Physics of Life Data Epidemiology

Lect 8: Beyond homogenous mixing

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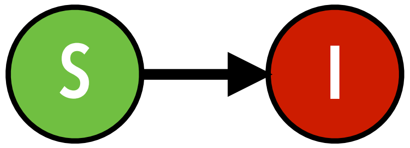
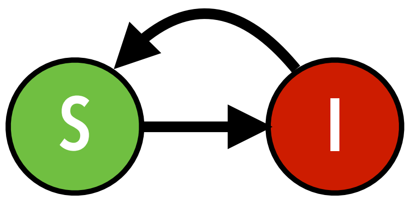
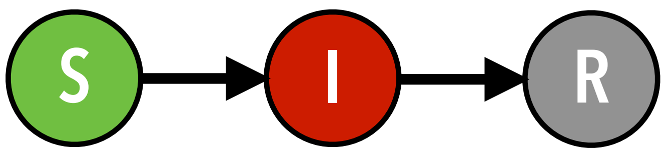
SI, SIS, SIR models



Epidemic expansion



Endemic circulation



Outbreak

SIR: Widely used in outbreak analysis because it captures all properties of an outbreak: initial exponential growth, peak, extinction before all individuals get infected

key simplifying assumptions:

-simplified disease natural history: only one infectious stage is considered; infectivity is constant form infectious to recovery; constant rate of transition from infectious to recovery

-homogenous mixing

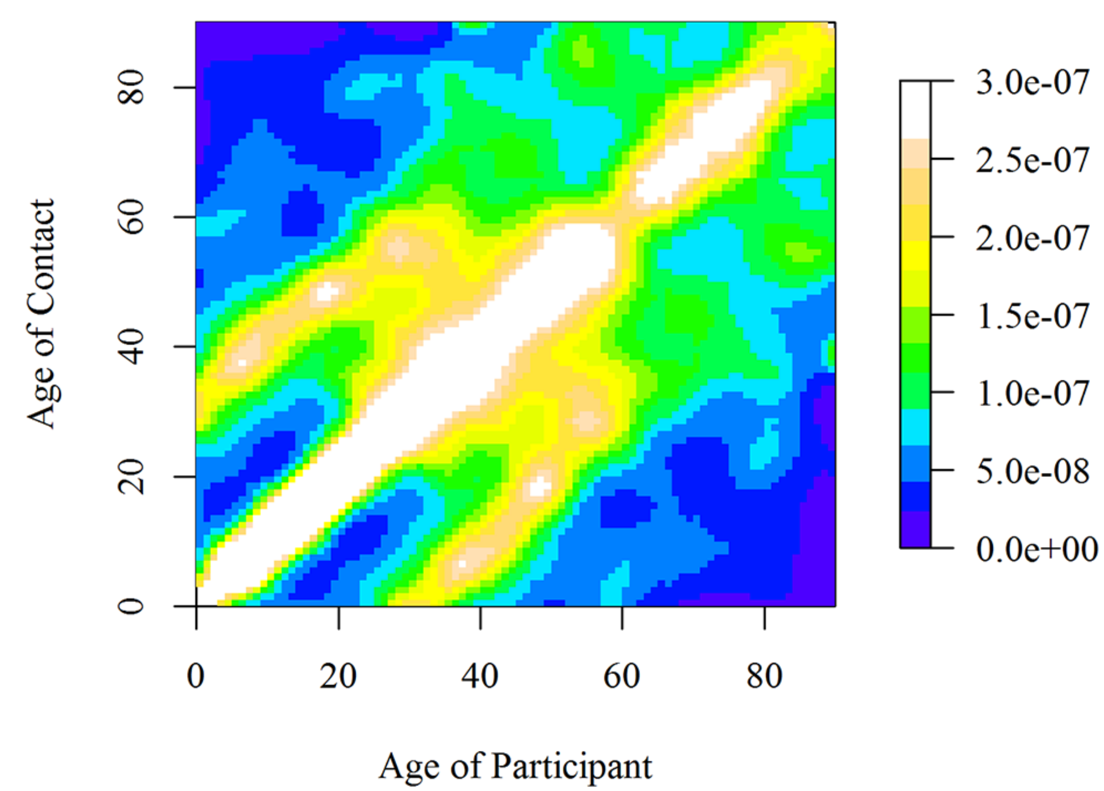
contact statistics

[POLYMOD STUDY, Mossong et al PLOS Med 2008]

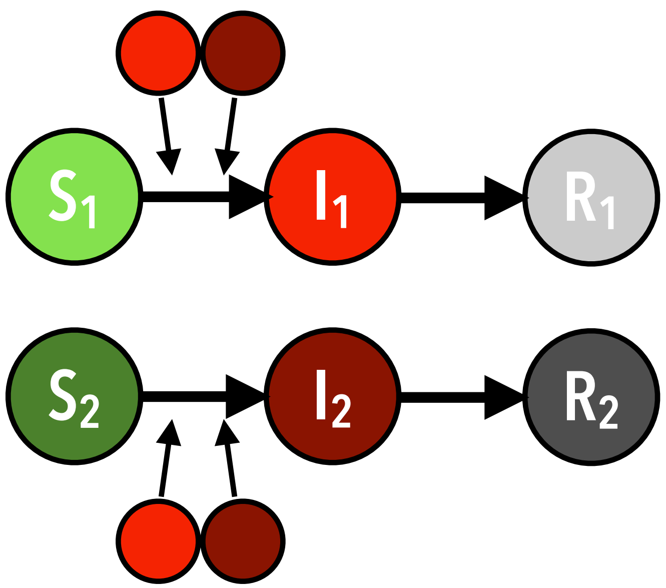
survey administered to thousands of people in many countries

how many people (by age bracket) did you met today? Did you met them at school, at work, at home in the community ?

typical European country



Risk structure models



$$\frac{ds_1}{dt} = -\beta_{11}s_1i_1 - \beta_{12}s_1i_2$$
$$\frac{di_1}{dt} = -\beta_{11}s_1i_1 - \beta_{12}s_1i_2 - \mu i_1$$
$$\frac{dr_1}{dt} = \mu i_1$$
$$\frac{ds_2}{dt} = -\beta_{21}s_2i_1 - \beta_{22}s_2i_2$$
$$\frac{di_2}{dt} = -\beta_{21}s_2i_1 - \beta_{22}s_2i_2 - \mu i_2$$
$$\frac{dr_2}{dt} = \mu i_2$$

Risk classes:

define classes according to infection risk (e.g. age).

This leads to non-homogeneous mixing

$$\begin{pmatrix}\beta_{11} & \beta_{12}\\ \beta_{21} & \beta_{22}\end{pmatrix} = \beta\begin{pmatrix}\alpha_1\gamma_1\langle k_{11}\rangle & \alpha_1\gamma_2\langle k_{12}\rangle\\ \alpha_2\gamma_1\langle k_{12}\rangle & \alpha_2\gamma_2\langle k_{22}\rangle\end{pmatrix}$$

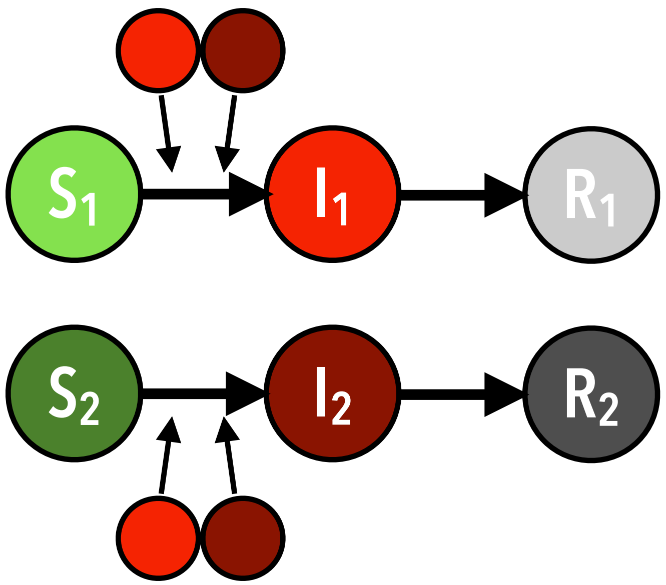
with

α_i = susceptibility

γ_i = infectiousness

$\langle k_{ij}\rangle$ = contacts between group i and j

Risk structure models



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Risk classes:

define classes according to infection risk (e.g. age).

This leads to non-homogeneous mixing

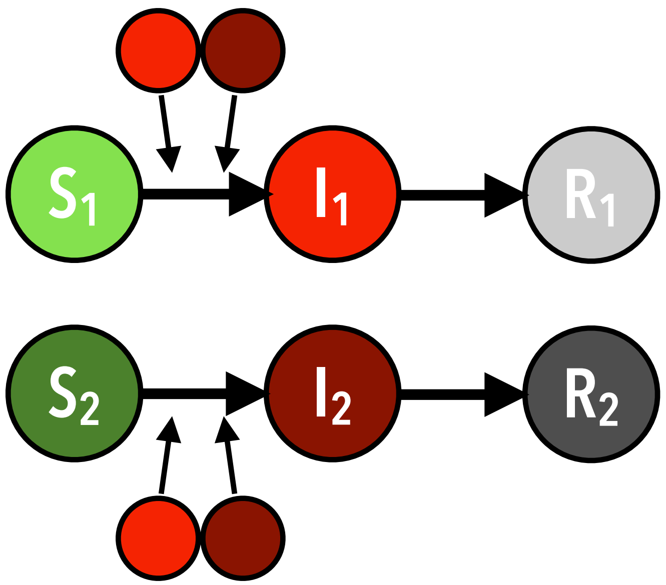
α_i = **susceptibility**

- Children are less susceptible to COVID-19
- Children tend to be more susceptible to flu

γ_i = **infectiousness**

- Children tend to develop milder symptoms for COVID-19. Infectiousness tends to be higher when symptoms are more severe→Children less infectious
- for flu no need to account explicitly for γ

Risk structure models



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dynamics in structured populations

we can compute the exponential growth from the dominant eigenvalue and R_0 from the exponential growth:

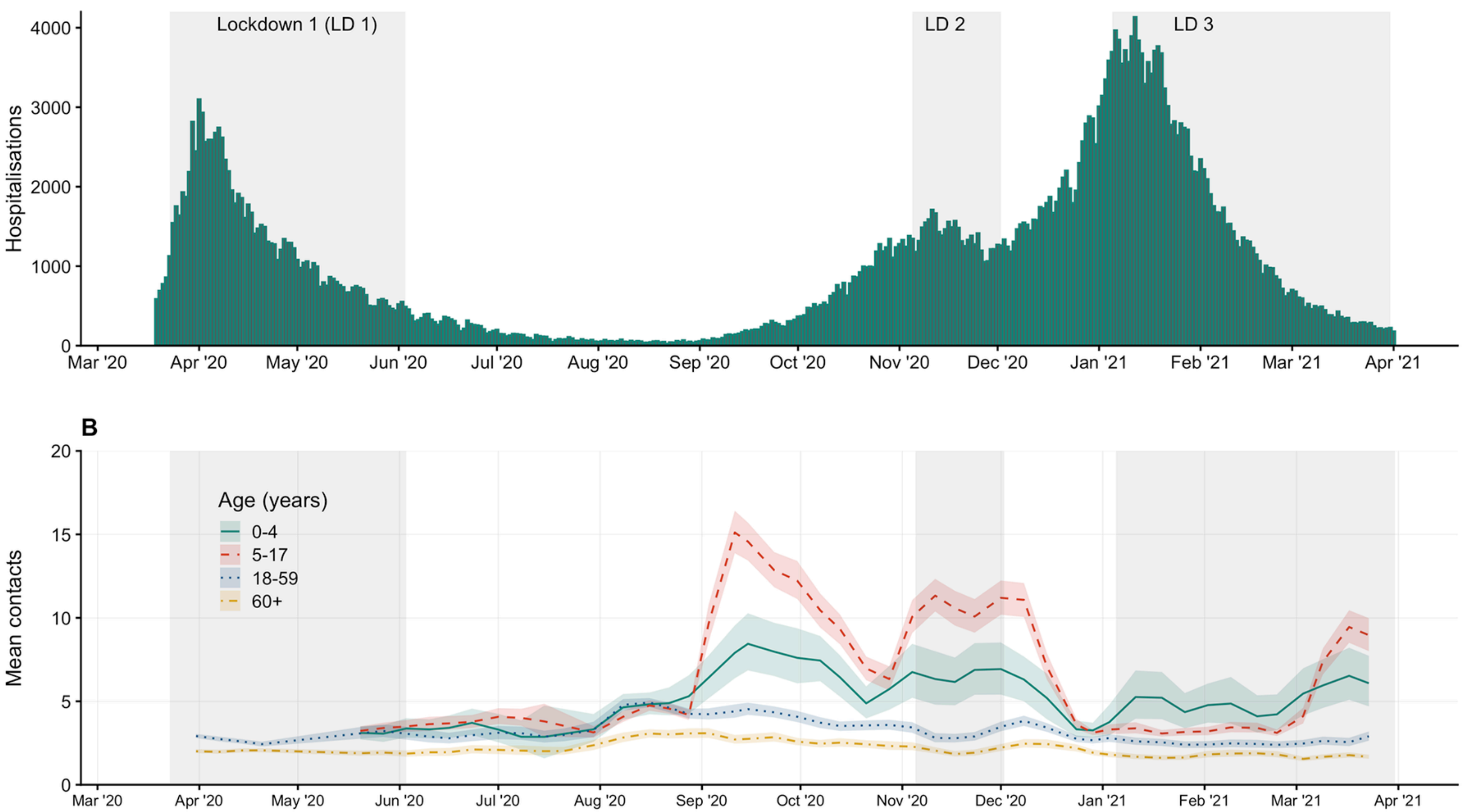
R_0 risk matrix (RM) > R_0 homogenous mixing (HM)

