

Probabilistic Behavior of Average Transmit Energy in Broadcast Systems with Precoding

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Abstract— In this paper, we introduce a shaping concept to reduce average transmit energy for channel inversion techniques in multiple antenna broadcast systems. Based on the probabilistic view point for the input constellations, the optimal probability distribution for the data vectors in the channel inversion techniques is found. We have also introduced a theoretical Selective Mapping (SLM) technique (based on random coding arguments) in channel inversion techniques to reduce the transmit average energy. The asymptotic gain that the SLM technique can provide is derived. The proposed technique results in a significant saving in energy as the corresponding shaping gain can be much higher than the upper bound of 1.53, known for the conventional shaping. The exact gain depends on the orthogonality defect of the channel.

I. INTRODUCTION

Recently, there has been a considerable interest in Multi-Input Multi-Output (MIMO) antenna systems due to achieving a high capacity as compared to single-antenna systems [1].

Multiuser MIMO systems can also exploit most of the advantages of multiple-antenna systems. In a MIMO broadcast system, the sum-capacity grows linearly with the minimum number of the transmit and receive antennas [2].

In a broadcast system, when an access point with multiple antennas is used to communicate with many users, each with one antenna, the communication problem is complicated by the fact that each user must decode its signal independently from the remaining users. As a simple precoding scheme for MIMO broadcast systems, the channel inversion technique (or zero-forcing beam-forming [3]) can be used at the transmitter to separate the data for different users. However, this method is vulnerable to the poor channel conditions which are due to the near-singularity of the channel matrix (having at least one small eigenvalue).

In [4], the authors have introduced a *vector perturbation technique* which has a good performance in terms of symbol error rate, but has a high complexity. In [5], the authors have used lattice-basis reduction to reduce the average transmitted energy of the method in [4] by reducing the second moment of the fundamental region of the transmitted lattice.

In this paper, we introduce a shaping concept for average transmit energy for the channel inversion techniques. By using the fact that the channel is not orthogonal, the gain that we can achieved is significantly higher than the regular shaping gains that can be achieved in methods like [4].

Relying on continuous approximation, a probability distribution along with a support region is defined for different input constellations. Based on the defined distribution, the corresponding theoretical average transmit energy is computed. For a given channel matrix, we have also found the optimal probability distribution for the data vectors in the channel inversion technique to minimize the average transmit energy.

We have also introduced a theoretical Selective Mapping (SLM) technique (based on random coding arguments) in channel inversion techniques to reduce the transmit average energy. The effect of applying this technique to the mentioned probability distribution on the average transmit energy is investigated. Using strong literature in quantization, the asymptotic gain that the SLM technique can provide is also derived.

The rest of the paper is organized as follows. The system model is introduced in Section II. In Section III, the average transmit energy for different probabilistic constellations is investigated. Section IV finds the optimal probability distribution for transmit data. Section V is devoted to introducing the theoretical SLM technique. Some asymptotic analysis for maximum gain of average transmit energy by using SLM technique is derived in Section VI. Finally, Section VII concludes the paper with some simulation results.

II. SYSTEM MODEL

A multiple antenna broadcast system with \tilde{N} transmit antennas and \tilde{M} single-antenna users is modeled as

$$\tilde{\mathbf{y}} = \sqrt{\frac{SNR}{\tilde{M}\tilde{E}_{sav}}} \tilde{\mathbf{H}}\tilde{\mathbf{x}} + \tilde{\mathbf{n}}, \quad (1)$$

where $\tilde{\mathbf{H}} = [\tilde{h}_{ij}]$ is the $\tilde{M} \times \tilde{N}$ channel matrix composed of independent, identically distributed complex Gaussian random elements with zero mean and unit variance, $\tilde{\mathbf{n}}$ is an $\tilde{M} \times 1$ complex additive white Gaussian noise vector with zero mean and unit variance, and $\tilde{\mathbf{x}}$ is an $\tilde{N} \times 1$ data vector with $E\{\|\tilde{\mathbf{x}}\|^2\} = 1$. The parameter SNR in (1) is the SNR per receive antenna.

