

Supplement to: Code Spectrum and Reliability Function: Binary Symmetric Channel

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A much simpler proof of Theorem 1 from [1] is presented below, using notation and formulas numeration of [1]. The text below replaces the subsection **General case** from §4 of [1, p. 11].

General case. In the general case for some ω we are interested in a pairs $(\mathbf{x}_i, \mathbf{x}_j)$ with $d_{ij} = \omega n$. But there may exist a pairs $(\mathbf{x}_k, \mathbf{x}_l)$ with $d_{kl} < \omega n$. Using the “cleaning” procedure [2] we show that the influence of such pairs $(\mathbf{x}_k, \mathbf{x}_l)$ on the value P_e is not large. It will allow us to reduce the general case to the model one.

Note that if

$$\frac{1}{n} \log X_{\max}(t, \omega) = o(1), \quad n \rightarrow \infty, \quad (\text{S.1})$$

then from (27) and (28) we get

$$\begin{aligned} & \frac{1}{n} \log \frac{1}{P_e} \leq -\log q + \\ & + \min_{0 \leq t \leq 1} \min_{\omega} \left\{ t \log \frac{q}{p} - \omega - (1 - \omega) h_2 \left(\frac{1}{2} - \frac{1 - 2t}{2(1 - \omega)} \right) - b(\omega) \right\} + o(1), \end{aligned} \quad (\text{S.2})$$

where $b(\omega) = n^{-1} \log B_{\omega n}$.

The minimum over t in the right-hand side of (S.2) is attained when

$$t(\omega) = \frac{\omega}{2} + (1 - \omega)p, \quad (\text{S.3})$$

and then (S.2) takes the form

$$\frac{1}{n} \log \frac{1}{P_e} \leq \min_{\omega} f(\omega) + o(1), \quad f(\omega) = \frac{\omega}{2} \log \frac{1}{4pq} - b(\omega). \quad (\text{S.4})$$

Let $f(\omega)$ attains its minimum (over all ω) at some ω_0 . By definition we have for any ω

$$\frac{\omega_0}{2} \log \frac{1}{4pq} - b(\omega_0) \leq \frac{\omega}{2} \log \frac{1}{4pq} - b(\omega). \quad (\text{S.5})$$

To avoid a superfluous awkwardness, we omit the remaining term $o(1)$ in the Theorem 2. Then there exists ω such that $\omega \leq G(\alpha, \tau)$ and $b(\omega) \geq \mu(R, \alpha, \omega)$. Denote ω^* the smallest $\omega \leq G(\alpha, \tau)$ for which we have $b(\omega) \geq \mu(R, \alpha, \omega)$.

We call $(\mathbf{x}_i, \mathbf{x}_j)$ a ω -pair if $d(\mathbf{x}_i, \mathbf{x}_j) = \omega n$. Then the total number of ω -pairs equals $M2^{nb(\omega)}$. We use $t = t(\omega_0)$ from (S.3), and say that a point \mathbf{y} is ω -covered if there exists a ω -pair $(\mathbf{x}_i, \mathbf{x}_j)$ such that $d(\mathbf{x}_i, \mathbf{y}) = d(\mathbf{x}_j, \mathbf{y}) = tn$. Then there are $M2^{nb(\omega)}Z(t(\omega_0), \omega)$ ω -covered points \mathbf{y} (taking into account the covering multiplicities). Introduce the set $\mathbf{Y}(\omega)$ of all ω -covered points \mathbf{y} . We set a small $\varepsilon > 0$ and perform a cleaning procedure. Consider the set $\mathbf{Y}(\omega_0)$ and exclude from it all points \mathbf{y} that are also ω -covered for any ω such that $|\omega - \omega_0| \geq \varepsilon$, i.e. consider the set of all ω_0 -covered points \mathbf{y} which are not ω -covered for any ω such that $|\omega - \omega_0| \geq \varepsilon$:

$$\mathbf{Y}'(\omega_0) = \mathbf{Y}(\omega_0) \setminus \bigcup_{|\omega - \omega_0| \geq \varepsilon} \mathbf{Y}(\omega). \quad (\text{S.6})$$

