Vietoris–Rips Complexes of Regular Polygons

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Vietoris–Rips Simplicial Complexes

Definition. For X a compact metric space and $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has vertex set X and a finite simplex $\sigma \subseteq X$ whenever $\mathrm{diam}(\sigma) < r$. Moreover, we denote the *Vietoris–Rips persistent homology barcode* of X by $\mathrm{H}^{\mathrm{VR}}(X)$.

The literature provides a variety of results that establish the importance of the VR Complex. Specifically, the persistent homology functor H^{VR} is "stable" in the following sense:

Theorem. For any compact metric space M and any finite subspace $X \subseteq M$, if X converges to M in the Gromov-Hausdorff distance, then $\mathrm{H}^{\mathrm{VR}}(X)$ converges to $\mathrm{H}^{\mathrm{VR}}(M)$ in the bottleneck distance [5].

However, there are very few infinite, continuous spaces whose barcodes are known. To this end, we were able to compute $\mathrm{H}^{\mathrm{VR}}(M)$ for $M = P_n$, the boundary of the regular n-sided polygon, equipped with the Euclidean metric of the plane.

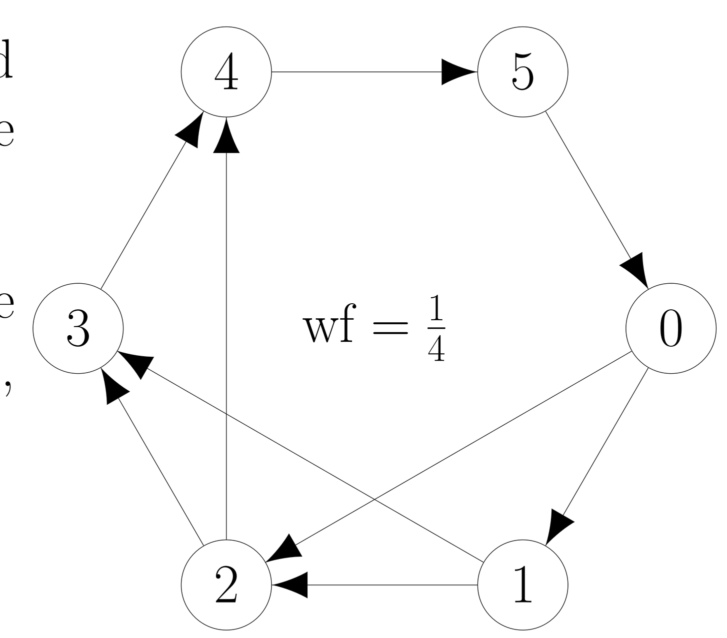
Cyclic Graphs and Winding Fraction

A major result of [2] is that the homotopy type of the clique complex of any *cyclic graph* is determined by its *winding fraction*. These details can be understood in the following informal manner:

Definition. A directed graph G is *cyclic* if its vertices can be placed in a cyclic order such that, whenever there is an edge $u \to w$, then there are also edges $u \to v \to w$ for all $u \prec v \prec w$.

Definition. For a cyclic graph G and a vertex v, define $f(v)$ to be the clockwise-most vertex w such that there exists an edge $v \to w$. Then, the *winding fraction* of G is

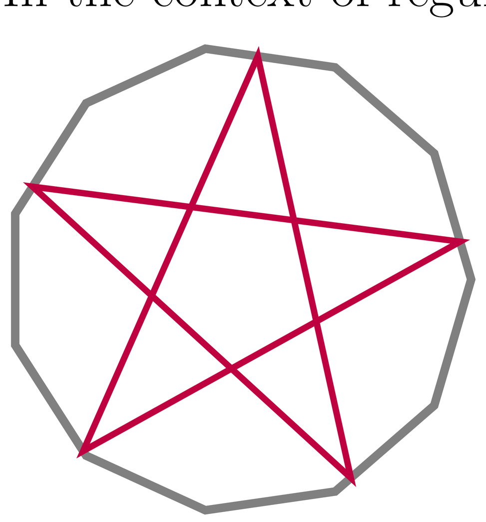
$$\mathrm{wf}(G) = \sup\left\{ \frac{\omega}{k} \;\middle|\; \begin{array}{l} G \text{ contains an } f\text{-periodic orbit of} \\ \text{length } k \text{ which "winds" } \omega \text{ times} \\ \text{around the center of } G. \end{array} \right\}.$$



This in mind, we reduce the task to a geometric task: When are cyclic graphs supported in P_n, and how does the winding fraction change over these regimes?

