Exercise 1: Bayesian Inference for Poisson model

A quality control engineer works at a factory that produces washing machines. The number of defective washing machines manufactured each day follows a Poisson distribution with an unknown rate parameter μ. The objective of the engineer is to estimate μ based on observed data and prior knowledge. Over the next 5 days, he observes the following number of defective products: $\mathbf{x} = (3, 0, 1, 5, 2)$.

(a) Assuming a *positive uniform* prior distribution for the parameter μ

- Determine and draw the posterior distribution for μ, given the data

- Evaluate mean, median and variance, both analytically and numerically in R

(b) Taking into account that historical data from similar factories suggest that the rate of defective washing machines is around 2 per day, with standard deviation of 1, model your belief using the suitable *Gamma* prior and then

- Determine and draw the posterior distribution for μ, given the data

- Evaluate mean, median and variance, both analytically and numerically in R.

(c) Evaluate a 95% credibility interval for the results obtained with different priors (you can also try with other priors). In addition, compare the results obtained using Gamma posterior distribution with a normal approximation that matches the same mean and standard deviation.

Exercise 2: Efficiency using Bayesian approach

A researcher `A` wants to evaluate the efficiency of detector 2 (Det2). For this purpose, he sets up the apparatus shown in the figure 1, where Det2 is sandwiched between Det1 and Det3. Let `n` be the number of signals recorded simultaneously by Det1 and Det3, and `r` be those also recorded by Det2, researcher `A` obtains `n` = 20 and `r` = 12.
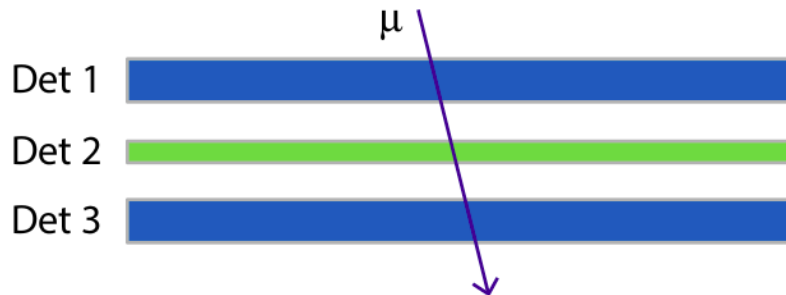

Figure 1:

Assuming a binomial model where `n` is the number of trials and `r` is the number of success out of `n` trials,

a) Evaluate the *mean* and the *standard deviation* using a Bayesian approach under the hypothesis of:

  – uniform prior $\sim \mathcal{U}(0,1)$
  – Jeffrey's prior $\sim \text{Beta}(1/2, 1/2)$

b) Plot the posterior distributions for both cases

Taking into account that the same detector has been studied by researcher B, who has performed only $n = 10$ measurements and has obtained $r = 10$ signals,

c) Evaluate the *mean*, the *standard deviation* and the *posterior* distribution using a uniform prior with the results of researcher B.

d) Repeat the computation of points a) and b) with the data of researcher A using as a prior the posterior obtained from point c).

e) Compute 95% credible interval using the posterior of the previous point d).

Exercise 3 - Bayesian Inference for Binomial model

• A coin is flipped $n = 28$ times with the following outcomes:

T, T, T, T, T, H, T, T, H, H, T, T, H, H, H, T, H, T, H, T, H, H, T, H, T, H, T, H

a) Assuming a flat prior, and a beta prior, plot the likelihood, prior and posterior distributions for the data set.

b) Evaluate the most probable value for the coin probability p and, integrating the posterior probability distribution, give an estimate for a 95% credibility interval.

c) Repeat the same analysis assuming a sequential analysis of the data. Show how the most probable value and the credibility interval change as a function of the number of coin tosses (i.e. from 1 to 28).

d) Do you get a different result, by analysing the data sequentially with respect to a one-step analysis (i.e. considering all the data as a whole) ?

Exercise 4 - Poll
A couple of days before an election in which four parties (A,B,C,D) compete, a poll is taken using a sample of 200 voters who express the following preferences 57, 31,45 and 67 for, respectively, parties A,B,C and D.
   Using a Bayesian approach, for all parties

• Calculate the expected percentage of votes and a 68% credibility interval by assuming as prior a

  – uniform prior
  – a prior constructed from the results obtained from another poll conducted the previous week on a sample of 100 voters who expressed the following preferences 32,14,26,28 for, respectively, parties A,B,C and D.