Each point $\mathbf{y} \in \mathbf{Y}'(\omega_0)$ can be ω -covered only if $|\omega - \omega_0| < \varepsilon$. We show that for an appropriate ε both sets $\mathbf{Y}(\omega_0)$ and $\mathbf{Y}'(\omega_0)$ have essentially the same cardinalities. Each ω -pair $(\mathbf{x}_i, \mathbf{x}_j)$ ω -covers the set $\mathbf{Z}_{ij}(t, \omega)$ with the cardinality $Z(t, \omega)$. We compare the values $\sum_{|\omega - \omega_0| \geq \varepsilon} 2^{nb(\omega)}Z(t, \omega)$ and $2^{nb(\omega_0)}Z(t, \omega_0)$ (see (S.6)). For that purpose consider the function

$$f(\omega) = \frac{1}{n} \log \frac{2^{nb(\omega)}Z(t, \omega)}{2^{nb(\omega_0)}Z(t, \omega_0)} = b(\omega) - b(\omega_0) + u(t, \omega) - u(t, \omega_0), \quad (\text{S.7})$$

where

$$u(t, \omega) = \omega + (1 - \omega)h_2 \left[\frac{1}{2} - \frac{(1 - \omega_0)(1 - 2p)}{2(1 - \omega)} \right].$$

Due to (S.5) we have $b(\omega) - b(\omega_0) \leq (\omega_0 - \omega)[\log(4pq)]/2$, and then for the function $f(\omega)$ from (S.7) we get

$$\begin{aligned} f(\omega) &\leq v(\omega) = -\left(\frac{1}{2} - p\right)(\omega_0 - \omega) \log \frac{q}{p} + \\ &+ (1 - \omega) \left[h_2 \left(\frac{1}{2} - \frac{(1 - 2p)(1 - \omega_0)}{2(1 - \omega)} \right) - h_2(p) \right], \\ v' &= \frac{1}{2} \log \frac{1}{4pq} + \frac{1}{2} \log \left[1 - \frac{(1 - 2p)^2(1 - \omega_0)^2}{(1 - \omega)^2} \right], \\ v'' &= -\frac{(1 - 2p)^2(1 - \omega_0)^2 \log_2 e}{(1 - \lambda)[(1 - \omega)^2 - (1 - 2p)^2(1 - \omega_0)^2]} < -\frac{(1 - 2p)^2}{3}. \end{aligned} \quad (\text{S.9})$$

Since $v(\omega_0) = v'(\omega_0) = 0$, then for any ω we have

$$f(\omega) \leq v(\omega) < -\frac{(1 - 2p)^2}{6}(\omega_0 - \omega)^2.$$

Now after simple calculations we have

$$\begin{aligned}
& \sum_{|\omega - \omega_0| \geq \varepsilon} 2^{nb(\omega)} Z(t, \omega) / [2^{nb(\omega_0)} Z(t, \omega_0)] = \sum_{|\omega - \omega_0| \geq \varepsilon} 2^{nf(\omega)} \leq \\
& \leq 2 \sum_{\omega - \omega_0 \geq \varepsilon} 2^{-(1-2p)^2(\omega_0 - \omega)^2 n/6} = 2 \sum_{i \geq \varepsilon n} 2^{-(1-2p)^2 i^2 / (6n)} \leq \\
& \leq 2 \left[1 + \frac{3}{(1-2p)^2 \varepsilon} \right] e^{-(1-2p)^2 \varepsilon^2 n/6} \leq \frac{6n^{-1/6}}{1-2p},
\end{aligned}$$

if we set

$$\varepsilon = \frac{2\sqrt{\ln n}}{(1-2p)\sqrt{n}}.$$

Therefore for $n^{1/6} \geq 12/(1-2p)$ we get

$$2^{nb(\omega_0)} Z(t, \omega_0) - \sum_{|\omega - \omega_0| \geq \varepsilon} 2^{nb(\omega)} Z(t, \omega) \geq \frac{1}{2} 2^{nb(\omega_0)} Z(t, \omega_0).$$

