

The Computational Complexity of $3k$ -CLIQUE

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Abstract: In this note, we show that the fastest deterministic and exact algorithm that solves the $3k$ -CLIQUE problem must run in $\Omega(n^{2k})$ time in the worst-case scenario on a classical computer, where n is the number of vertices in the graph.

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The $3k$ -CLIQUE problem is to locate a clique of size $3k$ or determine that no clique of size $3k$ exists in a given undirected graph G , where k is a positive integer that is not part of the input of the problem [3]. In this note, we show that the fastest deterministic and exact algorithm that solves $3k$ -CLIQUE must run in $\Omega(n^{2k})$ time in the worst-case scenario on a classical computer, where n is the number of vertices in the graph:

Let G be an undirected graph with n vertices. For every k -clique C in G , create a corresponding vertex $v(C)$ in an auxiliary graph G' . And for every two vertices $v(C_1)$ and $v(C_2)$ in G' , create an edge connecting them in G' if and only if $C_1 \cup C_2$ forms a $2k$ -clique in G . Then G' will have $O(n^k)$ vertices and $O(n^{2k})$ edges. Note that the 3-CLIQUE problem on G' is equivalent to the $3k$ -CLIQUE problem on G [3].

If an algorithm is able to consistently locate a 3-clique in G' in $o(n^{2k})$ time when G' is of size $\Theta(n^{2k})$ and has only a constant number of 3-cliques, that algorithm must use an oracle. Thus, it must take $\Omega(n^{2k})$ time in the worst-case scenario for any algorithm that does not use an oracle to solve the 3-CLIQUE problem on G' . Then since the 3-CLIQUE problem on G' is equivalent to the $3k$ -CLIQUE problem on G , it must also take $\Omega(n^{2k})$ time in the worst-case scenario for any algorithm that does not use an oracle to solve the $3k$ -CLIQUE problem on G . And this implies that $P \neq NP$ [1]. \square

This lower bound is confirmed by the fact that the fastest known deterministic and exact algorithm that solves $3k$ -CLIQUE was first published in 1985 and has a running-time of $\Theta(n^{\omega k})$, where $\omega \geq 2$ [1, 2, 3].

References

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