No-Regret Performative Binary Prediction with Strategic Experts

Anonymous Author(s)
Affiliation
Address
email

Abstract

In this paper, we study the well-known online binary prediction with expert advice framework under two additional considerations: 1) Experts act strategically and aim to maximize their influence on the algorithm's predictions by potentially misreporting their beliefs about the events, and 2) Predictions are performative, i.e., they can influence the state of the world and the binary events. We show that if the influence of the expert's predictions on outcomes is bounded, we can design no-regret algorithms where the regret is defined with respect to the true beliefs of the experts.

1 Introduction

Learning from a constant flow of information is one of the most prominent challenges in machine learning. In particular, online learning requires the learner to iteratively make decisions and at the time of making each decision, the outcome associated with it is unknown to the learner. The *experts problem* is perhaps the most well-known problem in online learning [1, 2, 3, 4]. In this problem, the learner aims to make predictions about a sequence of T binary events. To do so, the learner has access to the advice of K experts who each have internal beliefs about the likelihood of each event. At each round $t \in [T]$, the learner has to choose one among the advice of K experts and upon making her choice, the t-th binary event is realized and a loss bounded between zero and one is revealed. The goal of the learner is to have no regret, i.e., to perform as well as the best-fixed expert in hindsight.
In many applications, however, the experts are strategic and wish to be selected by the learner as often as possible. To this end, they may strategically misreport their beliefs about the events. For instance, FiveThirtyEight[1] aggregates different pollsters according to their past performance to make a single prediction for elections and sports matches. To do so, FiveThirtyEight maintains publicly available pollster ratings[2]. A low rating can be harmful to the pollster's credibility and adversely impact their revenue opportunities in the future. Therefore, instead of maximizing their expected performance by reporting their predictions truthfully, the pollsters may decide to take risks and report more extreme beliefs to climb higher on the leaderboard. Therefore, it is important to design algorithms that not only achieve *no-regret* but also motivate the experts to report their true beliefs (*incentive-compatible*). Otherwise, the quality of the learner's predictions may be harmed.
Moreover, in real-world applications, the predictions can influence the state of the world, i.e., the predictions are *performative*. For instance, the prediction of a high rate of inflation might result in people buying goods before their cash reserves depreciate too much, thereby causing inflation.

[1] https://fivethirtyeight.com/
[2] https://projects.fivethirtyeight.com/pollster-ratings/

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2 Preliminaries and related work

[5] considered an offline problem with a principal and an expert. The expert could be either a human or an AI system. The principal aims to elicit honest predictions about a binary event from the expert. The expert's goal is to report a prediction p such that the expected loss given by a proper loss function ℓ is minimized. In order to model the performative predictions, [5] assumed that there is a function $f : [0, 1] \to [0, 1]$ such that $b = f(p)$ where b is the expert's belief about the binary outcome. In other words, given the prediction p, the expert's belief over the outcome varies according to the function f. They assumed that f is only known to the expert (not the principal).

The loss function ℓ is called proper if $\mathbb{E}_{r\sim\text{Bern}(b)}\ell(b, r) \leq \mathbb{E}_{r\sim\text{Bern}(b)}\ell(p, r)$ for all $b \in [0, 1], r \in \{0, 1\}, p \neq b$ (Bern(b) denotes a Bernoulli distribution with probability of success b). It is called strictly proper if the inequality is strict for all $p \neq b$. [6] showed that ℓ is (strictly) proper if and only if there exists a (strictly) convex function $G : [0, 1] \to \bar{\mathbb{R}}$ such that the following holds for all $p \in [0, 1], r \in \{0, 1\}$:

$$\ell(p, r) = -G(p) + (p - r)G'(p).$$

A prediction p is performatively optimal if $p \in \arg\min_{p\in[0,1]} \mathbb{E}_{r\sim\text{Bern}(f(p))}\ell(p, r)$. The expert aims to report a performatively optimal prediction.

