Vietoris-Rips Complexes of Regular Polygons

Samir Chowdhury
The Ohio State University

Adam Jaffe
Stanford University

Joint work with
Henry Adams (Colorado State University)
Bonginkosi Sibanda (Brown University)

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Setup and overview

Definition

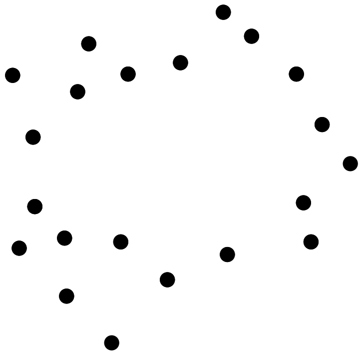
For metric space (X, d) and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathbf{VR}_<(X; r)$ is the set of all finite $\sigma \subseteq X$ with $\mathrm{diam}(\sigma) < r$.

Definition

For metric space (X, d) and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathbf{VR}_\leq(X; r)$ is the set of all finite $\sigma \subseteq X$ with $\mathrm{diam}(\sigma) \leq r$.

Remark

The *Vietoris–Rips simplicial complex* can be fully determined by the the underlying graph of its one skeleton, i.e the graph made by the zero and one dimensional simplices.

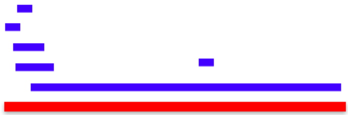


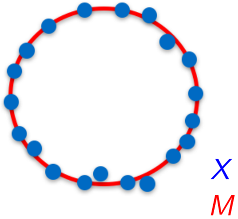
Setup and overview

Theorem (Chazal, de Silva, Oudot, 2013)

Suppose X, M are totally bounded metric spaces. Then for any $k \geq 0$,

$$d_B(dgm_k^{VR}(X), dgm_k^{VR}(M)) \leq 2d_{GH}(X, M)$$





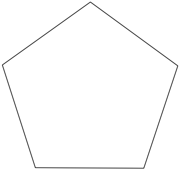
One application: X a uniform sample from a manifold M.

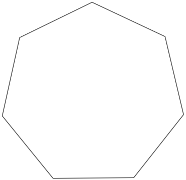
- Problem: $dgm_k^{VR}(M)$ is known for very few manifolds!
- Past work: circle (Adamaszek, Adams 2017), ellipses of small eccentricity (Adamaszek, Adams, Reddy 2018).

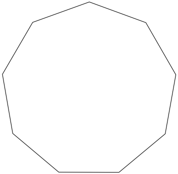
Regular Polygons

Definition

Given an integer $n \geq 3$, let the *regular n-gon* $P_n \subseteq \mathbb{R}^2$ be a set of n points equally spaced on S^1, with line segments connecting adjacent points together. We endow P_n with the Euclidean metric of $\mathbb{R}^2$.







Problem statement and strategy

- We want to describe the homotopy types and persistent homology of $\mathrm{VR}(P_n; r)$.
- Method of cyclic graphs (Adamaszek et al 2016) has been successful in the circle and ellipse case.
- First we quantify scales parameters for which $\mathrm{VR}(P_n; r)$ supports cyclic graphs.
- Theorem of Adamaszek, Adams, and Reddy asserts that $\mathrm{VR}(P_n; r) \simeq S^{2\ell+1}$ or $\mathrm{VR}(P_n; r) \simeq \bigvee^{P+F-1} S^{2\ell}$, depending on an invariant called the *winding fraction* of cyclic graphs supported on $\mathrm{VR}(P_n; r)$. Here P, F are integers depending on the geometry of P_n that we explain later.
- **Main result:** We characterize the scale parameters r at which $\mathrm{VR}(P_n; r)$ is homotopy equivalent to an odd sphere or a wedge of $P + F - 1$ even spheres. In the latter case, we precisely quantify P and F.

