$$

The proof of part (i) is similar to that of Theorem 3. Part (ii) can be proved by either using the definitions of  $\mathcal{R}^{C-NC}$  and  $\mathcal{R}^{NC-C}$  or more directly from the system structure (see Figs 3(a) and (b)) as follows. Given any C–NC system with rate tuple  $(R_1, R_2, R_3)$ , we can construct an NC–C system as follows: (1) combine the first two C–NC encoders to get the first NC–C encoder, (2) use the third C–NC encoder as the second NC–C encoder, and (3) use a null encoder with constant zero output as the third NC–C encoder. Then we have an NC–C system with rate tuple  $(R_1 + R_2, R_3, 0)$  and the same distortion tuple. Similarly, given any NC–C system, we can use a null encoder as the first C–NC encoder and combine the last two NC–C encoders to get a C–NC system. Part (iii) follows from part (ii).

The (sum-rate) JC-optimality property of a C–NC system with one-stage decoding frame-delay given by Corollary 3.2 automatically holds for an NC–C system with one-stage encoding frame-delay. This relationship allows one to focus on the performance of only C–NC systems instead of both C–NC and NC–C systems without loss of generality. This structural principle holds for the general multi-frame problem with multi-stage frame-delay, as discussed in Section VIII.

## VII. GENERAL C-NC RESULTS

The  $T$ -stage C-NC system with a  $k$ -stage decoding delay is a natural generalization of the 3-stage C-NC system with one-stage decoding delay. For  $j = 1, \dots, T$ , encoder- $j$  observes the current and all the past sources  $\mathbf{X}^j$  and encodes them at rate  $R_j$ . Decoder- $j$  observes all the messages sent by encoders one through  $(\min\{j + k, T\})$  and reconstructs  $\widehat{\mathbf{X}}_j$ . As an example, we present the diagram of a C-NC system with  $T = 4$  frames and  $k = 2$ -stage decoding delay in Fig. 6. Similar to Theorem 3 and Corollary 3.1, we have the following results.

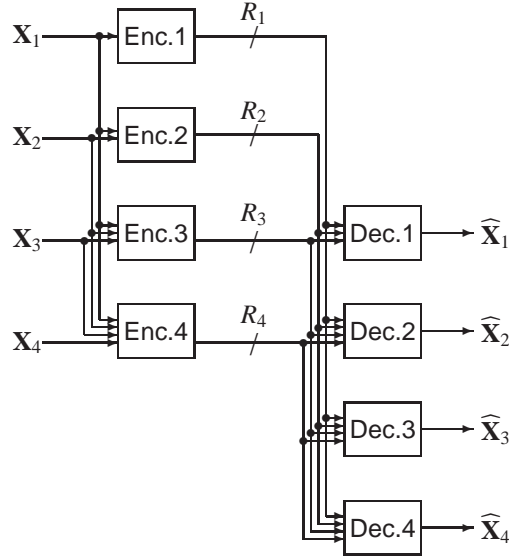


Fig. 6. A 4-stage C-NC system with a 2-stage decoding delay.

**Theorem 5** (*General C-NC rate region*) The rate region for the  $T$ -stage C-NC system with a  $k$ -stage decoding delay is given by:

$$\begin{aligned}
 \mathcal{R}_k^{C-NC} &= \{(\mathbf{R}, \mathbf{D}) \mid \exists U^{T-1}, \widehat{X}^T, g^{T-k-1}(\cdot), s.t. \\
 &R_j \geq I(X^j; U_j | U^{j-1}), \quad j = 1, \dots, (T-1) \\
 &R_T \geq I(X^T; \widehat{X}_{T-k}^T | U^{T-1}), \\
 &U_j - (X^j, U^{j-1}) - X_{j+1}^T, \quad j = 1, \dots, (T-1), \\
 &\widehat{X}_j = g_j(U^{j+k}), \quad j = 1, \dots, (T-k-1), \\
 &D_j \geq E[d_j(X_j, \widehat{X}^j)], \quad j = 1, \dots, T\}.
 \end{aligned}$$

with the following cardinality bounds

$$\text{for } j = 1, \dots, T-1, \quad |\mathcal{U}_j| \leq \prod_{k=1}^j |\mathcal{X}_k| \prod_{k=1}^{j-1} |\mathcal{U}_k| + 2T.$$

The cardinality bounds for the alphabets of the auxiliary random variables stated in Theorem 5 are obtained by a loose counting of constraints (see Appendix C.III). These bounds can be improved by some constants by a more careful counting of constraints. The first term  $\prod_{k=1}^j |\mathcal{X}_k| \prod_{k=1}^{j-1} |\mathcal{U}_k|$  comes from the Markov chain constraints. The second term  $2T$  comes from  $T$  rate constraints and  $T$  distortion constraints.

**Corollary 5.1** (*General C–NC sum-rate region*) The sum-rate region of the general C–NC system is given by  $\mathcal{R}_{k,sum}^{C-NC}(\mathbf{D}) = [R_{k,sum}^{C-NC}(\mathbf{D}), \infty)$ , where  $R_{k,sum}^{C-NC}(\mathbf{D})$  is the minimum value of  $I(X^T; \widehat{X}^T)$  subject to distortion constraints  $E[d_j(X_j, \widehat{X}^j)] \leq D_j, j = 1, \dots, T$  and Markov chain constraints

$$\widehat{X}_j - (X^{j+k}, \widehat{X}^{j-1}) - X_{j+k+1}^T, \quad j = 1, \dots, T - k - 1. \quad (7.14)$$

For general C–NC systems with increasing system frame-delays, the expressions of the minimum sum-rates contain the same objective function  $I(X^T; \widehat{X}^T)$  and distortion constraints  $E[d_j(X_j, \widehat{X}^j)] \leq D_j, j = 1, \dots, T$ , but with a decreasing number of Markov chain constraints. In the limit of maximum possible system frame-delay, equal to  $(T-1)$ , which is the same as in a JC system, we get the JC rate-distortion function with purely distortion (no Markov chain) constraints. When  $T = 2$ , a one-stage delayed C–NC system is trivial in terms of the sum-rate-distortion function because it reduces to that of a 2-stage JC system. Note that this reduction holds for *arbitrary source distributions and arbitrary distortion criteria*. So nontrivial C–NC systems must have at least  $T = 3$  frames. This is the motivation for choosing  $T = 3$  to start the discussion of delayed sequential coding systems in Section II-C. However, this type of reduction should be distinguished from the nontrivial reduction result of Corollary 3.2 which only holds for certain source distributions and distortion criteria.

Using the notation of directed information [19], [20]

$$I(A^N \rightarrow B^N) := \sum_{n=1}^N I(A^n; B_n | B^{n-1}),$$

and its generalization to  $k$ -directed information [21]

$$\begin{aligned} I_k(A^N \rightarrow B^N) &:= I(A^N; B^N) - \sum_{n=k+1}^N I(B^{n-k}; A_n | A^{n-1}) \\ &= I(A^N; B^N) - I(0^k B^{N-k} \rightarrow A^N), \end{aligned}$$

where  $0^k B^{N-k}$  is the  $N$ -length sequence  $(0, \dots, 0, B_1, \dots, B_{N-k})$ , we can write the objective function of the minimization problem in Corollary 5.1 as follows

$$I(X^T; \widehat{X}^T) = I_{k+1}(X^T \rightarrow \widehat{X}^T) + I(0^{k+1} \widehat{X}^{T-k-1} \rightarrow X^T). \quad (7.15)$$

The Markov chain constraints (7.14) are equivalent to the condition  $I(0^{k+1} \widehat{X}^{T-k-1} \rightarrow X^T) = 0$ . So the sum-rate can be reformulated as the minimum of the first term of (7.15) subject to the second term = 0 and the distortion constraints.

As the generalization of Corollary 3.2, we have the following result for  $k$ -th order Gauss-Markov sources where  $X_1, \dots, X_T$  form a  $k$ -th order Markov chain.

**Corollary 5.2** (*JC optimality of  $k$ -stage delayed C–NC systems for  $k$ -th order Gauss-Markov sources and MSE*)

$$R_{k,sum}^{C-NCGM}(\mathbf{D}) = R^{JCGM}(\mathbf{D})$$

for the distortion region  $\mathcal{D}^{JC}$ .

*Proof:* The proof is similar to that of Corollary 3.2. The JC rate-distortion function is achieved by the test channel (4.10) in the distortion region  $\mathcal{D}^{JC}$ . We will verify that the Markov chain  $\widehat{X}_j - (X^{j+k}, \widehat{X}^{j-1}) - X_{j+k+1}^T$  holds for  $j = 1, \dots, (T - k - 1)$ .

By the  $k$ -th order Markov property of the sources, we have  $X^j - X_{j+1}^{j+k} - X_{j+k+1}$ . The MMSE estimate of  $X_{j+k+1}$  based on  $X^{j+k}$  is given by

$$X_{j+k+1} = \sum_{m=1}^k a_m X_{j+m} + N \quad (7.16)$$

where  $N$  is a Gaussian random variable which is independent of  $X^{j+k}$ , and  $\{a_m\}$  are the coefficients of the MMSE estimate. By arguments which are similar to those used to show the independence of random variables in the proof of Corollary 3.2, it can be shown that  $N$  is independent of  $\widehat{X}^j$ . Therefore the best estimate of  $X_{j+k+1}$  based on  $\{X_{j+k}, \widehat{X}^j\}$  is still formula (7.16). It follows that the Markov chain  $(X^j, \widehat{X}^j) - X_{j+1}^{j+k} - X_{j+k+1}$  holds which in turn implies that  $\widehat{X}_j - (X^{j+k}, \widehat{X}^{j-1}) - X_{j+k+1}^T$  holds and completes the proof. ■

This corollary shows that for the  $k$ -th order Gauss-Markov sources, the JC sum-rate-MSE performance is achieved by the  $k$ -stage delayed C-NC system. Let  $\mathcal{D}_d$  denote the distortion region for which the  $d$ -stage delayed C-NC sum-rate matches the JC rate for  $k$ -th order Gauss-Markov sources and MSE. This region keeps expanding with delay,

$$\mathcal{D}_k \subseteq \mathcal{D}_{k+1} \subseteq \dots \subseteq \mathcal{D}_{T-1} = \{\mathbf{R}^+\}^T.$$

The last equality is because the JC system itself has  $(T-1)$ -stage delay.

