# Stochastic Partial Differential Equations: Theory and Numerical Simulations

Adam Q. Jaffe
Advised by: Lenya Ryzhik

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# Motivation

- Why study Stochastic Partial Differential Equations (SPDE)?
- In many (PDE) models with unknown parameters, it makes sense encode unknowns by randomness
- Examples:
  - Wave propagation in atmosphere
  - Fluid dynamics
  - Epidemiology
- Some Issues:
  - Lots of technical details
  - Hard to simulate in a computer

# Stochastic Ordinary Differential Equations

# SDEs from ODEs

- General ODE written as

$$\begin{cases} \frac{dU_t}{dt} = \mu(t, U_t) \\ t \in [0, T] \\ U_0 \in \mathbb{R} \end{cases} \tag{1}$$

- We have existence, uniqueness, and continuity under very mild assumptions on $\mu$
- Lots of numerical methods to solve these; any approach will (essentially) work
- Can we add simple random "noise" to a given ODE?

$$\frac{dU_t}{dt} = \mu(t, U_t) + \sigma(t, U_t) W(t) \tag{2}$$

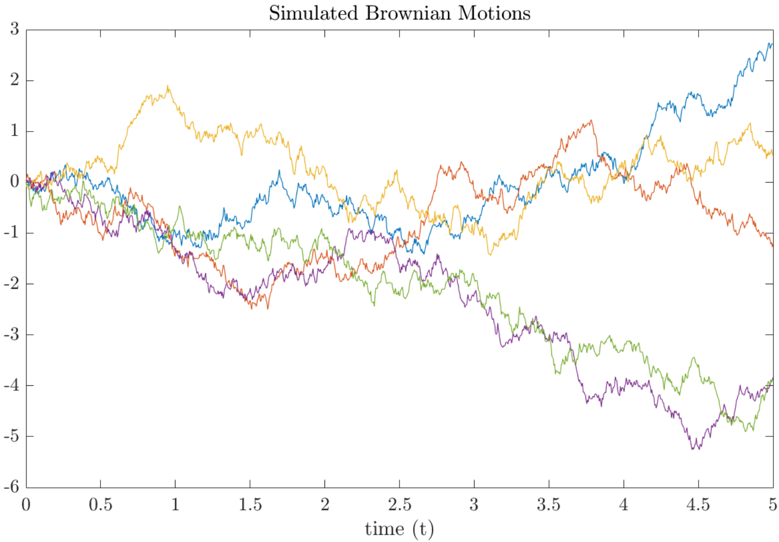
# Brownian Motion and White Noise

$$\{B_t : 0 \leq t \leq T\} \text{ or simply } B_t \tag{3}$$

- A random function
- Important properties: $B_t$ is...
  - almost surely continuous in $t$
  - almost surely nowhere differentiable in $t$
  - Limiting object of a random walk (CLT)
- "The derivative of Brownian motion is white noise."

$$W(t) = \frac{dB_t}{dt} \tag{4}$$

# Brownian Motion and White Noise



# SDEs from ODEs

- The equation

$$\frac{dU_t}{dt} = \mu(t, U_t) + \sigma(t, U_t)\frac{dB_t}{dt} \tag{5}$$

  in integral form is

$$U_t - U_0 = \int_0^t \mu(s, U_s)ds + \int_0^t \sigma(t, U_t)\frac{dB_s}{ds}ds \tag{6}$$

- Then reimagine the integral as

$$\int_0^t \sigma(t, U_t)\frac{dB_s}{ds}ds = \int_0^t \sigma(t, U_t)dB_s \tag{7}$$

- Can we interpret this in a Riemann-Stieljes-type sense, where $B_t$ acts as an "integrator"?

# Stochastic Integral

- For each sample path of $B_t$ and partition $0 = t_1 < \cdots < t_N = T$, write the Riemann-Stieljes sum

$$\int_0^T \sigma(t, U_t) dB_t \approx \sum_{j=0}^{N} \sigma(t_j^*, U_{t_j^*})(B_{t_{j+1}} - B_{t_j}) \tag{8}$$

- How to choose $t_j^* \in [t_j, t_{j+1})$? Surprisingly, this choice matters!
- $t_j^* = t_j$ leads to the *Ito integral*, $\int_0^T \sigma(t, U_t) dB_t$
- $t_j^* = \frac{1}{2}(t_j + t_{j+1})$ leads to the *Stratonovich integral*, $\int_0^T \sigma(t, U_t) \circ dB_t$

