

# The Computational Complexity of $3k$ -CLIQUE

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**Abstract:** In this note, we show that the fastest deterministic and exact algorithm that solves the  $3k$ -CLIQUE problem must run in  $\Omega(n^{2k})$  time in the worst-case scenario on a classical computer, where  $n$  is the number of vertices in the graph.

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The  $3k$ -CLIQUE problem is to determine whether or not a clique of size  $3k$  exists in a given undirected graph  $G$ , where  $k$  is a positive integer that is not part of the input of the problem [4]. In this note, we show that the fastest deterministic and exact algorithm that solves  $3k$ -CLIQUE must run in  $\Omega(n^{2k})$  time in the worst-case scenario on a classical computer, where  $n$  is the number of vertices in the graph:

Let  $G$  be an undirected graph with  $n$  vertices. For every  $k$ -clique  $C$  in  $G$ , create a corresponding vertex  $v(C)$  in an auxiliary graph  $G'$ . And for every two vertices  $v(C_1)$  and  $v(C_2)$  in  $G'$ , create an edge connecting them in  $G'$  if and only if  $C_1 \cup C_2$  forms a  $2k$ -clique in  $G$ . Then  $G'$  will have  $O(n^k)$  vertices and  $O(n^{2k})$  edges. Note that the  $3$ -CLIQUE problem on  $G'$  is equivalent to the  $3k$ -CLIQUE problem on  $G$  [4].

Let  $A = (a_{ij})$  be the adjacency matrix of  $G'$ . Then  $G'$  has no  $3$ -clique if and only if  $\sum_k a_{ik} \cdot a_{kj} = 0$  for each  $i, j$  such that  $a_{ij} = 1$  [2]. Clearly, if an algorithm is able to determine whether it is true that  $\sum_k a_{ik} \cdot a_{kj} = 0$  for each  $i, j$  such that  $a_{ij} = 1$  faster than the fastest algorithm that checks each equation  $\sum_k a_{ik} \cdot a_{kj} = 0$  individually for each  $i, j$  such that  $a_{ij} = 1$  until either an equation is found to be false or it is certain that all such equations are true, then that algorithm must use an oracle.

Hence, because there are adjacency matrices  $A$  of  $G'$  with  $\Theta(n^{2k})$  indices  $(i, j)$  such that  $a_{ij} = 1$  but only a constant number of indices  $(i, j)$  such that  $\sum_k a_{ik} \cdot a_{kj} \neq 0$ , any algorithm that does not use an oracle and determines whether it is true that  $\sum_k a_{ik} \cdot a_{kj} = 0$  for each  $i, j$  such that  $a_{ij} = 1$  must take  $\Omega(n^{2k})$  time in the worst-case scenario. Then since determining whether it is true that  $\sum_k a_{ik} \cdot a_{kj} = 0$  for each  $i, j$  such that  $a_{ij} = 1$  is equivalent to the  $3k$ -CLIQUE problem on  $G$ , it must

also take  $\Omega(n^{2k})$  time in the worst-case scenario for any deterministic and exact algorithm that does not use an oracle to solve the  $3k$ -CLIQUE problem on  $G$ . And this implies that  $P \neq NP$  [1].  $\square$

This lower bound is confirmed by the fact that the fastest known deterministic and exact algorithm that solves  $3k$ -CLIQUE was first published in 1985 and has a running-time of  $\Theta(n^{\omega k})$ , where  $\omega \geq 2$  [1, 3, 4].

## References

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