

A New Uncertainty-Bearing Floating-Point Arithmetic*

Chengpu Wang

40 Grossman Street, Melville, NY 11747, USA

Chengpu@gmail.com

Abstract

A new deterministic floating-point arithmetic called *precision arithmetic* is developed to track precision for arithmetic calculations. It uses a novel rounding scheme to avoid the excessive rounding error propagation of conventional floating-point arithmetic. Unlike interval arithmetic, its uncertainty tracking is based on statistics and the central limit theorem, with a much tighter bounding range. Its stable rounding error distribution is approximated by a truncated Gaussian distribution. Generic standards and systematic methods for comparing uncertainty-bearing arithmetics are discussed. The precision arithmetic is found to be superior to interval arithmetic in both uncertainty-tracking and uncertainty-bounding for normal usages.

The arithmetic code is published at <http://precisionarithmetic.sourceforge.net>.

Keywords: computer arithmetic, error analysis, interval arithmetic, multi-precision arithmetic, numerical algorithms.

AMS subject classifications: 65-00

*Submitted: May 20, 2010; Revised: November 10, 2010; July 18, 2011; September 1, 2012;

1 Introduction

1.1 Measurement Precision

Except for the simplest counting, scientific and engineering measurements never give completely precise results [?][?]. The precision of measured values ranges from an order-of-magnitude estimation of astronomical measurements to 10^{-2} to 10^{-4} of common measurements to 10^{-14} of state-of-art measurements of basic physics constants [?]. Such value which has uncertainty is called an *imprecise value*.

In scientific and engineering measurements, the uncertainty of a measurement x usually is characterized by the sample deviation δx [?][?][?]. In certain cases, such as raw reading from an ideal analog-to-digital converter, the uncertainty of a measurement x is given as a bounding range Δx^1 [?]. If $[x - \Delta x, x + \Delta x]$ crosses 0, x is neither positive nor negative for certainty due to the following two possibilities:

1. Either Δx is too large to give a precise measurement of x ;
2. Or x itself is a measurement of zero.

To distinguish which case it is, additional information is required so that the measurement $x \pm \Delta x$ itself is *insignificant* if $[x - \Delta x, x + \Delta x]$ crosses 0. An insignificant value also has conceptual difficulty in participating in many mathematical operations, such as calculating the square root or acting as a divisor.

$P \equiv \delta x/|x|$ is defined here as the (relative) *precision* of the measurement, whose inverse is commonly known as the *significance* [?][?]. Precision represents the reliable information content of a measurement. Finer precision means higher reliability and thus better reproducibility of the measurement [?][?]. Taking the traditional definition in measurement, precision in this paper does not mean the maximal bit count of significand as in the term “arbitrary precision arithmetic”² [?].

1.2 Problem of Conventional Floating-Point Arithmetic

The *conventional floating-point arithmetic* [?][?][?] assumes a constant and best-possible precision for each value all the time, and constantly generates artificial information during the calculation [?]. For example, the following calculation is carried out precisely in integer format:

$$\begin{aligned} 64919121 \times 205117922 - 159018721 \times 83739041 = \\ 13316075197586562 - 13316075197586561 = 1; \end{aligned} \quad (1.1)$$

If Formula (1.1) is carried out using conventional floating-point arithmetic:

$$\begin{aligned} 64919121 \times 205117922 - 159018721 \times 83739041 = \\ 64919121.000000000 \times 205117922.000000000 \\ - 159018721.000000000 \times 83739041.000000000 = \\ 13316075197586562. - 13316075197586560. = 2. = 2.0000000000000000; \end{aligned} \quad (1.2)$$

¹ x is normally an integer as the output of an ADC (Analog-to-Digital Converter). Ideally, Δx equals a half bit of ADC. Δx can be larger if the settle time is not long enough, or if the ADC is not ideal.

²Arbitrary precision integer means a digital integer which has arbitrary number of bits, while arbitrary precision arithmetic usually means fixed-point arithmetic [?] which has arbitrary fractional bits.

1. The multiplication results exceed the maximal significance of the 64-bit IEEE floating-point representation; so they are rounded off, generating rounding errors;
2. The normalization of the subtraction result amplifies the rounding error to most significant bit (MSB) by padding zeros.

Formula (1.2) is a showcase for the problem of conventional floating-point arithmetic. Because normalization happens after each arithmetic operation [?][?][?], such generation of rounding errors happens very frequently for addition and multiplication, and such amplification of rounding errors happens very frequently for subtraction and division. The accumulation of rounding errors is an intrinsic problem of conventional floating-point arithmetic [?], and in the majority of cases such accumulation is almost uncontrollable [?]. For example, because a rounding error from lower digits quickly propagates to higher digits, the 10^{-7} precision of the 32-bit IEEE floating-point format [?][?][?] is usually not fine enough for calculations involving input data of 10^{-2} to 10^{-4} precision.

Self-censored rules are developed to avoid such rounding error propagation [?][?], such as avoiding subtracting results of large multiplication, as in Formula (1.2). However, these rules are not enforceable, and in many cases are difficult to follow, e.g., even a most carefully crafted algorithm can result in numerical instability after extensive usage. Because the propagation speed of a rounding error depends on the nature of a calculation itself, e.g., generally faster in nonlinear algorithms than linear algorithms³ [?], propagation of rounding error in conventional floating-point arithmetic is very difficult to quantify generically [?]. Thus, it is difficult to tell if a calculation is improper or becomes excessive for a required result precision. In common practice, reasoning on an individual theoretical base is used to estimate the error and validity of calculation results, such as from the estimated transfer functions of the algorithms used in the calculation [?][?][?]. However, such analysis is both rare and generally very difficult to carry out in practice.

Today most experimental data are collected by an ADC (Analog-to-Digital Converter) [?]. The result obtained from an ADC is an integer with fixed uncertainty; thus, a smaller signal value has a coarser precision. When a waveform containing raw digitalized signals from ADC is converted into conventional floating-point representation, the information content of the digitalized waveform is distorted to favour small signals since all converted data now have the same and best possible precision. However, the effects of such distortion in signal processing are generally not clear.

What is needed is a floating-point arithmetic that tracks precision automatically. When the calculation is improper or becomes excessive, the results become insignificant. All existing uncertainty-bearing arithmetics are reviewed below.

1.3 Interval Arithmetic

Interval arithmetic [?][?][?][?][?][?] is currently a standard method to track calculation uncertainty. It ensures that the value x is absolutely bounded within its *bounding range* $[x] \equiv [\underline{x}, \bar{x}]$, in which \underline{x} and \bar{x} are lower and upper bounds for x , respectively. In this paper, interval arithmetic is simplified and tested as the following arithmetic formulas⁴

³A classic example is the contrast of the uncertainty propagation in the solutions for the 2nd-order linear differential equation vs. in those of Duffing equation (which has a x^3 term in addition to the x term in a corresponding 2nd-order linear differential equation).

⁴For the mathematical definition of interval arithmetic, please see [?].

[?]:

$$[x_1] + [x_2] = [\underline{x}_1 + \underline{x}_2, \bar{x}_1 + \bar{x}_2]; \quad (1.3)$$

$$[x_1] - [x_2] = [\underline{x}_1 - \bar{x}_2, \bar{x}_1 - \underline{x}_2]; \quad (1.4)$$

$$[x_1] \times [x_2] = [\min(\underline{x}_1 \underline{x}_2, \underline{x}_1 \bar{x}_2, \bar{x}_1 \underline{x}_2, \bar{x}_1 \bar{x}_2), \max(\underline{x}_1 \underline{x}_2, \underline{x}_1 \bar{x}_2, \bar{x}_1 \underline{x}_2, \bar{x}_1 \bar{x}_2)]; \quad (1.5)$$

$$0 \notin [x_2] : [x_1] / [x_2] = [x_1] \times [1/\bar{x}_2, 1/\underline{x}_2]; \quad (1.6)$$

If interval arithmetic is implemented using a floating-point representation with limited resolution, its resulting bounding range is widened further [?].

A basic problem is that the bounding range used by interval arithmetic is not compatible with usual scientific and engineering measurements, which instead use the statistical mean and deviations to characterize uncertainty [?][?]. Most measured values are well approximated by a Gaussian distribution [?][?][?], which has no limited bounding range. Let *bounding leakage* be defined as the possibility of the true value to be outside a bounding range. If a bounding range is defined using a statistical rule on bounding leakage, such as the $6\sigma - 10^{-9}$ rule for Gaussian distribution [?] (which says that the bounding leakage is about 10^{-9} for a bounding range of mean ± 6 -fold of standard deviations), there is no guarantee that the calculation result will also obey the $6\sigma - 10^{-9}$ rule using interval arithmetic, since interval arithmetic has no statistical foundation⁵.

Another problem is that interval arithmetic only provides the worst case of uncertainty propagation, so that it tends to over-estimate uncertainty in reality. For instance, in addition and subtraction, it gives the result when the two operands are +1 and -1 correlated respectively [?]. However, if the two operands are -1 and +1 correlated respectively instead, the actual bounding range after addition and subtraction reduces, which is called the best case in random interval arithmetic [?]. The vast overestimation of bounding ranges in these two worst cases prompts the development of affine arithmetic [?][?], which traces error sources using a first-order model. Being expensive in execution and depending on approximate modeling even for such basic operations as multiplication and division, affine arithmetic has not been widely used. In another approach, random interval arithmetic [?] reduces the uncertainty over-estimation of standard interval arithmetic by randomly choosing between the best-case and the worst-case intervals.

A third problem is that the results of interval arithmetic may depend strongly on the actual expression of an analytic function $f(x)$. For example, Formula (1.7), Formula (1.8) and Formula (1.9) are different expressions of the same $f(x)$; however, the correct result is obtained only through Formula (1.7), and uncertainty may be exaggerated in the other two forms, e.g., by 67-fold and 33-fold at input range $[0.49, 0.51]$ using Formula (1.8) and Formula (1.9), respectively. This is called the dependence problem of interval arithmetic [?].

$$f(x) = (x - 1/2)^2 - 1/4; \quad (1.7)$$

$$f(x) = x^2 - x; \quad (1.8)$$

$$f(x) = (x - 1)x; \quad (1.9)$$

⁵There is some attempt [?] to connect intervals in interval arithmetic to confidence interval or the equivalent so called p-box in statistics. Because this attempt seems to rely heavily on 1) specific properties of the uncertainty distribution within the interval and/or 2) specific properties of the functions upon which the interval arithmetic is used, this attempt does not seem to be generic. Anyway, this attempt seems to be outside the main course of interval arithmetic, which has no statistics in mind.

Interval arithmetic has very coarse and algorithm-specific precision but constant zero bounding leakage. It represents the other extreme from conventional floating-point arithmetic. To meet practical needs, a better uncertainty-bearing arithmetic should be based on statistical propagation of the rounding error, while also allowing reasonable bounding leakage for normal usages.

1.4 Statistical Propagation of Uncertainty

If each operand is regarded as a random variable, and the statistical correlation between the two operands is known, the resulting uncertainty is given by the *statistical propagation of uncertainty* [?][?], with the following arithmetic equations, in which σ is the deviation of a measured value x , P is its precision, and γ is the correlation between the two operands x_1 and x_2 :

$$(x_1 \pm \sigma_1) + (x_2 \pm \sigma_2) = (x_1 + x_2) \quad \pm \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\gamma}; \quad (1.10)$$

$$(x_1 \pm \sigma_1) - (x_2 \pm \sigma_2) = (x_1 - x_2) \quad \pm \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\gamma}; \quad (1.11)$$

$$(x_1 \pm \sigma_1) \times (x_2 \pm \sigma_2) = (x_1 \times x_2) \quad \pm |x_1 \times x_2| \sqrt{P_1^2 + P_2^2 + 2P_1P_2\gamma}; \quad (1.12)$$

$$(x_1 \pm \sigma_1)/(x_2 \pm \sigma_2) = (x_1/x_2) \quad \pm |x_1/x_2| \sqrt{P_1^2 + P_2^2 - 2P_1P_2\gamma}; \quad (1.13)$$

Tracking uncertainty propagation statistically seems an ideal solution. However, in practice, the correlation between two operands is generally not precisely known, so the direct use of statistical propagation of uncertainty is very limited. In this paper, as a proxy for statistical propagation of uncertainty, an *independence arithmetic* always assumes that no correlation exists between any two operands, whose arithmetic equations are Formula (1.10), Formula (1.11), Formula (1.12) and Formula (1.13), where $\gamma = 0$. Independence arithmetic is actually de facto arithmetic in engineering data processing, such as in the common belief that uncertainty after averaging reduces by the square root of number of measurements [?][?], or the ubiquitous Monte Carlo method⁶ [?][?], or calculating the mean and variance of a Taylor expansion [?].

1.5 Significance Arithmetic

Significance arithmetic [?] tries to track reliable bits in an imprecise value during the calculation. Except for two early attempts [?][?], significance arithmetic has not yet been implemented digitally. In these two attempts, the implementations of significance arithmetic are based on simple operating rules upon reliable bit counts, rather than on formal statistical approaches. They both treat the reliable bit counts as integers when applying their rules, while in reality a reliable bit count could be a fractional number, so they both can cause artificial quantum reduction of significance. Significance arithmetic is not widely practiced in scientific and engineering calculations [?].

Stochastic arithmetic [?][?], which can also be categorized as significance arithmetic, randomizes the least significant bits (LSB) of each of input floating-point values, repeats the same calculation multiple times, and then uses statistics to seek invariant

⁶Most but not all applications of Monte Carlo methods assume independence between any two random variables. In a minority of applications, a Monte Carlo method can be used to construct specified correlation between two random variables [?].

digits among the calculation results as significant digits. This approach may require too much calculation since the number of necessary repeats for each input is specific to each algorithm, especially when the algorithm contains branches. Its sampling approach may be more time-consuming and less accurate than direct statistical characterization [?], such as directly calculating the mean and deviation of the underlying distribution. It is based on modeling rounding errors in conventional floating-point arithmetic, which is quite complicated. A better approach may be to define arithmetic rules that make error tracking by probability easier.

As the mathematical foundation to significance arithmetic, when a uncertainty-bearing value is multiplied by a constant, the significance or relative precision still holds, while the absolute precision [?][?] scales with the constant. In this respect, fixed-point arithmetic [?], which assumes a fixed absolute precision, does not have a sounding mathematical foundation.

1.6 An Overview of This Paper

In this paper, a new floating-point arithmetic called *precision arithmetic* [?] is developed to track uncertainty during floating-point calculations, as described in Section 2. Generic standards and systematic methods for validating uncertainty-bearing arithmetics are discussed in Section 3. Precision arithmetic is compared with other uncertainty-bearing arithmetics in Section 4 to Section 8. A brief discussion is provided in Section 9.

2 Precision Arithmetic

2.1 Assumptions for Precision Arithmetic

As stated previously, the precision P is defined as the (relative) precision of a measurement in this paper. Precision arithmetic tracks uncertainty distribution during calculations using specially designed arithmetic rules. It has the *uncorrelated uncertainty assumption* as its basic assumption, presuming that the uncertainties of any two different values can be regarded as uncorrelated of each other. This assumption can be turned into a realistic statistical requirement for input data for precision arithmetic.

Because it is not realistic to track the actual uncertainty distributions, which may vary according to each specific algorithm, the objectives of precision arithmetic are to enclose the actual uncertainty distribution with a *bounding distribution*:

1. The bounding distribution is symmetric around an expected value which is the value given by mathematics when there is no uncertainty.
2. The bounding distribution is Gaussian, with deviations calculated by precision arithmetic.

As shown later in this paper, the objectives of precision arithmetic are extended from the central limit theorem [?].

In addition, precision arithmetic uses heavily the *scaling principle* which says that the result precision should not change when an imprecise value is either multiplied or divided by a non-zero constant. The scaling principle can be concluded from Formula (1.12) and Formula (1.13) for statistical propagation of uncertainty. It is also the foundation for significance arithmetic.

Related to the scaling principle, the *recovering principle* says that the imprecise result should restore the original imprecise value if mathematically the original value is restored conceptually, such as when an imprecise value is inverted twice. In precision arithmetic, the value of the imprecise value obeys the recovering principle, while it is questionable if the uncertainty of the imprecise value should be recovered.

2.2 The Uncorrelated Uncertainty Assumption

When there is a good estimation of the sources of uncertainty, the uncorrelated uncertainty assumption can be judged directly, e.g., if noise [?][?] is the major source of uncertainty, the uncorrelated uncertainty assumption is probably true. This criterion is necessary to ascertain repeated measurements of the same signal. Otherwise, the uncorrelated uncertainty assumption can be judged by the correlation and the respectively precisions of two measurements.

Let X , Y , and Z denote three mutually independent random variables [?] with variance $\sigma^2(X)$, $\sigma^2(Y)$ and $\sigma^2(Z)$, respectively. Let α denote a constant. Let $Cov()$ denote the covariance function. Let γ denote the correlation between $(X + Y)$ and $(\alpha X + Z)$. And let:

$$\eta_1^2 \equiv \frac{\sigma^2(Y)}{\sigma^2(X)}; \quad \eta_2^2 \equiv \frac{\sigma^2(Z)}{\sigma^2(\alpha X)} = \frac{\sigma^2(Z)}{\alpha^2 \sigma^2(X)}; \quad (2.1)$$

$$\gamma = \frac{Cov(X + Y, \alpha X + Z)}{\sqrt{\sigma^2(X + Y)} \sqrt{\sigma^2(\alpha X + Z)}} = \frac{\alpha/|\alpha|}{\sqrt{1 + \eta_1^2} \sqrt{1 + \eta_2^2}} \equiv \frac{\alpha/|\alpha|}{1 + \eta^2}; \quad (2.2)$$

Formula (2.2) gives the correlation γ between two random variables, each of which contains a completely uncorrelated part and a completely correlated part, with η being

the average ratio between these two parts. Formula (2.2) can also be interpreted reversely: if two random variables are correlated by γ , each of them can be viewed as containing a completely uncorrelated part and a completely correlated part, with η being the average ratio between these two parts.

One special application of Formula (2.2) is the correlation between a measured signal and its true signal, in which noise is the uncorrelated part between the two. Figure 1 shows the effect of noise on the most significant two bits of a 4-bit measured signal when $\eta = 1/4$. Its top chart shows a triangular waveform between 0 and 16 as a black line, and a white noise between -2 and +2, using the grey area. The measured signal is the sum of the triangle waveform and the noise. The middle chart of Figure 1 shows the values of the 3rd digit of the true signal as a black line, and the mean values of the 3rd bit of the measurement as a grey line. The 3rd bit is affected by the noise during its transition between 0 and 1. For example, when the signal is slightly below 8, only a small positive noise can turn the 3rd digit from 0 to 1. The bottom chart of Figure 1 shows the values of the 2nd digit of the signal and the measurement as a black line and a grey line, respectively. Figure 1 clearly shows that the correlation between the measurement and the true signal is less at the 2nd digit than at the 3rd digit. Quantitatively, according to Formula (2.2):

1. The overall measurement is 99.2% correlated to the signal with $\eta = 1/8$;
2. The 3rd digit of the measurement is 97.0% correlated to the signal with $\eta = 1/4$;
3. The 2nd digit of the measurement is 89.4% correlated to the signal with $\eta = 1/2$;
4. The 1st digit of the measurement is 70.7% correlated to the signal with $\eta = 1$;
5. The 0th digit of the measurement is 44.7% correlated to the signal with $\eta = 2$.

The above conclusion agrees with the common experiences that, below the noise level of measured signals, noises rather than true signals dominate each digit.

Similarly, while the correlated portion between two values has exactly the same value at each bit of the two values, the ratio of the uncorrelated portion to the correlated portion increases by 2-fold for each bit down from MSB of the two values, regardless of the nature of the uncorrelated portion. Quantitatively, let P denote the larger precision of the two values, and let η_P denote the ratio of the uncorrelated portion to the correlated portion at level of uncertainty; then η_P increases with decreased P according to Formula (2.3). According to Formula (2.2), if two significant values are overall correlated with γ , at the level of uncertainty the correlation between the two values decreases to γ_P according to Formula (2.4).

$$\eta_P = \frac{\eta}{P}, \quad P < 1; \quad (2.3)$$

$$\frac{1}{\gamma_P} - 1 = \left(\frac{1}{\gamma} - 1 \right) \frac{1}{P^2}, \quad P < 1; \quad (2.4)$$

Figure 2 plots the relation of γ vs. P for each given γ_P in Formula (2.4). When γ_P is less than a predefined maximal threshold (e.g., 2%, 5% or 10%), the two values can be deemed virtually uncorrelated of each other at the level of uncertainty. If the two values are independent of each other at their uncertainty levels, their uncertainties are uncorrelated of each other. Thus for each independence standard γ_P , there is a maximal allowed correlation between two values below which the uncorrelated uncertainty assumption of precision arithmetic holds. The maximal allowed correlation is a function of the larger precision of the two values according to Formula (2.4).

Figure 2 shows that for two precisely measured values, their correlation γ is allowed to be quite high. To be acceptable in precision arithmetic, each of the low-resolution values should contain enough noise in its uncertainty, so that they do not have much correction through the systematic error $[?][?]$. Thus, the uncertainty assumption uncertainty assumption has much weaker statistical requirement than the assumption for independence arithmetic, which requires the two values to be independent of each other.

It is tempting to add noise to otherwise unqualified values to make their uncertainties uncertainty assumption of each other. As an extreme case of this approach, if two values are constructed by adding noise to the same signal, they are 50% correlated at the uncertainty level so that they will not satisfy the uncorrelated uncertainty assumption⁷.

2.3 Precision Representation and Precision Round Up Rule

Let the content of a floating-point number be denoted as $S2^E$, in which S is the significand⁸ and E is the exponent of 2 of the floating-point number. In addition, the precision representation $S\sim 2^E$ contains a carry \sim to indicate its rounding error, which can be:

- $+$: The rounding error is positive;
- $-$: The rounding error is negative;
- $?$: The sign of the rounding error is unknown;
- $\#$: The precision value contains an error code. Each error code is generated due to a specific illegal arithmetic operation such as dividing by zero. An operand error code is directly transferred to the operation result. In this way, illegal operations can be traced back to the source.

Because there is only limited bits to hold S , a *round up* is needed, which proceeds according to the following round up rule:

- A value of $(2S)\sim 2^E$ is rounded up to $S\sim 2^{E+1}$.
- A value of $(2S+1)+2^E$ is rounded up to $(S+1)-2^{E+1}$.
- A value of $(2S+1)-2^E$ is rounded up to $S+2^{E+1}$.
- A value of $(2S+1)?2^E$ is rounded up to $(S+1)-2^{E+1}$.

Let the value before any rounding up be the original value, the round-up rule ensures that $S2^E$ is always the closest value with exponent E to the original value. After each round up, the original rounding error is reduced by half for the new significand. If the original significand is odd, the round up generates a new rounding error of $1/2$, which is added to the existing rounding error. Since the newly generated rounding error always cancels the existing rounding error, the rounding error range is limited to half bit of the significand, or the bounding range for the rounding error is $[-1/2, +1/2]$.

⁷The 50% curve in Figure 2 thus defines the maximal possible correlations between any two measured signals. This other conclusion of Formula (2.4) makes sense because the measurable correlation between two measurements should be limited by the precisions of their measurements.

⁸While “significand” is the official word $[?]$ to describe “The component of a binary floating-point number that consists of an explicit or implicit leading bit to the left of its implied binary point and a fraction field to the right”, “mantissa” is often unofficially used instead.

The precision arithmetic also tracks the rounding error bounding range R so that the precision representation becomes $S \sim R 2^E$.

If the initial \sim is wrong, it will be corrected by the first round up when S is odd, or R will be reduced to half after each round up when S is even. Hence, the precision round up process is stable and self-correcting.

