MATA02 - The Magic of Numbers

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Week 1 - Jan. 6th

No tutorials

Week 2 - Jan 13th

Today: Greatest Common Divisor (gcd) and Least Common Multiple (lcm)

↳ $\gcd(6,8)=2$ ↳ $\text{lcm}(6,8)=24$

Euclidean Algorithm: If $a=bq+r$, $0\le r<b$ then $\gcd(a,b)=\gcd(b,r)$

- $\gcd(a,0)=a$
- $\gcd(0,b)=b$
- $\text{lcm}(a,b)=\dfrac{ab}{\gcd(a,b)}$

Ex1: Use the Euclidean Algorithm to find $\gcd(51,96)$ and use it to find the $\text{lcm}(51,96)$

$96 = 51\cdot 1 + 45 \quad \rightarrow d=\gcd(96,51)$

$51 = 45\cdot 1 + 6 \quad \rightarrow d=\gcd(51,45)$

$45 = 6\cdot 7 + 3 \quad \rightarrow d=\gcd(45,6)$

$6 = \underline{\underline{3}}\cdot 2 + 0 \quad \rightarrow d=\gcd(6,3)$

↳ stop!

$\therefore \gcd(96,51)=\gcd(51,45)=\gcd(45,6)=\gcd(6,3)=3$

$\therefore \text{lcm}(96,51)=\dfrac{96\times 51}{\gcd(96,51)}=\dfrac{96\times 51}{3}=96\times 17=1632$

- If $a=ms$ and $b=mt$, then $\gcd(a,b)=m\times\gcd(s,t)$

Ex1: (Again!) but using this method.

$\gcd(51,96)=\gcd(3\cdot 17, 3\cdot 32)=3\cdot\gcd(17,32)=3\cdot 1=3$ ✓

Ex2: (T1-Q1) Use the Euclidean Algorithm to find gcd(366,150) and use it to find the lcm(366, 150)

$366 = 150 \cdot 2 + 66 \quad \rightarrow d = \gcd(366,150)$

$150 = 66 \cdot 2 + 18 \quad \rightarrow d = \gcd(150,66)$

$66 = 18 \cdot 3 + 12 \quad \rightarrow d = \gcd(66,18)$

$18 = 12 \cdot 1 + 6 \quad \rightarrow d = \gcd(18,12)$

$12 = \underline{\underline{6}} \cdot 2 + 0 \quad \rightarrow d = \gcd(12,6)$

$\therefore \gcd(366,150) = 6$

$\therefore \text{lcm}(366,150) = \dfrac{366 \times 150}{\gcd(366,150)} = \dfrac{366 \times 150}{6} = 61 \times 150 = 9150$

Ex3: (T1-Q3) Does the equation $12x + 20y = 90$ have a solution of integers x and y?

No :(Because on the left side we can factor out 4 → $4(3x+5y)$ however, 90 is <u>not</u> divisible by 4.

Ex4: Does the equation $11x + 1111 = 121 + 22y$ have a solution of integers x and y?

No :(. Because the right side is divisible by 11 → $11(11+2y)$, but the left side isn't as 1111 is not divisible by 11.

Ex5: (T1-Q4) If a and b are integers, and x is an integer such that $x^2+ax+b=0$. Show that $x \mid b$

↳ b is divisible by x

$x^2+ax+b=0$

$\Rightarrow b = -x^2 - ax$

$\Rightarrow b = -x\underbrace{(x+a)}$

↳ is an integer because x and a are integers

Because b is also an integer, then $x \mid b$. ∎

Ex6: Show that if $a \mid b$ and $b \mid a$, then $a=b$ or $a=-b$

$a \mid b$ means that there exist an $n \in \mathbb{Z}$ such that $an = b$ ①

$b \mid a$ means that there exist an $m \in \mathbb{Z}$ such that $bm = a$ ②

Use ① for ②: $(an)m = a \Rightarrow anm = a \Rightarrow nm = 1$.

