

# MATB41 - Multivariable Calculus

TA: Angela Zavaleta-Bernuy

Tutorials: TUT0007 (Wed. 11am-12pm)

TUT0006 (Thu. 10am-11am)

TUT0014 (Thu. 11am-12pm)

email: angela.zb@cs.toronto.edu

website: www.cs.toronto.edu/~angelazb

Week 1 (Sep 14th)

\* No tutorials \*

Week 2 (Sep 21st)

## Curved Lines

Parabola: (with vertex) at the origin

- $y = ax^2$  → open up when  $a > 0$  and down when  $a < 0$
- $x = by^2$  → open right when  $b > 0$  and left when  $b < 0$

\* Remember: We can replace  $x$  with  $x-m$  to shift right and  $y$  with  $y+n$  to shift up!

Ellipse: (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \text{intersections } (\pm a, 0) \text{ and } (0, \pm b)$$

When  $a=b$ , the curve is a circle of radius  $a=b$ .

Hyperbola: (with center) at the origin

$$\cdot \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 \quad \text{It has two curves that goes in opposite directions and do not touch the slant asymptote}$$

$$\cdot \left(\frac{y}{a}\right)^2 - \left(\frac{x}{b}\right)^2 = 1$$

Ex ① - Sketch the following curves:

a)  $x^2 + 3y^2 + 2x - 12y + 10 = 0$

$$\begin{aligned} (x^2 + 2x) + 3(y^2 - 4y) + 10 &= 0 \\ (x^2 + 2x + 1) - 1 + 3(y^2 - 4y + 4) - 12 + 10 &= 0 \\ (x+1)^2 + 3(y-2)^2 &= 3 \end{aligned}$$

$$\Rightarrow \frac{(x+1)^2}{3} + (y-2)^2 = 1 \quad \Rightarrow \left(\frac{x+1}{\sqrt{3}}\right)^2 + (y-2)^2 = 1$$

Okie, now what?! Another way to write the ellipse equation would be:

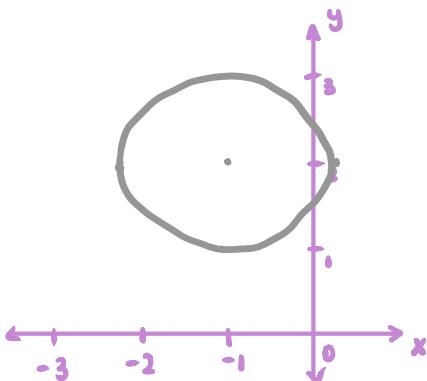
$$\left(\frac{x-m}{a}\right)^2 + \left(\frac{y-n}{b}\right)^2 = 1 \rightarrow \text{we just combined it with *}$$

So the new origin will be  $(-1, 2)$  as the ellipse will be shifted left by one and up by 2.

Now, let's find the end points

→ The horizontal max/min will be  $(-1 \pm \sqrt{3}, 2)$

→ The vertical max/min will be  $(-1, 2 \pm 1)$



b)  $\left(\frac{x-2}{4}\right)^2 - \left(\frac{y+2}{9}\right)^2 = 1$

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 1$$

The origin will be  $(2, -2)$ , we are shifting right 2 and 2 down.

Check the x-int :  $(y=0)$

$$\left(\frac{x-2}{2}\right)^2 - \frac{4}{9} = 1$$

$$\left(\frac{x-2}{2}\right)^2 = \frac{13}{9}$$

$$(x-2)^2 = \frac{52}{9}$$

$$x-2 = \pm \frac{2\sqrt{13}}{3}$$

$$x = 2 \pm \frac{2\sqrt{13}}{3}$$

Check the y-int :  $(x=0)$

$$1 - \left(\frac{y+2}{3}\right)^2 = 1$$

$$y = -2$$

Find the slant asymptote:

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 0$$

$$\frac{x-2}{2} = \frac{y+2}{3}$$

$$3x-6 = 2y+4$$

$$2y = 3x - 10$$

$$y = \frac{3x - 10}{2}$$

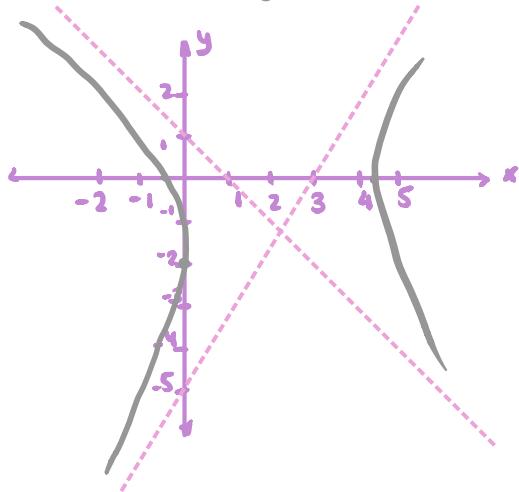
or

$$\frac{-x+2}{2} = \frac{y+2}{3}$$

$$-3x+6 = 2y+4$$

$$2y = -3x + 2$$

$$y = \frac{-3x + 2}{2}$$



## Curved Surfaces

**3D Sphere:** (with center) at the origin of radius  $R$

$$x^2 + y^2 + z^2 = R^2$$

**3D Ellipsoid:** (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \quad \text{3 points } (a, 0, 0), (0, b, 0), (0, 0, c)$$

We can replace  $x^2 + y^2$  with  $r^2$  to indicate the rotation.

