MATB44 - Tutorial 6 (IC 320 Fridays 10-11am)

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Week 1 - Sept. 6th

No tutorials

Week 2 - Sept. 13th

ODE: $F(t, x, x^{(1)}, \dots, x^{(k)}) = 0$

t - independent variable (time)
x - dependent variable (space)
x is a function of t i.e. $x(t)$

The order is the highest derivative that appears in F.

The system is linear if we can rewrite it to separate out the $x_\ell^{(j)}$'s as follows:

$$x_i^{(k)} = g_i(t) + \sum_{\ell=1}^{n} \sum_{j=1}^{k-1} f_{i,j,\ell}(t)\, x_\ell^{(j)}$$

Ex: $\underbrace{(4+t^2)x' + 2tx}_{} = 4t$

By product rule: $\frac{d}{dt}\left[(4+t^2)x\right] = 2tx + (4+t^2)x'$

$\Rightarrow \frac{d}{dt}\left[(4+t^2)x\right] = 4t$

$\Rightarrow (4+t^2)x = 2t^2 + C$

$\Rightarrow x(t) = \dfrac{2t^2 + C}{4+t^2}$

The system is homogeneous if $g_i(t) = 0$

If there is no direct dependance on t, the system is autonomous. Form $x' = f(x)$

Ex: Classify the following equations:

* $x' + \frac{1}{2}x = \frac{1}{2}e^{t/3}$ 1st order, linear

* $3 + xx' = t - x$ 1st order, non linear

* $x''+5x'+6x=0$ 2nd order, linear, homogeneous, autonomous
* $x'=t\sin x$ 1st order, non linear
* $x''-x=0$ 2nd order, linear, homogeneous, autonomous
* $tx'+2x=4t^2$ 1st order, linear
* $x'-2x+t=4$ 1st order, linear.

Ex: Solve $x'=x^2$ with $x_0=x(0)>0$

① $\frac{dx}{dt}=x^2 \Rightarrow \frac{dx}{x^2}=dt$

$$\Rightarrow \int \frac{dx}{x^2}=\int dt$$

$$\Rightarrow -\frac{1}{x}=t+C_1$$

$$\Rightarrow x=\frac{1}{-C_1-t}$$

When $x(0)=-\frac{1}{C_1}=x_0$ so $x=\frac{1}{\frac{1}{x_0}-t}$

② $f(x)=x^2$, a maximal nonzero interval is $(x_1,x_2)=(0,\infty)$

$$F(x)=\int_{x_0}^{x}\frac{dy}{y^2}=-\frac{1}{y}\Big|_{y=x_0}^{y=x}=-\frac{1}{x}+\frac{1}{x_0}$$

$\phi(t)=F^{-1}(t)$

$T_+=\lim_{x\to x_2}F(x)=\lim_{x\to\infty}-\frac{1}{x}+\frac{1}{x_0}=\frac{1}{x_0}$, so ϕ is defined for all $t>0$

$$t=F(F^{-1}(t))=\frac{-1}{F^{-1}(t)}+\frac{1}{x_0}$$

$$\Rightarrow \frac{1}{F^{-1}(t)}=\frac{1}{x_0}-t$$

$$\Rightarrow F^{-1}(t)=\frac{1}{\frac{1}{x_0}-t}$$

$$\Rightarrow \phi(t)=\frac{1}{\frac{1}{x_0}-t}$$

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Week 3 - Sept. 20th

Office Hours: Wed. 10-11 am IC404

A particular solution is a solution that has no arbitrary constant. A general solution is a k-parameter family of solutions that contain every particular solution

A separable first-order ODE is an ODE that can be rewritten as:

$$\boxed{\dot{x}(t,x) = g(t)f(x)} \qquad (\text{or } g(t)\,dt + f(x)\,dx = 0)$$

$$\Rightarrow \quad \frac{dx}{dt} = g(t)f(x)$$

$$\Rightarrow \quad \frac{dx}{f(x)} = g(t)\,dt$$

$$\Rightarrow \quad \boxed{\int \frac{dx}{f(x)} = \int g(t)\,dt}\ , \quad f(x) \neq 0$$

Ex: Show that the equation $\dot{x}(t,x) = \dfrac{t^2}{1-x^2}$ is separable and solve.