Main Result

Theorem

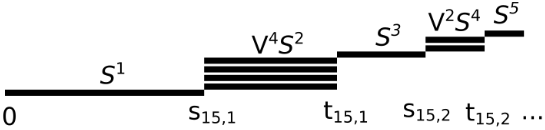
For fixed n, we have sequences of reals $\{s_{n,\ell}\}$ and $\{t_{n,\ell}\}$ that correspond to the first and last scale parameters for which an equilateral $(2\ell+1)$-star can be inscribed within P_n. Then:

$$\mathbf{VR}_<(P_n;r) \simeq \begin{cases} \bigvee^{q-1} S^{2\ell} & \text{when } s_{n,\ell} < r \leq t_{n,l} \\ S^{2\ell+1} & \text{when } t_{n,\ell} < r \leq s_{n,\ell+1} \end{cases}$$

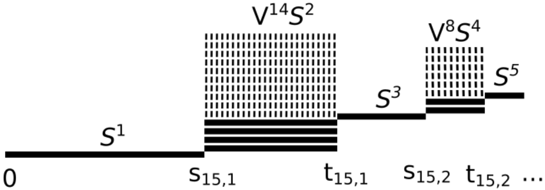
$$\mathbf{VR}_\leq(P_n;r) \simeq \begin{cases} \bigvee^{3q-1} S^{2\ell} & \text{when } s_{n,\ell} < r < t_{n,\ell} \\ S^{2\ell+1} & \text{when } t_{n,\ell} < r < s_{n,\ell+1}, \end{cases}$$

where $q = n/gcd(n, 2\ell+1)$. Furthermore, all of the above homological features are persistent, except for $2q$ copies of $S^{2\ell}$ during the even sphere regimes for $\leq$.

Main Result: Example

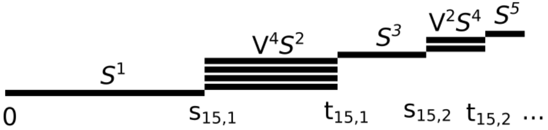


$\mathbf{VR}_<(P_{15}; r)$

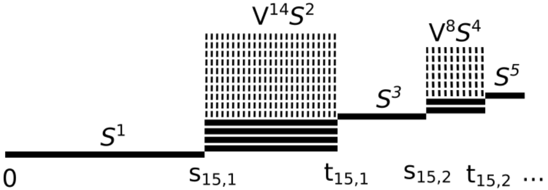


$\mathbf{VR}_\leq(P_{15}; r)$

Main Result: Example



$\mathbf{VR}_{<}(P_{15}; r)$



$\mathbf{VR}_{\leq}(P_{15}; r)$

Why do we get homology above dimension 1?

Intuition

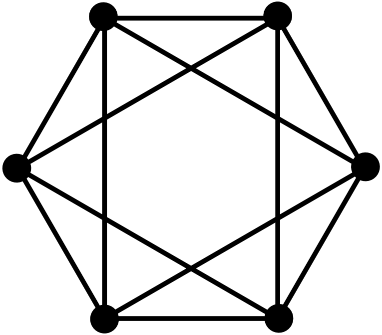
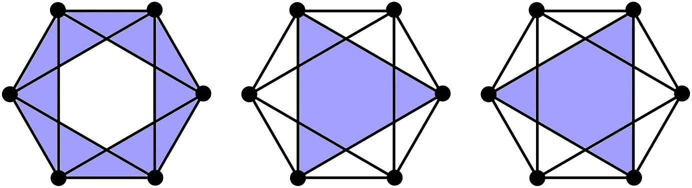
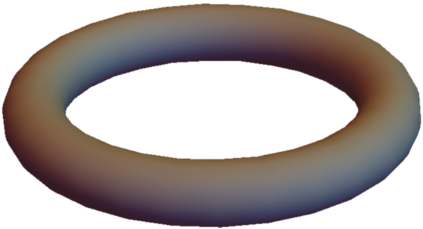
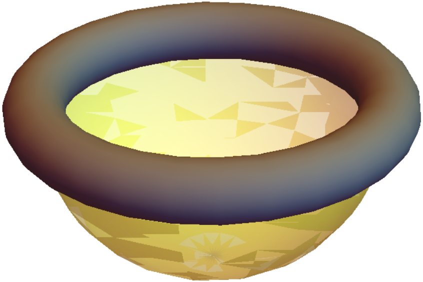


