MATB44 - Tutorial 6 (IC 320 Fridays 10-11am)

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Week 1 - Sept. 6th

No tutorials

Week 2 - Sept. 13th

ODE: $F(t, x, x^{(1)}, \dots, x^{(k)}) = 0$

t - independent variable (time)
x - dependent variable (space)
x is a function of t i.e. $x(t)$

The order is the highest derivative that appears in F.

The system is linear if we can rewrite it to separate out the $x_\ell^{(j)}$'s as follows:

$$x_i^{(k)} = g_i(t) + \sum_{\ell=1}^{n} \sum_{j=1}^{k-1} f_{i,j,\ell}(t)\, x_\ell^{(j)}$$

Ex: $\underbrace{(4+t^2)x' + 2tx}_{} = 4t$

By product rule: $\frac{d}{dt}\left[(4+t^2)x\right] = 2tx + (4+t^2)x'$

$\Rightarrow \frac{d}{dt}\left[(4+t^2)x\right] = 4t$

$\Rightarrow (4+t^2)x = 2t^2 + C$

$\Rightarrow x(t) = \frac{2t^2 + C}{4+t^2}$

The system is homogeneous if $g_i(t) = 0$

If there is no direct dependance on t, the system is autonomous. Form $x' = f(x)$

Ex: Classify the following equations:

* $x' + \frac{1}{2}x = \frac{1}{2}e^{t/3}$ 1st order, linear

* $3 + xx' = t - x$ 1st order, non linear

* $x''+5x'+6x=0$ — 2nd order, linear, homogeneous, autonomous
* $x'=t\sin x$ — 1st order, non linear
* $x''-x=0$ — 2nd order, linear, homogeneous, autonomous
* $tx'+2x=4t^2$ — 1st order, linear
* $x'-2x+t=4$ — 1st order, linear.

Ex: Solve $x'=x^2$ with $x_0=x(0)>0$

① $\frac{dx}{dt}=x^2 \Rightarrow \frac{dx}{x^2}=dt$

$$\Rightarrow \int \frac{dx}{x^2}=\int dt$$

$$\Rightarrow -\frac{1}{x}=t+C_1$$

$$\Rightarrow x=\frac{1}{-C_1-t}$$

When $x(0)=-\frac{1}{C_1}=x_0$ so $x=\frac{1}{\frac{1}{x_0}-t}$

② $f(x)=x^2$, a maximal nonzero interval is $(x_1,x_2)=(0,\infty)$

$$F(x)=\int_{x_0}^{x}\frac{dy}{y^2}=-\frac{1}{y}\Big|_{y=x_0}^{y=x}=-\frac{1}{x}+\frac{1}{x_0} \qquad \phi(t)=F^{-1}(t)$$

$T_+=\lim_{x\to x_2}F(x)=\lim_{x\to\infty}-\frac{1}{x}+\frac{1}{x_0}=\frac{1}{x_0}$, so ϕ is defined for all $t>0$

$$t=F(F^{-1}(t))=\frac{-1}{F^{-1}(t)}+\frac{1}{x_0}$$

$$\Rightarrow \frac{1}{F^{-1}(t)}=\frac{1}{x_0}-t$$

$$\Rightarrow F^{-1}(t)=\frac{1}{\frac{1}{x_0}-t}$$

$$\Rightarrow \phi(t)=\frac{1}{\frac{1}{x_0}-t}$$

■

Week 3 - Sept. 20th

Office Hours: Wed. 10-11 am 1C404

A particular solution is a solution that has no arbitrary constant. A general solution is a k-parameter family of solutions that contain every particular solution

A separable first-order ODE is an ODE that can be rewritten as:

$$\boxed{\dot{x}(t,x) = g(t)f(x)} \quad (\text{or } g(t)dt + f(x)dx = 0)$$

$$\Rightarrow \quad \frac{dx}{dt} = g(t)f(x)$$

$$\Rightarrow \quad \frac{dx}{f(x)} = g(t)\,dt$$

$$\Rightarrow \quad \boxed{\int \frac{dx}{f(x)} = \int g(t)\,dt}, \quad f(x) \neq 0$$

Ex: Show that the equation $\dot{x}(t,x) = \frac{t^2}{1-x^2}$ is separable and solve.

