Frugality Ratios And Improved Truthful Mechanisms for Vertex Cover

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Abstract. In *set-system auctions*, there is a task than can be completed by several overlapping teams of selfish agents, and the centre's goal is to hire one of these teams and pay as little as possible. Examples of this setting include shortest-path auctions and vertex-cover auctions. Recently, paper [15] introduced a new definition of *frugality ratio* for this problem. Informally, the "frugality ratio" is the ratio of the total payment of a mechanism to a desired payment bound. The ratio captures the extent to which the mechanism overpays, relative to perceived fair cost. In this paper, we propose a new truthful polynomial-time auction for the vertex cover problem. We bound its frugality ratio using the definition of [15]. We show that both the solution quality and the frugality ratio of our auction are within a constant factor of optimal; this is the first auction for this problem to have these properties. Moreover, we show how to transform any truthful auction into a frugal one while preserving the approximation ratio. Also, we consider two natural modifications of the definition of [15] and analyse the properties of the resulting payment bounds, such as monotonicity, computational hardness, robustness with respect to the draw resolution rule, etc. We study the relationships between our payment bounds, both for general set systems and for specific problems, such as path auctions and vertex cover auctions. We use these new definitions in the proof of our main result for vertex cover auctions via a bootstrapping technique, which may be of independent interest.

1 Introduction

In a *set system auction* there is a single buyer and many vendors that can provide various services. It is assumed that the buyer's requirements can be satisfied by various subsets of the vendors; these subsets are called the *feasible sets*. A widely-studied class of set-system auctions is *path auctions*, where each vendor is able to sell access to a link in a network, and the feasible sets are those sets whose links contain a path from a given source to a given destination; the study of these auctions has been initiated in the seminal paper by Nisan and Ronen [18] (see also [1, 9, 8, 6, 14, 7, 19]).

We assume that each vendor has a cost of providing his services, but submits a possibly larger *bid* to the auctioneer, so that if the bid is accepted, the vendor enjoys a profit of bid minus cost. Vendors want to maximise profit, while the buyer wants to minimise the amount he pays. In *truthful* auctions the vendors do however have an incentive to bid their true cost, as the centre promises to pay them premiums above their

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bids. A general topic of interest in mechanism design is the question of how much the centre will have to overpay in order to ensure truthful bids.

In the context of path auctions this topic was first addressed by Archer and Tardos [1]. They define the *frugality ratio* of a mechanism as the ratio between its total payment and the cost of the cheapest path disjoint from the path selected by the mechanism. They show that, for a large class of truthful mechanisms for this problem, the frugality ratio is as large as the number of edges in the shortest path. Talwar [20] extends this definition of frugality ratio to general set systems, and studies the frugality ratio of the classical VCG mechanism [21, 4, 13] for many specific set systems, such as minimum spanning trees and set covers.

While the definition of frugality ratio proposed by [1] is well-motivated and has been instrumental in studying truthful mechanisms for set systems, it is not completely satisfactory. Consider, for example, the graph of Figure 1 with the costs $c_{AB} = c_{BC} = c_{CD} = 0$, $c_{AC} = c_{BD} = 1$. This graph is 2-connected and the VCG payment to the winning path ABCD is bounded. However, the graph contains no A–D path that is disjoint from ABCD, and hence the frugality ratio of VCG on this graph remains undefined. At the same time, there is no *monopoly*, that is, there is no vendor that appears in all feasible sets. In auctions for other types of set systems, the requirement that there exist a feasible solution disjoint from the selected one is even more severe: for example, for vertex-cover auctions (where vendors correspond to the vertices of some underlying graph, and the feasible sets are vertex covers) the requirement means that the graph must be bipartite. To deal with this problem, Karlin et al. [15] suggest a better benchmark, which is defined for any monopoly-free set system. This quantity, which they denote by ν, intuitively corresponds to the value of a cheapest Nash equilibrium. Based on this new definition, the authors construct new mechanisms for the shortest path problem and show that the overpayment of these mechanisms is within a constant factor of optimal.

1.1   Our results

Vertex cover auctions We propose a truthful polynomial-time auction for vertex cover that outputs a solution whose cost is within a factor of 2 of optimal, and whose frugality ratio is at most 2Δ, where Δ is the maximum degree of the graph (Theorem 4). We complement this result by proving (Theorem 5) that for any Δ and n, there are graphs of maximum degree Δ and size $\Theta(n)$ for which *any* truthful mechanism has frugality ratio at least $\Delta/2$. This means that both the solution quality and the frugality ratio of our auction are within a constant factor of optimal. In particular, the frugality ratio is within a factor of 4 of optimal. To the best of our knowledge, this is the first auction for this problem that enjoys these properties. Moreover, we show how to transform any truthful mechanism for the vertex-cover problem into a frugal one while preserving the approximation ratio.

Frugality ratios Our vertex cover results naturally suggest two modifications of the definition of ν in [15]. These modifications can be made independently of each other, resulting in four different payment bounds TUmax, TUmin, NTUmax, and NTUmin, where NTUmin is equal to the original payment bound ν of in [15]. All four payment

bounds arise as Nash equilibria of certain games (see Appendix); the differences between them can be seen as "the price of initiative" and "the price of co-operation" (see Section 3). While our main result about vertex cover auctions (Theorem 4) is with respect to $\text{NTUmin} = \nu$, we make use of the new definitions by first comparing the payment of our mechanism to a weaker bound NTUmax, and then bootstrapping from this result to obtain the desired bound.

Inspired by this application, we embark on a further study of these payment bounds. Our results here are as follows:

1. We observe (Proposition 1) that the payment bounds we consider always obey a particular order that is independent of the choice of the set system and the cost vector, namely, $\text{TUmin} \leq \text{NTUmin} \leq \text{NTUmax} \leq \text{TUmax}$. We provide examples (Proposition 5 and Corollaries 1 and 2) showing that for the vertex cover problem any two consecutive bounds can differ by a factor of $n - 2$, where n is the number of agents. We then show (Theorem 2) that this separation is almost optimal for general set systems by proving that for any set system $\text{TUmax}/\text{TUmin} \leq n$. In contrast, we demonstrate (Theorem 3) that for path auctions $\text{TUmax}/\text{TUmin} \leq 2$. We provide examples (Propositions 2, 3 and 4) showing that this bound is tight. We see this as an argument for the study of vertex-cover auctions, as they appear to be more representative of the general team -selection problem than the widely studied path auctions.

2. We show (Theorem 1) that for any set system, if there is a cost vector for which TUmin and NTUmin differ by a factor of α, there is another cost vector that separates NTUmin and NTUmax by the same factor and vice versa; the same is true for the pairs $(\text{NTUmin}, \text{NTUmax})$ and $(\text{NTUmax}, \text{TUmax})$. This symmetry is quite surprising, since, e.g., TUmin and NTUmax are obtained from NTUmin by two very different transformations. This observation suggests that the four payment bounds should be studied in a unified framework; moreover, it leads us to believe that the bootstrapping technique of Theorem 4 may have other applications.

3. We evaluate the payment bounds introduced here with respect to a checklist of desirable features. In particular, we note that the payment bound $\nu = \text{NTUmin}$ of [15] exhibits some counterintuitive properties, such as nonmonotonicity with respect to adding a new feasible set (Proposition 7), and is NP-hard to compute (Theorem 6), while some of the other payment bounds do not suffer from these problems. This can be seen as an argument in favour of using weaker but efficiently computable bounds NTUmax and TUmax.

Related work

Vertex-cover auctions have been studied in the past by Talwar [20] and Calinescu [5]. Both of these papers are based on the definition of frugality ratio used in [1]; as mentioned before, this means that their results only apply to bipartite graphs. Talwar [20] shows that the frugality ratio of VCG is at most Δ. However, since finding the cheapest vertex cover is an NP-hard problem, the VCG mechanism is computationally infeasible. The first (and, to the best of our knowledge, only) paper to investigate polynomial-time truthful mechanisms for vertex cover is [5]. This paper studies an auction that is based

on the greedy allocation algorithm, which has an approximation ratio of $\log n$. While the main focus of [5] is the more general set cover problem, the results of [5] imply a frugality ratio of $2\Delta^2$ for vertex cover. Our results improve on those of [20] as our mechanism is polynomial-time computable, as well as on those of [5], as our mechanism has a better approximation ratio, and we prove a stronger bound on the frugality ratio; moreover, this bound also applies to the mechanism of [5].

2 Preliminaries

In most of this paper, we discuss auctions for set systems. A *set system* is a pair $(\mathcal{E}, \mathcal{F})$, where $\mathcal{E}$ is the *ground set*, $|\mathcal{E}| = n$, and $\mathcal{F}$ is a collection of *feasible sets*, which are subsets of $\mathcal{E}$. Two particular types of set systems are of interest to us — *shortest path* systems, in which the ground set consists of all edges of a network, and the feasible sets are paths between two specified vertices s and t, and *vertex cover* systems, in which the elements of the ground set are the vertices of a graph, and the feasible sets are vertex covers of this graph.

In set system auctions, each element e of the ground set is owned by an independent agent and has an associated non-negative cost c_e. The goal of the centre is to select (purchase) a feasible set. Each element e in the selected set incurs a cost of c_e. The elements that are not selected incur no costs.

The auction proceeds as follows: all elements of the ground set make their bids, the centre selects a feasible set based on the bids and makes payments to the agents. Formally, an auction is defined by an *allocation rule* $A : \mathbf{R}^n \mapsto \mathcal{F}$ and a *payment rule* $P : \mathbf{R}^n \mapsto \mathbf{R}^n$. The allocation rule takes as input a vector of bids and decides which of the sets in $\mathcal{F}$ should be selected. The payment rule also takes as input a vector of bids and decides how much to pay to each agent. The standard requirements are *individual rationality*, i.e., the payment to each agent should be at least as high as its incurred cost (0 for agents not in the selected set and c_e for agents in the selected set) and *incentive compatibility*, or *truthfulness*, i.e., each agent's dominant strategy is to bid its true cost.

An allocation rule is *monotone* if an agent cannot increase his chance of getting selected by raising its bid. Formally, for any bid vector $\mathbf{b}$ and any $e \in \mathcal{E}$, if $e \notin A(\mathbf{b})$ then $e \notin A(b_1, \ldots, b'_e, \ldots, b_n)$ for any $b'_e > b_e$. Given a monotone allocation rule A and a bid vector $\mathbf{b}$, the *threshold bid* t_e of an agent $e \in A(\mathbf{b})$ is the highest bid of this agent that still wins the auction, given that the bids of other participants remain the same. Formally, $t_e = \sup\{b'_e \in \mathbf{R} \mid e \in A(b_1, \ldots, b'_e, \ldots, b_n)\}$. It is well known (see, e.g. [18, 12]) that any auction that has a monotone allocation rule and pays each agent its threshold bid is truthful; conversely, any truthful auction has a monotone allocation rule.

The VCG mechanism is a truthful mechanism that maximises the "social welfare" and pays 0 to the losing agents. For set system auctions, this simply means picking a cheapest feasible set, paying each agent in the selected set its threshold bid, and paying 0 to all other agents. Note, however, that the VCG mechanism may be difficult to implement, since finding a cheapest feasible set may be intractable.

If U is a set of agents, $c(U)$ denotes $\sum_{w \in U} c_w$. Similarly, $b(U)$ denotes $\sum_{w \in U} b_w$.

3   Frugality ratios

We start by reproducing the definition of the quantity ν from [15, Definition 4].