final attack rate RM < final attack rate HM

When the population is structured we can design targeted interventions

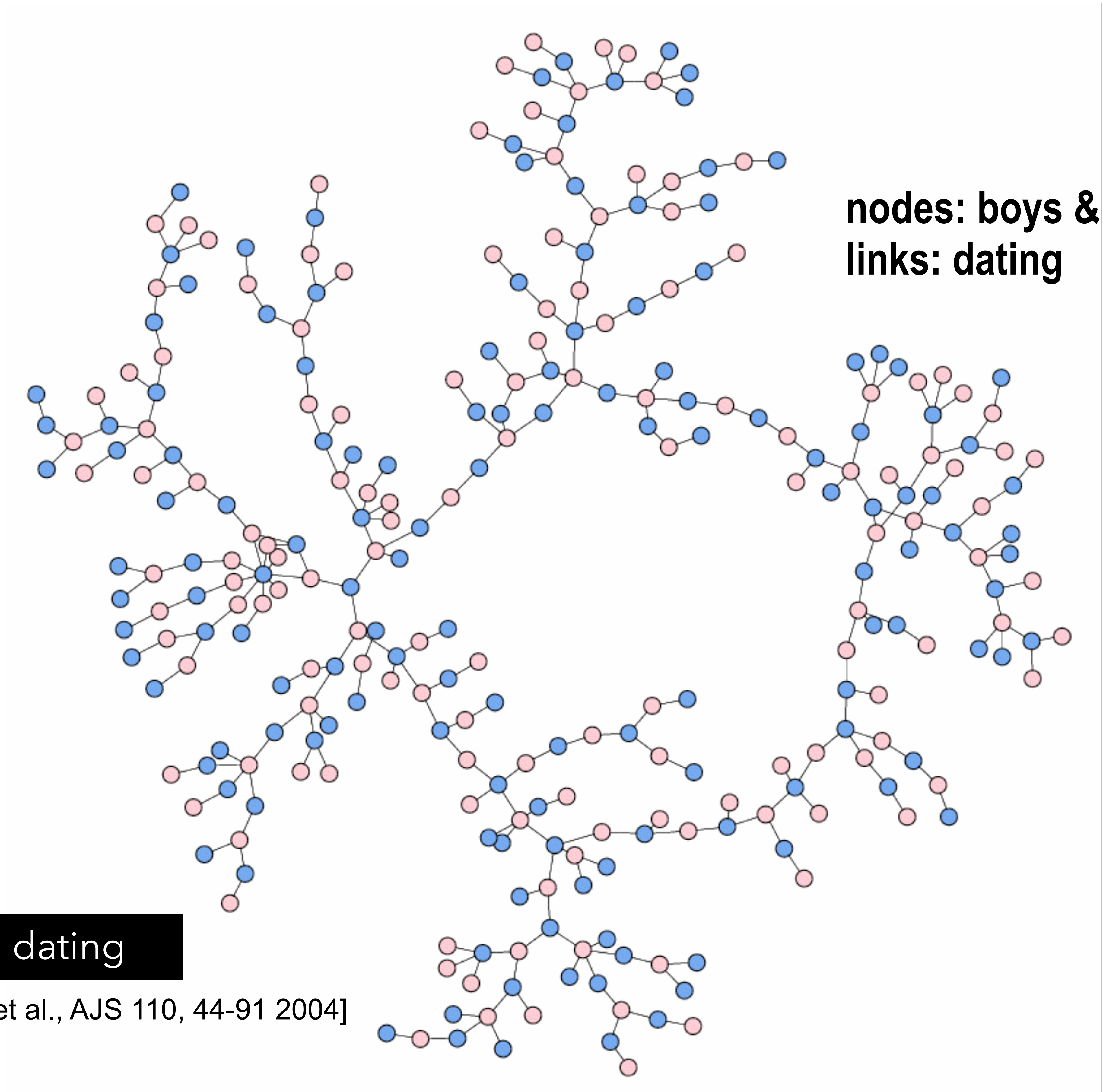
contact statistics

During the COIVD-19 pandemic: large scale contact surveys repeated in time
[**COMIX** Gimma et al, PLOS Med 2022, Verelst et al BMC Med 2021, http://www.socialcontactdata.org/socrates-comix/,
SocialCov Bosetti et al Euro Surveill 2021,
Koltai et al Sci Rep 2022, Zhang et al Science 2020, Feehan et al Nat Commun 2021]



[Gimma et al, PLOS Med 2022]

contact networks



nodes: boys & girls
links: dating

High school dating

[P. S. Bearman, et al., AJS 110, 44-91 2004]

contact networks

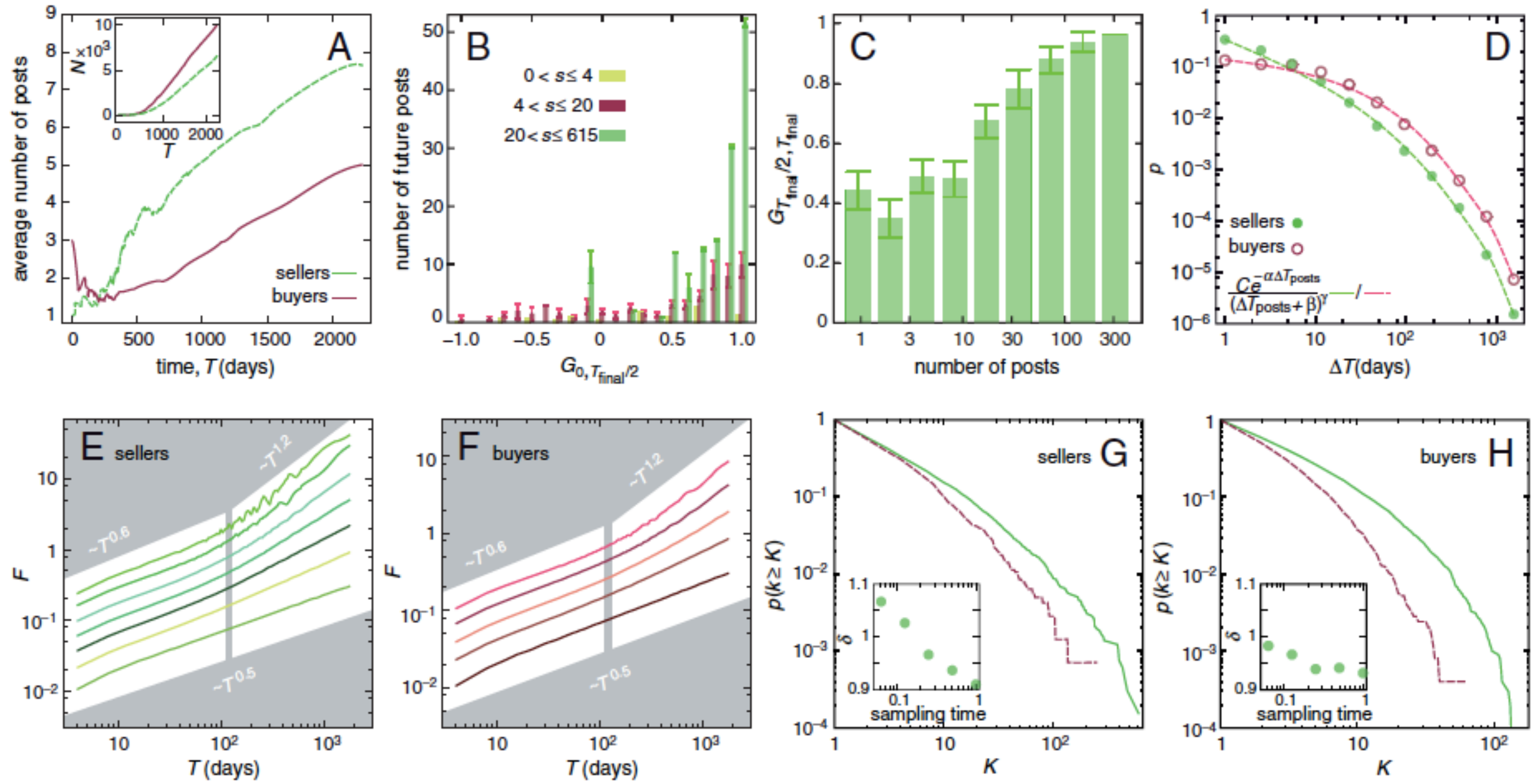


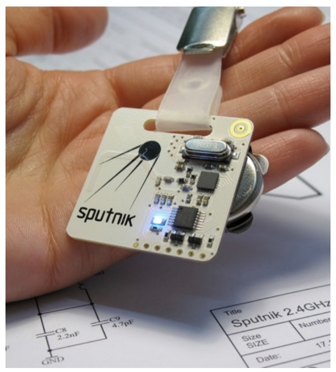
Fig. 1. Statistics of the dynamics of the community. (*A*) Time evolution of the average number of posts by sex buyers and about sex buyers. The *Inset* shows the growth in the number of sex sellers and sex buyers in the data. (*B*) The number of new posts according to the previous average grade at $T_{\mathrm{final}/2=1,116\ \mathrm{days}}$ for three different activity levels, or total number of posts, s. The R^2-values of these data are 0.19 ($0 < s \leq 4$), 0.29 ($4 < s \leq 20$), and 0.33 ($20 < s$). (*C*) The average future grade of sellers as a function of their number of contacts at half of the total sampling time (the data is logarithmically binned along the abscissa). (*D*) Shows the distribution of the time elapsed between two posts T_{posts} for buyers and sellers. Many posts were written during the same day, respectively, $p(T_{\mathrm{posts}} = 0) = 0.495$ and $p(T_{\mathrm{posts}} = 0) = 0.246$. The distributions are well fitted by $p(T_{\mathrm{posts}}) = C\exp(-\alpha T_{\mathrm{posts}}) = (T_{\mathrm{posts}} + \beta)^{\gamma}$, with: $C = 2.9 \pm 0.5$ days$^{\gamma}$, $\alpha = 0.0023 \pm 0.0001$ days^{-1}, $\beta = 3.1 \pm 0.4$ days, and $\gamma = 1.49 \pm 0.04$ (for sellers); and $C = 12 \pm 8$ days$^{\gamma}$, $\alpha = 0.0021 \pm 0.0002$ days^{-1}, $\beta = 18 \pm 4$ days, and $\gamma = 1.5 \pm 0.1$ (for buyers). (*E*) and (*F*) shows statistics the DFA fluctuation function as a function of the time-scale ΔT for sellers and buyers, resp. The different curves correspond to different activity levels—from bottom to top they represent less than 3, 3–7, 8–20, 21–54, 55–148, 149–403, and more than 403 posts (about sellers or from buyers) resp. *Black Lines* are inserted for reference. $T^{1/2}$ corresponds to uncorrelated interaction. (*G*) and (*H*) show degree distributions for sex sellers (*G*) and buyers (*H*) cumulative degree distributions for the full sampling time (*Solid Line*) and a yearlong window (starting one year after the full dataset; *Dashed Line*) for sex sellers and -buyers, resp. The *Insets* show the exponent of preferential attachment (Eq. 1).

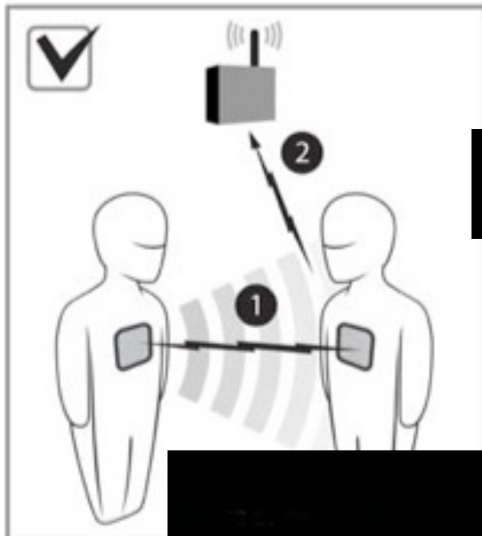
internet mediated prostitution

sexual contacts between 6,624 escorts and 10,106 sex buyers extracted from an online community

[LEC. Rocha, et al, PNAS 2009]

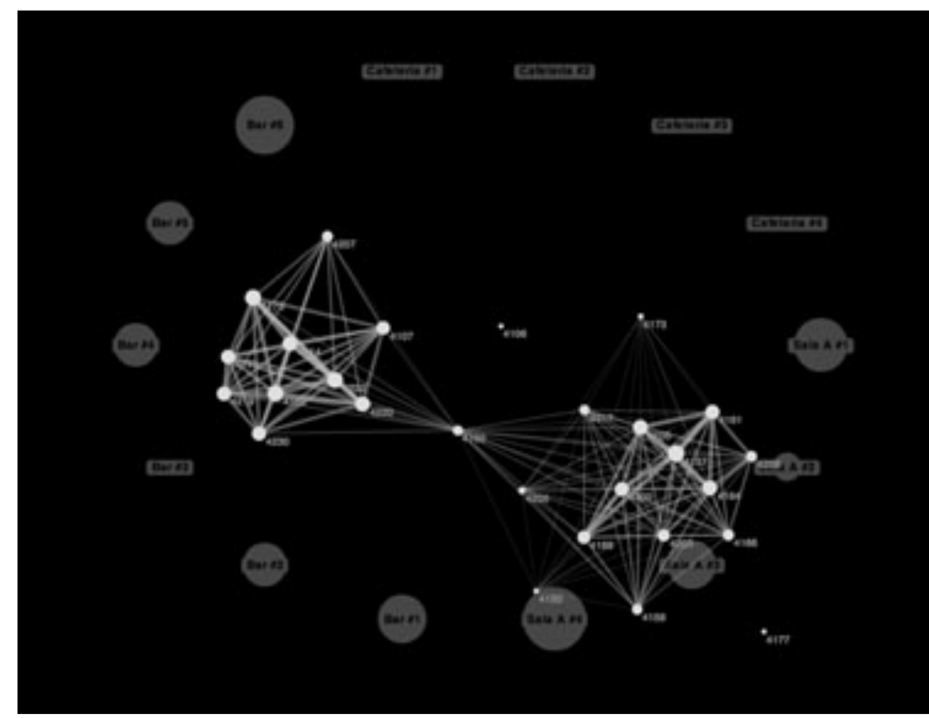
contact networks





face-to-face contacts

RFID technology



schools - workplaces - hospitals - museums - conferences -households - rural Africa