Let $f(x) = \dfrac{1}{1-x^2}$ and $g(t) = t^2$

$$\Rightarrow \int 1-x^2\,dx = \int t^2\,dt$$

$$x - \frac{x^3}{3} = \frac{t^3}{3} + C_1$$

$$3x - x^3 - t^3 = C \quad , C \in \mathbb{R}$$

Ex: Solve the equation $\dfrac{dx}{dt} = \dfrac{4t - t^3}{4 + x^3}$ and find the solution passing through the point $(0,1)$

$$\Rightarrow (4+x^3)\,dx = (4t - t^3)\,dt$$

$$\int (4+x^3)\,dx = \int (4t - t^3)\,dt$$

$$4x + \frac{x^4}{4} = 2t^2 - \frac{t^4}{4} + C_1$$

$$\Rightarrow 16x + x^4 - 8t^2 + t^4 = C \quad , C \in \mathbb{R},$$

When $(0,1) \rightarrow 16 + 1 = 17$

$$\therefore 16x + x^4 - 8t^2 + t^4 = 17$$

Let $z=f(x,y)$ be a function of x and y, $f(x,y)$ is homogeneous of order n if it can be written as $\boxed{f(x,y)=x^n g(u)}$, where $u=\frac{y}{x}$ or $f(x,y)=y^n g(u)$, where $u=\frac{x}{y}$.

$\boxed{P(x,y)dx+Q(x,y)dy=0}$ where $P(x,y)$ and $Q(x,y)$ are the homogeneous coefficients and can be solved by substituting $y=ux$, $dy=udx+xdu$

Ex: Find the general solution of $txx'=t^2+2x^2$

$$\Rightarrow \frac{1}{t^2}\left[txx'=t^2+2x^2\right]$$

$$\frac{x}{t}x'=1+2\frac{x^2}{t^2} \qquad \text{Let } u=\frac{x}{t} \quad \text{so} \quad x=ut$$
$$x'=u+tu'$$

$$\Rightarrow u(u+tu')=1+2u^2$$
$$u^2+tu'u=1+2u^2$$
$$tu'u=1+u^2$$
$$\frac{u'u}{1+u^2}=\frac{1}{t} \quad \rightarrow \text{OMG! This is separable!}$$

$$\int\frac{uu'}{1+u^2}=\int\frac{1}{t}dt \qquad \text{let } v=1+u^2$$
$$dv=2udu$$

$$\frac{1}{2}\int\frac{dv}{v}=\ln|t|+C_1$$
$$\frac{1}{2}\ln|1+u^2|=\ln|t|+C$$
$$\therefore \frac{1}{2}\ln\left|1+\frac{x^2}{t^2}\right|-\ln|t|=C$$

A differential expression $\boxed{P(x,y)dx+Q(x,y)dy}$ is called an exact differential if it is the total differential of some function $f(x,y)$.
i.e. if $\boxed{P(x,y)=\frac{\partial}{\partial x}f(x,y)}$ and $\boxed{Q(x,y)=\frac{\partial}{\partial y}f(x,y)}$.
If we can find $f(x,y)$, $f(x,y)=C$ is the 1-parameter family of solutions.

$P(x,y)dx+Q(x,y)dy=0$ is exact iff $\frac{\partial P(x,y)}{\partial y}=\frac{\partial Q(x,y)}{\partial x}$.