The only integers n and m that work for $nm=1$ are $n=m=1$ or $n=m=-1$

→ When $n=m=1 \Rightarrow a=b$, by ①

→ When $n=m=-1 \Rightarrow a=-b$, by ② $\boxtimes$

Week 3 - Jan 20th

Ex 1 (T2-Q1) Find integers x and y such that $6x+5y=4$

Let's do Euclid's Algo!

$6 = 5 \cdot 1 + 1$ → But wait! This line here might help:

$5 = 1 \cdot 4 + 1$

$1 = 1 \cdot 1 + 0$

Let's write it differently: $6 - 5 \cdot 1 = 1$

Now, we need the equation to equal 4: $(6 - 5 \cdot 1 = 1) \times 4$

$6 \cdot 4 - 5 \cdot 4 = 4$

The signs are different... $6 \cdot \underbrace{4}_{x} + 5 \cdot \underbrace{(-4)}_{y} = 4$

$\therefore$ In $6x+5y=4$, $x=4, y=-4$

Ex 2 Determine if the equations has integer solutions x and y.

eq 1 $\quad 1015x + 231y = 9$

eq 2 $\quad 1015x + 231y = 28$

① Find the gcd of $(1015, 231)$

$1015 = 231 \times 4 + 91 \quad (1)$

$231 = 91 \times 2 + 49 \quad (2)$

$91 = 49 \times 1 + 42 \quad (3)$

$49 = 42 \times 1 + 7 \quad (4)$

$42 = 7 \times 6 + 0$ → $\gcd(1015, 231) = 7$

So by definition $\exists m, n$ such that $1015m + 231n = 7$

Looking at eq 1 and eq 2, $7 \nmid 9$ and $7 \mid 28$. So only eq 2 has integer solution.

② Write gcd in terms of the values.

$$
\begin{aligned}
7 &= 49 - 42\times 1 && (4)\\
&= 49 - (91-49\times 1) && (3)\\
&= 49\times 2 - 91\\
&= (231-91\times 2)\times 2 - 91 && (2)\\
&= 231\times 2 - 91\times 4 - 91\\
&= 231\times 2 - 91\times 5\\
&= 231\times 2 - (1015-231\times 4)\times 5 && (1)\\
&= 231\times 2 - 1015\times 5 + 231\times 20\\
7 &= 231\times 22 + 1015\times(-5)
\end{aligned}
$$

③
$$
\begin{aligned}
4\times 7 &= 231\times 22\times 4 + 1015\times(-5)\times 4\\
28 &= 231\times \underbrace{88}_{y} + 1015\times\underbrace{(-20)}_{x}
\end{aligned}
$$

④ We want to find a general solution for $28 = 1015X + 231Y$

Theorem: If $d=\gcd(a,b)\mid c$ and x,y is an integer solution of $ax+by=c$, then so is $x+\frac{b}{d}t$, $y-\frac{a}{d}t$ for any integer t.

In this case, $d=7\mid 28$ and we have $28=\underbrace{1015}_{a}x+\underbrace{231}_{b}y$

We know $x=-20$ and $y=88$ in ③

The general solution has $X = -20+\frac{231}{7}t = -20+33t$

$Y = 88-\frac{1015}{7}t = 88-145t$

$\therefore$ The general solution is $1015(-20+33t)+231(88-145t)=28$

Ex3 (T2-Q2) Find all integers x and y such that $30x+8y=500$. For which solutions are x and y both positive?