A point p is a fixed point if $f(p) = p$. From the principal's perspective, fixed points (or approximately fixed points) are standards of honesty. If $|p - f(p)|$ is small, the principal is able to draw useful conclusions from the reports. We will use this notion for the online problem. If f is continuous, a fixed point p always exists. Moreover, if f is Lipschitz continuous with parameter $L_f < 1$, then the fixed point is unique.

[5] showed that fixed points are in general not performatively optimal. Moreover, they showed that if f is L_f-Lipschitz, G is L_G-Lipschitz, and G is γ-strongly convex, we have $|p - f(p)| \leq \frac{L_f L_G}{\gamma}$ where p is the performatively optimal prediction. Also, if $L_f < 1$, the fixed point p^* is unique and $|p - p^*| \leq \frac{L_f L_G}{\gamma(1-L_f)}$.

[7] also studied the binary prediction problem with performative predictions. In particular, they focused on two choices of f (p^* is the prior belief): 1) *Drift* model: $f(p) = \alpha p + (1 - \alpha)p^*$, and 2) *Reversion* model: $f(p) = \big(4(p - 0.5)^2\big)0.5 + \big(1 - 4(p - 0.5)^2\big)p^*$. The drift model is applicable to settings such as the prediction of inflation rate where predicting a high inflation rate results in people buying goods before their cash reserves depreciate too much and this in turn leads to inflation itself. In particular, large values of α correspond to self-fulfilling prophecies. The reversion model is applicable to settings where predicting an extreme value (e.g., close to 0 or 1) leads to the probability of outcome closer to 0.5, i.e., the event becomes less predictable. For instance, a prediction that one election candidate will win with near certainty might lead her fans not to show up to vote, resulting in a tighter election result. [7] showed that under proper scoring rules and the above two choices of f, the fixed point p^* is not the expert's performatively optimal report.

[8, 9, 10, 11] have studied the binary prediction problem with strategic experts. In particular, we use the incentive-compatible algorithm of [9] in our work and obtain $\mathcal{O}(\sqrt{T \ln K})$ regret bounds under some assumptions on ℓ and f.

3 Problem formulation

In this problem, there are K experts available and each expert makes probabilistic predictions about a sequence of T binary outcomes. At round $t \in [T]$, each expert $i \in [K]$ has a private prior belief $b_{i,t} \in [0, 1]$ about the outcome $r_t \in \{0, 1\}$, where r_t and $\{b_{i,t}\}_{i=1}^K$ are chosen adversarially. Moreover, each expert i has a function $f_{i,t} : [0, 1] \to [0, 1]$ that maps the prediction to the outcome, i.e., $f_{i,t}(p) = b$. Expert i reports $p_{i,t} \in [0, 1]$ as her prediction to the learner. Then, the learner makes her prediction $\sum_{i=1}^K \pi_{i,t}p_{i,t}$ (where $\sum_{i=1}^K \pi_{i,t} = 1$ and $\pi_{i,t} \geq 0\ \forall i \in [K]$) and upon committing to this prediction, the outcome $r_t \in \{0, 1\}$ is revealed, and the learner and expert i incur a loss of $\ell_t = \ell\big(\sum_{j=1}^K \pi_{j,t}f_{j,t}(\pi_t^T p_t), r_t\big)$ and $\ell_{i,t} = \ell\big(f_{i,t}(\pi_t^T p_t), r_t\big)$ respectively where $\ell : [0, 1] \times \{0, 1\} \to [0, 1]$ is a proper loss function (note that the losses are defined with respect to the true beliefs of the experts). From the perspective of expert i, the outcome r_t is sampled according to a Bernoulli distribution with probability of success $f_{i,t}(\pi_t^T p_t)$. The goal of the learner is to minimize the regret and also incentivize experts to (approximately) report their private beliefs at each round, i.e., the learner's algorithm should be designed such that the reported prediction of expert i,

$p_{i,t} = \arg\min_{p\in[0,1]} \Big(f_{i,t}\big(\sum_{j\neq i}^K \pi_{j,t}p_{j,t}+\pi_{i,t}p\big)\ell(p,1)+\big(1-f_{i,t}\big(\sum_{j\neq i}^K \pi_{j,t}p_{j,t}+\pi_{i,t}p\big)\big)\ell(p,0)\Big)$ is close to $f_{i,t}(\pi_t^T p_t)$.