To avoid using complex matrices, the system model (1) is represented by real matrices in (2) [6].

$$\Rightarrow \mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \quad (2)$$

where \mathbf{y} is the *received vector*, \mathbf{x} is the *input vector*, $\tilde{M} = 2\tilde{M}$, and $\tilde{N} = 2\tilde{N}$.

In broadcast systems, the receivers should decode their respective data independently and without any cooperation with each other. The main strategy in dealing with this restriction is to apply an appropriate precoding scheme at the transmitter. The simplest method in this category is using the channel inversion technique at the transmitter to separate the data for different users. When the number of transmit antennas equals the number of users, $\bar{M} = \bar{N} := M$, the transmitted signal is

$$\mathbf{s} = \mathbf{H}^{-1}\mathbf{u}. \quad (3)$$

As in [4], the normalized transmitted signal would be $\mathbf{x} = \frac{\mathbf{s}}{\sqrt{E\{\gamma\}}}$, where $\gamma = \|\mathbf{s}\|^2$.

The problem arises when \mathbf{H} is poorly conditioned and γ becomes very large, resulting in a high power consumption. This situation occurs when at least one of the eigenvalues of \mathbf{H} is very small which results in vectors with large norms as the columns of \mathbf{H}^{-1} . In other words, when $\det(\mathbf{H})$ is too small, the communication is not possible.

In a multiple antenna system, it is assumed that the data vector \mathbf{u} is selected from a constellation with discrete points. However, through this paper we investigate the probabilistic behavior of the transmitted signal \mathbf{x} . Assuming a large constellation, continuous approximation provides a probability distribution for each constellation, resulting in different $E\{\gamma\}$. The challenge is finding the best probability distribution with minimum $E\{\gamma\}$. Note that the expectation in $E\{\gamma\}$ is over \mathbf{u} and \mathbf{H}^{-1} is assumed to be constant.

III. AVERAGE TRANSMIT ENERGY OF PROBABILISTIC CONSTELLATIONS

In this section, we evaluate the average energy of transmit signals corresponding to different probability distributions for data vector $\mathbf{u} \in \mathbb{R}^M$ in a broadcast system with channel inversion, defined in (2).

Let $\mathbf{Q} := (\mathbf{H}^{-1})^T \mathbf{H}^{-1} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$, where \mathbf{U} is the unitary matrix of eigenvectors of \mathbf{Q} and $\mathbf{\Lambda}$ is the diagonal matrix of the corresponding eigenvalues, λ_i , $i = 1, \dots, M$. Assume $\mathbf{u} \in \mathbb{R}^M$ be a random vector with mean $E\{\mathbf{u}\} = \boldsymbol{\mu}$ and covariance $E\{\mathbf{u}\mathbf{u}^T\} = \boldsymbol{\Sigma} > 0$. The energy of the transmit signal is $\gamma = \mathbf{u}^T \mathbf{Q} \mathbf{u} = \mathbf{u}^T \mathbf{U} \mathbf{\Lambda} \mathbf{U}^T \mathbf{u}$. The average energy of the transmit signal can be written as [6]

$$E\{\gamma\} = \text{tr}(\mathbf{Q}\boldsymbol{\Sigma}) + \boldsymbol{\mu}^T \mathbf{Q} \boldsymbol{\mu}. \quad (4)$$

Assume that \mathbf{u} is a random vector in \mathbb{R}^M such that its elements are i.i.d random variables with a *uniform distribution* between $-A$ and A (*this case is equivalent to selecting the data vector \mathbf{u} from a QAM constellation*). In order to have an entropy of \mathcal{H} , we should have $\mathcal{H} = \log 2A$.

In [6], it is shown that the average energy of transmit vector corresponding to \mathbf{u} is

$$E_{cube} = E(\gamma) = \frac{A^2}{3} \sum_{i=1}^M \lambda_i = \frac{2^{2\mathcal{H}}}{12} \sum_{i=1}^M \lambda_i. \quad (5)$$

It can be shown that the average energy of the transmit signal corresponding to \mathbf{u} is [6]

$$E_{sphere} = E(\gamma) = \sigma^2 \sum_{i=1}^M \lambda_i = \frac{2^{2\mathcal{H}}}{2\pi e} \sum_{i=1}^M \lambda_i. \quad (6)$$

Note that when a random vector is uniformly selected in an M -dimensional $\mathcal{B}_M(0, \sqrt{M}\sigma)$, in the limit of $M \rightarrow \infty$, each dimension has a Gaussian distribution of $G(0, \sigma^2)$.