2.4 Probability Distribution of Rounding Errors

An ideal floating-point calculation is carried out conceptually to infinitesimal precision before it is rounded up to representation precision $[?][?][?]$. Thus, rounding up should be a process independent of any calculation, and it should be evaluated separately. To estimate the rounding error distribution within its bounding range $[-1/2, +1/2]$, a large number⁹ of positive random integers are converted into precision values and then rounded up once at a step time until each of them has a significand smaller than a predefined minimal significand threshold. The precision value at each step is compared with the original value for the rounding error. Figure 3 shows the result histogram of rounding errors for the minimal significand thresholds 0, 1, 4 and 16, respectively. When each significand bit has an equal chance to be either 0 or 1, the result distribution of the rounding errors is expected to be uniformly distributed within the range $[-1/2, +1/2]$ $[?]$. However, the precision round up rule changes this equal chance for a few lowest digits of a significand. So when the minimal significand threshold is smaller, the bias in rounding error distribution is larger, as shown in Figure 3, and the result distribution is close to uniform only when the minimal significand threshold is 4 and above.

2.5 Result Uncertainty For Addition and Subtraction

In floating-point arithmetic, rounding errors are uncertainties $[?][?][?]$. The precision round-up rule incorporates all randomness of an imprecise value into its carry and bounding range so that it preserves the uncorrelated uncertainty assumption between any two values. The uncorrelated uncertainty assumption suggests that the result rounding error distribution of addition is the convolution of the two operand rounding error distributions, while the result rounding error distribution of subtraction is the convolution of the first operand rounding error distribution and the mirror image of the second operand rounding error distribution $[?]$. Thus, when the exponents of two operands are equal, the results of addition and subtraction are:

$$S_1 \sim_1 R_1 2^E \pm S_2 \sim_2 R_2 2^E = (S_1 \pm S_2) \sim (R_1 + R_2) 2^E; \quad (2.5)$$

Table 1 shows the result \sim for addition, while Table 2 shows the result \sim for subtraction. It will be shown that the \sim immediately after a calculation is actually not important because the precision round up rule frequently is applied after each calculation in precision arithmetic as its normalization process.

Let $P_{\frac{1}{2}}(x)$ be the rounding error distribution after rounding up, which is uniformly distributed between $[-1/2, +1/2]$ according to Formula (2.6). Let $P_{\frac{n}{2}}(x)$ be the con-

⁹For each minimal significand threshold, 64K random integers are used. The actual number of random integers is not important as far as 1) it gives a stable empirical histogram, and 2) the random integers are uniformly distributed without repeat in values.

\sim_1 vs. \sim_2	$\sim_1 = \sim_2$	$\sim_1 \neq \sim_2$				
		$\sim_1 = ?$	$\sim_2 = ?$	$R_1 > R_2$	$R_1 < R_2$	$R_1 = R_2$
\sim	\sim_1	\sim_2	\sim_1	\sim_1	\sim_2	$?$

Table 1: Result \sim in $S_1 \sim_1 R_1 2^E + S_2 \sim_2 R_2 2^E = (S_1 + S_2) \sim (R_1 + R_2) 2^E$

\sim_1 vs. \sim_2	$\sim_1 = \sim_2$				$\sim_1 \neq \sim_2$	
	$\sim_1 = ?$	$R_1 > R_2$	$R_1 < R_2$	$R_1 = R_2$	$\sim_1 = ?$	$\sim_1 \neq ?$
\sim	$?$	\sim_1	$-\sim_2$	$?$	$-\sim_2$	\sim_1

Table 2: Result \sim in $S_1 \sim_1 R_1 2^E - S_2 \sim_2 R_2 2^E = (S_1 - S_2) \sim (R_1 + R_2) 2^E$.

volution of $P_{\frac{1}{2}}(x)$ according to Formula (2.7):

$$P_{\frac{1}{2}}(x) \equiv 1, \quad -1/2 \leq x \leq +1/2; \quad (2.6)$$

$$P_{\frac{n}{2}}(x) \equiv \int_{-\infty}^{+\infty} P_{\frac{1}{2}}(y) P_{\frac{n-1}{2}}(x-y) dy = \int_{-1/2}^{+1/2} P_{\frac{n-1}{2}}(x-y) dy, \quad n = 2, 3, 4, \dots; \quad (2.7)$$

Formula (2.7) shows that $P_{\frac{n}{2}}(x)$ has a bounding range of $R \equiv \frac{n}{2}$, in which case it is easy to prove that the deviation σ of $P_{\frac{n}{2}}(x)$ is determined by its bounding range R :

$$\sigma^2 = R/6; \quad (2.8)$$

Also, the same bounding range can be reached in any combination:

$$P_{\frac{m+n}{2}}(x) = \int_{-\infty}^{+\infty} P_{\frac{m}{2}}(y) P_{\frac{n}{2}}(x-y) dy; \quad (2.9)$$

In reality, $P_{\frac{1}{2}}(x)$ is not strictly uniformly distributed in its bounding range $[-1/2, +1/2]$. As the worst case, let $P_{1/2}(x)$ be the rounding error distribution with the minimal significand threshold of 0 in Figure 3. Figure 4 shows the rounding error distribution after addition and subtraction, in which:

- $R=1/2$: “1” for no addition or subtraction.
- $R=2/2$: “1+1” for addition once, and “1-1” for subtraction once.
- $R=3/2$: “1+1+1” for addition twice, “1-1-1” for subtraction twice, “1+1-1” for addition once then subtraction once, and “1-1+1” for subtraction once then addition once.

Figure 4 shows that the rounding error distributions for the same bounding range largely repeat each other, confirming Formula (2.9). Addition and subtraction have a slightly different result distribution due to uneven $P_{\frac{1}{2}}(x)$. In all cases, the distributions quickly approach Gaussian with the increase of bounding ranges.

Even for the worst-case $P_{\frac{1}{2}}(x)$, the deviation σ relates to the bounding range R empirically as $\sigma = 0.423005 R^{0.50000}$ with a reliable factor of 0.9999999, confirming Formula (2.8) empirically.

The Lyapunov form of the central limit theorem [?] states that if X_i is a random variable with mean μ_i and variance σ_i^2 for each i among a series of n mutually independent random variables, then with increased n , the sum $\sum_i^n X_i$ converges in distribution

to the Gaussian distribution with mean $\sum_i^n \mu_i$ and variance $\sum_i^n \sigma_i^2$. Applying the central limit theorem to precision arithmetic when a calculation ends with summation:

- $P_{n/2}(x)$ converges in distribution to a Gaussian distribution of mean 0 and deviation σ with an increased n
- Figure 4 shows that such convergence to Gaussian distribution is very quick.
- The stable rounding error distribution is *independent* of any initial rounding error distribution, so that we can extend the rounding error distribution to uncertainty distribution in general.

Because of rounding, $P_{\frac{n}{2}}(x)$ is extended to $P_R(x)$ for 2's fractional R , which is further extended to characterize uncertainty distribution in general.

2.6 Uncertainty Distribution

The probability density function $D_y(y)$ after linear transformation ($y = \alpha x + \beta$) of a generic probability density function $D_x(x)$ is [?]:

$$D_y(y) = D_x((y - \beta)/\alpha)/\alpha; \quad (2.10)$$

Letting $N(x)$ be the density function of a normal distribution, the density function of uncertainty distribution in precision arithmetic is thus:

$$\rho(y) \equiv N(y/\sigma)/\sigma, \quad y \in [-R, +R]; \quad (2.11)$$

The deviation δx and bounding range Δx of $S \sim R 2^E$ is:

$$\delta x = \sigma \times 2^E, \quad \sigma < \sqrt{\hat{R}/6}; \quad (2.12)$$

$$\Delta x = R \times 2^E, \quad R < \hat{R}; \quad (2.13)$$

$$\Delta x/\delta x = R/\sigma = \sqrt{6R}; \quad (2.14)$$

Using δx and Δx , Formula (2.11) becomes the probability density function $\rho(\tilde{y})$ in Formula (2.15):

$$\rho(\tilde{y}) = N(\tilde{y}/\delta x)/\delta x, \quad \tilde{y} \in [-\Delta x, +\Delta x]; \quad (2.15)$$

And the precision representation $S \sim R 2^E$ is interpreted as:.

$$x \pm \delta x = x + \tilde{y}, \quad \tilde{y} \in \rho(\tilde{y}); \quad (2.16)$$

2.7 Uncertainty Rounding and Normalization

According to Formula (2.10), when an imprecise value is rounded up once, its density function becomes $N(y/\frac{\sigma}{2})/\frac{\sigma}{2}, y \in [-\frac{R}{2}, +\frac{R}{2}]$, however the two ways to carry out rounding up can not reach this new distribution ideally:

- By range: When R is reduced to 1/2-fold, σ is reduced to $1/\sqrt{2}$ -fold. The new $\rho(y)$ becomes $N(y/\frac{\sigma}{\sqrt{2}})/\frac{\sigma}{\sqrt{2}}, y \in [-\frac{R}{2}, +\frac{R}{2}]$. This approach distorts the probability but retains strict bounding, which aligns with interval arithmetic.
- By deviation: When σ is reduced to 1/2-fold, R is reduced to 1/4-fold. The new $\rho(y)$ becomes $N(y/\frac{\sigma}{2})/\frac{\sigma}{2}, y \in [-\frac{R}{4}, +\frac{R}{4}]$. This approach ignores the probability distribution on the Gaussian tails outside $[-\frac{R}{4}, +\frac{R}{4}]$, but preserves the overall characteristics of the distribution.

Figure 5 compares these two ways of rounding up when the original rounding error range is $R=8$, in which $R=4$ is rounded up by range, while $R=2$ is rounded up by deviation. It clearly shows that rounding up by deviation results in a more similar rounding error distribution. Rounding up by deviation is required by the scaling principle, so it is used universally in precision arithmetic. Rounding up by deviation also introduces bounding leakage called *round-up leakage*. In Figure 5, the 8/2 distribution of the rounding error outside the range $[-2, +2]$ contributes to a round-up leakage of 0.05%.

Because the tail of the Gaussian distribution decreases by e^{-6Ry^2} , the round-up leakage decreases exponentially with the increased bounding range R . Smaller round-up leakage also means that the actual rounding error distribution becomes more similar to the rounding error distribution with increased bounding range R . When R is above a threshold \hat{R} , round-up leakage is small enough so that rounding up by deviation can be applied repeatedly. This is the *normalization* process in precision arithmetic. When $\hat{R} = 16$, the maximal normalization leakage is 10^{-6} , which is small enough for most applications, and which has a comparable bounding range as the de facto $6\sigma - 10^{-9}$ rule for negligible bounding leakages in statistics. In addition to limit the bit count for both R and S , normalization also enforces the correctness of carry sign \sim in precision representation $S \sim R 2^E$.

Because the precision round up rule only looks at the sign of the current rounding error, rounding up by either deviation or range will not change the rounding error distribution.

As an inverse operation to rounding up by deviation, a precision round-down rule is defined using the scaling principle. After rounding down once, $S \sim R 2^E$ becomes $(2S) \sim (4R)2^E - 1$. Round down reduces bounding leakage. To add or subtract two operands with different exponents, the operand with a larger exponent is first rounded down to the other exponent, and the result of addition or subtraction using Formula (2.5) is normalized afterwards.

2.8 Uncertainty Initiation

An integer S is initialized as $S2^0$.

A conventional 64-bit floating-point value $S2^E$ is usually initialized as $S? \frac{1}{2}2^E$ because the IEEE floating-point standard [?] guarantees accuracy to half bit of a significand.

A mean-deviation pair $(x \pm \delta x)$ of 64-bit conventional floating-point values is initialized as $S \sim R 2^E$ by:

1. rounding up δx until Formula (2.12) is satisfied;
2. obtaining R and E from final δx ;
3. rounding up x to E ; and
4. obtaining S and \sim from final x .

2.9 Result Uncertainty For Multiplication

After $S \sim R 2^E$ is multiplied by 2, both its range and deviation increase by 2-fold. If the scaling principle is applied, the result is $2S \sim 2^2 R 2^E$. When $S \sim R 2^E$ is normalized, the result is then normalized as $S \sim R 2^{E+1}$, and there is neither bound widening nor bounding leakage. Generally, the direct result of multiplying $S_1 \sim_1 R_1 2^{E_1}$ by $S_2 2^{E_2}$ is:

$$S_1 \sim_1 R_1 2^{E_1} \times S_2 2^{E_2} = (S_1 S_2) \sim_1 (S_2^2 R_1) 2^{E_1 + E_2}; \quad (2.17)$$

Because Formula (2.17) obeys scaling principle, the result uncertainty is still ρ -distributed.

According to uncorrelated uncertainty assumption, the product bounding range of multiplying $0\sim_1 R_1 2^{E_1}$ by $0\sim_2 R_2 2^{E_2}$ is $R_1 R_2 2^{E_1+E_2}$. Thus:

$$\begin{aligned} S_1 \sim_1 R_1 2^{E_1} \times S_2 \sim_2 R_2 2^{E_2} &= (S_1 S_2) 2^{E_1+E_2} + 0(\sim_1 \sim_2)(R_1 R_2) 2^{E_1+E_2} \\ &\quad + 0\sim_1(S_2^2 R_1) 2^{E_1+E_2} + 0\sim_2(S_1^2 R_2) 2^{E_1+E_2}; \end{aligned} \quad (2.18)$$

In Formula (2.18), both $0\sim_1(S_2^2 R_1) 2^{E_1+E_2}$ and $0\sim_2(S_1^2 R_2) 2^{E_1+E_2}$ are ρ -distributed. The probability density function for $0(\sim_1 \sim_2)(R_1 R_2) 2^{E_1+E_2}$ is calculated as:

$$\rho_{mul}(x) = \frac{d}{dx} \int_{x_1 \times x_2 < x} \frac{1}{\sqrt{2\pi}\delta x_1} e^{-\frac{x_1^2}{2(\delta x_1)^2}} \frac{1}{\sqrt{2\pi}\delta x_2} e^{-\frac{x_2^2}{2(\delta x_2)^2}} dx_1 dx_2; \quad (2.19)$$

Letting $\delta x \equiv (\delta x_1)(\delta x_2)$, $y_1 \equiv x_1/(\delta x_1)$, $y_2 \equiv x_2/(\delta x_2)$ and $y \equiv x/(\delta x)$, Formula (2.19) is simplified as:

$$\rho_{mul}(x) = \frac{1}{2\pi\delta x} \frac{d}{dy} \int_{y_1 \times y_2 < y} e^{-\frac{y_1^2 + y_2^2}{2}} dy_1 dy_2; \quad (2.20)$$

Using polar coordinate (r, θ) instead Cartesian coordinate (y_1, y_2) , Formula (2.20) is simplified as:

$$\begin{aligned} \rho_{mul}(x) &= \frac{1}{2\pi\delta x} \frac{d}{dy} \int_0^{2\pi} d\theta \int_0^{\frac{y}{2\sin(\theta)\cos(\theta)}} e^{-\frac{r^2}{2}} d\frac{r^2}{2} \\ &= \frac{1}{2\pi\delta x} \frac{d}{dy} \int_0^{2\pi} d\theta \left(1 - e^{-\frac{y}{\sin(2\theta)}}\right) = \frac{1}{2\pi\delta x} \int_0^\pi \frac{e^{-\frac{|y|}{\sin(\theta)}}}{\sin(\theta)} d\theta; \end{aligned} \quad (2.21)$$

Similarly, letting $\delta x \equiv \sqrt{(\delta x_1)^2 + (\delta x_2)^2}$, $y_1 \equiv x_1/(\delta x_1)$, $y_2 \equiv x_2/(\delta x_2)$ and $y \equiv x/\delta x$, the distribution for $0\sim R_1 2^E + 0\sim R_2 2^E = 0\sim(R_1 + R_2) 2^E$ is calculated as:

$$\begin{aligned} \rho_{add}(x) &= \frac{1}{2\pi\delta x} \frac{d}{dy} \int_0^{2\pi} d\theta \int_0^{\frac{y}{\sin(\theta) + \cos(\theta)}} e^{-\frac{r^2}{2}} r dr \\ &= \frac{1}{2\pi\delta x} \frac{d}{dy} \int_0^{2\pi} d\theta \left(1 - e^{-\frac{y^2}{2} \frac{1}{1 + \sin(2\theta)}}\right) = \frac{1}{2\pi\delta x} \int_0^{2\pi} d\theta |y| \frac{e^{-\frac{y^2}{2} \frac{1}{1 + \sin(2\theta)}}}{1 + \sin(2\theta)}; \end{aligned} \quad (2.22)$$

The result of Formula (2.22) is known as Formula (2.23). Because $2\sin(\theta)|_{\theta \in [0, \pi]}$ almost repeats $1 + \sin(\theta)|_{\theta \in [-\frac{\pi}{2}, \frac{3\pi}{2}]}$, the result of Formula (2.21) is estimated as Formula (2.24) and Formula (2.25):

$$y = \frac{x}{\sqrt{\delta x_1^2 + \delta x_2^2}} : \quad \rho_{add}(x) = \frac{1}{\sqrt{2\pi}\sqrt{\delta x_1^2 + \delta x_2^2}} e^{-\frac{y^2}{2}}; \quad (2.23)$$

$$y = \frac{x}{\delta x_1 \delta x_2} : \quad \rho_{mul}(x) = \frac{1}{\sqrt{2\pi}(\delta x_1 \delta x_2)} \frac{e^{-2|y|}}{\sqrt{|y|}}; \quad (2.24)$$

$$z \equiv 2\sqrt{y} = 2\sqrt{\frac{x}{\delta x_1 \delta x_2}} : \quad \rho_{mul}(x) = \frac{1}{\sqrt{2\pi}\sqrt{\delta x_1 \delta x_2}} e^{-\frac{z^2}{2}}; \quad (2.25)$$

Formula (2.25) shows that $0(\sim_1 \sim_2) R_1 R_2 @ (E_1 + E_2)$ is ρ -distributed with deviation $\sqrt{(\delta x_1)(\delta x_2)}$ or range $R_1 R_2$, which adds to the rest two ρ -distributed terms of Formula (2.18). Thus, the result of multiplication is also ρ -distributed.

The result precision P of multiplication is:

$$P^2 = \frac{R_1 R_2 + S_2^2 R_1 + S_1^2 R_2}{(S_1 S_2)^2} = P_1^2 + P_2^2 + \frac{1}{6} P_1^2 P_2^2; \quad (2.26)$$

Formula (2.26) shows that precision cannot be improved during multiplication. It is identical to Formula (1.12) except their cross term, representing difference in their statistical requirements.

2.10 Result Uncertainty For Division

The reverse of Formula (2.17) defines:

$$\frac{S_1 \sim_1 R_1 2^{E_1}}{S_2 2^{E_2}} = \frac{S_1}{S_2} \sim_1 \frac{R_1}{S_2^2} 2^{E_1 - E_2} \quad (2.27)$$

In Formula (2.27), the rounding error decreases by S_2 -fold, but the bounding range decreases by S_2^2 -fold, so there is bounding leakage. To limit the bounding leakage to acceptable level, the result is rounded down until normalized. In this case, rounding down the direct result is equivalent to round down the dividend $S_1 \sim R_1 2^{E_1}$ by the same amount before the division.

Using the methodology defined in [?], the probability density function for $1/(x + \delta x)$ is calculated as:

$$\rho_{inv}(y) = \frac{1}{\sqrt{2\pi}\delta x} \frac{e^{-\frac{(\frac{1}{y} - x)^2}{2(\delta x)^2}}}{y^2} \quad (2.28)$$

Letting $z \equiv (1/y - x)/\delta x$, whose range is $\Delta x/\delta x = \sqrt{6\bar{R}}$ according to Formula (2.14), the mean for $1/(x + \delta x)$ is calculated as:

$$\mu\left(\frac{1}{x}\right) = \frac{1}{\sqrt{2\pi}\delta x} \int_{|\frac{1}{y} - x| < \Delta x} y \rho_{inv}(y) dy = \frac{1}{x\sqrt{2\pi}} \int_{-\frac{\Delta x}{\delta x}}^{+\frac{\Delta x}{\delta x}} \frac{e^{-\frac{z^2}{2}}}{zP(x) + 1} dz; \quad (2.29)$$

Formula (2.29) diverges at $z = -1/P(x)$, which is excluded from the integration range when $\Delta x \ll |x|$. In such a case, replacing $\Delta x/\delta x$ with $\bar{R} \equiv \sqrt{6\bar{R}}$ for the integration range, the mean is calculated as:

$$M(j) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\bar{R}}^{+\bar{R}} z^j e^{-\frac{z^2}{2}} dz; \quad (2.30)$$

$$\mu\left(\frac{1}{x}\right) = \frac{1}{x} \sum_{j=0}^{+\infty} P(x)^j M(j); \quad (2.31)$$

\hat{R} has to be small enough so that Formula (2.31) can converge for non-zero $P(x)$.

The moment for uncertainty distribution is generally defined in Formula (2.30), which is 0 when j is odd. $M(2j)$ is calculated as:

$$\begin{aligned} M(2j) &= (2j-1)M(2j-2) - \frac{2}{\sqrt{2\pi}} \bar{R}^{2j-1} e^{-\frac{\bar{R}^2}{2}} \\ &= (2j-1)!!(M(0) - \frac{2}{\sqrt{2\pi}} e^{-\frac{\bar{R}^2}{2}} \sum_{k=1}^j \frac{\bar{R}^{2k-1}}{(2k-1)!!}); \end{aligned} \quad (2.32)$$

Formula (2.33) [?] shows that $M(2j)$ eventually becomes 0 for large enough j :

$$\begin{aligned} \lim_{j \rightarrow \infty} \sum_{k=1}^j \frac{\bar{R}^{2k-1}}{(2k-1)!!} &= e^{\frac{\bar{R}^2}{2}} \int_0^{+\bar{R}} e^{-\frac{z^2}{2}} dz; \\ \lim_{j \rightarrow \infty} M(2j) &= (2j-1)!!(M(0) - M(0)) = 0; \end{aligned} \quad (2.33)$$

Unfortunately, such convergence to 0 is very slow, and lower orders of $M(2j)$ actually increases exponentially according to Figure 6. Empirically:

$$j < 400 : \quad M(2j) = (2.4429 \hat{R}^{0.4998})^{2j}; \quad (2.34)$$

Specifically, $\hat{R} = 4$: $M(2j) \sim 4.8844^{2j}$. When $P(x) < 1/4.8844 = 0.20473$, $1/(x + \delta x)$ converges so that inversion can be defined statistically:

$$\mu\left(\frac{1}{x}\right) = \frac{1}{x}(1 + P(x)^2 + 3P(x)^4 + 15P(x)^6 + \dots); \quad (2.35)$$

However, Formula (2.35) does not obey the recovering principle, so that a better approach is to let the value of inversion to be $1/x$ and the inversion deviation defined around $1/x$, as:

$$\begin{aligned} \left(\delta \frac{1}{x}\right)^2 &\equiv \frac{1}{\sqrt{2\pi}\delta x} \int_{|\frac{1}{y}-x|<\Delta x} \left(y - \frac{1}{x}\right)^2 \rho_{inv}(y) dy \\ &= \frac{1}{x^2 \sqrt{2\pi}} \int_{-\bar{R}}^{+\bar{R}} e^{-\frac{z^2}{2}} \left(\frac{1}{zP(x)+1} - 1\right)^2 dz; \\ P\left(\frac{1}{x}\right)^2 &= \sum_{j=1}^{+\infty} P(x)^{2j} M(2j)(2j-1) = P(x)^2 + 9P(x)^4 + 75P(x)^6 + \dots; \end{aligned} \quad (2.36)$$

An equivalent way to calculate Formula (2.36) is to use Taylor expansion around $1/x$ in $\rho(y)$ space instead of in $\rho_{inv}(y)$ space:

$$\begin{aligned} \left(\delta \frac{1}{x}\right)^2 &\equiv \frac{1}{\sqrt{2\pi}\delta x} \int_{|\frac{1}{y}-x|<\Delta x} \left(y - \frac{1}{x}\right)^2 \rho(y)_{inv} dy \\ &= \frac{1}{\sqrt{2\pi}\delta x} \int_{|\frac{1}{y}-x|<\Delta x} \left(\frac{1}{y} - \frac{1}{x}\right)^2 \rho(y) dy = \frac{1}{x^2 \sqrt{2\pi}} \int_{-\bar{R}}^{+\bar{R}} e^{-\frac{z^2}{2}} \left(\frac{1}{zP(x)+1} - 1\right)^2 dz; \end{aligned} \quad (2.37)$$

Formula (2.36) obeys the recovering principle for uncertainty only approximately when $P(x)^2 \ll 1$.