Ex: Find the general solution of $y'=-\frac{1+2xy^2}{1+2x^2y}$

$$\Rightarrow (1+2x^2y)y'=-(1+2xy^2)$$
$$(1+2xy^2)+(1+2x^2y)y'=0$$

Is it exact? $P(x,y)=1+2xy^2$ $\qquad Q(x,y)=1+2x^2y$

$$\frac{\partial P(x,y)}{\partial y}=4xy \iff \frac{\partial Q(x,y)}{\partial x}=4xy \quad \checkmark$$

let's take $P(x,y)$, we know $P(x,y) = \frac{\partial}{\partial x} f(x,y)$

$$\Rightarrow \int P(x,y)\,dx = f(x,y)$$

$$\int 1+2xy^2\,dx = x + x^2y^2 + g(y) = f(x,y)$$

$$\Rightarrow \frac{\partial}{\partial y} f(x,y) = 2x^2y + g'(y) = Q(x,y) = 1 + 2x^2y$$

So $g'(y) = 1 \rightarrow g(y) = y$

$$\therefore\ x + x^2y^2 + y = C$$

Week 4 - Sept. 27th

An integrating factor (IF) will convert an inexact ODE $P(x,y)\,dx + Q(x,y)\,dy = 0$ into an exact ODE $IF\,P(x,y)\,dx + IF\,Q(x,y)\,dy = 0$

Given $\frac{dy}{dx} + P(x)y = Q(x)$ a known IF is $e^{\int P(x)dx}$

Ex: Find the general solution of $\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3}$

$P(x) = \frac{3}{x}$, $IF = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

$$\Rightarrow x^3\frac{dy}{dx} + x^3 \cdot \frac{3y}{x} = x^3 \frac{e^x}{x^3}$$

$$\Rightarrow x^3y' + 3x^2y = e^x$$

$$\Rightarrow \int (x^3y' + 3x^2y)\,dx = \int e^x\,dx$$

$$\Rightarrow x^3y = e^x + C$$

$$\Rightarrow y = \frac{e^x + C}{x^3}$$

Ex: Solve $6y' - 2y = xy^4$, $y(0) = -2$.

$$\Rightarrow 6y^{-4}y' - 2y^{-3} = x$$

let $u = y^{-3}$
$du = -3y^{-4}y'$

$$\Rightarrow -2u' - 2u = x \quad \text{or} \quad u' + u = -\frac{x}{2}$$

$P(x) = 1 \quad , \quad IF = e^{\int p\,dx} = e^x$

$\Rightarrow \quad u'e^x + ue^x = -\frac{e^x x}{2}$

$\Rightarrow \int (u'e^x + ue^x)\,dx = -\frac{1}{2}\int xe^x\,dx$

Side note:

$\int xe^x dx \qquad u = x \quad dv = e^x dx$
$\qquad\qquad\quad du = dx \quad v = e^x$

$= xe^x - \int e^x dx = xe^x - e^x + C.$

$\Rightarrow \quad ue^x = -\frac{1}{2}(xe^x - e^x) + C$

$\Rightarrow \quad u = -\frac{1}{2}(x-1) + Ce^{-x}$

$\Rightarrow \quad y^{-3} = -\frac{1}{2}(x-1) + Ce^{-x}$

$\Rightarrow \quad 2y^{-3} = (1-x) + 2Ce^{-x}$

$\Rightarrow \quad y^3 = \frac{2}{(1-x) + 2Ce^{-x}}$

IVP

$\Rightarrow (-2)^3 = \frac{2}{1+2C}$

$\Rightarrow -8 - 16C = 2$

$\Rightarrow \quad C = -\frac{5}{8}$

$\Rightarrow \quad y = \left[\frac{2}{\left[(1-x) - \frac{5}{4}e^{-x}\right]}\right]^{1/3}$

Bernoulli Equation: given $\frac{dy}{dx} + P(x)y = Q(x)y^n$.

multiply by $(1-n)y^{-n} \Rightarrow (1-n)y^{-1}\frac{dy}{dx} + (1-n)y^{1-n}P(x) = (1-n)\,Q(x)$

Substitute $u = y^{1-n}$
$du = (1-n)\,y^{-n}dy \quad \Rightarrow \frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$

$IF\ e^{\int (1-n)\,P(x)dx}$

Let X be a real vector space. A norm on X is a map $\|\cdot\|: X \to [0,\infty)$ s.t.