Let $f(x) = \frac{1}{1-x^2}$ and $g(t) = t^2$

$$\Rightarrow \int 1-x^2\,dx = \int t^2\,dt$$

$$x - \frac{x^3}{3} = \frac{t^3}{3} + C_1$$

$$3x - x^3 - t^3 = C, \quad C \in \mathbb{R}$$

Ex: Solve the equation $\frac{dx}{dt} = \frac{4t-t^3}{4+x^3}$ and find the solution passing through the point $(0,1)$

$$\Rightarrow (4+x^3)\,dx = (4t-t^3)\,dt$$

$$\int (4+x^3)\,dx = \int (4t-t^3)\,dt$$

$$4x + \frac{x^4}{4} = 2t^2 - \frac{t^4}{4} + C_1$$

$$\Rightarrow 16x + x^4 - 8t^2 + t^4 = C, \quad C \in \mathbb{R},$$

When $(0,1) \rightarrow 16 + 1 = 17$

$$\therefore 16x + x^4 - 8t^2 + t^4 = 17$$

Let $z=f(x,y)$ be a function of x and y, $f(x,y)$ is homogeneous of order n if it can be written as $\boxed{f(x,y)=x^n g(u)}$, where $u=\frac{y}{x}$ or $f(x,y)=y^n g(u)$, where $u=\frac{x}{y}$.

$\boxed{P(x,y)dx+Q(x,y)dy=0}$ where $P(x,y)$ and $Q(x,y)$ are the homogeneous coefficients and can be solved by substituting $y=ux$, $dy=udx+xdu$

Ex: Find the general solution of $txx'=t^2+2x^2$

$$\Rightarrow \frac{1}{t^2}\left[txx'=t^2+2x^2\right]$$

$$\frac{x}{t}x'=1+2\frac{x^2}{t^2}$$

Let $u=\frac{x}{t}$ so $x=ut$
$x'=u+tu'$

$$\Rightarrow u(u+tu')=1+2u^2$$
$$u^2+tu'u=1+2u^2$$
$$tu'u=1+u^2$$
$$\frac{u'u}{1+u^2}=\frac{1}{t}$$

→ OMG! This is separable!

$$\int\frac{uu'}{1+u^2}=\int\frac{1}{t}dt$$

Let $v=1+u^2$
$dv=2udu$

$$\frac{1}{2}\int\frac{dv}{v}=\ln|t|+C_1$$
$$\frac{1}{2}\ln|1+u^2|=\ln|t|+C$$
$$\therefore \frac{1}{2}\ln\left|1+\frac{x^2}{t^2}\right|-\ln|t|=C$$

A differential expression $\boxed{P(x,y)dx+Q(x,y)dy}$ is called an exact differential if it is the total differential of some function $f(x,y)$.
i.e. if $\boxed{P(x,y)=\frac{\partial}{\partial x}f(x,y)}$ and $\boxed{Q(x,y)=\frac{\partial}{\partial y}f(x,y)}$.

If we can find $f(x,y)$, $f(x,y)=C$ is the 1-parameter family of solutions.

$P(x,y)dx+Q(x,y)dy=0$ is exact iff $\frac{\partial P(x,y)}{\partial y}=\frac{\partial Q(x,y)}{\partial x}$.

Ex: Find the general solution of $y'=-\frac{1+2xy^2}{1+2x^2y}$

$$\Rightarrow (1+2x^2y)y'=-(1+2xy^2)$$
$$(1+2xy^2)+(1+2x^2y)y'=0$$

Is it exact? $P(x,y)=1+2xy^2$ $Q(x,y)=1+2x^2y$

$$\frac{\partial P(x,y)}{\partial y}=4xy \iff \frac{\partial Q(x,y)}{\partial x}=4xy \quad \checkmark$$

let's take $P(x,y)$, we know $P(x,y) = \frac{\partial}{\partial x} f(x,y)$

$$\Rightarrow \int P(x,y)\,dx = f(x,y)$$

$$\int 1+2xy^2\,dx = x + x^2y^2 + g(y) = f(x,y)$$

$$\Rightarrow \frac{\partial}{\partial y} f(x,y) = 2x^2y + g'(y) = Q(x,y) = 1 + 2x^2y$$