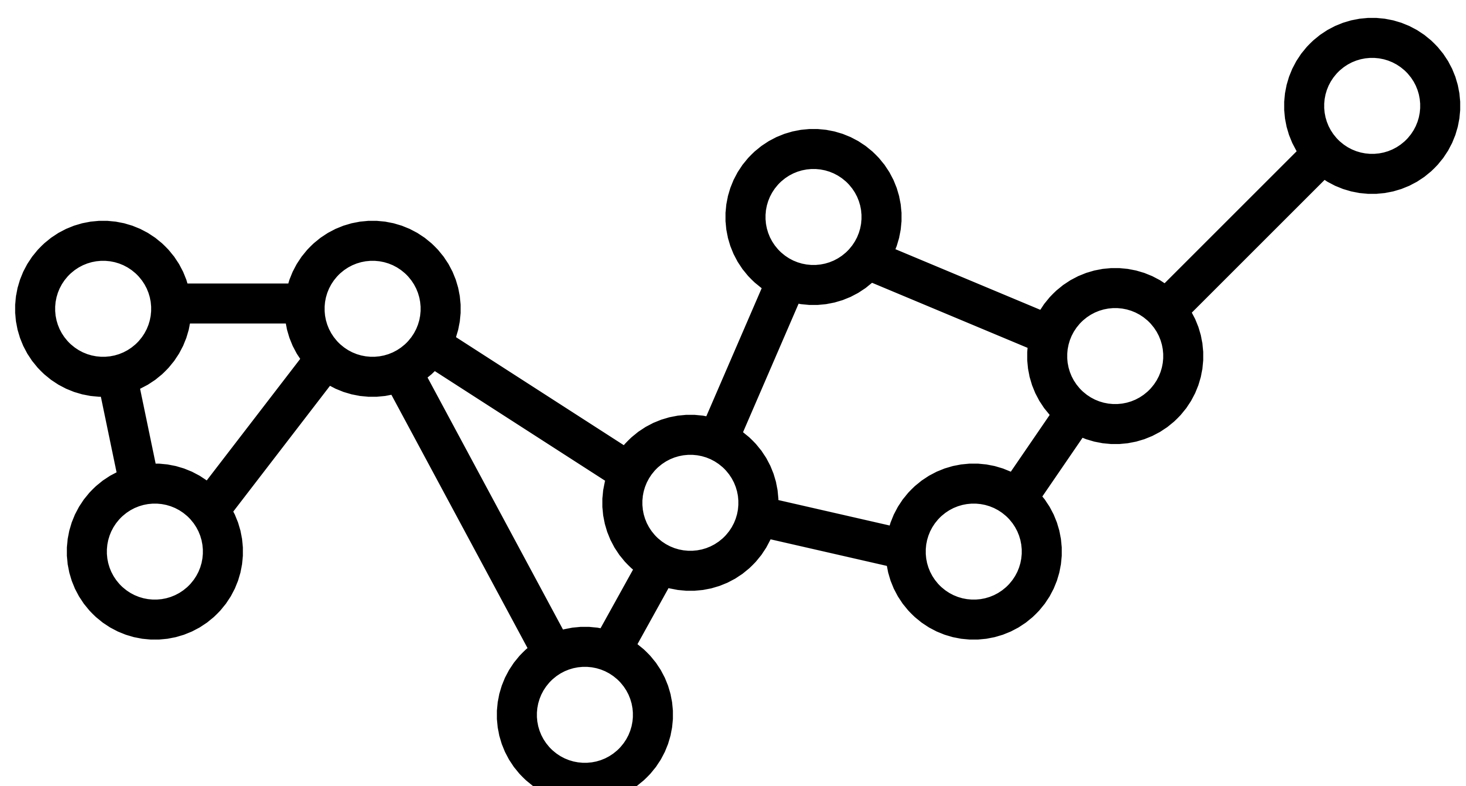
[Sociopatterns.org]

networks

[Networks, M.E.J. Newman, Oxford University Press (2018)]

[Dynamical processes on complex networks, A. Barrat, M. Barthélemy, A. Vespignani, Cambridge University Press, 2008]

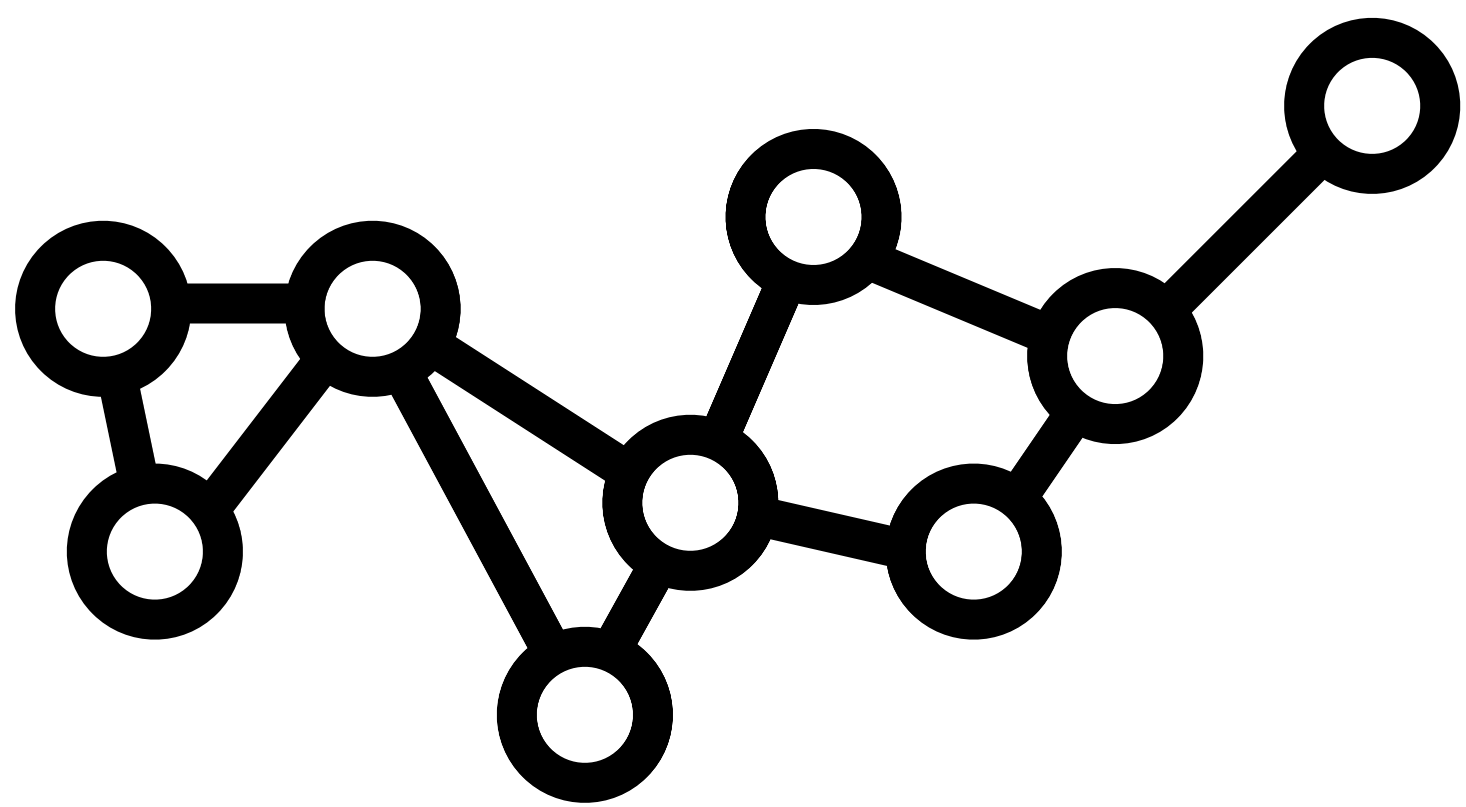
[Epidemic processes in complex networks, Romualdo Pastor-Satorras, Claudio Castellano, Piet Van Mieghem, and Alessandro Vespignani, Rev Mod Phys 87, 925, 2015]



networks

A Network (graph) G(V,E) is composed by a set of nodes (vertices) V and a set of links (edges) E

- Nodes: entities V = […,i,j,k,…]
- Links: relationships between entities E =[…,(i,j),(i,k),…]
- Number of nodes N
- Number of links L



networks

Links can be of different types and so networks:

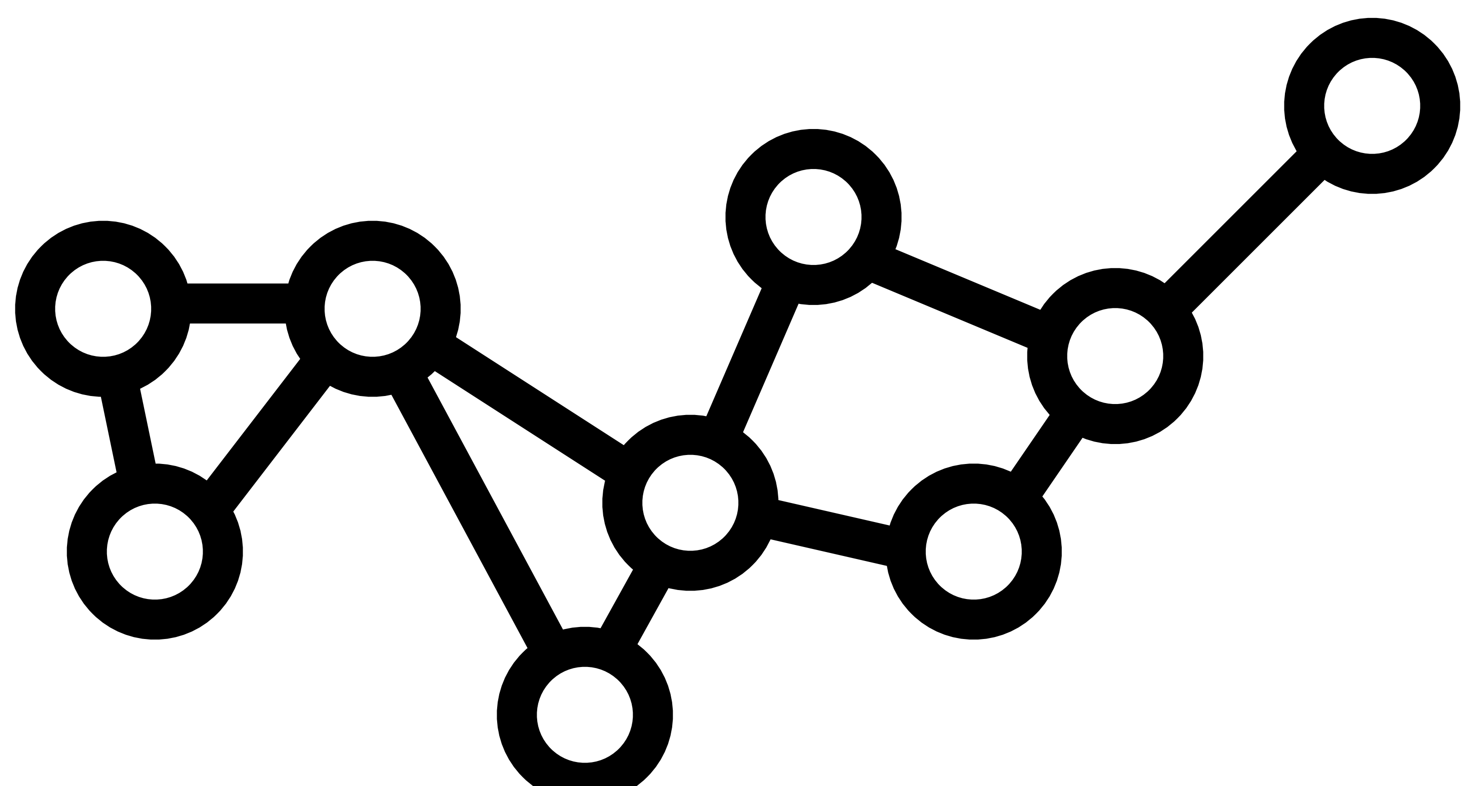
- Undirected/Directed
- Unweighted/Weighted

Network density (connectance)

- Fraction of links over all the possible pairs:

$$d = \frac{L}{N(N-1)}$$

- Real networks usually have a very low density ($L \ll N^2$) : sparse



networks

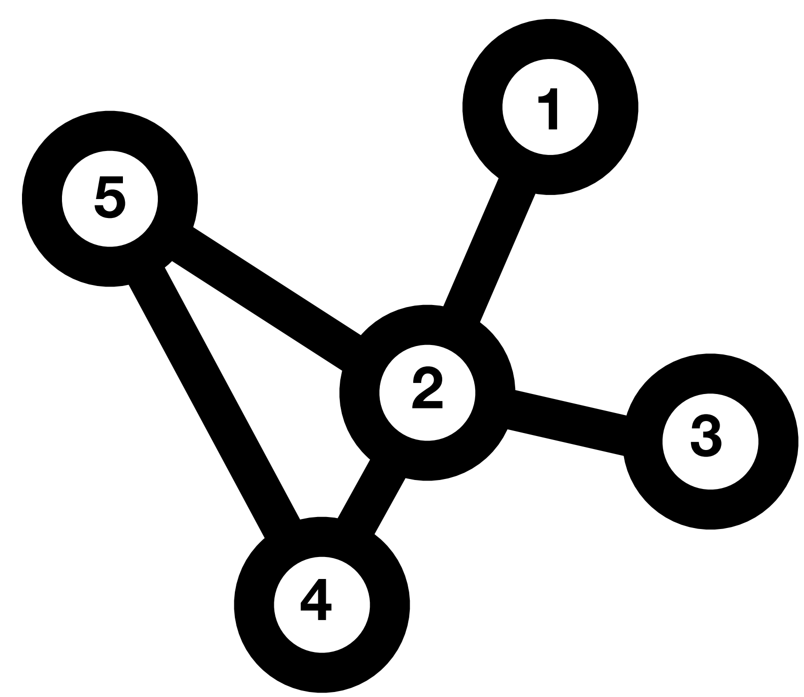
Representing a graph: the adjacency matrix A

- $a_{ij} = 1$ if a link between nodes i and j exists
- $a_{ij} = 0$ otherwise
- Symmetrical for undirected/unweighted graphs $a_{ij} = a_{ji}$

Mapping network physics to linear algebra

Not convenient for simulations for sparse networks, i.e. mostly composed by 0s (majority of real networks) → alternatives: adjacency lists, edge list.