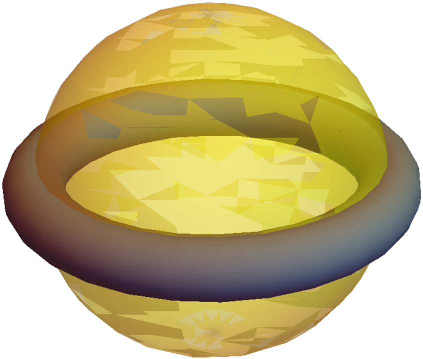
Figure: $\mathrm{VR}_{\leq}(6 \text{ points}; \frac{1}{3}) \simeq S^2$

Intuition









Intuition

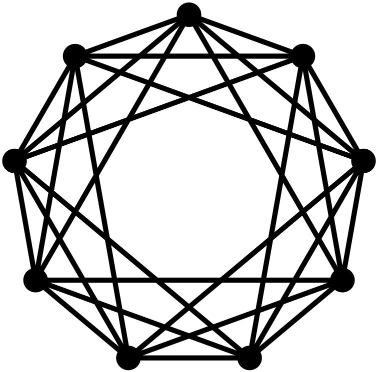
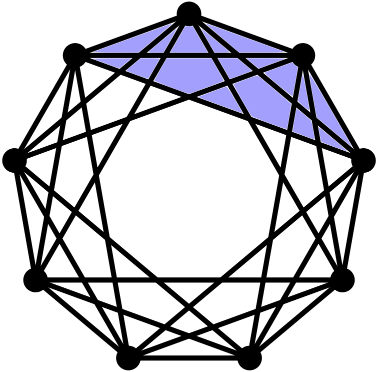
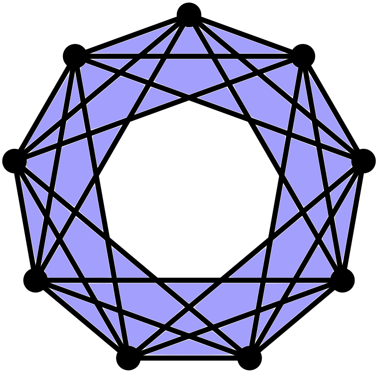


Figure: $\mathrm{VR}_{\leq}(9\text{ points}; \frac{1}{3}) \simeq \bigvee^2 S^2$

Intuition



Intuition



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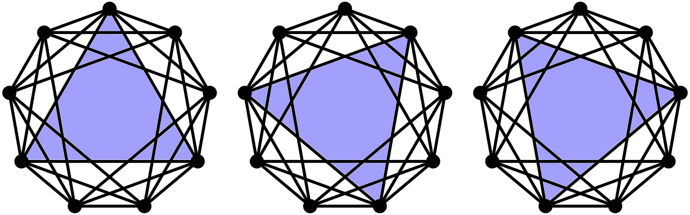
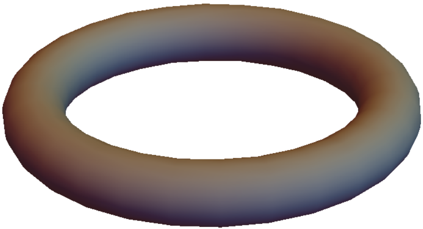
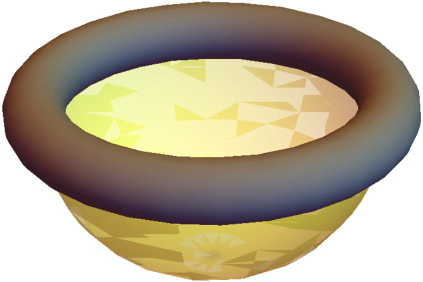
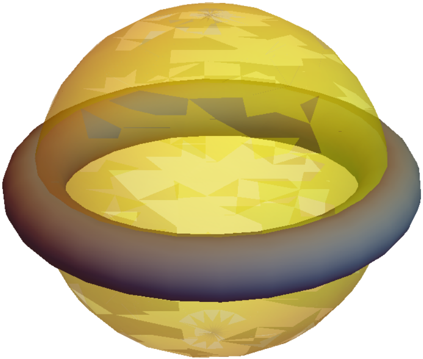
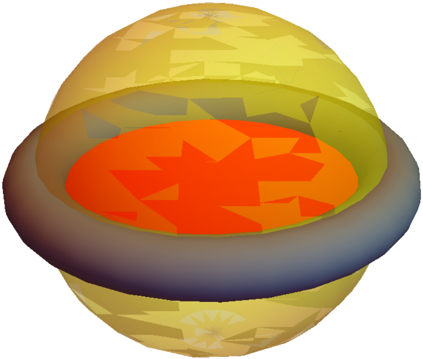