**Ex② - Sketch the following surfaces:**

a)  $x^2 + y^2 + \frac{z^2}{4} = 1$

$$r^2 + \frac{z^2}{4} = 1$$

$$4r^2 + z^2 = 4$$

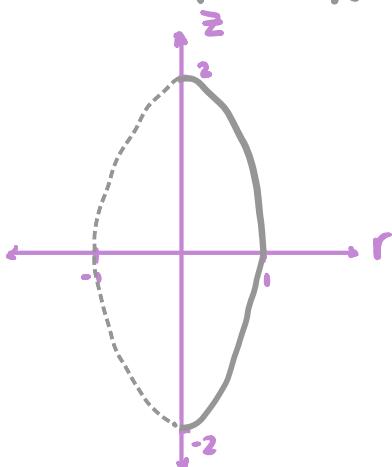
$$r^2 + \frac{z^2}{4} = 1$$

(center at  $(0, 0)$ )

$r$ -int:  $(\pm 1, 0)$

$z$ -int:  $(0, \pm 2)$

\* Use the ellipsoid formula



$r \geq 0$  because  
 $r = \sqrt{x^2 + y^2}$

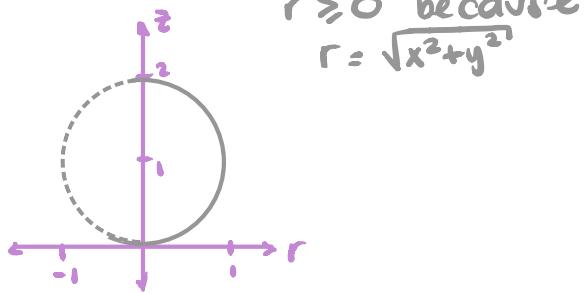
b)  $x^2 + y^2 + z^2 = 2z$

$$r^2 + z^2 - 2z = 0$$

$$r^2 + z^2 - 2z + 1 - 1 = 0$$

$$r^2 + (z-1)^2 = 1 \rightarrow \text{circle!}$$

Center  $(0, 1)$



$r \geq 0$  because  
 $r = \sqrt{x^2 + y^2}$

## Dot and Cross Product

let  $\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix} = u_1v_1 + u_2v_2 + \dots$

let  $\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left( \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$   
 $= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$

You can find the angle between two vectors using dot product:

Let the angle be called  $\theta$ , then  $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$

Ex ③ - Find the angle between  $\vec{u} = (1, -3, 1)$  and  $\vec{v} = (2, 1, 2)$  in  $\mathbb{R}^3$

First, we need the dot product and lengths:

$$\vec{u} \cdot \vec{v} = (1, -3, 1) \cdot (2, 1, 2) = 2 - 3 + 2 = 1$$

$$\|\vec{u}\| = \sqrt{1^2 + (-3)^2 + 1^2} = \sqrt{11}$$

$$\|\vec{v}\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

$$\Rightarrow \cos \theta = \frac{1}{3\sqrt{11}} \quad \therefore \quad \theta = \cos^{-1}\left(\frac{1}{3\sqrt{11}}\right)$$

## Line and Planes

Vector equation of a line in  $\mathbb{R}^3$ :  $\vec{l} = (a_1, a_2, a_3) + t[v_1, v_2, v_3]$   
 $\vec{l} = \vec{a} + t\vec{v}$

Parametric equation of a line in  $\mathbb{R}^3$ :  $x = a_1 + t v_1$   
 $y = a_2 + t v_2$   
 $z = a_3 + t v_3$

**Ex ④ - Find the equation of the line or plane**

a) The line through  $(1, -1, 2)$  and  $(3, 1, 9)$

The direction vector for the line is  $(3, 1, 9) - (1, -1, 2) = (2, 2, 7)$

$$\rightarrow \text{v. eq. } \vec{l} = (1, -1, 2) + t(2, 2, 7), t \in \mathbb{R}$$

$$\rightarrow \text{P. eq. } x = 1 + 2t, y = -1 + 2t, z = 2 + 7t, t \in \mathbb{R}$$

b) The plane through  $(1, -3, 1)$ ,  $(2, 1, 1)$ ,  $(1, 4, 0)$

let's find a pair of directional vectors  $\vec{v}_1$  and  $\vec{v}_2$

$$\vec{v}_1 = (2, 1, 1) - (1, -3, 1) = (1, 4, 0)$$

$$\vec{v}_2 = (1, 4, 0) - (1, -3, 1) = (0, 7, -1)$$

To get the equation of the plane, we need to find the normal which is  $\vec{v}_1 \times \vec{v}_2$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 0 \\ 0 & 7 & -1 \end{vmatrix} = (-4, 1, 7)$$

The plane then is  $-4x + y + 7z = d$

$$\text{Let's plug a point: } -4(2) + (1) + 7(1) = -8 + 1 + 7 = 0 = d$$

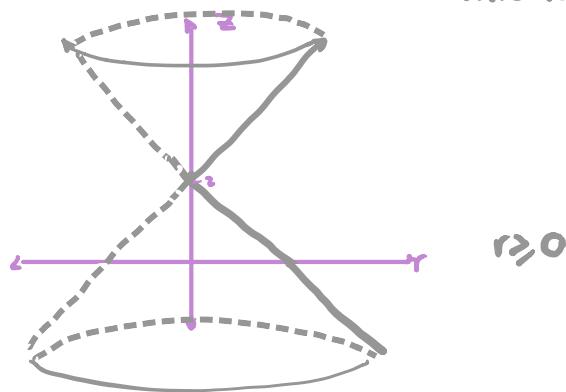
$\rightarrow$  The plane is  $-4x + y + 7z = 0$