### VIII. GENERAL NC-NC RESULTS

We can consider the general NC-NC systems with  $k_1$ -stage delay on the encoder side and  $k_2$ -stage delay on the decoder side. C-NC and NC-C systems are special cases when  $k_1 = 0$  and  $k_2 = 0$ , respectively. As an example, in Fig. 7, we present the diagram of an NC-NC system with one-stage encoding delay and one-stage decoding delay ( $T = 4, k_1 = k_2 = 1$ ). Although NC-NC systems appear to be structurally more complex, we can relate the rate-distortion region of NC-NC systems to that of the C-NC systems using structural arguments as in Section VI. Denoting the rate region of the NC-NC systems described above by  $\mathcal{R}_{k_1, k_2}^{NC-NC}$ , we have the following result which is similar to parts (ii) and (iii) of Theorem 4.

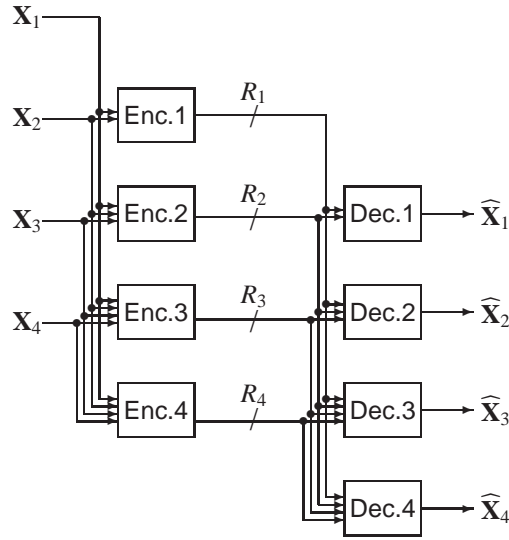


Fig. 7. A 4-stage NC-NC system with 1-stage encoding delay and 1-stage decoding delay. It has the same sum-rate-distortion performance as the system in Fig. 6.

**Theorem 6** (Relationship between general NC-NC and C-NC rate regions) For any distortion tuple  $\mathbf{D}$ ,

- (i)  $(R_1, \dots, R_T, \mathbf{D}) \in \mathcal{R}_{k_1, k_2}^{NC-NC} \Rightarrow (0, \dots, 0, R_1, \dots, R_{T-k_1-1}, \sum_{j=T-k_1}^T R_j, \mathbf{D}) \in \mathcal{R}_{k_1+k_2}^{C-NC}$
- (ii)  $(R_1, \dots, R_T, \mathbf{D}) \in \mathcal{R}_{k_1+k_2}^{C-NC} \Rightarrow (\sum_{j=1}^{k_1+1} R_j, R_{k_1+2}, \dots, R_T, 0, \dots, 0, \mathbf{D}) \in \mathcal{R}_{k_1, k_2}^{NC-NC}$ .

where both sequences of zeros contains  $k_1$  zeros.

This result can be proved by noting that the first NC–NC encoder can be replaced by the combination of the first  $k_1$  C–NC encoders, and the last C–NC encoder can be replaced by the combination of the last  $k_1$  NC–NC encoders, without affecting the reproduction of frames. As a consequence of this theorem, we have an exact equivalence between the sum-rates of the NC–NC and the C–NC systems.

**Corollary 6.1** (*Sum-rate equivalence between NC–NC and C–NC*) The minimum sum-rates of the  $(k_1, k_2)$ -stage delayed NC–NC systems and the  $(k_1 + k_2)$ -stage delayed C–NC systems are equal:

$$R_{k_1, k_2}^{NC-NC}(\mathbf{D}) = R_{k_1 + k_2}^{C-NC}(\mathbf{D}).$$

In conclusion, for any two delayed sequential coding systems, when the *sums* of the encoding frame-delay and decoding frame-delay are equal, they have the same sum-rate-distortion performance. For example, the NC–NC system in Fig. 7 has the same minimum sum-rate as the 2-stage delayed C–NC system in Fig. 6. Therefore we can always take the C–NC system as a representative of all the delayed sequential coding systems.

## IX. CONCLUDING REMARKS

In this paper, motivated by video coding applications, we studied the problem of sequential coding of correlated sources with encoding and/or decoding frame-delays and characterized the fundamental tradeoffs between individual frame rates, individual frame distortions, and encoding/decoding frame-delays in terms of single-letter information-theoretic quantities. Our characterization of the rate-distortion region was for multiple sources, general inter-frame source correlations, and general frame-specific and coupled single-letter fidelity criteria. The main message of this study is that even a single frame-delay holds potential for yielding significant performance improvements in sequential coding problems, sometimes even matching the joint coding performance.

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## APPENDIX I

### COROLLARY 1.2 PROOF

We first show that the right hand side of (3.6) is an upper bound of  $R_{sum}^{C-CGM}$  by defining the auxiliary random variables satisfying all the constraints in (3.4) and evaluating the objective function of (3.4). Then we show that the right hand side of (3.6) is also a lower bound of  $R_{sum}^{C-CGM}$  using information inequalities.

*Upper bound:* Due to the Markov chains in (3.4),

$$I(X^3; \widehat{X}^3) = I(X_1; \widehat{X}_1) + I(X^2; \widehat{X}_2 | \widehat{X}_1) + I(X^3; \widehat{X}_3 | \widehat{X}^2), \quad (\text{I.1})$$

where on the right hand side, each term corresponds to a stage of coding. We will sequentially define  $\widehat{X}_1, \widehat{X}_2, \widehat{X}_3$  and evaluate the expression stage by stage to highlight the structure of the optimal (achievable) coding scheme.

At first, since  $D_1 \leq \sigma_1^2$ , we can find a random variable  $\widehat{X}_1$  such that: (1)  $\widehat{X}_1 + Z_1 = X_1$ , (2)  $\widehat{X}_1$  and  $Z_1$  are independent Gaussian variables with variances  $(\sigma_1^2 - D_1)$  and  $D_1$  respectively, and (3) the Markov chain  $\widehat{X}_1 - X_1 - X_2^3$  holds. The MSE constraint  $E(X_1 - \widehat{X}_1)^2 \leq D_1$  is satisfied because  $E(Z_1^2) = D_1$ . Note that the distribution of  $(X_1, \widehat{X}_1)$  achieves the rate-distortion function for the Gaussian source  $X_1$  and we have

$$I(X_1; \widehat{X}_1) = \frac{1}{2} \log \left( \frac{\sigma_1^2}{D_1} \right). \quad (\text{I.2})$$

Since  $(X_1, X_2)$  are jointly Gaussian, we have

$$X_2 = \rho_1 \frac{\sigma_2}{\sigma_1} X_1 + N_1 = \rho_1 \frac{\sigma_2}{\sigma_1} \widehat{X}_1 + W_2,$$

where  $N_1$  is a Gaussian variable with variance  $(1 - \rho_1^2)\sigma_2^2$  which is independent of  $(\widehat{X}_1, Z_1)$ .  $W_2 := (\rho_1 \frac{\sigma_2}{\sigma_1} Z_1 + N_1)$  is the innovation from  $\widehat{X}_1$  to  $X_2$ , whose variance  $\sigma_{W_2}^2$  is given in (3.5). When  $D_2 \leq \sigma_{W_2}^2$ , we can find a random variable  $\widehat{W}_2$  such that: (1)  $\widehat{W}_2 + Z_2 = W_2$ , (2)  $\widehat{W}_2$  and  $Z_2$  are independent Gaussian variables with variances  $(\sigma_{W_2}^2 - D_2)$  and  $D_2$  respectively, and (3) the Markov chain  $\widehat{W}_2 - (X_2, \widehat{X}_1) - (X_1, X_3)$  holds. Define  $\widehat{X}_2 := (\rho_1 \frac{\sigma_2}{\sigma_1} \widehat{X}_1 + \widehat{W}_2)$ , which implies  $X_2 = (\widehat{X}_2 + Z_2)$ . The MSE constraint  $E(X_2 - \widehat{X}_2)^2 \leq D_2$  is satisfied because  $E(Z_2^2) = D_2$ . The Markov chain constraint  $\widehat{X}_2 - (X_2, \widehat{X}_1) - X_3$  is also satisfied. Note that the distribution of  $(W_2, \widehat{W}_2)$  achieves the rate-distortion function for the Gaussian source  $W_2$  and we have

$$I(X^2; \widehat{X}_2 | \widehat{X}_1) = I(X_2; \widehat{X}_2 | \widehat{X}_1) = \frac{1}{2} \log \left( \frac{\sigma_{W_2}^2}{D_2} \right), \quad (\text{I.3})$$

where the first step is because  $\widehat{X}_2 - (X_2, \widehat{X}_1) - X_1$  forms a Markov chain.

