

# The Computational Complexity of $3k$ -CLIQUE

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**Abstract:** In this note, we show that the fastest deterministic and exact algorithm that solves the  $3k$ -CLIQUE problem must run in  $\Omega(n^{2k})$  time in the worst-case scenario on a classical computer, where  $n$  is the number of vertices in the graph.

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The  $3k$ -CLIQUE problem is to determine whether a given undirected graph  $G$  contains a clique of size  $3k$ , where  $k$  is a positive integer that is not part of the input of the problem [3]. In this note, we show that the fastest deterministic and exact algorithm that solves  $3k$ -CLIQUE must run in  $\Omega(n^{2k})$  time in the worst-case scenario on a classical computer, where  $n$  is the number of vertices in the graph:

Let  $G$  be an undirected graph with  $n$  vertices. For every  $k$ -clique  $C$  in  $G$ , create a corresponding vertex  $v(C)$  in an auxiliary graph  $\tilde{G}$ . And for every two vertices  $v(C_1)$  and  $v(C_2)$  in  $\tilde{G}$ , create an edge connecting them in  $\tilde{G}$  if and only if  $C_1 \cup C_2$  forms a  $2k$ -clique in  $G$ . Note that the 3-CLIQUE problem on  $\tilde{G}$  is equivalent to the  $3k$ -CLIQUE problem on  $G$  [3].

Since graph  $\tilde{G}$  has  $\Theta(n^k)$  vertices and  $\Theta(n^{2k})$  edges in the worst-case scenario, it must take  $\Omega(n^{2k})$  time to solve 3-CLIQUE on  $\tilde{G}$  in the worst-case scenario. Thus, since the 3-CLIQUE problem on  $\tilde{G}$  is equivalent to the  $3k$ -CLIQUE problem on  $G$ , it must also take  $\Omega(n^{2k})$  time to solve the  $3k$ -CLIQUE problem on  $G$  in the worst-case scenario. And this implies that  $P \neq NP$  [1].  $\square$

This lower bound is confirmed by the fact that the fastest known deterministic and exact algorithm that solves  $3k$ -CLIQUE was first published in 1985 and has a running-time of  $\Theta(n^{\omega k})$ , where  $\omega \geq 2$  [1, 2, 3].

## References

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