In [4], a perturbation method is introduced to reduce the average energy of the transmit signal. The data is transmitted by judiciously adding an integer vector offset. If the data vector is selected from a QAM constellation, it is shown that the average transmit energy of perturbation technique would be [6]

$$E_{perturb} = E(\gamma) = \int_{\mathcal{V}(\mathbf{H}^{-1})} \|\mathbf{x}\|^2 dF(\mathbf{x}), \quad (7)$$

where $F(\mathbf{x})$ is the uniform distribution over $\mathcal{V}(\mathbf{H}^{-1})$ and $\mathcal{V}(\mathbf{H}^{-1})$ is the voronoi region of matrix \mathbf{H}^{-1} .

Since the probability distribution over the voronoi region is uniform, the average energy in (7) can be formulated by the second moment of \mathbf{H}^{-1} [7]. The dimensionless second moment of the voronoi region $\mathcal{V}(\mathbf{H}^{-1})$ is defined as

$$G(\mathcal{V}(\mathbf{H}^{-1})) = \frac{1}{M} \frac{\int_{\mathcal{V}(\mathbf{H}^{-1})} \|\mathbf{x}\|^2 dF(\mathbf{x})}{\text{Vol}(\mathcal{V}(\mathbf{H}^{-1}))^{\frac{2}{M}}} \quad (8)$$

Therefore,

$$E_{perturb} = MG(\mathcal{V}(\mathbf{H}^{-1})) (\text{Vol}(\mathcal{V}(\mathbf{H}^{-1})))^{\frac{2}{M}}, \quad (9)$$

where

$$\text{Vol}(\mathcal{V}(\mathbf{H}^{-1})) = \sqrt{\det(\mathbf{Q})} (2A)^M. \quad (10)$$

In [7], it is shown that $G(\mathcal{V}(\mathbf{H}^{-1})) \geq G_M$ and

$$\frac{1}{(M+2)\pi} \Gamma\left(\frac{M}{2} + 1\right)^{\frac{2}{M}} \leq G_M \leq \frac{1}{M\pi} \Gamma\left(\frac{M}{2} + 1\right)^{\frac{2}{M}} \Gamma\left(1 + \frac{2}{M}\right) \quad (11)$$

It is clear that $E_{perturb} \leq E_{cube}$ [4]. However, the relation between E_{sphere} and $E_{perturb}$ completely depends on the achieved voronoi region of \mathbf{H}^{-1} .

According to (5) and (6), $\mathcal{G}_{sphere} = \frac{E_{cube}}{E_{sphere}} = \frac{\pi e}{6}$, which is known as the shaping gain. The general purpose of constellation shaping is to reduce the average energy of the signals without reducing the minimum Euclidean distance. Since \mathbf{H}^{-1} is not orthogonal, the gain is significantly higher than the known upper bound of $\frac{\pi e}{6}$, conventional shaping gain. The exact gain also depends on the orthogonality defect of the channel. This gain can be achieved by introducing correlation among Gaussian elements of vector \mathbf{u} .

IV. OPTIMUM PROBABILISTIC DATA

In designing input constellations for communication systems, the objective is to reduce the average energy of the transmit signal, while keeping a fixed entropy or rate and minimum distance for the constellation.

Theorem 1: Let $\mathbf{u} \in \mathbb{R}^M$ be a random vector with mean $E\{\mathbf{u}\} = \boldsymbol{\mu}$ and covariance $E\{\mathbf{u}\mathbf{u}^T\} = \boldsymbol{\Sigma} > 0$ in a broadcast system introduced in (2). Let $\mathcal{H}(\mathbf{u})$ denote the entropy of the data vector \mathbf{u} . Then, a Gaussian random vector \mathbf{u} with $\boldsymbol{\mu} = \mathbf{0}$ and covariance matrix

$$\boldsymbol{\Sigma} = \sqrt[M]{\prod \lambda_i} \sigma^2 \mathbf{H} \mathbf{H}^T, \quad (12)$$

will minimize the energy of the transmit signal given a fixed maximum entropy per real dimension¹, \mathcal{H} , where σ^2 is the variance of a Gaussian random variable with entropy \mathcal{H} .