Similarly, we can define  $\widehat{X}_3$  such that when  $D_3 \leq \sigma_{W_3}^2$ ,

$$I(X^3; \widehat{X}_3 | \widehat{X}^2) = \frac{1}{2} \log \left( \frac{\sigma_{W_3}^2}{D_3} \right). \quad (\text{I.4})$$

Finally, combining (I.1)-(I.4) and (3.4), when  $\mathbf{D} \in \mathcal{D}^{C-C}$ , we have

$$R^{C-CGM}(\mathbf{D}) \leq \frac{1}{2} \log \left( \frac{\sigma_1^2}{D_3} \right) + \frac{1}{2} \log \left( \frac{\sigma_{W_2}^2}{D_2} \right) + \frac{1}{2} \log \left( \frac{\sigma_{W_3}^2}{D_3} \right).$$

*Lower bound:* For any choice of  $\widehat{X}^3$  satisfying the constraints in (3.4), we have

$$\begin{aligned} R^{C-CGM}(\mathbf{D}) &= \min [I(X_1; \widehat{X}_1) + I(X^2; \widehat{X}_2 | \widehat{X}_1) + I(X^3; \widehat{X}_3 | \widehat{X}^2)] \\ &\geq \min [I(X_1; \widehat{X}_1) + I(X_2; \widehat{X}_2 | \widehat{X}_1) + I(X_3; \widehat{X}_3 | \widehat{X}^2)] \\ &= \min [h(X_1) - h(X_1 | \widehat{X}_1) + h(X_2 | \widehat{X}_1) - h(X_2 | \widehat{X}^2) \\ &\quad + h(X_3 | \widehat{X}^2) - h(X_3 | \widehat{X}^3)] \\ &\geq h(X_1) + \min [h(X_2 | \widehat{X}_1) - h(X_1 | \widehat{X}_1) \\ &\quad + h(X_3 | \widehat{X}^2) - h(X_2 | \widehat{X}^2) - h(X_3 - \widehat{X}_3)] \\ &\geq \frac{1}{2} \log(2\pi e \sigma_1^2) - \frac{1}{2} \log(2\pi e D_3) + \min [h(X_2 | \widehat{X}_1) \\ &\quad - h(X_1 | \widehat{X}_1)] + \min [h(X_3 | \widehat{X}^2) - h(X_2 | \widehat{X}^2)], \end{aligned} \quad (\text{I.5})$$

where all the minimizations above are subject to all the constraints in (3.4). By Lemma 5 in [9], since the Markov chain  $\widehat{X}_1 - X_1 - X_2$  holds and  $h(X_1 | \widehat{X}_1) \leq \frac{1}{2} \log(2\pi e D_1)$  we have

$$\min [h(X_2 | \widehat{X}_1) - h(X_1 | \widehat{X}_1)] \geq \frac{1}{2} \log \left( \frac{\sigma_{W_2}^2}{D_1} \right). \quad (\text{I.6})$$

Similarly, since the Markov chain  $\widehat{X}^2 - X_2 - X_3$  holds and  $h(X_2 | \widehat{X}^2) \leq h(X_2 | \widehat{X}_2) \leq \frac{1}{2} \log(2\pi e D_2)$ , replacing  $(X_1, X_2, \widehat{X}_1)$  by  $(X_2, X_3, \widehat{X}^2)$  respectively in (I.6), we have

$$\min [h(X_3 | \widehat{X}^2) - h(X_2 | \widehat{X}^2)] \geq \frac{1}{2} \log \left( \frac{\sigma_{W_3}^2}{D_2} \right). \quad (\text{I.7})$$



Using (I.6) and (I.7) in (I.5), we have

$$\begin{aligned} R^{C-CGM}(\mathbf{D}) &\geq \frac{1}{2} \log \left( \frac{\sigma_1^2}{D_3} \right) + \frac{1}{2} \log \left( \frac{\sigma_{w_2}^2}{D_1} \right) + \frac{1}{2} \log \left( \frac{\sigma_{w_3}^2}{D_2} \right) \\ &= \frac{1}{2} \log \left( \frac{\sigma_1^2}{D_1} \right) + \frac{1}{2} \log \left( \frac{\sigma_{w_2}^2}{D_2} \right) + \frac{1}{2} \log \left( \frac{\sigma_{w_3}^2}{D_3} \right). \end{aligned}$$

In conclusion, the right hand side of the above formula is both an upper bound and a lower bound of  $R^{C-CGM}(\mathbf{D})$ , and is thus equal to  $R^{C-CGM}(\mathbf{D})$ . ■

## APPENDIX II

### THEOREM 3 FORWARD PROOF

For any tuple  $(\mathbf{R}, \mathbf{D})$  belonging to the right hand side of (5.11), there exist random variables  $U^2, \widehat{X}^3$  and a function  $g_1$  such that all the constraints in (5.11) are satisfied. We will describe the encoders and decoders with parameters  $(M^3, R'_2, R'_3, \epsilon_1)$  in Subsections I to III. In Subsection IV and V, we choose the values of the parameters and analyze the rates and distortions to show that (2.1) and (2.2) hold for every  $\epsilon > 0$  and sufficiently large  $n$ .

#### I Generation of codebooks

- 1) Randomly generate a codebook  $C_1$  consisting of  $M_1$  sequences (codewords) of length  $n$  drawn iid  $\sim \prod_{i=1}^n p_{U_1}(u_1(i))$ . Index the codewords by  $s_1 \in \{1, 2, \dots, M_1\}$ . Denote the  $s_1$ -th codeword by  $\mathbf{U}_1(s_1)$ .
- 2) Randomly generate a codebook  $C_2$ , independently of  $C_1$ , consisting of  $2^{nR'_2}$  sequences (codewords) of length  $n$  drawn iid  $\sim \prod_{i=1}^n p_{U_2}(u_2(i))$ . Index the codewords by  $s'_2 \in \{1, 2, \dots, 2^{nR'_2}\}$ . Denote the  $s'_2$ -th codeword by  $\mathbf{U}_2(s'_2)$ . Then randomly assign the indices of the codewords to one of  $M_2$  bins according to a uniform distribution on  $\{1, 2, \dots, M_2\}$ , where  $M_2 \leq 2^{nR'_2}$ . Let  $\mathcal{B}_2(s_2)$  denote the set of indices assigned to the  $s_2$ -th bin.
- 3) Randomly generate a codebook  $C_3$ , independently of  $(C_1, C_2)$ , consisting of  $2^{nR'_3}$  sequences (codewords) of length  $n$  drawn iid  $\sim \prod_{i=1}^n p_{\widehat{X}_2, \widehat{X}_3}(\widehat{x}_2(i), \widehat{x}_3(i))$ . Note that each component of each codeword is a tuple  $(\widehat{x}_2(i), \widehat{x}_3(i)) \in \widehat{X}_2 \times \widehat{X}_3$ . Index the codewords by  $s'_3 \in \{1, 2, \dots, 2^{nR'_3}\}$ . Denote the  $s'_3$ -th codeword by  $\widehat{\mathbf{X}}_2^3(s'_3)$ . Then randomly assign the indices to one of  $M_3$  bins according to a uniform distribution on  $\{1, 2, \dots, M_3\}$ , where  $M_3 \leq 2^{nR'_3}$ . Let  $\mathcal{B}_3(s_3)$  denote the set of indices assigned to the  $s_3$ -th bin.

Reveal all the codebooks to all the encoders and the decoders.

#### II Encoding

- 1) Given a source sequence  $\mathbf{X}_1$ , encoder-1 looks for a codeword  $\mathbf{U}_1(s_1)$  in  $C_1$  such that  $(\mathbf{X}_1, \mathbf{U}_1(s_1)) \in A_{\epsilon_1}^{*(n)}(p_{X_1 U_1})$  where  $\epsilon_1 > 0$  and  $A_{\epsilon_1}^{*(n)}(p_{X_1 U_1})$  is the  $\epsilon_1$ -strong typical set of length  $n$  with respect to the joint distribution  $p_{X_1, U_1}$  [16]. For simplicity, we will not indicate either the distribution or the length of sequence in the definition of a strong typical set if there is no ambiguity. If no such codeword can be found, set  $s_1 = 1$ . If more than one such codeword exists, pick the one with the smallest index  $s_1$ . Encoder-1 sends  $s_1$  as the message.
- 2) Given sequences  $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{U}_1(s_1)\}$ , encoder-2 looks for a codeword  $\mathbf{U}_2(s'_2)$  in  $C_2$  such that  $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{U}_1(s_1), \mathbf{U}_2(s'_2)) \in A_{\epsilon_1}^*$ . If no such codeword exists, set  $s'_2 = 1$ . If more than one such codeword exists, pick the one with the smallest  $s'_2$ . Encoder-2 sends the bin index  $s_2$  such that  $s'_2 \in \mathcal{B}_2(s_2)$ .
- 3) Given sequences  $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{U}_1(s_1), \mathbf{U}_2(s'_2)\}$ , encoder-3 looks for a codeword  $\widehat{\mathbf{X}}_2^3(s'_3)$  in codebook  $C_3$  such that  $(\mathbf{X}^3, \mathbf{U}_1(s_1), \mathbf{U}_2(s'_2), \widehat{\mathbf{X}}_2^3(s'_3)) \in A_{\epsilon_1}^*$ . If no such codeword exists, set  $s'_3 = 1$ . If more than one such codeword exists, pick the one with the smallest  $s'_3$ . Encoder-3 sends the bin index  $s_3$  such that  $s'_3 \in \mathcal{B}_3(s_3)$ .