So $g'(y) = 1 \rightarrow g(y) = y$

$$\therefore x + x^2y^2 + y = C$$

Week 4 - Sept. 27th

An integrating factor (IF) will convert an inexact ODE $P(x,y)\,dx + Q(x,y)\,dy = 0$ into an exact ODE $IF\,P(x,y)\,dx + IF\,Q(x,y)\,dy = 0$

Given $\frac{dy}{dx} + P(x)y = Q(x)$ a known IF is $e^{\int P(x)dx}$

Ex: Find the general solution of $\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3}$

$P(x) = \frac{3}{x}$, $IF = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

$$\Rightarrow x^3\frac{dy}{dx} + x^3 \cdot \frac{3y}{x} = x^3 \frac{e^x}{x^3}$$

$$\Rightarrow x^3y' + 3x^2y = e^x$$

$$\Rightarrow \int (x^3y' + 3x^2y)\,dx = \int e^x\,dx$$

$$\Rightarrow x^3y = e^x + C$$

$$\Rightarrow y = \frac{e^x + C}{x^3}$$

Ex: Solve $6y' - 2y = xy^4$, $y(0) = -2$.

$\Rightarrow 6y^{-4}y' - 2y^{-3} = x$ let $u = y^{-3}$
$du = -3y^{-4}y'$

$\Rightarrow -2u' - 2u = x$ or $u' + u = -\frac{x}{2}$

$P(x) = 1 \quad , \; IF = e^{\int p dx} = e^x$

$\Rightarrow \quad u'e^x + ue^x = -\frac{e^x x}{2}$

$\Rightarrow \int (u'e^x + ue^x) dx = -\frac{1}{2} \int x e^x dx$

$\Rightarrow \quad ue^x = -\frac{1}{2}(xe^x - e^x) + C$

$\Rightarrow \quad u = -\frac{1}{2}(x-1) + Ce^{-x}$

$\Rightarrow \quad y^{-3} = -\frac{1}{2}(x-1) + Ce^{-x}$

$\Rightarrow \quad 2y^{-3} = (1-x) + 2Ce^{-x}$

$\Rightarrow \quad y^3 = \frac{2}{(1-x) + 2Ce^{-x}}$

Side note:

$\int x e^x dx \qquad \begin{matrix} u = x & dv = e^x dx \\ du = dx & v = e^x \end{matrix}$

$= xe^x - \int e^x dx = xe^x - e^x + C.$

IVP

$\Rightarrow (-2)^3 = \frac{2}{1 + 2C}$

$\Rightarrow -8 - 16C = 2$

$\Rightarrow \quad C = -\frac{5}{8}$

$\Rightarrow \quad y = \left[\frac{2}{\left[(1-x) - \frac{5}{4}e^{-x}\right]} \right]^{1/3}$

Bernoulli Equation: given $\frac{dy}{dx} + P(x)y = Q(x)y^n$.

multiply by $(1-n)y^{-n} \Rightarrow (1-n)y^{-1}\frac{dy}{dx} + (1-n)y^{1-n}P(x) = (1-n)Q(x)$

Substitute $u = y^{1-n}$
$du = (1-n)y^{-n}dy \qquad \Rightarrow \frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$

$IF \; e^{\int (1-n)P(x)dx}$

Let X be a real vector space. A norm on X is a map $\|\cdot\|: X \to [0,\infty)$ s.t.
(i) $\|0\| = 0$, $\|x\| > 0$ for $x \neq 0$
(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for $\alpha \in \mathbb{R}$ and $x \in X$
(iii) $\|x+y\| \leq \|x\| + \|y\|$ for $x, y \in X$

Together $(X, \|\cdot\|)$ is a normed vector space. A sequence of vectors x_n converges to x if $\lim_{n\to\infty} \|x_n - x\| = 0$.

A contraction is a mapping $K: C \subseteq X \to C$ where there exists a contraction constant $\theta \in [0,1)$ s.t. $\|K(x)-K(y)\| \leq \theta \|x-y\|$, $x, y \in C$.

Note: $K^n(x) = K(K^{n-1}(x))$ $\quad K^0(x) = x$

Ex: Let $K(x) = 40 + \frac{x}{3}$. What is the contraction constant? Is $K(x)$ a contraction on $C = [0, 90]$? on $C = [0, 30]$?

