

# TIGHT BOUNDS ON THE COMPLEXITY OF RECOGNIZING ODD-RANKED ELEMENTS

SHRIPAD THITE

**ABSTRACT.** Let  $S = \langle s_1, s_2, s_3, \dots, s_n \rangle$  be a given vector of  $n$  real numbers. The *rank* of  $z \in \mathbb{R}$  with respect to  $S$  is defined as the number of elements  $s_i \in S$  such that  $s_i \leq z$ . We consider the following decision problem: determine whether the odd-numbered elements  $s_1, s_3, s_5, \dots$  are precisely the elements of  $S$  whose rank with respect to  $S$  is odd. We prove a bound of  $\Theta(n \log n)$  on the number of operations required to solve this problem in the algebraic computation tree model.

Let  $S = \langle s_1, s_2, s_3, \dots, s_n \rangle \in \mathbb{R}^n$  be a given vector. For an arbitrary real  $z$ , define the *rank* of  $z$  with respect to  $S$ , denoted by  $\text{rank}_S(z)$ , as the number of elements of  $S$  less than or equal to  $z$ . Thus, for instance, the largest element of  $S$  has rank  $n$ . Let  $\text{odd}(S)$  denote the set of elements of  $S$  whose rank with respect to  $S$  is odd.

We consider the following problem: determine whether the odd-numbered elements  $s_1, s_3, s_5, \dots$  are precisely the elements of  $S$  whose rank with respect to  $S$  is odd. Without loss of generality, we can assume that  $n$  is even because, otherwise, we can append an extra element  $+\infty$  without changing the answer.

Note that  $\text{odd}(S)$  has size  $n/2$  if and only if all  $n$  values  $s_i \in S$  are distinct; hence, the answer is ‘yes’ only if  $S$  is a vector of  $n$  distinct numbers.

We prove matching upper and lower bounds on the number of operations required to solve the problem in the algebraic computation tree (ACT) model (see Ben-Or [1]).

The following algorithm solves the problem using  $O(n \log n)$  comparisons. Sort  $S' = \langle s_1, s_3, s_5, \dots, s_{n-1} \rangle$  in non-decreasing order with an optimal sorting algorithm. Similarly, sort  $S$  in non-decreasing order. Then, scan the vector  $S'$  and the odd-numbered elements of  $S$  to decide whether the two are equal.

Next, we prove the matching lower bound.

For a vector  $S = \langle s_1, s_2, s_3, \dots, s_n \rangle$ , let  $\sigma(S)$  denote the permuted vector  $\langle s_{\sigma(1)}, s_{\sigma(2)}, s_{\sigma(3)}, \dots, s_{\sigma(n)} \rangle$ . We call a permutation  $\sigma$ , where  $\sigma(i)$  is odd if and only if  $i$  is odd, a *permissible* permutation.

**Lemma 1.** *There are  $\left(\left(\frac{n}{2}\right)!\right)^2$  permissible permutations of a vector of  $n$  elements.*

*Proof.* There are  $\frac{n}{2}!$  permutations of  $n$  elements that permute the  $n/2$  odd-numbered elements only, and  $\frac{n}{2}!$  that permute the  $n/2$  even-numbered elements only. A permissible permutation of  $n$  elements is any composition of two permutations, one that permutes the odd-numbered elements only and one that permutes the even-numbered elements only.  $\square$

**Observation 2.** *A permutation  $\sigma$  is permissible if and only if its inverse  $\sigma^{-1}$  is permissible.*

Let  $W \subset \mathbb{R}^n$  be the set of inputs for which the answer to the question posed in the problem is ‘yes’. Recall that every point in  $W$  corresponds to a set of  $n$  distinct real numbers.

**Lemma 3.** *For an arbitrary point  $X \in W$ , there is a unique permutation  $\sigma$  that sorts  $X$ , i.e., such that  $x_{\sigma(1)} < x_{\sigma(2)} < x_{\sigma(3)} < \dots < x_{\sigma(n)}$ . Moreover, such a permutation  $\sigma$  is permissible.*

*Proof.* The uniqueness of the sorting permutation  $\sigma$  follows because every point in  $W$  corresponds to a set of distinct reals. When  $X$  is sorted, the odd-ranked elements must occupy the odd-numbered positions of the sorted vector. Since  $X \in W$ , the odd-ranked elements are already in odd-numbered positions of the original vector  $X$ . Therefore, the permutation  $\sigma$  is permissible.  $\square$

Let  $\sigma_X$  denote the sorting permutation for  $X$ .

