CONSTRUCTION OF THE LEBESGUE MEASURE ON $\mathbb{R}^d$

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We provide a construction of the Lebesgue measure on $\mathbb{R}^d$ via the Carathéodory extension theorem. This material is based on the first few weeks of the Autumn 2018 lectures of STATS 310A / MATH 230A at Stanford, taught by Sourav Chatterjee.

First we give a precise statement of our main tool, whose proof can be found in any textbook on measure theory:

Theorem 1. (Carathéodory) Let Ω be a set with an algebra of subsets $\mathcal{A}$, and let $\mu : \mathcal{A} \to [0, \infty]$ be a measure on $\mathcal{A}$. Then, μ has an extension to $\sigma(\mathcal{A})$, the σ-algebra generated by $\mathcal{A}$. Moreover, if $(\Omega, \mathcal{A}, \mu)$ is σ-finite, then this extension is unique.

At this point it is clear that we are done if we can construct a suitable algebra and measure to which we can apply the above result. As a first step, let us consider the following collection of subsets of $\mathbb{R}^d$ (which we might call the "rectangles"):

$$\begin{aligned}\mathcal{R} = \{(a_1, b_1] \times \cdots \times (a_d, b_d] : -\infty \le a_i \le b_i < \infty\} \\ \cup \{(a_1, \infty) \times \cdots \times (a_d, \infty) : -\infty \le a_i < \infty\}\end{aligned}$$

Then let us define $\mathcal{A}$ as the collection of all sets of finite disjoint unions of elements of $\mathcal{R}$:

$$\mathcal{A} = \left\{ \bigcup_{i=1}^{k} R_i : R_1, R_2, \dots R_k \in \mathcal{R} \text{ disjoint} \right\}$$

Lemma 2. The collection $\mathcal{A}$ is an algebra.

Proof. To see $\emptyset \in \mathcal{A}$, simply take $a_i = b_i = 0$ in the definition of $\mathcal{R}$. Now we must check that $\mathcal{A}$ is closed under finite unions. To do this, we first consider the case of one union. Take any A_1 and A_2 in $\mathcal{A}$ and write:

$$A_1 = \bigcup_{i=1}^{k} R_i^{(1)} \text{ and } A_2 = \bigcup_{i=1}^{k} R_i^{(2)}.$$

To show $A_1 \cup A_2 \in \mathcal{A}$, we must check that it can be written as disjoint union of elements of $\mathcal{R}$. This is immediate if we prove the following claim: For any rectangles $R_1, R_2 \in \mathcal{R}$, the set $R_1 \cup R_2$ is in $\mathcal{A}$. Let us write

$$R_1 = I_1^{(1)} \times \cdots \times I_d^{(1)} \text{ and } R_2 = I_1^{(2)} \times \cdots \times I_d^{(2)}$$

Then we can write:

$$R_1 \cup R_2 = \bigcup_{\substack{J_1 \in S_1 \\ \cdots \\ J_d \in S_d}} \Big(J_1 \times \cdots \times J_d \Big),$$

where the subscript means each J_i is allowed to range over the set

$$S_i = \{ I_i^{(1)} \setminus I_i^{(2)}, I_i^{(2)} \setminus I_i^{(1)}, I_i^{(1)} \cap I_i^{(2)} \}.$$

Since every element of each S_i is an interval in $\mathbb{R}$, it is clear that $R_1 \cup R_2 \in \mathcal{A}$. Applying this result pairwise to the pieces of A_1 and A_2, we get $A_1 \cup A_2 \in \mathcal{A}$. Extending this by induction on the number of unions, we see that $\mathcal{A}$ is closed under finitely many unions. An identical argument, along with the observation

$$R_1 \cap R_2 = \Big(I_1^{(1)} \cap I_1^{(2)} \Big) \times \cdots \times \Big(I_d^{(1)} \cap I_d^{(2)} \Big),$$

shows that $\mathcal{A}$ is closed under finitely many intersections as well.

Finally, we prove that $\mathcal{A}$ is closed under complements. Take any $A \in \mathcal{A}$ and write $A = \bigcup_{i=1}^{k} R_i$ for $R_1, R_2, \ldots R_k \in \mathcal{R}$. Since $\mathcal{A}$ is closed under finite intersections, we can prove that $\mathbb{R}^d \setminus A = \bigcap_{i=1}^{k} (\mathbb{R}^d \setminus R_i)$ is in $\mathcal{A}$ by showing that any $R \in \mathcal{A}$ gives $\mathbb{R}^d \setminus R \in \mathcal{A}$. To do this, simply observe that the complement of $R = (a_1, b_1] \times \cdots \times (a_d, b_d]$ is

$$\mathbb{R}^d \setminus R = \Big((-\infty, a_1] \cup (b_1, \infty) \Big) \times \cdots \times \Big((-\infty, a_d] \cup (b_d, \infty) \Big).$$

