

MATA02 - The Magic of Numbers

TA: Angela Zavaleta-Bernuy

Tutorials: T2 (HW308 Thursdays 9-10 am)

T3 (AA206 Tuesdays 9-10 am)

T6 (MW160 Thursdays 10-11 am)

Office hours: IC404 Mondays 2-4 pm

Email: angela.zavaletabernuy@mail.utoronto.ca

Website: angelazb.github.io

Week 1 - Jan. 6th

No tutorials

Week 2 - Jan 13th

Today: Greatest Common Divisor (gcd) and Least Common Multiple (lcm)

$$\hookrightarrow \text{gcd}(6, 8) = 2$$

$$\hookrightarrow \text{lcm}(6, 8) = 24$$

Euclidean Algorithm: If $a = bq + r$, $0 \leq r < b$ then $\text{gcd}(a, b) = \text{gcd}(b, r)$

$$* \text{gcd}(a, 0) = a$$

$$* \text{gcd}(0, b) = b$$

$$* \text{lcm}(a, b) = \frac{ab}{\text{gcd}(a, b)}$$

Ex1: Use the Euclidean Algorithm to find $\text{gcd}(51, 96)$ and use it to find the $\text{lcm}(51, 96)$

$$\begin{aligned} 96 &= 51 \cdot 1 + 45 & \rightarrow d = \text{gcd}(96, 51) \\ 51 &= 45 \cdot 1 + 6 & \rightarrow d = \text{gcd}(51, 45) \\ 45 &= 6 \cdot 7 + 3 & \rightarrow d = \text{gcd}(45, 6) \\ 6 &= 3 \cdot 2 + 0 & \xrightarrow{\text{stop!}} d = \text{gcd}(6, 3) \end{aligned}$$

$$\therefore \text{gcd}(96, 51) = \text{gcd}(51, 45) = \text{gcd}(45, 6) = \text{gcd}(6, 3) = 3$$

$$\therefore \text{lcm}(96, 51) = \frac{96 \times 51}{\text{gcd}(96, 51)} = \frac{96 \times 51}{3} = 96 \times 17 = 1632$$

* If $a = ms$ and $b = mt$, then $\text{gcd}(a, b) = m \times \text{gcd}(s, t)$

Ex1: (Again!) but using this method.

$$\text{gcd}(51, 96) = \text{gcd}(3 \cdot 17, 3 \cdot 32) = 3 \cdot \text{gcd}(17, 32) = 3 \cdot 1 = 3 \quad \checkmark$$

Ex2: (T1-Q1) Use the Euclidean Algorithm to find $\gcd(366, 150)$ and use it to find the $\text{lcm}(366, 150)$

$$366 = 150 \cdot 2 + 66 \rightarrow d = \gcd(366, 150)$$

$$150 = 66 \cdot 2 + 18 \rightarrow d = \gcd(150, 66)$$

$$66 = 18 \cdot 3 + 12 \rightarrow d = \gcd(66, 18)$$

$$18 = 12 \cdot 1 + 6 \rightarrow d = \gcd(18, 12)$$

$$12 = 6 \cdot 2 + 0 \rightarrow d = \gcd(12, 6)$$

$$\therefore \gcd(366, 150) = 6$$

$$\therefore \text{lcm}(366, 150) = \frac{366 \times 150}{\gcd(366, 150)} = \frac{366 \times 150}{6} = 61 \times 150 = 9150$$

Ex3: (T1-Q3) Does the equation $12x+20y=90$ have a solution of integers x and y ?

No \because Because on the left side we can factor out 4 $\rightarrow 4(3x+5y)$ however, 90 is not divisible by 4.

Ex4: Does the equation $11x+1111=121+22y$ have a solution of integers x and y ?

No \because Because the right side is divisible by 11 $\rightarrow 11(11+2y)$, but the left side isn't as 1111 is not divisible by 11.

Ex5: (T1-Q4) If a and b are integers, and x is an integer such that $x^2+ax+b=0$. Show that $x|b$

$\hookrightarrow b$ is divisible by x

$$x^2+ax+b=0$$

$$\Rightarrow b = -x^2 - ax$$

$$\Rightarrow b = -x \underbrace{(x+a)}$$

\hookrightarrow is an integer because x and a are integers

Because b is also an integer, then $x|b$. \square

Ex6: Show that if $a|b$ and $b|a$, then $a=b$ or $a=-b$

$a|b$ means that there exist an $n \in \mathbb{Z}$ such that $an=b$ ①

$b|a$ means that there exist an $m \in \mathbb{Z}$ such that $bm=a$ ②

Use ① for ②: $(an)m=a \Rightarrow anm=a \Rightarrow nm=1$.

The only integers n and m that work for $nm=1$ are $n=m=1$ or $n=m=-1$

→ When $n=m=1 \Rightarrow a=b$, by ①

→ When $n=m=-1 \Rightarrow a=-b$, by ②

□

Week 3 - Jan 20th

Ex 1 (T2-Q1) Find integers x and y such that $6x+5y=4$

let's do Euclid's Algo!

$$\begin{aligned} 6 &= 5 \cdot 1 + 1 \\ 5 &= 1 \cdot 4 + 1 \\ 1 &= 1 \cdot 1 + 0 \end{aligned} \rightarrow \text{But wait! This line here might help:}$$

let's write it differently: $6 - 5 \cdot 1 = 1$

Now, we need the equation $(6 - 5 \cdot 1 = 1) \times 4$
to equal 4:

$$\underline{\underline{6}} \cdot \underline{\underline{4}} - \underline{\underline{5}} \cdot \underline{\underline{4}} = \underline{\underline{4}}$$

The signs are different...

$$6 \cdot \underbrace{4}_{x} + 5 \cdot \underbrace{(-4)}_{y} = 4$$

∴ In $6x+5y=4$, $x=4, y=-4$

Ex 2 Determine if the equations has integer solutions x and y .

$$\text{eq 1 } 1015x + 231y = 9$$

$$\text{eq 2 } 1015x + 231y = 28$$

① Find the gcd of $(1015, 231)$

$$\begin{aligned} 1015 &= 231 \times 4 + 91 & (1) \\ 231 &= 91 \times 2 + 49 & (2) \\ 91 &= 49 \times 1 + 42 & (3) \\ 49 &= 42 \times 1 + 7 & (4) \\ 42 &= 7 \times 6 + 0 & \rightarrow \gcd(1015, 231) = 7 \end{aligned}$$

so by definition $\exists m, n$ such that $1015m + 231n = 7$

Looking at eq 1 and eq 2, $7 \nmid 9$ and $7 \mid 28$. So only eq 2 has integer solution.

② Write gcd in terms of the values.

$$\begin{aligned}
 7 &= 49 - 42 \times 1 && (4) \\
 &= 49 - (91 - 49 \times 1) && (3) \\
 &= 49 \times 2 - 91 && \\
 &= (231 - 91 \times 2) \times 2 - 91 && (2) \\
 &= 231 \times 2 - 91 \times 4 - 91 \\
 &= 231 \times 2 - 91 \times 5 \\
 &= 231 \times 2 - (1015 - 231 \times 4) \times 5 && (1) \\
 &= 231 \times 2 - 1015 \times 5 + 231 \times 20 \\
 7 &= 231 \times 22 + 1015 \times (-5)
 \end{aligned}$$

③ $9 \times 7 = 231 \times 22 \times 4 + 1015 \times (-5) \times 4$
 $28 = 231 \times \underbrace{88}_y + 1015 \times \underbrace{(-20)}_x$

④ We want to find a general solution for $28 = 1015X + 231Y$

Theorem: If $d = \gcd(a, b) | c$ and x, y is an integer solution of $ax + by = c$, then so is $x + \frac{b}{d}t, y - \frac{a}{d}t$ for any integer t .

In this case, $d = 7 | 28$ and we have $28 = \underbrace{1015}_a X + \underbrace{231}_b Y$
We know $x = -20$ and $y = 88$ in ③

The general solution has $X = -20 + \frac{231}{7}t = -20 + 33t$

$$Y = 88 - \frac{1015}{7}t = 88 - 145t$$

$$\therefore \text{The general solution is } 1015(-20+33t) + 231(88-145t) = 28$$

Ex3 (T2-Q2) Find all integers x and y such that $30x + 8y = 500$. For which solutions are x and y both positive?