Let $(\mathcal{E}, \mathcal{F})$ be a set system and let S be the cheapest feasible set with respect to the true costs c_e where ties are broken lexicographically. Then $\nu(\mathbf{c})$ is the solution to the following optimisation problem.

Minimise $B = \sum_{e \in S} b_e$ subject to

(1) $b_e \geq c_e$ for all $e \in \mathcal{E}$

(2) $\sum_{e \in S \setminus T} b_e \leq \sum_{e \in T \setminus S} c_e$ for all $T \in \mathcal{F}$

(3) for every $e \in S$, there is a $T_e \in \mathcal{F}$ such that $e \notin T_e$ and $\sum_{e' \in S \setminus T_e} b_{e'} = \sum_{e' \in T_e \setminus S} c_{e'}$

The bound $\nu(c)$ can be seen as an outcome of a two-stage process, where first each agent $e \in S$ makes a bid b_e stating how much it wants to be paid, and then the centre decides whether to accept these bids. The behaviour of both parties is affected by the following considerations. From the centre's point of view, the set S must remain the most attractive choice, i.e., it must be among the cheapest feasible sets under the new costs $c_e' = c_e$ for $e \notin S$, $c_e' = b_e$ for $e \in S$ (condition (2)). The reason for that is that if (2) is violated for some set T, the centre would prefer T to S. On the other hand, no agent would agree to a payment that does not cover its costs (condition (1)), and moreover, each agent tries to maximise its profit by bidding as high as possible, i.e., none of the agents can increase its bid without violating condition (2) (condition (3)). The centre wants to minimise the total payout, so $\nu(\mathbf{c})$ corresponds to the best possible outcome from the centre's point of view.

This definition captures many important aspects of our intuition about 'fair' payments. However, it can be modified in two ways, both of which are still quite natural, but result in different payment bounds.

First, we can consider the worst rather than the best possible outcome for the centre. That is, we can consider the maximum total payment that the agents can extract by jointly selecting their bids subject to (1), (2), and (3). Such a bound corresponds to maximising B subject to (1), (2), and (3) rather than minimising it. If it is the agents who make the original bids (rather than the centre), this kind of bidding behaviour is plausible. On the other hand, in a game in which the centre proposes payments to the agents in S and the agents accept them as long as (1), (2) and (3) are satisfied, we would be likely to observe a total payment of $\nu(c)$. Hence, the difference between these two definitions can be seen as "the price of initiative".

Second, the agents may be able to make payments to each other. In this case, if they can extract more money from the centre by agreeing on a vector of bids that violates individual rationality (i.e., condition (1)) for some bidders, they might be willing to do so, as the agents who are paid below their costs will be compensated by other members of the group. The bids must still be realistic, i.e., they have to satisfy $b_e \geq 0$. The resulting change in payments can be seen as "the price of co-operation" and corresponds to replacing condition (1) with the following weaker condition (1^*):

$$b_e \geq 0 \text{ for all } e \in \mathcal{E}. \tag{1^*}$$

By considering all possible combinations of these modifications, we obtain four different payment bounds, namely

- $\mathrm{TUmin}(\mathbf{c})$, which is the solution to the optimisation problem "Minimise B" subject to (1^*), (2), and (3).
- $\mathrm{TUmax}(\mathbf{c})$, which is the solution to the optimisation problem "Maximise B" subject to (1^*), (2), and (3).
- $\mathrm{NTUmin}(\mathbf{c})$, which is the solution to the optimisation problem "Minimise B" subject to (1), (2), and (3).
- $\mathrm{NTUmax}(\mathbf{c})$, which is the solution to the optimisation problem "Maximise B" subject to (1), (2), (3).

The abbreviations TU and NTU correspond, respectively, to transferable utility and non-transferable utility, i.e., the agents' ability/inability to make payments to each other. Note that the quantity $\nu(\mathbf{c})$ from [15] is $\mathrm{NTUmin}(\mathbf{c})$.

The second modification (transferable utility) is more intuitively appealing in the context of the maximisation problem, as both assume some degree of co-operation between the agents. While the second modification can be made without the first, the resulting payment bound $\mathrm{TUmin}(\mathbf{c})$ is too strong to be a realistic benchmark, at least for general set systems. In particular, it can be smaller than the total cost of the cheapest solution (see Section 6). However, we provide the definition and some results about $\mathrm{TUmin}(\mathbf{c})$ in the paper, both for completeness and because we believe that it may help to understand which properties of the payment bounds are important for our proofs. Another possibility would be to introduce an additional constraint $\sum_{e \in S} b_e \geq \sum_{e \in S} c_e$ in the definition of $\mathrm{TUmin}(\mathbf{c})$ (note that this condition holds automatically for $\mathrm{TUmax}(\mathbf{c})$, as $\mathrm{TUmax}(\mathbf{c}) \geq \mathrm{NTUmax}(\mathbf{c})$); however, such a definition would have no direct economic interpretation, and some of our results (in particular, the ones in Section 4) would no longer be true.

Remark 1. For the payment bounds that are derived from maximisation problems, (i.e., $\mathrm{TUmax}(\mathbf{c})$ and $\mathrm{NTUmax}(\mathbf{c})$), constraints of type (3) are redundant and can be dropped. Hence, $\mathrm{TUmax}(\mathbf{c})$ and $\mathrm{NTUmax}(\mathbf{c})$ are solutions to linear programs, and therefore can be computed in polynomial time as long as we have a separation oracle for constraints in (2). In contrast, $\mathrm{NTUmin}(\mathbf{c})$ can be NP-hard to compute even if the size of $\mathcal{F}$ is polynomial (see Section 6).

The first and third inequalities in the following observation follow from the fact that condition (1^*) is strictly weaker than condition (1).

Proposition 1. $\mathrm{TUmin}(\mathbf{c}) \leq \mathrm{NTUmin}(\mathbf{c}) \leq \mathrm{NTUmax}(\mathbf{c}) \leq \mathrm{TUmax}(\mathbf{c})$.

Let $\mathcal{M}$ be a truthful mechanism for $(\mathcal{E}, \mathcal{F})$. Let $p_{\mathcal{M}}(\mathbf{c})$ denote the total payments of $\mathcal{M}$ when the actual costs are $\mathbf{c}$. A *frugality ratio* of $\mathcal{M}$ with respect to a payment bound

is the ratio between the payment of $\mathcal{M}$ and this payment bound. In particular,

$$\phi_{\mathrm{TUmin}}(\mathcal{M}) = \sup_{\mathbf{c}} p_{\mathcal{M}}(\mathbf{c})/\mathrm{TUmin}(\mathbf{c}),$$

$$\phi_{\mathrm{TUmax}}(\mathcal{M}) = \sup_{\mathbf{c}} p_{\mathcal{M}}(\mathbf{c})/\mathrm{TUmax}(\mathbf{c}),$$

$$\phi_{\mathrm{NTUmin}}(\mathcal{M}) = \sup_{\mathbf{c}} p_{\mathcal{M}}(\mathbf{c})/\mathrm{NTUmin}(\mathbf{c}),$$

$$\phi_{\mathrm{NTUmax}}(\mathcal{M}) = \sup_{\mathbf{c}} p_{\mathcal{M}}(\mathbf{c})/\mathrm{NTUmax}(\mathbf{c}).$$

We conclude this section by showing that there exist set systems and respective cost vectors for which all four payment bounds are different. In the next section, we quantify this difference, both for general set systems, and for specific types of set systems, such as path auctions or vertex cover auctions.

Example 1. Consider the shortest-path auction on the graph of Figure 1. The feasible sets are all paths from A to D. It can be verified, using the reasoning of Propositions 2 and 3 below, that for the cost vector $c_{AB} = c_{CD} = 2$, $c_{BC} = 1$, $c_{AC} = c_{BD} = 5$, we have

- TUmax$(\mathbf{c}) = 10$ (with the bid vector $b_{AB} = b_{CD} = 5$, $b_{BC} = 0$),
- NTUmax$(\mathbf{c}) = 9$ (with the bid vector $b_{AB} = b_{CD} = 4$, $b_{BC} = 1$),
- NTUmin$(\mathbf{c}) = 7$ (with the bid vector $b_{AB} = b_{CD} = 2$, $b_{BC} = 3$),
- TUmin$(\mathbf{c}) = 5$ (with the bid vector $b_{AB} = b_{CD} = 0$, $b_{BC} = 5$).

4   Comparing payment bounds

4.1   Path auctions

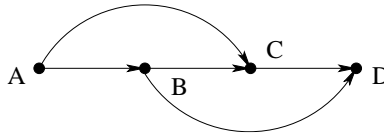We start by showing that for path auctions any two consecutive payment bounds can differ by at least a factor of 2.

Fig. 1. The diamond graph

Proposition 2. *There is an instance of the shortest-path problem for which we have* $\mathrm{NTUmax}(\mathbf{c})/\mathrm{NTUmin}(\mathbf{c}) \geq 2$.

Proof. This construction is due to David Kempe [16]. Consider the graph of Figure 1 with the edge costs $c_{AB} = c_{BC} = c_{CD} = 0$, $c_{AC} = c_{BD} = 1$. Under these costs, ABCD is the cheapest path. The inequalities in (2) are $b_{AB} + b_{BC} \leq c_{AC} = 1$, $b_{BC} + b_{CD} \leq c_{BD} = 1$. By condition (3), both of these inequalities must be tight (the former one is the only inequality involving b_{AB}, and the latter one is the only inequality involving b_{CD}). The inequalities in (1) are $b_{AB} \geq 0$, $b_{BC} \geq 0$, $b_{CD} \geq 0$. Now, if the goal is to maximise $b_{AB} + b_{BC} + b_{CD}$, the best choice is $b_{AB} = b_{CD} = 1$, $b_{BC} = 0$, so NTUmax($\mathbf{c}$) = 2. On the other hand, if the goal is to minimise $b_{AB} + b_{BC} + b_{CD}$, one should set $b_{AB} = b_{CD} = 0$, $b_{BC} = 1$, so NTUmin($\mathbf{c}$) = 1.

Proposition 3. *There is an instance of the shortest-path problem for which we have* TUmax($\mathbf{c}$)/NTUmax($\mathbf{c}$) ≥ 2.

Proof. Again, consider the graph of Figure 1. Let the edge costs be $c_{AB} = c_{CD} = 0$, $c_{BC} = 1$, $c_{AC} = c_{BD} = 1$. ABCD is the lexicographically-least cheapest path, so we can assume that $S = \{AB, BC, CD\}$. The inequalities in (2) are the same as in the previous example, and by the same argument both of them are, in fact, equalities. The inequalities in (1) are $b_{AB} \geq 0$, $b_{BC} \geq 1$, $b_{CD} \geq 0$. Our goal is to maximise $b_{AB} + b_{BC} + b_{CD}$. If we have to respect the inequalities in (1), we have to set $b_{AB} = b_{CD} = 0$, $b_{BC} = 1$, so NTUmax($\mathbf{c}$) = 1. Otherwise, we can set $b_{AB} = b_{CD} = 1$, $b_{BC} = 0$, so TUmax($\mathbf{c}$) ≥ 2.

Proposition 4. *There is an instance of the shortest-path problem for which we have* NTUmin($\mathbf{c}$)/TUmin($\mathbf{c}$) ≥ 2.