# Ito versus Stratonovich Calculus

- Ito Calculus
  - Does not "see into the future"
  - Is a martingale
  - The usual chain rule fails
  - Numerical methods are harder
- Stratonovich Calculus
  - Does not "see into the future"
  - Is not a martingale
  - Usual chain rule holds
  - Numerical methods are easier (Can use numerical methods for ODEs)

# SDE Existence and Uniqueness

- A general SDE is written as

$$U_t - U_0 = \int_0^t \mu(s, U_s) ds + \int_0^t \sigma(s, U_s)(\circ) dB_s. \tag{9}$$

Or, by the shorthand:

$$dU_t = \mu(t, U_t) dt + \sigma(t, U_t)(\circ) dB_t \tag{10}$$

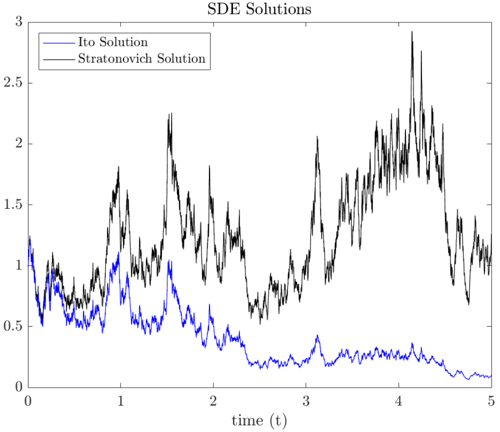
- Can interpret in the Ito or Stratonovich sense
- We want a solution defined on $t \in [0, T]$, with a possibly random initial condition $U_0$.

**Theorem**

*If $\mu, \sigma$ are Lipschitz continuous and satisfy a linear growth bound, then almost surely there exists a unique continuous function on $[0, T]$ which satisfies the SDE.*

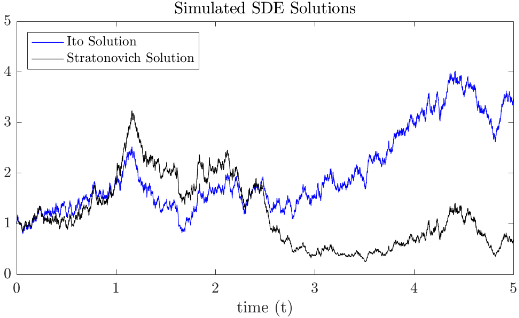
# SDE Example 1

- Consider the SDE $dU_t = U_t dB_t$ on $[0, T]$ with deterministic initial condition $U_0 = 1$.
- The Ito solution is $U_t = U_0 \exp(B_t - \frac{t}{2})$
- The Stratonovich solution is $U_t = U_0 \exp(B_t)$



# SDE Example 2

- Consider the SDE $dU_t = \exp(t - U_t^2)dt + \tanh(U_t)dB_t$ on $[0, T]$ with deterministic initial condition $U_0 = 1$.
- No explicit solutions exist!
- Need to resort to numerical simulations



# Stochastic Partial Differential Equations

# SPDEs from PDEs

- Consider the heat equation

$$\begin{cases} \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + g(t,x) \\ (t,x) \in [0,T] \times \mathbb{R} \\ U(0,x) = \phi(x) \end{cases} \tag{11}$$

- Its explicit solutions are given by

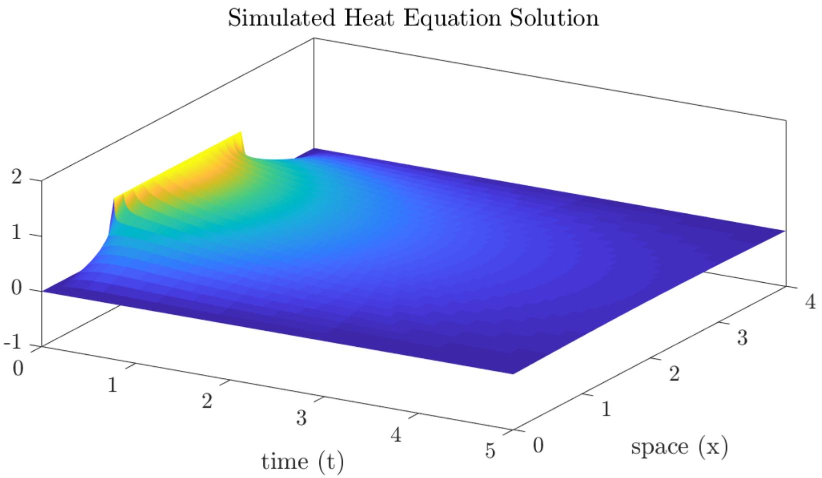
$$U(t,x) = \int_{\mathbb{R}} G(t, x-y)\phi(y)dy + \int_0^t \int_{\mathbb{R}} G(t-s, x-y)g(s,y)dyds \tag{12}$$

where $G$ is the usual heat kernel:

$$G(t,x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x|^2}{4t}\right) \tag{13}$$