① $30 = 8 \times 3 + 6 \quad (1)$
 $8 = 6 \times 1 + 2 \quad (2)$
 $6 = 2 \times 3 + 0 \quad \rightarrow \gcd(30, 8) = 2$

② $2 = 8 - 6 \times 1 \quad (2)$
 $= 8 - (30 - 8 \times 3) \times 1 \quad (1)$
 $= 8 - 30 \times 1 + 8 \times 3$
 $2 = 8 \times 4 + 30 \times (-1)$

③ $250 \times 2 = 8 \times 4 \times 250 + 30 \times (-1) \times 250$
 $500 = 8 \times \underbrace{1000}_y + 30 \times \underbrace{(-250)}_x$

$$\textcircled{4} \text{ General Solution: } x = -250 + \frac{8}{2} t = -250 + 4t$$

$$y = 1000 - \frac{30}{2} t = 1000 - 15t$$

$$x \geq 0 \rightarrow -250 + 4t \geq 0$$

$$4t \geq 250$$

$$t \geq \frac{125}{2} = 62.5$$

$$y \geq 0 \rightarrow 1000 - 15t \geq 0$$

$$1000 \geq 15t$$

$$\frac{200}{3} \geq t \approx 66.\bar{6}$$

So $62.5 \leq t \leq 66.\bar{6}$, t is an integer

$\therefore t = 63, 64, 65, 66$ to have x and y positive

Ex 4 (T2-Q4) Suppose the sophomores, juniors, and seniors in the tutorial decided to collect money to host a party. If each sophomore contributes \$25, each junior contributes \$18, each senior contributes \$10, \$450 will be collected. If there are 35 students, how many sophomores, juniors, and seniors are there?

$$\begin{aligned} x + y + z &= 35 & \textcircled{1} \\ 25x + 18y + 10z &= 450 & \textcircled{2} \end{aligned}$$

$$\textcircled{1} \times 10 \rightarrow 10x + 10y + 10z = 350 \quad \textcircled{3}$$

$$\textcircled{2} - \textcircled{3} \rightarrow 15x + 8y = 100$$

let's do Euclid's Alg. to find the gcd of 15 and 8

$$\begin{aligned} \textcircled{1} \quad 15 &= 8 \times 1 + 7 & (1) \\ 8 &= 7 \times 1 + 1 & (2) \\ 7 &= 1 \times 7 + 0 & \rightarrow \gcd(15, 8) = 1 \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad 1 &= 8 - 7 \times 1 & (2) \\ 1 &= 8 - (15 - 8 \times 1) \times 1 & (1) \\ 1 &= 8 \times 2 - 15 \times 1 \\ 1 &= 8 \times \underbrace{2}_{y} + 15 \times \underbrace{(-1)}_{x} \end{aligned}$$

\hookrightarrow but there cannot be a negative solution!

$$\begin{aligned} \textcircled{3} \quad 100 \times 1 &= 8 \times 2 \times 100 + 15 \times (-1) \times 100 \\ 100 &= 8 \times 200 + 15 \times (-100) \end{aligned}$$

④ General solution: $x = -100 + \frac{8t}{1} = -100 + 8t$
 $y = 200 - \frac{15t}{1} = 200 - 15t$

We need to find $x \geq 0, y \geq 0$

$$\begin{aligned} -100 + 8t &\geq 0 & 200 - 15t &\geq 0 \\ 8t &\geq 100 & 200 &\geq 15t \\ t &\geq \frac{25}{2} = 12.5 & 13.3 &\approx \frac{40}{3} \geq t \end{aligned}$$

$\Rightarrow 12.5 \leq t \leq 13.3$, and t is an integer.

$$\begin{aligned} \therefore t=13 &\rightarrow x = -100 + 8 \times 13 = -100 + 104 = 4 \\ &\rightarrow y = 200 - 15 \times 13 = 200 - 195 = 5 \end{aligned}$$

$$\begin{aligned} \rightarrow x+y+z &= 35 \Rightarrow 4+5+z=35 \\ &\Rightarrow z=26 \end{aligned}$$

Ex 5 (T2-Q5) Show there are infinitely primes of the form $4t-1$, where t is an integer

Let's do a proof by contradiction. Which means, we will assume there are only a finite number of primes of the form $4t-1$, we will call them p_1, p_2, \dots, p_r .

Let $N = p_1 p_2 \dots p_r$, and let's consider the number $4N-1$. By our initial assumption, $4N-1$ is not prime, so it has a prime factor!

We claim $4N-1$ has a prime factor of the form $4t-1$.

Now, let's see what happens when we multiply $4x+1$ and $4y+1$

$$\rightarrow (4x+1)(4y+1) = 16xy + 4x + 4y + 1 = 4(4xy + x + y) + 1$$

Which means, $4N-1$ must have a prime factor of form $4t-1$. Since 2 is not a factor.

So there is a p_i such that $p_i | 4N-1$. Then as $p_i | N$, we have $p_i | 1$. Which contradicts!

There are infinitely many primes of the form $4t-1$. \square

Week 4 - Jan 27th

If n is composite, then it has a prime divisor p such that $p \leq \sqrt{n}$.

To determine if n is prime, check whether each of the prime numbers up to \sqrt{n} divide n

Ex1 (T3-Q1) Which of the following numbers are prime?

(a) 407

$\sqrt{407} < 21$ which means that we only need to check for primes up to 20
primes up to 20 = {2, 3, 5, 7, 11, 13, 17, 19}

- $2 \mid 407$? Nope! $407 = 2 \times 203 + 1$
- $3 \mid 407$? Nope! $407 = 3 \times 135 + 2$
- $5 \mid 407$? Nope! $407 = 5 \times 81 + 2$
- $7 \mid 407$? Nope! $407 = 7 \times 58 + 1$
- $11 \mid 407$? Yes $407 = 11 \times 37$

$\therefore 407$ is not prime.

(b) 463

$\sqrt{463} < 22$ which means that we only need to check for primes up to 21
primes up to 21 = {2, 3, 5, 7, 11, 13, 17, 19}

- $2 \mid 463$? Nope! $463 = 2 \times 231 + 1$
- $3 \mid 463$? Nope! $463 = 3 \times 154 + 1$
- $5 \mid 463$? Nope! $463 = 5 \times 92 + 3$
- $7 \mid 463$? Nope! $463 = 7 \times 66 + 1$
- $11 \mid 463$? Nope! $463 = 11 \times 42 + 1$
- $13 \mid 463$? Nope! $463 = 13 \times 35 + 8$
- $17 \mid 463$? Nope! $463 = 17 \times 27 + 4$
- $19 \mid 463$? Nope! $463 = 19 \times 24 + 7$

$\therefore 463$ is prime.

Sieve of Eratosthenes: To find all prime numbers between a and b ($a \leq b$), for each prime p up to \sqrt{b} , cross out every multiple of p . The remaining numbers are primes.

Ex2 Find all primes between 235 and 265

$\sqrt{265} < 17$ which means that we only need to check for primes up to 16

Primes up to 16 = {2, 3, 5, 7, 11, 13}

235	236	237	238	239	240	241	242	243	244
245	246	247	248	249	250	251	252	253	254
255	256	257	258	259	260	261	262	263	264
265									

Step #1: Cross out all #'s that are divisible by 2

Step #2: Cross out all #'s that are divisible by 3

Step #3: Cross out all #'s that are divisible by 5

Step #4: Cross out all #'s that are divisible by 7

Step #5: Cross out all #'s that are divisible by 11

Step #6: Cross out all #'s that are divisible by 13

∴ The primes in between 235 and 265 are {239, 241, 251, 257, 263}

Ex3 (T3-Q2) Find all primes between 201 and 250

$\sqrt{250} < 16$ which means that we only need to check for primes up to 15

Primes up to 15 = {2, 3, 5, 7, 11, 13}

201	202	203	204	205	206	207	208	209	210
211	212	213	214	215	216	217	218	219	220
221	222	223	224	225	226	227	228	229	230
231	232	233	234	235	236	237	238	239	240
241	242	243	244	245	246	247	248	249	250

∴ The primes in between 201 and 250 are {211, 223, 227, 229, 233, 239, 241}

Ex4 (T3-Q3) Find the prime-power decomposition for each of the following.