### III Decoding

- 1) Given the received indices  $s^2$ , decoder-1 looks for a sequence  $\mathbf{U}_2(\hat{s}'_2)$  such that  $\hat{s}'_2 \in \mathcal{B}_2(s_2)$  and  $(\mathbf{U}_1(s_1), \mathbf{U}_2(\hat{s}'_2)) \in A_{\epsilon_1}^*$ . If more than one such sequence exists, pick the one with the smallest  $\hat{s}'_2$ . Generate the reproduction sequence  $\widehat{\mathbf{X}}_1$  by

$$\widehat{X}_1(i) = g_1(U_1(s_1, i), U_2(\hat{s}_2, i)), \quad i = 1, \dots, n,$$

where  $\widehat{X}_1(i)$ ,  $U_1(s_1, i)$ , and  $U_2(\hat{s}_2, i)$  are the  $i$ -th components of the sequences  $\widehat{\mathbf{X}}_1$ ,  $\mathbf{U}_1(s_1)$ , and  $\mathbf{U}_2(\hat{s}_2)$  respectively.

- 2) Given the received indices  $s^3$  and previously decoded index  $\hat{s}'_2$ , decoder-2 looks for a sequence  $\widehat{\mathbf{X}}_2^3(\hat{s}'_3)$  such that  $\hat{s}'_3 \in \mathcal{B}_3(s_3)$  and  $(\mathbf{U}_1(s_1), \mathbf{U}_2(\hat{s}'_2), \widehat{\mathbf{X}}_2^3(\hat{s}'_3)) \in A_{\epsilon_1}^*$ . If more than one such sequence exists, pick the one with the smallest  $\hat{s}'_3$ . Separate  $\widehat{\mathbf{X}}_2^3(\hat{s}'_3)$  (note that each component is a tuple) to get the reproduction sequences  $\widehat{\mathbf{X}}_2(\hat{s}'_3)$  and  $\widehat{\mathbf{X}}_3(\hat{s}'_3)$ . This decoder is conceptually the combination of decoder-2 and 3 in Fig. 3(a).

### IV Analysis of probabilities of error events

Let us consider the following “error events”  $\mathcal{E}_1$  through  $\mathcal{E}_{11}$ . If none of them happens, the decoders successfully reproduce what the encoders intend to send, and the expected distortions are closed to  $E[d_j(X_j, \widehat{X}^j)]$ , which is not greater than  $D_j$ . Otherwise, if any event happens, the decoders may make mistakes on reproduction, and we will bound the distortions by the worst case distortion  $d_{j,\max}$ .

- $\mathcal{E}_1$ : (Frame-1 not typical)  $\mathbf{X}_1 \notin A_{\epsilon_1}^*$ .

$Pr(\mathcal{E}_1) \rightarrow 0$  as  $n \rightarrow \infty$  by the strong law of large numbers.

- $\mathcal{E}_2$ : (Encoder-1 fails to find a codeword) Given any (deterministic) sequence  $\mathbf{x}_1 \in A_{\epsilon_1}^*$ ,  $\nexists s_1$  such that  $(\mathbf{x}_1, \mathbf{U}_1(s_1)) \in A_{\epsilon_1}^*$ .

By [16, Lemma 13.6.2, p.359], for any typical sequence  $\mathbf{x}_1$  and each codeword  $\mathbf{U}_1(s_1)$  which is randomly generated iid according to  $p_{U_1}$ , we have

$$2^{-n(I(X_1; U_1) + \epsilon_2)} \leq Pr((\mathbf{x}_1, \mathbf{U}_1(s_1)) \in A_{\epsilon_1}^*) \leq 2^{-n(I(X_1; U_1) - \epsilon_2)},$$

where  $\epsilon_2$  depends on  $\epsilon_1$  and  $n$ , and  $\epsilon_2 \rightarrow 0$  as  $\epsilon_1 \rightarrow 0$  and  $n \rightarrow \infty$ . Therefore we have

$$\begin{aligned} Pr(\mathcal{E}_2) &= (1 - Pr((\mathbf{x}_1, \mathbf{U}_1(1)) \in A_{\epsilon_1}^*))^{M_1} \\ &\leq \exp(-M_1 2^{-n(I(X_1; U_1) + \epsilon_2)}), \end{aligned}$$

where the inequality is because  $(1 - x)^n \leq \exp(-nx)$ . Let

$$M_1 := 2^{n(R_1 + \epsilon_1 + \epsilon_2)}.$$

Since  $R_1 \geq I(X_1; U_1)$ , we have

$$Pr(\mathcal{E}_2) \leq \exp(-2^{n(R_1 + \epsilon_1 - I(X_1; U_1))}) \leq \exp(-2^{n\epsilon_1})$$

which goes to zero as  $n \rightarrow \infty$ .

- $\mathcal{E}_3$ : (Message-1 not jointly typical with frame-2) Given any sequences  $(\mathbf{x}_1, \mathbf{u}_1(s_1)) \in A_{\epsilon_1}^*$ ,  $(\mathbf{x}_1, \mathbf{X}_2, \mathbf{u}_1(s_1)) \notin A_{\epsilon_1}^*$ .

Using the Markov lemma [16, Lemma 14.8.1, p.436], since the Markov chain  $U_1 - X_1 - X_2$  holds and  $\mathbf{X}_2$  is drawn iid  $\sim p_{X_2|X_1}$ ,  $Pr((\mathbf{x}_1, \mathbf{X}_2, \mathbf{u}_1(s_1)) \notin A_{\epsilon_1}^*) \leq \epsilon_1$  for  $n$  sufficiently large. Therefore  $Pr(\mathcal{E}_3) \rightarrow 0$  as  $\epsilon_1 \rightarrow 0$ , and  $n \rightarrow \infty$ .

- $\mathcal{E}_4$ : (Encoder-2 fails to find a codeword) Given any sequences  $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{u}_1(s_1)) \in A_{\epsilon_1}^*$ ,  $\nexists s'_2$  such that  $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{u}_1(s_1), \mathbf{U}_2(s'_2)) \in A_{\epsilon_1}^*$ .

By arguments which are similar to those used in the analysis of  $\mathcal{E}_2$ , we have

$$Pr(\mathcal{E}_4) \leq \exp\left(-2^{n(R'_2 - I(X^2 U_1; U_2) - \epsilon_3)}\right),$$

where  $\epsilon_3 \rightarrow 0$  as  $\epsilon_1 \rightarrow 0$  and  $n \rightarrow \infty$ . Let

$$R'_2 := I(X^2 U_1; U_2) + \epsilon_1 + \epsilon_3.$$

We have

$$Pr(\mathcal{E}_4) \leq \exp(-2^{n\epsilon_1}),$$

which goes to zero as  $n \rightarrow \infty$ .

- $\mathcal{E}_5$ : (*Encoder-2's bin size too large*) Given that  $s'_2 \in \mathcal{B}_2(s_2)$ , the cardinality of the  $s_2$ -th bin satisfies

$$|\mathcal{B}_2(s_2)| \geq \frac{2^{n(R'_2 + \epsilon_1)}}{M_2} + 1.$$

Because  $s'_2 \in \mathcal{B}_2(s_2)$  and the other  $(2^{nR'_2} - 1)$  codewords are randomly assigned,  $(|\mathcal{B}_2(s_2)| - 1)$  follows the binomial distribution with parameters  $(2^{nR'_2} - 1, 1/M_1)$ . We will use the following Chernoff bound [22, Thm 4.4(3), p.64]: For a binomial random variable  $X$  with parameters  $(n, p)$ , if  $a \geq 6np$ , then  $Pr(X \geq a) \leq 2^{-a}$ . When  $n\epsilon_1 > 3$  which guarantees  $2^{n\epsilon_1} > 6$ , taking  $a := 2^{R'_2 + \epsilon_1}/M_2$ , we have

$$Pr\left(|\mathcal{B}_2(s_2)| - 1 \geq \frac{2^{n(R'_2 + \epsilon_1)}}{M_2}\right) \leq 2^{-\frac{2^{n(R'_2 + \epsilon_1)}}{M_2}}.$$

Since  $M_2 \leq 2^{nR'_2}$ , we have  $Pr(\mathcal{E}_5) \rightarrow 0$  as  $n \rightarrow \infty$ .

- $\mathcal{E}_6$ : (*Decoder-1 fails to identify the correct codeword from the bin*) In the bin  $\mathcal{B}_2(s_2)$  whose size is not greater than  $(2^{n(R'_2 + \epsilon_1)}/M_2 + 1)$ , Given any sequence  $\mathbf{u}_1(s_1) \in A_{\epsilon_1}^*$ ,  $\exists \hat{s}'_2 \neq s'_2$  such that  $(\mathbf{u}_1(s_1), \mathbf{u}_2(\hat{s}'_2)) \in A_{\epsilon_1}^*$ .

