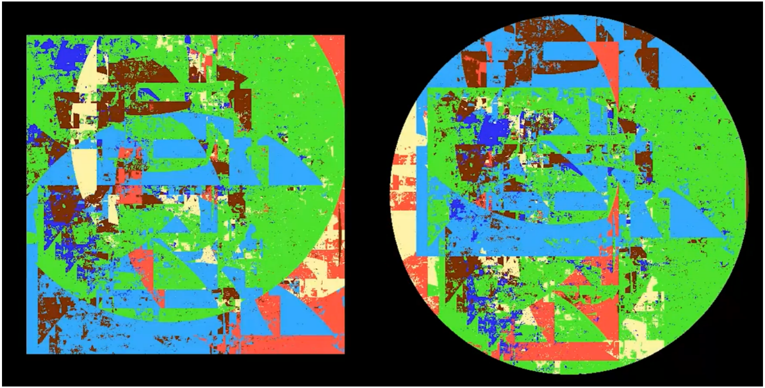
Finding a Smaller Upper Bound on the Number of Pieces to Square the Circle

Angela Chen (UCLA), advised by Andrew Marks (UCLA)

The Theorem It All Revolves Around

Theorem

It is possible to break a solid circle into finitely many pieces, and translate them to form a square of equal area.



Let's introduce some background behind this conclusion.

Background: Starting with an Easier Question

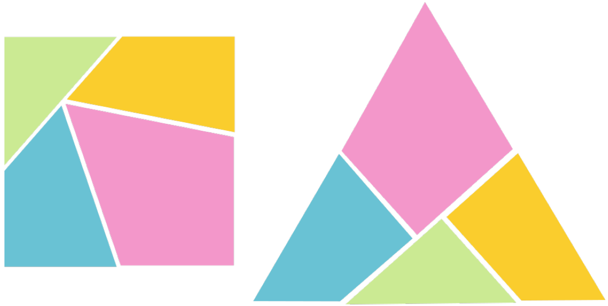
Can we break a polygon into finitely many pieces, and translate them to form a square of equal area?

Background: Starting with an Easier Question

Can we break a polygon into finitely many pieces, and translate them to form a square of equal area?

Yes! We call this "scissors congruence," or dissection congruence.

Definition: Two shapes are **dissection congruent** if we can cut the first shape into finitely many pieces which we can rearrange to get the second shape. (The boundaries of the pieces must overlap, but nothing else.)



Background: Dissection Congruence

Definition: Two shapes are **dissection congruent** if we can cut the first shape into finitely many pieces which we can rearrange to get the second shape. (The boundaries of the pieces must overlap, but nothing else.)

Background: Dissection Congruence

Definition: Two shapes are **dissection congruent** if we can cut the first shape into finitely many pieces which we can rearrange to get the second shape. (The boundaries of the pieces must overlap, but nothing else.)

If two polygons are dissection congruent, they have the same area.

Background: Dissection Congruence

Definition: Two shapes are **dissection congruent** if we can cut the first shape into finitely many pieces which we can rearrange to get the second shape. (The boundaries of the pieces must overlap, but nothing else.)

If two polygons are dissection congruent, they have the same area.

Theorem (Wallace-Bolyai-Gerwien, 1807, 1833)

Any two polygons of the same area are dissection congruent.

Tarski's Circle Squaring Problem

A circle and square are not dissection congruent, but let's consider a similar notion: are a circle and a square equidecomposable?

Tarski's Circle Squaring Problem

A circle and square are not dissection congruent, but let's consider a similar notion: are a circle and a square equidecomposable?

Question (Tarski's Circle Squaring Problem, 1925)

Are a solid two dimensional square and circle (necessarily of the same area) equidecomposable?

Tarski's Circle Squaring Problem, Answered

Theorem (Laczkovich, 1990)

It is possible to break a solid circle into finitely many pieces, and translate them to form a square of equal area.

Some main ideas from the proof (Laczkovich, Marks, Unger):
Work in the torus rather than the usual Cartesian plane.

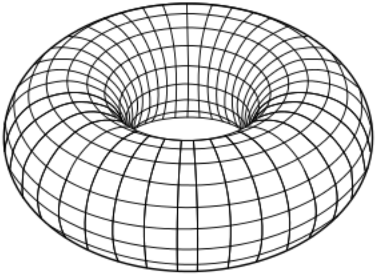
Make a flow between the circle and square.