(a) $437 = 19 \times 23$

(b) $709 = 709$

(c) $876 = 2^2 \times 3 \times 73$

- * An element x of a set is called a **unit** if there exists an element y such that $xy=1$
- * An **irreducible** is a non-unit element of A whose only positive divisors are itself, units, and products of itself and units.

Ex: Can we write a # in two different products of irreducibles, $A = 3t+1$?
 $\Rightarrow 100 = 4 \times 25 = 10 \times 10$

Ex5 (T3-Q4) Consider the set containing positive integers of the form $5t+1$. Find an element of B with two different irreducible decompositions in B .

$$t=1 \rightarrow 6 = 2 \times 3$$

$$t=2 \rightarrow 11 = 11$$

$$t=3 \rightarrow 16 = 2^4$$

$$t=4 \rightarrow 21 = 3 \times 7$$

$$t=5 \rightarrow 26 = 2 \times 13$$

$$t=6 \rightarrow 31 = 31$$

$$t=7 \rightarrow 36 = 2^2 \times 3^2$$

$$t=8 \rightarrow 41 = 41$$

$$t=9 \rightarrow 46 = 2 \times 23$$

$$t=10 \rightarrow 51 = 3 \times 17$$

$$t=11 \rightarrow 56 = 2^3 \times 7$$

:

Any common factors? $2^4, 3, 7$

$$2^4 \times 3 \times 7 = 336 = 335 + 1 = 5 \times 67 + 1$$

How can we decompose 336? Remember they need to be in B ($5t+1$ form)

$$\overbrace{2 \times 2 \times 2 \times 2}^{\textcircled{1}} \times \overbrace{3 \times 7}^{\textcircled{2}}$$

$$336 = 56 \times 6 = 16 \times 21$$

→ both are irreducible.

Ex5 (T3-Q5) Suppose n is composite, and let p be the smallest prime divisor of n . If $p > \sqrt[3]{n}$, show that n/p is prime.

Suppose n/p is not prime. Then $n/p = q \times r$ where $q, r \in \mathbb{Z}$. Since p is the smallest prime divisor of n , then $p \leq q$ and $p \leq r$.

$$\Rightarrow n = p \times q \times r \geq p^3 > (\sqrt[3]{n})^3 = n$$

$\hookrightarrow n > n$?! impossible!

$\therefore n/p$ is prime

□

Week 5 - Feb 3rd

Ex 1 (T4-Q1) Find the prime factorization of the product of 4460 and 6146

$$\begin{array}{c|c}
 4460 & 2 \\
 2230 & 2 \\
 1115 & 5 \\
 223 & 223 \rightarrow \text{Check: } 7, 11, 13 \\
 1 & \text{because } \sqrt{223}
 \end{array}$$

$$\begin{array}{c|c}
 6146 & 2 \\
 3073 & 7 \\
 439 & 439 \leftarrow \text{Check: } 11, 13, 17, 19 \\
 1 & \text{because } \sqrt{439}
 \end{array}$$

$$\text{So } 4460 = 2^3 \times 5 \times 223 \text{ and } 6146 = 2 \times 7 \times 439$$

$$\begin{aligned}
 \Rightarrow 4460 \times 6146 &= 2^3 \times 5 \times 223 \times 2 \times 7 \times 439 \\
 &= 2^3 \times 5 \times 7 \times 223 \times 439.
 \end{aligned}$$

Ex 2 (T4-Q2) Which of the following is the square of a fraction?

(a) $\frac{3^2 \times 5^3}{2^6}$ $\xrightarrow{\text{No.}}$ odd exponent.

(b) $\frac{3^2 \times 2^2}{5^4 \times 7^8}$ Yes! $\left(\frac{3 \times 2}{5^2 \times 7^4}\right)^2$

(c) $\frac{3^6 \times 2}{7^2 \times 2^3}$ Yes! $\left(\frac{3^3}{7 \times 2}\right)^2$

(d) $13^2 \times 3^4$ Yes! $(13 \times 3^2)^2$

Ex 3 (T4-Q3) Using prime factorizations, compute the gcd and lcm of 7200 and 4374.

$$\begin{array}{c|c}
 7200 & 2 \\
 3600 & 2 \\
 1800 & 2 \\
 900 & 2 \\
 450 & 2 \\
 225 & 3 \\
 75 & 3 \\
 25 & 5 \\
 5 & 5 \\
 1 &
 \end{array}$$

$$\begin{array}{c|c}
 4374 & 2 \\
 2187 & 3 \\
 729 & 3 \\
 243 & 3 \\
 81 & 3 \\
 27 & 3 \\
 9 & 3 \\
 3 & 3 \\
 1 &
 \end{array}$$

$$\begin{aligned}
 \rightarrow 7200 &= 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \\
 &= 2^6 \times 3^2 \times 5^2
 \end{aligned}$$

$$\begin{aligned}
 \rightarrow 4374 &= 2 \times 3 \\
 &= 2 \times 3^7
 \end{aligned}$$

For the gcd, we will consider the common prime factors and smallest exponents.

$$7200 = 2^5 \times 3^2 \times 5^2 \quad 4374 = 2 \times 3^7$$

$$\Rightarrow \gcd(7200, 4374) = 2 \times 3^2$$

For the lcm, we will consider all prime factors and biggest exponents.

$$7200 = 2^5 \times 3^2 \times 5^2 \quad 4374 = 2 \times 3^7$$

$$\Rightarrow \text{lcm}(7200, 4374) = 2^5 \times 3^7 \times 5^2$$

Euler's Number: $\phi(n)$ to get the number of numbers from 1 to n that are relatively prime.

* If p is prime, then $\phi(p) = p - 1$

* If p is prime, then $\phi(p^n) = (p-1)p^{n-1}$

* If m and n are relatively prime, then $\phi(mn) = \phi(m)\phi(n)$

* Let $n = p_1^{e_1} \cdots p_r^{e_r}$ be the prime factorization of n, then $\phi(n) = n \times \frac{p_1-1}{p_1} \times \cdots \times \frac{p_r-1}{p_r}$

Ex 4 Calculate the following:

$$(a) \phi(48) = \phi(2^4 \times 3) = 48 \times \frac{2-1}{2} \times \frac{3-1}{3} = 2^4 \times 3 \times \frac{1}{2} \times \frac{2}{3} = 2^4 = 16$$

$$(b) \phi(480) = \phi(2^5 \times 3 \times 5) = 480 \times \frac{2-1}{2} \times \frac{3-1}{3} \times \frac{5-1}{5} = 2^5 \times 3 \times 5 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} = 2^7 = 128$$

$$(c) \phi(4800) = \phi(2^6 \times 3 \times 5^2) = 4800 \times \frac{2-1}{2} \times \frac{3-1}{3} \times \frac{5-1}{5} = 2^6 \times 3 \times 5^2 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} = 2^8 \times 5 = 1280$$

$$(d) \phi(55) = \phi(5 \times 11) = \phi(5) \times \phi(11) = (5-1)(11-1) = 4 \times 10 = 40$$

$$= 55 \times \frac{5-1}{5} \times \frac{11-1}{11} = 5 \times 11 \times \frac{4}{5} \times \frac{10}{11} = 40$$

$$(e) \phi(89) = 89 - 1 = 88$$

$$(f) \phi(32) = \phi(2^5) = (2-1)2^{5-1} = 2^4 = 16$$

Ex 5 (T4-Q5) Find four values of n such that $\phi(n) = 8 = 2^3$

Remember $\phi(2^k) = 2^{k-1}$, $\phi(3) = 2$, $\phi(5) = 4 = 2^2$

Then for $\emptyset(n)=8$, $\emptyset(2^4)=2^3 \rightarrow \emptyset(16)$

$$\emptyset(3)\emptyset(5)=8 \rightarrow \emptyset(15)$$

$$\emptyset(3)\emptyset(2^3)=8 \rightarrow \emptyset(21)$$

$$\emptyset(2^2)\emptyset(5)=8 \rightarrow \emptyset(20)$$

$$\therefore \emptyset(16) = \emptyset(15) = \emptyset(21) = \emptyset(20) = 8.$$

Week b - Feb 10th

Ex1 (T5-Q1) Calculate the following:

(a) How many numbers between 1 and 287 are not multiples of 7 or 41?