Proof: The matrix \mathbf{Q} is positive semi-definite ($\boldsymbol{\mu}^T \mathbf{Q} \boldsymbol{\mu} \geq 0$ for any vector $\boldsymbol{\mu}$). Therefore, in order to minimize $E\{\gamma\}$ in (4), $E\{\mathbf{u}\} = \boldsymbol{\mu} = \mathbf{0}$.

Instead of minimizing $E\{\gamma\} = \text{tr}(\mathbf{Q}\boldsymbol{\Sigma})$ given $\mathcal{H}(\mathbf{u}) = \text{fixed}$, we consider the equivalent problem of maximization of $\mathcal{H}(\mathbf{u})$ given $E\{\gamma\} = \text{tr}(\mathbf{Q}\boldsymbol{\Sigma}) = \text{fixed}$. The constraint is on the values of $\boldsymbol{\Sigma}$. In [8], it has been shown that for a random vector \mathbf{u} with zero mean and covariance matrix $\boldsymbol{\Sigma}$

$$\mathcal{H}(\mathbf{u}) \leq \frac{1}{2} \ln(2\pi e)^M |\boldsymbol{\Sigma}|.$$

with equality iff \mathbf{u} is a Gaussian vector. Therefore, we are looking for a Gaussian random vector with zero mean and covariance $\boldsymbol{\Sigma}$ such that

$$\begin{aligned} \max \quad & \ln |\boldsymbol{\Sigma}| \\ \text{s.t.} \quad & \text{tr}(\mathbf{Q}\boldsymbol{\Sigma}) = K \\ & \boldsymbol{\Sigma} \succ 0. \end{aligned} \quad (13)$$

By considering $\mathbf{Q} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T$ and $\boldsymbol{\Sigma}' = \mathbf{U}^T \boldsymbol{\Sigma} \mathbf{U}$, it is easy to show that the optimization problem (13) is equivalent to

$$\begin{aligned} \max \quad & \ln |\boldsymbol{\Sigma}'| \\ \text{s.t.} \quad & \text{tr}(\boldsymbol{\Lambda}\boldsymbol{\Sigma}') = K \\ & \boldsymbol{\Sigma}' \succ 0, \end{aligned} \quad (14)$$

where K is the limiting value for the average energy.

The Hadamard inequality states that for a Hermitian positive definite matrix $\boldsymbol{\Sigma}' = [\sigma'_{ij}]$ we have $|\boldsymbol{\Sigma}'| \leq \prod_i \sigma'_{ii}$ with equality iff $\boldsymbol{\Sigma}'$ is a diagonal matrix. Therefore, in order to maximize the entropy, we assume the covariance matrix $\boldsymbol{\Sigma}'$ is a diagonal matrix with diagonal elements σ_i^2 . Hence, the optimization problem (14) can be written as

$$\begin{aligned} \max \quad & \sum_{i=1}^M \ln \sigma_i^2 \\ \text{s.t.} \quad & \sum_{i=1}^M (\lambda_i \sigma_i^2) = K. \end{aligned} \quad (15)$$

¹Without loss of generality, we assume that entropy per real dimension is the same.

If the entropy per real dimension is \mathcal{H} , for independent Gaussian random variables with variance σ_i^2 in (15), we have [6]

$$\sigma_i^2 = \frac{\sqrt[M]{\prod \lambda_i}}{\lambda_i} \sigma^2 \quad i = 1, \dots, M. \quad (16)$$

Therefore, the optimum solution of (13) would be

$$\boldsymbol{\Sigma} = \sqrt[M]{\prod \lambda_i} \sigma^2 \mathbf{U} \boldsymbol{\Lambda}^{-1} \mathbf{U}^T = \sqrt[M]{\prod \lambda_i} \sigma^2 \mathbf{H} \mathbf{H}^T. \quad (17)$$

This choice of $\boldsymbol{\Sigma}$ suggests that the minimum value of the average energy among transmit signals with different probability distributions is

$$E_{opt} = E\{\gamma\} = M \sqrt[M]{\prod \lambda_i} \sigma^2. \quad (18)$$

Note that the minimum value of the average energy of the transmit signal in (18) corresponds to the average energy of an M -dimensional Gaussian random vector with independent elements with zero mean and variance $\mathcal{R}_{eq}^2 = \sqrt[M]{\prod \lambda_i} \sigma^2$.