Cyclic Graphs

Definition

A directed graph G is *cyclic* if its vertices can be placed in a cyclic order such that, whenever there is an edge $v \to u$, then there are also edges $v \to w \to u$ for all $v \prec w \prec u$.

Definition

For a cyclic graph G and a vertex v, define $f(v)$ to be the clockwise-most point u such that there exists an edge $v \to u$.

Definition

The *winding fraction* of a cyclic graph G is

$$\mathrm{wf}(G) = \sup \left\{ \frac{\omega}{k} \;\middle|\; \begin{array}{l} G \text{ contains an } f\text{-periodic orbit of} \\ \text{length } k \text{ which “winds” } \omega \text{ times} \\ \text{around the center of } G. \end{array} \right\}.$$

Cyclic Graphs

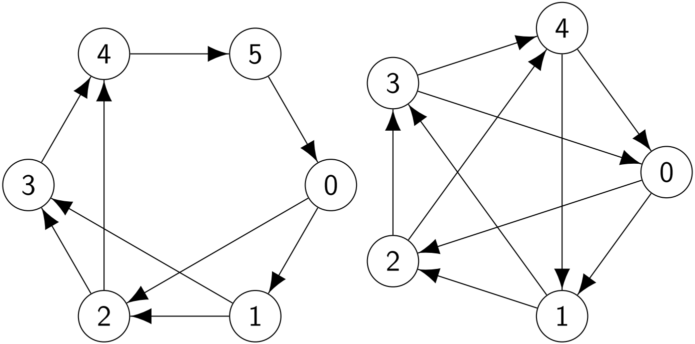


Figure: cyclic graphs of winding fraction $\frac{1}{4}$ (left) and $\frac{2}{5}$ (right).

Cyclic Graphs

Every vertex in a cyclic graph can be classified as exactly one of *fast*, *slow*, or *periodic* (to be defined later).

Theorem (Adamaszek, Adams, Reddy 2018)

Let G be a cyclic graph with P periodic orbits and F invariant sets of fast points. Then:

- *If* $\frac{\ell}{2\ell+1} < \mathrm{wf}(G) \leq \frac{\ell+1}{2\ell+3}$ *for some integer* $\ell \geq 0$, *then* $\mathrm{Cl}(G) \simeq S^{2\ell+1}$.
- *If* $\mathrm{wf}(G) = \frac{\ell}{2\ell+1}$, *then* $\mathrm{Cl}(G) \simeq \bigvee^{P+F-1} S^{2\ell}$.

Geometric Lemmas for Regular Polygons

Question

For which scale parameters $r > 0$ does $\mathbf{VR}(P_n; r)$ form a cyclic graph?

Answer

The graph $\mathbf{VR}(P_n; r)$ is cyclic up to the scale parameter shown.

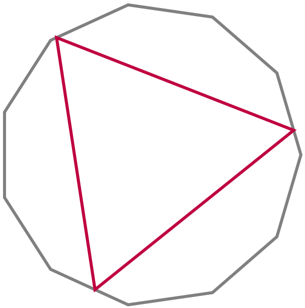
Remark

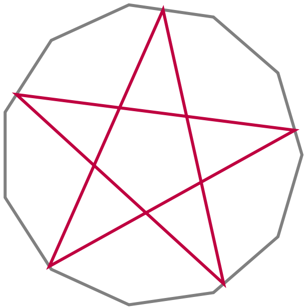
Since $r_3 = 0$, we conclude that $\mathbf{VR}(P_3; r)$ is not a cyclic graph for any $r > 0$.