Proof. This construction is also based on the graph of Figure 1. The edge costs are $c_{AB} = c_{CD} = 1$, $c_{BC} = 0$, $c_{AC} = c_{BD} = 1$. ABCD is the lexicographically least cheapest path, so we can assume that $S = \{AB, BC, CD\}$. Again, the inequalities in (2) are the same, and both are, in fact, equalities. The inequalities in (1) are $b_{AB} \geq 1$, $b_{BC} \geq 0$, $b_{CD} \geq 1$. Our goal is to minimise $b_{AB} + b_{BC} + b_{CD}$. If we have to respect the inequalities in (1), we have to set $b_{AB} = b_{CD} = 1$, $b_{BC} = 0$, so NTUmin($\mathbf{c}$) = 2. Otherwise, we can set $b_{AB} = b_{CD} = 0$, $b_{BC} = 1$, so TUmin($\mathbf{c}$) ≤ 1.

In Section 4.4 (Theorem 3), we show that the separation results in Propositions 2, 3, and 4 are optimal.

4.2   Connections between separation results

The separation results for path auctions are obtained on the same graph using very similar cost vectors. It turns out that this is not coincidental. Namely, we can prove the following theorem.

Theorem 1. *For any set system* $(\mathcal{E}, \mathcal{F})$,

$$\max_{\mathbf{c}} \frac{\text{TUmax}(\mathbf{c})}{\text{NTUmax}(\mathbf{c})} = \max_{\mathbf{c}} \frac{\text{NTUmax}(\mathbf{c})}{\text{NTUmin}(\mathbf{c})} = \max_{\mathbf{c}} \frac{\text{NTUmin}(\mathbf{c})}{\text{TUmin}(\mathbf{c})}.$$

The proof of the theorem follows directly from the four lemmas proved below; more precisely, the first equality in Theorem 1 is obtained by combining Lemmas 1 and 2, and the second equality is obtained by combining Lemmas 3 and 4.

Lemma 1. *For any cost vector* $\mathbf{c}$ *for* $(\mathcal{E}, \mathcal{F})$ *such that* $\mathrm{TUmax}(\mathbf{c})/\mathrm{NTUmax}(\mathbf{c}) = \alpha$, *there is a cost vector* $\mathbf{c}'$ *such that* $\mathrm{NTUmax}(\mathbf{c}')/\mathrm{NTUmin}(\mathbf{c}') \geq \alpha$.

Proof. Suppose that there is a cost vector $\mathbf{c}$ such that $\mathrm{TUmax}(\mathbf{c}) = X$, $\mathrm{NTUmax}(\mathbf{c}) = Y$, $X/Y = \alpha$. Assume without loss of generality that the winning set S consists of elements $1, \ldots, k$, and let $\mathbf{b}^1 = (b_1^1, \ldots, b_k^1)$ and $\mathbf{b}^2 = (b_1^2, \ldots, b_k^2)$ be the bid vectors that correspond to $\mathrm{TUmax}(\mathbf{c})$ and $\mathrm{NTUmax}(\mathbf{c})$, respectively.

Construct the cost vector $\mathbf{c}'$ by setting $c_i' = c_i$ for $i \notin S$, $c_i' = \min\{c_i, b_i^1\}$ for $i \in S$. Clearly, S is a cheapest set under $\mathbf{c}'$. Moreover, as the costs of elements outside of S remained the same, the right-hand sides of all constraints in (2) did not change, so any bid vector that satisfies (2) and (3) with respect to $\mathbf{c}$, also satisfies them with respect to $\mathbf{c}'$. We will construct two bid vectors $\mathbf{b}^3$ and $\mathbf{b}^4$ that satisfy conditions (1), (2), and (3) for the cost vector $\mathbf{c}'$, and have $\sum_{i \in S} b_i^3 = X$, $\sum_{i \in S} b_i^4 = Y$. As $\mathrm{NTUmax}(\mathbf{c}') \geq X$ and $\mathrm{NTUmin}(\mathbf{c}') \leq Y$, this implies the lemma.

We can set $b_i^3 = b_i^1$: this bid vector satisfies conditions (2) and (3) since $\mathbf{b}^1$ does, and we have $b_i^1 \geq \min\{c_i, b_i^1\} = c_i'$, which means that $\mathbf{b}^3$ satisfies condition (1). Furthermore, we can set $b_i^4 = b_i^2$. Again, $\mathbf{b}^4$ satisfies conditions (2) and (3) since $\mathbf{b}^2$ does, and since $\mathbf{b}^2$ satisfies condition (1), we have $b_i^2 \geq c_i \geq c_i'$, which means that $\mathbf{b}^4$ satisfies condition (1). $\square$

Lemma 2. *For any cost vector* $\mathbf{c}$ *for* $(\mathcal{E}, \mathcal{F})$ *such that* $\mathrm{NTUmax}(\mathbf{c})/\mathrm{NTUmin}(\mathbf{c}) = \alpha$, *there is a cost vector* $\mathbf{c}'$ *such that* $\mathrm{TUmax}(\mathbf{c}')/\mathrm{NTUmax}(\mathbf{c}') \geq \alpha$.

Proof. Suppose that there is a cost vector $\mathbf{c}$ such that $\mathrm{NTUmax}(\mathbf{c}) = X$, $\mathrm{NTUmin}(\mathbf{c}) = Y$, $X/Y = \alpha$. Again, assume that the winning set S consists of elements $1, \ldots, k$, and let $\mathbf{b}^1 = (b_1^1, \ldots, b_k^1)$ and $\mathbf{b}^2 = (b_1^2, \ldots, b_k^2)$ be the bid vectors that correspond to $\mathrm{NTUmax}(\mathbf{c})$ and $\mathrm{NTUmin}(\mathbf{c})$, respectively.

Construct the cost vector $\mathbf{c}'$ by setting $c_i' = c_i$ for $i \notin S$, $c_i' = b_i^2$ for $i \in S$. As $\mathbf{b}^2$ satisfies condition (2), S is a cheapest set under $\mathbf{c}'$. As in the previous construction, the right-hand sides of all constraints in (2) did not change. Let $\mathbf{b}^3$ be a bid vector that corresponds to $\mathrm{NTUmax}(\mathbf{c}')$. Let us prove that $\mathrm{NTUmax}(\mathbf{c}') = \sum_{i \in S} b_i^3 = Y$. Indeed, the bid vector $\mathbf{b}^3$ must satisfy $b_i^3 \geq c_i' = b_i^2$ for $i = 1, \ldots, k$ (condition (1)). Suppose that $b_i^3 > c_i'$ for some $i = 1, \ldots, k$, and consider the constraint in (2) that is tight for b_i^2. There is such a constraint, as $\mathbf{b}^2$ satisfies condition (3). Namely, for some T not containing i,

$$\sum_{j \in S \setminus T} b_j^2 = \sum_{j \in T \setminus S} c_j.$$

For every j appearing in the left-side of this constraint, we have $b_j^3 \geq b_j^2$ but $b_i^3 > b_i^2$, so the bid vector $\mathbf{b}^3$ violates this constraint. Hence, $b_i^3 = c_i' = b_i^2$ for all i and therefore $\mathrm{NTUmax}(\mathbf{c}') = \sum_{i \in S} b_i^3 = Y$.

On the other hand, we can construct a bid vector $\mathbf{b}^4$ that satisfies conditions (2) and (3) with respect to $\mathbf{c}'$ and has $\sum_{i \in S} b_i^4 = X$. Namely, we can set $b_i^4 = b_i^1$: as $\mathbf{b}^1$

satisfies conditions (2) and (3), so does $\mathbf{b}^4$. As $\mathrm{TUmax}(\mathbf{c}') \geq \sum_{i \in S} b_i^4$, this proves the lemma.

Lemma 3. *For any cost vector $\mathbf{c}$ for $(\mathcal{E}, \mathcal{F})$ such that $\mathrm{NTUmax}(\mathbf{c})/\mathrm{NTUmin}(\mathbf{c}) = \alpha$, there is a cost vector $\mathbf{c}'$ such that $\mathrm{NTUmin}(\mathbf{c}')/\mathrm{TUmin}(\mathbf{c}') \geq \alpha$.*

Proof. Suppose that there is a cost vector $\mathbf{c}$ such that $\mathrm{NTUmax}(\mathbf{c}) = X$, $\mathrm{NTUmin}(\mathbf{c}) = Y$, $X/Y = \alpha$. Again, assume that the winning set S consists of elements $1, \ldots, k$, and let $\mathbf{b}^1 = (b_1^1, \ldots, b_k^1)$ and $\mathbf{b}^2 = (b_1^2, \ldots, b_k^2)$ be the bid vectors that correspond to $\mathrm{NTUmax}(\mathbf{c})$ and $\mathrm{NTUmin}(\mathbf{c})$, respectively.

The cost vector $\mathbf{c}'$ is obtained by setting $c_i' = c_i$ for $i \notin S$, $c_i' = b_i^1$ for $i \in S$. Since $\mathbf{b}^1$ satisfies condition (2), S is a cheapest set under $\mathbf{c}'$, and the right-hand sides of all constraints in (2) did not change.

Let $\mathbf{b}^3$ be a bid vector that corresponds to $\mathrm{NTUmin}(\mathbf{c}')$. It is easy to see that $\mathrm{NTUmin}(\mathbf{c}') = \sum_{i \in S} b_i^3 = X$, since the bid vector $\mathbf{b}^3$ must satisfy $b_i^3 \geq c_i' = b_i^1$ for $i = 1, \ldots, k$ (condition (1)), and $\sum_{i \in S} b_i^1 = \mathrm{NTUmax}(\mathbf{c}) = X$. On the other hand, we can construct a bid vector $\mathbf{b}^4$ that satisfies conditions (2) and (3) with respect to $\mathbf{c}'$ and has $\sum_{i \in S} b_i^4 = Y$. Namely, we can set $b_i^4 = b_i^2$: as $\mathbf{b}^2$ satisfies conditions (2) and (3), so does $\mathbf{b}^4$. As $\mathrm{TUmin}(\mathbf{c}') \leq \sum_{i \in S} b_i^4$, this proves the lemma.

Lemma 4. *For any cost vector $\mathbf{c}$ for $(\mathcal{E}, \mathcal{F})$ such that $\mathrm{NTUmin}(\mathbf{c})/\mathrm{TUmin}(\mathbf{c}) = \alpha$, there is a cost vector $\mathbf{c}'$ such that $\mathrm{NTUmax}(\mathbf{c}')/\mathrm{NTUmin}(\mathbf{c}') \geq \alpha$.*

Proof. Suppose that there is a cost vector $\mathbf{c}$ such that $\mathrm{NTUmin}(\mathbf{c}) = X$, $\mathrm{TUmin}(\mathbf{c}) = Y$, $X/Y = \alpha$. Again, assume that the winning set S consists of elements $1, \ldots, k$, and let $\mathbf{b}^1 = (b_1^1, \ldots, b_k^1)$ and $\mathbf{b}^2 = (b_1^2, \ldots, b_k^2)$ be the bid vectors that correspond to $\mathrm{NTUmin}(\mathbf{c})$ and $\mathrm{TUmin}(\mathbf{c})$, respectively.

Construct the cost vector $\mathbf{c}'$ by setting $c_i' = c_i$ for $i \notin S$, $c_i' = \min\{c_i, b_i^2\}$ for $i \in S$. Clearly, S is a cheapest set under $\mathbf{c}'$. Moreover, as the costs of elements outside of S remained the same, the right-hand sides of all constraints in (2) did not change.

We will construct two bid vectors $\mathbf{b}^3$ and $\mathbf{b}^4$ that satisfy conditions (1), (2), and (3) for the cost vector $\mathbf{c}'$, and have $\sum_{i \in S} b_i^3 = X$, $\sum_{i \in S} b_i^4 = Y$. As $\mathrm{NTUmax}(\mathbf{c}') \geq X$ and $\mathrm{NTUmin}(\mathbf{c}') \leq Y$, this implies the lemma.