Distributing the $\times$'s across the $\cup$'s, we see that $\mathbb{R}^d \setminus R$ is the disjoint union of 2^d pieces, each of which is in $\mathcal{R}$. Therefore, $\mathcal{A}$ is closed under complements. $\square$

Next we define the map $\lambda : \mathcal{A} \to \mathbb{R}$ by the following rule: If $A \in \mathcal{A}$ consists of k disjoint pieces from $\mathcal{R}$, then we can write

$$A = \bigcup_{i=1}^{k} \Big((a_1^{(i)}, b_1^{(i)}] \times \cdots \times (a_d^{(i)}, b_d^{(i)}] \Big).$$

For A of this form, we define

$$\lambda(A) = \sum_{i=1}^{k} \left((b_1^{(i)} - a_1^{(i)}) \cdots (b_d^{(i)} - a_d^{(i)}) \right).$$

Lemma 3. The function λ is a measure on $\mathcal{A}$.

Proof. It is clear that we have $\lambda(\emptyset) = 0$, so the only difficult step is proving countable additivity. To do this, suppose that $A_1, A_2, \dots \in \mathcal{A}$ are disjoint with $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. Since $\bigcup_{i=1}^{\infty} A_i$ is a union of disjoint set in $\mathcal{R}$, we can prove $\lambda(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \lambda(A_i)$ by proving the result only for a single piece of $\bigcup_{i=1}^{\infty} A_i$. As such, suppose that $R_1, R_2, \dots$ is a sequence of disjoint elements of $\mathcal{R}$ with $\bigcup_{i=1}^{\infty} R_i = R \in \mathcal{R}$. For now, let us assume that R is finite. That is, we can write

$$R_i = (a_1^{(i)}, b_1^{(i)}] \times \cdots \times (a_d^{(i)}, b_d^{(i)}]$$

for all i, as well as $R = (a_1, b_1] \times \cdots \times (a_d, b_d]$. Then observe that, for any integer K, we have

$$(a_1, b_1] \cdots (a_d, b_d] \supseteq \bigcup_{i=1}^{K} \left((a_1^{(i)}, b_1^{(i)}] \cdots (a_d^{(i)}, b_d^{(i)}] \right).$$

Taking the Lebesgue measure of both sides, we get

$$(b_1 - a_1) \cdots (b_d - a_d) \geq \sum_{i=1}^{K} \left((b_1^{(i)} - a_1^{(i)}) \cdots (b_d^{(i)} - a_d^{(i)}) \right).$$

Then take the limit as $K \to \infty$ to get

$$(b_1 - a_1) \cdots (b_d - a_d) \geq \sum_{i=1}^{\infty} \left((b_1^{(i)} - a_1^{(i)}) \cdots (b_d^{(i)} - a_d^{(i)}) \right). \tag{1}$$

The converse is more tricky. We proceed by "thinning" R into a slightly smaller closed set and "thickening" $\bigcup_{i=1}^{\infty} R_i$ into a slightly larger open set. To do this, fix arbitrary positive reals $0 < \delta < \min_{1 \leq n \leq d}(b_n - a_n)$ and $\varepsilon > 0$. Also, let $c^{(i)}(n)$ represent the nth degree term of the product:

$$\left((b_1^{(i)} - a_1^{(i)}) + 1 \right) \cdots \left((b_d^{(i)} - a_d^{(i)}) + 1 \right).$$

Then let $\{a_i\}_{i=1}^{\infty}$ be the sequence given by:

$$a_i = \min \left\{ \frac{2^{-i}}{\max_{0 \leq n < d} c^{(i)}(n)}, 1 \right\},$$

which, in particuar, this gives the following bound, for any $n \in \{1, \dots d\}$

$$\sum_{i=1}^{\infty} a_i^n c^{(i)}(n) \leq 1.$$

We can then use this sequence get the following subset relation:

$$[a_1+\delta,b_1]\times\cdots\times[a_d+\delta,b_d]\subseteq\bigcup_{i=1}^{\infty}\Big((a_1^{(i)},b_1^{(i)}+a_i\varepsilon)\times\cdots\times(a_d^{(i)},b_d^{(i)}+a_i\varepsilon)\Big).$$

But notice that the right side is an open cover of the left side. And by the Heine-Borel theorem, the left is compact in $\mathbb{R}^d$, so there exists an integer K (and, if necessary, some reordering of the R_i's) that gives

$$[a_1+\delta,b_1]\times\cdots\times[a_d+\delta,b_d]\subseteq\bigcup_{i=1}^{K}\Big((a_1^{(i)},b_1^{(i)}+a_i\varepsilon)\times\cdots\times(a_d^{(i)},b_d^{(i)}+a_i\varepsilon)\Big).$$

Then we can change the endpoints slightly to get:

$$(a_1+\delta,b_1]\times\cdots\times(a_d+\delta,b_d]\subseteq\bigcup_{i=1}^{K}\Big((a_1^{(i)},b_1^{(i)}+a_i\varepsilon]\times\cdots\times(a_d^{(i)},b_d^{(i)}+a_i\varepsilon]\Big). \tag{2}$$