To define the regret metric, we also need to specify the benchmark. Consider the algorithms $\{\mathcal{A}_i\}_{i=1}^K$ where algorithm $\mathcal{A}_i$ always chooses expert i at all rounds. At round $t \in [T]$, expert i aims to report $p^*_{i,t}$ to algorithm $\mathcal{A}_i$ to minimize the following:

$$\mathbb{E}_{r_t\sim \text{Bern}\left(f_{i,t}(p_{i,t})\right)} \ell(p_{i,t}, r_t) = f_{i,t}(p_{i,t})\ell(p_{i,t}, 1) + (1 - f_{i,t}(p_{i,t}))\ell(p_{i,t}, 0)$$

Now, we can define the regret metric as follows:

$$R_T = \sum_{t=1}^T \ell\big(\sum_{j=1}^K \pi_{j,t} f_{j,t}(\pi_t^T p_t), r_t\big) - \min_{i\in[K]} \sum_{t=1}^T \ell(f_{i,t}(p^*_{i,t}), r_t).$$

[5] studied an offline framework, called the "prediction markets", that is very similar to our setting. To be precise, they considered a setting with K traders where each trader submits a single prediction and gets scored according to a proper scoring rule. Each player $i \in [K]$ has an associated number $\omega_i \in [0, 1]$ (such that $\sum_{i=1}^K \omega_i = 1$) that represents the fraction of overall capital in the market provided by player i (similar to the expert weights $\pi_{i,t}$ in our framework). Each player i is scored according to a strictly proper loss function ℓ and the player i aims to minimize their expected loss $\ell(p_i, f(\omega^T p))$. In the game, all players simultaneously provide their prediction $\{p_i\}_{i\in[K]}$ which is the pure strategy Nash equilibrium. Then, the binary event is sampled according to a Bernoulli distribution with success probability $b = f(\omega^T p)$. The authors proved that if f is L_f-Lipschitz, G is L_G-Lipschitz, and G is γ-strongly convex, we have $|f(\omega^T p) - p_i| \leq \frac{\omega_i L_f L_G}{\gamma}$ for all $i \in [K]$.

In our setting, each expert $i \in [K]$ has separate belief functions $\{f_{i,t}\}_{t\in[T]}$ that map the learner's prediction to their belief. At round $t \in [T]$, expert $i \in [K]$ aims to minimize her expected loss defined as follows:

$$\mathbb{E}_{r_t\sim \text{Bern}\left(f_{i,t}(\pi_t^T p_t)\right)} \ell(p_{i,t}, r_t) = f_{i,t}(\pi_t^T p_t)\ell(p_{i,t}, 1) + (1 - f_{i,t}(\pi_t^T p_t))\ell(p_{i,t}, 0).$$

Considering the fact that the loss of each player depends on the actions of others, this could be characterized as a game between the K experts where each expert aims to minimize her expected loss defined above. We first show the following result.

Theorem 1. *For quadratic loss function (i.e., $G(p) = p^2 - p$) and the drift model for $\{f_{i,t}\}_{i\in[K],t\in[T]}$ (i.e., $f_{i,t}(p) = \alpha p + (1 - \alpha)b_{i,t}$) with $\alpha < 0.5$, the aforementioned K-person game is convex.*

Therefore, we can use the result of [12] to conclude that the pure strategy Nash equilibrium $\{p_{i,t}\}_{i\in[K]}$ exists at each round $t \in [T]$ and is unique.

We provide the main result of the paper in the following theorem.