In other words, all points $\mathbf{y} \in \mathbf{Y}'(\omega_0)$ are, in total, ω -covered, at least, $2^{nb(\omega_0)} Z(t, \omega_0)/2$ times, and, moreover, each point $\mathbf{y} \in \mathbf{Y}'(\omega_0)$ can be ω -covered only if $|\omega - \omega_0| < \varepsilon$. Due to the formula (S.9) it means that the cardinalities of the sets $\mathbf{Y}(\omega_0)$ and $\mathbf{Y}'(\omega_0)$ have equal exponential order.

For each point $\mathbf{y} \in \mathbf{Y}'(\omega_0)$ consider the set $\mathbf{X}_t(\mathbf{y})$ defined in (19), i.e. the set of all codewords $\{\mathbf{x}_i\}$ such that $d(\mathbf{x}_i, \mathbf{y}) = t(\omega_0)n$. The codewords from $\mathbf{X}_t(\mathbf{y})$ satisfy also the condition $|d(\mathbf{x}_i, \mathbf{x}_j) - \omega_0 n| \leq \varepsilon n$, i.e. the set $\mathbf{X}_t(\mathbf{y})$ constitutes almost a simplex. It is clear that the number $|\mathbf{X}_t(\mathbf{y})|$ of such codewords is not exponential on n , i.e.

$$\log |\mathbf{X}_t(\mathbf{y})| = o(n), \quad \mathbf{y} \in \mathbf{Y}'(\omega_0), \quad n \rightarrow \infty. \quad (\text{S.9})$$

For accurateness the formula (S.9) is proved below. It follows from (S.9) that the condition (S.1) is satisfied with $X_{\max}(t, \omega_0) = \max_{i, \mathbf{y} \in \mathbf{Y}'(\omega_0)} |\mathbf{X}_i(\mathbf{y}, t, \omega_0)|$ (cf. (25)). Using the upper bound (S.4) and the inequality (S.5) we get

$$\begin{aligned}
\frac{1}{n} \log \frac{1}{P_e} & \leq f(\omega_0) + o(1) \leq f(\omega^*) + o(1) \leq \max_{\omega \leq G(\alpha, \tau)} g(\omega) + o(1), \\
g(\omega) & = \frac{\omega}{2} \log \frac{1}{4pq} - \mu(R, \alpha, \omega),
\end{aligned} \quad (\text{S.10})$$

from which the desired upper bound (11) follows.

It remains us to prove the relation (S.9). If $\omega^* \geq \omega_1$ then (S.9) immediately follows from [1, proposition 4]. In the general case (S.9) follows from the lemma.

L e m m a. Let $\mathcal{C} = \{\mathbf{x}_1, \dots, \mathbf{x}_M\}$ be a code such that for some ω the relation holds

$$\max_{i \neq j} |d(\mathbf{x}_i, \mathbf{x}_j) - \omega n| = o(n), \quad n \rightarrow \infty.$$

Then

$$n^{-1} \ln M \rightarrow 0, \quad n \rightarrow \infty. \quad (\text{S.11})$$

P r o o f. If $\mathbf{x}_i, \mathbf{x}_j$ are binary codewords then for their Hamming and Euclidean distances we have $d_H(\mathbf{x}_i, \mathbf{x}_j) = \|\mathbf{x}_i - \mathbf{x}_j\|^2$. Without loss of generality we may assume that all codewords $\{\mathbf{x}_i\}$ have the same Hamming weight An . Then a binary code $\{\mathbf{x}_i\}$ of the length n can be considered as an Euclidean code $\{\mathbf{x}_i\} \subset S^n(\sqrt{An})$. For the Euclidean case the relation (S.11) has been proved in [3, Lemma 2]. \blacktriangle

It finishes the upper bound (11) proof. \blacktriangle

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