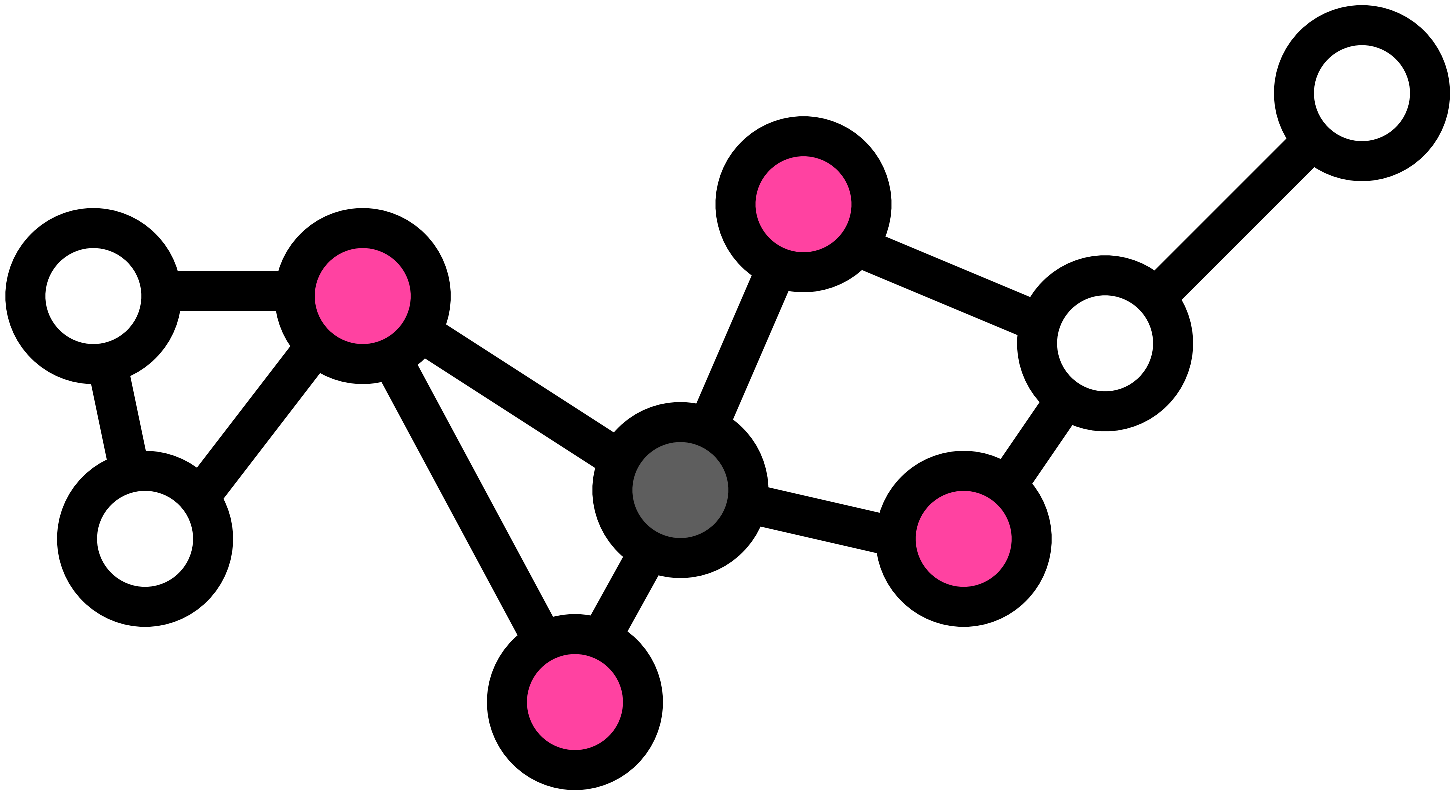
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$



networks

Neighbourhood of node i: the set of nodes connected to i

Degree of node i, k_i: Number of neighbours of node i. $k_i = \sum_j a_{ij}$

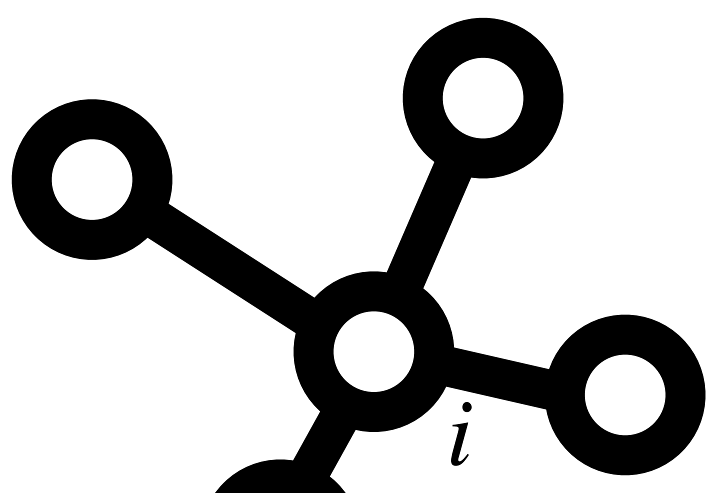


networks

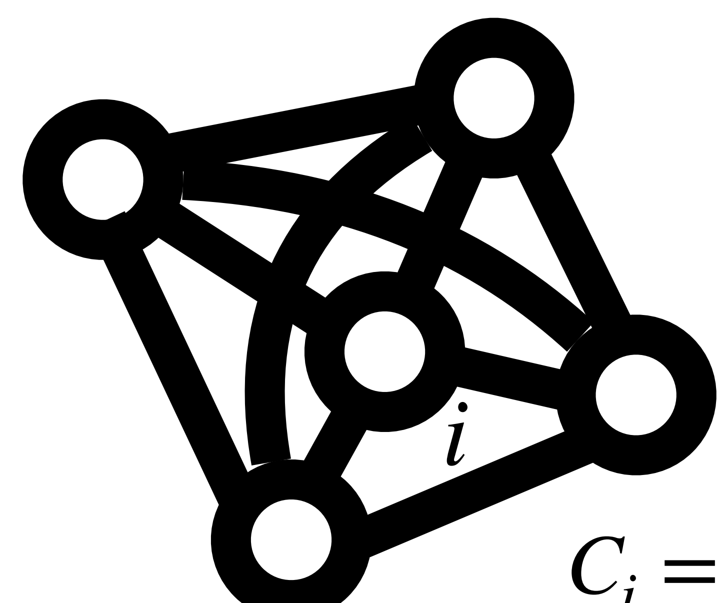
clustering coefficient of node i (C_i): number of edges connecting nodes of the neighbour of i divided by the maximum number of edges possible

$$C_i = \frac{E_i}{k_i(k_i - 1)/2},$$ where E_i is the number of edges in the neighbour of i

Average clustering coefficient $\langle C \rangle = \frac{1}{N} \sum_i C_i$



$C_i = 0$



$C_i = 1$

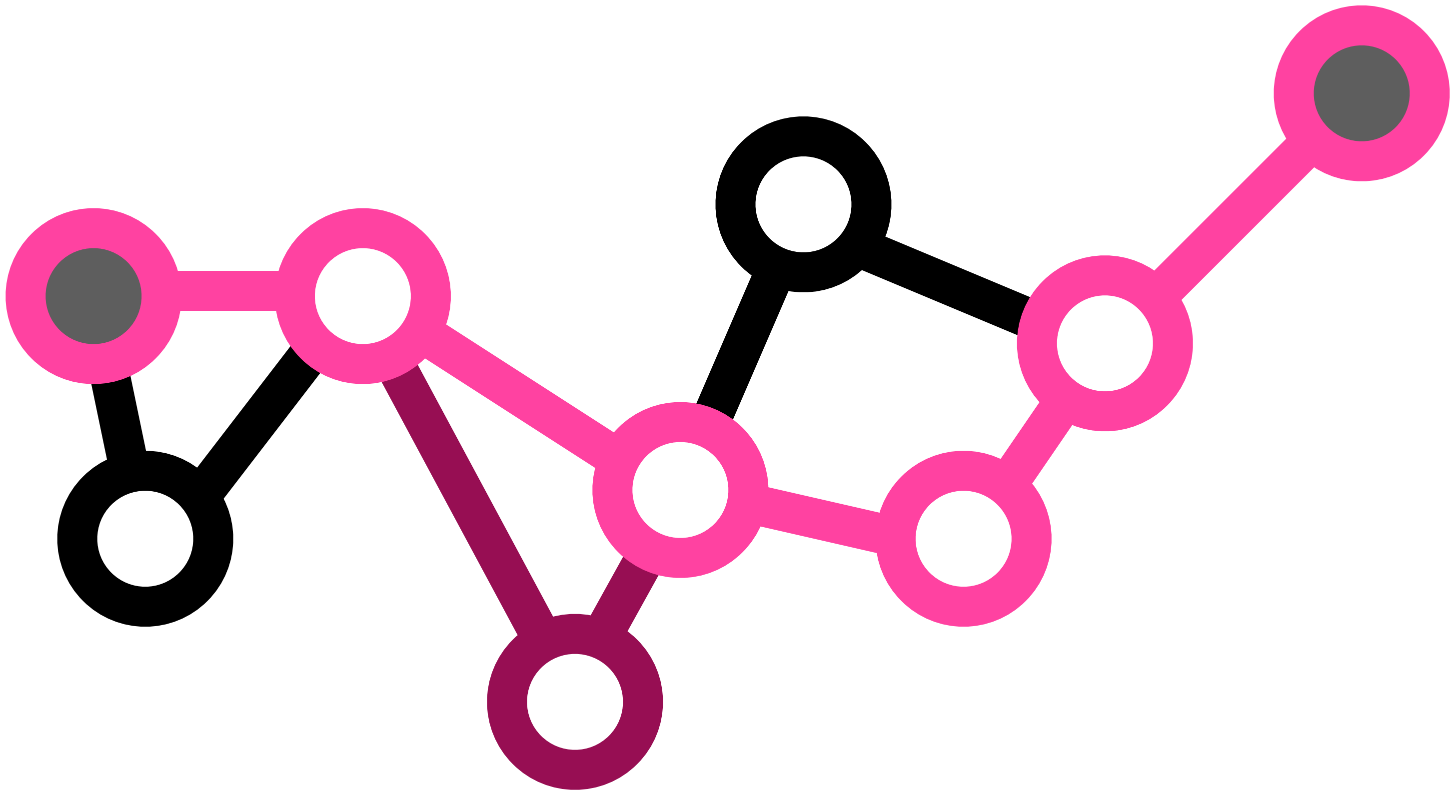
networks

Path: if it is possible to go from node i to node j following links

Shortest Path: a path covering the minimum number of links from *i* to *j*

Distance l_{ij}: length of the shortest path between *i,j*

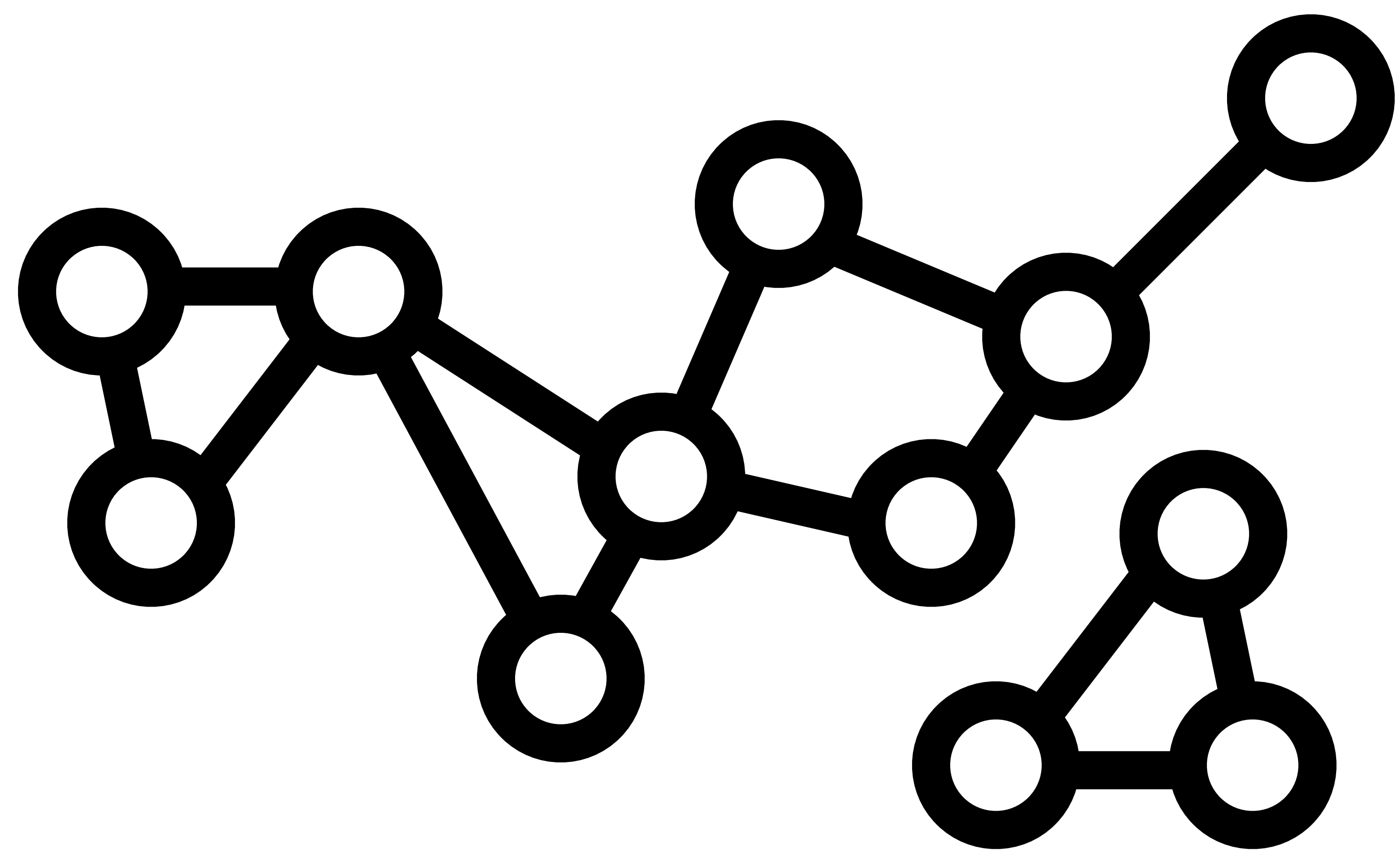
Diameter: The shortest path of maximum length in the network, i.e. $l_{\max} = \max_{ij} l_{ij}$



networks

connected network: when every possible couple of nodes is reachable through a path

connected component: connected part of a network



networks

The adjacency matrix encodes the whole information of the network. However, the information needs to be decrypted.

We characterize the network in terms of topological properties, and we study the implications of these properties on the dynamical processes on the top of the network.