## Level Set

**Ex ⑤ - Draw the level set of  $f(x, y, z) = z^2 - x^2 - y^2 - 4z$  for  $f = -4$**

$\hookrightarrow$  this is in 4D though

$$\begin{aligned} -4 &= z^2 - x^2 - y^2 - 4z \\ x^2 + y^2 &= z^2 - 4z + 4 \\ r^2 &= (z-2)^2 \\ r &= |z-2| \end{aligned}$$



**Week 3 (Sep 28th)**

## Limits and Continuity

$f(x, y)$  is continuous at the 2D point  $\vec{a}$  if  $\lim_{(x,y) \rightarrow \vec{a}} f(x, y) = f(\vec{a})$

$(x,y) \rightarrow \vec{a}$

It's not too hard to show when a limit DNE in  $\mathbb{R}^2$  at  $(0,0)$ :  
 Try different curves in terms of  $x$  or  $y$ , if they approach to different values at  $(0,0)$

To show that the limit exists we can use the Squeeze Theorem:

To attain  $\lim_{(x,y) \rightarrow \vec{a}} f(x,y)$ , can try to find  $g(x,y)$  and  $h(x,y)$  so that:

$$1. g(x,y) \leq f(x,y) \leq h(x,y) \text{ near the point } \vec{a}$$

$$2. \lim_{(x,y) \rightarrow \vec{a}} g(x,y) = L = \lim_{(x,y) \rightarrow \vec{a}} h(x,y)$$

Then we conclude  $\lim_{(x,y) \rightarrow \vec{a}} f(x,y) = L$

Ex ② - Decide whether the function has a limit at  $(0,0)$

$$a) \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \text{DNE}$$

$$\text{Restrict to } x=0, \lim_{(0,y) \rightarrow (0,0)} \frac{-y^2}{y^2} = -1$$

$$\text{Restrict to } y=0, \lim_{(x,0) \rightarrow (0,0)} \frac{x^2}{x^2} = 1$$

$$b) \lim_{(x,y) \rightarrow (0,0)} \frac{|x|}{\sqrt{x^2+y^2}} = \text{DNE}$$

$$\text{Restrict to } y=0, \lim_{(x,0) \rightarrow (0,0)} \frac{|x|}{\sqrt{x^2}} = 1$$

$$\text{Restrict to } x=0, \lim_{(0,y) \rightarrow (0,0)} \frac{|x|}{\sqrt{y}} = \frac{0}{\sqrt{y}} = 0$$

$$c) \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{\underset{\textcircled{1}}{x^3}}{\underset{\textcircled{2}}{x^2 + y^2}} - \frac{\underset{\textcircled{2}}{y^3}}{\underset{\textcircled{2}}{x^2 + y^2}} = 0$$

$$\textcircled{1} \quad x^2 \leq x^2 + y^2 \rightarrow 0 \leq \frac{x^2}{x^2 + y^2} \leq 1$$

$$\rightarrow 0 \cdot x \leq \frac{x^3}{x^2 + y^2} \leq x$$

$$\rightarrow \lim_{(x,y) \rightarrow (0,0)} 0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2+y^2} \leq \lim_{(x,y) \rightarrow (0,0)} x$$

$$\rightarrow 0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2+y^2} \leq 0 \\ \Rightarrow 0$$

② Same as ①

$$y^2 \leq x^2+y^2 \rightarrow 0 \leq \frac{y^2}{x^2+y^2} \leq 1$$

$$\rightarrow 0 \cdot \leq \frac{y^3}{x^2+y^2} \leq$$

$$\rightarrow \lim_{(x,y) \rightarrow (0,0)} y \cdot 0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{y^3}{x^2+y^2} \leq \lim_{(x,y) \rightarrow (0,0)} y$$

$$\rightarrow 0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{y^3}{x^2+y^2} \leq 0 \\ \Rightarrow 0$$

d)  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

We know  $|\sin \theta| \leq 1$  and we saw in class  $|\sin \theta| \leq \theta$

$$\Rightarrow |\frac{\sin(x^2+y^2)}{x^2+y^2}| \leq \frac{x^2+y^2}{x^2+y^2} = 1$$

Recall  $\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$ , what if we let  $t = x^2+y^2$

$$\Rightarrow \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1 = \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

Ex③ - Find if the limit of  $f$  exists at  $(0,0)$ , if it does, find  $f(0,0)$  for  $f$  to be continuous.

$$f(x,y) = \frac{x^3-x^2-2x^2y+xy^2-y^2-2y^3}{x^2+y^2}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2(x-1-2y)+y^2(x-1-2y)}{x^2+y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2+y^2)(x-1-2y)}{x^2+y^2}$$

$$= \lim_{(x,y) \rightarrow (0,0)} x-1-2y \\ = -1$$

$$\Rightarrow f(0,0) = -1$$

$f(\vec{x})$  is homogeneous of degree  $k$  if for every  $\vec{x} \in \mathbb{R}^n$  and every scalar  $c > 0$  we have:

$$f(cx) = c^k f(x)$$

Ex ④ - Show if  $f$  is homogeneous

a)  $f(x, y) = 8x^2y^2 - 9x^4$

$$\begin{aligned} f(cx, cy) &= 8(cx)^2(cy)^2 - 9(cx)^4 \\ &= 8c^2x^2c^2y^2 - 9c^4x^4 \\ &= 8c^4x^2y^2 - 9c^4x^4 \\ &= c^4(8x^2y^2 - 9x^4) \\ &= c^4 f(x, y) \end{aligned}$$

$\Rightarrow f$  is homogeneous of degree 4.

b)  $f(x, y) = \frac{x^2}{y^2} + xy + \frac{y^2}{x^2}$

$$\begin{aligned} f(cx, cy) &= \frac{(cx)^2}{(cy)^2} + (cx)(cy) + \frac{(cy)^2}{(cx)^2} \\ &= \frac{c^2x^2}{c^2y^2} + c^2xy + \frac{c^2y^2}{c^2x^2} \\ &= \frac{x^2}{y^2} + c^2xy + \frac{y^2}{x^2} \quad \Rightarrow \text{cannot factor } c. \end{aligned}$$

$\Rightarrow f$  is not homogeneous.