Way ①:

- Multiples of 7 between 1 and 287 : $\frac{287}{7} = 41$

- Multiples of 41 between 1 and 287 : $\frac{287}{41} = 7$

- Multiples of 7 and 41 between 1 and 287 : $\frac{287}{7 \times 41} = 1$

$$\Rightarrow \underbrace{287}_{\text{total #s between 1 and 287}} - \underbrace{41}_{\text{mults of 7}} - \underbrace{7}_{\text{mults of 41}} + 1 = 240$$

mults of 7 mults of 41

mvlt

of both 7 and 41

Way ②: ONLY because $287 = 7 \times 41$ and $\gcd(7, 41) = 1$, the numbers that are not multiples of 7 or 41 are numbers relatively prime to 287.

$$\Rightarrow \emptyset(287) = \emptyset(7 \times 41) = \emptyset(7)\emptyset(41) = (7-1)(41-1) = 6 \times 40 = 240$$

\therefore There are 240 numbers between 1 and 287 that are not multiples of 7 or 41.

(b) How many numbers between 1 and 363 are not multiples of 3 or 11?

Way ①:

- Multiples of 3 between 1 and 363 : $\frac{363}{3} = 121$

- Multiples of 11 between 1 and 363 : $\frac{363}{11} = 33$

- Multiples of 3 and 11 between 1 and 363 : $\frac{363}{3 \times 11} = 11$

$$\Rightarrow 363 - 121 - 33 + 11 = 220$$

Way ①: ONLY because $363 = 3 \times 11^2$ and $\gcd(3, 11) = 1$, the numbers that are not multiples of 3 or 11 are numbers relatively prime to 363

$$\Rightarrow \phi(363) = \phi(3 \times 11^2) = \phi(3)\phi(11^2) = (3-1)(11-1)11 = 2 \times 10 \times 11 = 220$$

\therefore There are 220 numbers between 1 and 363 that are not multiples of 3 or 11.

(c) How many numbers between 1 and 78 are not multiples of 2, 3 or 13?

Way ①:

- Multiples of 2 between 1 and 78: $\frac{78}{2} = 39$
- Multiples of 3 between 1 and 78: $\frac{78}{3} = 26$
- Multiples of 13 between 1 and 78: $\frac{78}{13} = 6$
- Multiples of 2 and 3 between 1 and 78: $\frac{78}{2 \times 3} = 13$
- Multiples of 2 and 13 between 1 and 78: $\frac{78}{2 \times 13} = 3$
- Multiples of 3 and 13 between 1 and 78: $\frac{78}{3 \times 13} = 2$
- Multiples of 2, 3 and 13 between 1 and 78: $\frac{78}{2 \times 3 \times 13} = 1$

$$\Rightarrow 78 - \underbrace{39}_{\substack{\text{Total #s} \\ \text{between} \\ 1 \text{ and } 78}} - \underbrace{26}_{\substack{\text{mults} \\ \text{of 2}}} - \underbrace{6}_{\substack{\text{mults} \\ \text{of 3}}} + \underbrace{13}_{\substack{\text{mults} \\ \text{of 13}}} + \underbrace{3}_{\substack{\text{mults} \\ \text{of both 2} \\ \text{and 3}}} + \underbrace{2}_{\substack{\text{mults} \\ \text{of both 3} \\ \text{and 13}}} - 1 = 24$$

mults of 2 mults of 3 mults of 13
 mults of both 2 and 3 mults of both 3 and 13

Way ②: ONLY because $78 = 2 \times 3 \times 13$ and 2, 3, and 13 are primes, the numbers that are not multiples of 2 or 3 or 13 are numbers relatively prime to 78

$$\Rightarrow \phi(78) = \phi(2 \times 3 \times 13) = \phi(2)\phi(3)\phi(13) = (2-1)(3-1)(13-1) = 1 \times 2 \times 12 = 24$$

\therefore There are 24 numbers between 1 and 78 that are not multiples of 2, 3 or 13.

Ex2 (T5-Q2) Calculate the following:

(a) How many numbers between 1 and 154 are not multiples of 7 or 11?

- Multiples of 7 between 1 and 154: $\frac{154}{7} = 22$

- Multiples of 11 between 1 and 154: $\frac{154}{11} = 14$
- Multiples of 7 and 11 between 1 and 154: $\frac{154}{7 \times 11} = 2$

$$\Rightarrow 154 - 22 - 14 + 2 = 120$$

\therefore There are 120 numbers between 1 and 154 that are not multiples of 7 or 11.

Modular Arithmetic

$a+b$ in mod n is equal to the remainder of $a+b$ after dividing by n.

i.e. $a \equiv b \pmod{n}$ when $n \mid (a-b)$

Ex3 Calculate the following:

$$(a) 2+5 \pmod{4} \rightarrow 2+5 \equiv 3 \pmod{4}$$

$$(b) 15+13 \pmod{16} \rightarrow 15+13 \equiv 12 \pmod{16}$$

$$(c) 22+8 \pmod{29} \rightarrow 22+8 \equiv 1 \pmod{29}$$

$$(d) 21+8 \pmod{29} \rightarrow 21+8 \equiv 0 \pmod{29}$$

Ex4 Make an addition table for mod 4 arithmetic

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Annotations:

- $3+1 = 2+2 = 1+3 = 4 \Rightarrow 4 \equiv 0 \pmod{4}$
- $3+2 = 2+3 = 5 \Rightarrow 5 \equiv 1 \pmod{4}$
- $3+3 = 6 \Rightarrow 6 \equiv 2 \pmod{4}$

Ex5 (T5-Q4) Make an addition table for mod 9 arithmetic

+	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8	0
2	2	3	4	5	6	7	8	0	1
3	3	4	5	6	7	8	0	1	2
4	4	5	6	7	8	0	1	2	3
5	5	6	7	8	0	1	2	3	4
6	6	7	8	0	1	2	3	4	5
7	7	8	0	1	2	3	4	5	6
8	8	0	1	2	3	4	5	6	7

Ex b (T5-Q5) Suppose today is Saturday. What day of the week will it be 200 days from now?

$$200 \pmod{7} \equiv 4 \quad \text{or} \quad 200 \equiv 4 \pmod{7}$$

∴ It will be 4 days after Saturday (Wednesday)

Ex 7 Suppose we are in April. What month will it be 37 months from now?

$$37 \pmod{12} \equiv 1 \quad \text{or} \quad 37 \equiv 1 \pmod{12}$$

∴ It will be 1 month after April (May)

Week 7 - Feb 17th

Reading week - No tutorial

Week 8 - Feb 24th

Ex1 (T6-Q1) Calculate the following

(a) $-35 \pmod{80} \rightarrow -35 \equiv 80 - 35 \equiv 45 \pmod{80}$

(b) $-37 \pmod{42} \rightarrow -37 \equiv 42 - 37 \equiv 5 \pmod{42}$

Ex2 (T6-Q2) Calculate the following

(a) $34 - 63 \pmod{73}$

* Way ① : Change the numbers to their mod equivalent and then compute

$$34 - 63 \equiv 34 + (-63) \equiv 34 + (73 - 63) \equiv 34 + 10 \equiv 44 \pmod{73}$$

Way ② : Compute first and then find their mod equivalent.

$$34 - 63 \equiv -29 \equiv 73 - 29 \equiv 44 \pmod{73}$$

(b) $61 - 40 \pmod{71}$

Way ① : $61 - 40 \equiv 61 + (71 - 40) \equiv 61 + 31 \equiv 92 \equiv 21 \pmod{71}$

Way ② : $61 - 40 \equiv 21 \pmod{71}$

(c) $5 - 8 \pmod{75}$

Way ① : $5 - 8 \equiv 5 + (75 - 8) \equiv 5 + 67 \equiv 72 \pmod{75}$

Way ② : $5 - 8 \equiv -3 \equiv 72 \pmod{75}$

$$(d) 81 - 72 \pmod{85}$$

Way ①: $81 - 72 \equiv 81 + (85 - 72) \equiv 81 + 13 \equiv 94 \equiv 9 \pmod{85}$

Way ②: $81 - 72 \equiv 9 \pmod{85}$

Ex3 (Tb - Q3) Calculate the following

(a) $3 \times 4 \pmod{88} \rightarrow 3 \times 4 \equiv 12 \pmod{88}$

(c) $11 \times 14 \pmod{90} \rightarrow 11 \times 14 \equiv 154 \equiv 64 \pmod{90}$

Ex4 (Tb - Q4) Calculate the following (Always show steps!)