Stars Inscribed in Regular Polygons

In the context of regular polygons, we are able to answer the above questions completely.



Lemma. There exist scales $\{r_n\}_{n=3}^{\infty}$ such that $\mathrm{VR}(P_n; r)$ is a cyclic graph for all scales $0 \leq r < r_n$. Moreover, we have an explicit formula for these scales.

Definition. A $(2\ell+1)$*-pointed star inscribed in* $\mathrm{VR}(P_n; r)$ is an f-periodic orbit of size $2\ell + 1$ which "winds" ℓ times around its center.

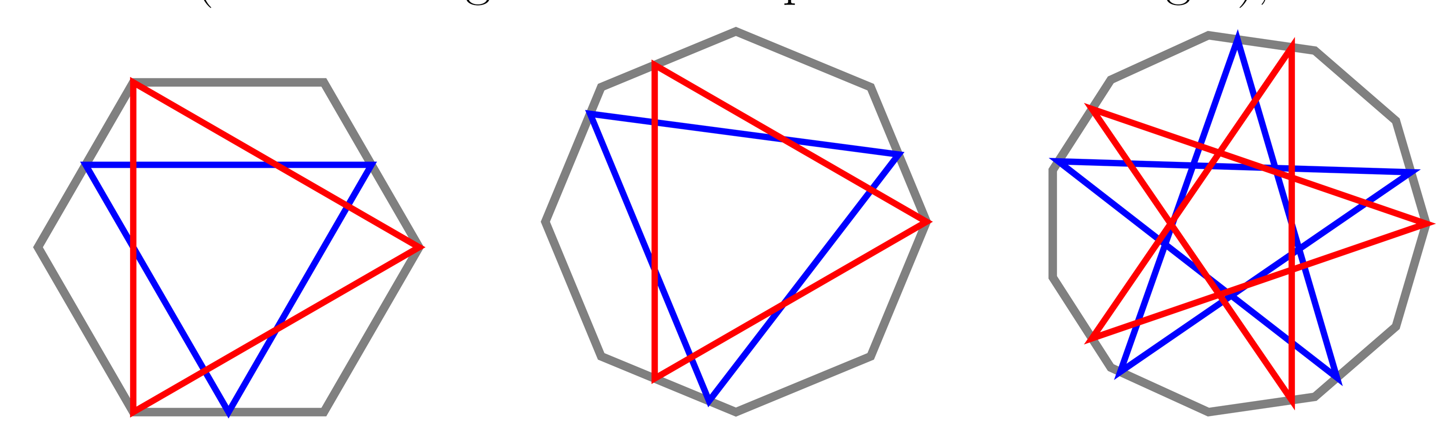
Definition. Let $s_{n,\ell}$ and $t_{n,\ell}$ be the smallest and largest scale parameters $r \geq 0$ for which a $(2\ell + 1)$-pointed star can be inscribed in $\mathrm{VR}(P_n; r)$.

Note that the winding fraction of $\mathrm{VR}(P_n; r)$ is equal to $\ell/(2\ell + 1)$ whenever $r \in (s_{n,\ell}, t_{n,\ell})$. In this case, the homotopy type is a wedge sum of 2ℓ-dimensional spheres, the number of which depends on the number of stars that can be inscribed into $\mathrm{VR}(P_n; r)$. For $r \notin [s_{n,\ell}, t_{n,\ell}]$, the homotopy type is simply an odd sphere.

Stars Inscribed in Regular Polygons (continued)

Lemma. For any basepoint $v \in P_n$, there exists a unique scale $r \geq 0$ such that a unique $(2\ell + 1)$-pointed star containing v can be inscribed in $\mathrm{VR}(P_n; r)$, if and only if $n \geq 4\ell + 2$.