We can set $b_i^3 = b_i^1$. Indeed, the vector $\mathbf{b}^3$ satisfies conditions (2) and (3) since $\mathbf{b}^1$ does. Also, since $\mathbf{b}^1$ satisfies condition (1), we have $b_i^1 \geq c_i \geq c_i'$, i.e., $\mathbf{b}^3$ satisfies condition (1) with respect to $\mathbf{c}'$. On the other hand, we can set $b_i^4 = b_i^2$: the vector $\mathbf{b}^4$ satisfies conditions (2) and (3) since $\mathbf{b}^2$ does, and it satisfies condition (1), since $b_i^2 \geq c_i'$.

4.3 Vertex-cover auctions

In contrast to the case of path auctions, for vertex-cover auctions the gap between $\mathrm{NTUmin}(\mathbf{c})$ and $\mathrm{NTUmax}(\mathbf{c})$ (and hence between $\mathrm{NTUmax}(\mathbf{c})$ and $\mathrm{TUmax}(\mathbf{c})$, and between $\mathrm{TUmin}(\mathbf{c})$ and $\mathrm{NTUmin}(\mathbf{c})$) can be proportional to the size of the graph.

Proposition 5. *For any $n \geq 3$, there is a an n-vertex graph and a cost vector $\mathbf{c}$ for which $\mathrm{TUmax}(\mathbf{c})/\mathrm{NTUmax}(\mathbf{c}) \geq n - 2$.*
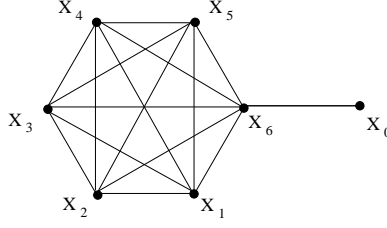
Fig. 2. Graph that separates payment bounds for vertex cover, $n = 7$

Proof. The underlying graph consists of an $(n-1)$-clique on the vertices $X_1, \ldots, X_{n-1}$, and an extra vertex X_0 adjacent to X_{n-1}. The costs are $c_{X_1} = c_{X_2} = \cdots = c_{X_{n-2}} = 0$, $c_{X_0} = c_{X_{n-1}} = 1$. We can assume that $S = \{X_0, X_1, \ldots, X_{n-2}\}$ (this is the lexicographically first vertex cover of cost 1). For this set system, the constraints in (2) are $b_{X_i} + b_{X_0} \leq c_{X_{n-1}} = 1$ for $i = 1, \ldots, n-2$. Clearly, we can satisfy conditions (2) and (3) by setting $b_{X_i} = 1$ for $i = 1, \ldots, n-2$, $b_{X_0} = 0$. Hence, $\mathrm{TUmax}(\mathbf{c}) \geq n - 2$. For $\mathrm{NTUmax}(\mathbf{c})$, there is an additional constraint $b_{X_0} \geq 1$, so the best we can do is to set $b_{X_i} = 0$ for $i = 1, \ldots, n-2$, $b_{X_0} = 1$, which implies $\mathrm{NTUmax}(\mathbf{c}) = 1$.

Combining Proposition 5 with Lemmas 1 and 3, we derive the following corollaries.

Corollary 1. *For any $n \geq 3$, there is an instance of the vertex cover problem on an n-vertex graph for which for which* $\mathrm{NTUmax}(\mathbf{c})/\mathrm{NTUmin}(\mathbf{c}) \geq n - 2$.

Corollary 2. *For any $n \geq 3$, there is an instance of the vertex cover problem on an n-vertex graph for which* $\mathrm{NTUmin}(\mathbf{c})/\mathrm{TUmin}(\mathbf{c}) \geq n - 2$.

4.4   Upper bounds

It turns out that the lower bound proved in the previous subsection is almost tight. More precisely, the following theorem shows that no two payment bounds can differ by more than a factor of n; moreover, this is the case not just for the vertex cover problem, but for general set systems. We bound the gap between $\mathrm{TUmax}(\mathbf{c})$ and $\mathrm{TUmin}(\mathbf{c})$. Since $\mathrm{TUmin}(\mathbf{c}) \leq \mathrm{NTUmin}(\mathbf{c}) \leq \mathrm{NTUmax}(\mathbf{c}) \leq \mathrm{TUmax}(\mathbf{c})$, this bound applies to any pair of payment bounds.

Theorem 2. *For any set system $(\mathcal{E}, \mathcal{F})$ and any cost vector $\mathbf{c}$, we have*

$$\mathrm{TUmax}(\mathbf{c})/\mathrm{TUmin}(\mathbf{c}) \leq n$$

Proof. Assume without loss of generality that the winning set S consists of elements $1, \ldots, k$. Let $c_1, \ldots, c_k$ be the true costs of elements in S, let $b'_1, \ldots, b'_k$ be their bids that correspond to $\mathrm{TUmin}(\mathbf{c})$, and let $b''_1, \ldots, b''_k$ be their bids that correspond to $\mathrm{TUmax}(\mathbf{c})$.

Consider the conditions (2) and (3) for S. One can pick a subset $\mathcal{L}$ of at most k inequalities in (2) so that for each $i = 1, \ldots, k$ there is at least one inequality in $\mathcal{L}$ that is tight for b'_i. Suppose that the jth inequality in $\mathcal{L}$ is of the form

$$b_{i_1} + \cdots + b_{i_t} \leq c(T_j \setminus S).$$

For b_i', all inequalities in $\mathcal{L}$ are, in fact equalities. Hence, by adding up all of them we obtain

$$k \sum_{i=1,\ldots,k} b_i' \geq \sum_{j=1,\ldots,k} c(T_j \setminus S).$$

On the other hand, all these inequalities appear in condition (2), so they must hold for b_i'', i.e.,

$$\sum_{i=1,\ldots,k} b_i'' \leq \sum_{j=1,\ldots,k} c(T_j \setminus S).$$

Combining these two inequalities, we obtain

$$n\mathrm{TUmin}(\mathbf{c}) \geq k\mathrm{TUmin}(\mathbf{c}) \geq \mathrm{TUmax}(\mathbf{c}).$$

Remark 2. The final line of the proof of Theorem 2 shows that, in fact, the upper bound on $\mathrm{TUmax}(\mathbf{c})/\mathrm{TUmin}(\mathbf{c})$ can be strengthened to the size of the winning set, k. Note that in Proposition 5, as well as in Corollaries 1 and 2, $k = n - 1$, so these results do not contradict each other.

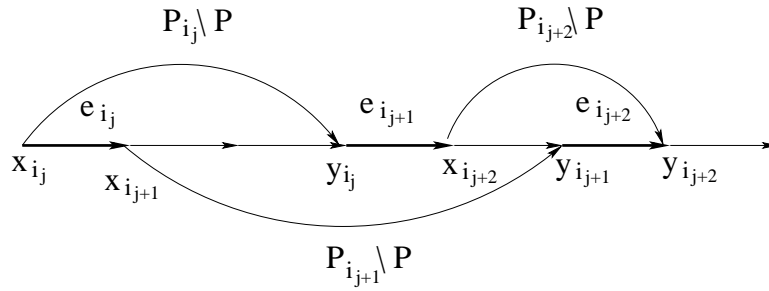For path auctions, this upper bound can be improved to 2, matching the lower bounds of Section 4.1.

Fig. 3. Proof of Theorem 3: constraints for $\hat{P}_{i_j}$ and $\hat{P}_{i_{j+2}}$ do not overlap

Theorem 3. *For any instance of the shortest path problem,* $\mathrm{TUmax}(\mathbf{c}) \leq 2\,\mathrm{TUmin}(\mathbf{c})$.

Proof. Given a network (G, s, t), assume without loss of generality that the lexicographically-least cheapest s–t path, P, in G is $\{e_1, \ldots, e_k\}$, where $e_1 = (s, v_1), e_2 = (v_1, v_2), \ldots, e_k = (v_{k-1}, t)$. Let $c_1, \ldots, c_k$ be the true costs of $e_1, \ldots, e_k$, and let $\mathbf{b}' = (b_1', \ldots, b_k')$ and $\mathbf{b}'' = (b_1'', \ldots, b_k'')$ be bid vectors that correspond to $\mathrm{TUmin}(\mathbf{c})$ and $\mathrm{TUmax}(\mathbf{c})$, respectively.

For any $i = 1, \ldots, k$, there is a constraint in (2) that is tight for b_i' with respect to the bid vector $\mathbf{b}'$, i.e., an s–t path P_i that avoids e_i and satisfies $b(P \setminus P_i) = c(P_i \setminus P)$. We can assume without loss of generality that P_i coincides with P up to some vertex x_i, then deviates from P to avoid e_i, and finally returns to P at a vertex y_i and coincides

with P from then on (clearly, it might happen that $s = x_i$ or $t = y_i$). Indeed, if P_i deviates from P more than once, one of these deviations is not necessary to avoid e_i and can be replaced with the respective segment of P without increasing the cost of P_i. Among all paths of this form, let $\hat{P}_i$ be the one with the largest value of y_i, i.e., the "rightmost" one. This path corresponds to an inequality I_i of the form $b'_{x_{i+1}} + \cdots + b'_{y_i} \leq c(\hat{P}_i \setminus P)$.

As in the proof of Theorem 2, we construct a set of tight constraints $\mathcal{L}$ such that every variable b'_i appears in at least one of these constraints; however, now we have to be more careful about the choice of constraints in $\mathcal{L}$. We construct $\mathcal{L}$ inductively as follows. Start by setting $\mathcal{L} = \{I_1\}$. At the jth step, suppose that all variables up to (but not including) b'_{i_j} appear in at least one inequality in $\mathcal{L}$. Add I_{i_j} to $\mathcal{L}$.

Note that for any j we have $y_{i_{j+1}} > y_{i_j}$. This is because the inequalities added to $\mathcal{L}$ during steps 1—j did not cover $b'_{i_{j+1}}$. See Figure 3. Since $y_{i_{j+2}} > y_{i_{j+1}}$, we must also have $x_{i_{j+2}} > y_{i_j}$: otherwise, $\hat{P}_{i_{j+1}}$ would not be the "rightmost" constraint for $b'_{i_{j+1}}$. Therefore, the variables in $I_{i_{j+2}}$ and I_{i_j}, do not overlap, and hence no b'_i can appear in more than two inequalities in $\mathcal{L}$.

Now we follow the argument of the proof of Theorem 2 to finish. By adding up all of the (tight) inequalities in $\mathcal{L}$ for b'_i we obtain $2\sum_{i=1,\ldots,k} b'_i \geq \sum_{j=1,\ldots,k} c(\hat{P}_j \setminus P)$. On the other hand, all these inequalities appear in condition (2), so they must hold for b''_i, i.e., $\sum_{i=1,\ldots,k} b''_i \leq \sum_{j=1,\ldots,k} c(\hat{P}_j \setminus P)$, so $\mathrm{TUmax}(\mathbf{c}) \leq 2\mathrm{TUmin}(\mathbf{c})$.

5 Truthful mechanisms for vertex cover

Recall that for a vertex-cover auction on a graph $G = (V, E)$, an *allocation rule* is an algorithm that takes as input a bid b_v for each vertex and returns a vertex cover $\hat{S}$ of G. As explained in Section 2, we can combine a monotone allocation rule with threshold payments to obtain a truthful auction.

Two natural examples of monotone allocation rules are A_{opt}, i.e., the algorithm that finds an optimal vertex cover, and the greedy algorithm A_{GR}. However, A_{opt} cannot be guaranteed to run in polynomial time unless $P = NP$ and A_{GR} has approximation ratio of $\log n$.