(i) $\|0\| = 0$, $\|x\| > 0$ for $x \neq 0$
(ii) $\|\alpha x\| = |\alpha|\,\|x\|$ for $\alpha \in \mathbb{R}$ and $x \in X$
(iii) $\|x+y\| \leq \|x\| + \|y\|$ for $x, y \in X$

Together $(X, \|\cdot\|)$ is a normed vector space. A sequence of vectors x_n converges to x if $\lim_{n\to\infty} \|x_n - x\| = 0$.

A contraction is a mapping $K: C \subseteq X \to C$ where there exists a contraction constant $\theta \in [0,1)$ s.t. $\|K(x)-K(y)\| \leq \theta \|x-y\|$, $x, y \in C$.

Note: $K^n(x) = K(K^{n-1}(x))$ $\quad K^0(x) = x$

Ex: Let $K(x) = 40 + \frac{x}{3}$. What is the contraction constant? Is $K(x)$ a contraction on $C = [0, 90]$? on $C = [0, 30]$?

$$\|K(x) - K(y)\| = \|40 + \frac{x}{3} - 40 - \frac{y}{3}\| = \|\frac{x}{3} - \frac{y}{3}\| = \frac{1}{3}\|x-y\|$$

$\therefore \theta = \frac{1}{3}$

$C[0,90]$ ✓ $\quad K(0) = 40 \in [0,90]$

$C[0,30]$ ✗ $\quad K(0) = 40 \notin [0,30]$

Consider an initial value problem (IVP) $\dot{x} = f(t,x)$, $x(t_0) = x_0$, where $x, t \in \mathbb{R}$ and $f \in C(U, \mathbb{R})$ where $U \subseteq \mathbb{R}^2$ is an open subset of $\mathbb{R}^2$ and $(t_0, x_0) \in U$.

Let's define Picard Iteration by a map $K: C(U,\mathbb{R}) \to C(U, \mathbb{R})$

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

and the Picard iterates.

$x_0(t) = x_0$ (the constant function through the scalar x_0)

$$x_1(t) = K(x_0)(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\, ds$$

$$x_2(t) = K^2(x_0)(t) = K(x_1)(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))\, ds.$$

$\vdots$

$$x_m(t) = K^m(x_0)(t) = K(x_{m-1})(t) = x_0 + \int_{t_0}^{t} f(s, x_{m-1}(s))\, ds$$

Ex: Calculate the Picard Iterates x_0, x_1, x_2. for $f(t,x) = 3 - 2x$, $x(t_0) = x_0$

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

$$K^{m+1}(x)(t) = x_0 + \int_{t_0}^{t} f(s, x_m(s))\, ds = x_0 + \int_{t_0}^{t} (3 - 2x_m(s))\, ds.$$

$$\Rightarrow x_0(t) = x_0$$

$$\Rightarrow x_1(t) = K(x_0)(t) = x_0 + \int_{t_0}^{t} 3 - 2x_0(s)\,ds$$

$$= x_0 + \int_{t_0}^{t} 3 - 2x_0\,ds$$

$$= x_0 + (3-2x_0)(t-t_0)$$

$$= x_0 + t(3-2x_0) - t_0(3-2x_0)$$

$$\Rightarrow x_2(t) = K^2(x_0(t)) = x_0 + \int_{t_0}^{t} 3 - 2x_1(s)\,ds$$

$$= x_0 + \int_{t_0}^{t} 3 - 2\Big(x_0 + s(3-2x_0) - t_0(3-2x_0)\Big)\,ds$$

$$= x_0 + \left[3 - 2x_0 + 2t_0(3-2x_0)\right](t-t_0) - 2(3-2x_0)\int_{t_0}^{t} s\,ds$$

$$= x_0 + (3-2x_0)(1+2t_0)(t-t_0) - (3-2x_0)\,s^2\Big|_{t_0}^{t}$$

$$= x_0 + (3-2x_0)(1+2t_0)(t-t_0) - (3-2x_0)(t^2-t_0).$$