**Observation 4.** *If  $\sigma_X$  is a permissible permutation, then  $X \in W$ .*

**Lemma 5.** *For every permissible permutation  $\sigma$ , there is a point  $X \in W$  such that  $\sigma = \sigma_X$ .*

*Proof.* Set  $X = \langle \sigma^{-1}(1), \sigma^{-1}(2), \sigma^{-1}(3), \dots, \sigma^{-1}(n) \rangle$ . We have,

$$\begin{aligned} \sigma(X) &= \langle \sigma(\sigma^{-1}(1)), \sigma(\sigma^{-1}(2)), \sigma(\sigma^{-1}(3)), \dots, \sigma(\sigma^{-1}(n)) \rangle \\ &= \langle 1, 2, 3, \dots, n \rangle \end{aligned}$$

Therefore,  $\sigma(X)$  is sorted, and by Lemma 3, it is the unique permutation that sorts  $X$ ; hence,  $\sigma = \sigma_X$ .

It remains to show that the point  $X$  that we chose belongs to  $W$ . The set of real numbers represented by  $X$  is  $\{1, 2, 3, \dots, n\}$ . Since  $\sigma$  is permissible, so is  $\sigma^{-1}$  by Observation 2; hence,  $\sigma^{-1}(i)$  is odd if and only if  $i$  is odd. Therefore, the  $i$ th component of the vector  $X$  is odd if and only if  $i$  is odd, which means that  $X \in W$ .  $\square$

**Lemma 6.** *For every two points  $X, Y \in W$  such that  $\sigma_X \neq \sigma_Y$ , the two points  $X$  and  $Y$  lie in different connected components of  $W$ .*

*Proof.* Since  $X, Y \in W$ , both  $\sigma_X$  and  $\sigma_Y$  are permissible permutations, by Lemma 3.

For every point  $A = \langle a_1, a_2, a_3, \dots, a_n \rangle \in W$  such that

$$a_{\sigma_X(1)} < a_{\sigma_X(2)} < a_{\sigma_X(3)} < \dots < a_{\sigma_X(n)}$$

we have  $\sigma_A = \sigma_X$ . Since  $\sigma_X$  is permissible, so is  $\sigma_A$ ; by Observation 4, this implies that  $A \in W$ . Additionally,  $A$  is in the same connected component of  $W$  as  $X$  because every convex combination  $B$  of  $A$  and  $X$  satisfies  $\sigma_B = \sigma_X$ .

On the other hand, since  $\sigma_Y \neq \sigma_X$ , there exists an  $i$  in the range  $1 \leq i \leq n-1$  such that  $y_{\sigma_X(i)} \geq y_{\sigma_X(i+1)}$ . Then,  $X$  and  $Y$  cannot be in the same connected component of  $W$  because they are separated by the hyperplane  $y_{\sigma_X(i)} = y_{\sigma_X(i+1)}$ ; every point  $P$  on this hyperplane lies outside  $W$  because it corresponds to an input where  $\text{odd}(P)$  has fewer than  $n/2$  elements.

We have thus shown that the region  $R_X$  where

$$R_X = \{ \langle a_1, a_2, a_3, \dots, a_n \rangle \in W : a_{\sigma_X(1)} < a_{\sigma_X(2)} < a_{\sigma_X(3)} < \dots < a_{\sigma_X(n)} \}$$

is a maximal connected component of  $W$  containing  $X$  ( $R_X$  also happens to be convex); since  $\sigma_Y \neq \sigma_X$ , the region  $R_X$  does not contain  $Y$ .  $\square$

**Theorem 7.** *The set  $W$  has  $\left(\left(\frac{n}{2}\right)!\right)^2$  connected components.*

*Proof.* The set  $W$  can be partitioned such that each part corresponds to a permissible permutation  $\sigma$ ; by Lemma 5,  $\sigma = \sigma_X$  for some  $X \in W$ . By Lemma 1,  $W$  is partitioned into  $\left(\left(\frac{n}{2}\right)!\right)^2$  parts. By Lemma 6, every two distinct permissible permutations  $\sigma$  and  $\sigma'$  correspond to two different connected components of  $W$ , one consisting of all points  $X \in W$  for which  $\sigma_X = \sigma$  and the other consisting of all points  $Y \in W$  for which  $\sigma_Y = \sigma'$ .  $\square$

**Corollary 8.** *Every algebraic computation tree that decides the membership problem in  $W$  must have depth  $\Omega(n \log n)$ .*

*Proof.* Ben-Or [1] has proved that the minimum height of an algebraic computation tree deciding membership in  $W$  is  $\Omega(\log \#W)$  where  $\#W$  is the number of connected components of  $W$ . By Theorem 7, such a tree must have depth  $\Omega(n \log n)$ .  $\square$

*Acknowledgments.* Thanks to Mark de Berg, Jeff Erickson, Sarel Har-Peled, and Jan Vahrenhold for fruitful discussions.

## REFERENCES

- [1] “Lower Bounds for Algebraic Computation Trees”. Michael Ben-Or. In *Proc. ACM Symposium on Theory of Computing*, pp. 80–86, 1983.

*Current address:* Department of Mathematics and Computer Science, Technische Universiteit Eindhoven, Postbus 513, 5600 MB Eindhoven, The Netherlands; Email: sthite@win.tue.nl