The sign of the representation $S \sim R 2^E$ becomes negation of itself after inversion. Dividing an operand by itself results in precise 1.

2.11 Result Uncertainty For Square and Square Root

A special case of multiplication is between an operand and itself. If x is $N(x)$ distributed, x^2 is χ^2 -distributed with freedom 1 [?], which has mean 1 and variance of 2. $N(x)$ and $\chi^2(x)$ have quite different characteristics, e.g., $\chi^2(x)$ only roughly resembles half of $N(x)$. The bounding goal of precision arithmetic extends χ^2 distribution to

the other side of the mathematically expected value, and absorb the square of mean of the χ^2 distribution into final variance:

$$(S_1 \sim R_1 2^{E_1})^2 \equiv S_1^2 2^{2E_1} + 0 \sim 4R_1 S_1^2 2^{2E_1} + 0 \sim 3R_1^2 2^{2E_1}; \quad (2.38)$$

The result precision of square is:

$$P^2 = \frac{4R_1 S_1^2 + 3R_1^2}{S_1^4} = 4P_1^2 + 3P_1^4; \quad (2.39)$$

Similar to Formula (2.37), with $z = (y - x)/\delta x$ the deviation for x^2 is calculated as Formula (2.40), which confirms Formula (2.39):

$$\begin{aligned} (\delta x^2)^2 &= \frac{1}{\sqrt{2\pi}\delta x} \int_{|y^2 - x^2| < \Delta x} (y^2 - x^2)^2 \rho(y) dy \\ &= \frac{x^4}{\sqrt{2\pi}} \int_{-\bar{R}}^{+\bar{R}} e^{-\frac{z^2}{2}} ((1 + zP(x))^2 - 1)^2 dz; \end{aligned} \quad (2.40)$$

$$P(x^2)^2 = 4M(2)P(x)^2 + 4M(3)P(x)^3 + M(4)P(x)^4 = 4P(x)^2 + 3P(x)^4;$$

And the deviation for \sqrt{x} is calculated as Formula (2.41).

$$\begin{aligned} (\delta\sqrt{x})^2 &= \frac{1}{\sqrt{2\pi}\delta x} \int_{|\sqrt{y} - \sqrt{x}| < \Delta x} (\sqrt{y} - \sqrt{x})^2 \rho(y) dy \\ &= \frac{|x|}{\sqrt{2\pi}} \int_{-\bar{R}}^{+\bar{R}} e^{-\frac{z^2}{2}} (\sqrt{1 + zP(x)} - 1)^2 dz \\ &= \frac{|x|}{\sqrt{2\pi}} \int_{-\bar{R}}^{+\bar{R}} e^{-\frac{z^2}{2}} \left(\frac{zP(x)}{2} - \frac{z^2 P(x)^2}{8} + \frac{z^3 P(x)^3}{16} + \dots \right)^2 dz; \\ P(\sqrt{x})^2 &= \frac{1}{4}P(x)^2 + \frac{15}{64}P(x)^4 + \dots; \end{aligned} \quad (2.41)$$

The above Taylor-expansion method can be extended to calculate power x^n and root $\sqrt[n]{x}$ for any integer n .

The combination of Formula (2.40) and Formula (2.41) as Formula (2.42) shows again that the uncertainty deviation obeys the recovery principle only approximately when $P(x)^2 \ll 1$. Furthermore, the result of $\sqrt{x^2}$ and $(\sqrt{x})^2$ are different in Formula (2.42), showing dependency problem. The reason why the uncertainty deviation can not obey the recovery principle strictly is not clear at this moment.

$$\begin{aligned} P(\sqrt{x^2})^2 &= \frac{1}{4}(4P(x)^2 + 3P(x)^4) + \frac{15}{64}(4P(x)^2 + 3P(x)^4) + \dots \\ &= P(x)^2 + \frac{9}{2}P(x)^4 + \dots; \\ P((\sqrt{x})^2)^2 &= 4\left(\frac{1}{4}P(x)^2 + \frac{15}{64}P(x)^4 + \dots\right) + 3\left(\frac{1}{4}P(x)^2 + \frac{15}{64}P(x)^4 + \dots\right)^2; \\ &= P(x)^2 + \frac{9}{8}P(x)^4 + \dots; \end{aligned} \quad (2.42)$$

2.12 Result Uncertainty For Function Evaluation

Extending Formula (2.37), the uncertainty of the function $f(x)$ at $(x \pm \delta x)$ is evaluated by the set $\{f(x + \tilde{y}) - f(x)\}$, as Formula (2.43), in which \tilde{y} and $\rho(\tilde{y})$ are defined by

Formula (2.15).

$$(\delta f)^2 \equiv \int (f(x + \tilde{y}) - f(x))^2 \rho(\tilde{y}) d\tilde{y}; \quad (2.43)$$

For example, Formula (2.44) provides uncertainty for generic polynomial to J -th order.

$$\begin{aligned} C_j^k &\equiv \frac{j!}{k!(j-k)!}; \\ (\delta \sum_{j=0}^J a_j x^j)^2 &= \int (\sum_{j=0}^J a_j (x + \tilde{y})^j - \sum_{j=0}^J a_j x^j)^2 \rho(\tilde{y}) d\tilde{y} \\ &= \int (\sum_{j=1}^J a_j \sum_{k=1}^j C_j^k \tilde{y}^k x^{j-k})^2 \rho(\tilde{y}) d\tilde{y} \\ &= \sum_{j_1=1}^J \sum_{j_2=1}^J a_{j_1} a_{j_2} \sum_{k_1=1}^{j_1} \sum_{k_2=1}^{j_2} C_{j_1}^{k_1} C_{j_2}^{k_2} M(k_1 + k_2) (\delta x)^{k_1 + k_2} x^{j_1 + j_2 - k_1 - k_2}; \end{aligned} \quad (2.44)$$

If the function $f(x)$ is Taylor expandable at x , $f(x + \tilde{y}) - f(x)$ is calculated according to Formula (2.45), in which $f^{(n)}$ denotes the n th derivatives of $f(x)$ at x . $(\delta f)^2$ is given by Formula (2.46), in which $M(j)$ is defined by Formula (2.30) and (2.30).

$$f(x + \tilde{y}) - f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(x)}{n!} \tilde{y}^n; \quad (2.45)$$

$$(\delta f)^2 = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{f^{(n)}(x)}{n!} \frac{f^{(j)}(x)}{j!} (\delta x)^{n+j} M(n+j) - f(x)^2; \quad (2.46)$$

Let $P(f(x)) \equiv \delta f(x)/|f(x)|$ be defined as the precision for $f(x)$; and let α be a constant. According to Formula (2.46):

$$\delta(\alpha x) = \alpha(\delta x); \quad P(\alpha x) = P(x); \quad (2.47)$$

$$(\delta \frac{1}{x})^2 = \frac{1}{x^2} \sum_{j=1}^{\infty} (2j-1) M(2j) P(x)^{2j}; \quad P(\frac{1}{x})^2 = \sum_{j=1}^{\infty} (2j-1) M(2j) P(x)^{2j}; \quad (2.48)$$

$$(\delta x^2)^2 = 4x^2 (\delta x)^2 + 3(\delta x)^4; \quad P(x^2)^2 = 4P(x)^2 + 3P(x)^4; \quad (2.49)$$

Formula (2.47) confirm the scaling principle, while Formula (2.49) and (2.48) confirm Formula (2.39) and (2.36), respectively.

Due to uncorrelated uncertainty assumption, Taylor expansion can also be used to find the result $(\delta f)^2$ of the function $f(x_1, x_2)$, in which $f^{(m,n)}$ denotes the m th and n th partial derivatives of x_1 and x_2 , respectively; and the uncorrelated uncertainty assumption between x_1 and x_2 leads to independence between the random variables \tilde{y}_1 and \tilde{y}_2 in Formula (2.51):

$$f(x_1 + \tilde{y}_1, x_2 + \tilde{y}_2) - f(x_1, x_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{f^{(m,n)}}{m!n!} \tilde{y}_1^m \tilde{y}_2^n - f(x_1, x_2); \quad (2.50)$$

$$(\delta f)^2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{f^{(m,n)}}{m!n!} \frac{f^{(i,j)}}{i!j!} (\delta x_1)^{m+i} (\delta x_2)^{n+j} M(m+i) M(n+j) - f(x_1, x_2)^2; \quad (2.51)$$

Such an approach can be extended to a function of an arbitrary number of input variables. Formula (2.51) shows that an input contributes to the result uncertainty in more than one way, in the same way as Δx appears in more than one term in the Taylor expansion of $\Delta f(x, y, z)$.

According to Formula (2.51):

$$\delta(x_1 \pm x_2)^2 = (\delta x_1)^2 + (\delta x_2)^2; \quad (2.52)$$

$$\delta(x_1 \times x_2)^2 = (\delta x_1)^2 x_2^2 + x_1^2 (\delta x_2)^2 + (\delta x_1)^2 (\delta x_2)^2; \quad (2.53)$$

$$P(x_1 \times x_2)^2 = P(x_1)^2 + P(x_2)^2 + \frac{1}{6} P(x_1)^2 P(x_2)^2; \quad (2.54)$$

$$\delta\left(\frac{x_1}{x_2}\right)^2 = \frac{(\delta x_1)^2}{x_2^2} + x_1^2 \left(\delta \frac{1}{x_2}\right)^2 + (\delta x_1)^2 \left(\delta \frac{1}{x_2}\right)^2 \simeq \frac{\delta(x_1 \times x_2)^2}{x_2^4}; \quad (2.55)$$

Formula (2.52) and (2.54) confirm Formula (2.5) and (2.26), respectively.

If $f(x)$ is a black-box function, because the normal distribution $N(x)$ is well known, standard methods exist to divide the range $[x - \Delta x, x + \Delta x]$ into equal-probability quantiles [?], and δf can be found numerically by sampling. For example, a κ -point monotonic sampling requires that 1) each numerically monotonic region contains at least κ consecutive sampling points; and 2) the whole range $[x - \Delta x, x + \Delta x]$ has been divided into monotonic regions only. When κ is sufficiently large, the chance of missing a peak or a valley is small enough so that the sampling is fair enough. The range $[x - \Delta x, x + \Delta x]$ is first divided into κ equal-probability quantiles, and at each additional step, each quantile is further divided into an equal number of sub quantiles until both sampling requirements are met. Then δf^2 is calculated as the sum of 1) the sample variance and 2) the square of the sample mean.

2.13 Dependence Problem

Formula (2.46) and its multi-dimension extensions such as Formula (2.51) accurately account for all contribution to the result uncertainty within $f(x)$, providing a clean and deterministic solution for $(\delta f)^2$, e.g., it gives the same result for Formula (1.7), Formula (1.8) and Formula (1.9). Therefore, precision arithmetic has no expression-based dependence problem.

It is tempting to define basic arithmetic operations as Formula (2.52), Formula (2.53) and Formula (2.55), and apply them progressively to calculate $f(x)$, similar to how basic arithmetic operations are used in conventional floating arithmetic. However, such an approach may apply the uncorrelated uncertainty assumption wrongly between a value and its mathematical expression, such as between x and x^2 , so that it may result in the dependence problem similar to that of interval arithmetic [?]. For example, for such a use of precision arithmetic, only Formula (1.7) gives the correct result $4(x - \frac{1}{2})^2(\delta x)^2 + 3(\delta x)^4$, while Formula (1.8) over-estimates the result uncertainty by $4x(\delta x)^2$, which has the largest fold of over-estimation at $x = \frac{1}{2}$ when $\delta x < 1$. Let x , y and z be three values satisfying the uncorrelated uncertainty assumption. Functions $f(x, y)$ and $g(x, z)$ are correlated through x , and they need to be tested for the uncorrelated uncertainty assumption before they can be used to calculate $h(f, g)$ using precision arithmetic. For example, using precision arithmetic, the correlation γ between $(x \pm \delta x)$ and $(x \pm \delta x)^2$ is calculated by Formula (2.56), which shows that γ increases with decreased precision $P(x) \equiv \delta x/|x|$ of x , in contrast with how the uncorrelated uncertainty assumption favors finer precision P in Formula (2.4). After applying Formula (2.4), the correlation on the uncertainty level γ_P is no less than $\frac{16}{19}$,

so that precision arithmetic rejects calculating Formula (1.8) progressively using the basic arithmetic operations.

$$\gamma = \frac{\int ((x + \tilde{y})^2 - x^2) ((x + \tilde{y}) - x) \rho(\tilde{y}) d\tilde{y}}{\sqrt{\int ((x + \tilde{y})^2 - x^2)^2 \rho(\tilde{y}) d\tilde{y}} \sqrt{\int ((x + \tilde{y}) - x)^2 \rho(\tilde{y}) d\tilde{y}}} = \frac{1}{\sqrt{1 + \frac{3}{4}P(x)^2}} \quad (2.56)$$

In other words, converting a numerical algorithm from using conventional floating-point arithmetic to using precision arithmetic may be more complicated than directly replacing the variable types and the arithmetic being used. To avoid the dependence problem, the safest approach is to obtain an analytic form of the final expression of an algorithm before applying Formula (2.46) and its multi-dimension extensions, similar to how symbolic calculations are currently used in affine arithmetic [?].

2.14 Conditional Execution

Conditional execution based on the comparison relation between two values is frequently used in practical algorithms [?]. When each value has associated uncertainty, the comparison relation between two values becomes quite different. This is particularly true for interval arithmetic, in which a value can be anywhere inside its bounding range [?]. In precision arithmetic, each value has a mathematically expected value plus a well-defined bounding distribution for uncertainty. The comparison relation between two imprecise values in precision representation can be defined either by their mathematically expected values, or by their statistical comparison relations based on confidence [?].

However, the usage of condition execution in a traditional algorithm needs to be re-evaluated conceptually with uncertainty statistics in mind when upgrading an algorithm to use precision arithmetic, because most conditional executions are created to optimize implementation. For example, LU decomposition [?] carefully chooses the sequence of execution to minimize rounding errors, so that it introduces additional dependence problem due to conditional execution, e.g., small value change of a matrix item can result in different conditional execution path and large result difference. In other words, to solve the linear equation $Ax = b$, in which A is a matrix, x and b are two vectors, with the uncertainty of A^{-1} analytically solvable using Taylor expansion (as demonstrated in Section 5), precision arithmetic prefers to solve it as $x = A^{-1}b$ than to use the LU decomposition method.

2.15 Calculation Inside Uncertainty

Formula (2.57) shows that the current choice of \hat{R} calculates 0-bit inside uncertainty, e.g., 1 with precision 10^{-3} is represented as 1024 in S . In contrast, all other arithmetic represents the value as $2^{53}/2^{53}$. While calculating many bits inside uncertainty does not seem meaningful according to significance arithmetic [?], not calculating at all inside uncertainty may not be an optimal approach either. Thus, Formula (2.12) is modified as Formula (2.58), in which χ is a small constant positive integer, to introduce the χ -bit calculation inside uncertainty by providing an altered interpretation of the precision for $S \sim R 2^E$.

$$\begin{aligned} 1/3 \pm 0.001 + 2/3 \pm 0.001 &= 341 + 6.29 \cdot 2^{-10} + 683 - 6.29 \cdot 2^{-10} \\ &= 1024 \pm 12.6 \cdot 2^{-10} = 1 \pm 0.001\sqrt{2}; \end{aligned} \quad (2.57)$$

$$\delta x = \sqrt{R/6 \cdot 2^{-\chi}} 2^{E+\chi}; \quad (2.58)$$

Table 3 shows examples of precision arithmetic with different χ for $\hat{R} = 16$, e.g., with $\chi = 2$, precision arithmetic represents the expected value of 1.000 ± 0.001 as $2^{12}/2^{12}$. χ will be set to 4 empirically later in this paper.

χ	0	1	2
$0.5 \pm 0.001 =$	$512?6.29 \ 2^{-10}$	$1024?12.58 \ 2^{-11}$	$2048?25.16 \ 2^{-12}$
$1 \pm 0.001 =$	$1024?6.29 \ 2^{-10}$	$2048?12.58 \ 2^{-11}$	$4096?25.16 \ 2^{-12}$
$1 \pm 0.002 =$	$512?6.29 \ 2^{-9}$	$1024?12.58 \ 2^{-10}$	$2048?25.16 \ 2^{-10}$

Table 3: Examples of precision arithmetic with different χ for $\hat{R} = 16$, in which χ stands for bits calculated inside uncertainty.

The limited calculation inside uncertainty does not necessarily mean that precision arithmetic has a larger calculation error. In Formula eqn: addition with 0-bit inside uncertainty, the mathematically expected value for the result is precisely 1, even though the mathematically expected values of the two operands for addition are not precisely 1/3 and 2/3 after the uncertainty initiation, respectively.

2.16 Implementation

The conventional 64-bit floating-point standard IEEE-754 [?][?] has:

- 11 bits for storing exponent E ;
- 53 bits for storing significand S (with a hidden MSB).
- 1 bit for storing sign;

To be a super set of the conventional 64-bit floating-point standard, an 80-bit implementation of precision arithmetic has:

- 11 bits for storing exponent E ;
- 53 bits for storing significand S (without using the hidden MSB);
- 1 bit for storing sign;
- 2 bits for storing carry \sim ;
- 13 bits to store the bounding range R as a fixed-point value.

Precision arithmetic is implemented in C++. With heavy additional codes to count for statistics and to detect implementation errors, it runs about seven times slower than the implementation of interval arithmetic using Formula (1.3), (1.4), (1.5) and (1.6). It is probably faster in speed than the implementation of interval arithmetic without the dependence problem [?]. With code weight trimming and optimization, its speed is expected to be improved at least threefold. Unlike conventional floating-point arithmetic, it only calculates a limited number of significand bits, e.g., 12 bits instead of all the 53 bits when the precision is 10^{-3} and χ is 2. Its slowest but very frequent operation is to find the position of the highest non-zero significand digit, which can be found instantly with a decoder [?]. Thus, future hardware optimization can also improve the speed of precision arithmetic by another estimated tenfold.

2.17 Alternative Form of Precision Arithmetic

Because the uncorrelated uncertainty assumption can lead directly to 1) the Gaussian distribution as the underlying distribution for rounding errors, and 2) Formula (2.46) and its multi-dimension extensions such as Formula (2.51) for generic Taylor expansion, an *alternative form of precision arithmetic* is to represent each uncertainty-bearing value as $x \pm \delta x$ in Formula (2.16). The bounding range is then calculated from δx as the confidence interval [?] for any required upper limit on bounding leakage, e.g., if the required bounding leakage is 10^{-9} or less, the bounding interval is $[x - 6\delta x, x + 6\delta x]$. This alternative form of precision arithmetic is *not* adopted in this paper for the following reasons:

- For the actual numerical calculation, if conventional floating-point arithmetic is used separately for x and δx , then x and δx will be contaminated by unspecified amount of rounding errors. Because the calculation for δx is more complex than that for x , δx probably contains more rounding error than x . Thus, the current form of precision arithmetic defines its own floating-point representation for $x \pm \delta x$ as $S \sim R \cdot 2^E$.
- Another effect of using conventional floating-point arithmetic for x and δx is to calculate many bits inside uncertainty, whose validity is not clear at this moment. In contrast, as demonstrated by Table 3, the current form of precision arithmetic controls the number of bits calculated inside uncertainty.

However, the alternative form could be valuable in theoretical discussions of precision arithmetic.

2.18 Types of Uncertainties Included in Precision Arithmetic

There are four sources of result uncertainty after a calculation [?][?]:

- input uncertainties
- rounding errors
- truncation errors
- modelling errors

As described previously, both input uncertainties and rounding errors are included in the uncertainty specification of precision arithmetic.

In many cases, because a numerical algorithm approaches its analytic counterpart only after infinite execution, a good numerical algorithm should have an estimator of the *truncation error* toward its analytic counterpart, such as the Cauchy remainder estimator for Taylor expansion [?], or the residual error for numerical integration [?]. Using conventional floating-point arithmetic, a subjective upper limit is chosen for the truncation error, to stop the numerical algorithm at limited execution [?]. However, such arbitrary upper limit may not be achievable with the amount of rounding errors accumulated during calculation, so that such upper limit may actually give a falsely small result precision. Because precision arithmetic tracks rounding errors of a calculation efficiently, it can be used to search for the optimal execution termination point for the numerical algorithm when the truncation error is no longer significant, which is named as the *truncation rule* in this paper. In other words, using precision arithmetic, the result precision of a calculation is determined by the inputs and the

calculation itself. Section 8 will provide such a case of applying truncation rule to numerical integration.

Modelling errors arise when an approximate analytic solution is used, or when a real problem is simplified to obtain the solution. For example, Section 4 demonstrates that the discrete Fourier transformation is only an approximation for the mathematically defined Fourier transformation. Conceptually, modelling errors originate from mathematics, so they are outside the domain for precision arithmetic.

3 Standards and Methods for Comparing Uncertainty-Bearing Arithmetic

3.1 Comparing Standards and Methods

Algorithms each with a known analytic result are used to characterize uncertainty-bearing arithmetic. The difference between the arithmetic result and the analytic result is defined as the *value error*. The question is whether the uncertainty bounding range or the uncertainty deviation is enough to cover the value error with an increased amount of calculation for any input. Corresponding to two different goals for uncertainty-bearing, there are actually two different sets of measurements to characterize an uncertainty-bearing arithmetic:

- The ratio of the absolute value error to the uncertainty deviation is defined as the *tracking ratio* for each output value. An ideal uncertainty-tracking arithmetic should have an average tracking ratio close to 1.
- The ratio of the absolute value error to the uncertainty bounding range is defined as the *bounding ratio* for each output value. An ideal uncertainty-bounding arithmetic should have a maximal bounding ratio either 1 or less than but close to 1. If the maximal bounding ratio is larger than 1, *bounding leakage* measures the probability for errors to be outside uncertainty bounding range.

In both cases, all measurements should be stable for an algorithm so that they should not change significantly for different input deviation, input data, or the amount of calculation. For example, if different branches of conditional executions contain very different amounts of calculations, such stability is crucial for obtaining a valid estimation of result precision.

Without dependency problem, the tracking ratios for precision arithmetic is expected to be normal distributed, and the average tracking ratio is a constant $\frac{2}{\sqrt{2\pi}} \simeq 0.7979$ when the bounding leakage is ignored.

3.2 Comparing Uncertainty-Bearing Arithmetics

Precision arithmetic tracks both the uncertainty bounding range and the uncertainty deviation, so it can be evaluated for both goals. Independence arithmetic has no uncertainty bounding range, while interval arithmetic has no uncertainty deviation. To be able to compare all the three arithmetics, $[x - 6\delta x, x + 6\delta x]$ is used artificially as the bounding range for an average value x with deviation δx for independence arithmetic, and vice versa for interval arithmetic.