Theorem 2. *Let the loss function be quadratic (i.e., $G(p) = p^2 - p$) and assume the drift model for $\{f_{i,t}\}_{i\in[K],t\in[T]}$ (i.e., $f_{i,t}(p) = \alpha p+(1-\alpha)b_{i,t}$). Using the WSU algorithm of [9], if $\alpha = \mathcal{O}(\sqrt{\frac{\ln K}{T}})$, the regret bound is $\mathcal{O}(\sqrt{T \ln K})$, i.e., performative predictions do not lead to worse regret guarantees.*

Proof. For a fixed $i \in [K]$ and $t \in [T]$, we have:

$$\frac{d}{dp_{i,t}}\mathbb{E}_{r_t\sim \text{Bern}\left(f_{i,t}(\pi_t^T p_t)\right)} \ell(p_{i,t}, r_t) = -\alpha\pi_{i,t}(2p_{i,t} - 1) + 2(p_{i,t} - \alpha\pi_t^T p_t - (1 - \alpha)b_{i,t}) = 0$$

$$\big(2(1 - \alpha\pi_{i,t}) - 2\alpha\pi_{i,t}\big)p_{i,t} = 2(1 - \alpha)b_{i,t} - \alpha\pi_{i,t} + 2\alpha \sum_{j\neq i} \pi_{j,t}p_{j,t}$$

$$p_{i,t} = \frac{1 - \alpha}{1 - 2\alpha\pi_{i,t}} b_{i,t} + \frac{\alpha}{1 - 2\alpha\pi_{i,t}} (\sum_{j\neq i} \pi_{j,t}p_{j,t} - 0.5\pi_{i,t})$$

Therefore, $p_{i,t} = \min\{0, \frac{1-\alpha}{1-2\alpha\pi_{i,t}} b_{i,t} + \frac{\alpha}{1-2\alpha\pi_{i,t}} (\sum_{j\neq i} \pi_{j,t}p_{j,t} - 0.5\pi_{i,t})\}$. Using the optimality condition, we can write:

$$\big(- \alpha\pi_{i,t}(2p_{i,t} - 1) + 2(p_{i,t} - f_{i,t}(\pi_t^T p_t))\big)(f_{i,t}(\pi_t^T p_t) - p_{i,t}) \geq 0$$

$$2(f_{i,t}(\pi_t^T p_t) - p_{i,t})^2 \leq -\alpha\pi_{i,t}(2p_{i,t} - 1)(f_{i,t}(\pi_t^T p_t) - p_{i,t}) \leq \alpha\pi_{i,t} \underbrace{|2p_{i,t} - 1|}_{\leq 1} |f_{i,t}(\pi_t^T p_t) - p_{i,t}|$$

Thus, $|f_{i,t}(\pi_t^T p_t) - p_{i,t}| \leq \frac{\alpha \pi_{i,t}}{2}$ holds.
On the other hand, we have:

$$\begin{aligned}
\frac{d}{dp_{i,t}} \mathbb{E}_{r_t \sim \text{Bern}\left(f_{i,t}(p_{i,t})\right)} \ell(p_{i,t}, r_t) = 2(p_{i,t} - \alpha p_{i,t} - (1-\alpha) b_{i,t}) - \alpha(2p_{i,t} - 1) = 0 \\
\big(2(1-\alpha) - 2\alpha\big) p_{i,t} = 2(1-\alpha) b_{i,t} - \alpha \\
p_{i,t} = \frac{1-\alpha}{1-2\alpha} b_{i,t} - \frac{0.5\alpha}{1-2\alpha}
\end{aligned}$$

So, $p^*_{i,t} = \min\{0, \frac{1-\alpha}{1-2\alpha} b_{i,t} - \frac{0.5\alpha}{1-2\alpha}\}$. We can use the optimality conditions here as well to write:

$$\begin{aligned}
&\big(2(p^*_{i,t} - f_{i,t}(p^*_{i,t}))) - \alpha(2p^*_{i,t} - 1)\big)(f_{i,t}(p^*_{i,t}) - p^*_{i,t}) \geq 0 \\
&2(f_{i,t}(p^*_{i,t}) - p^*_{i,t})^2 \leq -\alpha(2p^*_{i,t} - 1)(f_{i,t}(p^*_{i,t}) - p^*_{i,t}) \leq \alpha \underbrace{|2p^*_{i,t} - 1|}_{\leq 1} |f_{i,t}(p^*_{i,t}) - p^*_{i,t}|
\end{aligned}$$