Another approximation algorithm for vertex cover, which has approximation ratio 2, is the *local ratio* algorithm A_{LR} [2, 3]. This algorithm considers the edges of G one by one. Given an edge $e = (u, v)$, it computes $\epsilon = \min\{b_u, b_v\}$ and sets $b_u = b_u - \epsilon$, $b_v = b_v - \epsilon$. After all edges have been processed, A_{LR} returns the set of vertices $\{v \mid b_v = 0\}$. It is not hard to check that if the order in which the edges are considered is independent of the bids, then this algorithm is monotone as well. Hence, we can use it to construct a truthful auction that is guaranteed to select a vertex cover whose cost is within a factor of 2 from the optimal.

However, while the quality of the solution produced by A_{LR} is much better than that of A_{GR}, we still need to show that its total payment is not too high. In the next subsection, we bound the frugality ratio of A_{LR} (and, more generally, all algorithms that satisfy the condition of *local optimality*, defined later) by 2Δ, where Δ is the maximum degree of G. We then prove a matching lower bound showing that for some graphs the frugality ratio of any truthful auction is at least $(\Delta - 1)/4$.

5.1   Upper bound

We say that an allocation rule is *locally optimal* if whenever $b_v > \sum_{w \sim v} b_w$, the vertex v is not chosen.

Claim. The algorithms A_{opt}, A_{GR}, and A_{LR} are locally optimal.

Theorem 4. *Any vertex cover auction $\mathcal{M}$ that has a locally optimal and monotone allocation rule and pays each agent its threshold bid has frugality ratio $\phi_{\mathrm{NTUmin}}(\mathcal{M}) \leq 2\Delta$.*

To prove Theorem 4, we first show that the total payment of any locally optimal mechanism does not exceed $\Delta c(V)$. We then demonstrate that $\mathrm{NTUmin}(\mathbf{c}) \geq c(V)/2$. By combining these two results, the theorem follows.

Lemma 5. *Consider a graph $G = (V, E)$ with maximum degree Δ. Let $\mathcal{M}$ be a vertex-cover auction on G that satisfies the conditions of Theorem 4. Then for any cost vector $\mathbf{c}$, the total payment of $\mathcal{M}$ satisfies $p_{\mathcal{M}}(\mathbf{c}) \leq \Delta c(V)$.*

Proof. First note that any such auction is truthful, so we can assume that each agent's bid is equal to its cost. Let $\hat{S}$ be the vertex cover selected by $\mathcal{M}$. Then $p_{\mathcal{M}}(\mathbf{c}) = \sum_{v \in \hat{S}} t_v \leq \sum_{v \in \hat{S}} \sum_{w \sim v} c_w \leq \sum_{w \in V} \Delta c_w = \Delta c(V)$.

We now derive a lower bound on $\mathrm{TUmax}(\mathbf{c})$; while not essential for the proof of Theorem 4, it helps us build the intuition necessary for that proof.

Lemma 6. *For a vertex cover instance $G = (V, E)$ in which S is the lexicographically-least minimum vertex cover, $\mathrm{TUmax}(\mathbf{c}) \geq c(V \setminus S)$*

Proof. For a vertex w with at least one neighbour in S, let $d(w)$ denote the number of neighbours that w has in S. Consider the bid vector $\mathbf{b}$ in which, for each $v \in S$, $b_v = \sum_{w \sim v, w \notin S} c_w / d(w)$. Then $\sum_{v \in S} b_v = \sum_{v \in S} \sum_{w \sim v, w \notin S} c_w / d(w) = \sum_{w \notin S} c_w = c(V \setminus S)$. To finish we want to show that $\mathbf{b}$ is feasible in the sense that it satisfies (2). Consider a vertex cover T, and extend the bid vector $\mathbf{b}$ by assigning $b_v = c_v$ for $v \notin S$. Then

$$b(T) = c(T \setminus S) + b(S \cap T) \geq c(T \setminus S) + \sum_{v \in S \cap T} \sum_{w \in \overline{S} \cap \overline{T}: w \sim v} c_w / d(w),$$

and since all edges between $\overline{S} \cap \overline{T}$ and S go to $S \cap T$, the right-hand-side is equal to

$$c(T \setminus S) + \sum_{w \in \overline{S} \cap \overline{T}} c_w = c(T \setminus S) + c(\overline{S} \cap \overline{T}) = c(V \setminus S) = b(S).$$

Next, we prove a lower bound on $\mathrm{NTUmax}(\mathbf{c})$; we will then use it to obtain a lower bound on $\mathrm{NTUmin}(\mathbf{c})$.

Lemma 7. *For a vertex cover instance $G = (V, E)$ in which S is the lexicographically-least minimum vertex cover, $\mathrm{NTUmax}(\mathbf{c}) \geq c(V \setminus S)$*

Proof. If $c(S) \geq c(V \setminus S)$, by condition (1) we are done. Therefore, for the rest of the proof we assume that $c(S) < c(V \setminus S)$. We show how to construct a bid vector $(b_e)_{e \in S}$ that satisfies conditions (1) and (2) such that $b(S) \geq c(V \setminus S)$; clearly, this implies $\mathrm{NTUmax}(\mathbf{c}) \geq c(V \setminus S)$.

Recall that a network flow problem is described by a directed graph $\Gamma = (V_\Gamma, E_\Gamma)$, a source node $s \in V_\Gamma$, a sink node $t \in V_\Gamma$, and a vector of capacity constraints a_e, $e \in E_\Gamma$. Consider a network (V_Γ, E_Γ) such that $V_\Gamma = V \cup \{s, t\}$, $E_\Gamma = E_1 \cup E_2 \cup E_3$, where $E_1 = \{(s, v) \mid v \in S\}$, $E_2 = \{(v, w) \mid v \in S, w \in V \setminus S, (v, w) \in E\}$, $E_3 = \{(w, t) \mid w \in V \setminus S\}$. Since S is a vertex cover for G, no edge of E can have both of its endpoints in $V \setminus S$, and by construction, E_2 contains no edges with both endpoints in S. Therefore, the graph (V, E_2) is bipartite with parts $(S, V \setminus S)$.

Set the capacity constraints for $e \in E_\Gamma$ as follows: $a_{(s,v)} = c_v$, $a_{(w,t)} = c_w$, $a_{(v,w)} = +\infty$ for all $v \in S$, $w \in V \setminus S$. Recall that a *cut* is a partition of the vertices in V_Γ into two sets C_1 and C_2 so that $s \in C_1$, $t \in C_2$; we denote such cut by $C = (C_1, C_2)$. Abusing notation, we write $e = (u, v) \in C$ if $u \in C_1, v \in C_2$ or $u \in C_2, v \in C_1$, and say that an edge $e = (u, v)$ *crosses* the cut C. The *size* of a cut C is computed as $sz(C) = \sum_{(v,w) \in C} a_{(v,w)}$. We have $sz(s, V \cup \{t\}) = c(S)$, $sz(\{s\} \cup V, t) = c(V \setminus S)$.

Let $C_{\min} = (\{s\} \cup S' \cup W', \{t\} \cup S'' \cup W'')$ be a minimum cut in Γ, where $S', S'' \subseteq S$, $W', W'' \subseteq V \setminus S$. As $sz(C_{\min}) \leq sz(s, V \cup \{t\}) = c(S) < +\infty$, and any edge in E_2 has infinite capacity, no edge $(u, v) \in E_2$ crosses $C_{\min}$.

Consider the network $\Gamma' = (V_{\Gamma'}, E_{\Gamma'})$, where $V_{\Gamma'} = \{s\} \cup S' \cup W' \cup \{t\}$, $E_{\Gamma'} = \{(u, v) \in E_\Gamma \mid u, v \in V_{\Gamma'}\}$. Clearly, $C' = (\{s\} \cup S' \cup W', \{t\})$ is a minimum cut in Γ' (otherwise, there would exist a smaller cut for Γ). As $sz(C') = c(W')$, we have $c(S') \geq c(W')$.

Now, consider the network $\Gamma'' = (V_{\Gamma''}, E_{\Gamma''})$, where $V_{\Gamma''} = \{s\} \cup S'' \cup W'' \cup \{t\}$, $E_{\Gamma'} = \{(u, v) \in E_\Gamma \mid u, v \in V_{\Gamma''}\}$. Similarly, $C'' = (\{s\}, S'' \cup W'' \cup \{t\})$ is a minimum cut in Γ'', $sz(C'') = c(S'')$. As the size of a maximum flow from s to t is equal to that of a minimum cut separating s and t, there exists a flow $\mathcal{F} = (f_e)_{e \in E_{\Gamma''}}$ of size $c(S'')$. This flow has to saturate all edges between s and S'', i.e., $f_{(s,v)} = c_v$ for all $v \in S''$. Now, increase the capacities of all edges between s and S'' to $+\infty$. In the modified network, the size of a minimum cut (and hence a maximum flow) is $c(W'')$, and a maximum flow $\mathcal{F}' = (f'_e)_{e \in E_{\Gamma''}}$ can be constructed by greedily augmenting $\mathcal{F}$.

Set $b_v = c_v$ for all $v \in S'$, $b_v = f'_{(s,v)}$ for all $v \in S''$. As $\mathcal{F}'$ is constructed by augmenting $\mathcal{F}$, we have $b_v \geq c_v$ for all $v \in S$, i.e., condition (1) is satisfied.

Now, let us check that no vertex cover $T \subseteq V$ can violate condition (2). Set $T_1 = T \cap S'$, $T_2 = T \cap S''$, $T_3 = T \cap W'$, $T_4 = T \cap W''$; our goal is to show that $b(S' \setminus T_1) + b(S'' \setminus T_2) \leq c(T_3) + c(T_4)$. Consider all edges $(u, v) \in E$ such that $u \in S' \setminus T_1$. If $(u, v) \in E_2$ then $v \in T_3$ (no edge in E_2 can cross the cut), and if $u, v \in S$ then $v \in T_1 \cup T_2$. Hence, $T_1 \cup T_3 \cup S''$ is a vertex cover for G, and therefore $c(T_1) + c(T_3) + c(S'') \geq c(S) = c(T_1) + c(S' \setminus T_1) + c(S'')$. Consequently, $c(T_3) \geq c(S' \setminus T_1) = b(S' \setminus T_1)$. Now, consider the vertices in $S'' \setminus T_2$. Any edge in E_2 that starts in one of these vertices has to end in T_4 (this edge has to be covered by T, and it cannot go across the cut). Therefore, the total flow out of $S'' \setminus T_2$ is at most the total flow out of T_4, i.e., $b(S'' \setminus T_2) \leq c(T_4)$. Hence, $b(S' \setminus T_1) + b(S'' \setminus T_2) \leq c(T_3) + c(T_4)$.
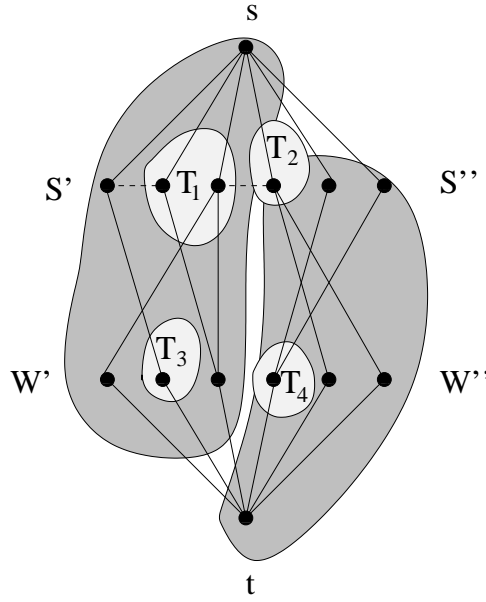
Fig. 4. Proof of Lemma 7. Dashed lines correspond to edges in $E \setminus E_2$

Finally, we derive a lower bound on the payment bound that is of interest to us, namely, $\text{NTUmin}(\mathbf{c})$.