Geometric Lemmas for Regular Polygons

Definition

In a cyclic graph G, an f-periodic orbit which has length $2\ell + 1$ and which "winds" ℓ times around the center of G is called an *inscribed equilateral* $(2\ell + 1)$-*pointed star*, or simply a $(2\ell + 1)$-*star*.





Geometric Lemmas for Regular Polygons

Definition

Let $s_{n,\ell}$ and $t_{n,\ell}$ be the smallest and largest scale parameters $r > 0$ for which a $(2\ell + 1)$-star can be inscribed into P_n.

Remark

The winding fraction of $\mathbf{VR}(P_n; r)$ equals $\frac{\ell}{2\ell+1}$ for all scales $r \in (s_{n,\ell}, t_{n,\ell})$.

Geometric Lemmas for Regular Polygons

Lemma

For any integers $\ell \geq 1$ and $n \geq 3$, there exists a unique $(2\ell + 1)$-star inscribed in P_n containing any given basepoint if and only if $n \geq 4\ell + 2$.

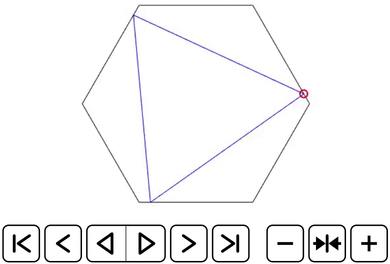
Definition

For $\ell \geq 1$, $n \geq 4\ell + 2$, and $x \in P_n$, denote the unique inscribed $(2\ell + 1)$-star containing x by $S_{2\ell+1}(x)$, and its side length by $s_{2\ell+1}(x)$.

Lemma

The function $s_{2\ell+1} : P_n \to \mathbb{R}$ is continuous.

Geometric Lemmas for Regular Polygons



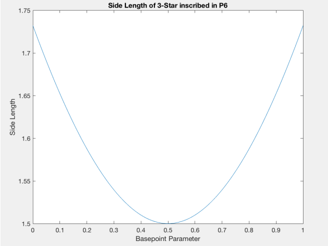
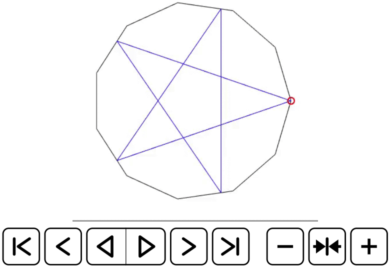


Figure: $n = 6$ and $2\ell + 1 = 3$



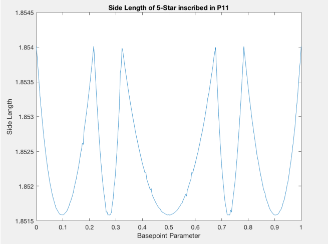


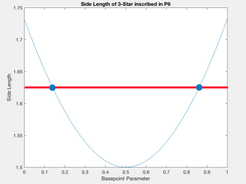
Figure: $n = 11$ and $2\ell + 1 = 5$

Geometric Lemmas for Regular Polygons

Definition

Given an integer $n \geq 3$ and a real $r > 0$, let $\ell \geq 0$ be the largest integer satisfying $n \geq 4\ell + 2$. Then any point $x \in P_n$ can be classified as one of:

- *fast*, if $s_{2\ell+1}(x) < r$
- *slow*, if $s_{2\ell+1}(x) > r$
- *periodic*, if $s_{2\ell+1}(x) = r$



Definition

The integer F, the number of invariant sets of fast points in $\mathbf{VR}(P_n; r)$, is equal to the number of connected components in $s^{-1}_{2\ell+1}((-\infty, r))$, divided by $2\ell + 1$.

Definition

The integer P, the number of periodic orbits in $\mathbf{VR}(P_n; r)$, is equal to the cardinality of $s^{-1}_{2\ell+1}(\{r\})$, divided by $2\ell + 1$.