By arguments which are similar to those used in the analysis of  $\mathcal{E}_2$ , we have

$$Pr((\mathbf{u}_1(s_1), \mathbf{u}_2(\hat{s}'_2)) \in A_{\epsilon_1}^*) \leq 2^{-n(I(U_1; U_2) - \epsilon_4)},$$

where  $\epsilon_4 \rightarrow 0$  as  $\epsilon_1 \rightarrow 0$  and  $n \rightarrow \infty$ . By the union bound,

$$\begin{aligned} Pr(\mathcal{E}_6) &\leq (|\mathcal{B}_2(s_2)| - 1) 2^{-n(I(U_1; U_2) - \epsilon_4)} \\ &\leq 2^{n(R'_2 + \epsilon_1 - I(U_1; U_2) + \epsilon_4)} / M_2. \end{aligned}$$

Recall that

$$R'_2 = I(X^2 U_1; U_2) + \epsilon_1 + \epsilon_3.$$

Let

$$M_2 := 2^{n(R_2 + 3\epsilon_1 + \epsilon_3 + \epsilon_4)}.$$

Due to the fact that  $R_2 \geq I(X^2; U_2|U_1)$ , we can simplify the bound to

$$Pr(\mathcal{E}_6) \leq 2^{n(I(X^2; U_2|U_1) - R_2 - \epsilon_1)} \leq 2^{-n\epsilon_1},$$

which goes to zero as  $n \rightarrow \infty$ .

- $\mathcal{E}_7$ : (*Frame-3 not jointly typical with previous messages*) Given any sequences  $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{u}_1(s_1), \mathbf{u}_2(s'_2)) \in A_{\epsilon_1}^*$ ,  $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{u}_1(s_1), \mathbf{u}_2(s'_2), \mathbf{X}_3) \notin A_{\epsilon_1}^*$ .

By arguments which are similar to those used in the analysis of  $\mathcal{E}_3$ , the Markov chain  $U^2 - X^2 - X_3$  implies that  $Pr(\mathcal{E}_7) \rightarrow 0$  as  $\epsilon_1 \rightarrow 0$  and  $n \rightarrow \infty$ .

- $\mathcal{E}_8$ : (*Encoder-3 fails to find a codeword*) Given any sequences  $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{u}_1(s_1), \mathbf{u}_2(s'_2)) \in A_{\epsilon_1}^*$ ,  $\nexists s'_3$  such that  $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{u}_1(s_1), \mathbf{u}_2(s'_2), \widehat{\mathbf{X}}_2^3(s'_3)) \in A_{\epsilon_1}^*$ .

By arguments which are similar to those used in the analysis of  $\mathcal{E}_2$ , we have

$$Pr(\mathcal{E}_8) \leq \exp\left(-2^{n(R'_3 - I(X^3 U^2; \widehat{X}_2^3) - \epsilon_5)}\right),$$

where  $\epsilon_5 \rightarrow 0$  as  $\epsilon_1 \rightarrow 0$  and  $n \rightarrow \infty$ . Let

$$R'_3 := I(X^3 U^2; \widehat{X}_2^3) + \epsilon_1 + \epsilon_5.$$

We have

$$Pr(\mathcal{E}_8) \leq \exp(-2^{n\epsilon_1}),$$

which goes to zero when  $n \rightarrow \infty$ .

- $\mathcal{E}_9$ : (*Encoder-3's bin size too large*) Given that  $s'_3 \in \mathcal{B}_3(s_3)$ , the cardinality of the bin satisfies

$$|\mathcal{B}_3(s_3)| > \frac{2^{n(R'_3 + \epsilon_1)}}{M_3} + 1.$$

By arguments which are similar to those used in the analysis of  $\mathcal{E}_5$ , we can argue that  $Pr(\mathcal{E}_9) \rightarrow 0$  as  $n \rightarrow \infty$ .

- $\mathcal{E}_{10}$ : (*Decoder-2 fails to identify the correct codeword from the bin*) In the bin  $\mathcal{B}_3(s_3)$  whose size is not greater than  $(2^{n(R'_3 + \epsilon_1)}/M_3 + 1)$ , given any sequences  $(\mathbf{u}_1(s_1), \mathbf{u}_2(s'_2)) \in A_{\epsilon_1}^*$ ,  $\exists \hat{s}'_3 \neq s'_3$  such that  $(\mathbf{u}_1(s_1), \mathbf{u}_2(s'_2), \widehat{\mathbf{X}}_2^3(\hat{s}'_3)) \in A_{\epsilon_1}^*$ .

By arguments which are similar to those used in the analysis of  $\mathcal{E}_6$ , we have

$$Pr(\mathcal{E}_{10}) \leq 2^{n(I(X^3; \widehat{X}_2^3|U^2) + 2\epsilon_1 + \epsilon_5 + \epsilon_6)}/M_3,$$

where  $\epsilon_6 \rightarrow 0$  as  $\epsilon_1 \rightarrow 0$  and  $n \rightarrow \infty$ . Let

$$M_3 := 2^{n(R_3 + 3\epsilon_1 + \epsilon_5 + \epsilon_6)}.$$

Due to the fact that  $R_3 \geq I(X^3; \widehat{X}_2^3|U^2)$ , we have

$$Pr(\mathcal{E}_{10}) \leq 2^{n\epsilon_1},$$

which goes to zero as  $n \rightarrow \infty$ .

- $\mathcal{E}_{11}$ : (*Reproduction of frame-1 not jointly typical with other sequences*) Given any sequences  $(\mathbf{x}^3, \widehat{\mathbf{x}}_2^3(s'_3), \mathbf{u}_1(s_1), \mathbf{u}_2(s'_2)) \in A_{\epsilon_1}^*$  and a correct decoding  $\hat{s}'_2 = s'_2$ ,  $(\mathbf{x}^3, \widehat{\mathbf{x}}_2^3(s'_3), \mathbf{u}_1(s_1), \mathbf{u}_2(s'_2), \widehat{\mathbf{x}}_1) \notin A_{\epsilon_1}^*$ .

Although  $\widehat{\mathbf{x}}_1$  depends on  $(\mathbf{u}_1(s_1), \mathbf{u}_2(s'_2))$  deterministically by the function  $g_1$ , we can regard  $p_{\widehat{\mathbf{x}}_1|U_1, U_2}$  as a degraded probability distribution and use the Markov lemma and the trivial Markov chain  $(X^3, \widehat{X}_2^3) - U^2 - g_1(U^2)$  to show that  $Pr(\mathcal{E}_{11}) \rightarrow 0$  as  $\epsilon_1 \rightarrow 0$  and  $n \rightarrow \infty$ .

### V Analysis of the distortions

Consider the union of all the above events  $\mathcal{E} := \bigcup_{i=1}^{11} \mathcal{E}_i$ . When the codebooks are randomly generated according to Subsection I, since  $Pr(\mathcal{E}_i)$  vanishes for  $i = 1, \dots, 11$  as  $\epsilon_1 \rightarrow 0$ , and  $n \rightarrow \infty$ ,  $Pr(\mathcal{E})$  also vanishes. Therefore there must exists a sequence of codebooks  $\{(C_{1,l}, C_{2,l}, C_{3,l})\}_{l=1}^\infty$  for which  $Pr(\mathcal{E}) \rightarrow 0$  (the randomness comes from the generation of source sequences). We will focus on these codebooks in the following discussion.

In the case that  $\mathcal{E}$  does not happen, all the sequences are jointly  $\epsilon_1$ -strong typical:  $(\mathbf{X}^3, \mathbf{U}_1(s_1), \mathbf{U}_2(s'_2), \widehat{\mathbf{X}}_1, \widehat{\mathbf{X}}_2^3(s'_3)) \in A_{\epsilon_1}^*$ , and the decoded indices are correct:  $\hat{s}'_2 = s'_2, \hat{s}'_3 = s'_3$ . Since the expected distortion is a continuous function of the joint distribution, strong typicality implies distortion typicality. In other words, we have

$$|E[d_j^{(n)}(\mathbf{X}_j, \widehat{\mathbf{X}}^j)|\mathcal{E}^c] - E[d_j(X_j, \widehat{X}^j)]| < \epsilon_{d_j}, \quad j = 1, 2, 3,$$

where  $\epsilon_{d_j} \rightarrow 0$  as  $\epsilon_1 \rightarrow 0$ . Since  $D_j \geq E[d_j(X_j, \widehat{X}^j)]$ , we have  $E[d_j^{(n)}(\mathbf{X}_j, \widehat{\mathbf{X}}^j)|\mathcal{E}^c] \leq D_j + \epsilon_{d_j}$ ,  $j = 1, 2, 3$ .

In the case that  $\mathcal{E}$  does happen, by the definition of  $d_{j,\max}$ , we have  $E[d_j^{(n)}(\mathbf{X}_j, \widehat{\mathbf{X}}^j)|\mathcal{E}] \leq d_{j,\max}$ ,  $j = 1, 2, 3$ .