Lemma 8. *For a vertex cover instance $G = (V, E)$ in which S is the lexicographically-least minimum vertex cover,* $\text{NTUmin}(\mathbf{c}) \geq c(V \setminus S)$

Proof. Suppose for contradiction that $\mathbf{c}$ is a cost vector with $\text{NTUmin}(\mathbf{c}) < c(V \setminus S)$. Let $\mathbf{b}$ be the corresponding bid vector and let $\mathbf{c}'$ be a new cost vector with $c'_e = b_e$ for $e \in S$ and $c'_e = c_e$ for $e \notin S$. Condition (2) guarantees that S is an optimal solution to the cost vector $\mathbf{c}'$. Now compute a bid vector $\mathbf{b}'$ corresponding to $\text{NTUmax}(\mathbf{c}')$. We claim that $b'_e = c'_e$ for any $e \in S$. Indeed, suppose that $b'_e > c'_e$ for some $e \in S$ ($b'_e = c'_e$ for $e \notin S$ by construction). As $\mathbf{b}$ satisfies conditions (1)–(3), among the inequalities in (2) there is one that is tight for e and the bid vector $\mathbf{b}$. That is, $b(S \setminus T) = c(T \setminus S)$. By the construction of $\mathbf{c}'$, $c'(S \setminus T) = c'(T \setminus S)$. Now since $b'_f \geq c'_f$ for all $f \in S$, $b'_e > c'_e$ implies $b'(S \setminus T) > c'(S \setminus T) = c'(T \setminus S)$. But this violates (2). So we now know $\mathbf{b}' = \mathbf{c}'$. Hence, we have $\text{NTUmax}(\mathbf{c}') = \sum_{e \in S} b_e = \text{NTUmin}(\mathbf{c}) < c(V \setminus S)$, giving a contradiction to the fact that $\text{NTUmax}(\mathbf{c}') \geq c'(V \setminus S)$ which we proved in Lemma 7.

As $\text{NTUmin}(\mathbf{c})$ satisfies condition (1), we have $\text{NTUmin}(\mathbf{c}) \geq c(S)$. Together will Lemma 8, this implies $\text{NTUmin}(\mathbf{c}) \geq \max\{c(V \setminus S), c(S)\} \geq c(V)/2$. Combined with Lemma 5, this completes the proof of Theorem 4.

Remark 3. As $\text{NTUmin}(\mathbf{c}) \leq \text{NTUmax}(\mathbf{c}) \leq \text{TUmax}(\mathbf{c})$, our bound of 2Δ extends to the smaller frugality ratios that we consider, i.e., $\phi_{\text{NTUmax}}(\mathcal{M})$ and $\phi_{\text{TUmax}}(\mathcal{M})$. It

is not clear whether it extends to the larger frugality ratio $\phi_{\mathrm{TUmin}}(\mathcal{M})$. However, the frugality ratio $\phi_{\mathrm{TUmin}}(\mathcal{M})$ is not realistic because the payment bound $\mathrm{TUmin}(\mathbf{c})$ is inappropriately low – we show in Section 6 that $\mathrm{TUmin}(\mathbf{c})$ can be significantly smaller than the total cost of a cheapest vertex cover.

Extensions We can also apply our results to monotone vertex-cover algorithms that do not necessarily output locally-optimal solutions. To do so, we simply take the vertex cover produced by any such algorithm and transform it into a locally-optimal one, considering the vertices in lexicographic order and replacing a vertex v with its neighbours whenever $b_v > \sum_{u \sim v} b_u$. Note that if a vertex u has been added to the vertex cover during this process, it means that it has a neighbour whose bid is higher than b_u, so after one pass all vertices in the vertex cover satisfy $b_v \leq \sum_{u \sim v} b_u$. This procedure is monotone in bids, and it can only decrease the cost of the vertex cover. Therefore, using it on top of a monotone allocation rule with approximation ratio α, we obtain a monotone locally-optimal allocation rule with approximation ratio α. Combining it with threshold payments, we get an auction with $\phi_{\mathrm{NTUmin}} \leq 2\Delta$. Since any truthful auction has a monotone allocation rule, this procedure transforms any truthful mechanism for the vertex-cover problem into a frugal one while preserving the approximation ratio.

5.2 Lower bound

In this subsection, we prove that the upper bound of Theorem 4 is essentially optimal. Our proof uses the techniques of [8], where the authors prove a similar result for shortest-path auctions.

Theorem 5. *For any $\Delta > 0$ and any n, there exist a graph G of maximum degree Δ and size $N > n$ such that for any truthful mechanism $\mathcal{M}$ on G we have $\phi_{\mathrm{NTUmin}}(\mathcal{M}) \geq \Delta/2$.*

Proof. Given n and Δ, set $k = \lceil n/2\Delta \rceil$. Let G be the graph that consists of k blocks $B_1, \ldots, B_k$ of size 2Δ each, where each B_i is a complete bipartite graph with parts L_i and R_i, $|L_i| = |R_i| = \Delta$.

We will consider two families of cost vectors for G. Under a cost vector $\mathbf{x} \in X$, each block B_i has one vertex of cost 1; all other vertices cost 0. Under a cost vector $\mathbf{y} \in Y$, there is one block that has two vertices of cost 1, one in each part, all other blocks have one vertex of cost 1, and all other vertices cost 0. Clearly, $|X| = (2\Delta)^k$, $|Y| = k(2\Delta)^{k-1}\Delta^2$. We will now construct a bipartite graph W with the vertex set $X \cup Y$ as follows.

Consider a cost vector $\mathbf{y} \in Y$ that has two vertices of cost 1 in B_i; let these vertices be $v_l \in L_i$ and $v_r \in R_i$. By changing the cost of either of these vertices to 0, we obtain a cost vector in X. Let $\mathbf{x}_l$ and $\mathbf{x}_r$ be the cost vectors obtained by changing the cost of v_l and v_r, respectively. The vertex cover chosen by $\mathcal{M}(\mathbf{y})$ must contain either all vertices in L_i or all vertices in R_i. In the former case, we put in W an edge from $\mathbf{y}$ to $\mathbf{x}_l$ and in the latter case we put in W an edge from $\mathbf{y}$ to $\mathbf{x}_r$ (if the vertex cover includes all of B_i, W contains both of these edges).

The graph W has at least $k(2\Delta)^{k-1}\Delta^2$ edges, so there must exist an $\mathbf{x} \in X$ of degree at least $k\Delta/2$. Let $y_1, \ldots, y_{k\Delta/2}$ be the other endpoints of the edges incident to $\mathbf{x}$, and for each $i = 1, \ldots, k\Delta/2$, let v_i be the vertex whose cost is different under $\mathbf{x}$ and $\mathbf{y}_i$; note that all v_i are distinct.

It is not hard to see that $\mathrm{NTUmin}(\mathbf{x}) \leq k$: any cheapest vertex cover contains the all-0 part of each block, and we can satisfy conditions (1)–(3) by letting one of the vertices in the all-0 part of each block to bid 1, while all other the vertices in the cheapest set bid 0.

On the other hand, by monotonicity of $\mathcal{M}$ we have $v_i \in \mathcal{M}(\mathbf{x})$ for $i = 1, \ldots, k\Delta/2$ (v_i is in the winning set under $\mathbf{y}_i$, and $\mathbf{x}$ is obtained from $\mathbf{y}_i$ by decreasing the cost of v_i), and moreover, the threshold bid of each v_i is at least 1, so the total payment of $\mathcal{M}$ on $\mathbf{x}$ is at least $k\Delta/2$. Hence, $\phi_{\mathrm{NTUmin}}(\mathcal{M}) \geq \mathcal{M}(\mathbf{x})/\mathrm{NTUmin}(\mathbf{x}) \geq \Delta/2$.

Remark 4. The lower bound of Theorem 5 can be generalised to randomised mechanisms, where a randomised mechanism is considered to be truthful if it can be represented as a probability distribution over truthful mechanisms. In this case, instead of choosing the vertex $x \in X$ with the highest degree, we put both $(\mathbf{y}, \mathbf{x}_l)$ and $(\mathbf{y}, \mathbf{x}_r)$ into W, label each edge with the probability that the respective part of the block is chosen, and pick $\mathbf{x} \in X$ with the highest weighted degree. The argument can be further extended to a more permissive definition of truthfulness for randomised mechanisms, but this discussion is beyond the scope of this paper.

6   Properties of the payment bounds

In this section we consider several desirable properties of payment bounds and evaluate the four payment bounds proposed in this paper with respect to them. The particular properties that we are interested in are independence of the choice of S (Section 6.3), monotonicity (Section 6.4.1), computational hardness (Section 6.4.2), and the relationship with other reasonable bounds, such as the total cost of the cheapest set (Section 6.1), or the total VCG payment (Section 6.2).

6.1   Comparison with total cost

Our first requirement is that a payment bound should not be less that the total cost of the selected set. Payment bounds are used to evaluate the performance of set-system auctions. The latter have to satisfy individual rationality, i.e., the payment to each agent must be at least as large as its incurred costs; it is only reasonable to require the payment bound to satisfy the same requirement.

Clearly, $\mathrm{NTUmax}(\mathbf{c})$ and $\mathrm{NTUmin}(\mathbf{c})$ satisfy this requirement due to condition (1), and so does $\mathrm{TUmax}(\mathbf{c})$, since $\mathrm{TUmax}(\mathbf{c}) \geq \mathrm{NTUmax}(\mathbf{c})$. However, $\mathrm{TUmin}(\mathbf{c})$ fails this test. The example of Proposition 4 shows that for path auctions, $\mathrm{TUmin}(\mathbf{c})$ can be smaller than the total cost by a factor of 2. Moreover, there are set systems and cost vectors for which $\mathrm{TUmin}(\mathbf{c})$ is smaller than the cost of the cheapest set S by a factor of $\Omega(n)$. Consider, for example, the vertex-cover auction for the graph of Proposition 5 with the costs $c_{X_1} = \cdots = c_{X_{n-2}} = c_{X_{n-1}} = 1$, $c_{X_0} = 0$. The cost of a cheapest vertex cover is $n - 2$, and the lexicographically first vertex cover of cost $n - 2$ is

$\{X_0, X_1, \ldots, X_{n-2}\}$. The constraints in (2) are $b_{X_i} + b_{X_0} \leq c_{X_{n-1}} = 1$. Clearly, we can satisfy conditions (2) and (3) by setting $b_{X_1} = \cdots = b_{X_{n-2}} = 0$, $b_{X_0} = 1$, which means that $\mathrm{TUmin}(\mathbf{c}) \leq 1$. This observation suggests that the payment bound $\mathrm{TUmin}(\mathbf{c})$ is too strong to be realistic, since it can be substantially lower than the cost of the cheapest feasible set.