(a) $8^2 \pmod{11}$

Way ①: $8^2 \equiv (8-11)^2 \equiv (-3)^2 \equiv 9 \pmod{11}$

Way ②: $8^2 \equiv 64 \equiv 9 \pmod{11}$

(b) $2^5 \pmod{12}$

$2^5 \equiv 32 \equiv 8 \pmod{12}$

(c) $9^9 \pmod{13}$

Way ①: $9^9 \equiv (-4)^9 \rightarrow$ is this any better? No :)

Way ②: $9^9 \equiv 9^8 \cdot 9 \equiv (9^2)^4 \cdot 9 \equiv (81)^4 \cdot 9 \equiv 3^4 \cdot 9 \equiv 81 \times 9 \equiv 3 \times 9 \equiv 27 \equiv 1 \pmod{13}$

(d) $5^{12} \pmod{14}$

$5^{12} \equiv (5^2)^6 \equiv 25^6 \rightarrow 25 \equiv 11 \equiv -3 \pmod{14}$

Way ①: $\equiv 11^6 \equiv (11^2)^3 \equiv 121^3 \rightarrow 121 \equiv 9 \equiv -5 \pmod{14}$

*①A: $\equiv 9^3 \equiv 9^2 \times 9 \equiv 81 \times 9 \xrightarrow{81 \equiv 11 \pmod{14}} 11 \times 9 \equiv 99 \equiv 1 \pmod{14}$

①B: $\equiv (-5)^3 \equiv (-5)^2 \times (-5) \equiv 25 \times (-5) = 11 \times (-5) \equiv -55 \equiv 1 \pmod{14}$

Way ②: $\equiv (-3)^6$

②A: $\equiv (-3)^4 \cdot (-3)^2 \equiv 81 \times 9 \xrightarrow{81 \equiv 11 \pmod{14}} 11 \times 9 \equiv 99 \equiv 1 \pmod{14}$

②B: $\equiv (-3)^3 \cdot (-3)^3 \equiv (-27) \cdot (-27) \equiv (-1) \cdot (-1) \equiv 1 \pmod{14}$

Ex5 Make a multiplication table for mod 4 arithmetic

x	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Annotations:

- $2 \times 2 \equiv 4 \equiv 0 \pmod{4}$
- $2 \times 3 \equiv 3 \times 2 \equiv 6 \equiv 2 \pmod{4}$
- $3 \times 3 \equiv 9 \equiv 1 \pmod{4}$

Ex6 (T6-Q5) Make a multiplication table for mod 9 arithmetic

x	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	4	6	8	1	3	5	7
3	0	3	6	0	3	6	0	3	6
4	0	4	8	3	7	2	6	1	5
5	0	5	1	6	2	7	3	8	4
6	0	6	3	0	6	3	0	6	3
7	0	7	5	3	1	8	6	4	2
8	0	8	7	6	5	4	3	2	1

→ Symmetric?

Ex7 (T6-Q6) Suppose today is Sunday. What day of the week was it 150 days ago?

$$150 = 7 \times 21 + 3 \quad 150 \equiv 3 \pmod{7}$$

3 days ago from today (Sunday) → Thursday!

Ex8 (T6-Q7) Suppose you have 17 quarters and each package of M&Ms costs 95 cents. If you buy as many as possible, how much money will you have left over?

$$75 \times 25 = 425 \text{ cents or } \$4.25$$

$$425 = 95 \times 4 + 45 \quad 425 \equiv 45 \pmod{95}$$

You can get 4 packages and have \$0.45 left.

Week 9 - March 2nd

Ex1 (T6-Q1) Find the smallest nonnegative integer x such that

(a) $x \equiv 30 \pmod{13}$

$$4 \equiv 30 \pmod{13} \rightarrow x=4$$

(b) $x \equiv -19 \pmod{13}$

$$-6 \equiv 7 \pmod{13} \rightarrow x=7$$

Ex 2 (T7-Q2) Find the largest negative integer x s.t.

(a) $x \equiv 38 \pmod{42}$

$$38 - 42 \equiv -4 \pmod{42} \rightarrow x = -4$$

(b) $x \equiv 40 \pmod{42}$

$$40 - 42 \equiv -2 \pmod{42} \rightarrow x = -2$$

Ex 3 (T7-Q4) Compute the following:

(a) $27 \times 26 \pmod{29}$

$$27 \times 26 \equiv (-2) \times (-3) \equiv 6 \pmod{29}$$

(b) $45678904 \times 45678905 \pmod{45678903}$

$$45678904 \times 45678905 \equiv (1) \times (2) \equiv 2 \pmod{45678903}$$

(c) $85 \times 87 \pmod{86}$

$$85 \times 87 \equiv (-1) \times (1) \equiv -1 \equiv 85 \pmod{86}$$

(d) $71 \times 75 \pmod{73}$

$$71 \times 75 \equiv (-2) \times (2) \equiv -4 \equiv 69 \pmod{73}$$

(e) $35 \times 52 \pmod{16}$

$$35 \times 52 \equiv 3 \times 4 \equiv 12 \pmod{16}$$

(f) $3456782 \times 2145781 \pmod{3}$

3456780 is divisible by 3. Why? $(3+4+5+6+7+8+0) = 33 \rightarrow 3|33$.
2145780 is divisible by 3. Why? $(2+1+4+5+7+8+0) = 27 \rightarrow 3|27$

$$(3456780+2) \times (2145780+1) \equiv (0+2) \times (0+1) \equiv 2 \pmod{3}$$

Cancellation

If $ac \equiv bc \pmod{n}$ and $\gcd(c, n) = 1$, then $a \equiv b \pmod{n}$

If $ac \equiv bc \pmod{n}$ and $\gcd(c, n) = d$, then $a \equiv b \pmod{\frac{n}{d}}$

Ex 4 (T7-Q5) Solve the following:

(a) $3x \equiv 2 \pmod{16}$

$$2 \equiv 18 \pmod{16} \rightarrow 3x \equiv 18 \pmod{16}$$

$$\text{Because } \gcd(3, 16) = 1, \quad \frac{3}{3}x \equiv \frac{18}{3} \pmod{16}$$

$$\Rightarrow x \equiv b \pmod{16}$$

$$(b) \quad 5x \equiv 2 \pmod{16}$$

$$\underbrace{2 \equiv 18 \equiv 34 \equiv 50 \equiv 6 \pmod{16}}_{\text{Find the divisible by 5 one}} \rightarrow 5x \equiv 50 \pmod{16}$$

$$\text{Because } \gcd(5, 16) = 1, \quad \frac{5}{5}x \equiv \frac{50}{5} \pmod{16}$$

$$\Rightarrow x \equiv 10 \pmod{16}$$

$$(c) \quad 2x \equiv 4 \pmod{12}$$

$$\text{Because } \gcd(2, 12) = 2, \quad \frac{2x}{2} \equiv \frac{4}{2} \pmod{\frac{12}{2}}$$

$$\Rightarrow x \equiv 2 \pmod{6}$$

Ex 5 (T7-Qb) Prove that no integer of the form $9t+4$ is the sum of three cubes.

$$\exists a, b, c \text{ s.t. } a^3 + b^3 + c^3 = 9t + 4$$

$$\Rightarrow a^3 + b^3 + c^3 \equiv 4 \pmod{9}$$

$$\rightarrow 0^3 \equiv 0 \pmod{9}$$

$$\rightarrow 3^3 \equiv 0 \pmod{9}$$

$$\rightarrow 6^3 \equiv 0 \pmod{9}$$

$$\rightarrow 1^3 \equiv 1 \pmod{9}$$

$$\rightarrow 4^3 \equiv 1 \pmod{9}$$

$$\rightarrow 7^3 \equiv 1 \pmod{9}$$

$$\rightarrow 2^3 \equiv 8 \pmod{9}$$

$$\rightarrow 5^3 \equiv 8 \pmod{9}$$

$$\rightarrow 8^3 \equiv 8 \pmod{9}$$

$$\text{So } x^3 \equiv 0, 1, 8 \pmod{9}$$

There is no combination that will give 4.