Definition 2: Channel Shaping Gain - According to (6) and (18),

$$\mathcal{G}_H = \frac{\text{Arithmetic Mean}(\lambda_1, \dots, \lambda_N)}{\text{Geometric Mean}(\lambda_1, \dots, \lambda_N)}, \quad (19)$$

where the geometric mean of a data set is always smaller than or equal to the set's arithmetic mean (the two means are equal if and only if all members of the data set are equal). On the other hand, without the channel matrix, we have the conventional shaping gain. However, the presence of \mathbf{H}^{-1} will affect the shaping gain by the *Channel Shaping Gain*, \mathcal{G}_H , defined in (19).

V. SELECTIVE MAPPING

The idea of Selective Mapping (SLM) is to generate a large set of data vectors that represent the same information, where the data vector resulting in the lowest transmit energy will be selected for transmission.

In the sequel, we use a random coding argument to explain the SLM method, its analysis, and the maximum theoretical gain that can be achieved. In the system model (2), the entropy is fixed. In order to provide multiple choices for the SLM method, the entropy is increased such that for each data vector there are N choices available. Then, among each set of N points the one corresponding to the lowest transmit energy is selected. Assume that the entropy is increased to \mathcal{H}' . In each dimension, the number of points is increased by a factor of $2^{\mathcal{H}' - \mathcal{H}}$. Therefore, we have

$$N = \left(2^{\mathcal{H}' - \mathcal{H}}\right)^M. \quad (20)$$

In the SLM method, N i.i.d. samples of \mathbf{u} are generated, $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N\}$, and \mathbf{s}_l with the lowest transmit energy is selected for transmission. In other words, $\gamma_l = \min\{\gamma_1, \gamma_2, \dots, \gamma_N\}$. We are looking for the probabilistic behavior of γ_l .

The energy $\gamma = \mathbf{u}^T \mathbf{Q} \mathbf{u}$ is a quadratic expression of random vector \mathbf{u} . There are a lot of research on the probabilistic

relations between γ and \mathbf{u} [9, and ref. therein]. In order statistic references [10, and ref. therein], the relations between the probability behavior of \mathbf{u}_l and \mathbf{u} are discussed. It is shown that [9]

$$F_{\gamma_l}(x) = 1 - (1 - F_\gamma(x))^N, \quad (21)$$

$$E\{\gamma_l\} = \int_0^\infty (1 - F_\gamma(x))^N dx, \quad (22)$$

where $F_\gamma(x)$ is the probability distribution of γ . It is clear that $(1 - F_\gamma(x))^N \leq 1 - F_\gamma(x)$. Therefore,

$$\int_0^\infty (1 - F_\gamma(x))^N dx \leq \int_0^\infty (1 - F_\gamma(x)) dx, \quad (23)$$

which shows that applying SLM method will reduce the average energy with a fixed probability distribution. On the other hand, by applying SLM method on the mentioned cases in Section III, we have

$$E_{opt-SLM} < E_{sphere-SLM} < E_{cube-SLM}. \quad (24)$$

VI. ASYMPTOTIC ANALYSIS

In the proposed method, N i.i.d. samples of \mathbf{u} are generated, $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N\}$, and among the corresponding transmit vectors, $\mathbf{s}_i = \mathbf{H}^{-1}\mathbf{u}_i$, the vector \mathbf{s}_l with the lowest transmit energy is selected for transmission. In other words, in the SLM method, we are looking for

$$\min_{1 \leq i \leq N} \|\mathbf{s}_i\|^2, \quad (25)$$

where $\|\cdot\|$ represents the square norm.