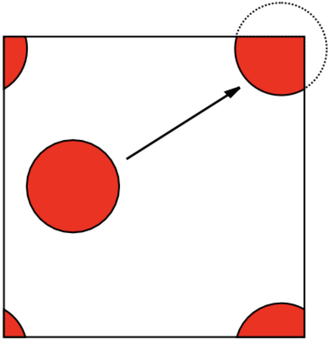
Estimate the discrepancy.

Some proof ideas: Working in the torus

We limit the points of our circle and square to be within $[0,1) \times [0,1)$. If any point goes beyond the edges, it wraps around to the other side.

This helps us with translating pieces, since we can work on one "canvas" and don't have to worry about pieces ending up too far away.





Some proof ideas: Flow of the characteristic functions

We give points inside the circle a "weight" of 1, and points outside the circle a "weight" of 0 (the *characteristic function* of the circle).

Using concepts from graph theory, we can construct a function (called a *flow*) that "moves" these weights to other points. So, we can construct a flow that takes the characteristic function of a circle as input and outputs the characteristic function of a square.

Some proof ideas: Flow of the characteristic functions

We give points inside the circle a “weight” of 1, and points outside the circle a “weight” of 0 (the *characteristic function* of the circle).

Using concepts from graph theory, we can construct a function (called a *flow*) that “moves” these weights to other points. So, we can construct a flow that takes the characteristic function of a circle as input and outputs the characteristic function of a square.

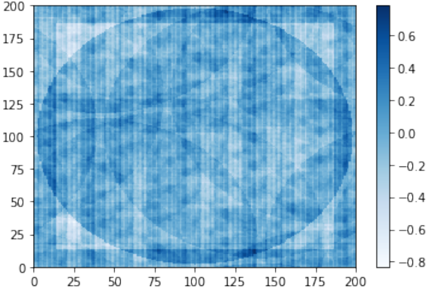
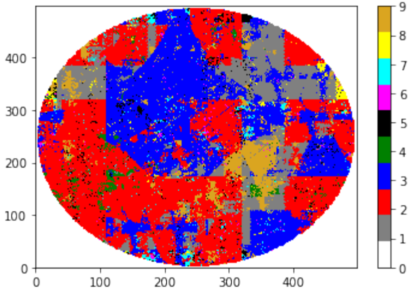
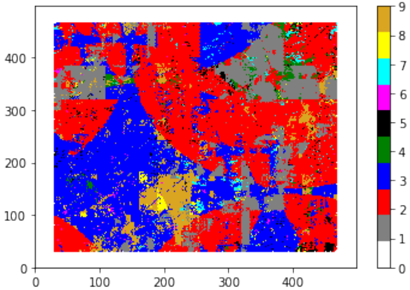


Figure: Visualization of the flow function’s change of weights

Some proof ideas: Flow of the characteristic functions

The flow function from the circle to the square, simulated:





Some proof ideas: Estimating the discrepancy

To iron out the wrinkles in the flow proof, we need to consider *discrepancy*, a measure of the deviation of a point set from a uniform distribution.

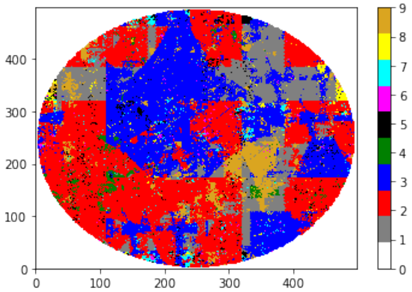
Think of discrepancy as how far something is from being “ideal”.

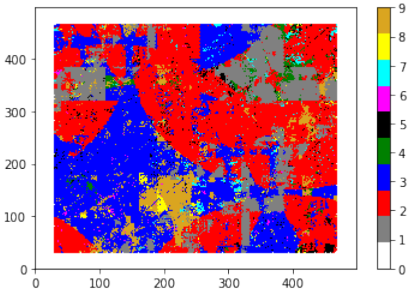
We need an upper bound on the discrepancy of the set of points in the circle in order for the flow proof to work!

Number of Pieces Needed

Our question: How many pieces do we need to break the circle into in order to translate them to form a square?

Initial estimates were around 10^{200} pieces, then decreased to 10^{50} pieces.





Looking at computer simulations, it seems like we could get less than 10 pieces...

The Fruits of My Labor

To decrease the number of pieces needed, we need to decrease the upper bound on the discrepancy. After going through the proof in detail and tracking constants throughout it...