Week 10 - March 9th

Ex1 (Q4-T8) Compute the following

$$(a) 6^{12} \pmod{22}$$

$$6^{12} \equiv (6^2)^6 \equiv (16^2)^3 \pmod{22}$$

$$b^2 \equiv 36 \equiv 14 \equiv -8 \pmod{22}$$

$$b^4 \equiv (b^2)^2 \equiv (-8)^2 \equiv 64 \equiv 20 \equiv -2 \pmod{22}$$

$$\rightarrow b^{12} \equiv (b^4)^3 \equiv (-2)^3 \equiv -8 \equiv 14 \pmod{22}$$

(b) $10^8 \pmod{18}$

$$10^2 \equiv 100 \equiv 10 \pmod{18}$$

$$10^4 \equiv (10^2)^2 \equiv 10^2 \equiv 100 \equiv 10 \pmod{18}$$

$$\rightarrow 10^8 \equiv (10^4)^2 \equiv 10^2 \equiv 100 \equiv 10 \pmod{18}$$

Ex2 (Q1-T8) Compute the following:

(a) $\frac{1}{10} \pmod{13}$ Because $\gcd(10, 13) = 1$, $\frac{1}{10} \pmod{13}$ exists.

Way ① We need to find what # multiple of 10 has $1 \pmod{13}$

$$10 \times 4 \equiv 40 \equiv 1 \pmod{13} \rightarrow \frac{1}{10} \equiv 4 \pmod{13}$$

Way ② Euclidean Algo!

$$13 = 10 \times 1 + 3$$

$$10 = 3 \times 3 + 1$$

$$1 = 10 - 3 \times 3$$

$$1 = 10 - (13 - 10 \times 1) \times 3$$

$$1 = 10 - 13 \times 3 + 10 \times 3$$

$$1 = 10 \times 4 - \underbrace{13 \times 3}_{\text{divisible by 13}}$$

$$1 \equiv 10 \times 4 \pmod{13}$$

$$\rightarrow \frac{1}{10} \equiv 4 \pmod{13}$$

(c) $\frac{1}{40} \pmod{41}$ Because $\gcd(40, 41) = 1$, $\frac{1}{40} \pmod{41}$ exists.

$$40 \equiv (-1) \pmod{41} \rightarrow (-1)(-1) \equiv 1 \equiv (40)(40) \pmod{41}$$

$$\rightarrow \frac{1}{40} \equiv 40 \pmod{41}$$

(d) $\frac{1}{122} \pmod{501}$ Because $\gcd(122, 501) = 1$, $\frac{1}{122} \pmod{501}$ exists.

$$501 = 122 \times 4 + 13$$

$$122 = 13 \times 9 + 5$$

$$13 = 5 \times 2 + 3$$

$$5 = 3 \times 1 + 2$$

$$3 = 2 \times 1 + 1$$

$$1 = 3 - 2 \times 1$$

$$= 3 - (5 - 3 \times 1) \times 1$$

$$= 3 - 5 + 3$$

$$= 3 \times 2 - 5$$

$$\begin{aligned}
&= (13 - 5 \times 2) \times 2 - 5 \\
&= 13 \times 2 - 5 \times 5 \\
&= 13 \times 2 - (122 - 13 \times 9) \times 5 \\
&= 13 \times 2 - 122 \times 5 + 13 \times 45 \\
&= 13 \times 47 - 122 \times 5 \\
&= (501 - 122 \times 4) \times 47 - 122 \times 5 \\
&= 501 \times 47 - 122 \times 188 - 122 \times 5 \\
&= 501 \times 47 - 122 \times 193
\end{aligned}$$

$$\rightarrow 1 \equiv 501 \times 47 - 122 \times 193 \equiv 122 \times (-193) \equiv 122 \times 308 \pmod{501}$$

$$\frac{1}{122} \equiv 308 \pmod{501}$$

(f) $\frac{4}{12} \pmod{41}$ Because $\gcd(12, 41) = 1$, $\frac{4}{12} \pmod{41}$ exists.

$$41 = 12 \times 3 + 5$$

$$12 = 5 \times 2 + 2$$

$\overbrace{\quad}^{\text{need to}} \text{get to } 4$

$$\begin{aligned}
2 &= 12 - 5 \times 2 \\
&= 12 - (41 - 12 \times 3) \times 2 \\
&= 12 - 41 \times 2 + 12 \times 6 \\
(2 &= 12 \times 7 - 41 \times 2) \times 2
\end{aligned}$$

$$4 = 12 \times 14 - 41 \times 4$$

$$\rightarrow 4 \equiv 12 \times 14 - 41 \times 4 \equiv 12 \times 14 \pmod{41}$$

$$\therefore \frac{4}{12} \equiv 14 \pmod{41}$$

Ex3 (Q2-T8) Solve the following

(a) $2x \equiv 3 \pmod{21}$ $\gcd(2, 21) = 1 \rightarrow \frac{1}{2} \pmod{21}$ exists.

$$1 \equiv 22 \pmod{21} \rightarrow \frac{1}{2} \equiv 11 \pmod{21}$$

$$x \equiv 3 \times 11 \equiv 33 \equiv 12 \pmod{21}$$

(b) $6x \equiv 9 \pmod{21}$ $\gcd(6, 21) = 3$, so we need to use cancellation

$\rightarrow 2x \equiv 3 \pmod{7}$ $\gcd(2, 7) = 1 \rightarrow \frac{1}{2} \pmod{7}$ exists.

$$1 \equiv 8 \pmod{7} \rightarrow 8 = 2 \times 4 \rightarrow \frac{1}{2} \equiv 4 \pmod{7}$$

$$\rightarrow x \equiv 3 \times 4 \equiv 12 \equiv 5 \pmod{7}$$

Ex4 (Q3-T8) Find all the reciprocals of non-zero values mod 13

If $\frac{1}{a} \equiv b$, then $\frac{1}{b} \equiv a$ and $\frac{1}{-a} \equiv -b$

$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Want to find a pair of #'s that multiplied together, its product is $1 \text{ mod } 13$

$$1 \times 1 \equiv 1 \text{ mod } 13$$

$$12 \times 12 \equiv 144 \equiv 1 \text{ mod } 13$$

$$2 \times 7 \equiv 14 \equiv 1 \text{ mod } 13$$

$$7 \times 2 \equiv 14 \equiv 1 \text{ mod } 13$$

$$3 \times 9 \equiv 27 \equiv 1 \text{ mod } 13$$

$$9 \times 3 \equiv 27 \equiv 1 \text{ mod } 13$$

$$4 \times 10 \equiv 40 \equiv 1 \text{ mod } 13$$

$$10 \times 4 \equiv 40 \equiv 1 \text{ mod } 13$$

$$5 \times 8 \equiv 40 \equiv 1 \text{ mod } 13$$

$$8 \times 5 \equiv 40 \equiv 1 \text{ mod } 13$$

$$6 \times 11 \equiv 66 \equiv 1 \text{ mod } 13$$

$$11 \times 6 \equiv 66 \equiv 1 \text{ mod } 13$$

x $1/x$

1 1

2 7

3 9

4 10

5 8

6 11

7 2

8 5

9 3

10 4

11 6

12 12

Week 11 - March 16th

Fermat's Theorem: If p is prime and $p \nmid a$ then $a^{p-1} \equiv 1 \text{ mod } p$.

