Tutorial & Practical 7: Nonparametric Regression

Question 1 Penalized spline regression

Consider the ordinary nonparametric regression model

$$y_i = f(x_i) + \epsilon_i; \quad 1 \leq i \leq n$$

where $y_i \in \mathbb{R}$, $x_i \in \mathbb{R}$, $\epsilon_i \in \mathbb{R} \sim \mathcal{N}(0, \sigma^2)$ and are i.i.d. For approximating f we propose to use the linear spline model.

1. Provide the form of this linear spline model and define the set of linear parameters
2. Derive its matrix form and the associated penalized spline fitting criterion
3. To what this criterion corresponds when considering the coefficients of the truncated linear functions as random with $\text{cov}(\mathbf{u}) = \sigma_u^2 I$ where $\mathbf{u}$ is the vector of coefficients of the truncated linear function.
4. Derive the expression for the penalized least squares estimator of the unknown parameters of the model
5. Provide the associated expression for the best fitted values
6. Provide a model form which the estimation of the coefficients of the polynomial functions is straightforwardly obtained
7. Derive the estimator of these parameters and their covariance

Question 2 Kernel regression

1. Provide the expression of the Nadaraya-Watson kernel estimator and discuss its singularity condition
2. Is the Nadaraya-Watson estimator with a Gaussian Kernel differentiable?
3. Is the Nadaraya-Watson estimator with the Epanechnikov Kernel differentiable?

Question 3 Linear smoothers

Let's consider a set of observations generated according to the model

$$y_i = f(x_i) + \epsilon_i; \quad 1 \leq i \leq n$$

where $f : [0, 1] \to \mathbb{R}$, the pairs (x_i, y_i) , $i = 1, ..., n$ are observed and ϵ_i are i.i.d. with $E(\epsilon_i) = 0$ and $E(\epsilon_i^2) = \sigma_\epsilon^2$.

Assuming $f \in L_2\left[0,1\right]$ and an orthonormal basis $\{\rho_j\}_{j=1}^{\infty}$ of $L_2\left[0,1\right]$ then

$$f(x) = \sum_{j=1}^{\infty} \theta_j \rho_j(x),$$

where

$$\theta_j = \int_0^1 f(x)\rho_j(x)dx.$$

The projection estimator for f is given by

$$\hat{f}_{nN}(x) = \sum_{j=1}^{N} \hat{\theta}_j \rho_j(x),$$

where

$$\hat{\theta}_j = \frac{1}{n}\sum_{i=1}^{n} y_i \rho_j(x_i).$$

1. Which parameter plays the role of the smoothing parameter in the projection estimator
2. Show that $\hat{f}_{nN}$ is a linear estimator in y_i's
3. Show that $E\left(\hat{\theta}_j\right) = \theta_j + r_j$
4. Show that $E\left[\left(\hat{\theta}_j - \theta_j\right)^2\right] = \frac{\sigma_\epsilon^2}{n} + r_j^2$
5. Provide the expression of the MISE $= E\left[\|\hat{f}_{nN} - f\|_2^2\right]$
6. Discuss the effect of N on the MISE