①
$$
\begin{aligned}
30 &= 8\times 3 + 6 && (1)\\
8 &= 6\times 1 + 2 && (2)\\
6 &= 2\times 3 + 0 && \rightarrow \gcd(30,8)=2
\end{aligned}
$$

②
$$
\begin{aligned}
2 &= 8 - 6\times 1 && (2)\\
&= 8 - (30-8\times 3)\times 1 && (1)\\
&= 8 - 30\times 1 + 8\times 3\\
2 &= 8\times 4 + 30\times(-1)
\end{aligned}
$$

③
$$
\begin{aligned}
250\times 2 &= 8\times 4\times 250 + 30\times(-1)\times 250\\
500 &= 8\times\underbrace{1000}_{y} + 30\times\underbrace{(-250)}_{x}
\end{aligned}
$$

④ General solution: $x = -250 + \frac{8}{2}t = -250 + 4t$

$y = 1000 - \frac{30}{2}t = 1000 - 15t$

$x \geq 0 \rightarrow -250 + 4t \geq 0$

$4t \geq 250$

$t \geq \frac{125}{2} = 62.5$

$y \geq 0 \rightarrow 1000 - 15t \geq 0$

$1000 \geq 15t$

$\frac{200}{3} \geq t \approx 66.\overline{6}$

So $62.5 \leq t \leq 66.\overline{6}$, t is an integer

$\therefore$ $t = 63, 64, 65, 66$ to have x and y positive

Ex 4 (T2-Q4) Suppose the sophomores, juniors, and seniors in the tutorial decided to collect money to host a party. If each sophomore contributes \$25, each junior contributes \$18, each senior contributes \$10, \$450 will be collected. If there are 35 students, how many sophomores, juniors, and seniors are there?

$x + y + z = 35$ ①

$25x + 18y + 10z = 450$ ②

① $\times 10 \rightarrow 10x + 10y + 10z = 350$ ③

② - ③ $\rightarrow 15x + 8y = 100$

let's do Euclid's Alg. to find the gcd of 15 and 8

①
$15 = 8 \times 1 + 7$ (1)

$8 = 7 \times 1 + 1$ (2)

$7 = 1 \times 7 + 0$ $\rightarrow \gcd(15, 8) = 1$

②
$1 = 8 - 7 \times 1$ (2)

$1 = 8 - (15 - 8 \times 1) \times 1$ (1)

$1 = 8 \times 2 - 15 \times 1$

$1 = 8 \times \underbrace{2}_{y} + 15 \times \underbrace{(-1)}_{x}$

↳ but there cannot be a negative solution!

③
$100 \times 1 = 8 \times 2 \times 100 + 15 \times (-1) \times 100$

$100 = 8 \times 200 + 15 \times (-100)$

④ General solution: $x = -100 + \frac{8t}{1} = -100+8t$

$y = 200 - \frac{15t}{1} = 200 - 15t$

We need to find $x \geq 0$, $y \geq 0$

$-100 + 8t \geq 0$ $\quad$ $200 - 15t \geq 0$

$8t \geq 100$ $\quad$ $200 \geq 15t$

$t \geq \frac{25}{2} = 12.5$ $\quad$ $13.\bar{3} \approx \frac{40}{3} \geq t$

$\Rightarrow 12.5 \leq t \leq 13.\bar{3}$, and t is an integer.

$\therefore t = 13 \rightarrow x = -100 + 8 \times 13 = -100 + 104 = 4$

$\rightarrow y = 200 - 15 \times 13 = 200 - 195 = 5$

$\rightarrow x + y + z = 35 \Rightarrow 4 + 5 + z = 35$

$\Rightarrow z = 26$

Ex 5 (T2-Q5) Show there are infinitely primes of the form $4t-1$, where t is an integer

Let's do a proof by contradiction. Which means, we will assume there are only a finite number of primes of the form $4t-1$, we will call them $p_1, p_2, \cdots, p_r$.

Let $N = p_1 p_2 \cdots p_r$, and let's consider the number $4N-1$. By our initial assumption, $4N-1$ is not prime, so it has a prime factor!

We claim $4N-1$ has a prime factor of the form $4t-1$.

Now, let's see what happens when we multiply $4x+1$ and $4y+1$

$\rightarrow (4x+1)(4y+1) = 16xy + 4x + 4y + 1 = 4(4xy + x + y) + 1$

Which means, $4N-1$ must have a prime factor of form $4t-1$. Since 2 is not a factor.