$\|K(x) - K(y)\| = \|40 + \frac{x}{3} - 40 - \frac{y}{3}\| = \|\frac{x}{3} - \frac{y}{3}\| = \frac{1}{3}\|x-y\|$

$\therefore \theta = \frac{1}{3}$

$C[0,90]$ ✓ $\quad K(0) = 40 \in [0,90]$

$C[0,30]$ X $\quad K(0) = 40 \notin [0,30]$

Consider an initial value problem (IVP) $\dot{x} = f(t,x)$, $x(t_0) = x_0$, where $x, t \in \mathbb{R}$ and $f \in C(U, \mathbb{R})$ where $U \subseteq \mathbb{R}^2$ is an open subset of $\mathbb{R}^2$ and $(t_0, x_0) \in U$.

Let's define Picard Iteration by a map $K: C(U, \mathbb{R}) \to C(U, \mathbb{R})$

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

and the Picard iterates.

$x_0(t) = x_0$ (the constant function through the scalar x_0)

$$x_1(t) = K(x_0)(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\, ds$$

$$x_2(t) = K^2(x_0)(t) = K(x_1)(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))\, ds.$$

$\vdots$

$$x_m(t) = K^m(x_0)(t) = K(x_{m-1})(t) = x_0 + \int_{t_0}^{t} f(s, x_{m-1}(s))\, ds$$

Ex: Calculate the Picard Iterates x_0, x_1, x_2. for $f(t,x) = 3 - 2x$, $x(t_0) = x_0$

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

$$K^{m+1}(x)(t) = x_0 + \int_{t_0}^{t} f(s, x_m(s))\, ds = x_0 + \int_{t_0}^{t} (3 - 2x_m(s))\, ds.$$

$\Rightarrow x_0(t) = x_0$

$\Rightarrow x_1(t) = K(x_0)(t) = x_0 + \int_{t_0}^{t} 3 - 2x_0(s)\,ds$

$= x_0 + \int_{t_0}^{t} 3 - 2x_0\,ds$

$= x_0 + (3-2x_0)(t-t_0)$

$= x_0 + t(3-2x_0) - t_0(3-2x_0)$

$\Rightarrow x_2(t) = K^2(x_0(t)) = x_0 + \int_{t_0}^{t} 3 - 2x_1(s)ds$

$= x_0 + \int_{t_0}^{t} 3 - 2\Big(x_0 + s(3-2x_0) - t_0(3-2x_0)\Big)\,ds$

$= x_0 + \big[3 - 2x_0 + 2t_0(3-2x_0)\big](t-t_0) - 2(3-2x_0)\int_{t_0}^{t} s\,ds$

$= x_0 + (3-2x_0)(1+2t_0)(t-t_0) - (3-2x_0)\,s^2\Big|_{t_0}^{t}$

$= x_0 + (3-2x_0)(1+2t_0)(t-t_0) - (3-2x_0)(t^2 - t_0)$

Week 5 - Oct. 4th

$\dot{x} = f(t,x)$, $x(t_0) = x_0$, $x: \mathbb{R} \to \mathbb{R}^n$

Let $K: C(U, \mathbb{R}^n) \to C(U, \mathbb{R}^n)$, $U \subseteq \mathbb{R}^{n+1}$ an open set

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds.$$

Then the sequence $x_0(t) = x_0$, $x_m = K(x_{m-1})$ converges to the solution x.

Ex: $\dot{x}(t) = \begin{pmatrix} t+y \\ y^2+z^2 \end{pmatrix}$, $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{matrix} y(0)=1 \\ z(0)=0 \end{matrix}$

$$x_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \int_0^t s + (y(0))\,ds \\ \int_0^t (y(0))^2 + (z(0))^2\,ds \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \int_0^t s+1\,ds \\ \int_0^t ds \end{pmatrix} = \begin{pmatrix} \frac{t^2}{2} + t + 1 \\ t \end{pmatrix} \begin{matrix} \to y_1 \\ \to z_1 \end{matrix}$$

$$x_2(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \int_0^t s + \left(\frac{s^2}{2} + s + 1\right) ds^{(1)} \\ \int_0^t \left(\frac{s^2}{2} + s + 1\right)^2 + s^2\,ds^{(2)} \end{pmatrix} = \begin{pmatrix} 1 + \frac{t^3}{6} + t^2 + t \\ \frac{t^5}{20} + \frac{t^4}{4} + t^3 + t^2 + t \end{pmatrix}$$

