Stochastic Simulation (MIE1613H) - Homework 2

Due: Feb 24th, 2021

- Submit your homework to Quercus in PDF format by the above deadline. Late submissions are penalized 10% each day.
- At the top of your homework include your name, student number, department, and program.
- You must include both the Python source code (including comments to make it easy to follow) and the output.
- You may discuss the assignment with other students, but each student must solve the problems, write the code and the solutions individually.
- **Full mark will be given to answers that are correct and clearly explained. Write a brief and clear explanation of your solution for each problem.**

Problem 1. **(10 Pts.)** (Chapter 4, Exercise 8) In the simulation of the Asian option, the sample mean of 10,000 replications was 2.198270479 and the standard deviation was 4.770393202. *Approximately* how many replications would it take to decrease the relative error to less than 1%?

Note: The relative error of a sample mean is the standard error divided by the mean. Use the definition of the relative error and the provided estimates to obtain an estimate for the required number of replications.

Problem 2. (20 Pts.) (Down-and-out call option) Another variation of European options are barrier options. Denote the stock price at time t by $X(t)$ and assume that it is modelled as a Geometric Brownian Motion (GBM). A down-and-out call option with barrier B and strike price K has payoff

$$I\left\{\min_{0\le t\le T} X(t) > B\right\}(X(T)-K)^+,$$

where $I\{A\}$ is the indicator function of event A. This means that if the value of the asset falls below \$$B$ before the option matures then the option is worthless. Hence, the value of the option is

$$E\left[e^{-rT} I\left\{\min_{0\le t\le T} X(t) > B\right\}(X(T)-K)^+\right].$$

Using the same parameters as in the Asian option example, i.e., $T = 1$, $X(0) = \$50$; $K = \$55$; $r = 0.05$ and $\sigma^2 = (0.3)^2$, estimate the value of this option for barriers $B = 35, 40, 45$ and provide an intuitive reason for the effect of increasing the barrier on the value of the option. Use $m = 64$ steps when discretizing the GBM and use $n = 40,000$ replications. Report a 95% confidence interval for your estimates.

Problem 3. (20 Pts.) (Chapter 4, Exercise 4) Beginning with the PythonSim event-based $M/G/1$ simulation, implement the changes necessary to make it an $M/G/s$ simulation (a single queue with

s servers). Keep the arrival rate at $\lambda = 1$ and use average service time $\tau = 0.8 \times s$, and simulate the system for $s = 1, 2, 3$.

(a) Report the estimated expected number of customers in the system (including customers in the queue and service), expected system time, and expected number of busy servers in each case.

(b) Compare the results and state clearly what you observe. What you're doing is comparing queues with the same service capacity, but with 1 fast server as compared to 2 or more slow servers.

HINT: The attribute "NumberOfUnits" of the Resource object returns the number of available units for any instance of the object.

Problem 4. (20 Pts.) (Chapter 4, Exercise 5) Modify the PythonSim event-based simulation of the $M/G/1$ queue to simulate a $M/G/1/c$ retrial queue. This means that customers who arrive to find $c < \infty$ customers in the system (including the customer in service) leave immediately, but arrive again after an exponentially distributed amount of time with mean MeanTR. Do we need the arrival rate λ to be smaller than service rate $1/\tau$ for the system to reach steady-state? Explain your answer using numerical evidence from the simulation model.

HINT: The existence of retrial customers should not affect the arrival process for first-time arrivals.

Problem 5. (20 Pts.) A long-term care home in Toronto is plannig to test all of its 170 staff (nurses and caregivers) before they start their shift using the new COVID-19 Rapid Test. Consider the 7AM shift and assume that staff arrive between 6:30AM and 7AM for their shift with arrival times uniformly distributed during the 30-minute period prior to the shift. Test times take on average 15 minutes and can be modelled using an Erlang distribution with 4 phases.

(a) Provide a 95% Confidence Interval for the expected average waiting time (excluding the test time) of staff when there are 10 paralell servers (testing stations) available. Assume that staff wait in a single First-Come, First-Serverd queue before being tested. Use 100 replications.

(b) Approximately how many servers are required to keep the average waiting time below 15 minutes?

HINT: You may want to schedule the arrival events for all 170 staff to the calendar at the beginning of each replication.

Problem 6. (10 Pts.) Consider the $AR(1)$ model described in Section 3.3. of the textbook. Derive an expression for the asymptotic bias β of the sample mean $\bar{Y}(m)$. **HINT**: Start with the definition of asymptotic bias from Eq. (5.2).