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Eg:

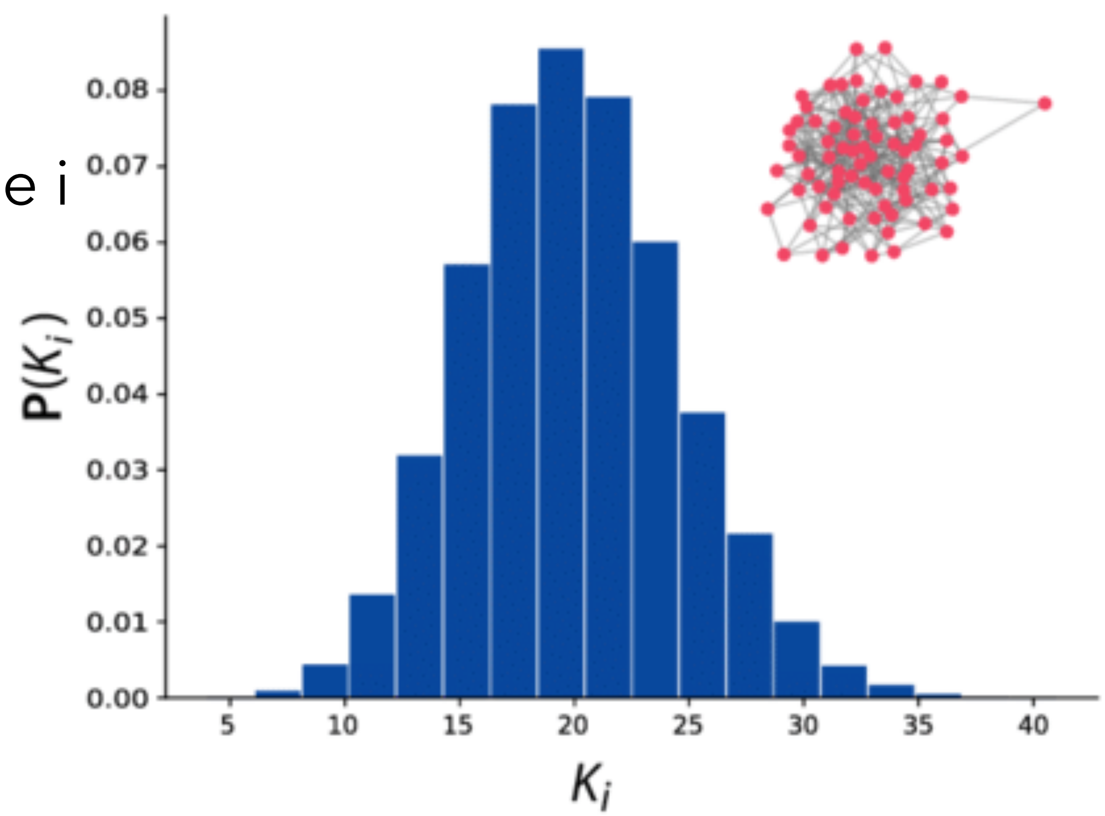
Degree of node i, ki: Number of neighbors of node i

What is the distribution of the degrees, $P(k)$?

Moments of the distribution?

e.g. average degree:

$$\langle k \rangle = \frac{1}{N}\sum_{i=1}^{N} k_i \text{ or } \langle k \rangle = \frac{2L}{N} = d(N-1)$$



networks

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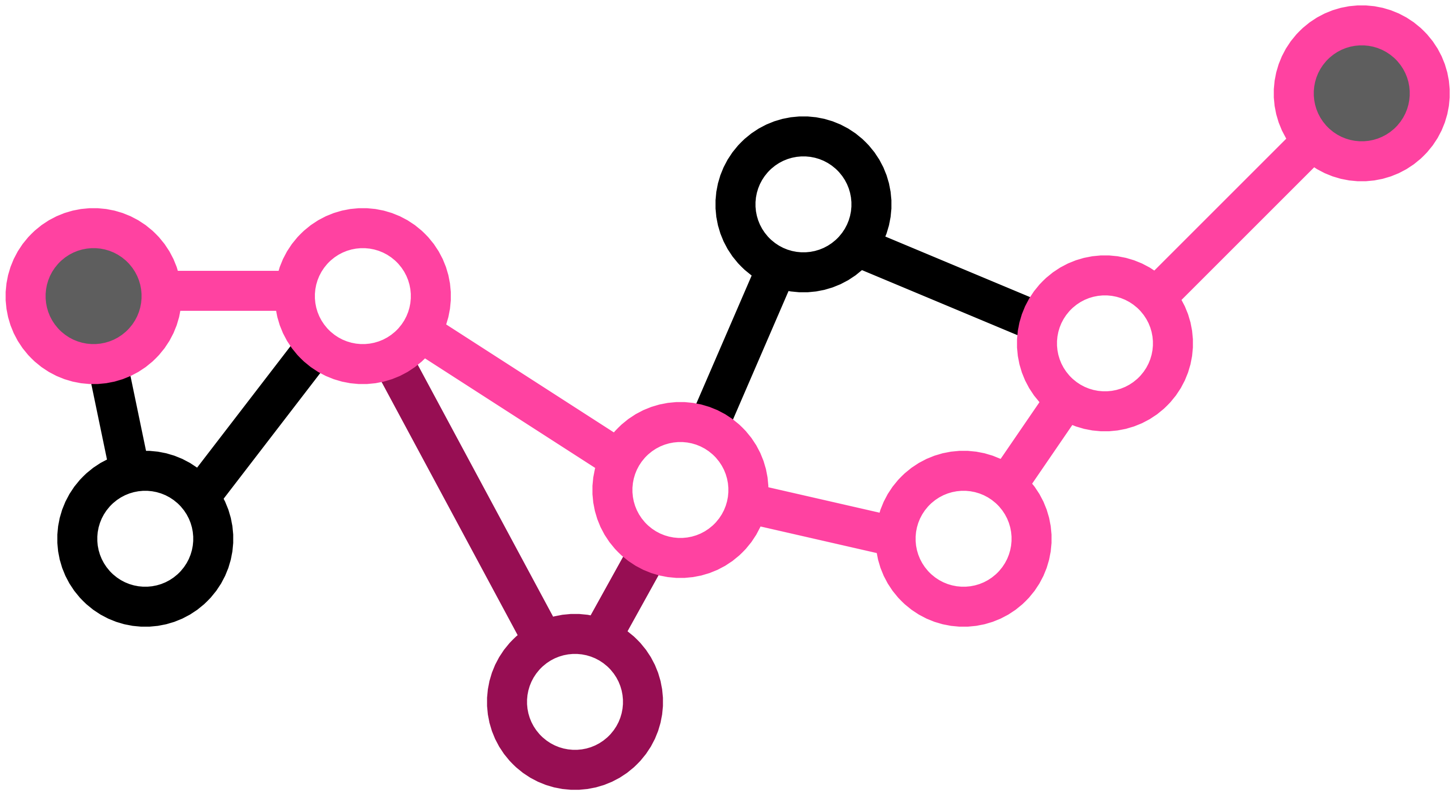
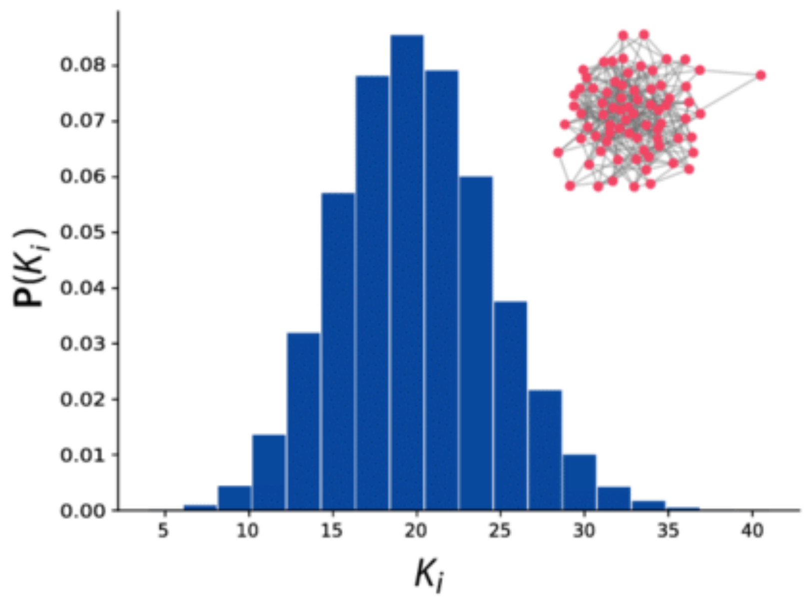
Network models: synthetic random networks (network ensembles) with specific properties



Erdös and Rényi Model

[Erdős, P.; Rényi, A. Publicationes Mathematicae. 6: 290–297 (1959)]



Links between nodes are drawn at random

Graph G with $G(N, p)$

- N number of nodes
- p probability of connection

Algorithm:

```
- Create an empty graph with N nodes
- Connect each possible couple of nodes
  with probability p
- Avoid self-loops and multiple edges
```

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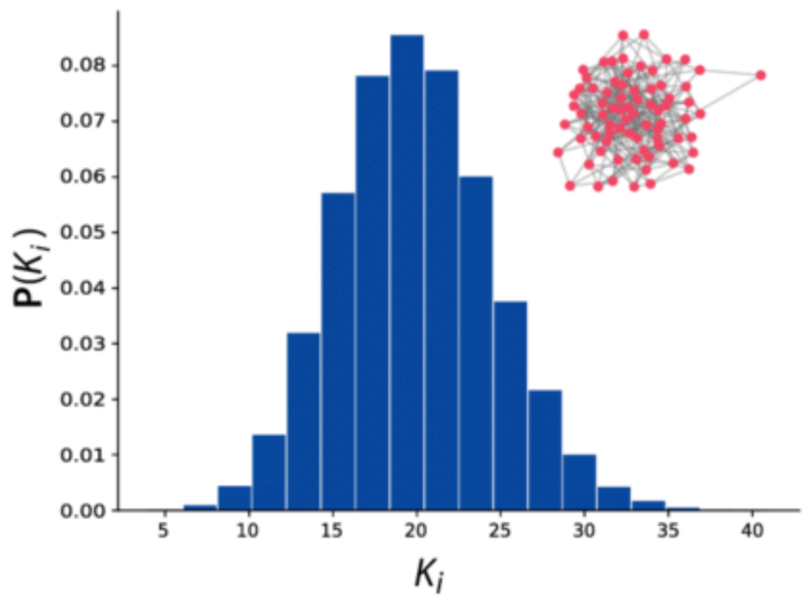

How degrees are distributed over the network?

P(k) = ?

If links are drawn at random with probability p

Probability that a node has k neighbours p_k is given by a binomial distribution:

$$p_k = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

Average degree $\langle k \rangle = p(N-1)$

Variance $\sigma_k^2 = p(1-p)(N-1)$

For sparse networks we have $k \ll N$. Then the Binomial (N,p) distribution can be approximated by a Poisson distribution with ($\lambda = pN$)

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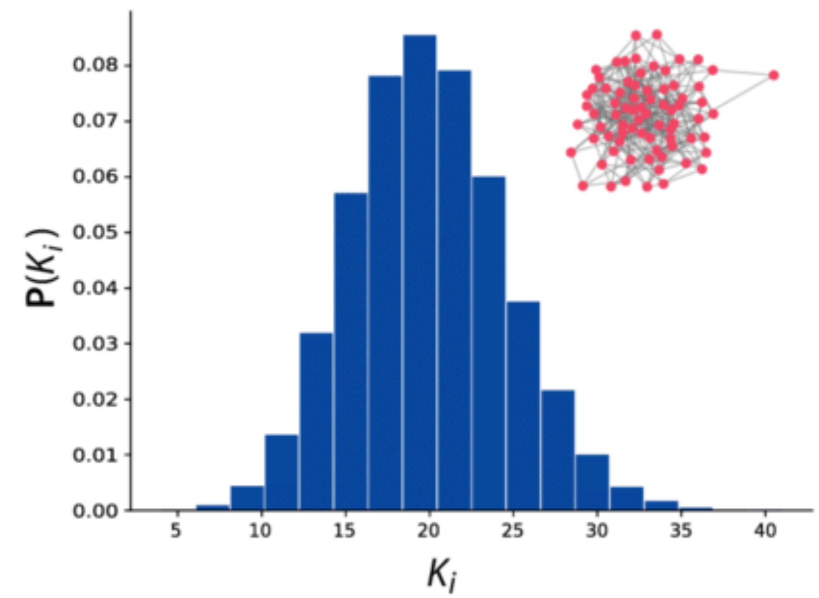

a node is connected to k nodes

at a 2 steps distance it has $k(k-1)$ nodes

at a 3 steps distance it has $k(k-1)^2$ nodes

node within a distance l:

$$k + k(k-1) + \ldots k(k-1)^{l-1} = k\frac{(k-1)^l - 1}{k-2} \simeq (k-1)^l$$

$$(k-1)^l = N - 1 \Rightarrow l \simeq \frac{\log N}{\log k}$$

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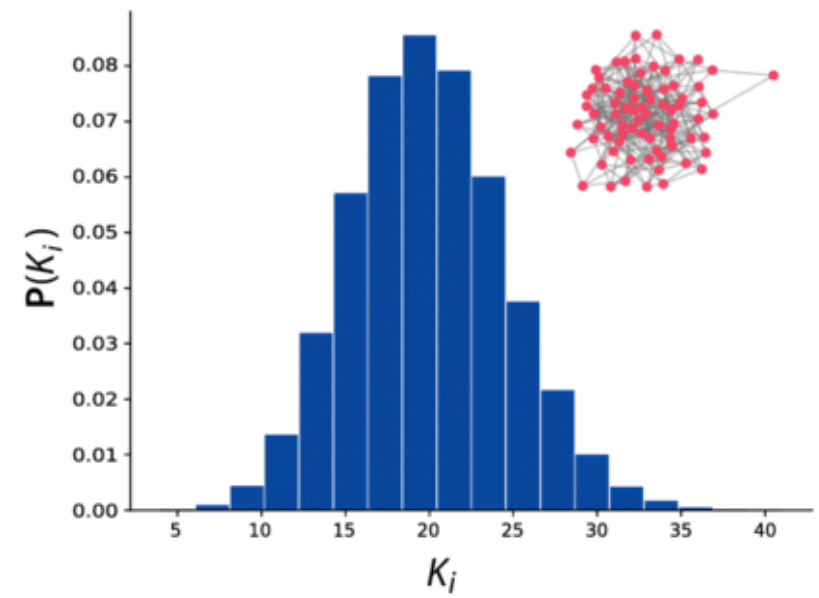