Nevertheless, some of the positive results proved in [15] for $\mathrm{NTUmin}(\mathbf{c})$ go through for $\mathrm{TUmin}(\mathbf{c})$ as well. In particular, one can show that if the feasible sets are the bases of a monopoly-free matroid, then $\phi_{\mathrm{TUmin}}(\mathrm{VCG}) = 1$. To show that $\phi_{\mathrm{TUmin}}(\mathrm{VCG})$ is at most 1, one must prove that the VCG payment is at most $\mathrm{TUmin}(\mathbf{c})$. This is shown for $\mathrm{NTUmin}(\mathbf{c})$ in the first paragraph of the proof of Theorem 5 in [15]. Their argument does not use condition (1) at all, so it also applies to $\mathrm{TUmin}(\mathbf{c})$. On the other hand, $\phi_{\mathrm{TUmin}}(\mathrm{VCG}) \geq 1$ since $\phi_{\mathrm{TUmin}}(\mathrm{VCG}) \geq \phi_{\mathrm{NTUmin}}(\mathrm{VCG})$ and $\phi_{\mathrm{NTUmin}}(\mathrm{VCG}) \geq 1$ by Proposition 7 of [15] (and also by Proposition 6 below).

6.2   Comparison with VCG payments

Another measure of suitability for payment bounds is that they should not result in frugality ratios that are less then 1 for well-known truthful mechanisms. If this is indeed the case, the payment bound may be too weak, as it becomes too easy to design mechanisms that perform well with respect to it. It particular, a reasonable requirement is that a payment bound should not exceed the total payment of the classical VCG mechanism.

The following proposition shows that $\mathrm{NTUmax}(\mathbf{c})$, and therefore also $\mathrm{NTUmin}(\mathbf{c})$ and $\mathrm{TUmin}(\mathbf{c})$, do not exceed the VCG payment $p_{\mathrm{VCG}}(\mathbf{c})$. The proof essentially follows the argument of Proposition 7 of [15].

Proposition 6. $\phi_{\mathrm{NTUmax}}(\mathrm{VCG}) \geq 1$.

Proof. Suppose $e \in S$. The VCG payment p_e is $\min_{T:e \notin T}\{c_e + c(T) - c(S)\}$. Let T_e be the feasible set T which achieves the minimum so $p_e = c(T_e) - c(S - e)$. But constraint (2) gives $b(S - T) \leq c(T - S)$ for all T so since $e \notin T_e$, $b_e + b(S - T_e - e) \leq c(T_e - S)$ so

$$b_e \leq c(T_e - S) - b(S - T_e - e). \tag{1}$$

Now by (1), $b(S - T_e - e) \geq c(S - T_e - e)$, so (1) gives

$$b_e \leq c(T_e - S) + c(T_e \cap S) - c(S - T_e - e) - c(T_e \cap S).$$

Since $e \notin T_e \cap S$, we get $b_e \leq c(T_e) - c(S - e) = p_e$.

Proposition 6 shows that none of the payment bounds $\mathrm{TUmin}(\mathbf{c})$, $\mathrm{NTUmin}(\mathbf{c})$ and $\mathrm{NTUmax}(\mathbf{c})$ exceeds the payment of VCG. However, the payment bound $\mathrm{TUmax}(\mathbf{c})$ can be larger that the total VCG payment. In particular, for the instance in Proposition 5, the VCG payment is smaller than $\mathrm{TUmax}(\mathbf{c})$ by a factor of $n - 2$. We have already seen that $\mathrm{TUmax}(\mathbf{c}) \geq n - 2$. On the other hand, under VCG, the threshold bid of any X_i, $i = 1, \ldots, n - 2$, is 0: if any such vertex bids above 0, it is deleted from the winning set together with X_0 and replaced with X_{n-1}. Similarly, the threshold bid of X_0 is 1, because if X_0 bids above 1, it can be replaced with X_{n-1}. So the VCG payment is 1.

This result is not surprising: the definition of $\mathrm{TUmax}(\mathbf{c})$ implicitly assumes there is co-operation between the agents, while the computation of VCG payments does not take into account any interaction between them. Indeed, co-operation enables the agents to extract higher payments under VCG. That is, VCG is not group-strategyproof. This suggests that as a payment bound, $\mathrm{TUmax}(\mathbf{c})$ may be too liberal, at least in a context where there is little or no co-operation between agents. Perhaps $\mathrm{TUmax}(\mathbf{c})$ can be a good benchmark for measuring the performance of mechanisms designed for agents that can form coalitions or make side payments to each other, in particular, group-strategyproof mechanisms.

Another setting in which bounding ϕ_{TUmax} is still of some interest is when, for the underlying problem, the optimal allocation and VCG payments are NP-hard to compute. In this case, finding a *polynomial-time computable* mechanism with good frugality ratio with respect to $\mathrm{TUmax}(\mathbf{c})$ is a non-trivial task, while bounding the frugality ratio with respect to more challenging payment bounds could be too difficult. To illustrate this point, compare the proofs of Lemma 6 and Lemma 7: both require some effort, but the latter is much more difficult than the former.

6.3   The choice of S

All payment bounds defined in this paper correspond to the total bid of all elements in the cheapest feasible set, where ties are broken lexicographically. While this definition ensures that our payment bounds are well-defined, the particular choice of the draw-resolution rule appears arbitrary, and one might wonder if our payment bounds are sufficiently robust to be independent of this choice. It turns out that is indeed the case for $\mathrm{NTUmin}(\mathbf{c})$ and $\mathrm{NTUmax}(\mathbf{c})$, i.e., these bounds do not depend on the draw-resolution rule. To see this, observe that given two feasible sets S_1 and S_2 of the same cost, if we choose S_1 as the basis for our definition, all vertices in $S_1 \setminus S_2$ would have to bid their true cost, since otherwise S_2 would become cheaper than S_1. Hence, any bid vector for S_1 can only have $b_e \neq c_e$ for $e \in S_1 \cap S_2$, and hence constitutes a valid bid vector for S_2 and vice versa.

However, for $\mathrm{TUmin}(\mathbf{c})$ and $\mathrm{TUmax}(\mathbf{c})$ this is not the case. For example, consider the set system

$$\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}, \mathcal{F} = \{S_1 = \{e_1, e_2\}, S_2 = \{e_2, e_3, e_4\}, S_3 = \{e_4, e_5\}\}$$

with the costs $c_1 = 2$, $c_2 = c_3 = c_4 = 1$, $c_5 = 3$. The cheapest sets are S_1 and S_2. If the definition of $\mathrm{TUmax}(\mathbf{c})$ is based on S_1, we have $\mathrm{TUmax}(\mathbf{c}) \leq 4$, as the total bid of the elements in S_1 cannot exceed the total cost of S_3. On the other hand, if the definition of $\mathrm{TUmax}(\mathbf{c})$ is based on S_2, we have $\mathrm{TUmax}(\mathbf{c}) \geq 5$, as we can set $b_2 = 3, b_3 = 0, b_4 = 2$. Similarly, if the definition of $\mathrm{TUmin}(\mathbf{c})$ is based on S_1, the inequalities in (2) are $b_1 \leq 2$, $b_1 + b_2 \leq 4$, resulting in $\mathrm{TUmin}(\mathbf{c}) = 4$, and if the definition of $\mathrm{TUmin}(\mathbf{c})$ is based on S_2, we have $\mathrm{TUmin}(\mathbf{c}) \leq 3$, as we can set $b_2 = 1$, $b_3 = 2, b_4 = 0$.

6.4   Negative results for $\mathrm{NTUmin}(\mathbf{c})$ and $\mathrm{TUmin}(\mathbf{c})$

The results in [15] and our vertex cover results are proved for the frugality ratio ϕ_{NTUmin}. Indeed, it can be argued that ϕ_{NTUmin} is the "best" definition of frugality ratio, because

among all reasonable payment bounds (i.e., ones that are at least as large as the cost of the cheapest feasible set), it is most demanding of the algorithm. However, $\mathrm{NTUmin}(\mathbf{c})$ is not always the easiest or the most natural payment bound to work with. In this subsection, we discuss several disadvantages of $\mathrm{NTUmin}(\mathbf{c})$ (and also $\mathrm{TUmin}(\mathbf{c})$) as compared to $\mathrm{NTUmax}(\mathbf{c})$ and $\mathrm{TUmax}(\mathbf{c})$.

6.4.1   Nonmonotonicity   The first problem with $\mathrm{NTUmin}(\mathbf{c})$ is that it is not monotone with respect to $\mathcal{F}$, i.e., it may increase when one adds a feasible set to $\mathcal{F}$. (It is, however, monotone in the sense that if an agent's cost increases, it is less likely to be chosen.) Intuitively, a good payment bound should satisfy this monotonicity requirement, as adding a feasible set increases the competition, so it should drive the prices down. Note that this indeed the case for $\mathrm{NTUmax}(\mathbf{c})$ and $\mathrm{TUmax}(\mathbf{c})$ since a new feasible set adds a constraint in (2), thus limiting the solution space for the respective linear program.

Proposition 7. *Adding a feasible set to $\mathcal{F}$ can increase* $\mathrm{NTUmin}(\mathbf{c})$ *by a factor of* $\Omega(n)$.

Proof. Let $\mathcal{E} = \{x, xx, y_1, \ldots, y_n, z_1, \ldots, z_n\}$. Set $Y = \{y_1, \ldots, y_n\}$, $S = Y \cup \{x\}$, $T_i = Y \setminus \{y_i\} \cup \{z_i\}$, $i = 1, \ldots, n$, and suppose that $\mathcal{F} = \{S, T_1, \ldots, T_n\}$. The costs are $c_x = 0$, $c_{xx} = 0$, $c_{y_i} = 0$, $c_{z_i} = 1$ for $i = 1, \ldots, n$. Note that S is the cheapest feasible set. Let $\mathcal{F}' = \mathcal{F} \cup \{T_0\}$, where $T_0 = Y \cup \{xx\}$. For $\mathcal{F}$, the bid vector $b_{y_1} = \cdots = b_{y_n} = 0$, $b_x = 1$ satisfies (1), (2), and (3), so $\mathrm{NTUmin}(\mathbf{c}) \leq 1$. For $\mathcal{F}'$, any optimal solution has $b_x = 0$ (by constraint in (2) with T_0). Condition (3) for y_i implies $b_x + b_{y_i} = c_{z_i} = 1$, so $b_{y_i} = 1$ and $\mathrm{NTUmin}(\mathbf{c}) = n$.

For path auctions, it has been shown [17] that $\mathrm{NTUmin}(\mathbf{c})$ is non-monotone in a slightly different sense, i.e., with respect to adding a new edge (agent) rather than a new feasible set (a team of existing agents). We present this example here for completeness.
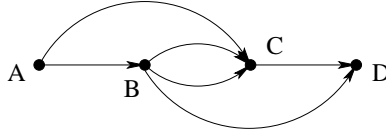

Fig. 5. Nonmonotonicity of $\mathrm{NTUmin}(\mathbf{c})$ for path auctions

Proposition 8. *For shortest path auctions, adding an edge to the graph can increase* $\mathrm{NTUmin}(\mathbf{c})$ *by a factor of* 2.

Proof. Consider the graph of Figure 1 with the edge costs $c_{AB} = c_{BC} = c_{CD} = 0$, $c_{AC} = c_{BD} = 1$. In this graph, $ABCD$ is the shortest path, and it is easy to see that $\mathrm{NTUmin}(\mathbf{c}) = 1$ with the bid vector $b_{AB} = b_{CD} = 0$, $b_{BC} = 1$. Now suppose that we

add a new edge $\widehat{BC}$ of cost 0 between B and C, obtaining the graph of Figure 5. We can assume that the original shortest path $ABCD$ is the lexicographically first shortest path in the new graph, so it gets selected. However, now we have a new constraint in (2), namely, $b_{BC} \leq c_{\widehat{BC}} = 0$, so we have $\mathrm{NTUmin}(\mathbf{c}) = 2$ with the bid vector $b_{AB} = b_{CD} = 1$, $b_{BC} = 0$.