Thus, $|f_{i,t}(p^*_{i,t}) - p^*_{i,t}| \leq \frac{\alpha}{2}$ holds. Putting the above results together, we can bound $|p_{i,t} - f_{i,t}(p^*_{i,t})|$ as follows:

$$\begin{aligned}
|p_{i,t} - f_{i,t}(p^*_{i,t})| &= |p_{i,t} - p^*_{i,t} + p^*_{i,t} - f_{i,t}(p^*_{i,t})| \\
&\leq |p_{i,t} - p^*_{i,t}| + |f_{i,t}(p^*_{i,t}) - p^*_{i,t}| \\
&\leq |\frac{1-\alpha}{1-2\alpha\pi_{i,t}} b_{i,t} + \frac{\alpha}{1-2\alpha\pi_{i,t}} (\sum_{j \neq i} \pi_{j,t} p_{j,t} - 0.5\pi_{i,t}) - \frac{1-\alpha}{1-2\alpha} b_{i,t} + \frac{0.5\alpha}{1-2\alpha}| + \frac{\alpha}{2} \\
&= |\frac{1-\alpha}{(1-2\alpha\pi_{i,t})(1-2\alpha)} \big(1 - 2\alpha - 1 + 2\alpha\pi_{i,t}\big) b_{i,t} + \alpha\big(\frac{\sum_{j \neq i} \pi_{j,t} p_{j,t} - 0.5\pi_{i,t}}{1-2\alpha\pi_{i,t}} + \frac{0.5}{1-2\alpha}\big)| + \frac{\alpha}{2} \\
&\leq \frac{2\alpha(1-\alpha)(1-\pi_{i,t})}{(1-2\alpha\pi_{i,t})(1-2\alpha)} b_{i,t} + \alpha\big(\frac{\sum_{j \neq i} \pi_{j,t} p_{j,t} - 0.5\pi_{i,t}}{1-2\alpha\pi_{i,t}} + \frac{0.5}{1-2\alpha} + \frac{1}{2}\big).
\end{aligned}$$

Assuming $\alpha < 0.25$, we have:

$$|p_{i,t} - f_{i,t}(p^*_{i,t})| \leq 8\alpha b_{i,t} + 3.5\alpha = (8b_{i,t} + 3.5)\alpha \leq 11.5\alpha.$$

Using the regret bound of the WSU algorithm of [9], we have:

$$\sum_{t=1}^{T} \ell(\sum_{j=1}^{K} \pi_{j,t} p_{j,t}, r_t) - \min_{i \in [K]} \sum_{t=1}^{T} \ell(p_{i,t}, r_t) \leq \mathcal{O}(\sqrt{T \ln K}).$$

For quadratic loss function $\ell(p, r) = (p - r)^2$, the function is 2-Lipschitz. Therefore, the following inequalities hold for all $t \in [T]$ and $i \in [K]$:

$$\begin{aligned}
|\ell(f_{i,t}(\pi_t^T p_t), r_t) - \ell(p_{i,t}, r_t)| \leq 2|f_{i,t}(\pi_t^T p_t) - p_{i,t}| \leq \alpha\pi_{i,t} \leq \alpha, \\
|\ell(p_{i,t}, r_t) - \ell(f_{i,t}(p^*_{i,t}), r_t)| \leq 2|p_{i,t} - f_{i,t}(p^*_{i,t})| \leq 23\alpha.
\end{aligned}$$

Putting the above results together, the regret bound is $\mathcal{O}(\sqrt{T \ln K}) + \mathcal{O}(\alpha T)$. Therefore, plugging in the value of α, we obtain the $\mathcal{O}(\sqrt{T \ln K})$ regret bound as stated. $\square$

4 Conclusion

In this paper, we studied the binary prediction problem with expert advice under the assumption that experts are strategic and the predictions are performative. We showed that for a particular choice of the loss function and the belief mapping function, if the influence of the predictions on outcomes is bounded, the standard $\mathcal{O}(\sqrt{T \ln K})$ regret bound could be obtained, i.e., the performative prediction setting does not lead to worse regret guarantees. It is interesting to study this problem under more general choices of the loss function and the belief mapping function and see if our results could be extended.