(1) $\int_0^t \frac{s^2}{2} + 2s + 1\,ds = \left[\frac{s^3}{6} + s^2 + s\right]_0^t$

(2) $\int_0^t \left(\frac{s^4}{4} + s^3 + 2s^2 + 2s + 1\right) + s^2 ds = \left[\frac{s^5}{20} + \frac{s^4}{4} + s^3 + s^2 + s\right]_0^t$

Ex: $\dot{x}(t) = \begin{pmatrix} 2z+t^2 \\ t+y \end{pmatrix}$, $x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{matrix} y(1)=1 \\ z(1)=1 \end{matrix}$

$$x_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_1^t 2+s^2 ds \\ \int_1^t s+1 ds \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \left[2s+\frac{s^3}{3}\right]_1^t \\ \left[\frac{s^2}{2}+s\right]_1^t \end{pmatrix} = \begin{pmatrix} 1+2t+\frac{t^3}{3}-2-\frac{1}{3} \\ 1+\frac{t^2}{2}+t-\frac{1}{2}-1 \end{pmatrix}$$

$$= \begin{pmatrix} 2t+\frac{t^3}{3}-\frac{4}{3} \\ \frac{t^2}{2}+t-\frac{1}{2} \end{pmatrix}$$

$$x_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_1^t s^2+2s-1+s^2 ds \\ \int_1^t s+2s+\frac{s^3}{3}-\frac{4}{3} ds \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_1^t 2s^2+2s-1 ds \\ \int_1^t 3s+\frac{s^3}{3}-\frac{4}{3} ds \end{pmatrix}$$

$$= \begin{pmatrix} 1+\left[\frac{2s^3}{3}+s^2-s\right]_1^t \\ 1+\left[\frac{3s^2}{2}+\frac{s^4}{12}-\frac{4s}{3}\right]_1^t \end{pmatrix} = \begin{pmatrix} 1+\frac{2t^3}{3}+t^2-t-\frac{2}{3}-1+1 \\ 1+\frac{3t^2}{2}+\frac{t^4}{12}-\frac{4t}{3}-\frac{3}{2}-\frac{1}{12}+\frac{4}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2t^3}{3}+t^2-t+\frac{1}{3} \\ \frac{3t^2}{2}+\frac{t^4}{12}-\frac{4t}{3}+\frac{3}{4} \end{pmatrix}$$

Picard-Lindelöf

Ex: $\dot{x} = x^{1/3}+t$, $x(1)=0$

$V_0 = [0,2] \times [-1,1]$

$$\frac{|f(t,x)-f(t,y)|}{|x-y|} = \frac{|x^{1/3}+t-y^{1/3}-t|}{|x-y|} = \frac{|x^{1/3}-y^{1/3}|}{|x-y|} \to \infty$$

If we let $y=0$, then $\frac{|x^{1/3}|}{|x|} = \frac{1}{|x^{2/3}|} \to \infty$

Thus, we cannot show that there exists a unique local solution.

Complex Numbers:

$\mathbb{C}$ is defined by the numbers defined by $z = \underbrace{x}_{Re(z)} + \underbrace{y}_{Im(z)}i$, where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$

A complex number can be represented in polar form $z = r(\cos\theta + i\sin\theta)$

Modulus of $z \rightarrow |z| = \sqrt{x^2+y^2}$

$e^{iz} = \cos z + i\sin z$

$\theta = \operatorname{Arg} z = \arctan \frac{y}{x}$

Ex: Find the modulus and argument of $z = 1+i$, then write z in polar form

$|z| = \sqrt{1+1} = \sqrt{2}$

$\theta = \arg z = \arctan(1) = \frac{\pi}{4}$

$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \xrightarrow{\text{check!}} = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \frac{2}{2} + i\frac{2}{2} = 1+i$ ☺