Geometric Lemmas for Regular Polygons

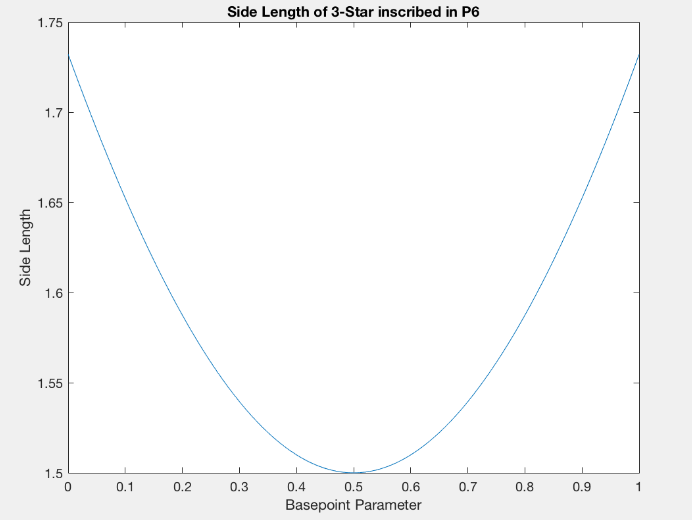


Figure: $n = 6$ and $2\ell + 1 = 3$

Geometric Lemmas for Regular Polygons

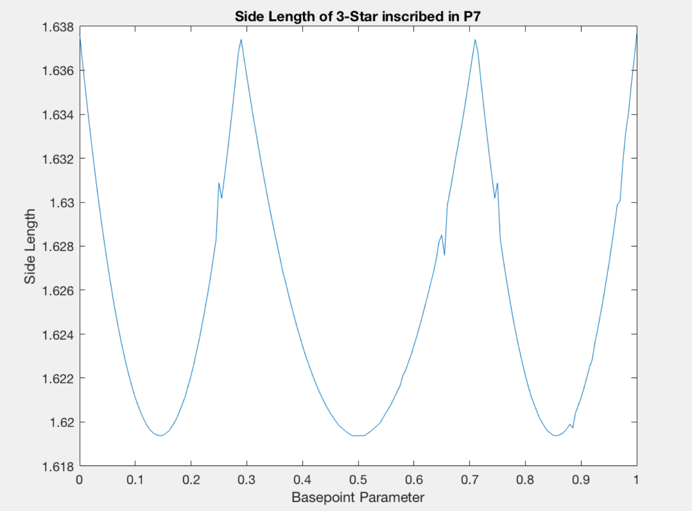


Figure: $n = 7$ and $2\ell + 1 = 3$

Geometric Lemmas for Regular Polygons

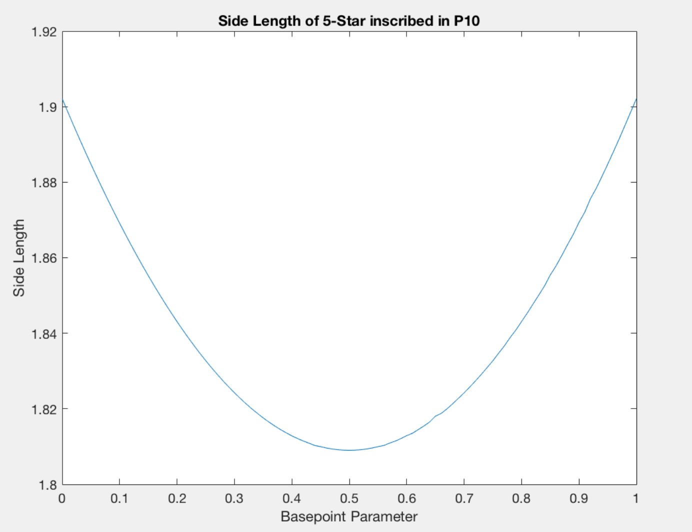


Figure: $n = 10$ and $2\ell + 1 = 5$

Geometric Lemmas for Regular Polygons

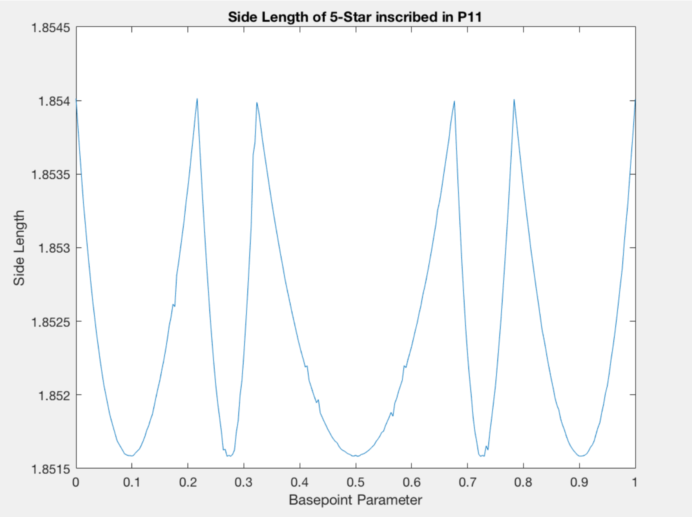


Figure: $n = 11$ and $2\ell + 1 = 5$

Geometric Lemmas for Regular Polygons

Question

How many distinct equilateral $(2\ell+1)$-stars of side length r can be inscribed into P_n?

Answer

The number of equilateral $(2\ell+1)$-stars of side length r that can be inscribed into P_n is equal to:

$$\begin{cases} n/gcd(n, 2\ell+1) & \text{if } r = s_{n,\ell} \text{ or } t_{n,\ell} \\ 2n/gcd(n, 2\ell+1) & \text{if } s_{n,\ell} < r < t_{n,\ell} \\ 0 & \text{otherwise} \end{cases}$$

Main Result

Theorem

For $r \in (0, r_n)$ we have:

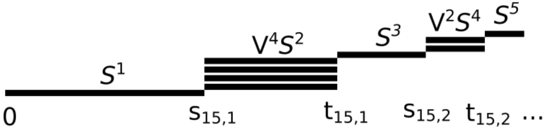
$$\mathbf{VR}_<(P_n; r) \simeq \begin{cases} \bigvee^{q-1} S^{2\ell} & \text{when } s_{n,\ell} < r \leq t_{n,l} \\ S^{2\ell+1} & \text{when } t_{n,\ell} < r \leq s_{n,\ell+1} \end{cases}$$