Readily, we see that the Lebesgue measure of the left side of (2) is

$$\lambda\Big([a_1+\delta,b_1]\times\cdots\times[a_d+\delta,b_d]\Big)=(b_1-a_1-\delta)\cdots(b_d-a_d-\delta). \tag{3}$$

Likewise, the Lebesgue measure of the right side can be bounded as follows:

(4)
$$\begin{aligned}\lambda\Big(\bigcup_{i=1}^{K}\Big((a_1^{(i)},b_1^{(i)}+a_i\varepsilon]\times\cdots\times(a_d^{(i)},b_d^{(i)}+a_i\varepsilon]\Big)\Big)&=\sum_{i=1}^{K}\Big((b_1^{(i)}-a_1^{(i)})+a_i\varepsilon\Big)\cdots\Big((b_d^{(i)}-a_d^{(i)})+a_i\varepsilon\Big)\\&=\sum_{i=1}^{K}\Big(c^{(i)}(d)+a_i\varepsilon c^{(i)}(d-1)+\cdots+a_i^d\varepsilon^d c^{(i)}(0)\Big)\\&=\sum_{i=1}^{K}c^{(i)}(d)+\varepsilon\sum_{i=1}^{K}a_ic^{(i)}(d-1)+\cdots+\varepsilon^d\sum_{i=1}^{K}a_i^dc^{(i)}(0)\\&\le\sum_{i=1}^{K}c^{(i)}(d)+\varepsilon+\cdots+\varepsilon^d,\end{aligned}$$

where the last inequality follows by the construction of the sequence a_i. Also notice that $c^{(i)}(d)=(b_1^{(i)}-a_1^{(i)})\cdots(b_d^{(i)}-a_d^{(i)})$. Then combining (2), (3), and (4), we get:

$$(b_1-a_1-\delta)\cdots(b_d-a_d-\delta)\le\sum_{i=1}^{K}\Big((b_1^{(i)}-a_1^{(i)})\cdots(b_d^{(i)}-a_d^{(i)})\Big)+\varepsilon+\cdots+\varepsilon^d.$$

Taking the limit as $\delta\to0$ and $\varepsilon\to0$, this implies

$$(b_1-a_1)\cdots(b_d-a_d)\le\sum_{i=1}^{\infty}\Big((b_1^{(i)}-a_1^{(i)})\cdots(b_d^{(i)}-a_d^{(i)})\Big). \tag{5}$$

Combining (1) and (5), we have proven:

$$(b_1 - a_1)\cdots(b_d - a_d) = \sum_{i=1}^{\infty}\left((b_1^{(i)} - a_1^{(i)})\cdots(b_d^{(i)} - a_d^{(i)})\right),$$

which is the desired countable additivity. Finally, note that the outstanding case of R being infinite is readily proved by adapting the proof above of the statment

$$(b_1 - a_1)\cdots(b_d - a_d) \le \sum_{i=1}^{\infty}\left((b_1^{(i)} - a_1^{(i)})\cdots(b_d^{(i)} - a_d^{(i)})\right).$$

If the projection of R onto dimension n is of the form $(-\infty, b_n]$, then we perform the "shrinking" process by fixing some function $\tilde{a}_n : (0,\infty) \to (-\infty, b_n)$ and then replacing $[a_n + \delta, b_n]$ with $[\tilde{a}_n(\delta), b_n]$. (For example, take $\tilde{a}_n(\delta) = b_n - 1/\delta$.) In this way, the measure of the left side goes to ∞ as $\delta \to 0$, as needed. In the case that one of the projections of R is of the form (a_n, ∞), then we create a functon $\tilde{b}_n : (0,\infty) \to (a_n, \infty)$ (for example, $\tilde{b}_n(\delta) = a_n + 1/\delta$), which satisfies the necessary limit.

□

Lastly, we are able to make the following observation:

Lemma 4. The space $(\mathbb{R}^d, \mathcal{A}, \lambda)$ is σ-finite.

Proof. Let $B = [0,1) \times \cdots \times [0,1)$ be the unit cube in $\mathbb{R}^d$, which clearly satisfies $\lambda(B) = 1$. Then we can write:

$$\mathbb{R}^d = \bigcup_{z \in \mathbb{Z}^d} (B + z).$$

This is a countable union of λ-finite pieces comprising $\mathbb{R}^d$. □

This proves exactly the desired result: λ is a σ-finite measure on the algebra $\mathcal{A}$, so, by the Carathéodory Extension Theorem, there exists a unique extension of λ to $\sigma(\mathcal{A})$. Moreover, since the set of rectangles $\mathcal{R}$ generates the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$, we also know $\mathcal{B}(\mathbb{R}^d) \subseteq \sigma(\mathcal{A})$. (Indeed, this containment is not strict, but that takes some work to prove.) Hence, we have at last constructed the familiar measure space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda)$.