As stated previously:

- Independence arithmetic assumes that any two operands are independent of each other, which may not be true in most cases.
- Precision arithmetic assumes that the uncertainties of any two operands are independent of each other, but allows the two operands themselves to be correlated.
- Interval arithmetic has the worst-case assumption because it needs to have zero bounding leakage unconditionally.

The statistical assumption of precision arithmetic is weaker than that of independence arithmetic but stronger than that of interval arithmetic, so after executing the same

algorithm on the same input data, the output deviation and the bounding range of precision arithmetic are expected to be larger than those of independence arithmetic but smaller than those of interval arithmetic.

- According to Formula (2.5) and Formula (2.8), the result deviation of addition and subtraction by precision arithmetic propagates in the same way as that of independence arithmetic, while the result bounding range propagates in the same way as that of interval arithmetic. Hence addition and subtraction cannot differentiate the three arithmetics.
- According to Formula (2.53) and Formula (2.55), the result precision of multiplication and division by precision arithmetic is always larger than that by independence arithmetic. However, if both operands have precisions much less than 1, the result precision of multiplication and division is very close to that of independence arithmetic. Thus, the result of precision arithmetic should be much closer to that of independence arithmetic.
- The uncertainty distribution of precision arithmetic is a truncated Gaussian distribution according to Formula (2.15). When an imprecise value is multiplied by a constant, because its uncertainty bounding range and its uncertainty distribution deviation cannot be scaled linearly simultaneously according to Formula (2.12) and Formula (2.13), precision arithmetic chooses to preserve the distribution deviation rather than the bounding range, thus introducing bounding leakages. Figure 5 suggests that the bounding range of precision arithmetic should be much narrower than that of interval arithmetic, while the shape of Gaussian distribution suggests that such introduced bounding leakage should be small when the truncation range is much larger than the distribution deviation, e.g., less than 10^{-6} for the chosen normalization method whose truncation range is about ± 5 deviations.
- Formula (2.46) and its multi-dimensional expansions such as Formula (2.51) are mathematically strict so that precision arithmetic has no dependence problem on expression differences. In contrast, there seems no similar solution for generic Taylor expansion using interval arithmetic, because there seems no general analytic solution to find maxima and minima for generic polynomial at any range [?]. In this respect, precision arithmetic is mathematically simpler than interval arithmetic.

3.3 Comparing Algorithms for Tests

Algorithms of completely different nature with each representative for its category are needed to test the generic applicability of uncertainty-bearing arithmetic.

An algorithm can be categorized by comparing the amount of its input and output data:

- *Transforming*: A transforming algorithm has about equal amounts of input and output data. The information contained in the data remains about the same after transforming. The Discrete Fourier Transformation is a typical transforming algorithm, which contains exactly the same amount of input and output data, and its output data can be transformed back to the input data using essentially the same algorithm. Matrix inversion is another such reversible algorithm. For reversible transformations, a unique requirement for uncertainty-bearing arithmetic is to introduce the least amount of additional uncertainty after forward

and reverse transformation, which provides an objective testing standard for a uncertainty-bearing arithmetic. A test of uncertainty-bearing arithmetic using FFT algorithms is provided in Section 4, and a test of matrix inversion is provided in Section 5.

- *Generating*: A generating algorithm has much more output data than input data. Solving differential equations numerically and generating a numerical table of a specific function are two typical generating algorithms. The generating algorithm codes mathematical knowledge into data, so there is an increase of information in the output data. In generating algorithms, input uncertainty should also be considered when deciding if the result is good enough so that the calculation can stop. Some generating algorithms are theoretical calculations which involve no imprecise input so that all result uncertainty is due to rounding errors. Section 6 demonstrates such an algorithm, which calculates a table of the sine function using trigonometric relations and two precise input data, $\sin(0) = 0$ and $\sin(\pi/2) = 1$.
- *Reducing*: A reducing algorithm has much less output data than input data such as numerical integration and statistical characterization of a data set. Some information of the data is lost while other information is extracted during reducing. Conventional wisdom is that a reducing algorithm generally benefits from a larger input data set [?]. Such a notion needs to be re-evaluated when uncertainty accumulates during calculation. A test of uncertainty-bearing arithmetic using numerical integration is provided in Section 8.

Other relations between the input and output can also be used to categorize an algorithm.

- In an *expressive* algorithm, each output is implemented as an analytic mathematical expression of inputs. Formula (2.46) and Formula (2.51) of precision arithmetic are powerful tools to solve expressive algorithms using precision arithmetic.
- In a *progressive* algorithm, each output is based on partial inputs and previously generated outputs. If an output depends on the state which is defined by previous inputs and outputs, the algorithm is also progressive. Most practical algorithms are progressive. Even if there may be an expected analytic mathematical expression between its input and output, an algorithm may not be expressive due to its progressive implementation. The dependence problem usually exists in a progressive algorithm.

3.4 Input Data to Use

To test input data of any precision, a precise input value can be cast to any specific input deviation using precision representation. There exist two ways of implementing such a casting:

- A *clean* signal is obtained by directly casting a perfect signal to a specific precision. Such casting may contain systematic rounding errors. For instance, if a perfect sine signal repeats 2^n times in 2^{n+2} samples, the signal contains only values $0, \pm 1$ and $\pm 1/\sqrt{2}$, with each value repeated multiple times in the signal. The symmetry of the arithmetic may be tested by the output symmetry of clean input signals, e.g., in the discrete Fourier transformation, the frequency space should be conjugately symmetrical [?] for a clean signal in signal space.

- A *noisy* signal is obtained by adding Gaussian noise of the same deviation as the input deviation to a perfect signal before casting. It represents a realistic signal, and it should be used in validating arithmetic on uncertainty propagation.

4 Comparison Using FFT

4.1 Frequency Response of DFT (Discrete Fourier Transformation)

Each testing algorithm needs to come under careful scrutiny. One important issue here is whether the digital implementation of the algorithm is faithful for the original analytic algorithm. For example, the DFT is only faithful for continuous Fourier transformation at certain frequencies, and it has a different degree of faithfulness for other frequencies. This is called the frequency response of the DFT in this paper.

For each signal sequence $h[k], k = 0, 1 \dots N-1$, in which N is a positive integer, the DFT $H[n], n = 0, 1 \dots N-1$ and its reverse transformation is given by Formula (4.1) [?], in which k is the *index frequency* for the DFT:

$$H[n] = \sum_{k=0}^{N-1} h[k] e^{i2\pi \frac{k}{N} n}; \quad h[k] = \frac{1}{N} \sum_{n=0}^{N-1} H[n] e^{-i2\pi \frac{n}{N} k}; \quad (4.1)$$

The $H[n]$ of a pure sine signal $h[k] = \sin(2\pi f k/N)$ is calculated by Formula (4.2), in which f is the frequency of the sine wave. When f is an index frequency for $H[n]$, Formula (4.2) becomes Formula (4.3). Otherwise, the general solution for Formula (4.2) is Formula (4.4), which approaches (4.3) when f approaches its closest integer F , or Formula (4.5) when f approaches $F \pm 1/2$.

$$H[n] = \frac{\sum_{k=0}^{N-1} e^{i2\pi(n+f)\frac{k}{N}} - \sum_{k=0}^{N-1} e^{i2\pi(n-f)\frac{k}{N}}}{2i}; \quad (4.2)$$

$$H[n] = i\delta_{n,F}N/2; \quad (4.3)$$

$$H[n] = \frac{1}{2} \frac{\sin(2\pi f - 2\pi \frac{f}{N}) + \sin(2\pi \frac{f}{N}) - \sin(2\pi f)e^{-i2\pi \frac{n}{N}}}{\cos(2\pi \frac{n}{N}) - \cos(2\pi \frac{f}{N})}; \quad (4.4)$$

$$H[n] = N/\pi; \quad (4.5)$$

The DFT $H[n]$ of the signal $h[k]$ is the digital implementation of the continuous Fourier transformation $H(s)$ of the signal $h(t)$ [?], in which $H(s) = i\delta(s - f)$ for $h[k] = \sin(2\pi f)$. From Formula (4.4), when the signal frequency of the original signal falls between two index frequencies of the transformation, the peak is lower and wider with a wrong phase, depending on the fractional frequency $|f - F|$. Thus, the DFT is only faithful for signal components with exactly one of the index frequencies of the transform, and it suppresses and widens unfaithful signal components, each of which has a phase different from its closest faithful representation, with the phase of a sine wave distorted toward that of a cosine wave, and vice versa. Examples of unfaithful representations of fractional frequency by the DFT are shown in Figure 7.

Due to its width, a frequency component in an unfaithful transformation may interact with other frequency components of the Discrete Fourier spectrum, thus sabotaging the whole idea of using the Fourier Transformation to decompose a signal into independent frequency components. Because the reverse DFT mathematically restores the original $\{h[k]\}$ for any $\{H[n]\}$, it exaggerates and narrows all unfaithful signal components correspondingly. This means that the common method of signal processing in the Fourier space [?][?][?] may generate artefacts due to its uniform treatment of faithful and unfaithful signal components, which probably coexist in reality. Unlike aliasing [?][?][?], unfaithful representation of the DFT has an equal presence in the

whole frequency range so that it cannot be avoided by sampling the original signal differently.

An unfaithful representation arises from the implied assumption of the DFT. The continuous Fourier transformation has an infinitive signal range so that:

$$h(t) \Leftrightarrow H(s) : h(t - \tau) \Leftrightarrow H(s)e^{i2\pi s\tau}; \quad (4.6)$$

As an analog, the DFT $G[n]$ of the signal $h[k], k = 1 \dots N$ can be calculated mathematically from the DFT $H[n]$ of $h[k], k = 0 \dots N - 1$:

$$G[n] = (H[n] + h[N] - h[0])e^{i2\pi n/N}; \quad (4.7)$$

Applying Formula (4.6) to Formula (4.7) results in Formula (4.8).

$$h[N] = h[0]; \quad (4.8)$$

Thus, the DFT has an implied assumption that the signal $h[k]$ repeats itself outside the region of $[0, N - 1]$ [?]. For an unfaithful frequency, $h[N - 1]$ and $h[N]$ are discontinuous in regard to signal periodicity, resulting in larger peak width, lower peak height, and the wrong phase.

The most convenient signals to test uncertainty-bearing arithmetic are perfect sine or cosine signals with index frequencies. A linear signal with the slope $\lambda, h[k] = \lambda k$, provides a generic test for input frequencies other than index frequencies, whose Fourier spectrum is:

$$H[n] = -\lambda \frac{N}{2} \left(1 + \frac{i}{\tan(\pi n/N)} \right); \quad (4.9)$$

4.2 FFT (Fast Fourier Transformation)

When $N = 2^L$, in which L is a positive integer, the generalized Danielson-Lanczos lemma [?] can be applied to the DFT as FFT [?], in which $m = L, L - 1, \dots, 1, 0$ indicates progress of the transformation, and j is the bit-reverse of n :

$$m = L : H[n, \frac{k}{2^m}] = h[j], k, n = 0, 1 \dots N - 1; \quad (4.10)$$

$$m = L - 1 \dots 0 : H[n, \frac{k}{2^m}] = H[n, \frac{k}{2^{m+1}}] + H[n, \frac{k}{2^{m+1}}] \exp(+i2\pi \frac{n}{2^{L-m}}); \quad (4.11)$$

$$m = 0 : H[n] = H[n, \frac{k}{2^m}]; \quad (4.12)$$

Thus, each output value is obtained after applying Formula (4.11) L times. L is called FFT order in this paper.

The calculation of the term $\exp(i2\pi \frac{n}{2^{L-m}})$ in Formula (4.11) can be simplified. Let $<<$ denote a bit left-shift operation and let $\&$ denote a bitwise AND operation:

$$\varphi[n] \equiv \exp(i2\pi \frac{n}{2^L}) : \exp(i2\pi \frac{n}{2^{L-m}}) = \varphi[(n << m) \& ((1 << L) - 1)]; \quad (4.13)$$

It is important to have an accurate phase factor array $\varphi[n]$ when tracking the FFT calculation error. The accuracy of $\varphi[n]$ can be checked rigidly within itself by trigonometric relations so that no significant error is introduced from trigonometric functions.

Formula (4.11) always sums up two mutually independent operands, so the error propagation in a FFT algorithm is precisely tracked by independence arithmetic, and

the dependency problem should not be a concern for interval arithmetic and precision arithmetic.

FFT is one of the most widely used algorithms [?]. By providing a balanced usage of addition, subtraction and multiplication involving trigonometric functions, it serves as one of the most important benchmarks in testing processors for overall mathematical performance [?]. Since all three uncertainty-bearing arithmetics are generic in nature without special optimization for FFT algorithms, the testing result using FFT algorithms should be generic for expressive algorithms. FFT algorithms provide a good linear platform to test any uncertainty-bearing arithmetic, with 1) a clearly defined value measuring the amount of calculation, 2) a known error propagation mechanism, 3) no conditional execution in the algorithm, and 4) using only basic arithmetic operations without the dependence problem.

4.3 Evaluating Calculation Inside Uncertainty

Figure 8 shows the output deviations and value errors for a noisy sine signal after forward FFT. It shows that the output deviations using precision arithmetic are slightly larger than the output deviations using independence arithmetic, but much less than those using interval arithmetic. For a fixed input deviation, the output deviation using independence arithmetic is a constant for each FFT. Because the value and uncertainty interact with each other through normalization in precision arithmetic, output deviations of Formula (4.11) are no longer a constant. One interesting consequence is that only in precision arithmetic the output deviations for a noisy input signal are larger than those for a corresponding clean input signal.

Figure 8 shows that the value errors calculated using precision arithmetic are comparable to those using conventional floating-point arithmetic, and they are both comparable to the output deviations using either precision arithmetic or independence arithmetic. In other words, the result of calculating 2-bit or 53-bit into uncertainty are quite comparable so that the limited calculation inside uncertainty is reasonable.

Figure 9 compares the output value errors of precision arithmetic calculating different bits inside uncertainty. With no calculation inside uncertainty, the output value errors exist only on four levels. Such quantum distribution is reduced noticeably by the 2-bit calculation inside uncertainty, and is further reduced by the 4-bit calculation inside uncertainty. Compared with Figure 8, Figure 9 shows that the result using precision arithmetic with the 4-bit calculation inside uncertainty approaches that using independence arithmetic so that the 4-bit calculation inside uncertainty seems sufficient. Precision arithmetic with the 4-bit calculation inside uncertainty is used for further tests.

4.4 Evaluating Uncertainty Distribution

Precision arithmetic tracks all increases of rounding errors, but it cannot track decreases of the rounding error due to mutual cancellations during arithmetic operations. Hence the uncertainty distribution provided by precision arithmetic serves as the bounding distribution for value errors, and the actual distribution could be narrower than the bounding distribution. FFT provides a good test for such probability bounding. Its forward and reverse algorithms are identical except for a constant so that they result in exactly the same bounding probability distributions. On the other hand, the forward FFT condenses a sine signal into only two non-zero imaginary values by mutual cancellation of signal components, while the reverse FFT spreads only two

non-zero imaginary values to construct a sine signal. Thus, the forward FFT is more sensitive to calculation errors than the reverse FFT, and should have a broader actual uncertainty distribution.

Because value errors are only observable as errors of the result significand, while precision arithmetic has limited bits calculated inside uncertainty, theoretical rounding error distributions obtained from Formula 2-16 are integrated around LSB of the significand, to result in theoretical probability densities of the value error significands for different bounding ranges. The occurrence frequencies of the output value error significands are counted for different bounding ranges and normalized as measured probability densities. Figure 10 and Figure 11 show theoretical and empirical distribution of significand errors for different bounding ranges on noisy sine signals for the forward and reverse FFT, respectively. They show that the uncertainty distribution of the forward FFT is very close to the bounding distribution, while the uncertainty distribution of the reverse FFT is indeed narrower. In contrast, for the linear input signal, the forward and reverse FFT have almost identical value tracking ratios distribution. Thus, the previous hypothesis on the distribution width of value tracking ratios is validated empirically in this case.

Another way to measure the empirical probability distribution is to normalize the output value error with the corresponding output uncertainty deviation according to Formula 2-11. If output value errors are Gaussian-distributed with the deviation given precisely by the corresponding output uncertainty deviation, then the normalized histogram should be normal-distributed. Figure 12 shows that the normalized histograms using independence arithmetic are best fit by a Gaussian distribution with the deviation of 0.98 and the mean of 0.06. If there were no rounding errors, all the results should have zero value errors. Indeed, in Figure 12, the measured histograms have a much larger population at where the value errors are zero so that the measured deviation is expected to be less than 1. The reason for the small but non-zero mean for the measured probability distribution is not clear at this moment. Figure 13 shows that the measured normalized histograms using precision arithmetic are very similar to those using independence arithmetic, and they are all well fitted by a Gaussian distribution which is very close to a normal distribution.

4.5 Evaluating Uncertainty-Tracking

Figure 14 shows that for the same input deviation, the output deviations of the forward FFT increase exponentially with the FFT order using all three arithmetics. Figure 15 shows that for the same FFT order, the output deviations of the forward FFT increase linearly with the input deviation using all three arithmetics. The output deviation does not change with input frequency so that all data of the same input deviation and the same FFT order but with different input frequencies can be pooled together during analysis. The trends in Figure 14 and Figure 15 are modeled by Formula (4.14), in which L is the FFT order, δx is the input deviation, δy is the average output deviation, and α and β are empirical fitting constants:

$$\delta y = \alpha \beta^L \delta x; \quad (4.14)$$

β measures the propagation speed of the deviation with an increased amount of calculation in Formula (4.14). It is called *propagation base rate*. Unless β is close to 1, β dominates α in fitting, thus determining characteristics of Formula (4.14).

It turns out that Formula (4.14) is a very good fit for both average output deviations and value errors for all three arithmetics, such as demonstrated in Figure 16.

Because uncertainty-tracking is a competition between error propagation and uncertainty propagation, the average output tracking ratio for the forward FFT is expected to fit Formula (4.15) and Formula (4.16), in which z is the average output tracking ratio, L is the FFT order, $(\alpha_{dev}, \beta_{dev})$ and $(\alpha_{err}, \beta_{err})$ are fitting parameters of Formula (4.14) for average output deviations and value errors, respectively:

$$z = \alpha\beta^L; \quad (4.15)$$

$$\alpha = \alpha_{err}/\alpha_{dev}; \quad \beta = \beta_{err}/\beta_{dev}; \quad (4.16)$$

The estimated average output tracking ratio can then be compared with the measured ones to evaluate the predictability of the uncertainty-tracking mechanism. One example of measured average output tracking ratios is shown in Figure 17, which shows that the average output tracking ratios using precision arithmetic are a constant despite that both average output uncertainty deviations and value errors increase linearly with the input deviation and exponentially with the FFT order. Formula (4.14) and Formula (4.15) are found empirically to be a good fit for any FFT algorithm with any input signal using any arithmetic.

The Reverse FFT algorithm is identical to the Forward FFT algorithm, except when:

- The Reverse FFT algorithm uses constant $(-i)$ instead of $(+i)$ in Formula (4.11).
- The Reverse FFT algorithm divides the result further by 2^L .

Thus, the average output deviations and value errors of the reverse FFT algorithm are expected to obey Formula (4.14) and Formula (4.17), in which $(\alpha_{for}, \beta_{for})$ are corresponding fitting parameters of Formula (4.14) for the forward FFT, while the average output tracking ratios are expected to obey Formula (4.16) with the same α and β as those of the forward FFT.

$$\alpha = \alpha_{for}; \quad \beta = \beta_{for}/2; \quad (4.17)$$

The Round-trip FFT is the forward FFT followed by the reverse FFT, with the output of the forward FFT as input to the reverse FFT. Thus, both its average output deviations and value errors are expected to fit Formula (4.14) and Formula (4.18), in which $(\alpha_{for}, \beta_{for})$ and $(\alpha_{rev}, \beta_{rev})$ are corresponding fitting parameters of Formula (4.14) for the forward FFT and the reverse FFT, respectively. Its tracking ratios are expected to fit Formula (4.15) and Formula (4.18), in which $(\alpha_{for}, \beta_{for})$ and $(\alpha_{rev}, \beta_{rev})$ are corresponding fitting parameters of Formula (4.15) for the forward FFT and the reverse FFT, respectively.

$$\alpha = \alpha_{for}\alpha_{rev}; \quad \beta = \beta_{for}\beta_{rev}; \quad (4.18)$$

Figure 18, Figure 19 and Figure 20 show the fitting of β for independent, precision and interval arithmetic for all the three algorithms, respectively. These three figures show that all measured β make no distinction between input signals for any algorithms using any arithmetic, e.g., there is no difference between the real part and the imaginary part for a sine signal. The estimated β for average tracking ratios is obtained from Formula (4.16). The estimated β for average uncertainty deviations and value errors for the reverse FFT and the roundtrip FFT are obtained from Formula (4.17) and Formula (4.18), respectively. The estimated β for average uncertainty deviations for the forward FFT is $\sqrt{2}$, which will be demonstrated later. The measured β and the estimated β agree well with each other in all cases. This confirms that uncertainty-tracking is a simple competition between the error propagation and uncertainty propagation:

- Figure 18 confirms that independence arithmetic is ideal for uncertainty-tracking for FFT algorithms: 1) β for tracking ratios is a constant 1; and 2) β for both the average output deviations and value errors is both 1 for the round-trip FFT because the result signal after the round-trip FFT should be restored as the original signal. Thus, theoretical β for the forward FFT and the reverse FFT are $\sqrt{2}$ and $1/\sqrt{2}$, respectively.
- Precision arithmetic has β for average output deviations slightly larger than those of value errors, resulting in β for average output tracking ratios to be a constant slightly less than 1. Its β for average output deviations is slightly larger than the corresponding β of independence arithmetic, so its average output deviations propagate slightly faster with an increased FFT order than those of independent arithmetic. Such slightly faster increase with the amount calculation is anticipated by the difference between Formula (2.53) and Formula (1.12) with $\gamma = 0$.
- The β for average output deviations using interval arithmetic is always much larger than β for average output value errors, resulting in β for average output tracking ratios of about 0.62 for the forward and reverse FFT, and about $0.39 \cong 0.62^2$ for the roundtrip FFT. Consequently, using interval arithmetic, the average output deviations propagate much faster with the amount of calculations than the value error does. Such fast propagation of uncertainty ranges is intrinsic to interval arithmetic due to its worst-case assumption.

Figure 21 shows that for the forward FFT, the measured average output tracking ratios using either precision arithmetic or independence arithmetic are approximately constant of 0.8 in both cases, regardless of the FFT order. In contrast, Figure 21 shows that using interval arithmetic the measured average output tracking ratios decrease exponentially with the FFT order L . Such trends of average tracking ratios hold for all three FFT algorithms and all input signals. Thus, in this case, the direct uncertainty tracking provided by precision arithmetic is better than the indirect uncertainty tracking provided by interval arithmetic.

Figure 22 shows that using precision arithmetic, each average output uncertainty deviation equals the corresponding input uncertainty deviation for all FFT orders after a round-trip operation. Thus, after each round-trip operation, precision arithmetic restores the original signal and the corresponding uncertainty for FFT. Such behavior seems ideal for a reversible algorithm. In contrast, Figure 23 shows that using interval arithmetic, the average output uncertainty deviations increase exponentially with FFT orders, which means the undesirable broadening of uncertainty in the restored signal after a round-trip operation.