Ex1 Compute the following

(a) $5^2 \text{ mod } 3 \rightarrow 3-1=2$

Checklist:

- Is p prime?
- $p \nmid a$?

Check:

- ✓ 3 is prime
- ✓ $3 \nmid 5$

} we can

use
Fermat's!

$$\Rightarrow 5^2 \equiv 1 \text{ mod } 3$$

We could also solve it straight forward to check it works!

$$5^2 \equiv 25 \equiv 1 \pmod{3} \rightarrow 25 = 3 \times 8 + 1 \quad \text{OK}$$

(b) (T9-Q1) $16^{28} \pmod{29}$

Check: By Fermat's Theorem,

- ✓ 29 is prime
- ✓ $29 \nmid 16$

$$16^{28} \equiv 1 \pmod{29}$$

(c) (T9-Q1) $14^{10} \pmod{5}$

Check: $10-1=9 \neq \text{oh no...}$

- ✓ 5 is prime } This tells us we can use Fermat's Theorem,
✓ $5 \nmid 14$ } but the exponent of 14 is NOT 5-1. Now what?!

Well: $14^{10} \equiv (14^4)^2 14^2$

found it! now we can compute everything separate.

$$\Rightarrow 14^4 \equiv 1 \pmod{5}, \text{ by Fermat's Theorem}$$

$$\Rightarrow 14 \equiv 4 \equiv -1 \pmod{5}$$

$$\therefore 14^{10} \equiv (14^4)^2 14^2 \equiv (1)^2 (-1)^2 \equiv 1 \pmod{5}$$

(d) (T9-Q1) $7^{436} \pmod{5}$

Check: We can use Fermat's but we need to
✓ 5 is prime } find a way to bring 7^4 in.
✓ $5 \nmid 7$

$$7^{436} \equiv (7^4)^{109} \equiv 1^{109} \equiv 1 \pmod{5}, \text{ by Fermat's Theorem.}$$

(e) (T9-Q1) $5^{251} \pmod{7}$

Check: We can use Fermat's but we need to
✓ 7 is prime } find a way to bring 5^6 in.
✓ $7 \nmid 5$

Let's do some division! $251 = 6 \times 41 + 5 \Rightarrow 5^{251} \equiv (5^6)^{41} \times 5^5$

$$\Rightarrow 5^6 \equiv 1 \pmod{7}, \text{ by Fermat's Theorem.}$$

$$\Rightarrow 5^5 \equiv 5^4 5 \equiv (-2)^4 5 \equiv 16 \times 5 \equiv 2 \times 5 \equiv 10 \equiv 3 \pmod{7}$$

$$\therefore 5^{251} \equiv (5^6)^{41} \times 5^5 \equiv (1)^{41} \times 3 \equiv 3 \pmod{7}$$

Ex2 (T9-Q2) Use the Mod 7 Power table below, find all solutions to the following:

(a) $x^3 \equiv 5 \pmod{7}$

Look at all cubes: this col.
None give you 5.

(b) $x^4 \equiv 2 \pmod{7}$

Look at this col:
Two values give you 2: 2, 5

(c) $x^5 \equiv 1 \pmod{7}$

Look at this col.
1 gives you 1

(d) $x^5 \equiv 4 \pmod{7}$

Look at this col.
2 gives you 4

b	1	2	3	4	5	6
1^b	1	1	1	1	1	1
2^b	2	4	1	2	4	1
3^b	3	2	6	4	5	1
4^b	4	2	1	4	2	1
5^b	5	4	6	2	3	1
6^b	6	1	6	1	6	1

Modular Roots: We say a is k -th root of b ($\text{mod } n$) if $a^k \equiv b \pmod{n}$

Unique Roots: Let p be a prime. If $\gcd(k, p-1) = 1$, then the equation $x^k \equiv b \pmod{p}$ has exactly one solution. If $\gcd(k, p-1) \neq 1$, then the equation $x^k \equiv b \pmod{p}$ has either no solution or multiple solutions.

If $\gcd(k, p-1) = 1$, then $x^k \equiv b \pmod{p}$ has solution b^y , where y is the solution to $ky \equiv 1 \pmod{p-1}$

Ex3 (T9-Q4) Solve the following

(a) $x^9 \equiv 2 \pmod{17}$

① $\gcd(9, 16) = 1$: unique solution

We know $x^9 \equiv 2 \pmod{17}$ has a solution 2^y where y is the solution of $9y \equiv 1 \pmod{16}$

② Solve $9y \equiv 1 \pmod{16}$

Euclidean: $16 = 9 \times 1 + 7$

$9 = 7 \times 1 + 2$

$7 = 2 \times 3 + 1$

$1 = 7 - 2 \times 3$

$1 = 7 - (9 - 7 \times 1) \times 3$

$1 = 7 \times 4 - 9 \times 3$

$1 = (16 - 9 \times 1) \times 4 - 9 \times 3$

$1 = 16 \times 4 + 9 \times (-7)$

$$\Rightarrow 9(-7) \equiv 1 \pmod{16} \Rightarrow 9(-7) \equiv 9 \times (16-7) \equiv 9 \times 9 \equiv 1 \pmod{16}$$

$$\Rightarrow y = 9$$

③ Solve $2^y \equiv 2^9$ which is the solution of $x^9 \equiv 2 \pmod{17}$

i.e. solve $2^9 \pmod{17}$

$$2^9 \equiv (2^4)^2 \cdot 2 \equiv (16)^2 \cdot 2 \equiv (-1)^2 \cdot (2) \equiv 2 \pmod{17}$$

(b) $x^5 \equiv 3 \pmod{47}$

① $\gcd(5, 46) = 1$: unique solution

We know $x^5 \equiv 3 \pmod{47}$ has a solution 3^y where y is the solution of $5y \equiv 1 \pmod{46}$

② Solve $5y \equiv 1 \pmod{46}$

Euclidean: $46 = 5 \times 9 + 1 \rightarrow 1 = 46 - 5 \times 9$
 $1 = 46 + 5 \times (-9)$

$$\Rightarrow 5 \times (-9) \equiv 1 \pmod{46} \Rightarrow 5 \times (-9) \equiv 5 \times (46-9) \equiv 5 \times 37 \equiv 1 \pmod{46}$$

$$\Rightarrow y = 37$$

③ Solve $3^y \equiv 3^{37}$ which is the solution of $x^5 \equiv 3 \pmod{47}$

i.e. solve $3^{37} \pmod{47}$

$$3^{37} \equiv 3^{36} \cdot 3 \equiv 3^{32} \cdot 3^4 \cdot 3 \pmod{47}$$

$$3^2 \equiv 9 \pmod{47}$$

$$3^4 \equiv 81 \equiv 34 \equiv -13 \pmod{47}$$

$$3^8 \equiv (3^4)^2 \equiv (-13)^2 \equiv 169 \equiv 28 \equiv -19 \pmod{47}$$

$$3^{16} \equiv (3^8)^2 \equiv (-19)^2 \equiv 361 \equiv 32 \equiv -15 \pmod{47}$$

$$3^{32} \equiv (3^{16})^2 \equiv (-15)^2 \equiv 225 \equiv 37 \equiv -10 \pmod{47}$$

$$\Rightarrow 3^{37} \equiv 3^{32} \cdot 3^4 \cdot 3 \equiv (-10) \cdot (-13) \cdot (3) \equiv (130)(3) \equiv (36)(3) \equiv 108 \equiv 14 \pmod{47}$$

Week 12 - March 23rd

Euler's Theorem: Let n be a positive integer. If n and a are relatively prime, then $a^{\phi(n)} \equiv 1 \pmod{n}$

Ex 1 Compute the following:

(a) (T10-Q1) $3^{20} \pmod{25}$

Checklist:

$\gcd(n, a) = 1$? $\gcd(25, 3) = 1$
 $\emptyset(n)$? $\emptyset(25) = \emptyset(5^2) = 5^{2-1}(5-1) = 5 \times 4 = 20 \rightarrow \text{exponent of } 3!$

By Euler's Thm, $3^{20} \equiv 3^{\emptyset(25)} \equiv 1 \pmod{25}$

(b) $4^{18} \pmod{27}$

Checklist:

$\gcd(n, a) = 1$? $\gcd(27, 4) = 1$
 $\emptyset(n)$? $\emptyset(27) = \emptyset(3^3) = 3^{3-1}(3-1) = 9 \times 2 = 18 \rightarrow \text{exponent of } 4!$

By Euler's Thm, $4^{18} \equiv 4^{\emptyset(27)} \equiv 1 \pmod{27}$

(c) $9^{780} \pmod{7}$

* Check:
 $\checkmark 7 \text{ is prime}$ $\checkmark 7 \nmid 9$ } We can use Fermat's but we need to find a way to bring 9^b in.