Let $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_N$, be i.i.d. \mathbb{R}^M -valued random variables with distribution Q , i.e.

$$Q(\mathbf{v}) = \mathbb{P}\{s_{i_1} \leq v_1, \dots, s_{i_M} \leq v_M\} \quad i = 1, \dots, N. \quad (26)$$

For any region \mathcal{R} , the probability $Q(\mathcal{R})$ is the probability that there is at least one code point in the region \mathcal{R} , i.e.

$$Q(\mathcal{R}) = \int_{\mathcal{R}} Q(d\mathbf{y}).$$

Define the r^{th} order transmit energy as

$$\gamma_{r,N}^Q = \min_{1 \leq i \leq N} \|\mathbf{s}_i\|^r, \quad (27)$$

where based on our previous notation $\gamma_l = \gamma_{2,N}^Q$. In this section, the asymptotic probabilistic behavior of $\gamma_{2,N}^Q$, when $N \rightarrow \infty$, is investigated. The following lemmas will help us in proving our main results.

Lemma 3:

$$E\left\{N^{\frac{r}{M}} \gamma_{r,N}^Q\right\} = \int_{\mathbb{R}_+} \left(1 - Q\left(\mathcal{B}_M(0, \frac{v^{\frac{1}{r}}}{N^{\frac{1}{M}}})\right)\right)^N dv. \quad (28)$$

Proof: See [6]. ■

Lemma 4: For any $\rho > 0$, we can define $\mathcal{A}_\rho = \{v \in \mathbb{R}_+; v^{1/r}/N^{1/M} \leq \rho\}$ and set

$$g_\rho := \inf_{\delta \in (0, \rho)} \frac{Q(\mathcal{B}_M(0, \delta))}{\lambda(\mathcal{B}_M(0, \delta))}.$$

Then,

$$\int_{\mathcal{A}_\rho} \left(1 - Q\left(\mathcal{B}_M(0, \frac{v^{\frac{1}{r}}}{N^{\frac{1}{M}}})\right)\right)^N dv \leq \int_{\mathcal{A}_\rho} \exp\left(-B_M g_\rho v^{M/r}\right) dv. \quad (29)$$

Note that λ is the M -dimensional Lebesgue measure, which here it is defined as the M -dimensional volume of a region.

Proof: See [6]. ■

Theorem 5: Let $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_N$, be i.i.d. \mathbb{R}^M -valued random variables with distribution Q . Then,

$$\lim_{N \rightarrow \infty} E\left\{N^{\frac{r}{M}} \gamma_{r,N}^Q\right\} = B_M^{-\frac{r}{M}} \Gamma\left(1 + \frac{r}{M}\right) g_\rho^{-\frac{r}{M}}. \quad (30)$$

Proof: According to Lemma 3, we have

$$E\left\{N^{\frac{r}{M}} \gamma_{r,N}^Q\right\} = \int_{\mathbb{R}_+} \left(1 - Q\left(\mathcal{B}_M(0, \frac{v^{\frac{1}{r}}}{N^{\frac{1}{M}}})\right)\right)^N dv. \quad (31)$$

For any $\rho > 0$ define $\bar{\mathcal{A}}_\rho$ as the complement region of \mathcal{A}_ρ . Therefore,

$$\begin{aligned} E\left\{N^{\frac{r}{M}} \gamma_{r,N}^Q\right\} &= \int_{\mathcal{A}_\rho} \left(1 - Q\left(\mathcal{B}_M(0, \frac{v^{\frac{1}{r}}}{N^{\frac{1}{M}}})\right)\right)^N dv \\ &\quad + \int_{\bar{\mathcal{A}}_\rho} \left(1 - Q\left(\mathcal{B}_M(0, \frac{v^{\frac{1}{r}}}{N^{\frac{1}{M}}})\right)\right)^N dv. \end{aligned} \quad (32)$$

Based on Lemma 4,

$$\begin{aligned} &\int_{\mathcal{A}_\rho} \left(1 - Q\left(\mathcal{B}_M(0, \frac{v^{\frac{1}{r}}}{N^{\frac{1}{M}}})\right)\right)^N dv \\ &\leq \int_{\mathcal{A}_\rho} \exp\left(-B_M g_\rho v^{M/r}\right) dv. \end{aligned} \quad (33)$$