Therefore the expected distortion for the  $j$ -th frame is,

$$\begin{aligned} E[d_j^{(n)}(\mathbf{X}_j, \widehat{\mathbf{X}}^j)] &= E[d_j^{(n)}(\mathbf{X}_j, \widehat{\mathbf{X}}^j)|\mathcal{E}]Pr(\mathcal{E}) \\ &\quad + E[d_j^{(n)}(\mathbf{X}_j, \widehat{\mathbf{X}}^j)|\mathcal{E}^c](1 - Pr(\mathcal{E})) \\ &\leq d_{j,\max}Pr(\mathcal{E}) + E[d_j(\mathbf{X}_j, \widehat{\mathbf{X}}^j)|\mathcal{E}^c] \\ &\leq d_{j,\max}Pr(\mathcal{E}) + D_j + \epsilon_{d_j} \end{aligned}$$

When  $\epsilon_1 \rightarrow 0$ , and  $n \rightarrow \infty$ , for codebooks  $\{(C_{1,l}, C_{2,l}, C_{3,l})\}_{l=1}^\infty$ ,  $Pr(\mathcal{E})$  and all the  $\epsilon$  variables vanish. Therefore  $\forall \epsilon > 0$ , by driving the variables to their limits, we can always find  $\epsilon_1 > 0$  for sufficiently large  $n$ , such that

$$\begin{aligned} \frac{1}{n} \log M_1 - R_1 &= \epsilon_1 + \epsilon_2 < \epsilon, \\ \frac{1}{n} \log M_2 - R_2 &= 3\epsilon_1 + \epsilon_3 + \epsilon_4 < \epsilon, \\ \frac{1}{n} \log M_3 - R_3 &= 3\epsilon_1 + \epsilon_5 + \epsilon_6 < \epsilon, \\ E[d_j^{(n)}(\mathbf{X}_j, \widehat{\mathbf{X}}^j)] - D_j &\leq d_{j,\max}Pr(\mathcal{E}) + \epsilon_{d_j} < \epsilon. \end{aligned}$$

Therefore (2.1) and (2.2) hold, which completes the proof. ■

### APPENDIX III

#### THEOREM 3 CONVERSE PROOF

##### I Information equalities

If a rate-distortion-tuple  $(\mathbf{R}, \mathbf{D}) = (R_1, \dots, R_T, D_1, \dots, D_T)$  is admissible for the 3-stage C-NC system, then  $\forall \epsilon > 0$ , there exists  $N(\epsilon)$ , such that  $\forall n > N(\epsilon)$  we have blocklength  $n$  encoders and decoders  $\{f_1^{(n)}, f_2^{(n)}, f_3^{(n)}, g_1^{(n)}, g_2^{(n)}, g_3^{(n)}\}$  satisfying

$$\begin{aligned} E[d_j(\mathbf{X}_j, \widehat{\mathbf{X}}^j)] &\leq D_j + \epsilon, \\ \frac{1}{n} \log M_j &\leq R_j + \epsilon, \quad j = 1, \dots, T. \end{aligned}$$

Denote the messages sent by the three ( $T = 3$ ) encoders respectively by  $S_1, S_2$ , and  $S_3$ , and define the auxiliary random variables by  $U_j(i) := (S_j, X_j(i-))$ ,  $j = 1, 2$ . Due to the structure of the system we have the following Markov chains

$$\begin{aligned} \mathbf{X}_2^3 - \mathbf{X}_1 - S_1, \\ \mathbf{X}_3 - \mathbf{X}^2 - S^2 - \widehat{\mathbf{X}}_1, \end{aligned}$$

which are readily verified. For the first coding rate, we have

$$\begin{aligned}
n(R_1 + \epsilon) &\geq H(S_1) \\
&= H(S_1) - H(S_1|\mathbf{X}_1) \\
&= I(S_1; \mathbf{X}_1) \\
&= \sum_{i=1}^n I(X_1(i); S_1|X_1(i-)) \\
&\stackrel{(a)}{=} \sum_{i=1}^n I(X_1(i); S_1, X_1(i-)) \\
&= \sum_{i=1}^n I(X_1(i); U_1(i))
\end{aligned}$$

Step (a) is because  $(X_1(i))_{i=1}^n$  are iid. The Markov chains  $X_2^3(i) - X_1(i) - U_1(i)$  can be verified to hold for each  $i = 1, \dots, n$ .

In the next stage,

$$\begin{aligned}
n(R_2 + \epsilon) &\geq H(S_2) \\
&\geq H(S_2|S_1) \\
&= H(S_2|S_1) - H(S_2|S_1, \mathbf{X}^2) \\
&= I(S_2; \mathbf{X}^2|S_1) \\
&= \sum_{i=1}^n I(X^2(i); S_2|U_1(i), X_2(i-)) \\
&\stackrel{(b)}{=} \sum_{i=1}^n I(X^2(i); S_2, X_2(i-)|U_1(i)) \\
&= \sum_{i=1}^n I(X^2(i); U_2(i)|U_1(i))
\end{aligned}$$

Step (b) is because the Markov chain  $X^2(i) - U_1(i) - X_2(i-)$  holds for each  $i$ . For each  $i$ ,  $\widehat{X}_1(i)$  is a deterministic function of  $S^2$ , which is itself a deterministic function of  $U^2(i)$ . Therefore there exists a function  $g_{1,i}$  such that  $\widehat{X}_1(i) = g_{1,i}(U^2(i))$  for each  $i$ . The Markov chain  $X_3(i) - (X^2(i), U_1(i)) - U_2(i)$  can also be verified to hold for each  $i = 1, \dots, n$ .

In the final stage,

$$\begin{aligned}
n(R_3 + \epsilon) &\geq H(S_3) \\
&\geq H(S_3|S^2) \\
&= H(S_3|S^2) - H(S_3|S^2, \mathbf{X}^3) \\
&= I(S_3; \mathbf{X}^3|S^2) \\
&= \sum_{i=1}^n I(X^3(i); S_3|U^2(i), X_3(i-)) \\
&\stackrel{(c)}{=} \sum_{i=1}^n I(X^3(i); S_3, X_3(i-)|U^2(i)) \\
&\stackrel{(d)}{\geq} \sum_{i=1}^n I(X^3(i); \widehat{X}_2^3(i)|U^2(i))
\end{aligned}$$

where step (c) is because the Markov chain  $X^3(i) - U^2(i) - X_3(i-)$  holds for each  $i$ . Step (d) is because  $\widehat{X}_2^3(i)$  is a deterministic function of  $\{S_1, S_2, S_3\} \subseteq \{S_3, U_1(i), U_2(i)\}$  for each  $i = 1, \dots, n$ .

Hence we have shown that for any admissible rate-distortion tuple  $(\mathbf{R}, \mathbf{D})$ ,  $\forall \epsilon > 0, \exists N(\epsilon)$  such that for all  $n > N(\epsilon)$ ,

$$\begin{aligned}
R_1 + \epsilon &\geq \frac{1}{n} \sum_{i=1}^n I(X_1(i); U_1(i)), \\
R_2 + \epsilon &\geq \frac{1}{n} \sum_{i=1}^n I(X^2(i); U_2(i)|U_1(i)), \\
R_3 + \epsilon &\geq \frac{1}{n} \sum_{i=1}^n I(X^3(i); \widehat{X}_2^3(i)|U^2(i)), \\
D_j + \epsilon &\geq E[d_j^{(n)}(\mathbf{X}_j, \widehat{\mathbf{X}}^j)], \quad j = 1, \dots, T, \\
\widehat{X}_1(i) &= g_{1,i}(U^2(i)), i = 1, \dots, n
\end{aligned}$$

and the Markov chains  $X_2^3(i) - X_1(i) - U_1(i)$  and  $X_3(i) - (X^2(i), U_1(i)) - U_2(i)$  hold for each  $i$ . Note that the Markov chains imply that

$$\sum_{i=1}^n I(U_1(i); X_2^3(i)|X_1(i)) = 0, \quad (\text{III.8})$$

$$\sum_{i=1}^n I(U_2(i); X_3(i)|X^2(i), U_1(i)) = 0. \quad (\text{III.9})$$



## II Time-sharing

We introduce a timesharing random variable  $Q$  taking values in  $\{1, \dots, n\}$  equally likely, which is independent of all the other random variables. We have

$$\begin{aligned}
 R_1 + \epsilon &\geq \frac{1}{n} \sum_{i=1}^n I(X_1(i); U_1(i)) \\
 &= \frac{1}{n} \sum_{i=1}^n I(X_1(i); U_1(i) | Q = i) \\
 &= I(X_1(Q); U_1(Q) | Q) \\
 &= I(X_1(Q); U_1(Q), Q)
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 R_2 + \epsilon &\geq I(X^2(Q); U_2(Q) | U_1(Q), Q), \\
 R_3 + \epsilon &\geq I(X^3(Q); \widehat{X}_2^3(Q) | U^2(Q), Q), \\
 D_j + \epsilon &\geq E[d_j(X_j(Q), \widehat{X}^j(Q))].
 \end{aligned}$$

Now define  $U_1 := (U_1(Q), Q)$ ,  $U_2 := U_2(Q)$ ,  $X_j := X_j(Q)$ ,  $\widehat{X}_j := \widehat{X}_j(Q)$  for  $j = 1, \dots, T$ . Also define deterministic functions  $g_1$  as follows,

$$g_1(U^2) = g_1(U^2(Q), Q) := g_{1,Q}(U^2(Q)) = \widehat{X}_1(Q) = \widehat{X}_1,$$

which are consistent with the definitions of  $\{U^2, \widehat{X}_1\}$ . Then we have the inequalities

$$R_1 + \epsilon \geq I(X_1; U_1), \tag{III.10}$$

$$R_2 + \epsilon \geq I(X^2; U_2 | U_1), \tag{III.11}$$

$$R_3 + \epsilon \geq I(X^3; \widehat{X}_2^3 | U^2), \tag{III.12}$$

$$D_j + \epsilon \geq E[d_j(X_j, \widehat{X}^j)], \quad j = 1, \dots, T. \tag{III.13}$$

Concerning the Markov chains, note that

$$\begin{aligned}
 I(U_1; X_2^3 | X_1) &= I(U_1(Q), Q; X_2^3(Q) | X_1(Q)) \\
 &= I(Q; X_2^3(Q) | X_1(Q)) \\
 &\quad + I(U_1(Q); X_2^3(Q) | X_1(Q), Q).
 \end{aligned}$$

The first term is zero because  $Q$  is designed to be independent of all other random variables. The second term is zero because of Equation (III.8). Hence the Markov chain  $U_1 - X_1 - X_2^3$  holds. Furthermore, we have

$$I(U_2; X_3 | X^2, U_1) = I(U_2(Q); X_3(Q) | X^2(Q), U_1(Q), Q) = 0,$$

because of Equation (III.9). Hence the Markov chain  $U_2 - (X^2, U_1) - X_3$  holds.