$$\mathbf{VR}_\leq(P_n; r) \simeq \begin{cases} \bigvee^{3q-1} S^{2\ell} & \text{when } s_{n,\ell} < r < t_{n,\ell} \\ S^{2\ell+1} & \text{when } t_{n,\ell} < r < s_{n,\ell+1}, \end{cases}$$

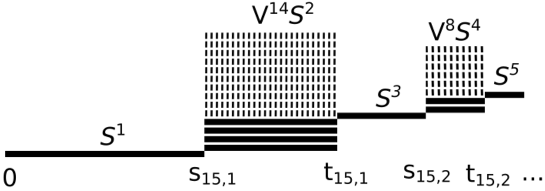
where $q = n/gcd(n, 2\ell + 1)$. Furthermore,

- *For $s_{n,\ell} < r < \tilde{r} \leq t_{n,\ell}$ or $t_{n,\ell} < r < \tilde{r} \leq s_{n,\ell+1}$, inclusion $\mathbf{VR}_<(P_n; r) \hookrightarrow \mathbf{VR}_<(P_n; \tilde{r})$ is a homotopy equivalence.*
- *For $t_{n,\ell} < r < \tilde{r} < s_{n,\ell+1}$, inclusion $\mathbf{VR}_\leq(P_n; r) \hookrightarrow \mathbf{VR}_\leq(P_n; \tilde{r})$ is a homotopy equivalence.*
- *For $s_{n,\ell} \leq r < \tilde{r} \leq t_{n,\ell}$, inclusion $\mathbf{VR}_\leq(P_n; r) \hookrightarrow \mathbf{VR}_\leq(P_n; \tilde{r})$ induces a rank $q - 1$ map on 2ℓ-dimensional homology $H_{2\ell}(-; \mathbb{F})$ for any field $\mathbb{F}$.*

Main Result: Example



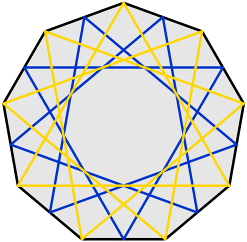
$\mathbf{VR}_{<}(P_{15}; r)$



$\mathbf{VR}_{\leq}(P_{15}; r)$

Future Work

- Finish paper and post to arXiv



Thank you!