Note that the integral in (33) is limited (The proof is easy and it is similar to the approach in [11]). As $N \rightarrow \infty$, we have $\mathcal{A}_\rho \rightarrow \mathbb{R}_+$. Therefore,

$$\begin{aligned} &\int_{\mathcal{A}_\rho} \left(1 - Q\left(\mathcal{B}_M(0, \frac{v^{\frac{1}{r}}}{N^{\frac{1}{M}}})\right)\right)^N dv \rightarrow \\ &\int_{\mathbb{R}_+} \exp\left(-B_M g_\rho v^{M/r}\right) dv, \end{aligned}$$

and

$$\int_{\bar{\mathcal{A}}_\rho} \left(1 - Q\left(\mathcal{B}_M(0, \frac{v^{\frac{1}{r}}}{N^{\frac{1}{M}}})\right)\right)^N dv \rightarrow 0. \quad (34)$$

In [12], it is shown that

$$\int_{\mathbb{R}_+} \exp\left(-B_M g_\rho v^{M/r}\right) dv = B_M^{-\frac{r}{M}} \Gamma\left(1 + \frac{r}{M}\right) g_\rho^{-\frac{r}{M}}. \quad (35)$$

Therefore, we have

$$\lim_{N \rightarrow \infty} E\left\{N^{\frac{r}{M}} \gamma_{r,N}^Q\right\} = B_M^{-\frac{r}{M}} \Gamma\left(1 + \frac{r}{M}\right) g_\rho^{-\frac{r}{M}}. \quad (36)$$

■

Now, consider the special case of a uniform distribution.

Theorem 6: Let $\mathcal{R} \subset \mathbb{R}^M$ be a compact set with $\lambda(\mathcal{R}) > 0$ and let s_1, \dots, s_N be i.i.d. random variables with uniform distribution over \mathcal{R} . Then,

$$\lim_{N \rightarrow \infty} E \left\{ N^{\frac{r}{M}} \gamma_{r,N}^Q \right\} = B_M^{-\frac{r}{M}} \Gamma(1 + \frac{r}{M}) \lambda(\mathcal{R})^{\frac{r}{M}}. \quad (37)$$

Proof: See [6]. ■

Note that in both cases of independent Gaussian random variables and the optimal case, when $M \rightarrow \infty$, we have a uniform distribution over an oval or a ball (with the same volume). Therefore, according to Theorem 6, these two methods have the same average transmit energy when the number of points is large enough.

Corollary 1: In a broadcast system, applying SLM method to the optimal case (Theorem 1) or independent Gaussian case (Case II) will result in equal values for the average transmit energy when the number of points in the SLM method is large enough.

The expression in (37) is the average transmit energy for large N . We can approximately say that

$$E_{SLM} = B_M^{-\frac{2}{M}} \Gamma(1 + \frac{2}{M}) N^{-\frac{2}{M}} \lambda(\mathcal{R})^{\frac{2}{M}}. \quad (38)$$

For $M \geq 2$, we have, $\lambda(\mathcal{R}) = B_M \mathcal{R}'_{eq}{}^{M/2}$ and $\mathcal{R}'_{eq} = N \mathcal{R}_{eq}$. Therefore,

$$E_{SLM} = \Gamma(1 + \frac{2}{M}) \mathcal{R}_{eq}^2, \quad (39)$$

and the maximum gain that we can achieve is

$$\mathcal{G}_{SLM} = \frac{M}{\Gamma(1 + \frac{2}{M})}.$$

We must emphasize that in our random coding argument the probability of the event that two different code words have the same transmit data vector is negligible. In the case of this event, we have an error in our broadcast system. However, since the probability of this event is small, the average transmit energy would not change.

VII. SIMULATION RESULTS

In this section, a broadcast system with different probability distribution for transmit data is presented. In this system, we have considered $M = 4$. The entropy per each real dimension is considered $\mathcal{H} = 6$. We generated 1000 different random channel matrix, and for each sample we have found the average transmit energy. The average transmit energy for a broadcast system with uniform distribution is used as the benchmark for comparison. The gain of different methods compared to this benchmark is computed and the complementary cumulative density function for these gains is calculated and depicted in Fig. 1. In this simulation, the number of points for SLM is $N = 256$. By increasing N to 4096, it is observed that the gap between curves for SLM methods with optimal PDF and Gaussian PDF is decreased.