Theorem (7.1). *Let $f : I \to I$ be a piecewise strictly monotonic function on two continuous intervals $I_1 \sqcup I_2 = I$, and suppose there is a constant $0 < C \leq 1$ such that*

$$\left|\frac{f(x)-f(y)}{x-y}\right| \geq C \quad \forall x, y \in I_1, \forall x, y \in I_2, x \neq y \tag{1}$$

Let $H_f = \{(x, y) : x \in I, 0 \leq y \leq f(x)\}$. Then for every finite subset of I^2,

$$S = \{(x_n, y_n) : n = 1, ..., N\}$$

and for every $m \in \mathbb{N}$, we have

$$D_2(S; H_f) \leq \max\left(8D_2(S), \frac{216}{C}E_m\right) \tag{2}$$

where

$$E_m = \frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h}\left\{\frac{1}{N}\left|\sum_{n=1}^{N} e^{2\pi hi(f(x_n)-y_n)}\right| + \left|\int_0^1 e^{2\pi hif(x_n)}\, dx - \frac{1}{N}\sum_{n=1}^{N} e^{2\pi hif(x_n)}\right|\right\}$$

The constants in Theorem 9.1 are as follows: C is the constant from Lemma 9.3, so

$$C = \max\left(\frac{b}{\pi} + b^2 + d, \frac{4}{\sqrt{c}}, 2\right)$$

C_1 is the constant from Equation 8.1, 8.2, 8.3,

$$C_1 = \frac{(2+|\ln c|)^{2+2\epsilon}\left(\frac{2}{(\ln 2)^{\epsilon}\epsilon}\right)^2\left(\frac{363}{140\cdot 2^{1+\epsilon}} + \frac{\ln(7)^{-\epsilon}}{\epsilon}\right)^2}{1-\frac{\pi^4}{9}c}$$

C_2 is the constant from Equation 6.2, i.e. Wagon 9.15, so

$$C_2 = \frac{(2+|\ln c|)^{2+2\epsilon}\left(\frac{2}{(\ln 2)^{\epsilon}\epsilon}\right)^2\left(\frac{363}{140\cdot 2^{1+\epsilon}} + \frac{\ln(7)^{-\epsilon}}{\epsilon}\right)}{1-\frac{2\pi^2}{3}c}$$

$$C_3 = C_2 + 4098CC_1$$
$$C_4 = (256C + 2C^2)C_2$$
$$C_5 = 2C_4 + C_3$$
$$C_6 = 1 + C_5\left(\ell(C) + \frac{4}{3}\right)^{6+6\epsilon}$$

Substituting these values into C_6 gives us

$$\begin{aligned}
C_6 &= 1 + [2C_2(256C + 2C^2) + C_2 + 4098CC_1]\left(\ell(C) + \frac{4}{3}\right)^{6+6\epsilon} \\
&= 1 + [C_2(4C^2 + 512C + 1) + 4098CC_1]\left(\ell(C) + \frac{4}{3}\right)^{6+6\epsilon} \\
&= 1 + \Bigg[\frac{(2+|\ln c|)^{2+2\epsilon}\left(\frac{2}{(\ln 2)^{\epsilon}\epsilon}\right)^2\left(\frac{363}{140\cdot 2^{1+\epsilon}} + \frac{\ln(7)^{-\epsilon}}{\epsilon}\right)}{1-\frac{2\pi^2}{3}c}(4C^2 + 512C + 1) \\
&\quad + 4098C\frac{(2+|\ln c|)^{2+2\epsilon}\left(\frac{2}{(\ln 2)^{\epsilon}\epsilon}\right)^2\left(\frac{363}{140\cdot 2^{1+\epsilon}} + \frac{\ln(7)^{-\epsilon}}{\epsilon}\right)^2}{1-\frac{\pi^4}{9}c}\Bigg]\left(\ell(C) + \frac{4}{3}\right)^{6+6\epsilon}
\end{aligned}$$

The Fruits of My Labor

To decrease the number of pieces needed, we need to decrease the upper bound on the discrepancy. After going through the proof in detail and tracking constants throughout it...