So there is a p_i such that $p_i \mid 4N-1$. Then as $p_i \mid N$, we have $p_i \mid 1$. Which contradicts!

There are infinitely many primes of the form $4t-1$. ☒

Week 4 - Jan 27th

If n is composite, then it has a prime divisor p such that $p \leq \sqrt{n}$.

To determine if n is prime, check whether each of the prime numbers up to $\sqrt{n}$ divide n

Ex1 (T3-Q1) Which of the following numbers are prime?

(a) 407

$\sqrt{407} < 21$ which means that we only need to check for primes up to 20

primes up to 20 $= \{2, 3, 5, 7, 11, 13, 17, 19\}$

- $2 \mid 407$? Nope! $407 = 2 \times 203 + 1$
- $3 \mid 407$? Nope! $407 = 3 \times 135 + 2$
- $5 \mid 407$? Nope! $407 = 5 \times 81 + 2$
- $7 \mid 407$? Nope! $407 = 7 \times 58 + 1$
- $11 \mid 407$? Yes $407 = 11 \times 37$

$\therefore$ 407 is not prime.

(b) 463

$\sqrt{463} < 22$ which means that we only need to check for primes up to 21

primes up to 21 $= \{2, 3, 5, 7, 11, 13, 17, 19\}$

- $2 \mid 463$? Nope! $463 = 2 \times 231 + 1$
- $3 \mid 463$? Nope! $463 = 3 \times 154 + 1$
- $5 \mid 463$? Nope! $463 = 5 \times 92 + 3$
- $7 \mid 463$? Nope! $463 = 7 \times 66 + 1$
- $11 \mid 463$? Nope! $463 = 11 \times 42 + 1$
- $13 \mid 463$? Nope! $463 = 13 \times 35 + 8$
- $17 \mid 463$? Nope! $463 = 17 \times 27 + 4$
- $19 \mid 463$? Nope! $463 = 19 \times 24 + 7$

$\therefore$ 463 is prime.

Sieve of Eratosthenes: To find all prime numbers between a and b $(a \leq b)$, for each prime p up to $\sqrt{b}$, cross out every multiple of p. The remaining numbers are primes.

Ex2 Find all primes between 235 and 265

$\sqrt{265} < 17$ which means that we only need to check for primes up to 16

primes up to 16 = {2, 3, 5, 7, 11, 13}

~~235~~	~~236~~	~~237~~	~~238~~	239	~~240~~	241	~~242~~	~~243~~	~~244~~
~~245~~	~~246~~	~~247~~	~~248~~	~~249~~	~~250~~	251	~~252~~	~~253~~	~~254~~
~~255~~	~~256~~	257	~~258~~	~~259~~	~~260~~	~~261~~	~~262~~	263	~~264~~
~~265~~									

Step #1: Cross out all #s that are divisible by 2

Step #2: Cross out all #s that are divisible by 3

Step #3: Cross out all #s that are divisible by 5

Step #4: Cross out all #s that are divisible by 7

Step #5: Cross out all #s that are divisible by 11

Step #6: Cross out all #s that are divisible by 13

∴ The primes in between 235 and 265 are {239, 241, 251, 257, 263}

Ex3 (T3-Q2) Find all primes between 201 and 250

$\sqrt{250} < 16$ which means that we only need to check for primes up to 15

primes up to 15 = {2, 3, 5, 7, 11, 13}

~~201~~	~~202~~	~~203~~	~~204~~	~~205~~	~~206~~	~~207~~	~~208~~	~~209~~	~~210~~
211	~~212~~	~~213~~	~~214~~	~~215~~	~~216~~	~~217~~	~~218~~	~~219~~	~~220~~
~~221~~	~~222~~	223	~~224~~	~~225~~	~~226~~	227	~~228~~	229	~~230~~
~~231~~	~~232~~	233	~~234~~	~~235~~	~~236~~	~~237~~	~~238~~	239	~~240~~
241	~~242~~	~~243~~	~~244~~	~~245~~	~~246~~	~~247~~	~~248~~	~~249~~	~~250~~

∴ The primes in between 201 and 250 are {211, 223, 227, 229, 233, 239, 241}

Ex 4 (T3-Q3) Find the prime-power decomposition for each of the following.