# SPDEs from PDEs

- Consider the heat equation with $g(t, x) = 0$, and initial condition $\phi(x) = \mathbf{1}_{[1,3]}(x)$.



# SPDEs from PDEs

- Lesson from SDEs: Easiest to work in integral form. Write

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + f(U(t,x))W(t,x) \tag{14}$$

as

$$U(t,x) = \int_{\mathbb{R}} G(t, x-y)\phi(y)dy \tag{15}$$

$$+ \int_0^t \int_{\mathbb{R}} G(t-s, x-y) f(U(s,y))W(s,y)dyds \tag{16}$$

- Let's come up with a precise definition of

$$\int_0^t \int_{\mathbb{R}} G(t-s, x-y)W(s,y)dyds = \int_0^t \int_{\mathbb{R}} G(t-s, x-y)dB_{s,y} \tag{17}$$

- Use Riemann-Stieljes-type sums, and choose between Ito and Stratonovich integral
- We use Ito calculus from now on

# SPDE Existence and Uniqueness

- A general stochastic heat equation is written as

$$U(t,x) = \int_{\mathbb{R}} G(t,x-y)\phi(y)dy + \int_0^t \int_{\mathbb{R}} G(t-s,x-y)f(U(s,y))dB_{s,y} \tag{18}$$

Or, by the shorthand:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + f(U(t,x))W(t,x) \tag{19}$$

- We impose an inital condition $U(0,x) = \phi(x)$.

**Theorem**

*If $f$ is Lipschitz continuous and the initial condition $\phi(x)$ is compactly supported, then almost surely there exists a unique continuous function on $[0,T] \times \mathbb{R}$ which satisfies the SPDE.*

# SPDE Existence and Uniqueness

## Theorem

*If $f$ is Lipschitz continuous and the initial condition $\phi(x)$ is compactly supported, then almost surely there exists a unique continuous function on $[0, T] \times \mathbb{R}$ which satisfies the SPDE.*

## Proof Sketch.

(Existence) By Picard's iteration method: Set

$$U_0(t,x) = \phi(x) \tag{20}$$

$$U_{n+1}(t,x) = \int_{\mathbb{R}} G(t, x-y)\phi(y)dy + \int_0^t \int_{\mathbb{R}} G(t-s, x-y) f(U_n(s,y)) dB_{s,y} \tag{21}$$

Each $U_n(t,x)$ is continuous in $t$ and $x$. □

## Proof Sketch (continued).

Define the supremum of the $L^2(\Omega)$ norm of the adjacent differences

$$z_n(t) = \sup_{x\in\mathbb{R}} \sup_{0\le s\le t} \mathbb{E}\left[\left|U_{n+1}(s,x) - U_n(s,x)\right|^2\right]. \tag{22}$$

These are bounded as

$$z_n(t) \le C_1 \int_0^t z_{n-1}(s)ds. \tag{23}$$

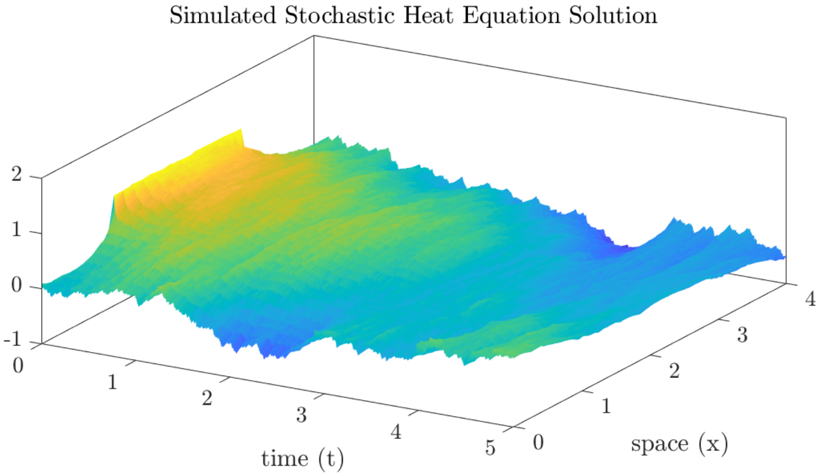
By induction:

$$z_n(t) \le C_2 \frac{(C_1 t)^{n+1}}{(n+1)!} \le C_2 \frac{(C_1 T)^{n+1}}{(n+1)!} \tag{24}$$