4.6 Evaluating Uncertainty-Bounding

While uncertainty tracking is the result of the propagation competition between average output deviations and average values errors with increased amount of calculations, uncertainty bounding is the result of the propagation competition between output bounding ranges and maximal value errors, both of which still fit Formula (4.14) well using any arithmetic experimentally. Formula (4.15) and Formula (4.16) can be used to estimate the maximal bounding ratio as well. For example, Figure 24 shows that the maximal output bounding ratios using precision arithmetic fit Formula (4.15) well. Unlike average output tracking ratios in Figure 21, the maximal output bounding ratios increase slowly with the FFT order using either precision arithmetic

or independent arithmetic. In contrast, interval arithmetic has its maximal bounding ratios decreasing exponentially with the increased FFT order for all algorithms while keeping its bounding leakages at constant 0. Detailed analysis shows that in interval arithmetic, β for the maximal uncertainty bounding ranges exceeds β for the maximal value error, suggesting the source of over-estimating uncertainty range with the increased amount of calculations. Defining empirical *deviation leakage* as the frequency of the value errors to be outside the range of $\text{mean} \pm \text{deviation}$, Figure 25 shows that the deviation leakages is roughly a constant using precision arithmetic, suggesting the statistical nature of uncertainty bounding using precision arithmetic. Whether precision arithmetic is better than interval arithmetic in uncertainty bounding depends on the statistical requirements for the uncertainty bounding:

- In the situation when absolute bounding is required, interval arithmetic is the only choice.
- In the range estimation [?] involving low-resolution measurements whose sources of uncertainty are unclear, interval arithmetic is a better choice because the independence uncertainty assumption of precision arithmetic may not be satisfied.
- Otherwise, precision arithmetic should be more suitable for normal usages.

5 Comparison Using Matrix Inversion

5.1 Uncertainty Propagation in Matrix Determinant

Let vector $[p_1, p_2 \dots p_n]_n$ denote a permutation of the vector $(1, 2 \dots n)$ [?]. Let $\$[p_1, p_2 \dots p_n]_n$ denote the permutation sign of $[p_1, p_2 \dots p_n]_n$ [?]. For a n -by- n square matrix M with the element $x_{i,j}$, $i, j = 1, 2 \dots n$, let its determinant be defined as Formula (5.1) [?] and let the sub-determinant at index (i, j) be defined as Formula (5.2) [?]:

$$|M| \equiv \sum_{[p_1 \dots p_n]_n} \$[p_1 \dots p_n]_n \prod_k x_{k, p_k}; \quad (5.1)$$

$$|M|_{i,j} \equiv \sum_{[p_1 \dots p_n]_n}^{p_i=j} \$[p_1 \dots p_n]_n \prod_k^{k \neq i} x_{k, p_k}; \quad (5.2)$$

$(-1)^{i+j}|M|_{(i,j)}$ is the determinant of the $(n-1)$ -by- $(n-1)$ matrix that results from deleting the row i and column j of M [?]. Formula (5.3) holds for the arbitrary row index i or the arbitrary column index j [?]:

$$|M| = \sum_{j=1}^n |M|_{i,j} x_{i,j} = \sum_{i=1}^n |M|_{i,j} x_{i,j}; \quad (5.3)$$

Assuming $p_1, p_2 \in \{1, 2 \dots n\}$, let $[p_1, p_2]_n$ denote the length-2 unordered permutation which satisfies $p_1 \neq p_2$, and let $\langle p_1, p_2 \rangle_n$ denote the length-2 ordered permutation which satisfies $p_1 < p_2$. Letting $\langle i_1, i_2 \rangle_n$ be an arbitrary ordered permutation, Formula (5.3) can be applied to $M|_{i,j}$, as:

$$|M|_{\langle i_1, i_2 \rangle_n [j_1, j_2]_n} \equiv \sum_{[p_1 \dots p_n]_n}^{p_{i_1}=j_1, p_{i_2}=j_2} \$[p_1 \dots p_n]_n \prod_k^{k \neq i_1, k \neq i_2} x_{k, p_k}; \quad (5.4)$$

$$|M| = \sum_{j_1} x_{i_1, j_1} |M|_{i_1, j_1} = \sum_{j_1} \sum_{j_2}^{i_2 \neq i_1, j_2 \neq j_1} x_{i_1, j_1} x_{i_2, j_2} |M|_{\langle i_1, i_2 \rangle_n [j_1, j_2]_n}; \quad (5.5)$$

Because $|M|_{\langle i_1, i_2 \rangle_n [j_1, j_2]_n}$ relates to the determinant of the $(n-2)$ -by- $(n-2)$ matrix that results from deleting the row i_1 and i_2 , and the column j_1 and j_2 of M . This leads to Formula (5.6).

$$||M|_{\langle i_1, i_2 \rangle_n [j_1, j_2]_n}| = ||M|_{\langle i_1, i_2 \rangle_n [j_2, j_1]_n}|; \quad (5.6)$$

The definition of a sub-determinant can be extended to Formula (5.7), in which $m \in \{1, 2 \dots n\}$. Formula (5.5) can be generalized as Formula (5.8), in which $m \in \{1, 2 \dots n\}$ and $\langle i_1 \dots i_m \rangle_n$ is an arbitrary ordered permutation. Formula (5.8) can be viewed as the extension for both Formula (5.3) and Formula (5.1).

$$|M|_{\langle i_1 \dots i_m \rangle_n [j_1 \dots j_m]_n} \equiv \sum_{[p_1 \dots p_n]_n}^{p_{i_k}=j_k, k \in \{1 \dots m\}} \$[p_1 \dots p_n]_n \prod_{k \in \{1 \dots n\}}^{k \notin \{i_1 \dots i_m\}} x_{k, p_k}; \quad (5.7)$$

$$|M| = \sum_{[j_1 \dots j_m]_n} |M|_{\langle i_1 \dots i_m \rangle_n [j_1 \dots j_m]_n} \prod_{k=1}^m x_{i_k, j_k}; \quad (5.8)$$

According to the basic assumption of precision arithmetic, the uncertainty of each element $x_{i,j}$ is independently and symmetrically distributed. Let $\tilde{y}_{i,j}$ denote a random variable at the index (i, j) symmetrically distributed with the deviation $\delta x_{i,j}$. Let $|\widetilde{M}|$ denote the determinant of the matrix \widetilde{M} whose element is $(x_{i,j} + \tilde{y}_{i,j})$. Applying Taylor expansion to Formula (5.8) results in Formula (5.9), which results in Formula (5.10) after applying Formula (2.43):

$$|\widetilde{M}| - |M| = \sum_{m=1}^n \sum_{<i_1 \dots i_m>_n} \sum_{[j_1 \dots j_m]_n} |M_{<i_1 \dots i_m>_n [j_1 \dots j_m]_n}| \prod_{k=1}^m \tilde{y}_{i_k, j_k}; \quad (5.9)$$

$$\delta |M|^2 = \sum_{m=1}^n \sum_{<i_1 \dots i_m>_n} \sum_{[j_1 \dots j_m]_n} |M_{<i_1 \dots i_m>_n [j_1 \dots j_m]_n}|^2 \prod_{k=1}^m \delta x_{i_k, j_k}^2; \quad (5.10)$$

Defining $|M_{<>_n <>_n}| \equiv |M|$, Formula (5.11) is an recursive form of Formula (5.10):

$$\begin{aligned} \delta |M_{<p_1 \dots p_k>_n <q_1 \dots q_k>_n}|^2 &= \sum_{p_i} \sum_{q_j} \delta x_{p_i, q_j}^2 \\ &(|M_{<p_1 \dots p_i \dots p_k>_n <q_1 \dots q_j \dots q_k>_n}|^2 + \delta |M_{<p_1 \dots p_i \dots p_k>_n <q_1 \dots q_j \dots q_k>_n}|^2); \end{aligned} \quad (5.11)$$

When using Formula (5.3) to calculate determinant in conventional floating-point arithmetic:

- The input uncertainty can not be accounted for.
- One path is chosen out of many possible paths, such as selecting a different sub-determinant to start with.
- Because of the rounding error, each path may result in a different result even if all elements of the determinant are precise, and the spread of all results is expected to be inversely proportional to the stability of the matrix [?].

In another word, using conventional floating-point arithmetic, the calculation of determinant is one leap of faith. Instead, Formula (5.11) shows that the result uncertainty is the aggregation of uncertainties from all possible path of Formula (5.3). To accounts for all such uncertainties, Formula (5.11) starts from all 1x1 sub-determinants, and constructs all sub-determinants whose size is 1 larger, until reaches the determinant itself. Thus, uncertainty-bearing calculation should be order-of-magnitude more complex and time-consuming than the correspond calculation using conventional floating-point arithmetic.

The element $z_{i,j}$ at the index (i, j) of the inverted matrix M^{-1} is calculated as [?]:

$$z_{i,j} = \frac{|M_{j,i}|}{|M|}; \quad (5.12)$$

Formula (5.12) shows that the uncertainty of the matrix determinant $|M|$ propagates to every element of the inverted matrix M^{-1} . Instead, the matrix which consists of the element $|M_{j,i}|$ at the index (i, j) is defined as the adjugate matrix M^A [?], whose elements are not directly affected by M^{-1} . M^A is recommended to replace M^{-1} whenever the application allows [?].

5.2 Matrix Testing Algorithm

A matrix \widehat{M} is constructed using random integers between $[-16384, +16384]$. Its adjugate matrix \widehat{M}^A and its determinant $|\widehat{M}|$ are calculated precisely using integer arithmetic. \widehat{M} , $|\widehat{M}|$ and \widehat{M}^A are all scaled proportionally as M , $|M|$ and M^A so that the elements of M are 2's fractional numbers randomly distributed between $[-1, +1]$. The scaled matrix M is called a clean testing matrix. M^{-1} is calculated from $|M|$ and M^A using Formula (5.12). Floating-point arithmetic is used to calculate M^A and M^{-1} from M , and the results are compared with the corresponding precise results for value errors. Gaussian noises corresponding to different deviations between 10^{-17} and 10^{-1} may be added to each clean testing matrix, to result in noisy testing matrix. Each combination of matrix size and input deviation is tested by 32 different noisy matrices.

5.3 Testing Matrix Stability

Each matrix has a different stability [?], which means how stable the inverted matrix is in regard to small value changes of the original matrix elements. It is well known that more mutual cancellations in Formula (5.1) mean less stability of the matrix [?][?], with the Hilbert matrix [?] being the most famous unstable matrix. The condition number has been defined to quantify the stability of a matrix [?]. Even though the definition of the condition number excludes the effects of rounding errors, in reality most calculations are done numerically using conventional floating-point arithmetic so that the combination effect of rounding errors and matrix instability cannot be avoided in practice. When a matrix is unstable, the result is more error prone due to rounding errors of conventional floating-point arithmetic [?]. Consequently, there are no general means to avoid the mysterious and nasty “numerical instability” in numerical applications due to rounding errors [?]. For example, the numerical value of the calculated condition number of a matrix may have already been a victim of “numerical instability”, and there is no sure way to judge this suspicion, so this value may not be very useful in judging the stability of the matrix in practice. On the other hand, the rounding errors of conventional floating-point arithmetic can be used to test the stability of a matrix. Rounding errors effectively change the item values of a matrix, so they produce a larger effect on a less stable matrix. If the inverted matrix and the adjugate matrix are calculated using conventional floating-point arithmetic, larger value errors indicate that the matrix is less stable.

Precision arithmetic accounts for all rounding error with stable characterization of result uncertainties. More mutual cancellations in Formula (5.1) will result in a smaller absolute value related to the uncertainty deviation of the determinant. Thus, the precision of the determinant $|M|$ of a matrix M calculated using precision arithmetic measures the amount of mutual cancellations, and it may measure the stability of a matrix. Particularly, if $|M|$ is of coarser precision, then each element of M^{-1} should tend to have a larger value error, according to Formula (5.12). This hypothesis is confirmed by Figure 26, which shows a good linear relation between the precision of $|M|$ and the average value error of its inverted matrix M^{-1} , regardless of the matrix size. The maximal output values errors are related to the precision of $|M|$ in the same fashion. In contrast, Figure 27 shows that the value errors of the adjugate matrix M^A do not depend noticeably on the precision of $|M|$. Thus, the precision of the denominator in Formula (5.12) determines the overall stability in matrix inversion, confirming the validity of common advice to avoid matrix inversion operations in

general [?].

Such a linear relation between the precision and the value error also extends to the calculation of the adjugate matrix. Let the relative value error be defined as the ratio of the value error divided by the expected value. The relative error is expected to correspond to the result precision linearly. Figure 28 compares each precision of the sub-matrix determinant $|M_{j,i}|$ with the corresponding relative error of the element at the index (i, j) of the adjugate matrix M^A of the clean matrix of different sizes. It shows that larger relative errors of adjugate matrix elements indeed correspond to coarser precisions of the sub-matrix determinant.

While each condition number [?] only gives the result sensitivity to one matrix element, Formula (5.10) contains the result sensitivity to any matrix element, any combination of matrix elements, as well as the aggregated result uncertainty deviation. Therefore, Formula (5.10) and Formula (5.11) may be better than the condition numbers for describing matrix stability.

5.4 Testing Uncertainty Propagation in Adjugate Matrix

When the adjugate matrix is calculated using precision arithmetic, Figure 29 shows that the average output deviations for the adjugate matrix increase linearly with the input deviation, which is in good agreement with Formula (4.14). Such relation is also true for maximal and average output values errors. Formula (4.14) is expected to describe the general value error propagation for linear algorithms in which L is the amount of calculations [?]. The question is what value L should be when calculating the adjugate matrix of a square matrix of size N . Figure 29 suggests that L increases with N^2 for the average output precision and average output error¹⁰.

Figure 30 shows that the average output tracking ratio of the adjugate matrix using precision arithmetic is approximately a constant of 0.8. Figure 30 is very similar to Figure 17. Similar to the maximal output bounding ratios of FFT algorithms, the maximal output bounding ratios for the adjugate matrix using precision also obey Formula (4.15) well, with β of 1.005, meaning a slow increase with the matrix size. Added to the similarity is the normalized uncertainty distribution shown in Figure 31, which is very similar to Figure 13. Even though FFT and the calculating adjugate matrix are two very different sets of linear transformational algorithms, their uncertainty propagation characteristics are remarkably similar even in quantitative details. This similarity indicates that precision arithmetic is a generic arithmetic for linear algorithms.

5.5 Calibration

Because $|M_{j,i}|$ and $|M|$ are not independent of each other, M^{-1} calculated by Formula (5.12) contains the dependency problem. Figure 31 shows that the tracking ratios for the adjugate matrix and the inverted matrix are both standard distributed, while they are exponentially distributed when the inverted matrix is inverted again. Because the inverted matrix has the same tracking ratio distribution as that of the adjugated

¹⁰The amount of calculation L does not mean the calculation complexity using the Big O notation [?]. It is just a measurement of how output uncertainty increases with a dimension of calculation according to (4.14) [?]. For example, any sorting algorithm will not change the uncertainty distribution, so that L is always 0 regardless the calculation complexity for the sorting algorithm. The measured calculation time suggests calculation complexity of $O(2^N)$ for using Formula (5.11) to calculate the matrix determinant.

matrix, which has no dependency problem, the inverted matrix contains hardly any dependency problem. In contrast, Figure 31 shows that the double inverted matrix is severely affected by the dependency problem, such that its tracking ratio increases with matrix size as shown in Figure 32. Figure 33 shows that average tracking ratios for different matrix sizes follows a same exponential distribution, but with different extend, e.g., the distribution for matrix size 4 has yet reaches stable distribution beyond 2.5, which causes the increase of the average tracking ratio with the matrix size as shown in Figure 32.

Applying the same algorithms twice results in so much differences, which shows that the dependency problem has been embedded in the data, and which shows the importance of calibration.

6 Comparison Using Recursive Calculation of Sine Values

Starting from Formula (6.1), Formula (6.2) and Formula (6.3) can be used recursively to calculate the phase array $\varphi[n]$ in Formula (4.13).

$$\sin(0) = \cos\left(\frac{\pi}{2}\right) = 0; \quad \sin\left(\frac{\pi}{2}\right) = \cos(0) = 1; \quad (6.1)$$

$$\sin\left(\frac{\alpha + \beta}{2}\right) = \sqrt{\frac{1 - \cos(\alpha + \beta)}{2}} = \sqrt{\frac{1 - \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)}{2}}; \quad (6.2)$$

$$\cos\left(\frac{\alpha + \beta}{2}\right) = \sqrt{\frac{1 + \cos(\alpha + \beta)}{2}} = \sqrt{\frac{1 + \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}{2}}; \quad (6.3)$$

This algorithm is very different from both FFT and matrix inversion in nature because Formula (6.2) and Formula (6.3) are no longer linear, and the test presents a pure theoretical calculation without input uncertainty. The recursion iteration count L is a good measurement for the amount of calculations. Each repeated use of Formula (6.2) and Formula (6.3) accumulates calculation errors to the next usage so that both value errors and uncertainty are expected to increase with L . Each recursion iteration L corresponds to 2^{L-2} outputs, which enables statistical analysis for large L .

Figure 34 shows that both average output value errors and the corresponding average output deviation increase exponentially with the recursion count for all three arithmetics, and Figure 35 shows that in response to the increased amount of calculations:

- The average tracking ratio for precision arithmetic is a constant about 0.25;
- The maximal output bounding ratio for precision arithmetic increases slowly;
- The average tracking ratio for interval arithmetic decreases exponentially; and
- The maximal output bounding ratio for interval arithmetic remains roughly a constant.

Unlike FFT algorithms, the initial precise sine values participate in every stage of the recursion, which results in few small output deviations at each recursion. Detailed inspection shows that the maximal output bounding ratios for interval arithmetic are all obtained from small output deviations, and bounding ratios using interval arithmetic in general decrease exponentially with the amount of calculations. Thus, the result uncertainty propagation characteristics of the regressive calculation of sine values are very similar to those of both FFT and the calculating adjugate matrix; even though all these algorithms are quite different in nature. This may indicate again that the stability of precision arithmetic is generic, regardless of the algorithms used.

7 Validation Using Taylor Expansion

When a Taylor expansion is implemented using conventional floating-point arithmetic, the rounding errors are ignored, so that the result of a higher order of expansion is assumed to be more precise, because the Cauchy estimator of the expansion, which gives an upper bound for the remainder of the expansion, decreases with the order of the expansion for analytic expressions. A subjective upper limit is chosen for the Cauchy estimator, to stop the expansion at limited order [?]. However, such arbitrary upper limit may not be achievable with the amount of rounding errors accumulated during calculation, so that such upper limit may actually gives a false expansion precision.

Using precision arithmetic, the rounding errors as well as the input uncertainties are all accounted for, so that the maximal expansion order when applying a Taylor expansion of Formula (2.45) or Formula (2.50) is no longer subjective. Formula (2.44) is decomposed into the contribution of each successive term for Tylor expansion, as Formula (7.1):

$$\begin{aligned}
(\delta \sum_{j=0}^{J+1} a_j x^j)^2 &= \int (\sum_{j=0}^J a_j (x + \tilde{y})^j - \sum_{j=0}^J a_j x^j + a_{J+1} (x + \tilde{y})^{J+1} - a_{J+1} x^{J+1})^2 \rho(\tilde{y}) d\tilde{y} \\
&= \int (\sum_{j=0}^J a_j (x + \tilde{y})^j - \sum_{j=0}^J a_j x^j)^2 \rho(\tilde{y}) d\tilde{y} + a_{J+1}^2 \int (\sum_{k=1}^{J+1} C_{J+1}^k \tilde{y}^k x^{J+1-k})^2 \rho(\tilde{y}) d\tilde{y} \\
&\quad + 2 \int (\sum_{j=0}^J a_j \sum_{k=1}^j C_j^k \tilde{y}^k x^{j-k}) (a_{J+1} \sum_{k=1}^{J+1} C_{J+1}^k \tilde{y}^k x^{J+1-k}) \rho(\tilde{y}) d\tilde{y} \\
(\delta \sum_{j=0}^{J+1} a_j x^j)^2 - (\delta \sum_{j=0}^J a_j x^j)^2 &= \sum_{k_1=1}^{J+1} \sum_{k_2=1}^{J+1} a_{J+1}^2 C_{J+1}^{k_1} C_{J+1}^{k_2} M(k_1 + k_2) (\delta x)^{k_1 + k_2} x^{2J+2-k_1-k_2} \\
&\quad + 2 \sum_{k_1=1}^{J+1} \sum_{j=0}^J \sum_{k_2=1}^j a_j a_{J+1} C_{J+1}^{k_1} C_j^{k_2} M(k_1 + k_2) (\delta x)^{k_1 + k_2} x^{J+1+j-k_1-k_2}; \quad (7.1)
\end{aligned}$$

Applying Formula (7.1) to Taylor expansion:

1. Formula (7.1) provides the deviation at n -th expansion order, which becomes stabilized when the delta deviation at n -th expansion order (which is the contribution of the n -th expansion order to the deviation) is much less than the deviation at n -th expansion order.
2. The value resolution of precision arithmetic is the deviation divided by 2^b , in which b is the constant bits calculated inside uncertainty.
3. The maximal expansion order of a Taylor expansion is reached when the Cauchy estimator is less than the value resolution of precision arithmetic, after which the changes in Cauchy estimator is no longer detectable. Ideally, the Taylor expansion reminder should also become zero when the expansion order is larger than the maximal expansion order.

Formula (7.1) also shows that the deviation of Taylor expansion may decrease at certain expansion order. For example, at $x = 1 \pm \delta x$, $1 - 2x + x^2$ is equivalent to y^2 at $y = 0 \pm \delta x$, thus it has smaller result variance than $1 - 2x$ at $x = 1 \pm \delta x$.

Formula (7.2) provides an example test in Taylor expansion, in which n is a positive

integer.

$$f_n(x) = \sum_{j=0}^n (-x)^j; \quad \lim_{n \rightarrow \infty} f_n(x) = 1/(1+x); \quad (7.2)$$

In Formula (7.2), the absolute value of $(n+1)$ th term in the expansion is the Cauchy remainder estimator of the n th order expansion. Formula (7.2) is analytic when $|x|$ is less than 1, and a smaller value $|x|$ means faster convergence to the correct value $1/(1+x)$.

Using Formula (7.2) as a test case, Figure 36 confirms the above Taylor expansion process using precision arithmetic with 0-bit calculated inside uncertainty and with input uncertainty at 10^{-3} . For smaller $|x|$, in addition to faster decrease of both reminder and Cauchy estimator, delta deviation also decreases faster, thus deviation reaches its stable values faster. Once the maximal expansion order is reached, the reminder also becomes to zero. Figure 36 repeats the above process with 4-bit calculated inside uncertainty, which only differs from Figure 36 on the deviation threshold for the maximal expansion order: the threshold is deviation/16 instead deviation itself, and the maximal expansion order is significantly larger.

When input has larger uncertainty, deviation reaches to its stable value much slower, which is show in Figure 38 for 0-bit calculated inside uncertainty:

- When $x = 0.75$, deviation barely reaches its stable value when the Cauchy estimator reaches deviation.
- When $x = 0.875$, deviation has not reaches its stable value when the Cauchy estimator reaches deviation, and reminder does not become zero at the maximal expansion order but a few orders beyond.
- When $x = 0.9375$, deviation has no stable value and becomes negative eventually. Nevertheless, reminder becomes zero beyond the maximal expansion order.

In contrast, with 4-bit calculated inside uncertainty as shown in Figure 39:

- When $x = 0.75$, the maximal expansion order is reached later when the deviation is stabilized.
- When $x = 0.875$, the maximal expansion order is reached later when the deviation is stabilized, however reminder still does not become zero at the maximal expansion order but a few orders beyond.
- When $x = 0.9375$, deviation has no stable value and becomes negative eventually, after which the precision representation becomes undefined. Because Cauchy estimator never reaches deviation/16, the maximal expansion order is not defined either.