(a) $437 = 19 \times 23$

(b) $709 = 709$

(c) $876 = 2^2 \times 3 \times 73$

- ★ An element x of a set is called a **unit** if there exists an element y such that $xy = 1$

- ★ An **irreducible** is a non-unit element of A whose only positive divisors are itself, units, and products of itself and units.

Ex: Can we write a # in two different products of irreducibles, $A = 3t+1$?
$\Rightarrow 100 = 4 \times 25 = 10 \times 10$

Ex 5 (T3-Q4) Consider the set containing positive integers of the form $5t+1$. Find an element of B with two different irreducible decompositions in B.

$t=1 \to 6 = 2 \times 3$
$t=2 \to 11 = 11$
$t=3 \to 16 = 2^4$
$t=4 \to 21 = 3 \times 7$
$t=5 \to 26 = 2 \times 13$
$t=6 \to 31 = 31$
$t=7 \to 36 = 2^2 \times 3^2$
$t=8 \to 41 = 41$
$t=9 \to 46 = 2 \times 23$
$t=10 \to 51 = 3 \times 17$
$t=11 \to 56 = 2^3 \times 7$
$\vdots$

Any common factors? $2^4, 3, 7$

$2^4 \times 3 \times 7 = 336 = 335 + 1 = 5 \times 67 + 1$

How can we decompose 336? Remember they need to be in B ($5t+1$ form)

$2 \times 2 \times 2 \times 2 \times 3 \times 7$ ② ② ① ①

$336 = 56 \times 6 = 16 \times 21$
$\to$ both are irreducible.

Ex 5 (T3-Q5) Suppose n is composite, and let p be the smallest prime divisor of n. If $p > \sqrt[3]{n}$, show that n/p is prime.

Suppose n/p is not prime. Then $n/p = q \times r$ where $q, r \in \mathbb{Z}$. Since p is the smallest prime divisor of n, then $p \le q$ and $p \le r$.

$\Rightarrow n = p \times q \times r \geq p^3 > (\sqrt[3]{n})^3 = n$

$\hookrightarrow n > n$?! impossible!

$\therefore n/p$ is prime $\blacksquare$

Week 5 - Feb 3rd

Ex 1 (T4-Q1) Find the prime factorization of the product of 4460 and 6146

$$\begin{array}{r|l} 4460 & 2 \\ 2230 & 2 \\ 1115 & 5 \\ 223 & 223 \\ 1 & \end{array}$$

$223 \rightarrow$ Check: 7, 11, 13 because $< \sqrt{223}$

$$\begin{array}{r|l} 6146 & 2 \\ 3073 & 7 \\ 439 & 439 \\ 1 & \end{array}$$

$439 \leftarrow$ Check: 11, 13, 17, 19 because $\sqrt{439}$

So $4460 = 2^2 \times 5 \times 223$ and $6146 = 2 \times 7 \times 439$

$\Rightarrow 4460 \times 6146 = 2^2 \times 5 \times 223 \times 2 \times 7 \times 439$
$= 2^3 \times 5 \times 7 \times 223 \times 439.$

Ex 2 (T4-Q2) Which of the following is the square of a fraction?

(a) $\frac{3^2 \times 5^3}{2^6}$ NO. odd exponent.