Since this is summable, we get that $\{U_n(t,x)\}$ has an $L^2(\Omega)$-limit function $U(t,x)$. Since the convergence is uniform in $t$, the limit function $U(t,x)$ is continuous. Now check that $U$ satisfies that SPDE and is unique. □

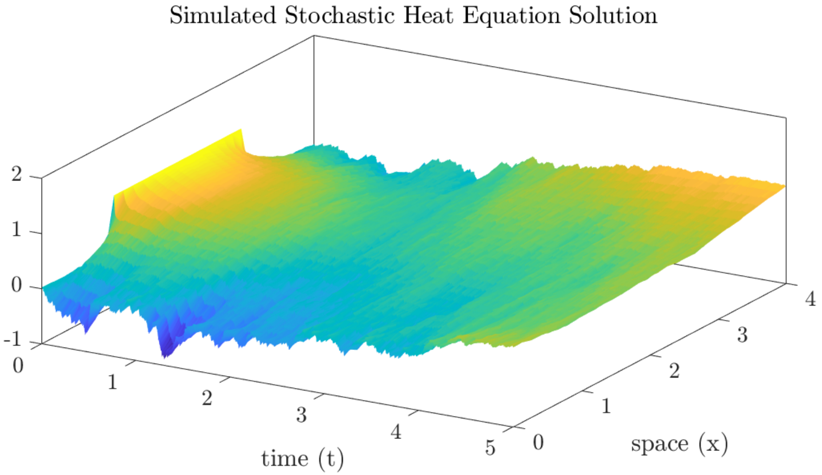
# SPDE Example 1

- Consider the stochastic heat equation with $f(u) = 1$, and deterministic initial condition $\phi(x) = \mathbf{1}_{[1,3]}(x)$.



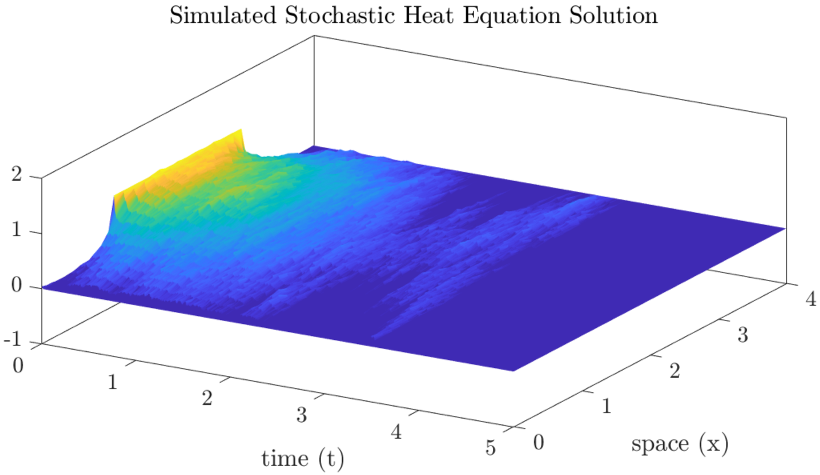
# SPDE Example 2

- Consider the stochastic heat equation with $f(u) = |1 - u|$, and deterministic initial condition $\phi(x) = \mathbf{1}_{[1,3]}(x)$.



# SPDE Example 3

- Consider the stochastic heat equation with $f(u) = \mathbf{1}_{[0,\infty)}(u)$, and deterministic initial condition $\phi(x) = \mathbf{1}_{[1,3]}(x)$.



# Research Questions

- "Smoothness" of solutions to S(P)DE
- Existence and Uniqueness of SPDE solutions other than the stochastic heat equation
- Relationship between Ito and Stratonovich calculus in the SPDE case
- How to simulate S(P)DE efficiently and precisely with a computer?
- Long-time behavior of certain SPDE models of interest

Thank you!