Week 4 (Oct. 5th)

## Partial derivatives

Ex ① - Find the partial derivatives of  $f(x, y) = y \sin(xy) + xe^{-y^2}$

$$\frac{\partial f}{\partial x} = y^2 \cos(xy) + e^{-y^2}$$

$$\frac{\partial f}{\partial y} = \sin(xy) + xy \cos(xy) - 2xye^{-y^2}$$

A function, whose partial derivatives exist and are continuous, is said to be of class  $C^1$ .

## Directional Derivative

The directional derivative towards the direction  $\vec{u}$  at the point  $\vec{a}$ :

$$D_{\vec{u}} f(\vec{a}) = \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h} \quad \text{where } \vec{u} \text{ is a unit vector.}$$

If  $f$  is differentiable, then all directional derivatives exist and

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} \quad \text{where } \vec{u} \text{ is a unit vector.}$$

\* If  $f$  is  $C^1$  at  $\vec{a}$ , then  $f$  is differentiable at  $\vec{a}$  ( $\nabla f$  exists)

**Ex ②** - At the point  $(3, 1, 2)$ , find the directional derivative of  $f(x, y, z) = xy^3z^2$  along the vector  $\vec{u} = (1, 3, 4)$

$f$  is  $C^1 \rightarrow$  its partial derivatives exist and are continuous.

$$\nabla f(x, y, z) = (y^3z^2, 3xy^2z^2, 2xy^3z)$$

$$\nabla f(3, 1, 2) = (4, 36, 12)$$

$$\Rightarrow D_{\vec{u}} f(p) = \frac{(4, 36, 12)(1, 3, 4)}{\sqrt{1+9+16}} = \frac{4+108+48}{\sqrt{26}} = \frac{160}{\sqrt{26}}$$

## Tangent Planes

A tangent plane is given by  $\nabla g(a, b, c) \cdot ((x, y, z) - (a, b, c)) = 0$

**Ex ③** - Compute an equation for the tangent plane at the point  $p$  to the graph of the function  $z = f(x, y)$

$$p = (1, 1, 1) \text{ and } xy + yz + zx = 3$$

$$g(x, y, z) = xy + yz + zx - 3$$

$$\nabla g(x, y, z) = (y+z, x+z, y+x)$$

$$\nabla g(1, 1, 1) = (2, 2, 2)$$

$$\Rightarrow (2, 2, 2)((x, y, z) - (1, 1, 1)) = 0$$

$$(2, 2, 2)(x-1, y-1, z-1) = 0$$

$$2x-2 + 2y-2 + 2z-2 = 0$$

$$x+y+z = 3$$

## Differentiation

$f$  is differentiable if the gradient vector  $\nabla f(\vec{a})$  satisfies

$$\lim_{\vec{h} \rightarrow 0} \frac{f(\vec{a} + \vec{h}) - f(\vec{a}) - \nabla f(\vec{a}) \cdot \vec{h}}{|\vec{h}|}$$

**Ex ④** - Find all directional derivatives and show whether the function is differentiable at  $(0, 0)$

$$f(x, y) = \begin{cases} \frac{3x^2y + 5xy^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$\vec{a} = (0, 0)$$

$$\begin{aligned} D_{\vec{u}} f &= \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(ha, hb) - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{3(ha)^2(hb) + 5(ha)(hb)^2}{(ha)^2 + (hb)^2} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot h^3 \frac{3a^2b + 5ab^2}{h^2(a^2 + b^2)} \\ &= \lim_{h \rightarrow 0} \frac{3a^2b + 5ab^2}{a^2 + b^2} \end{aligned}$$

$$\rightarrow \vec{u} = (1, 0), D_{\vec{u}} f = \frac{3(1)(0) + 5(1)(0)}{(1)^2 + (0)^2} = 0 \rightarrow \frac{\partial f}{\partial x} = 0$$

$$\rightarrow \vec{u} = (0, 1), D_{\vec{u}} f = \frac{3(0)(1) + 5(0)(1)}{(0)^2 + (1)^2} = 0 \rightarrow \frac{\partial f}{\partial y} = 0$$

$$\nabla f(\vec{a}) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (0, 0)$$

Now, let's apply the definition of differentiability

$$\begin{aligned} \lim_{\vec{h} \rightarrow 0} \frac{f(\vec{a} + \vec{h}) - f(\vec{a}) - \nabla f(\vec{a}) \cdot \vec{h}}{|\vec{h}|} &= \lim_{\vec{h} \rightarrow 0} \frac{f(\vec{h})}{|\vec{h}|}, \text{ let } \vec{h} = (h_1, h_2) \\ &= \lim_{(h_1, h_2) \rightarrow (0, 0)} \frac{1}{\sqrt{h_1^2 + h_2^2}} \cdot \frac{3h_1^2h_2 + 5h_1h_2^2}{h_1^2 + h_2^2} \end{aligned}$$

Restrict  $h_1 = 0$

$$\lim_{h_2 \rightarrow 0} \frac{1}{\sqrt{h_2^2}} \cdot \frac{3(0)h_2 + 5(0)h_2^2}{h_2^2} = 0$$

Restrict  $h_1 = h_2$

$$\lim_{h_2 \rightarrow 0} \frac{1}{\sqrt{2h_2^2}} \cdot \frac{3h_2^3 + 5h_2^3}{2h_2^2} = \lim_{h_2 \rightarrow 0} \frac{8h_2^3}{2\sqrt{2}h_2^3} = \frac{4}{\sqrt{2}}$$

$\therefore f$  is not differentiable at  $(0, 0)$

## Week 5 (Oct. 19th)

If  $f$  is  $C^2$ , then  $f_{xy} = f_{yx}$   $\left[ \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \right]$

Ex① - At  $(x,y) = (0,0)$ , find  $\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$  and  $\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)$ . Are they equal?