Theorem (7.1). *Let $f : I \to I$ be a piecewise strictly monotonic function on two continuous intervals $I_1 \sqcup I_2 = I$, and suppose there is a constant $0 < C \leq 1$ such that*

$$\left|\frac{f(x)-f(y)}{x-y}\right| \geq C \quad \forall x, y \in I_1, \forall x, y \in I_2, x \neq y \tag{1}$$

Let $H_f = \{(x, y) : x \in I, 0 \leq y \leq f(x)\}$. Then for every finite subset of I^2,

$$S = \{(x_n, y_n) : n = 1, ..., N\}$$

and for every $m \in \mathbb{N}$, we have

$$D_2(S; H_f) \leq \max\left(8D_2(S), \frac{216}{C}E_m\right) \tag{2}$$

where

$$E_m = \frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h}\left\{\frac{1}{N}\left|\sum_{n=1}^{N} e^{2\pi hi(f(x_n)-y_n)}\right| + \left|\int_0^1 e^{2\pi hif(x_n)}\,dx - \frac{1}{N}\sum_{n=1}^{N} e^{2\pi hif(x_n)}\right|\right\}$$

The constants in Theorem 9.1 are as follows: C is the constant from Lemma 9.3, so

$$C = \max\left(\frac{b}{\pi} + b^2 + d, \frac{4}{\sqrt{c}}, 2\right)$$

C_1 is the constant from Equation 8.1, 8.2, 8.3,

$$C_1 = \frac{(2+|\ln c|)^{2+2\epsilon}\left(\frac{2}{(\ln 2)^{\epsilon}\epsilon}\right)^2\left(\frac{363}{140\cdot 2^{1+\epsilon}} + \frac{\ln(7)^{-\epsilon}}{\epsilon}\right)^2}{1 - \frac{\pi^4}{9}c}$$

C_2 is the constant from Equation 6.2, i.e. Wagon 9.15, so

$$C_2 = \frac{(2+|\ln c|)^{2+2\epsilon}\left(\frac{2}{(\ln 2)^{\epsilon}\epsilon}\right)^2\left(\frac{363}{140\cdot 2^{1+\epsilon}} + \frac{\ln(7)^{-\epsilon}}{\epsilon}\right)}{1 - \frac{2\pi^2}{3}c}$$

$$\begin{aligned}
C_3 &= C_2 + 4098CC_1 \\
C_4 &= (256C + 2C^2)C_2 \\
C_5 &= 2C_4 + C_3 \\
C_6 &= 1 + C_5\left(\ell(C) + \frac{4}{3}\right)^{6+6\epsilon}
\end{aligned}$$

Substituting these values into C_6 gives us

$$\begin{aligned}
C_6 &= 1 + [2C_2(256C + 2C^2) + C_2 + 4098CC_1]\left(\ell(C) + \frac{4}{3}\right)^{6+6\epsilon} \\
&= 1 + [C_2(4C^2 + 512C + 1) + 4098CC_1]\left(\ell(C) + \frac{4}{3}\right)^{6+6\epsilon} \\
&= 1 + \left[\frac{(2+|\ln c|)^{2+2\epsilon}\left(\frac{2}{(\ln 2)^{\epsilon}\epsilon}\right)^2\left(\frac{363}{140\cdot 2^{1+\epsilon}} + \frac{\ln(7)^{-\epsilon}}{\epsilon}\right)}{1 - \frac{2\pi^2}{3}c}(4C^2 + 512C + 1)\right. \\
&\quad \left. + 4098C\frac{(2+|\ln c|)^{2+2\epsilon}\left(\frac{2}{(\ln 2)^{\epsilon}\epsilon}\right)^2\left(\frac{363}{140\cdot 2^{1+\epsilon}} + \frac{\ln(7)^{-\epsilon}}{\epsilon}\right)^2}{1 - \frac{\pi^4}{9}c}\right]\left(\ell(C) + \frac{4}{3}\right)^{6+6\epsilon}
\end{aligned}$$

We've reduced the number of pieces to 3.32×10^{48}.

Disclaimer: This is a work in progress.

Thanks!

References and Other Materials

My code:
`https://github.com/angelajc1/Circle-Squaring-Code`

Marks, A., Montalban, A., & Unger, S. (2022).
Squaring the Circle. Miller Institute.
Retrieved May 12, 2023.

Laczkovich, M. (1990).
Equidecomposability and discrepancy; a solution of Tarski's circle-squaring problem. Journal für die reine und angewandte Mathematik, Vol 404, pg 77-117.
Retrieved May 4, 2022 from `https://eudml.org/doc/153197`

Tomkowicz, G. & Wagon, S. (2016).
The Banach–Tarski Paradox: Second Edition, Ch. 9.
Retrieved August 7, 2022.