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Appendix

A Missing proofs

A.1 Proof of Theorem 1

Given that ℓ is a (strictly) proper loss function, for $p \in [0,1]$ and $r \in \{0,1\}$, we have:

$$\ell(p, r) = -G(p) + (p - r)G'(p).$$

where G is a (strictly) convex function. In order to guarantee that experts converge to the Nash equilibrium, we have to ensure that the loss of each expert is convex. Taking the derivative of the expected loss with respect to $p_{i,t}$, we have:

$$\begin{aligned}
\frac{d}{dp_{i,t}}\mathbb{E}_{r_t\sim\text{Bern}\left(f_{i,t}(\pi_t^T p_t)\right)}\ell(p_{i,t}, r_t) &= \pi_{i,t} f'_{i,t}(\pi_t^T p_t)\ell(p_{i,t}, 1) + f_{i,t}(\pi_t^T p_t)\ell'(p_{i,t}, 1) \\
&- \pi_{i,t} f'_{i,t}(\pi_t^T p_t)\ell(p_{i,t}, 0) + (1 - f_{i,t}(\pi_t^T p_t))\ell'(p_{i,t}, 0) \\
&= f_{i,t}(\pi_t^T p_t)\big(\ell'(p_{i,t}, 1) - \ell'(p_{i,t}, 0)\big) + \ell'(p_{i,t}, 0) \\
&+ \pi_{i,t} f'_{i,t}(\pi_t^T p_t)\big(\ell(p_{i,t}, 1) - \ell(p_{i,t}, 0)\big) \\
&= -\pi_{i,t} f'_{i,t}(\pi_t^T p_t)G'(p_{i,t}) - f_{i,t}(\pi_t^T p_t)G''(p_{i,t}) + p_{i,t}G''(p_{i,t})
\end{aligned}$$

Taking the second derivative of the expected loss, we can write:

$$\begin{aligned}
\frac{d^2}{dp_{i,t}^2}\mathbb{E}_{r_t\sim\text{Bern}(f_{i,t}(\pi_t^T p_t))}\ell(p_{i,t}, r_t) &= -\pi_{i,t}^2 f''_{i,t}(\pi_t^T p_t)G'(p_{i,t}) - \pi_{i,t} f'_{i,t}(\pi_t^T p_t)G''(p_{i,t}) + p_{i,t}G'''(p_{i,t}) \\
&+ G''(p_{i,t}) - \pi_{i,t} f'_{i,t}(\pi_t^T p_t)G''(p_{i,t}) - f_{i,t}(\pi_t^T p_t)G'''(p_{i,t}) \\
&= \big(p_{i,t} - f_{i,t}(\pi_t^T p_t)\big)G'''(p_{i,t}) + \big(1 - 2\pi_{i,t} f'_{i,t}(\pi_t^T p_t)\big)G''(p_{i,t}) \\
&- \pi_{i,t}^2 f''_{i,t}(\pi_t^T p_t)G'(p_{i,t}).
\end{aligned}$$

Given that G is convex, we have $G''(p) \geq 0$. Therefore, if $f_{i,t}$ is α-Lipschitz where $\alpha < 0.5$, we can argue that the second term in the second derivative $\big(1 - 2\pi_{i,t} f'_{i,t}(\pi_t^T p_t)\big)G''(p_{i,t})$ is non-negative. Assuming that the third derivative of G (G''') and the second derivative of $f_{i,t}$ ($f''_{i,t}$) are zero (i.e., G is a polynomial of degree at most 2 and f is an affine function), we can make sure that the other two terms are zero. For instance, for quadratic loss functions, we have $G(p) = p^2 - p$. If we consider the drift model with $\alpha < 0.5$, $\mathbb{E}_{r_t\sim\text{Bern}\left(f_{i,t}(\pi_t^T p_t)\right)}\ell(p_{i,t}, r_t)$ is a convex function of $p_{i,t}$.