Is  $f(x,y)$   $C^2$ ?

$$f(x,y) = \begin{cases} \frac{ax^4}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Time to find directional derivatives?

$$\begin{aligned} \vec{a} &= (0,0) \\ \vec{u} &= (a,b) \end{aligned} \quad D_{\vec{u}} f = \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} f(ha, hb)$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \frac{2(ha)^4}{(ha)^2 + (hb)^2}$$

$$= \lim_{h \rightarrow 0} \frac{2h^4a^4}{h(n^2a^2 + b^2)} \quad (n^2a^2 + b^2)$$

$$= \lim_{h \rightarrow 0} \frac{2h^4a^4}{h^3(a^2 + b^2)}$$

$$= 0$$

$$\Rightarrow \frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$$

Now, let's find the partial derivatives.

$$\begin{aligned} * \frac{\partial f}{\partial x}(x,y) &= \frac{8x^3(x^2+y^2) - 2x(2x^4)}{(x^2+y^2)^2} = \frac{8x^5 + 8x^3y^2 - 4x^5}{(x^2+y^2)^2} = \frac{4x^5 + 8x^3y^2}{(x^2+y^2)^2} \\ &= \frac{4x^3(x^2+2y^2)}{(x^2+y^2)^2} \end{aligned}$$

$$\Rightarrow g(x,y) = \frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{4x^3(x^2+2y^2)}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$* \frac{\partial f}{\partial y}(x,y) = \frac{0(x^2+y^2) - 2y(2x^4)}{(x^2+y^2)^2} = \frac{-4x^4y}{(x^2+y^2)^2}$$

$$\Rightarrow h(x,y) = \frac{\partial f}{\partial y}(x,y) = \begin{cases} \frac{-4x^4y}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial y}(x,y) \\ 0 \end{array} \right. , \quad (x,y) = (0,0)$$

Time to finally find  $f_{xy}$  and  $f_{yx}$  at  $(0,0)$

\* We need to find  $\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}(0,0)\right)$ , aka  $\frac{\partial g}{\partial y}(0,0)$

We can find the directional derivative at  $(0,0)$  where  $\vec{u} = (0,1)$

$$\begin{aligned} \frac{\partial g}{\partial y}(0,0) &= \lim_{h \rightarrow 0} \frac{g(\vec{a} + h\vec{u}) - g(\vec{a})}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} g(0,h) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \times 0 \\ &= 0 \end{aligned}$$

\* We need to find  $\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}(0,0)\right)$ , aka  $\frac{\partial k}{\partial x}(0,0)$

We can find the directional derivative at  $(0,0)$  where  $\vec{u} = (1,0)$

$$\begin{aligned} \frac{\partial k}{\partial x}(0,0) &= \lim_{h \rightarrow 0} \frac{k(\vec{a} + h\vec{u}) - k(\vec{a})}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} k(h,0) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \times 0 \\ &= 0 \end{aligned}$$

$$\therefore \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(0,0)\right) = 0 = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(0,0)\right)$$

But, is it  $C^2$ ? At  $(0,0)$ , yes!

$$\begin{aligned} * \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) &= \frac{[0(x^2+2y^2) + 4y(4x^3)](x^2+y^2)^2 - 2(2y)(x^2+y^2)4x^3(x^2+2y^2)}{(x^2+y^2)^4} \\ &= \frac{16x^3y(x^2+y^2) - 16x^3y(x^2+2y^2)}{(x^2+y^2)^3} \\ &= \frac{16x^3y(x^2+y^2 - x^2 - 2y^2)}{(x^2+y^2)^3} = \frac{-16x^3y^3}{(x^2+y^2)^3} \end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \begin{cases} \frac{-16x^3y^3}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$* \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y}(x,y) \right) = -16x^3y \frac{(x^2+y^2)^2 + 4x^4y(2)(2x)(x^2+y^2)}{(x^2+y^2)^4}$$

$$= -16x^3y \frac{(x^2+y^2) + 16x^5y}{(x^2+y^2)^4} = \frac{-16x^3y^3}{(x^2+y^2)^4}$$

$$\Rightarrow \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x}(x,y) \right) = \begin{cases} \frac{-16x^3y^3}{(x^2+y^2)^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

So  $f(x,y)$  is not  $C^2$

## Critical Points

Hessian form: ( $H_f$ )

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) & \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \\ \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) & \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) \end{bmatrix}$$

- \* If  $H_f$  is positive definite, then the Critical Point is a local minimum
- \* If  $H_f$  is negative definite, then the Critical Point is a local maximum
- \* Otherwise, the Critical Point is a saddle point.