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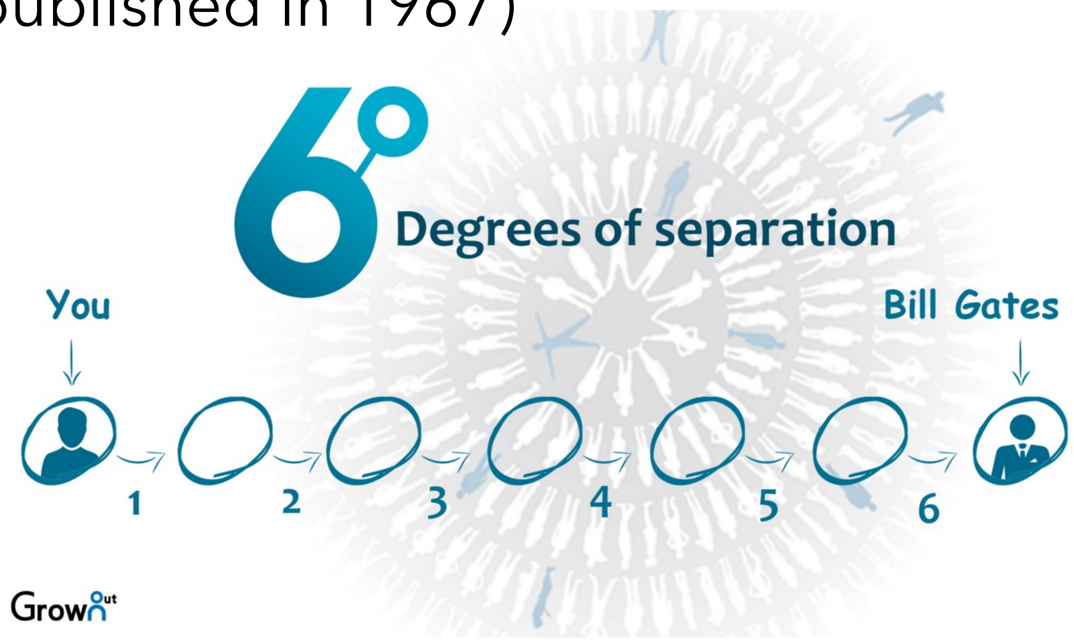
$$(k-1)^l = N - 1 \Rightarrow l \simeq \frac{\log N}{\log k}$$

diameter of the order of $\log N$

- **Cover the entire network in few steps**
- **efficient diffusion and spreading of a disease**

key property of social networks

Six degrees of separation (First introduce in a novel in 1929, Milgram experiment published in 1967)



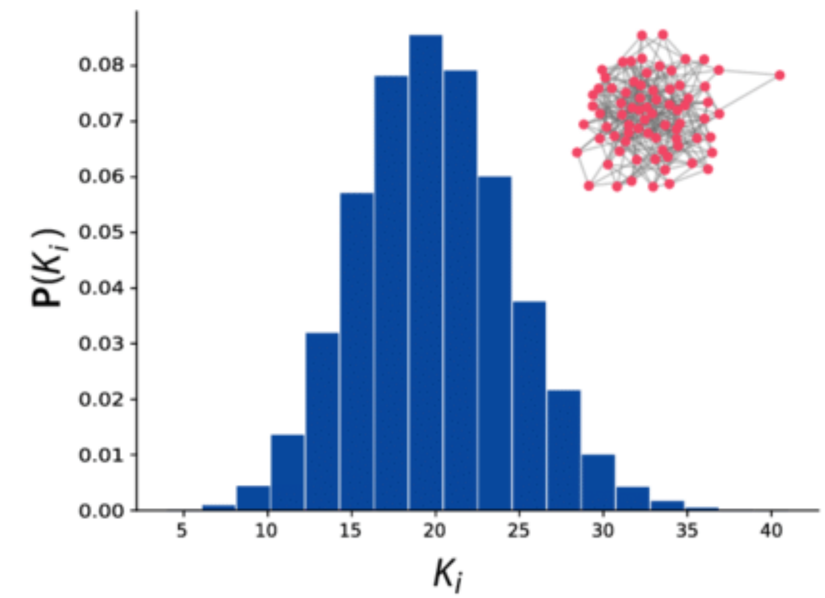
From: https://medium.com/ent101/six-degrees-of-separation-9d6f33854e9c

Real networks are smaller than one would expect

Erdös and Rényi Model

[Erdős, P.; Rényi, A. Publicationes Mathematicae. 6: 290–297 (1959)]

$$\langle C \rangle = p = \frac{\langle k \rangle}{N}$$



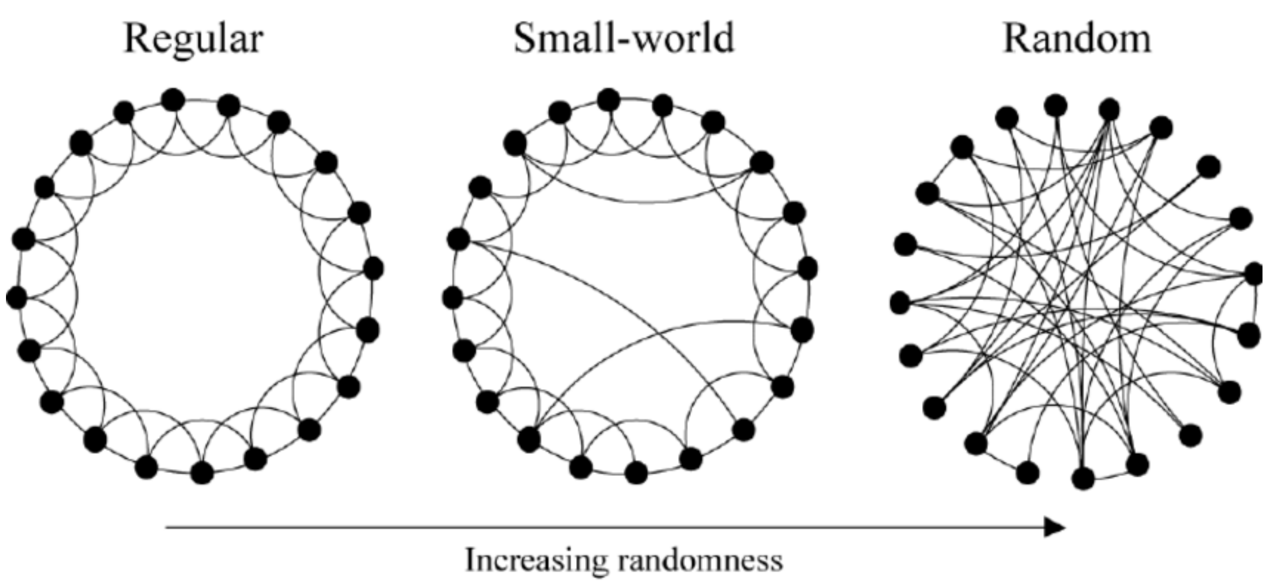
Real networks are smaller than one would expect

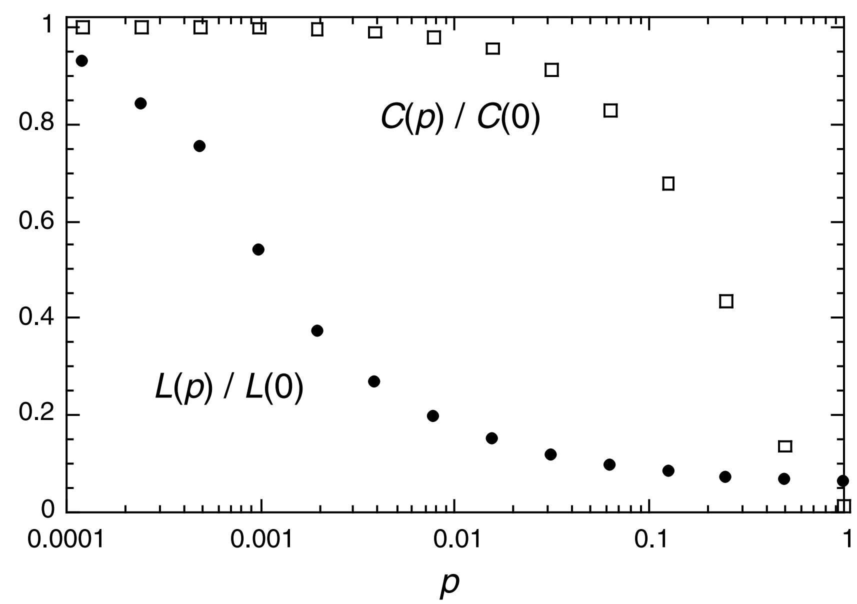
Erdös and Rényi Model realistically describes small world networks .. but it does not capture another property of real networks, **high clustering**

my friends are in general friends among each other

Small world network

[Watts, D., Strogatz, S. Collective dynamics of ‘small-world’ networks. *Nature* **393,** 440–442 (1998). https://doi.org/10.1038/30918]



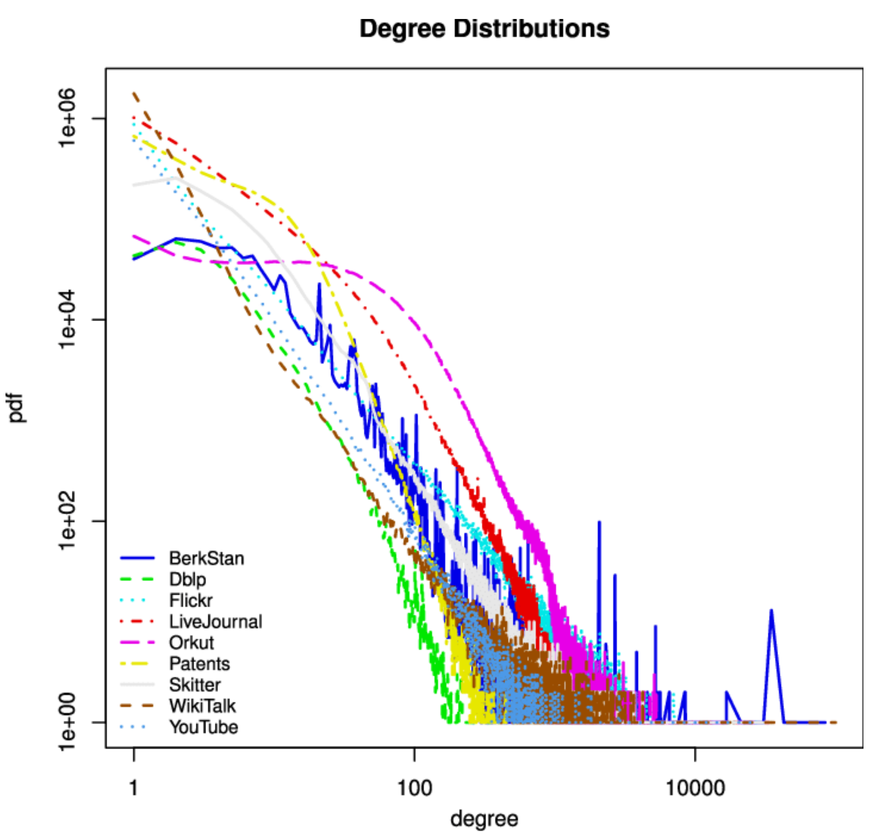


rewiring few links is enough to shorten the diameter of the graph.

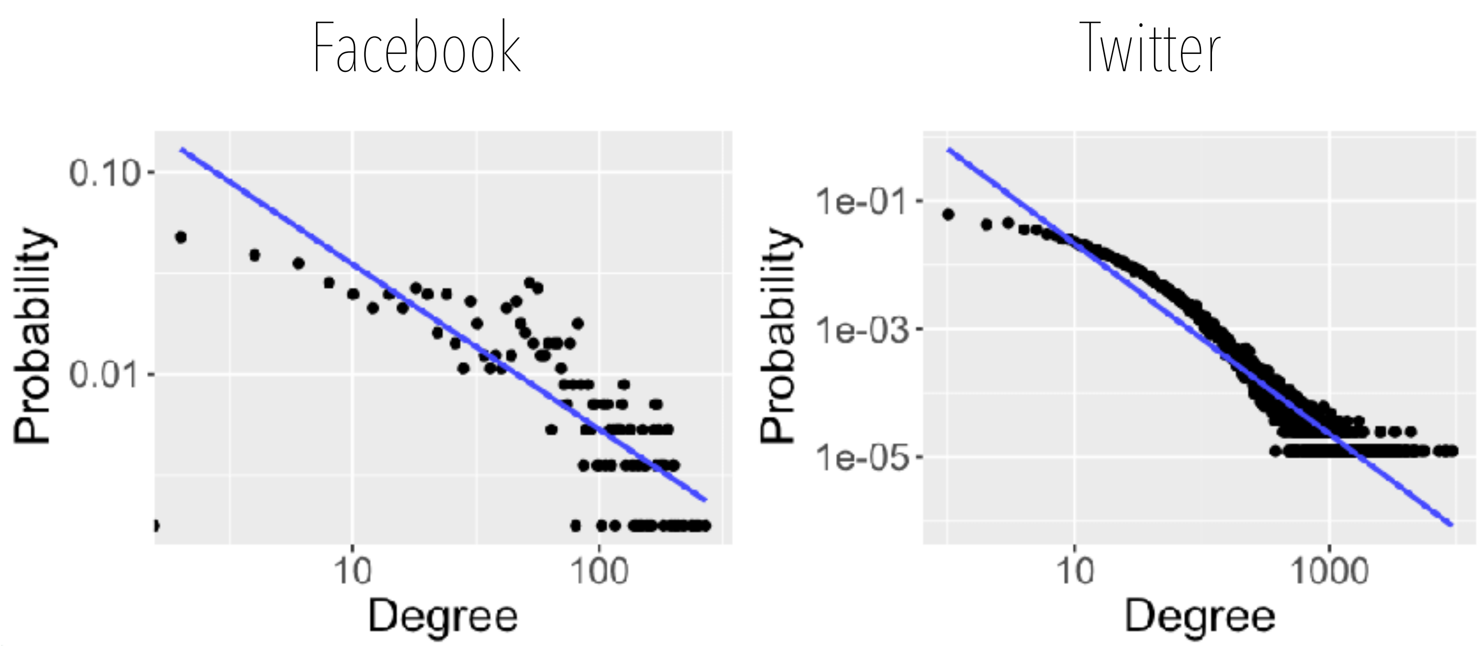
At a local level the transition to small world is almost undetectable

another key property of social networks

What is the degree distribution of real networks?



From Aksu, Hidayet et al. Distributed Core View Materialization and Maintenance for Large Dynamic Graphs. Knowledge and Data Engineering, IEEE Transactions on. 26. 2439-2452 (2014).

