Tutorial & Practical 9: Bootstrap Methods

Question 1

In this question we explore the bias reduction performance of a bootstrap estimator of a third power of the mean of a population assumed to be a scalar $\theta_0 = \theta(F_0) = \mu^3$ where

$$\mu = \int x dF_0(x)$$

Let $X = \{x_1, ..., x_n\}$ be a sample drawn from F_0 where $x_i \in \mathbb{R}$, used for the estimation of θ_0.

1. Provide the form of the nonparametric estimator obtained from the empirical distribution F_1.
2. Derive the expression of the bias $b_1 = \mathbb{E}\left(\hat{\theta} - \theta_0\right)$.
3. Derive the expression of the bootstrap estimate of b_1.
4. Use this expression to derive the bootstrap bias-reduced estimate $\hat{\theta}_1$ of θ
5. Derive the expression of the bias $b_2 = \mathbb{E}\left(\hat{\theta}_1 - \theta_0\right)$
6. Compare b_1 and b_2

Question 2

In this question we explore the bias reduction performance of a bootstrap estimator of a third power of the mean of a Normal population $N(\mu, \sigma^2)$ when the parameters are estimated using the maximum likelihood estimator from a set $X = \{x_1, ..., x_n\} \sim F_0$, $x_i \in \mathbb{R}$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$$

1. Provide the form of the nonparametric estimator $\hat{\theta}$ obtained from the empirical distribution F_1 and its associated bias b_1.
2. Derive the expression of the bootstrap bias-reduced estimate $\hat{\theta}$ of θ
3. Derive the expression of the bias associated with $\hat{\theta}_1$, b_2 and compare it with b_1
4. What is the value of b_2 when considering $\tilde{\sigma}^2$

   $$\tilde{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$$

   instead of $\hat{\sigma}^2$

Question 3

Suppose that we estimate the distribution of $\hat{\theta}-\theta$ by the bootstrap distribution $\hat{\theta}^{*}-\hat{\theta}$. Denote the $\alpha-$percentile of $\hat{\theta}^{*}-\hat{\theta}$ by $H^{-1}(\alpha)$. Derive the interval for θ that results from inverting the relation

$$\hat{H}^{-1}(\alpha) \leq \hat{\theta} - \theta \leq \hat{H}^{-1}(1-\alpha)$$