Let  $A$  be  $n \times n$  and symmetric:

- \*  $A$  is positive definite iff  $\det(A_k) > 0$  for  $k=1, 2, \dots, n$ .
- \*  $A$  is negative definite iff  $(-1)^k \det(A_k) > 0$  for  $k=1, 2, \dots, n$

Ex ② - Find the critical points of  $f(x,y) = x^3 - 12xy + 8y^3$

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 - 12y$$

$$\frac{\partial f}{\partial y}(x,y) = -12x + 24y^2$$

$$\begin{cases} 3x^2 - 12y = 0 \rightarrow 3x^2 = 12y \\ -12x + 24y^2 = 0 \end{cases} \quad y = \frac{x^2}{4}$$

$$\rightarrow -12x + 24 \left( \frac{x^2}{4} \right)^2 = 0$$

$$-12x + 24x^4 = 0$$

$$x\left(\frac{3}{2}x^3 - 12\right) = 0 \rightarrow x=0 \text{ or } \frac{3}{2}x^3 = 12 \\ \Rightarrow x^3 = 8 \rightarrow x=2$$

$\therefore$  CPs are  $(0,0)$  and  $(2,1)$

$$\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = 6x$$

$$\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = -12$$

$$\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = -12$$

$$\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right) = 48y$$

$$\Rightarrow H_f(x,y) = \begin{bmatrix} 6x & -12 \\ -12 & 48y \end{bmatrix}$$

① For  $(0,0)$

$$H_f(0,0) = \begin{bmatrix} 0 & -12 \\ -12 & 0 \end{bmatrix}$$

$$\rightarrow \det(A_1) = \det([0]) = 0$$

$$\rightarrow \det(A_2) = \det\left(\begin{bmatrix} 0 & -12 \\ -12 & 0 \end{bmatrix}\right) = -144 < 0$$

$\therefore$  saddle point

② For  $(2,1)$

$$H_f(2,1) = \begin{bmatrix} 12 & -12 \\ -12 & 48 \end{bmatrix}$$

$$\rightarrow \det(A_1) = \det([12]) = 12 > 0$$

$$\rightarrow \det(A_2) = \det\left(\begin{bmatrix} 12 & -12 \\ -12 & 48 \end{bmatrix}\right) = 576 - 144 = 432 > 0$$

$\therefore$  local min.

Ex ③- Find the global extrema for  $f(x,y) = 6x^2 - 8x + 2y^2 - 5$  on the closed disk  $x^2 + y^2 \leq 1$ .

The extrema can be at the interior or boundary.

$$\text{Interior: } \frac{\partial f}{\partial x}(x,y) = 12x - 8 \quad \frac{\partial f}{\partial y}(x,y) = 4y$$

$$\begin{cases} 12x - 8 = 0 \\ 4y = 0 \end{cases} \rightarrow \left(\frac{2}{3}, 0\right) \text{ is the only CP.}$$

Since  $\left(\frac{2}{3}\right)^2 + (0)^2 = \frac{4}{9} \leq 1$ , the point is inside the disk.

Boundary: it happens when  $x^2 + y^2 = 1$ . and  $-1 \leq x \leq 1$   
 $y^2 = 1 - x^2$ , let's substitute this

$$\begin{aligned} g(x) &= 6x^2 - 8x + 2(1 - x^2) - 5 \\ &= 6x^2 - 8x + 2 - 2x^2 - 5 \\ &= 4x^2 - 8x - 3 \end{aligned}$$

$$\begin{aligned} g'(x) &= 8x - 8 = 0 \\ 8x &= 8 \\ x &= 1 \quad \rightarrow \quad y^2 = 1 - 1 = 0, \quad (1, 0) \\ &\rightarrow y = 0, \quad 0 = 1 - x^2 \rightarrow (-1, 0) \end{aligned}$$

$$\begin{aligned} \therefore f\left(\frac{2}{3}, 0\right) &= -\frac{23}{2}, \quad \text{global min} \\ f(-1, 0) &= 9 \quad , \quad \text{global max} \\ f(1, 0) &= -7 \end{aligned}$$

## Lagrange Multipliers

Ex①-Use Lagrange multipliers to find the constrained critical points of  $f$  subject to the given constraints.

$$(a) f(x, y) = xy, \quad 4x^2 + 9y^2 = 32 \rightarrow g(x, y) = 4x^2 + 9y^2 - 32$$

$$h(x, y, \lambda) = xy - \lambda(4x^2 + 9y^2 - 32)$$

$$\begin{cases} h_x = y - 8x\lambda = 0 \rightarrow \lambda = \frac{y}{8x} \\ h_y = x - 18y\lambda = 0 \rightarrow \lambda = \frac{x}{18y} \end{cases} \quad \left[ \begin{array}{l} \frac{y}{8x} = \frac{x}{18y} \rightarrow 18y^2 = 8x^2 \rightarrow 4x^2 = 9y^2 \\ \text{or } \lambda = 0 \text{ (contradiction)} \end{array} \right]$$

$$h_\lambda = 32 - 4x^2 - 9y^2 \quad \longrightarrow \quad 32 - 4x^2 - 9y^2 = 0$$

$$8x^2 = 32$$

$$x^2 = 4 \quad \longrightarrow \quad 16 = 9y^2$$

$$x = \pm 2 \quad y = \pm \frac{4}{3}$$

$$\therefore \text{CPs are } \left(2, \frac{4}{3}\right), \left(2, -\frac{4}{3}\right), \left(-2, \frac{4}{3}\right), \left(-2, -\frac{4}{3}\right)$$

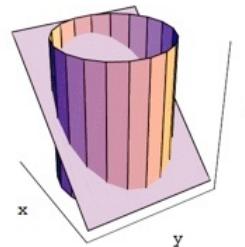
## Week 6 (Oct 29th)

(b) The cylinder  $x^2 + y^2 = 1$  intersects the plane  $x+z=1$  in an ellipse. Find the point on that ellipse that is the furthest from the origin.