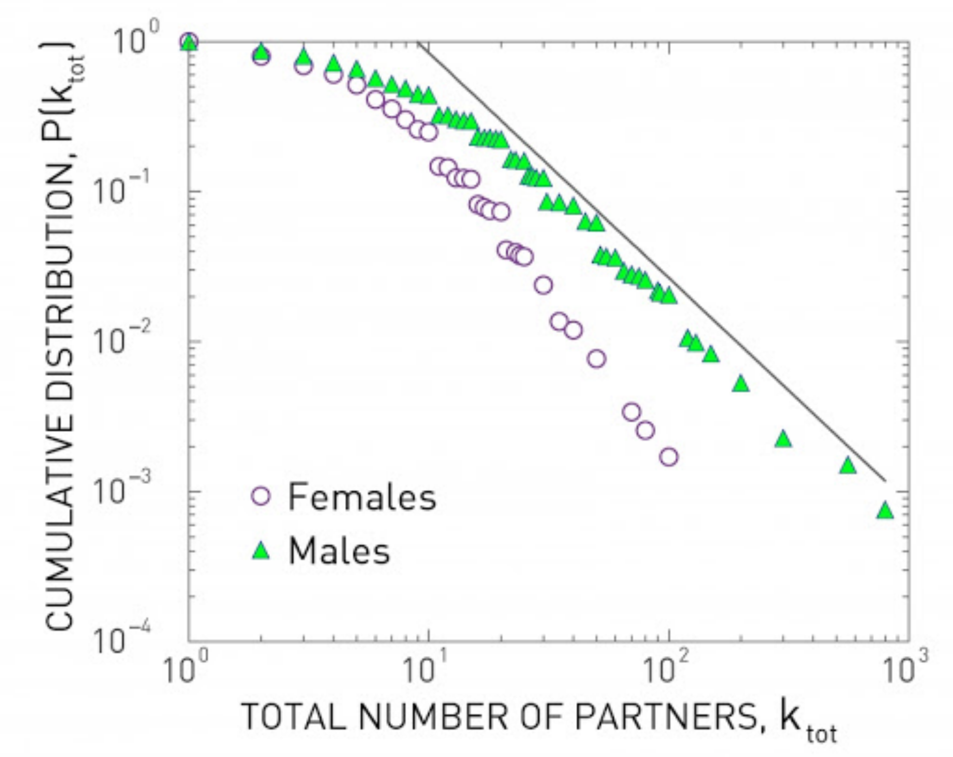


From Wilson, James & Uminsky, David. (2017). The power of A/B testing under interference.

another key property of social networks

What is the degree distribution of real networks?

Sexual partners 1996 study in Sweden

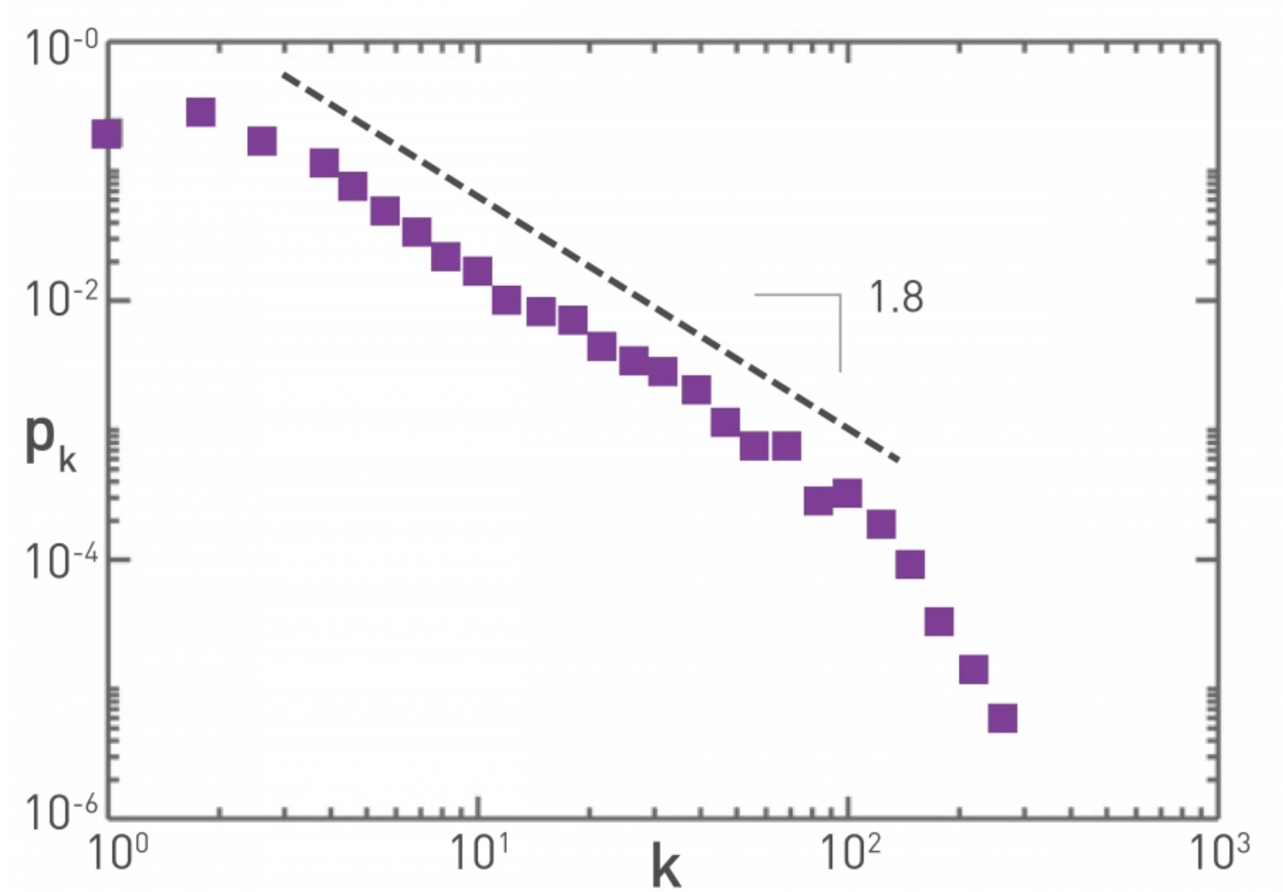


[B. Lewin. (ed.), Sex i Sverige. Om sexuallivet i Sverige 1996 [Sex in Sweden. On the Sexual Life in Sweden 1996]. National Institute of Public Health, Stockholm, 1998.]

another key property of social networks

What is the degree distribution of real networks?

Air transportation network



[Barrat et al PNAS 2004]

another key property of social networks

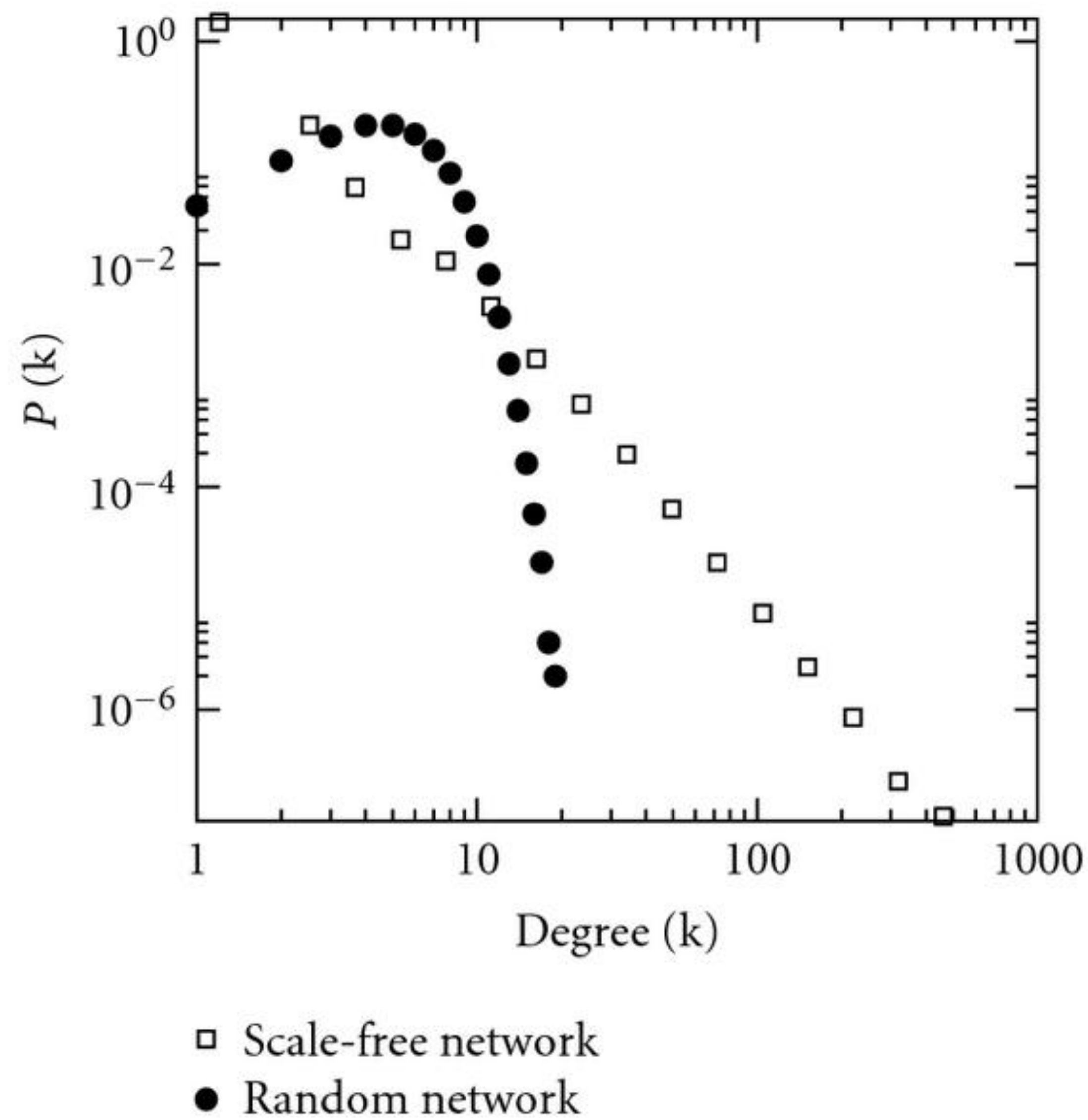
What is the degree distribution of real networks?

Real networks are not Poisson

In most contexts real networks are highly heterogenous

Heavy-tailed distributions:

i.e. power-law $P(k) \sim k^{-\gamma}$



[Danon, Leon et al. (2011). Networks and the Epidemiology of Infectious Disease. Interdisciplinary perspectives on infectious diseases. 2011. 284909. 10.1155/2011/284909]

Power low degree distribution

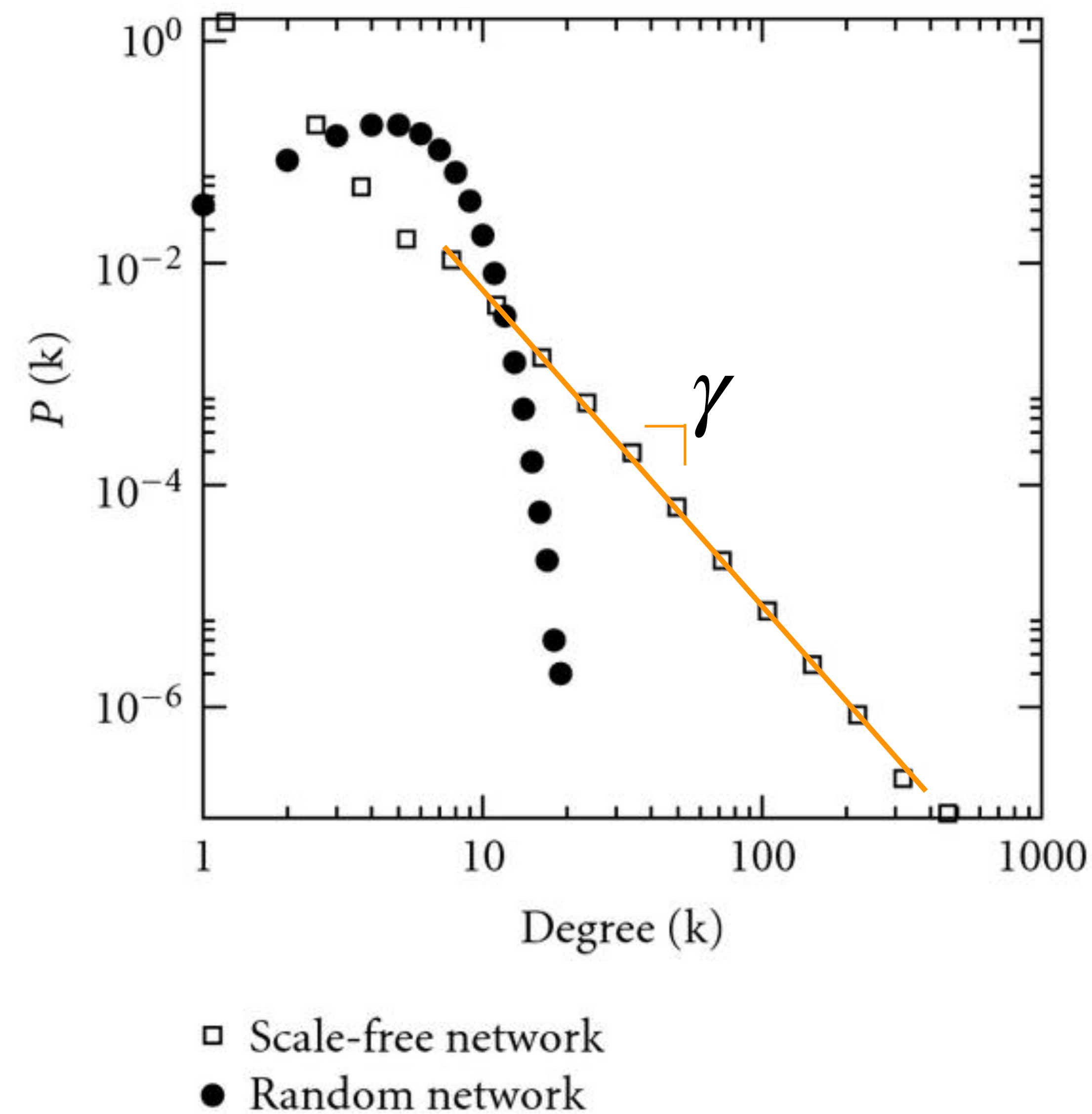
Heavy-tailed distributions: i.e. power-law

$P(k) \sim k^{-\gamma}$

- γ slope of the distribution.
- Most real networks have small values of γ i.e. $\gamma \leq 3$

In power low networks:

- Most of the nodes have a very low connectivity: less than a random net
- The probability of having very large degrees is not zero → *Hubs*
- hubs connect make shortcuts. In some cases Ultra-Small-World $\langle l \rangle \sim \ln(\ln(N))$. Important for epidemic spreading: shortcuts for spreading; super-spreaders



[Danon, Leon et al. (2011). Networks and the Epidemiology of Infectious Disease. Interdisciplinary perspectives on infectious diseases. 2011. 284909. 10.1155/2011/284909]

Power low degree distribution

Power-law degree distribution: $P(k) = C_0 k^{-\gamma}$ with $C_0 = (\gamma - 1)k_{min}^{\gamma-1}$

The general nth-moment of $P(k)$ is $\langle k^n \rangle = \int_{k_{min}}^{\infty} k^n P(k) dk = \int_{k_{min}}^{\infty} C_0 k^{n-\gamma} dk$

→ it converges only if $\gamma - 1 > n$

Remember that $\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2$

If $\gamma < 2$ both $\langle k \rangle$ and $\langle k^2 \rangle$ diverge with $N \to \infty$

If $2 < \gamma < 3$ the average degree $\langle k \rangle \to c$ but $\langle k^2 \rangle \to \infty$ as $N \to \infty$ BUT

$\sigma^2 \to \infty$ → **scale free network**

$\langle k \rangle$ becomes less relevant

Barabási-Albert Model

[Emergence of Scaling in Random Networks. Albert-László Barabási, Réka Albert. Science. 286, 5439, pp. 509-512, (1999)]

How to create a random scale-free network?