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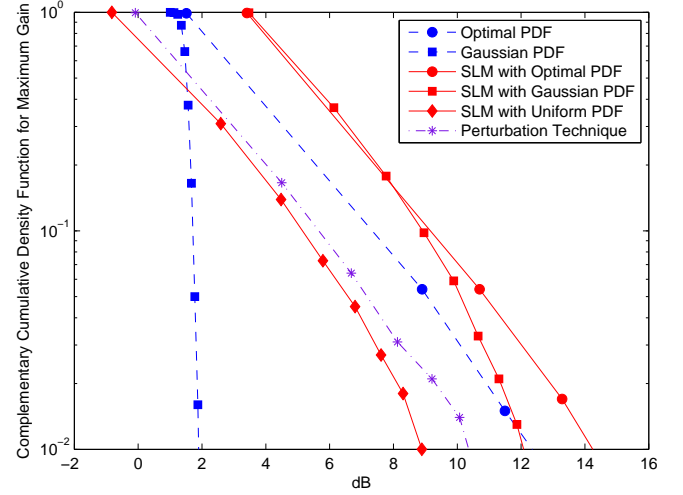


Fig. 1. Maximum Gain of Average Transmitting Energy for Different Methods vs. that in a Broadcast System with Uniform PDF

REFERENCES

- [1] E. Telatar, "Capacity of multi-antenna gaussian channels," *European Trans. on Telecomm. ETT*, vol. 10, no. 6, pp. 585–596, November 1999.
- [2] S. Vishwanath, N. Jindal, and A. Goldsmith, "Duality, achievable rates and sum capacity of Gaussian MIMO broadcast channels," *IEEE Trans. Info. Theory*, pp. 2658–2658, August 2003.
- [3] G. Caire and S. Shamai, "On the achievable throughput of a multiple-antenna Gaussian broadcast channel," *IEEE Trans. Info. Theory*, pp. 1691–1706, July 2003.
- [4] C. B. Peel, B. M. Hochwald, and A. L. Swindlehurst, "A vector-perturbation technique for near-capacity multiple-antenna multi-user communications-Part II: Perturbation," *IEEE Trans. on Comm.*, vol. 53, no. 3, Mar. 2005.
- [5] M. Taherzadeh, A. Mobasher, and A. K. Khandani, "Communication over MIMO broadcast channels using lattice-basis reduction," *Accepted for publication in IEEE Trans. on Info. Theory*, 2006.
- [6] A. Mobasher and A. K. Khandani, "Probabilistic Behavior of Average Transmit Energy in Multiple-Antenna Broadcast Systems with Precoding," Department of E&CE, University of Waterloo, Tech. Rep. UW-E&CE 2007-02, 2007, available via the WWW site at <http://www.cst.uwaterloo.ca/~amin>.
- [7] J. H. Conway and N. J. A. Sloane, "Voronoi, regions of lattices, second moments of polytopes, and quantization," *IEEE Trans. on Info. Theory*, vol. 28, no. 2, pp. 211–226, Mar. 1982.
- [8] A. Dembo, T. M. Cover, and J. A. Thomas, "Information theoretic inequalities," *IEEE Trans. on Info. Theory*, vol. 37, no. 6, pp. 1501–1518, Nov. 1991.
- [9] A. M. Mathai and S. B. Provost, *Quadratic Forms in Random Variables: Theory and Applications*, ser. Statistics: textbooks and monographs. New York: Marcel Dekker, Inc., 1992, vol. 126.
- [10] H. A. David and H. N. Nagaraja, *Order Statistics*, 3rd ed., ser. Wiley Series in Probability and Statistics. Hoboken, N.J., Wiley-Interscience, 2003.
- [11] P. Cohort, "Limit theorems for random normalized distortion," *Annals of Applied Probability*, vol. 14, no. 1, p. 118143, Mar. 2004.
- [12] J. Hoffmann-Jørgensen, *Probability with a view toward statistics*. Chapman and Hall, 1994, vol. 1.