Judged from the above simple cases of Taylor expansion, calculating inside uncertainty does not bring any benefit.

8 Validation of Precision Arithmetic Using Numerical Integration

In numerical integration over the variable x using conventional floating-point arithmetic, a finer sampling of the function to be integrated $f(x)$ is associated with a better result [?], and it is assumed that $f(x)$ can be sampled at infinitive fine intervals of x . In reality, floating-point arithmetic has limited significant bits, so that rounding errors will increase with finer sampling of $f(x)$. However, such limitation of numerical integration due to rounding errors is seldom studied seriously. In this paper:

1. The function to be integrated is treated as a black-box function.
2. The numerical integration is carried out using the rectangular rule [?].
3. The residual error is estimated locally as the difference between using the rectangular rule and using the trapezoidal rule [?].
4. The sampling is localized using simplest depth-first binary-tree search algorithm.
5. The sampling stops when the residual error is no longer significant.

Specifically, for each integration interval $[x_{start}, x_{end}]$, define:

$$x_{mid} \equiv (x_{start} + x_{end})/2; \quad (8.1)$$

$$f_{err} \equiv (f(x_{start}) + f(x_{end}))/2 - f(x_{mid}); \quad (8.2)$$

$$f_{\Delta} \equiv f(x_{mid})(x_{end} - x_{start}); \quad (8.3)$$

If f_{err} becomes insignificant, the interval $[x_{start}, x_{end}]$ is considered to be fine enough, and f_{Δ} is added to the total integration. Otherwise, the search continues on the intervals $[x_{start}, x_{mid}]$ and $[x_{mid}, x_{end}]$, which is the next depth for searching. This searching algorithm is very adaptive, with the local search depth depending only on how $f(x)$ changes locally. However, such adaptation to the local change of $f(x)$ brings one weakness to this searching algorithm: when $f(0) = f'(0) = 0$, the algorithm spends the majority of the execution time around $x = 0$, searching in tiny intervals of great depth, and adding tiny significant values to the result each time. This weakness is called zero trap here. It cannot be removed by simply offsetting $f(x)$ by a constant because doing so will change the precision of each sampling of $f(x)$, and increase the output uncertainty deviation. For a proof-of-principle demonstration, zero trap is avoided in this paper.

Formula (8.4) provides an example test for the above simple algorithm, in which n is a positive integer.

$$\frac{4^{n+1} - 10^{-6(n+1)}}{n+1} = \int_{10^{-6}}^4 x^n dx; \quad (8.4)$$

Table 4 shows that the result of numerical integration is very comparable to the expected value. It shows that the above integration algorithm introduces no broadening of result uncertainty, so the above algorithm always selects optimal integration intervals when calculating the best possible result for a numerical integration. Tests of integration using different polynomials with different integration ranges all confirm the above result.

One thing worth noticing in Table 4 is that even though Formula (8.3) consistently underestimates integration for each integration interval $[x_{start}, x_{end}]$, the final underestimation is quite small and comparable to the uncertainty deviation. This example shows that the bias inside the uncertainty range has insignificant contribution to the final result using precision arithmetic.

Power n	Search Depth	$\delta \left(\int_{10^{-6}}^4 x^n dx \right)$	$\int_{10^{-6}}^4 x^n dx - \frac{4^{n+1} - 10^{-6(n+1)}}{n+1}$
2	[25, 47]	1.32×10^{-14}	-0.705×10^{-14}
3	[25, 47]	2.52×10^{-14}	-1.42×10^{-14}
4	[26, 47]	1.16×10^{-13}	-1.13×10^{-13}
5	[26, 48]	5.08×10^{-13}	-6.82×10^{-13}
6	[26, 48]	1.92×10^{-12}	-2.72×10^{-12}

Table 4: Uncertainty deviation and value error of numerical integration vs. expected results using precision arithmetic for different power function. The search range is deepest near 10^{-6} .

9 Conclusion and Discussion

9.1 Summary

The starting point of precision arithmetic is the uncorrelated uncertainty assumption, which requires that its low-significance input data not be overwhelmed by systematic errors, and all of its input data not have confused identities. Figure 2 quantifies the statistical requirements for input data to precision arithmetic.

Due to the uncorrelated uncertainty assumption and central limit theorem, the rounding errors of precision arithmetic are shown to be bounded by a Gaussian distribution with a truncated range. The rounding error distribution is extended to describe the uncertainty distribution in general, with the uncertainty deviation of a single precision value given by Formula (2.15), and the result uncertainty deviation of a function given by Formula (2.46) and its multi-dimension extensions such as Formula (2.51).

Formula (4.14) is shown to describe the general uncertainty deviation propagation in precision arithmetic. The average tracking ratios and the maximal bounding ratio using precision arithmetic are shown to be independent of input precision, and stable for the amount of calculations for a few very different applications. In contrast, both average tracking ratios and the maximal bounding ratio using interval arithmetic are shown to decrease exponentially with the amount of calculations in all tests. Such stability is the major reason why precision arithmetic is better than interval arithmetic in all tests done so far.

Precision arithmetic is quite different from conventional arithmetic. Due to the dependence problem when precision arithmetic is applied incorrectly, precision arithmetic has much less operational freedom than conventional arithmetic and may require extensive symbolic calculations. Also, the comparison relation in conventional arithmetic needs to be re-evaluated in precision arithmetic.

9.2 Improving Precision Arithmetic

Figure 2 uses a cut-off for the test of the uncorrelated uncertainty assumption among two uncertainty-bearing values. A better approach is to associate the amount of the dependence problem with the amount of correlation between the uncertainties of the two values.

There are actually three different ways to round up $(2S + 1) \cdot 4R \cdot 2^E$:

1. always round up $(2S + 1) \cdot 4R \cdot 2^E$ to $(S + 1) - R \cdot 2^{E+1}$;
2. always round up $(2S + 1) \cdot 4R \cdot 2^E$ to $S + R \cdot 2^{E+1}$;
3. randomly round up $(2S + 1) \cdot 4R \cdot 2^E$ to either $(S + 1) - R \cdot 2^{E+1}$ or $S + R \cdot 2^{E+1}$.

The first method results in slightly slower loss of significand than the second method, while the third method changes precision arithmetic from deterministic to stochastic. Because no empirical difference has been detected among these three different rounding up methods, the first method is chosen in this paper. Further study is required to distinguish the different rounding up methods.

The objectives of precision arithmetic need to be studied further. For example, Formula (2.43) has rejected the effect of uncertainty on the expected value by incorporating the value shift due to uncertainty as increase of variance, such as in the case of calculating $f(x) = x^2$. The effect of such asymmetrical broadening is unclear.

The number of bits to be calculated inside uncertainty also needs to be studied further. For example, when limited bits are calculated inside uncertainty, adding

insignificant higher order term of a Taylor expansion may decrease the value error while increasing the uncertainty deviation, which may call for an optimal bits to be calculated inside uncertainty for the truncation rule.

Because precision arithmetic is based on generic concepts, it is targeted to be a generic arithmetic for both uncertainty-tracking and uncertainty-bounding. However, it seems a worthwhile alternative to interval arithmetic and the de facto independence arithmetic. Before applying it generally, precision arithmetic still needs more ground-work and testing. It should be tested further in other problems such as improper integrations, solutions to linear equations, and solutions to differential equations.

9.3 Acknowledgements

As an independent researcher, the author of this paper feels indebted to encouragements and valuable discussions with Dr. Zhong Zhong from Brookhaven National Laboratory, Prof. Hui Cao from Yale University, Dr. Anthony Begley from *Physics Reviews B*, the organizers of *AMCS 2005*, with Prof. Hamid R. Arabnia from University of Georgia in particular, and the organizers of *NKS Mathematica Forum 2007*, with Dr. Stephen Wolfram in particular. Finally, the author of this paper is very grateful for the editors and reviewers of *Reliable Computing* for their tremendous help in shaping this unusual paper from unusual source, with managing editor, Prof. Rolph Baker Kearfott in particular.

10 Figures

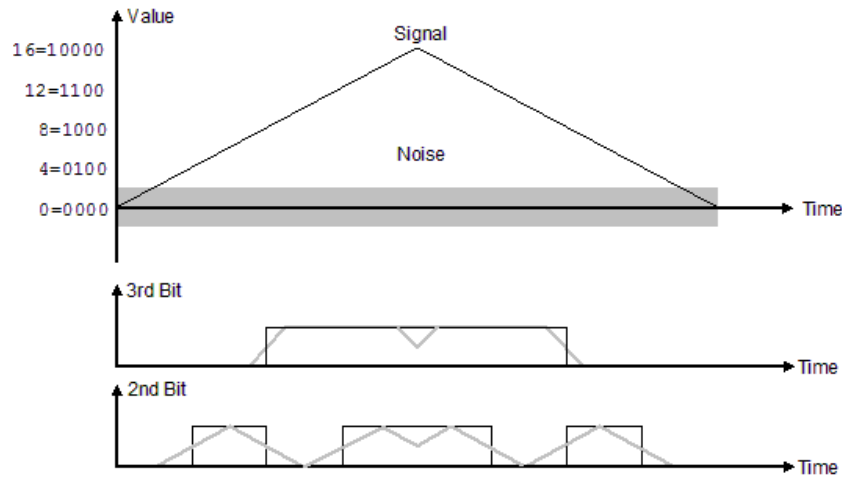


Figure 1: Effect of noise on bit values of a measured value. The triangular wave signal and the added white noise are shown at top using the thin black line and the grey area, respectively. The values are measured by a theoretical 4-bit Digital-to-Analog Converter in ideal condition, assuming LSB is the 0th bit. The measured 3rd and 2nd bits without the added noise are shown using thin black lines, while the mean values of the measured 3rd and 2nd bits with the added noise are shown using thin grey lines.

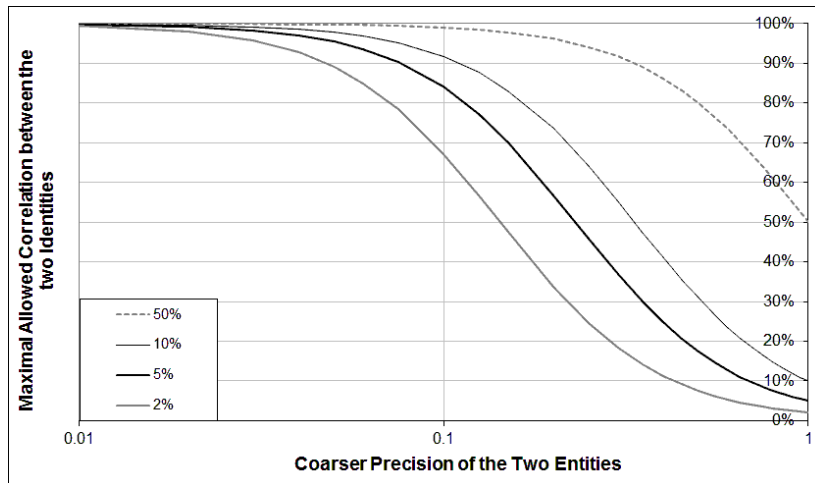


Figure 2: Allowed maximal correlation between two values vs. input precisions and independence standard (as shown in legend) for the independence uncertainty assumption of precision arithmetic to be true.

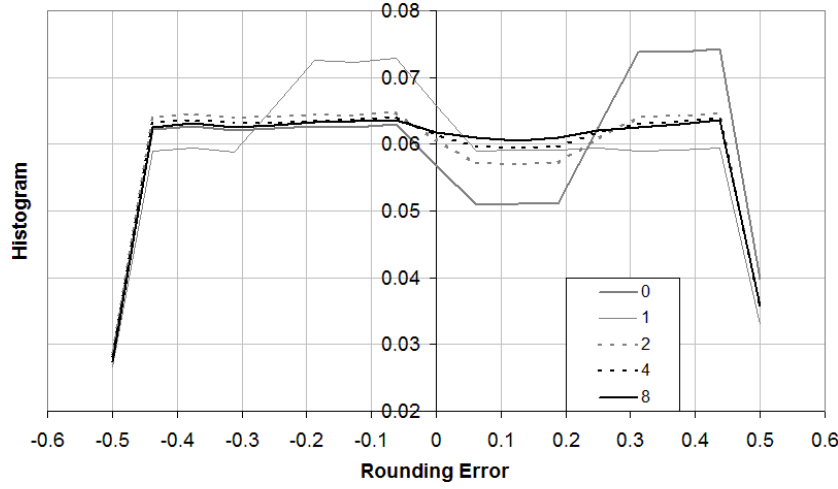


Figure 3: Measured probability distribution of rounding errors of precision round-up rule for the minimal significant thresholds 0, 1, 2, 4, and 8 respectively. Mathematically the probability is usually defined either in range $(-1/2, +1/2]$ or in range $[-1/2, +1/2)$, but not in range $[-1/2, +1/2]$. Because $-1/2$ and $+1/2$ in bounding range have different meaning in precision representation, the probability range is defined as $[-1/2, +1/2]$, which introduces the artificially smaller count of histogram in sections containing either $-1/2$ or $+1/2$.

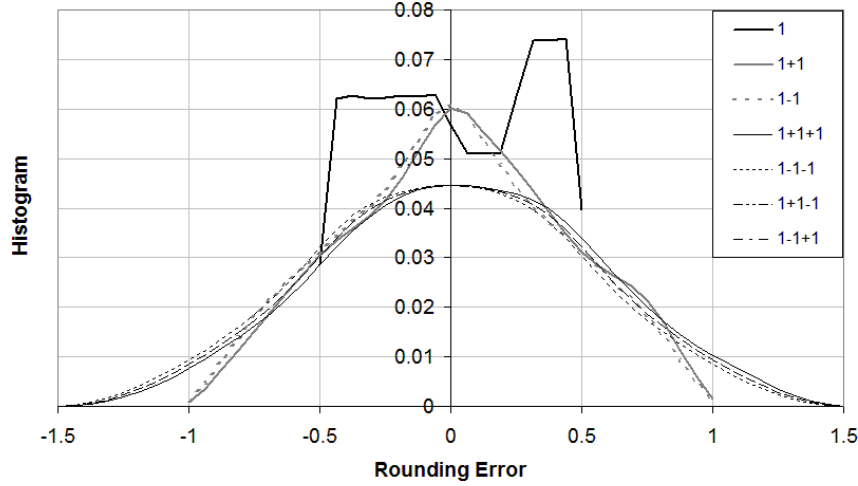


Figure 4: Measured probability distribution of the rounding error after addition and subtraction. In the legend, “1” for measured rounding error distribution for the minimal significant thresholds 0, “1+1” for addition once and “1-1” for subtraction once using the rounding error distribution of “1”, while “1+1+1” for addition twice, “1-1-1” for subtraction twice, “1+1-1” for addition once then subtraction once, and “1-1+1” for subtraction once then addition once.

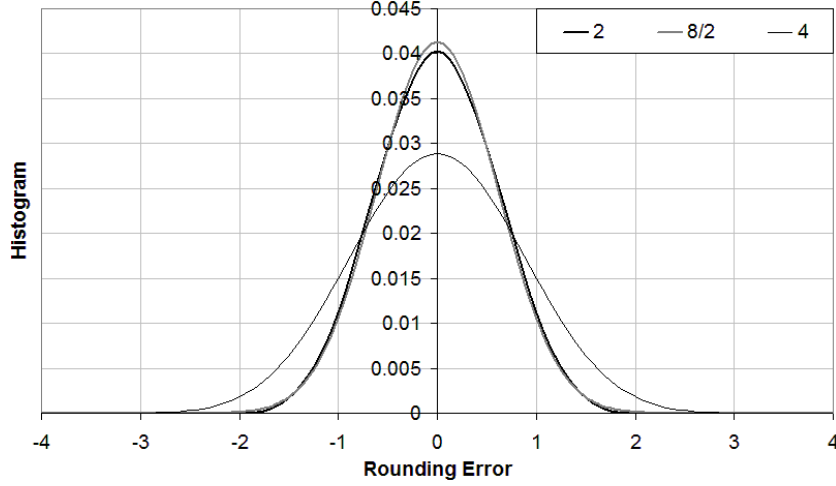


Figure 5: The result rounding error distribution $R = 8/2$ after the original error distribution $R = 8$ is rounded up once. The $R = 8/2$ distribution is compared with the $R = 4$ distribution and the $R = 2$ distribution, which have the same bounding range and deviation, respectively.

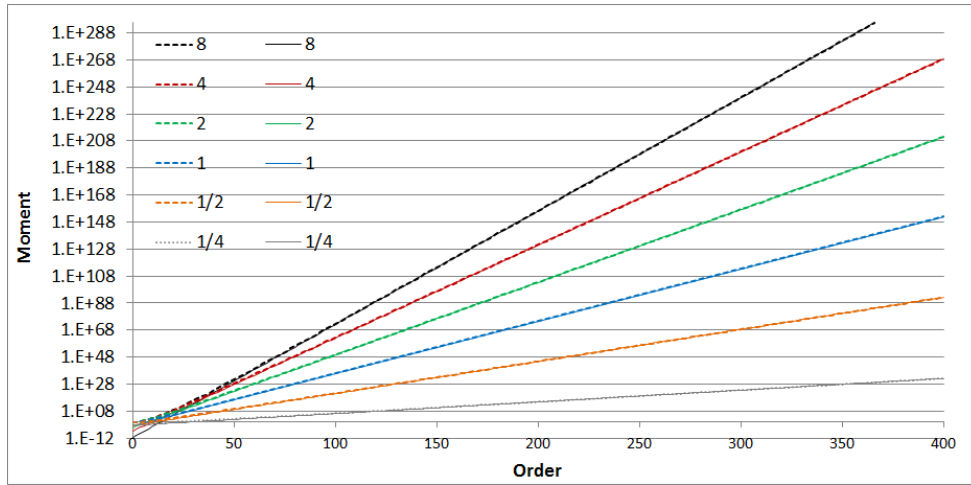


Figure 6: The moments up to 200-order for rounding error distribution for different bounding range \hat{R} of the legend. The moments for each bounding range \hat{R} are drawn in one color using dashed line, and exponentially fitted, whose fitting lines are displayed in the same color using thin solid lines.

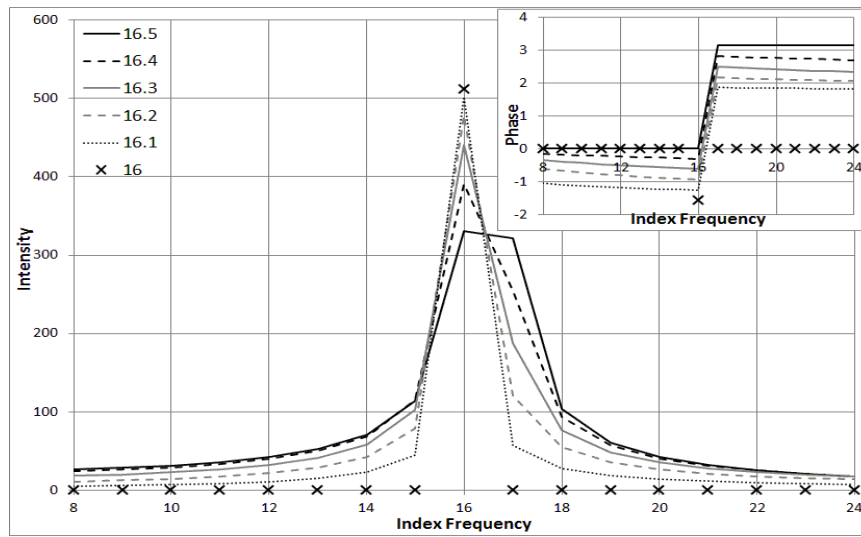


Figure 7: Unfaithful representations of perfect sine signals in the Discrete Fourier Transformation. The calculation is done on 1024 samples using FFT on a series of perfect sine signals having amplitude of 1 and slightly different frequencies as shown in legends. In the drawing, x axis shows frequency, y axis shows either intensity or phase (inlet). A faithful representation is also included for comparison, whose phase is $\pi/2$ at the index frequency, and undetermined at other frequencies.

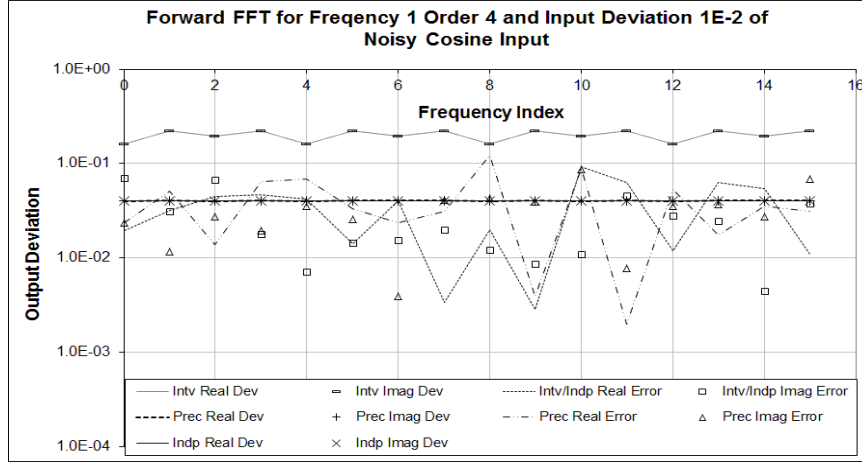


Figure 8: The output deviations and value errors of the forward FFT on a noisy sine signal of FFT order 4, index frequency 1 and input deviation 10^{-2} . In the legend, “Intv” means interval arithmetic, “Indp” means independence arithmetic, “Prec” means precision arithmetic, “Dev” means output uncertainty deviations, “Error” means output value errors, “Real” means real part, and “Imag” means imaginary part. Because both interval arithmetic and independence arithmetic using conventional floating arithmetic for underlying calculations, they have the same value errors.

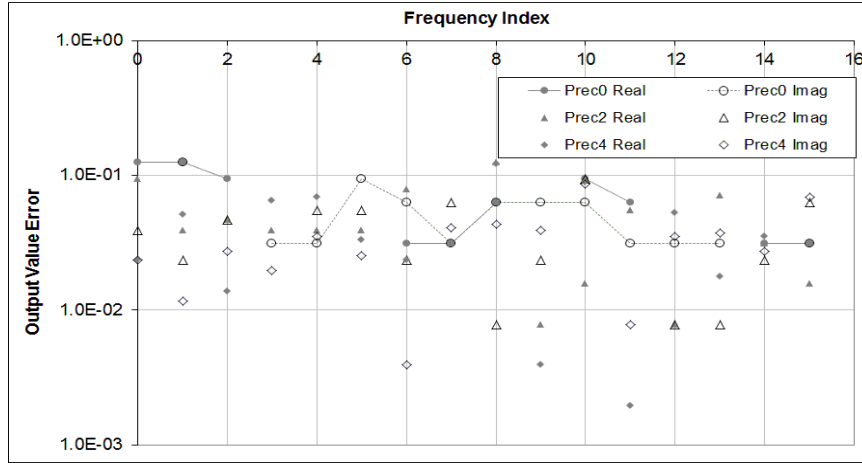


Figure 9: The output value errors of the forward FFT on a noisy sine signal of index frequency 1 and input deviation 10^{-2} using precision arithmetic with different bit inside uncertainty. In the legend, “Prec0” means precision arithmetic with 0-bit calculated inside uncertainty, “Prec2” means precision arithmetic with 2-bit calculated inside uncertainty, and “Prec4” means precision arithmetic with 4-bit calculated inside uncertainty.