Generative algorithm:

- At each time-step a new node enters the network and connects with pre-existing nodes at random
- **preferential attachment**: the probability that the entering node connects with pre-existing nodes is proportional to their degree

Result: $P(k) \sim k^{-3}$

Barabási-Albert Model

[Emergence of Scaling in Random Networks. Albert-László Barabási, Réka Albert. Science. 286, 5439, pp. 509-512, (1999)]

Preferential attachment is a reformulation of the **Rich get richer** or **Matthew effect** (‘60s)

Based on

- the Price’s model (prob of getting a new citation for a paper proportional to its citations, *cumulative advantage*)

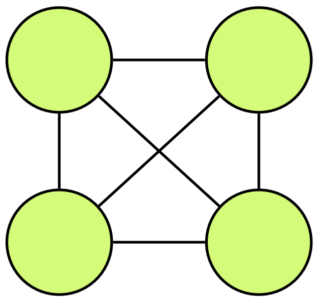
[Journal of the American Society for Information Science. 27(5): 292–306, 1976]

- Simon model in probability theory

[Simon, Herbert A. (1955). "On a Class of Skew Distribution Functions". Biometrika. Oxford University Press (OUP). 42 (3–4): 425–440. doi:10.1093/biomet/42.3-4.425. ISSN 0006-3444]

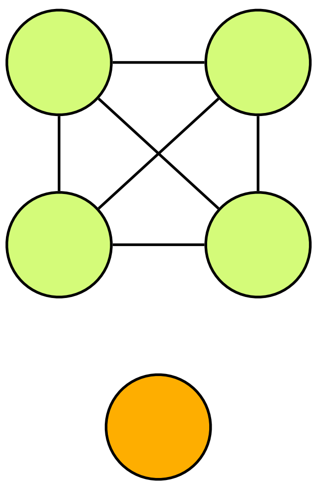
Probability to attract a new link at time t proportional to degree at time t: $\Pi(k_i) = \dfrac{k_i}{\sum_j k_j}$

Barabási-Albert Model: algorithm



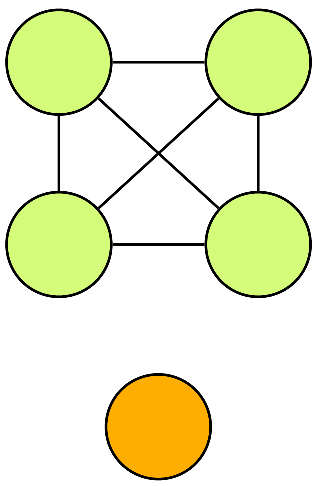
- Start with a clique of m_0 nodes

Barabási-Albert Model: algorithm



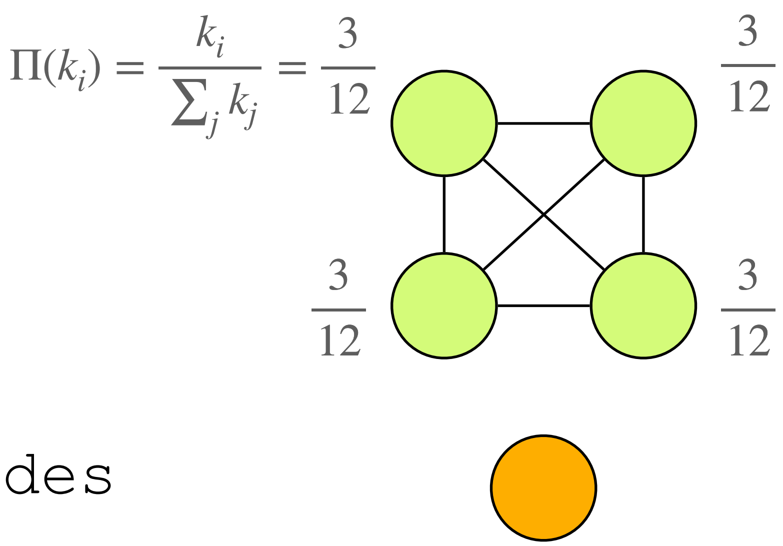
- Start with a clique of m_0 nodes
- At each time-step t:
    - Add a new node to the network

Barabási-Albert Model: algorithm



- Start with a clique of m_0 nodes
- At each time-step t:
  - Add a new node to the network
    - Create m (i.e. m=2) links between the new node and the existing ones according to the preferential attachment

Barabási-Albert Model: algorithm



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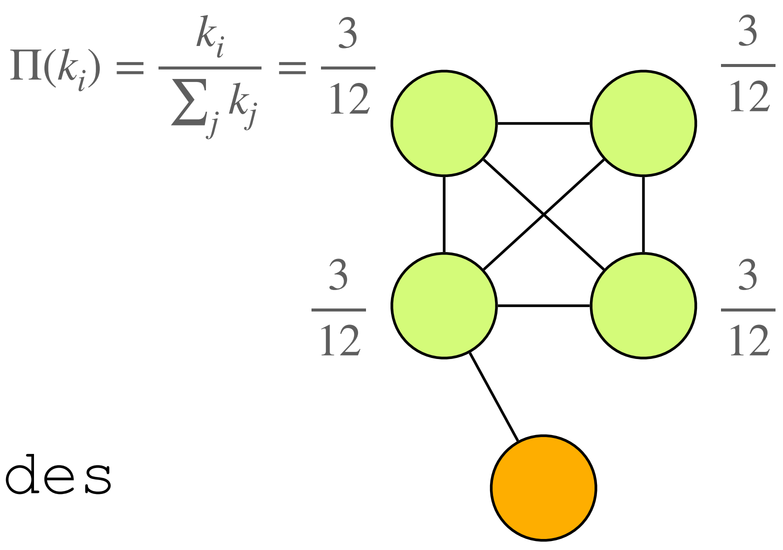

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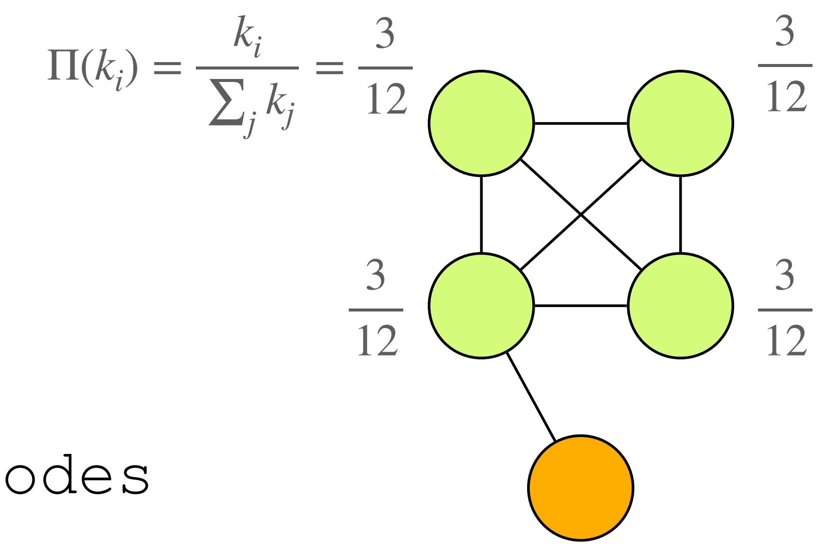

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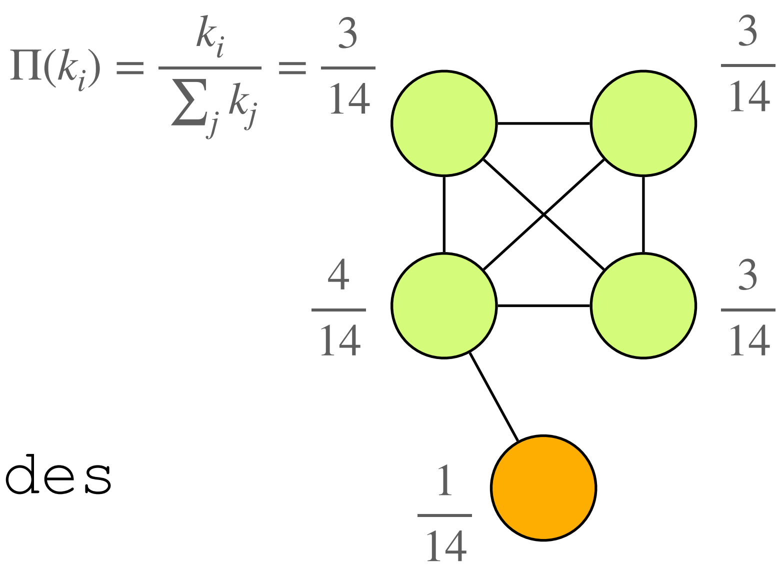
    - (Remember to update the connection probability after each link)

Barabási-Albert Model: algorithm



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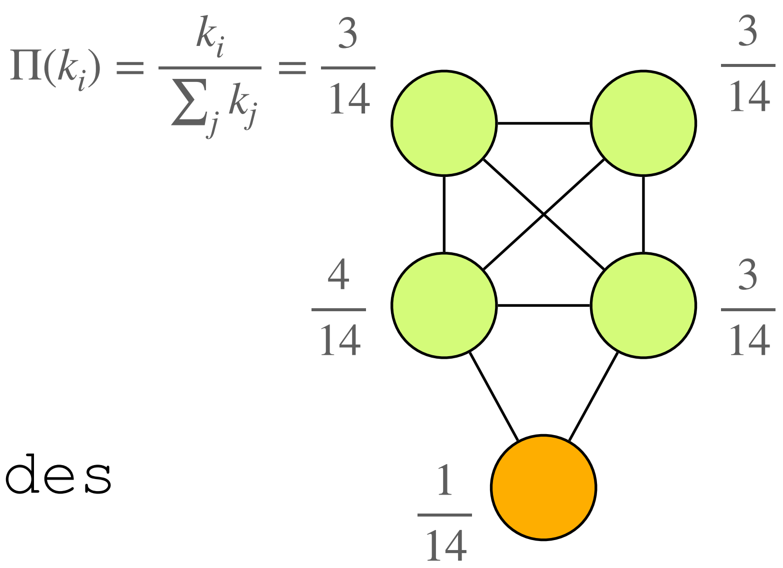
Barabási-Albert Model: algorithm



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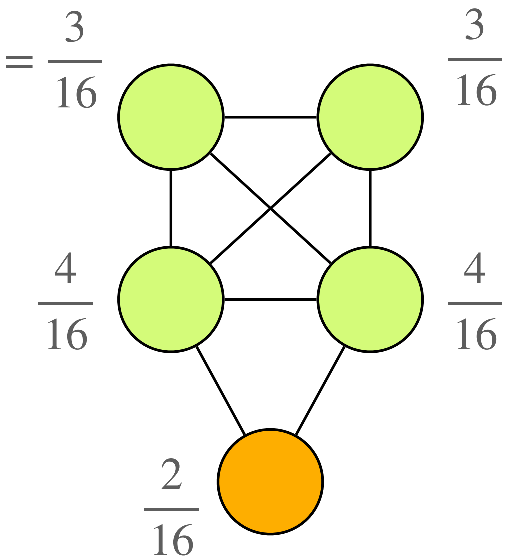
Barabási-Albert Model: algorithm

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j} = \frac{3}{16}$$



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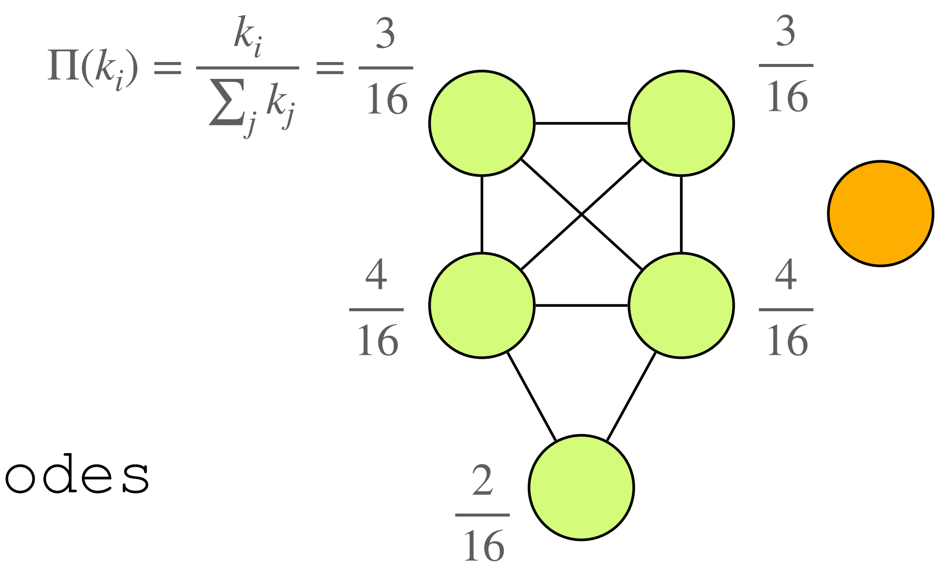
Barabási-Albert Model: algorithm



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- At each time-step t:
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    - Create m (i.e. m=2) links between the new node and the existing ones according to the preferential attachment
    - (Remember to update the connection probability after each link)
- Repeat until size N is reached

Barabási-Albert Model: algorithm

- Degree distribution $P(k) = \dfrac{2m(m+1)}{k(k+1)(k+2)} \simeq k^{-3}$ for large *k*
- $\gamma = 3$ independent from *m* and m_0
- $k_{max} \sim N^{\frac{1}{2}}$
- $\langle k \rangle \to c$ but $\langle k^2 \rangle \to \infty$ with N
- $\langle l \rangle \sim \dfrac{\ln(N)}{\ln(\ln(N))}$ small-world