(b) $\frac{3^2 \times 2^2}{5^4 \times 7^8}$ Yes! $\left(\frac{3 \times 2}{5^2 \times 7^4}\right)^{4th}$

(c) $\frac{3^6 \times 2}{7^2 \times 2^3}$ Yes! $\frac{3^3}{7 \times 2}$

(d) $13^2 \times 3^4$ Yes! (13×42)

Ex 3 (T4-Q3) Using prime factorizations, compute the gcd and lcm of 7200 and 4374.

$$\begin{array}{r|l} 7200 & 2 \\ 3600 & 2 \\ 1800 & 2 \\ 900 & 2 \\ 450 & 2 \\ 225 & 3 \\ 75 & 3 \\ 25 & 5 \\ 5 & 5 \\ 1 & \end{array} \qquad \begin{array}{r|l} 4374 & 2 \\ 2187 & 3 \\ 729 & 3 \\ 243 & 3 \\ 81 & 3 \\ 27 & 3 \\ 9 & 3 \\ 3 & 3 \\ 1 & \end{array}$$

$\rightarrow 7200 = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5$
$= 2^5 \times 3^2 \times 5^2$

$\rightarrow 4374 = 2 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$
$= 2 \times 3^7$

For the gcd, we will consider the common prime factors and smallest exponents.

$7200 = 2^5 \times 3^2 \times 5^2$ $\quad$ $4374 = 2 \times 3^7$

$\Rightarrow gcd(7200, 4374) = 2 \times 3^2$

For the lcm, we will consider all prime factors and biggest exponents.

$7200 = 2^5 \times 3^2 \times 5^2$ $\quad$ $4374 = 2 \times 3^7$

$\Rightarrow lcm(7200, 4374) = 2^5 \times 3^7 \times 5^2$

Euler's Number: $\varnothing(n)$ to get the number of numbers from 1 to n that are relatively prime.

* If p is prime, then $\varnothing(p) = p-1$
* If p is prime, then $\varnothing(p^n) = (p-1)p^{n-1}$
* If m and n are relatively prime, then $\varnothing(mn) = \varnothing(m)\varnothing(n)$
* Let $n = p_1^{e_1} \cdots p_r^{e_r}$ be the prime factorization of n, then $\varnothing(n) = n \times \frac{p_1 - 1}{p_1} \times \cdots \times \frac{p_r - 1}{p_r}$

__Ex 4__ Calculate the following:

(a) $\varnothing(48) = \varnothing(2^4 \times 3) = 48 \times \frac{2-1}{2} \times \frac{3-1}{3} = 2^4 \times 3 \times \frac{1}{2} \times \frac{2}{3} = 2^4 = 16$

(b) $\varnothing(480) = \varnothing(2^5 \times 3 \times 5) = 480 \times \frac{2-1}{2} \times \frac{3-1}{3} \times \frac{5-1}{5} = 2^5 \times 3 \times 5 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} = 2^7 = 128$

(c) $\varnothing(4800) = \varnothing(2^6 \times 3 \times 5^2) = 4800 \times \frac{2-1}{2} \times \frac{3-1}{3} \times \frac{5-1}{5} = 2^6 \times 3 \times 5^2 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} = 2^8 \times 5 = 1280$

(d) $\varnothing(55) = \varnothing(5 \times 11) = \varnothing(5) \times \varnothing(11) = (5-1)(11-1) = 4 \times 10 = 40$

$= 55 \times \frac{5-1}{5} \times \frac{11-1}{11} = 5 \times 11 \times \frac{4}{5} \times \frac{10}{11} = 40$

(e) $\varnothing(89) = 89 - 1 = 88$

(f) $\varnothing(32) = \varnothing(2^5) = (2-1)2^{5-1} = 2^4 = 16$

__Ex 5__ (T4-Q5) Find four values of n such that $\varnothing(n) = 8 = 2^3$

Remember $\varnothing(2^k) = 2^{k-1}$, $\varnothing(3) = 2$, $\varnothing(5) = 4 = 2^2$

Then for $\phi(n)=8$, $\phi(2^4)=2^3 \longrightarrow \phi(16)$

$\phi(3)\phi(5)=8 \longrightarrow \phi(15)$

$\phi(3)\phi(2^3)=8 \longrightarrow \phi(24)$

$\phi(2^2)\phi(5)=8 \longrightarrow \phi(20)$

$\therefore \phi(16)=\phi(15)=\phi(24)=\phi(20)=8$.