### III Cardinality bounds on the alphabet of auxiliary random variables

Till now we have shown that for any admissible rate-distortion tuple  $(\mathbf{R}, \mathbf{D})$ ,  $\forall \epsilon > 0$ , for sufficiently large  $n$ , inequalities (III.10)-(III.13) and the Markov chains  $U_1 - X_1 - X_2^3$  and  $U_2 - (X^2, U_1) - X_3$  hold. The definition of  $U_j(i) = (S_j, X_j(1), \dots, X_j(i-1))$  guarantees that  $U_j(i)$  has a finite alphabet, although its cardinality grows with  $n$ . Therefore  $U_1 = (U_1(Q), Q)$ ,  $U_2 = (U_2(Q), Q)$  also have the finite alphabets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  whose cardinalities grow with  $n$ . In this section, we will use the Carathéodory theorem to find new random variables  $U_1^*$  and  $U_2^{**}$  with smaller alphabets whose sizes are independent of  $n$ , such that inequalities (III.10)-(III.13) and the Markov chains still hold even if  $\{U_1, U_2\}$  are replaced by  $\{U_1^*, U_2^{**}\}$ .

Observe that we can define functionals  $\{f_{x_1}\}_{x_1 \in \mathcal{X}_1}, f_{R_j}, f_{d_j}, j = 1, 2, 3$  as follows. Note that they depend on  $u_1$ , conditional distributions conditioned on  $U_1$  and the function  $g_1$ .

$$\begin{aligned} p_{X_1}(x_1) &= \sum_{u_1 \in \mathcal{U}_1} p_{U_1}(u_1) p_{X_1|U_1}(x_1|u_1) \\ &=: \sum_{u_1 \in \mathcal{U}_1} p_{U_1}(u_1) f_{x_1}(u_1, p_{X_1|U_1}), \forall x_1 \in \mathcal{X}_1, \end{aligned} \quad (\text{III.14})$$

$$\begin{aligned} I(X_1; U_1) &= H(X_1) - \sum_{u_1 \in \mathcal{U}_1} p_{U_1}(u_1) H(X_1|U_1 = u_1) \\ &=: \sum_{u_1 \in \mathcal{U}_1} p_{U_1}(u_1) f_{R_1}(u_1, p_{X_1|U_1}), \end{aligned} \quad (\text{III.15})$$

$$\begin{aligned} I(X^2; U_2|U_1) &= \sum_{u_1 \in \mathcal{U}_1} p_{U_1}(u_1) I(X^2; U_2|U_1 = u_1) \\ &=: \sum_{u_1 \in \mathcal{U}_1} p_{U_1}(u_1) f_{R_2}(u_1, p_{X^2 U_2|U_1}), \end{aligned} \quad (\text{III.16})$$

$$\begin{aligned} I(X^3; \widehat{X}_2^3|U_1, U_2) &= \sum_{u_1 \in \mathcal{U}_1} p_{U_1}(u_1) I(X^3; \widehat{X}_2^3|U_1 = u_1, U_2) \\ &=: \sum_{u_1 \in \mathcal{U}_1} p_{U_1}(u_1) f_{R_3}(u_1, p_{X^3 U_2 \widehat{X}_3|U_1}), \end{aligned} \quad (\text{III.17})$$

$$\begin{aligned} E[d_1(X_1, \widehat{X}_1)] &= \sum_{u_1 \in \mathcal{U}_1} p_{U_1}(u_1) E[d_1(X_1, g_1(u_1, U_2))|U_1 = u_1] \\ &=: \sum_{u_1 \in \mathcal{U}_1} p_{U_1}(u_1) f_{d_1}(u_1, p_{X_1 U_2|U_1}, g_1), \end{aligned} \quad (\text{III.18})$$

$$\begin{aligned} E[d_2(X_2, \widehat{X}_2^2)] &= \sum_{u_1 \in \mathcal{U}_1} p_{U_1}(u_1) E[d_2(X_2, g_1(u_1, U_2), \widehat{X}_2^2|U_1 = u_1)] \\ &=: \sum_{u_1 \in \mathcal{U}_1} p_{U_1}(u_1) f_{d_2}(u_1, p_{X_2 U_2 \widehat{X}_2^2|U_1}, g_1), \end{aligned} \quad (\text{III.19})$$

$$\begin{aligned} E[d_3(X_3, \widehat{X}_2^3)] &= \sum_{u_1 \in \mathcal{U}_1} p_{U_1}(u_1) E[d_3(X_3, g_1(u_1, U_2), \widehat{X}_2^3|U_1 = u_1)] \\ &=: \sum_{u_1 \in \mathcal{U}_1} p_{U_1}(u_1) f_{d_3}(u_1, p_{X_3 U_2 \widehat{X}_2^3|U_1}, g_1). \end{aligned} \quad (\text{III.20})$$

We try to find a new random variable  $U_1^*$  to replace  $U_1$  such that all the quantities in the above equations need to be preserved. Because the Markov chains  $U_1 - X_1 - X_2^3$  and  $U_2 - (X^2, U_1) - X_3$  hold, we can write the joint distribution as follows,

$$p_{X^3 U^2 \hat{X}_2^3} = p_{U_1} p_{X_1|U_1} p_{X_2 X_3|X_1} p_{U_2|X^2 U_1} p_{\hat{X}_2^3|X^3 U^2}.$$

Fixing  $p_{X_1|U_1}, p_{X_2 X_3|X_1}, p_{U_2|X^2 U_1}, p_{\hat{X}_2^3|X^3 U^2}$  and  $g_1$ , the functionals  $\{f_{x_1}\}_{x_1 \in X_1}, f_{R_j}, f_{d_j}, j = 1, 2, 3$  become functions depending solely on  $u_1$ .

Since  $p_{X_1}(x_1)$  is a probability mass function which always adds up to 1, we only care about  $(|X_1| - 1)$  out of  $|X_1|$  equations (III.14). Suppose  $\{x_{1,1}, \dots, x_{1,|X_1|-1}\} \subset X_1$  are  $(|X_1| - 1)$  different elements of interest. Consider a set of  $k = |X_1| + 5$  dimensional vectors consisting of  $|\mathcal{U}_1|$  elements

$$\begin{aligned} \mathcal{A} = \{ & (f_{x_{1,1}}(u_1), \dots, f_{x_{1,|X_1|-1}}(u_1), f_{R_1}(u_1), f_{R_2}(u_1), \\ & f_{R_3}(u_1), f_{d_1}(u_1), f_{d_2}(u_1), f_{d_3}(u_1)) \}_{u_1 \in \mathcal{U}_1}. \end{aligned}$$

According to the above equations, the vector

$$\begin{aligned} \mathbf{a} = & (p_{X_1}(x_{1,1}), \dots, p_{X_1}(x_{1,|X_1|-1}), I(X_1; U_1), I(X^2; U_2|U_1), \\ & I(X^3; \hat{X}_3|U^2), E[d_1(X_1, \hat{X}_1)], E[d_2(X_2, \hat{X}^2)], E[d_3(X_3, \hat{X}^3)]) \end{aligned}$$

is in the convex hull of set  $\mathcal{A}$ . By the Carathéodory theorem [23], there exist  $(k + 1)$  vectors in  $\mathcal{A}$ , such that  $\mathbf{a}$  can be expressed by the convex combination of these vectors. Hence there exists  $\mathcal{U}_1^* \subset \mathcal{U}_1$  satisfying  $|\mathcal{U}_1^*| = k + 1$ , and coefficients  $\{\alpha_{u_1}\}_{u_1 \in \mathcal{U}_1^*}$  satisfying  $\sum \alpha_{u_1} = 1$  such that

$$\begin{aligned} p_{X_1}(x_1) &= \sum_{u_1 \in \mathcal{U}_1^*} \alpha_{u_1} f_{x_1}(u_1), \forall x_1 \in X_1, \\ I(X_1; U_1) &= \sum_{u_1 \in \mathcal{U}_1^*} \alpha_{u_1} f_{R_1}(u_1), \\ &\dots \\ E[d_3(X_3, \hat{X}^3)] &= \sum_{u_1 \in \mathcal{U}_1^*} \alpha_{u_1} f_{d_3}(u_1). \end{aligned}$$

Replacing  $U_1$  by a new random variable  $U_1^*$  on the alphabet  $\mathcal{U}_1^*$  with  $Pr(U_1^* = u_1) = \alpha_{u_1}$ , fixing the conditional distributions  $p_{X_1|U_1^*} = p_{X_1|U_1}, p_{U_2^*|X^2 U_1^*} = p_{U_2|X^2 U_1}, p_{\hat{X}_2^3|X^3 U_1^*} = p_{\hat{X}_2^3|X^3 U_1}$  and the function  $g_1$ , we preserve the marginal distribution of  $X_1$ , all the mutual informations and expected distortions in equations (III.15) - (III.20). The progress is the new random variable  $U_1^*$  takes value in a smaller alphabet  $\mathcal{U}_1^*$  whose size is independent of  $n$ .