Week 6 - Oct. 11th

Ex: Midterm Q4. Find a 1-parameter family of solutions to the following equation:

$$xy' + 3y - \sin x^3 = 0$$

Rewrite it $y' + \frac{3}{x}y - \frac{\sin x^3}{x} = 0$ → 1st Order, linear

$IF = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$

$\Rightarrow x^3y' + 3x^2y - x^2\sin x^3 = 0$

$f(x,y) = \int x^3 dy + C(x) = x^3y + C(x)$

$f(x,y) = \int 3x^2y - x^2\sin x^3\, dx + C(y) = x^3y - \frac{1}{3}\cos x^3$

$\therefore x^3y + \frac{1}{3}\cos x^3 = C.$

Second order differential equations:

Homogeneous with constant coefficients

$ay'' + by' + cy = 0$ where a, b, c are constants.

Characteristic equation $ar^2 + br + c = 0$, find roots r_1 and r_2

Then $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is a general solution

Ex: Find the general solution of $y'' + 5y' + 6y = 0$

$r^2 + 5r + 6 = 0 = (r+2)(r+3) = 0$
$r_1 = -2$ and $r_2 = -3$

Then $y = c_1 e^{-2t} + c_2 e^{-3t}$, $c_1, c_2 \in \mathbb{R}$

Ex: Find the solution of the initial value problem $y'' + 5y' + 6y = 0$ $y(0) = 2$, $y'(0) = 3$

$t = 0$, $y(0) = c_1 + c_2 = 2$

$y' = -2c_1 e^{-2t} - 3c_2 e^{-3t} \rightarrow y'(0) = -2c_1 - 3c_2 = 3$

$\begin{cases} c_1 + c_2 = 2 \\ -2c_1 - 3c_2 = 3 \end{cases}$ $-c_2 = 7 \rightarrow c_2 = -7$ and $c_1 = 9$

$\therefore y = 9e^{-2t} - 7e^{-3t}$

Ex: Find the solution of the initial value problem $4y'' - 8y' + 3y = 0$, $y(0) = 2$, $y'(0) = \frac{1}{2}$

$4r^2 - 8r + 3 = 0 = (2r-1)(2r-3)$
$r_1 = \frac{1}{2}$, $r_2 = \frac{3}{2}$

General sol: $y = c_1 e^{t/2} + c_2 e^{3t/2}$ $y' = \frac{c_1}{2} e^{t/2} + \frac{3c_2}{2} e^{3t/2}$

$y(0) = c_1 + c_2 = 2$

$y'(0) = \frac{c_1}{2} + \frac{3c_2}{2} = \frac{1}{2}$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ $c_1 = 2 - c_2$, $c_1 = 1 - 3c_2$ $c_2 = -\frac{1}{2}$, $c_1 = \frac{5}{2}$

$\therefore y = \frac{5}{2} e^{t/2} - \frac{1}{2} e^{3t/2}$

★ Ex: Find the solution of $y^{(4)} + y''' - 7y'' - y' + 6y = 0$ that satisfies the initial condition $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$, $y'''(0) = -1$

$r^4+r^3-7r^2-r+6=0$ Roots are $r_1=1, r_2=-1, r_3=2, r_4=-3$

General solution $y=c_1e^t+c_2e^{-t}+c_3e^{2t}+c_4e^{-3t}$

$y'=c_1e^t-c_2e^{-t}+2c_3e^{2t}-3c_4e^{-3t}$

$y''=c_1e^t+c_2e^{-t}+4c_3e^{2t}+9c_4e^{-3t}$

$y'''=c_1e^t-c_2e^{-t}+8c_3e^{2t}-27c_4e^{-3t}$

$$\left.\begin{aligned} y(0) &= c_1+c_2+c_3+c_4=1 \\ y'(0) &= c_1-c_2+2c_3-3c_4=0 \\ y''(0) &= c_1+c_2+4c_3+9c_4=-2 \\ y'''(0) &= c_1-c_2+8c_3-27c_4=-1 \end{aligned}\right\} \quad c_1=\frac{11}{8},\ c_2=\frac{5}{12},\ c_3=\frac{-2}{3},\ c_4=-\frac{1}{8}$$

$\therefore\ y=\frac{11}{8}e^t+\frac{5}{12}e^{-t}-\frac{2}{3}e^{2t}-\frac{1}{8}e^{-3t}$