Remark 5. It is not hard to modify the example of Proposition 8 so that the underlying graph has no multiple edges.

Remark 6. We can also show that $\mathrm{NTUmin}(\mathbf{c})$ is non-monotone for vertex cover. In this case, adding a new feasible set corresponds to *deleting* edges from the graph. It turns out that deleting a single edge can increase $\mathrm{NTUmin}(\mathbf{c})$ by a factor of $n - 2$; the construction is similar to that of Proposition 5.

6.4.2   NP-Hardness Another problem with $\mathrm{NTUmin}(\mathbf{c})$ is that it is NP-hard to compute even if the number of feasible sets is polynomial in n. Again, this puts it at a disadvantage compared to $\mathrm{NTUmax}(\mathbf{c})$ and $\mathrm{TUmax}(\mathbf{c})$ (see Remark 1).

Theorem 6. *Computing* $\mathrm{NTUmin}(\mathbf{c})$ *is NP-hard.*

Proof. We reduce EXACT COVER BY 3-SETS(X3C) to our problem. An instance of X3C is given by a universe $G = \{g_1, \ldots, g_n\}$ and a collection of subsets $C_1, \ldots, C_m$, $C_i \subset G, |C_i| = 3$, where the goal is to decide whether one can cover G by $n/3$ of these sets. Observe that if this is indeed the case, each element of G is contained in exactly one set of the cover.

Lemma 9. *Consider a minimisation problem P of the following form:*
Minimise $\sum_{i=1,\ldots,n} b_i$ *under conditions*

(1) $b_i \geq 0$ for all $i = 1, \ldots, n$
(2) for any $j = 1, \ldots, k$ we have $\sum_{b_i \in S_j} b_i \leq a_j$, where $S_j \subseteq \{b_1, \ldots, b_n\}$
(3) for each b_j, one of the constraints in (2) involving it is tight.

For any such P, one can construct a set system S and a vector of costs c such that $\mathrm{NTUmin}(\mathbf{c})$ *is the optimal solution to P.*

Proof. The construction is straightforward: there is an element of cost 0 for each b_i, an element of cost a_j for each a_j, the feasible solutions are $\{b_1, \ldots, b_n\}$, or any set obtained from $\{b_1, \ldots, b_n\}$ by replacing the elements in S_j by a_j.

By this lemma, all we have to do to prove Theorem 6 is to show how to solve X3C by using the solution to a minimisation problem of the form given in Lemma 9. We do this as follows. For each C_i, we introduce 4 variables x_i, $\bar{x}_i$, a_i, and b_i. Also, for each element g_j of G there is a variable d_j. We use the following set of constraints:

- In (1), we have constraints $x_i \geq 0$, $\bar{x}_i \geq 0$, $a_i \geq 0$, $b_i \geq 0$, $d_j \geq 0$ for all $i = 1, \ldots, m, j = 1, \ldots, n$.

- In (2), for all $i = 1, \ldots, m$, we have the following 5 constraints: $x_i + \bar{x}_i \leq 1$

  $x_i + a_i \leq 1$

  $\bar{x}_i + a_i \leq 1$

  $x_i + b_i \leq 1$

  $\bar{x}_i + b_i \leq 1$.

  Also, for all $j = 1, \ldots, n$ we have a constraint of the form $x_{i_1} + \cdots + x_{i_k} + d_j \leq 1$, where $C_{i_1}, \ldots, C_{i_k}$ are the sets that contain g_j.

The goal is to minimize $z = \sum_i (x_i + \bar{x}_i + a_i + b_i) + \sum_j d_j$.

Observe that for each j, there is only one constraint involving d_j, so by condition (3) it must be tight.

Consider the two constraints involving a_i. One of them must be tight, and therefore $x_i + \bar{x}_i + a_i + b_i \geq x_i + \bar{x}_i + a_i \geq 1$. Hence, for any feasible solution to (1)–(3) we have $z \geq m$. Now, suppose that there is an exact set cover. Set $d_j = 0$ for $j = 1, \ldots, n$. Also, if C_i is included in this cover, set $x_i = 1, \bar{x}_i = a_i = b_i = 0$, otherwise set $\bar{x}_i = 1$, $x_i = a_i = b_i = 0$. Clearly, all inequalities in (2) are satisfied (we use the fact that each element is covered exactly once), and for each variable, one of the constraints involving it is tight. This assignment results in $z = m$.

Conversely, suppose there is a feasible solution with $z = m$. As each addend of the form $x_i + \bar{x}_i + a_i + b_i$ contributes at least 1, we have $x_i + \bar{x}_i + a_i + b_i = 1$ for all i, $d_j = 0$ for all j. We will now show that for each i, either $x_i = 1$ and $\bar{x}_i = 0$, or $x_i = 0$ and $\bar{x}_i = 1$. For the sake of contradiction, suppose that $x_i = \delta < 1, \bar{x}_i = \delta' < 1$. As one of the constraints involving a_i must be tight, we have $a_i \geq \min\{1 - \delta, 1 - \delta'\}$. Similarly, $b_i \geq \min\{1 - \delta, 1 - \delta'\}$. Hence, $x_i + \bar{x}_i + a_i + b_i = 1 = \delta + \delta' + 2\min\{1 - \delta, 1 - \delta'\} > 1$. To finish the proof, note that for each $j = 1, \ldots, m$ we have $x_{i_1} + \cdots + x_{i_k} + d_j = 1$ and $d_j = 0$, so the subsets that correspond to $x_i = 1$ constitute a set cover.

Remark 7. In the proofs of Proposition 7 and Theorem 6 all constraints in (1) are of the form $b_e \geq 0$. Hence, the same results are true for $\mathrm{TUmin}(\mathbf{c})$.

Remark 8. For shortest-path auctions, the size of $\mathcal{F}$ can be superpolynomial. However, there is a polynomial-time separation oracle for constraints in (2) (it is easy to construct one based on any algorithm for finding shortest paths), so one can compute $\mathrm{NTUmax}(\mathbf{c})$ and $\mathrm{TUmax}(\mathbf{c})$ for this problem in polynomial time. We do not know if computing $\mathrm{NTUmin}(\mathbf{c})$ for shortest-path auctions is NP-hard; this was suggested as an open problem in [15].

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A   Nash equilibria and frugality ratios

Karlin et al. [15], argue that the payment bound ν can be viewed as the total payment in a Nash equilibrium of a certain game. In this section, we build on this intuition to justify the four payment bounds introduced above. We consider two variants of a game that differ in how profit is shared between the winning players. We will call these variants the TU game and the NTU game (standing for "transferable utility" and "non-transferable utility" respectively). We then show that $\mathrm{NTUmax}(\mathbf{c})$ and $\mathrm{NTUmin}(\mathbf{c})$ correspond to the worst and the best Nash equilibrium of the NTU game, and $\mathrm{TUmax}(\mathbf{c})$ and $\mathrm{TUmin}(\mathbf{c})$ correspond to the worst and the best Nash equilibrium of the TU game. $\mathrm{NTUmin}(\mathbf{c})$ corresponds to the payment bound ν of [15].

In both versions, the players are the elements of the ground set $\mathcal{E}$. Each player has an associated cost that is known to all parties. The game starts by the centre selecting a cheapest feasible set $S \in \mathcal{F}$ (with respect to the true costs), resolving ties lexicographically. Then the elements of S are allowed to make bids, and the centre decides whether or not to accept them. Intuitively, S ought to be able to win the auction, and we seek bids from S that are low enough to win, and high enough that no member of S has an incentive to raise his bid (because that would cause him to lose).

Given that S is supposed to win, we modify the game to rule out behaviour such as elements of S bidding unnecessarily high and losing. One way to enforce the requirement that S wins is via fines. If S is not the among the cheapest sets with respect to the bids (where the new cost of a set T is the sum of the total cost of $T \setminus S$ and the total bid of $S \setminus T$), the centre rejects the solution and every element of S who bids above its true cost pays a fine of size $C = \max_{e \in \mathcal{E}} c_e$, while other elements pay 0. Otherwise, members of S are paid their bids (which may then be shared amongst members of S). This ensures that in a Nash equilibrium, the resulting bids are never rejected as a result of S not being the cheapest feasible set.

In the NTU game, we assume that players cannot make payments to each other, i.e., the utility of each player in S is exactly the difference between his bid and his true cost. In particular, this means that no agent will bid below his true cost, which is captured by condition (1). In a Nash equilibrium, S is the cheapest set with respect to the bids, which is captured by condition (2). Now, suppose that condition (3) is not satisfied for some bidder e. Then the vector of bids is not a Nash equilibrium: e can benefit from increasing his bids by a small amount. Conversely, any vector of bids that satisfies (1), (2) and (3) is a Nash equilibrium: no player wants to decrease its bid, as it would lower the payment it receives, and no player can increase its bid, as it would violate (2) and will cause this bidder to pay a fine. As $\mathrm{NTUmin}(\mathbf{c})$ minimises $\sum_{e \in S} b_e$ under conditions (1), (2), and (3), and $\mathrm{NTUmax}(\mathbf{c})$ maximises it, these are, respectively, the best and the worst Nash equilibrium, from the centre's point of view.

In the TU game, the players in S redistribute the profits among themselves in equal shares, i.e., each player's utility is the difference between the total payment to S and the total cost of S, divided by the size of S. We noted in Section 6.1 that when S is *required* to be the winning set, this may result in Nash equilibria where members of S make a loss collectively, and not just individually as a result of condition (1) not applying. (Recall that we do assume that agents' bids are non-negative; condition (1^*).) $\mathrm{TUmin}(\mathbf{c})$ thus represents a situation in which "winners" are being coerced into accepting a loss-making contract.

$\mathrm{TUmax}(\mathbf{c})$ does not have the above problem, since it is larger than the other payment bounds, so members of S will not make a loss. The meaning of conditions (2) and (3) remains the same: the agents do not want the centre to reject their bid, and no agent can improve the total payoff by raising their bid. Note that we are not allowing coalitions (see remark 9), i.e., coordinated deviations by two or more players: even though the players share the profits, they cannot make joint decisions about their strategies. Similarly to the NTU game, it is easy to see that $\mathrm{TUmax}(\mathbf{c})$ and $\mathrm{TUmin}(\mathbf{c})$ are, respectively, the worst and the best Nash equilibria of this game from the centre's viewpoint.

Remark 9. Allowing payment redistribution within a set is different from allowing players to form coalitions (as in, e.g., the definition of strong Nash equilibrium): in the latter case, players are allowed to make joint decisions about their bids, but they cannot make payments to each other.

Remark 10. Both of our games are different from the one implicitly suggested by [15], as we do not allow bids from the players not in the winning set, i.e., our games are NOT true first-price auctions. The reason for this choice is that $\mathrm{NTUmax}(\mathbf{c})$ and $\mathrm{TUmax}(\mathbf{c})$ do not have a good characterisation in terms of first-price auctions, neither when players are allowed to form coalitions or redistribute payments, nor when they are not allowed to do so. One of the reasons for that is that an agent not in the cheapest set may bid above his true cost at a Nash equilibrium; while the agent itself does not benefit from it, it has no disincentive to do so, and it increases the total payment. Hence, the worst Nash equilibrium in a first-price auction may be much worse than $\mathrm{NTUmax}(\mathbf{c})$ or $\mathrm{TUmax}(\mathbf{c})$. We can penalise the agents not in the selected set for bidding above their true value, but this restriction does not seem natural in the context of first-price auctions, and essentially reduces them to the game considered above.