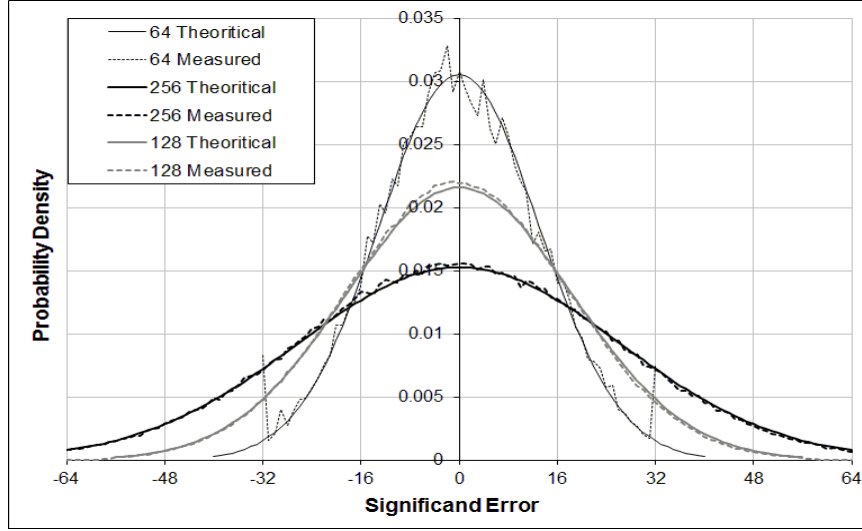


Figure 10: The theoretical and empirical distribution of significand errors for different bounding ranges (as shown in legend) using precision arithmetic for the forward FFT on noisy sine signals.

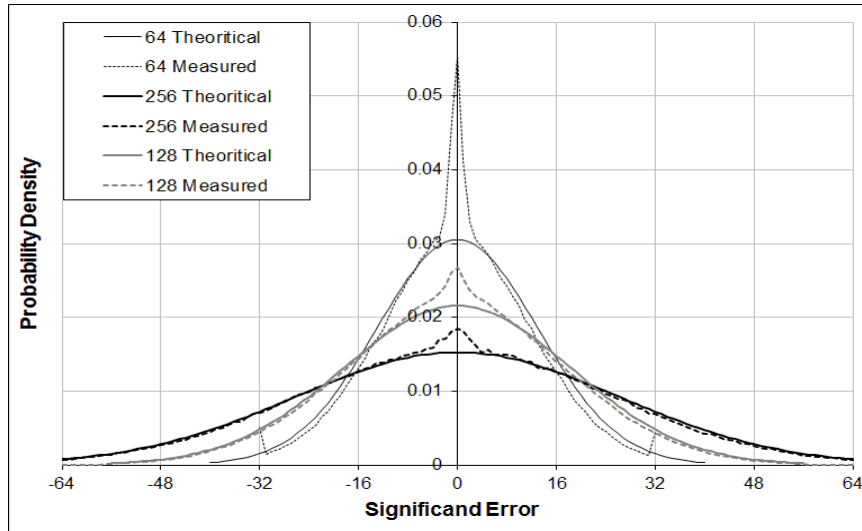


Figure 11: The theoretical and empirical distribution of significand errors for different bounding ranges (as shown in legend) using precision arithmetic for the reverse FFT on noisy sine signals.

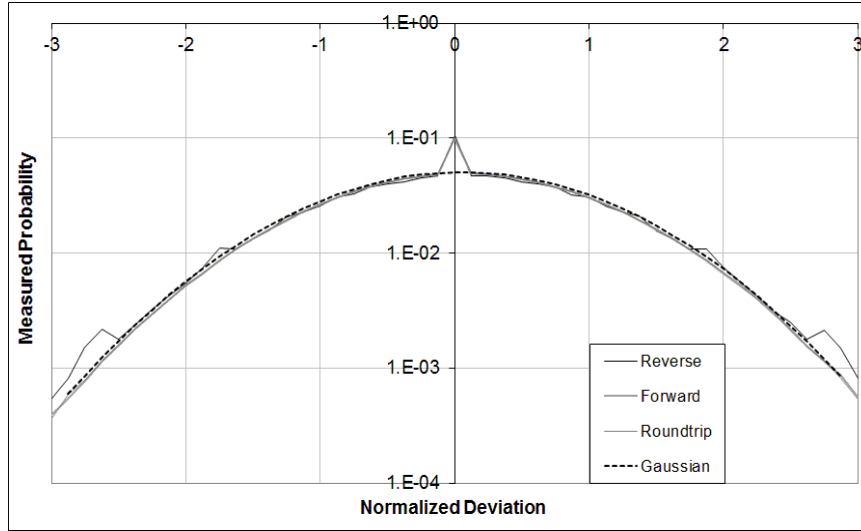


Figure 12: The measured tracking ratio distributions using independence arithmetic for FFT algorithms (as shown in legend). They are best fitted by a Gaussian distribution with the mean of 0.06 and deviation of 0.98.

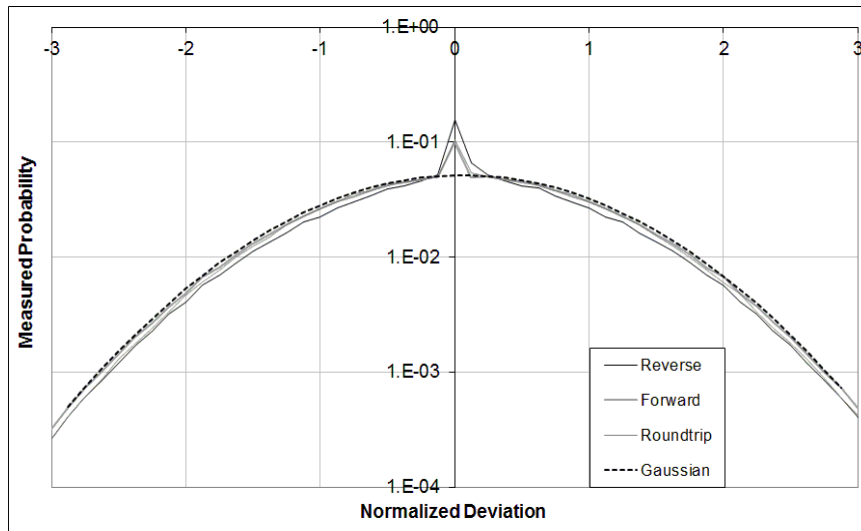


Figure 13: The measured tracking ratio distributions using precision arithmetic for FFT algorithms (as shown in legend). They are best fitted by a Gaussian distribution with the mean of 0.06 and deviation of 0.97.

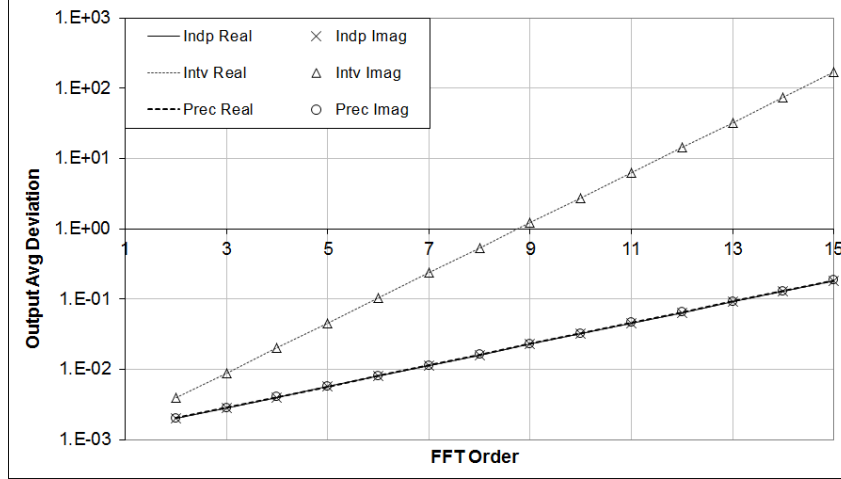


Figure 14: For the same input deviation of 10^{-3} , the empirical average output deviations of the forward FFT increase exponentially with the FFT order for all uncertainty-bearing arithmetics. In the legend, “Intv” means interval arithmetic, “Indp” means independence arithmetic, “Prec” means precision arithmetic, “Real” means real part, and “Imag” means imaginary part.

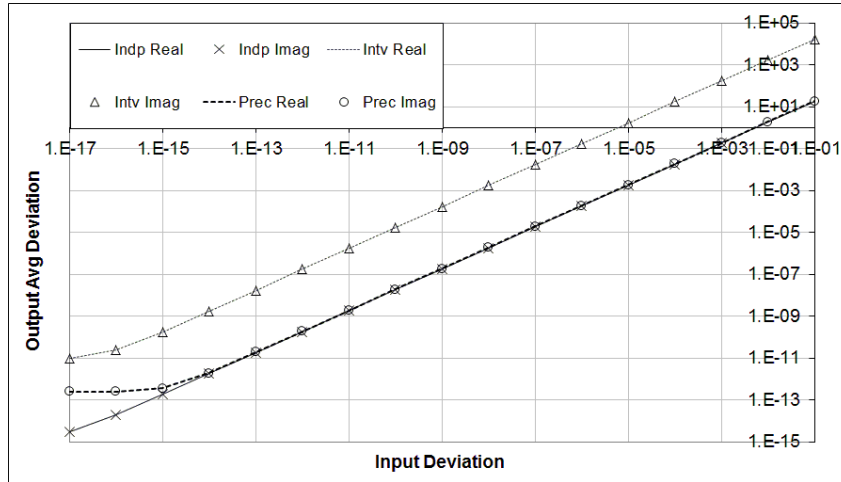


Figure 15: For the same order of the FFT calculation of 15, the empirical average output deviations of the forward FFT increases linearly with the input deviation for all uncertainty-bearing arithmetics. In the legend, “Intv” means interval arithmetic, “Indp” means independence arithmetic, “Prec” means precision arithmetic, “Real” means real part, and “Imag” means imaginary part.

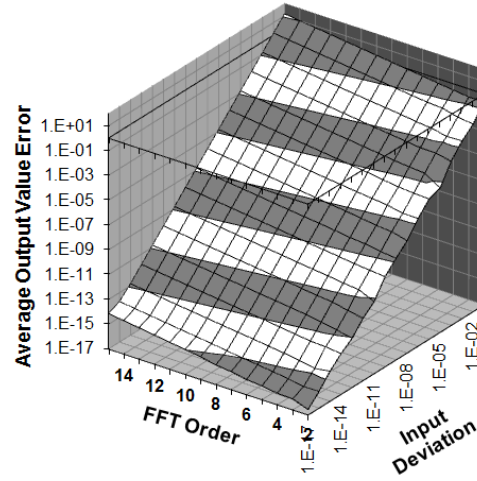


Figure 16: The empirical average output value errors using precision arithmetic increase exponentially with the FFT order and linearly with the input deviation, respectively.

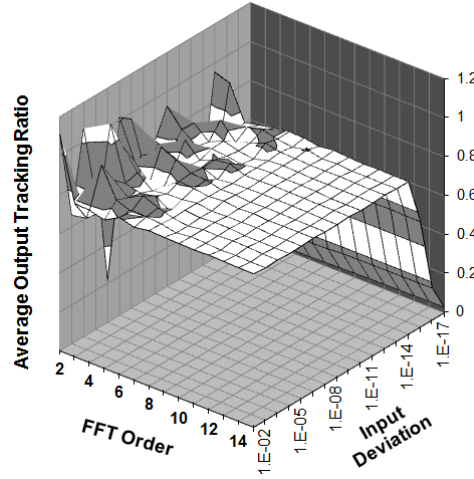


Figure 17: The empirical average output tracking ratios using precision arithmetic is a constant when the input deviation is larger than 10^{-14} and the FFT order is more than 5 for forward FFT algorithms. Because the precision of conventional floating-point representation is at 10^{-16} , adding Gaussian noises with the deviation of 10^{-17} should have little effect on the input data. For the same reason, the output tracking ratios are stable only when the input deviation is more than 10^{-14} . When the FFT order is 2, a FFT calculation actually involves no arithmetic calculation between input data. For the same reason, when the FFT order is less than 5, there is not enough arithmetic calculation for the result tracking ratios to reach equilibrium.

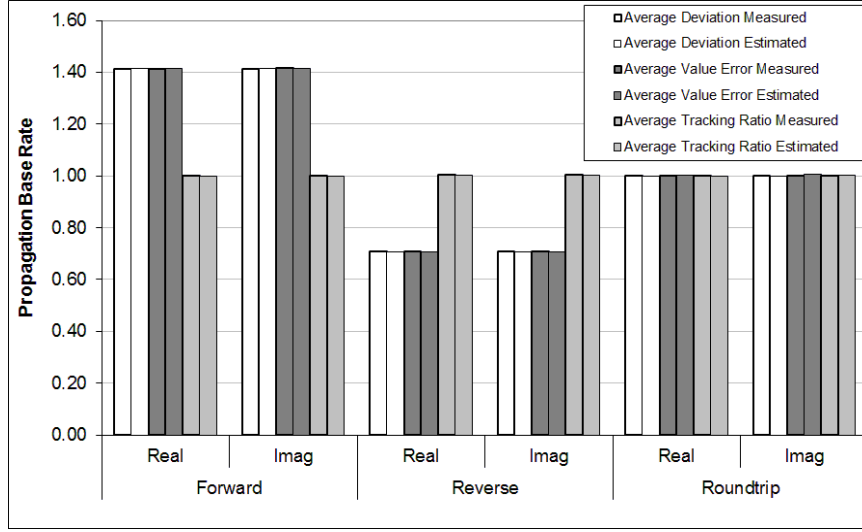


Figure 18: Empirical and theoretical β for fitting average output deviations, value errors and tracking ratios for forward, reverse and roundtrip FFT using independence arithmetic on noisy sine signals. In the chart, “Real” means real part, and “Imag” means imaginary part.

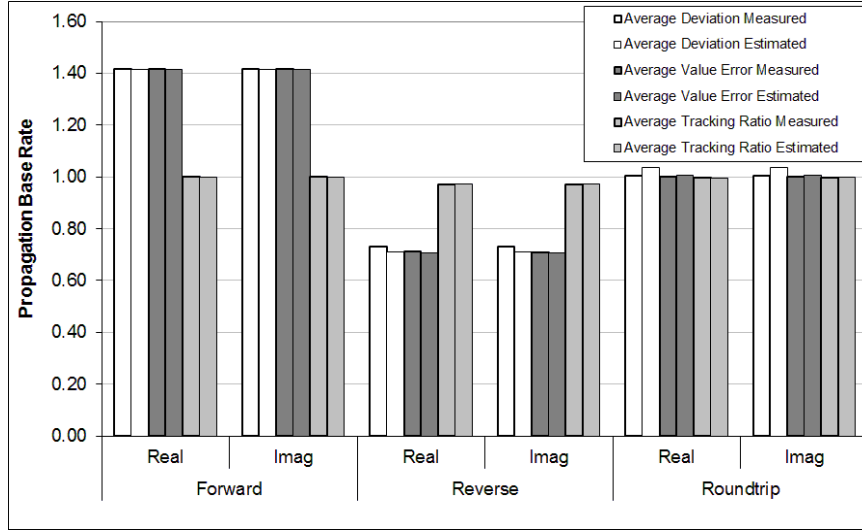


Figure 19: Empirical and theoretical β for fitting average output deviations, value errors and tracking ratios for forward, reverse and roundtrip FFT using precision arithmetic on noisy sine signals. In the chart, “Real” means real part, and “Imag” means imaginary part.

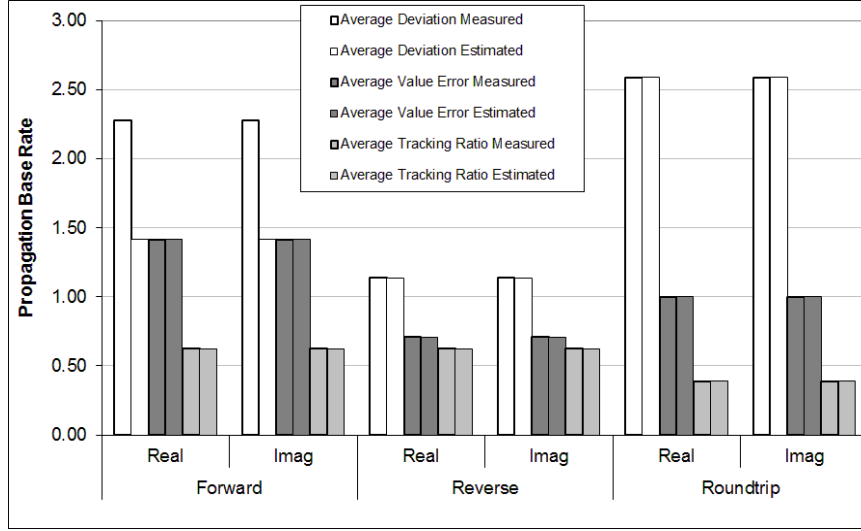


Figure 20: Empirical and theoretical β for fitting average output deviations, value errors and tracking ratios for forward, reverse and roundtrip FFT using interval arithmetic on noisy sine signals. In the chart, “Real” means real part, and “Imag” means imaginary part.

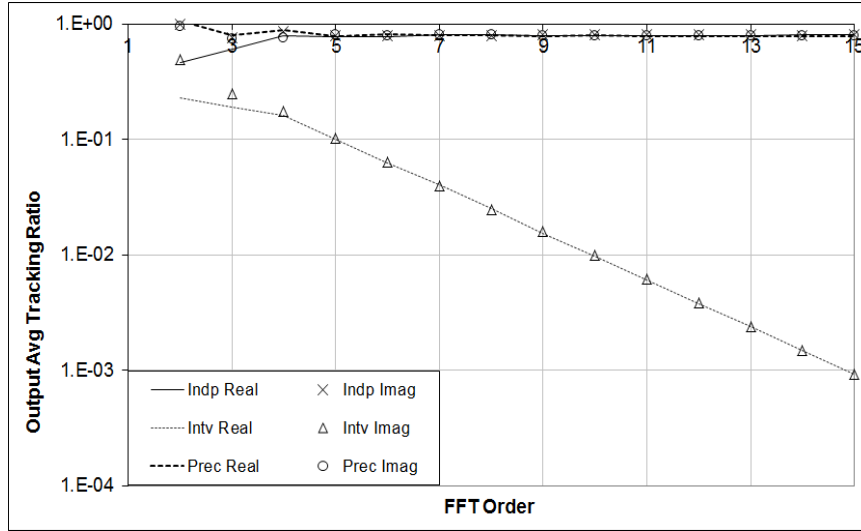


Figure 21: The empirical output average tracking ratios vs. the FFT order of the forward FFT for all three arithmetics when the input uncertainty deviation is 10^{-3} . In the legend, “Intv” means interval arithmetic, “Indp” means independence arithmetic, “Prec” means precision arithmetic, “Real” means real part, and “Imag” means imaginary part.

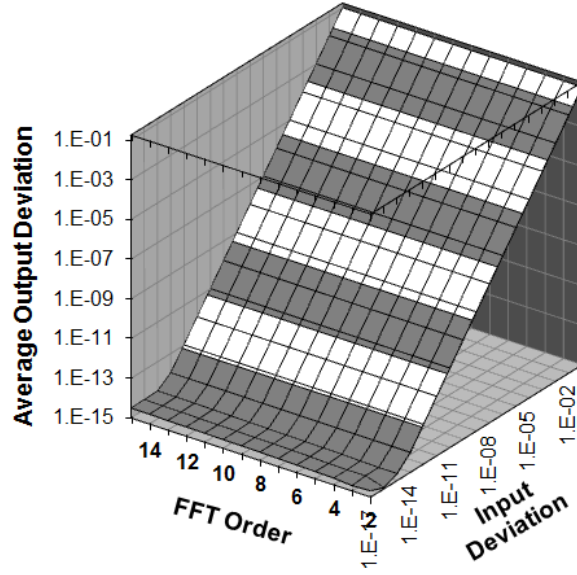


Figure 22: The empirical average output deviations vs. the FFT order and input deviations using precision arithmetic for the round-trip FFT algorithm.

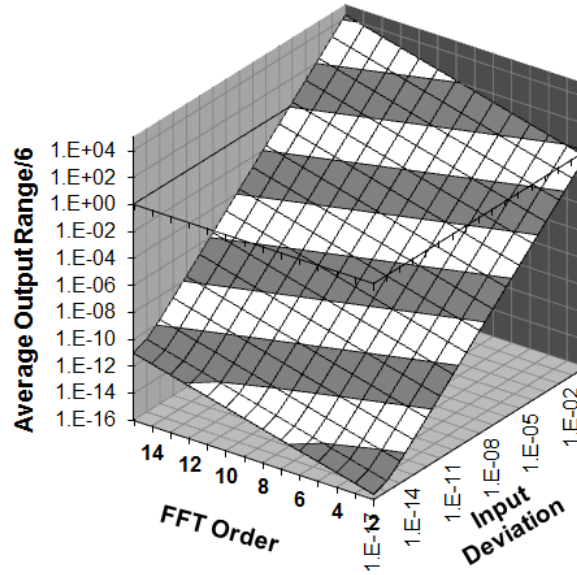


Figure 23: The empirical average output deviations vs. the FFT order and input deviations using interval arithmetic for the round-trip FFT algorithm.

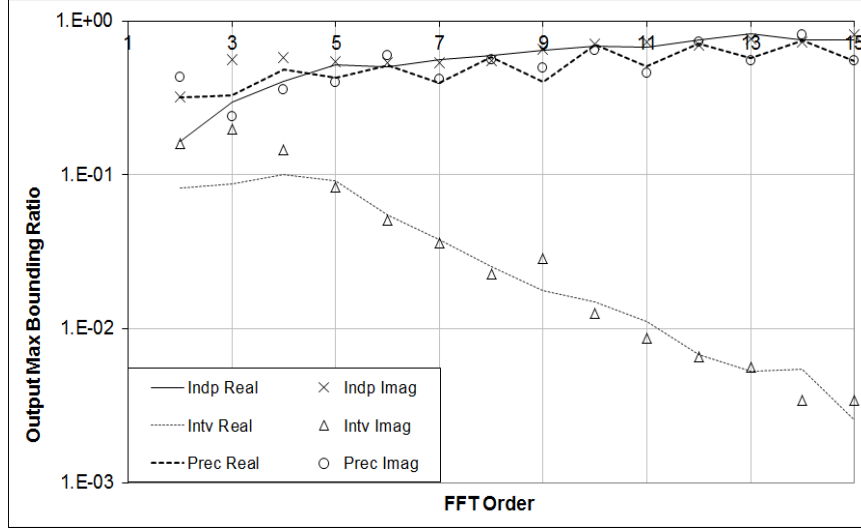


Figure 24: The empirical maximal output bounding ratios vs. the FFT order of the forward FFT for all three arithmetics. In the legend, “Intv” means interval arithmetic, “Indp” means independence arithmetic, “Prec” means precision arithmetic, “Real” means real part, and “Imag” means imaginary part.

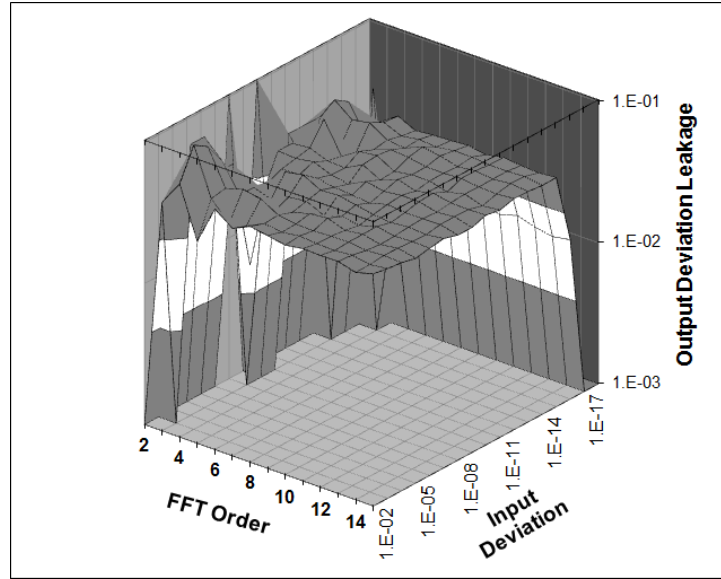


Figure 25: The empirical deviation leakages vs. the FFT order and input deviations using precision arithmetic for the forward FFT algorithm.