$9^{780} \equiv (9^6)^{130} \equiv 1^{130} \equiv 1 \pmod{7}$, by Fermat's Theorem

* Checklist:

$\gcd(7, 9) = 1$?
 $\emptyset(7)$? $\emptyset(7) = 7 - 1 = 6 \rightarrow 780 = 130 \times 6$

$\Rightarrow 9^6 \equiv 9^{\emptyset(7)} \equiv 1 \pmod{7}$

By Euler's Thm, $9^{780} \equiv (9^6)^{130} \equiv (1)^{130} \equiv 1 \pmod{7}$

(d) (T9-Q1) $7^{436} \pmod{5}$

* Check:
 $\checkmark 5 \text{ is prime}$ $\checkmark 5 \nmid 7$ } We can use Fermat's but we need to find a way to bring 7^4 in.

$7^{436} \equiv (7^4)^{109} \equiv 1^{109} \equiv 1 \pmod{5}$, by Fermat's Theorem.

* Checklist:

$\gcd(5, 7) = 1$?
 $\emptyset(5)$? $\emptyset(5) = 5 - 1 = 4 \rightarrow 436 = 109 \times 4$

$\Rightarrow 7^4 \equiv 7^{\emptyset(5)} \equiv 1 \pmod{5}$

By Euler's Thm, $7^{436} \equiv (7^4)^{109} \equiv (1)^{109} \equiv 1 \pmod{5}$

* If a and n are relatively prime, $a^{\emptyset(n)} \equiv 1 \pmod{n}$, so if m divides $\emptyset(n)$ with remainder r , then $a^m \equiv a^r \pmod{n}$

(e) (T10-Q1) $5^{19} \pmod{28}$

Check list:

$\gcd(28, 5) = 1$?

$\emptyset(28)$?

$$\emptyset(28) = \emptyset(2^2 \times 7) = \emptyset(2^2) \emptyset(7) = 2 \times 6 = 12$$

\Rightarrow Do the division algorithm with $\emptyset(28) = 12$ and 19

$$19 = 12 \times 1 + \overset{r}{\textcircled{7}} \rightarrow * 5^{19} \equiv 5^7 \pmod{28}$$

By Euler's Thm, $5^{19} \equiv 5^7 \equiv 5^6 \times 5 \equiv 1 \times 5 \equiv 5 \pmod{28}$,

$$\cdot 5^6 \equiv (5^2)^3 \equiv (25)^3 \equiv (-3)^3 \equiv -27 \equiv 1 \pmod{28}$$

(f) (T10-Q1) $4^{386} \pmod{51}$

Check list:

$\gcd(51, 4) = 1$?

$\emptyset(51)$?

$$\emptyset(51) = \emptyset(3 \times 17) = \emptyset(3) \times \emptyset(17) = 2 \times 16 = 32$$

\Rightarrow Do the division algorithm with $\emptyset(51) = 32$ and 386

$$386 = 32 \times 12 + \overset{r}{\textcircled{2}} \rightarrow * 4^{386} \equiv 4^2 \pmod{51}$$

By Euler's Thm, $4^{386} \equiv 4^2 \equiv 16 \pmod{51}$,

Unique Roots: If $\gcd(k, \emptyset(n)) = 1$ and $\gcd(b, n) = 1$, then the equation $x^k \equiv b \pmod{n}$ has exactly one solution.

If $\gcd(k, \emptyset(n)) = 1$ and $\gcd(b, n) = 1$, then the equation $x^k \equiv b \pmod{n}$ has solution b^y , where y is the solution to $ky \equiv 1 \pmod{\emptyset(n)}$

Ex2 (T10-Q3) Which of the following have a unique sol?

(a) $x^3 \equiv 5 \pmod{18}$

$$k=3, b=5, n=18$$

$$\emptyset(18) = \emptyset(2 \times 3^2) = \emptyset(2) \times \emptyset(3^2) = (2-1) 3^1 (3-1) = 6$$

① $\gcd(3, 6) = 3 \neq 1$ and $\gcd(5, 18) = 1 \Rightarrow$ not unique solution!

(b) $x^4 \equiv 6 \pmod{18}$

$$k=4, b=6, n=18$$

$$\emptyset(18) = \emptyset(2 \times 3^2) = \emptyset(2) \times \emptyset(3^2) = (2-1) 3^1 (3-1) = 6$$

① $\gcd(4, 6) = 2 \neq 1$ and $\gcd(6, 18) = 6 \neq 1 \Rightarrow$ not unique solution!

$$(c) x^5 \equiv 7 \pmod{18}$$

$$k=5, b=7, n=18, \phi(18)=6$$

① $\gcd(5, 6) = 1$ and $\gcd(7, 18) = 1 \Rightarrow$ unique solution!

We know $x^5 \equiv 7 \pmod{18}$ has a solution 7^y where y is the solution of $5y \equiv 1 \pmod{6}$

② Solve $5y \equiv 1 \pmod{6}$

$$\begin{aligned} 6 &= 5 \times 1 + 1 \rightarrow 1 = 6 - 5 \times 1 \\ &\quad 1 = 6 \times 1 + 5 \times (-1) \end{aligned}$$

$$\Rightarrow 5 \times (-1) \equiv 5 \times 5 \equiv 25 \equiv 1 \pmod{6}$$

$$\rightarrow y = 5$$

③ Solve $7^y = 7^5$ which is the solution of $x^5 \equiv 7 \pmod{18}$

i.e. Solve $7^5 \pmod{18}$

$$7^2 \equiv 49 \equiv 13 \equiv -5 \pmod{18}$$

$$7^4 \equiv (7^2)^2 \equiv (-5)^2 \equiv 25 \equiv 7 \pmod{18}$$

$$\Rightarrow 7^5 \equiv 7^4 \times 7 \equiv 7 \times 7 \equiv 13 \pmod{18} \quad \text{i.e. } x = 13 \pmod{18}$$

$$(d) x^5 \equiv 5 \pmod{18}$$

$$k=5, b=5, n=18, \phi(18)=6$$

① $\gcd(5, 6) = 1$ and $\gcd(5, 18) = 1 \Rightarrow$ unique solution!

We know $x^5 \equiv 5 \pmod{18}$ has a solution 5^y where y is the solution of $5y \equiv 1 \pmod{6}$

② Solve $5y \equiv 1 \pmod{6}$

$$\begin{aligned} 6 &= 5 \times 1 + 1 \rightarrow 1 = 6 - 5 \times 1 \\ &\quad 1 = 6 \times 1 + 5 \times (-1) \end{aligned}$$

$$\Rightarrow 5 \times (-1) \equiv 5 \times 5 \equiv 25 \equiv 1 \pmod{6}$$

$$\rightarrow y = 5$$

③ Solve $5^y = 5^5$ which is the solution of $x^5 \equiv 5 \pmod{18}$

i.e. Solve $5^5 \pmod{18}$

$$5^2 \equiv 25 \equiv 7 \pmod{18}$$

$$5^4 \equiv (5^2)^2 \equiv 7^2 \equiv 49 \equiv 13 \pmod{18}$$

$$\Rightarrow 5^5 \equiv 5^4 \times 5 \equiv 13 \times 5 \equiv 65 \equiv 11 \pmod{18} \quad \text{i.e. } x = 11 \pmod{18}$$