We need to maximize the distance function (aka  $f(x,y,z) = x^2 + y^2 + z^2$ ) subject to 2 constraints.

$$h(x,y,z, \lambda, \mu) = x^2 + y^2 + z^2 - \lambda(x^2 + y^2 - 1) - \mu(x + z - 1)$$

$$\begin{cases} h_x = 2x - 2x\lambda - \mu & ① \\ h_y = 2y - 2y\lambda & ② \\ h_z = 2z - \mu & ③ \\ h_\lambda = 1 - x^2 - y^2 & ④ \\ h_\mu = 1 - x - z & ⑤ \end{cases}$$



$$\begin{aligned} ② \quad 2y &= 2y\lambda \\ \Rightarrow 2y(1-\lambda) &= 0 \\ y &= 0, \lambda = 1 \end{aligned}$$

\* If  $\lambda = 1$ :

$$\begin{aligned} ① \quad \mu &= 2x(1-\lambda) = 0 \\ ③ \quad 2z &= \mu = 0 \rightarrow z = 0 \\ ⑤ \quad x &= 1 \\ ④ \quad 1+y^2 &= 1 \rightarrow y = 0 \end{aligned}$$

\* If  $y = 0$ :

$$\begin{aligned} ④ \quad x^2 &= 1 \rightarrow x = \pm 1 \\ ⑤ \quad -1+z &= 1 \rightarrow z = 2 \quad (x=-1) \\ 1+z &= 1 \rightarrow z = 0 \quad (x=1) \quad (\text{We know}) \end{aligned}$$

$$\therefore \text{CPs } (1, 0, 0) \text{ and } (-1, 0, 2)$$

Based on the distance function:

$$(1, 0, 0) \rightarrow \sqrt{1^2} = 1$$

$$(-1, 0, 2) \rightarrow \sqrt{(-1)^2 + (0)^2 + (2)^2} = \sqrt{5}$$

$\therefore$  The furthest point from the origin is  $(-1, 0, 2)$ .

Since the distance function is continuous and the intersection is compact in  $\mathbb{R}^3$ , the Extreme Value Theorem (EVT) ensures the existence of extreme.

### Chain Rule

$$(f(g(x)))' = f'(g(x)) g'(x)$$

Now, let's say  $y = y(t)$  and  $t = t(x)$

$$\stackrel{1D}{=} \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\stackrel{2D}{=} \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{dy}{dt} \cdot \frac{dt}{dx} \right) \quad \text{product rule!}$$

$$= \frac{d}{dx} \left( \frac{dy}{dt} \right) \frac{dt}{dx} + \frac{dy}{dt} \cdot \frac{d}{dx} \left( \frac{dt}{dx} \right) = \frac{d^2t}{dx^2}$$

## ODEs - Euler Equation

Second Order differential equations:

Homogeneous with constant coefficients

$$ay'' + by' + cy = 0 \quad \text{where } a, b, c \text{ are constants.}$$

Characteristic equation  $ar^2 + br + c = 0$ , find roots  $r_1$  and  $r_2$

Then  $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$  is a general solution

Ex ② - Let  $y'' + 5y' + 6y = 0$

(a) Find the general solution of the equation

$$r^2 + 5r + 6 = 0 = (r+2)(r+3) = 0$$

$$r_1 = -2 \text{ and } r_2 = -3$$

$$\text{Then } y = c_1 e^{-2t} + c_2 e^{-3t}, c_1, c_2 \in \mathbb{R}$$

(b) Find the solution of the initial value problem  $y(0)=2, y'(0)=3$

$$t=0, y(0) = c_1 + c_2 = 2$$

$$y' = -2c_1 e^{-2t} - 3c_2 e^{-3t} \rightarrow y'(0) = -2c_1 - 3c_2 = 3$$

$$\begin{cases} c_1 + c_2 = 2 \\ -2c_1 - 3c_2 = 3 \end{cases} \rightarrow c_2 = 7 \rightarrow c_2 = -7 \text{ and } c_1 = 9$$

$$\therefore y = 9e^{-2t} - 7e^{-3t}$$

Ex ③ - Find the solution of the initial value problem  $4y'' - 8y' + 3y = 0,$

$$y(0) = 2, y'(0) = \frac{1}{2}$$

$$4r^2 - 8r + 3 = 0 = (2r-1)(2r-3)$$

$$r_1 = \frac{1}{2}, r_2 = \frac{3}{2}$$

$$\text{General sol: } y = c_1 e^{t/2} + c_2 e^{3t/2}$$

$$y' = \frac{c_1}{2} e^{t/2} + \frac{3c_2}{2} e^{3t/2}$$

$$y(0) = c_1 + c_2 = 2$$

$$y'(0) = \frac{c_1}{2} + \frac{3c_2}{2} = \frac{1}{2}$$

$$\left. \begin{array}{l} c_1 = 2 - c_2 \\ c_1 = 1 - 3c_2 \end{array} \right\}$$

$$c_2 = -\frac{1}{2}, c_1 = \frac{5}{2}$$

$$\therefore y = \frac{5}{2} e^{t/2} - \frac{1}{2} e^{3t/2}$$

## Week 7 (Nov 2nd)

Ex ① - Consider  $u=u(x,y)$ . Find the general solution to the PDE  $\partial u_x - u_y = 10$   
with the initial condition  $u(x,0) = 5x$

$$a=2, b=-1$$

$$\begin{aligned} \text{Let } s &= ax+by, t = bx-ay \\ s &= 2x-y \quad t = -x-2y \end{aligned}$$

$$\begin{cases} \frac{\partial s}{\partial x} = 2 & \frac{\partial s}{\partial y} = -1 \\ \frac{\partial t}{\partial x} = -1 & \frac{\partial t}{\partial y} = -2 \end{cases}$$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial x} = 2 \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial y} = - \frac{\partial u}{\partial s} - 2 \frac{\partial u}{\partial t}$$