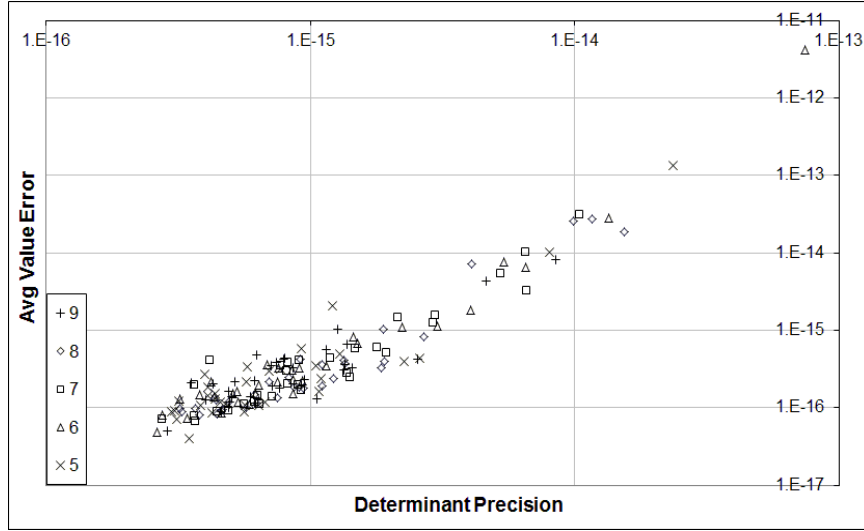


Figure 26: The empirical average value errors of the inverted matrix using conventional floating-point arithmetic vs. matrix determinant precision using precision arithmetic for clean matrices of different sizes (as shown in legend).

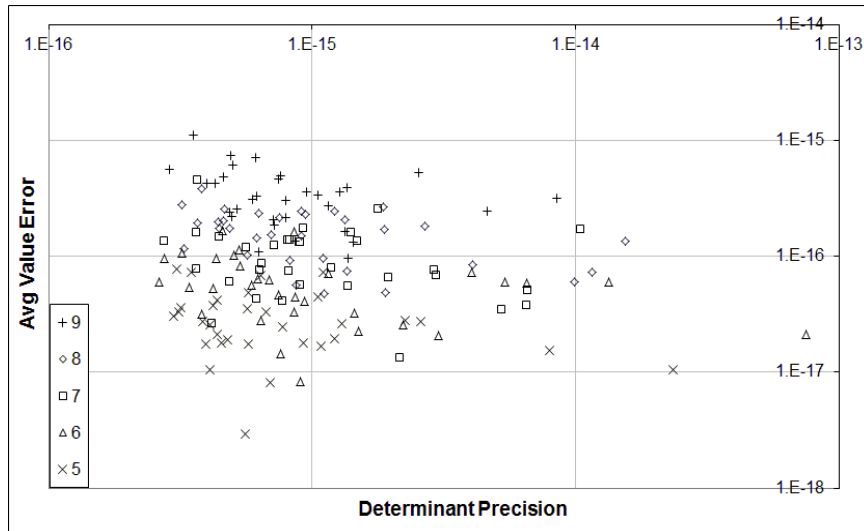


Figure 27: The empirical average value errors of the adjugate matrix using conventional floating-point arithmetic vs. matrix determinant precision using precision arithmetic for clean matrices of different sizes (as shown in legend).

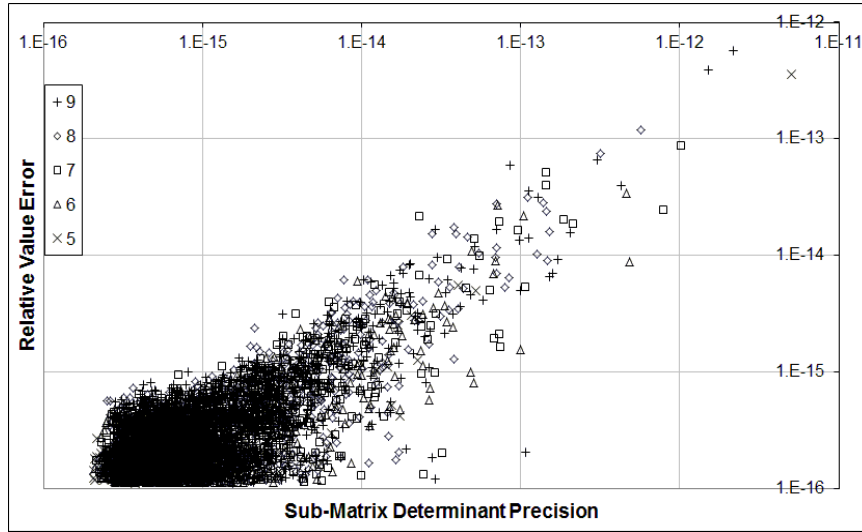


Figure 28: Empirical relative value errors of the adjugate matrix using conventional floating-point arithmetic vs. corresponding sub-matrix determinant precision using precision arithmetic for clean matrices of different sizes (as shown in legend).

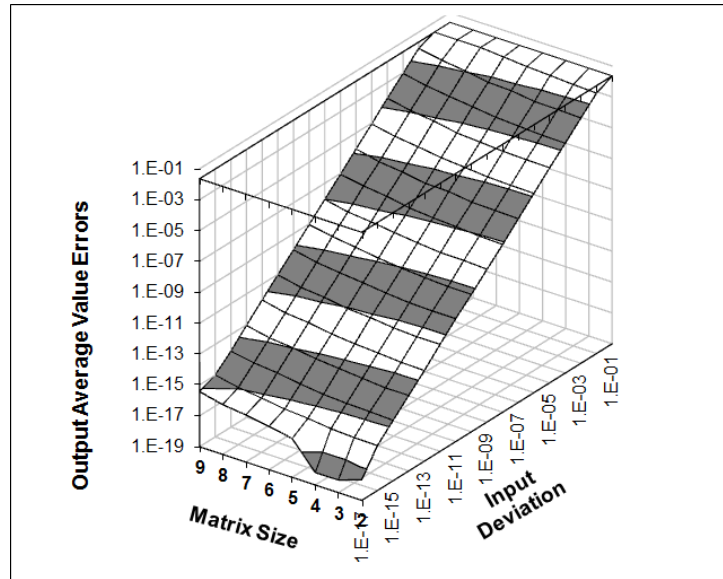


Figure 29: Using precision arithmetic, the average output deviations of the adjugate matrix vs. input precision and the matrix size.

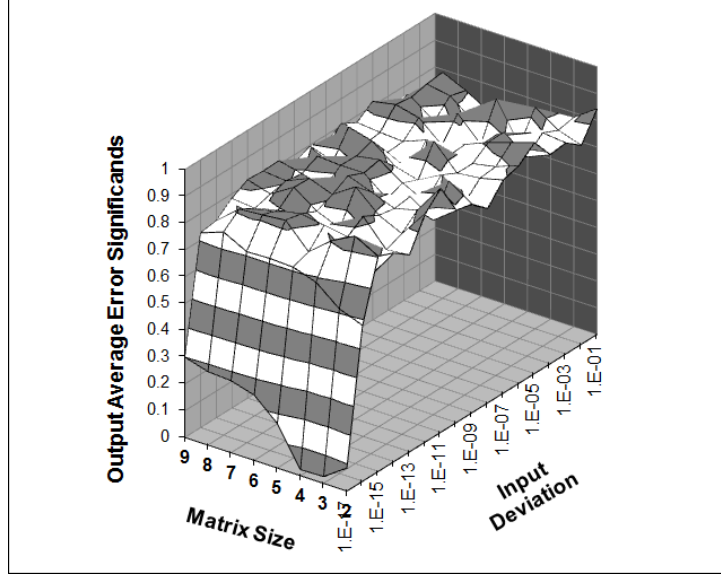


Figure 30: Using precision arithmetic, the average output tracking ratios of the adjugate matrix vs. input precision and the matrix size.

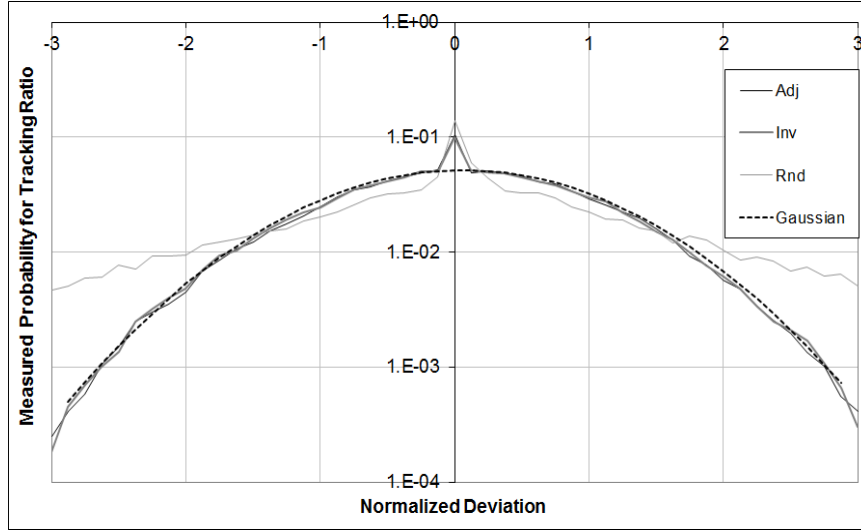


Figure 31: The measured tracking ratio distributions using precision arithmetic for matrix calculations of matrix size 9. They are best fitted by a Gaussian distribution with the mean of 0.06 and deviation of 0.96. In the legend, “Adj” means calculating adjugate M^A , “Inv” means calculating inverted M^{-1} , and “Rnd” means calculating double inverted $(M^{-1})^{-1}$.

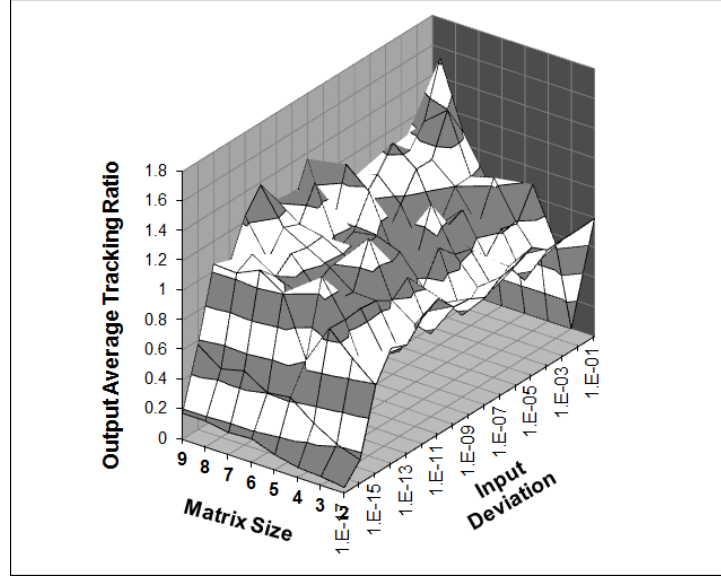


Figure 32: Using precision arithmetic, the average output tracking ratios of the double inverted matrix vs. input precision and the matrix size.

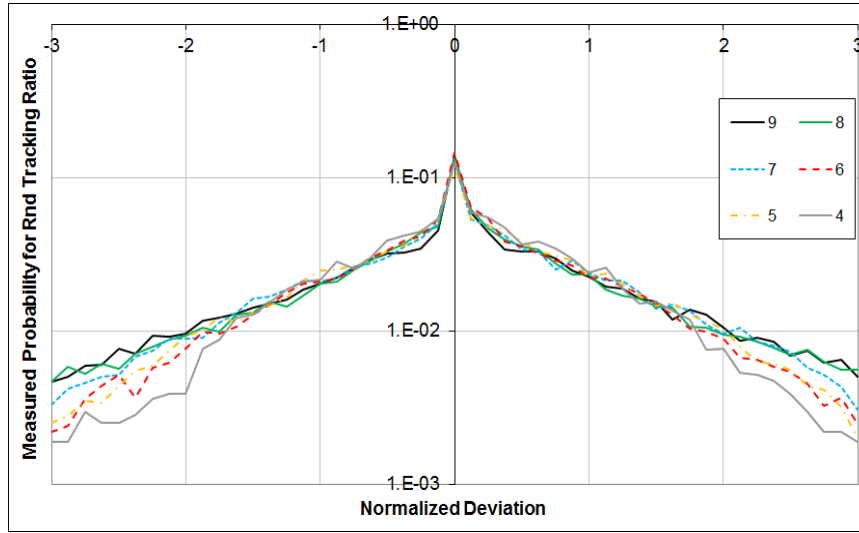


Figure 33: The measured tracking ratio distributions using precision arithmetic for matrix calculations of matrix size 9. They are best fitted by a Gaussian distribution with the mean of 0.06 and deviation of 0.96. In the legend, “Adj” means calculating adjugate M^A , “Inv” means calculating inverted M^{-1} , and “Rnd” means calculating double inverted $(M^{-1})^{-1}$.

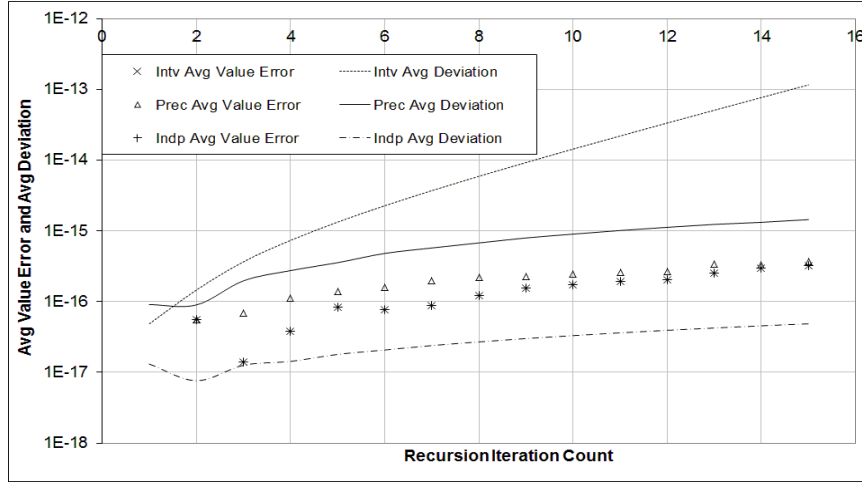


Figure 34: The empirical output average value errors and corresponding average output deviations vs. the recursion iteration count of the regressive calculation of sine values using interval arithmetic, precision arithmetic and independence arithmetic. The x-axis indicates the recursion iteration count L , while the y-axis indicates either the average value errors or average uncertainty deviations. In the legend, “Intv” means interval arithmetic, “Indp” means independence arithmetic, and “Prec” means precision arithmetic.

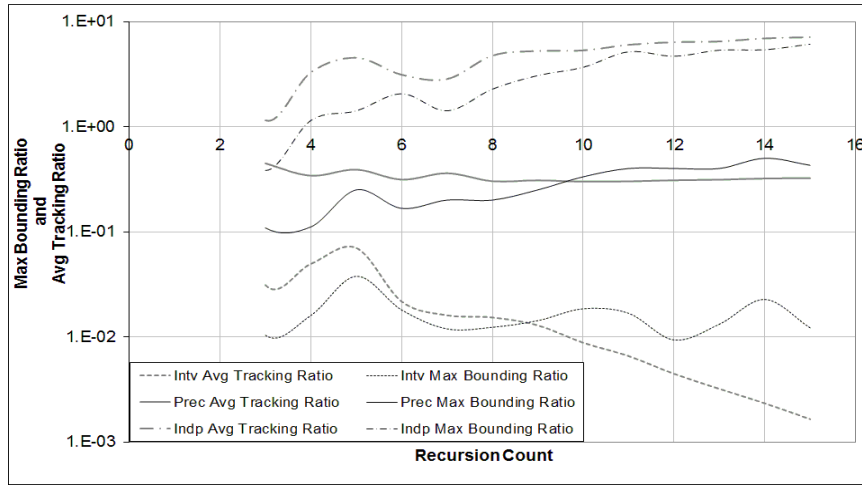


Figure 35: The empirical output maximal bounding ratios and average tracking ratios vs. the recursion iteration count of the regressive calculation of sine values using interval and precision arithmetics. In the legend, “Intv” means interval arithmetic, “Indp” means independence arithmetic, and “Prec” means precision arithmetic.

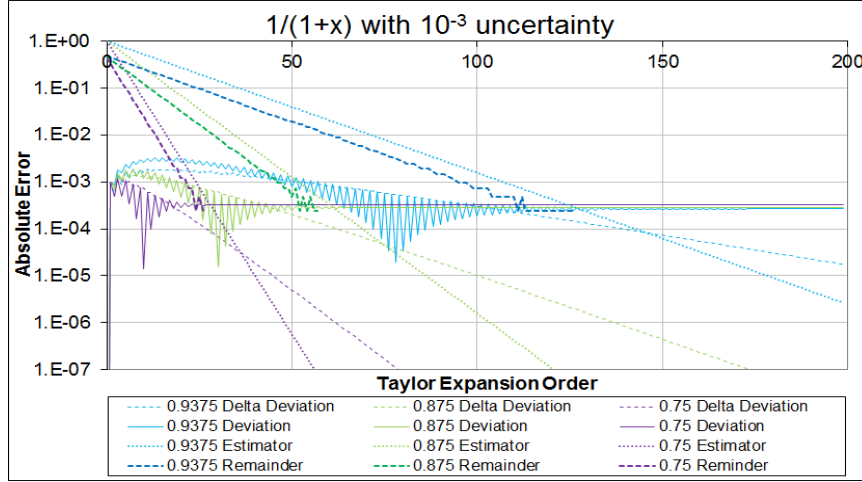


Figure 36: The delta deviation, deviations, Cauchy estimator and remainders of a Taylor expansion vs. the expansion orders for different input value with 10^{-3} input uncertainty using precision arithmetic with 0-bit calculated inside uncertainty. All the values in the figure are calculated using Formula (7.1). Different inputs are displayed using different color.

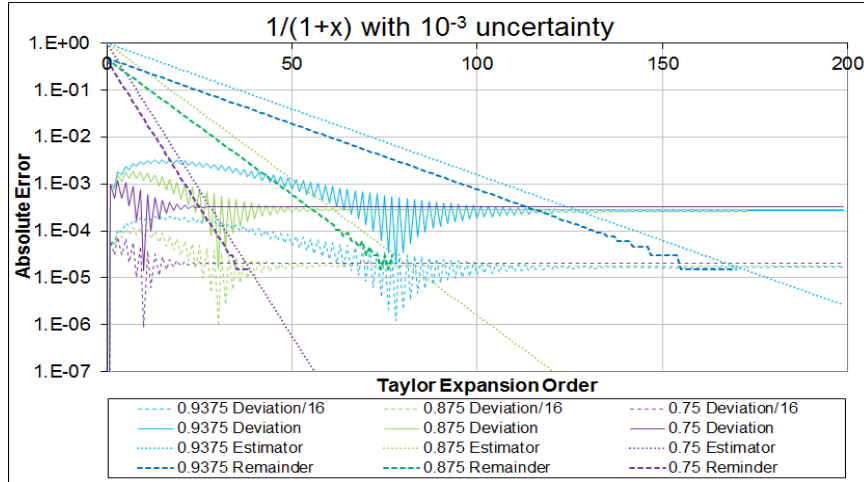


Figure 37: The deviations, deviations adjusted by $1/16$, Cauchy estimator and remainders of a Taylor expansion vs. the expansion orders for different input value with 10^{-3} input uncertainty using precision arithmetic with 4-bit calculated inside uncertainty. All the values in the figure are calculated using Formula (7.1). Different inputs are displayed using different color.

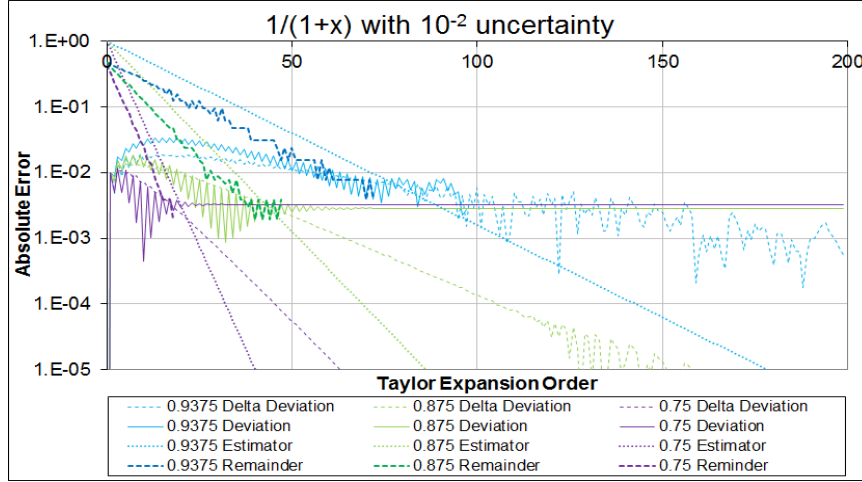


Figure 38: The delta deviation, deviations, Cauchy estimator and remainders of a Taylor expansion vs. the expansion orders for different input value with 10^{-2} input uncertainty using precision arithmetic with 0-bit calculated inside uncertainty. All the values in the figure are calculated using Formula (7.1). Different inputs are displayed using different color.

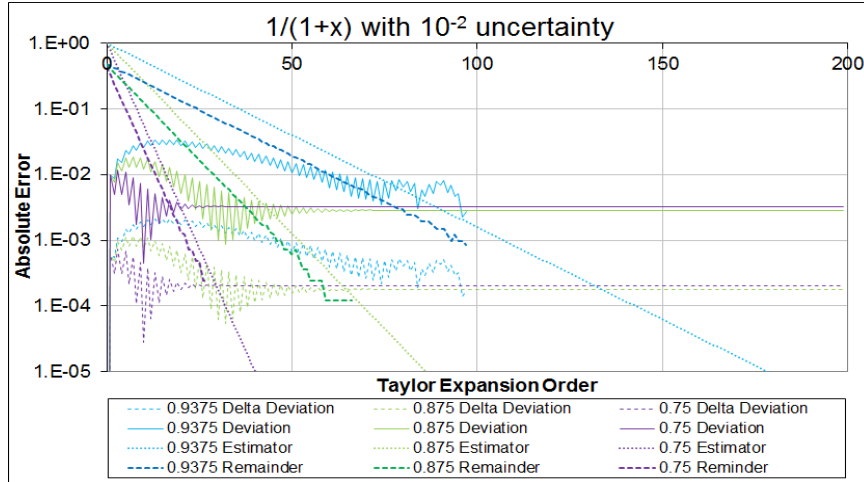


Figure 39: The deviations, deviations adjusted by $1/16$, Cauchy estimator and remainders of a Taylor expansion vs. the expansion orders for different input value with 10^{-2} input uncertainty using precision arithmetic with 4-bit calculated inside uncertainty. All the values in the figure are calculated using Formula (7.1). Different inputs are displayed using different color.