Note that because of the statistical structure of the joint distribution

$$p_{X^3 U_1^* \hat{X}_2^3} = p_{U_1^*} p_{X_1|U_1^*} p_{X_2 X_3|X_1} p_{U_2^*|X^2 U_1^*} p_{\hat{X}_2^3|X^3 U_1^*},$$

the Markov chains  $U_1^* - X_1 - X_2^3$  and  $U_2^* - (X^2, U_1^*) - X_3$  hold. Because the marginal distribution  $p_{X_1}$  is not changed, the joint distribution  $p_{X^3}$  also remains unchanged and consisting with the requirement of the problem. However, the distribution of  $U_2$  and  $\hat{X}_3$  is possibly changed. So we used  $U_2^*$  and  $\hat{X}_3^*$  to indicate the corresponding random variables associated with  $U_1^*$ . They still take values in alphabets  $\mathcal{U}_2$  and  $\hat{\mathcal{X}}^3$ . The function  $g_1$  is unchanged, which means that  $g_1(u_1, u_2) = g_1(u_1^*, u_2^*)$  as long as  $(u_1, u_2) = (u_1^*, u_2^*)$ . But the domain of  $g_1$  shrinks from  $\mathcal{U}_1 \times \mathcal{U}_2$  to  $\mathcal{U}_1^* \times \mathcal{U}_2$ .

Till now the alphabet  $\mathcal{U}_1$  is reduced to  $\mathcal{U}_1^*$  whose cardinality

$$|\mathcal{U}_1^*| = k + 1 = |X_1| + 6$$

is independent of  $n$ , while all the rate and distortion constraints and the Markov chains still hold. Then we start to deal with the alphabet  $\mathcal{U}_2$ .

Similar to the equations (III.14) to (III.20), we can define functionals  $f_{x_1 x_2 u_1^*}$  for all  $(x_1, x_2, u_1^*) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{U}_1^*$  and  $f_{R_2'}, f_{R_3'}, f_{d_1'}, f_{d_2'}, f_{d_3'}$ , such that

$$p_{X^2 U_1^*}(x_1, x_2, u_1^*) =: \sum_{u_2 \in \mathcal{U}_2} p_{U_2^*}(u_2) f_{x_1 x_2 u_1^*}(u_2, p_{X^2 U_1^*|U_2^*}), \quad \forall (x_1, x_2, u_1^*) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{U}_1^*, \quad (\text{III.21})$$

$$I(X^2; U_2^*|U_1^*) =: \sum_{u_2 \in \mathcal{U}_2^*} p_{U_2^*}(u_2) f_{R_2'}(u_2, p_{X^2 U_1^*|U_2^*}), \quad (\text{III.22})$$

$$I(X^3; \widehat{X}_3^*|U^{2*}) =: \sum_{u_2 \in \mathcal{U}_2^*} p_{U_2^*}(u_2) f_{R_3'}(u_2, p_{X^3 U_1^* \widehat{X}_3^*|U_2^*}), \quad (\text{III.23})$$

$$E[d_2(X_1, \widehat{X}_1^*)] =: \sum_{u_2 \in \mathcal{U}_2^*} p_{U_2^*}(u_2) f_{d_1'}(u_2, p_{X_1 U_1^*|U_2^*}, g_1), \quad (\text{III.24})$$

$$E[d_2(X_2, \widehat{X}_2^*)] =: \sum_{u_2 \in \mathcal{U}_2^*} p_{U_2^*}(u_2) f_{d_2'}(u_2, p_{X_2 U_1^* \widehat{X}_2^*|U_2^*}, g_1), \quad (\text{III.25})$$

$$E[d_3(X_3, \widehat{X}_3^*)] =: \sum_{u_2 \in \mathcal{U}_2^*} p_{U_2^*}(u_2) f_{d_3'}(u_2, p_{X_3 U_1^* \widehat{X}_3^*|U_2^*}, g_1). \quad (\text{III.26})$$

Because the Markov chain  $U_2^* - (X^2, U_1^*) - X_3$  holds, the joint distribution can be written as follows,

$$p_{X^3 U_1^* \widehat{X}_3^*} = p_{U_2^*} p_{X^2 U_1^*|U_2^*} p_{X_3|X^2 U_1^*} p_{\widehat{X}_3^*|X^3 U_2^*}.$$

Fixing  $p_{X^2 U_1^*|U_2^*}$ ,  $p_{X_3|X^2 U_1^*}$ ,  $p_{\widehat{X}_3^*|X^3 U_2^*}$  and  $g_1$ , the functionals become functions depending solely on  $u_2$ . Then following the same method, we can replace  $U_2^*$  by  $U_2^{**}$  and preserve the marginal distribution  $p_{X_1 X_2 U_1^*}$ , the mutual informations and expected distortions in equations (III.22)-(III.26). Because altogether  $|\mathcal{X}_1||\mathcal{X}_2||\mathcal{U}_1^*| + 4$  quantities should be preserved, one can limit the cardinality of alphabet by

$$|\mathcal{U}_2^{**}| = |\mathcal{X}_1||\mathcal{X}_2||\mathcal{U}_1^*| + 5 = |\mathcal{X}_1|^2|\mathcal{X}_2| + 6|\mathcal{X}_1||\mathcal{X}_2| + 5.$$

In addition, by the statistical structure of the joint distribution, the Markov chain  $U_2^{**} - (X^2, U_1^*) - X_3$  can be verified to hold. Finally, because the values of  $(U_1^*, U_2^{**})$  never influence the performance of the system, we can relabel them by  $\{1, 2, \dots, |\mathcal{U}_1^*|\} \times \{1, 2, \dots, |\mathcal{U}_2^{**}|\}$  such that their values do not depend on the original large size alphabets  $\mathcal{U}_1, \mathcal{U}_2$ . We completely discard the old auxiliary random variables and rename the new random variables  $\{U_1^*, U_2^{**}\}$  by  $\{U_1, U_2\}$  to continue the proof.

Up to now we showed for any admissible rate-distortion tuple  $(\mathbf{R}, \mathbf{D})$ ,  $\forall \epsilon > 0$ , for all  $n > N(\epsilon)$ , we can find  $(U^2, \widehat{X}^3, g_1)$  satisfying

$$R_1 + \epsilon \geq I(X_1; U_1), \quad (\text{III.27})$$

$$R_2 + \epsilon \geq I(X^2; U_2|U_1), \quad (\text{III.28})$$

$$R_3 + \epsilon \geq I(X^3; \widehat{X}_3^2|U^2), \quad (\text{III.29})$$

$$D_j + \epsilon \geq E[d_j(X_j, \widehat{X}^j)], \quad j = 1, \dots, T, \quad (\text{III.30})$$

$$\widehat{X}_1 = g_1(U^2),$$

$$|\mathcal{U}_1| = |\mathcal{X}_1| + 6,$$

$$|\mathcal{U}_2| = |\mathcal{X}_1|^2|\mathcal{X}_2| + 6|\mathcal{X}_1||\mathcal{X}_2| + 5,$$

and the Markov chains  $U_1 - X_1 - X_2^3$  and  $U_2 - (X^2, U_1) - X_3$  hold, which implies

$$I(U_1; X_2^3 | X_1) = 0, \quad (\text{III.31})$$

$$I(U_2; X_3 | X^2, U_1) = 0. \quad (\text{III.32})$$

#### IV Taking limits

Note that for each  $(\epsilon, n)$ ,  $|\mathcal{U}_j|$  is finite and independent of  $(\epsilon, n)$  for  $j = 1, 2$ . Therefore the conditional distribution  $p_{U^2, \widehat{X}_2^3 | X^3, \epsilon, n}(u^2, \widehat{x}_2^3 | x^3)$  is a finite dimensional stochastic matrix taking values in a compact set, and  $g_{1, \epsilon, n}$  has only a finite number of possibilities.

Let  $\{\epsilon_l\}_{l=1}^\infty$  be any sequence of real numbers such that  $\epsilon_l > 0$  and  $\epsilon_l \rightarrow 0$  as  $l \rightarrow \infty$ . Let  $\{n_l\}$  be any sequence of blocklengths where  $\forall l, n_l > N(\epsilon_l)$ . Since  $g_{1, \epsilon, n}$  takes values in a finite set,  $\exists g_1^*$  such that there exists a subsequence  $\{\epsilon_{l_i}\}_{i=1}^\infty$  such that for each  $\epsilon$  in this subsequence,  $g_{1, \epsilon, n} \equiv g_1^*$ .

Since  $p_{U^2, \widehat{X}_2^3 | X^3, \epsilon, n}$  lives in a compact set, there exists again a subsequence of  $\{p_{U^2, \widehat{X}_2^3 | X^3, \epsilon_{l_i}, n_{l_i}}\}$  which converges to a limit  $p_{U^{*2}, \widehat{X}_2^{*3} | X^3}$ . Denote the auxiliary random variables derived from the limit distribution by  $(U^{*2}, \widehat{X}_2^{*3})$ . Due to the continuity of conditional mutual information and expectation with respect to probability distributions, (III.27) - (III.32) become

$$\begin{aligned} R_1 &\geq I(X_1; U_1^*), \\ R_2 &\geq I(X^2; U_2^* | U_1^*), \\ R_3 &\geq I(X^3; \widehat{X}_2^{*3} | U^{*2}), \\ D_j &\geq E[d_j(X_j, \widehat{X}^{*j})], \quad j = 1, 2, 3, \\ I(U_1^*; X_2^3 | X_1) &= 0, \\ I(U_2^*; X_3 | X^2, U_1^*) &= 0, \end{aligned}$$

where  $\widehat{X}_1^* := g_1^*(U^{*2})$ . The last two equalities imply that the Markov chains  $U_1^* - X_1 - X_2^3$  and  $U_2^* - (X^2, U_1^*) - X_3$  hold. Therefore  $(\mathbf{R}, \mathbf{D})$  belongs to the right hand side of (5.11).  $\blacksquare$

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