Lemma. For $\ell \geq 1$ and $n \geq 4\ell + 2$, the function $s_{2\ell+1} : P_n \to \mathbb{R}$, assigning to each point $v \in P_n$ the scale $r \geq 0$ above (the side length of the unique star containing v), is continuous.



Geometric realizations of $s_{6,1}$ and $t_{6,1}$, $s_{8,1}$ and $t_{8,1}$, and $s_{11,2}$ and $t_{11,2}$.

Lemma. For any $(2\ell + 1)$-pointed star S inscribed in $\mathrm{VR}(P_n; r)$, the number of vertices of S coinciding with vertices of P_n is equal to either 0 or $\gcd(n, 2\ell + 1)$.

Corollary. For a fixed scale $r \in (s_{n,\ell}, t_{n,\ell})$, the number of $(2\ell + 1)$-pointed stars that can be inscribed in $\mathrm{VR}(P_n; r)$ is equal to $2n/\gcd(n, 2\ell + 1)$.

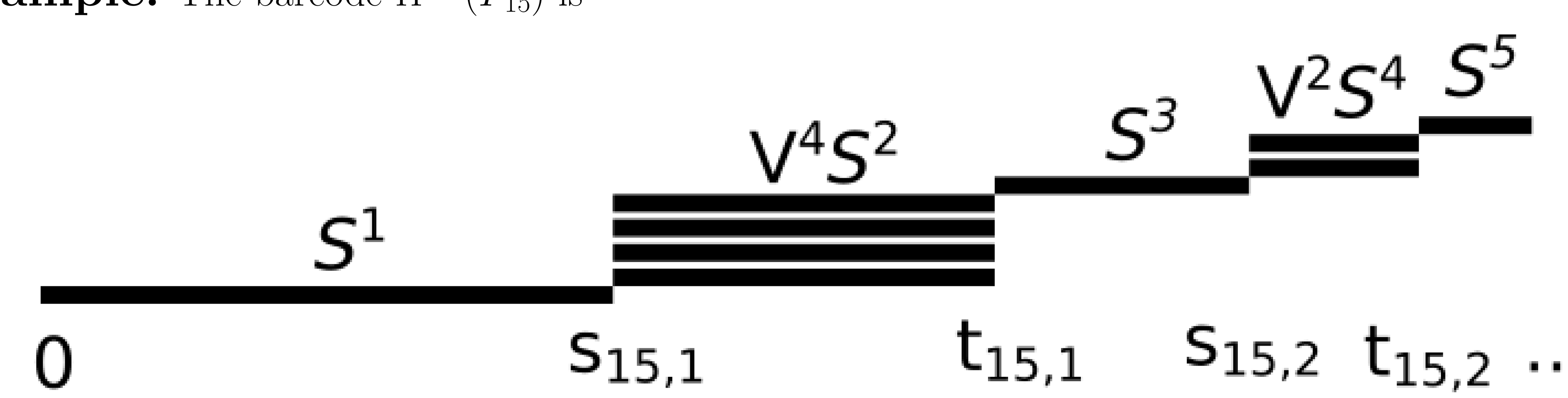
Persistent Homology of $\mathrm{VR}(P_n; r)$

Theorem. Denoting $q_\ell = n/\gcd(n, 2\ell + 1)$, we have:

$$\mathrm{VR}(P_n; r) \simeq \begin{cases} \bigvee^{q_\ell - 1} S^{2\ell} & \text{when } s_{n,\ell} < r \leq t_{n,\ell} \\ S^{2\ell+1} & \text{when } t_{n,\ell} < r \leq s_{n,\ell+1} \end{cases} \quad \text{for some } \ell \in \mathbb{N}$$

and all of the summands in question are persistent.

Example. The barcode $\mathrm{H}^{\mathrm{VR}}(P_{15})$ is



References

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[5] Frédéric Chazal, Vin de Silva, and Steve Oudot. Persistence stability for geometric complexes. *Geometriae Dedicata*, pages 1–22, 2013.