$$\Rightarrow \partial u_x - u_y = 10$$

$$2 \left( 2 \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} \right) - \left( - \frac{\partial u}{\partial s} - 2 \frac{\partial u}{\partial t} \right) = 10$$

$$5 \frac{\partial u}{\partial s} = 10$$

$$\frac{\partial u}{\partial s} = 2$$

$$\int \partial u = \int 2 \partial s$$

$$u = 2s + f(t)$$

$$\Rightarrow u(x,y) = 2(2x-y) + f(-x-2y) \quad \text{G.S.}$$

$$\text{I.V.P. Let } u(x,0) = 5x$$

$$\Rightarrow u(x,0) = 4x + f(-x) = 5x$$

$$f(-x) = x \rightarrow f(-x-2y) = x+2y$$

$$\therefore u(x,y) = 2(2x-y) + x+2y = 5x$$

Ex ② - Consider  $u=u(x,y)$ . Find the general solution to the PDE  $u_x - u_y + u^2 = 0$   
with the initial condition  $u(x,0) = \frac{1}{3x}$

$$a=1, b=-1$$

$$\text{Let } s = x-y, t = -x-y$$

$$\begin{cases} \frac{\partial s}{\partial x} = 1 & \frac{\partial s}{\partial y} = -1 \\ \frac{\partial t}{\partial x} = -1 & \frac{\partial t}{\partial y} = -1 \end{cases}$$

$$u_x : \frac{\partial u}{\partial x} = \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial x} = \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t}$$

$$u_y : \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial y} = -\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t}$$

$$\Rightarrow u_x - u_y + u^2 = 0$$

$$\left( \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} \right) - \left( -\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} \right) + u^2 = 0$$

$$2 \frac{\partial u}{\partial s} = -u^2$$

$$\int -2 \frac{\partial u}{u^2} = \int ds$$

$$\frac{2}{u} = s + f(t)$$

$$u = \frac{2}{s + f(t)}$$

$$\Rightarrow u(x, y) = \frac{2}{(x-y) + f(-x-y)} \quad \text{G.S.}$$

IVP. Let  $u(x, 0) = \frac{1}{3x}$

$$\begin{aligned} u(x, 0) &= \frac{2}{x + f(-x)} = \frac{1}{3x} \\ 6x &= x + f(-x) \\ f(-x) &= 5x \\ f(x) &= -5x \end{aligned}$$

$$u(x, y) = \frac{2}{x-y-5(-x-y)} = \frac{2}{6x+4y} = \frac{1}{3x+2y}$$

Ex@- Let  $u = u(x, y)$ ,  $x = s^2 + 3t$ ,  $y = s^2t$ . Find  $\frac{\partial u}{\partial s^2}$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\left\{ \begin{array}{l} \frac{\partial x}{\partial s} = 2s \\ \frac{\partial y}{\partial s} = 2st \\ \frac{\partial x}{\partial t} = 3 \\ \frac{\partial y}{\partial t} = s^2 \end{array} \right.$$

$$\begin{aligned}\frac{\partial}{\partial s} \left( \frac{\partial u}{\partial s} \right) &= \frac{\partial}{\partial s} \left( \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} \right) \\ &= \frac{\partial}{\partial s} \left( \frac{\partial u}{\partial x} \right) \frac{\partial x}{\partial s} + \frac{\partial u}{\partial x} \cdot \frac{\partial}{\partial s} \left( \frac{\partial x}{\partial s} \right) + \frac{\partial}{\partial s} \left( \frac{\partial u}{\partial y} \right) \frac{\partial y}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial}{\partial s} \left( \frac{\partial y}{\partial s} \right)\end{aligned}$$

\*  $\frac{\partial}{\partial s} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) \frac{\partial x}{\partial s} + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) \frac{\partial y}{\partial s} = 2s u_{xx} + 2st u_{xy}$

\*  $\frac{\partial}{\partial s} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) \frac{\partial x}{\partial s} + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) \frac{\partial y}{\partial s} = 2s u_{xy} + 2st u_{yy}$

$$\begin{aligned}\Rightarrow \frac{\partial}{\partial s} \left( \frac{\partial u}{\partial s} \right) &= (2s u_{xx} + 2st u_{xy}) (2s) + u_x (2) + (2s u_{xy} + 2st u_{yy}) (2st) + u_y (2t) \\ &= 4s^2 u_{xx} + 4st u_{xy} + 2u_x + 4s^2 t u_{xy} + 4s^2 t^2 u_{yy} + 2tu_y \\ &= 4s^2 u_{xx} + 4st u_{xy} (1+s) + 4s^2 t^2 u_{yy} + 2u_x + 2tu_y\end{aligned}$$

## Week 8 (Nov 9th)

Ex① - Find the general solution  $u=u(x,t)$  to the PDE.

$u_{xx} - u_{xt} - 6u_{tt} = 0$  with the change of variables:  $v=2x-t$ ,  $w=3x+t$

$u=u(v,w)$  and  $v=v(x,t)$ ,  $w=w(x,t)$

$$\begin{cases} \frac{\partial v}{\partial x} = 2 & \frac{\partial w}{\partial x} = 3 \\ \frac{\partial v}{\partial t} = -1 & \frac{\partial w}{\partial t} = 1 \end{cases}$$

By Chain Rule:

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial x} = 2 \frac{\partial u}{\partial v} + 3 \frac{\partial u}{\partial w}$$

$$\begin{aligned}u_{xx} &= \frac{\partial}{\partial x} \left( 2 \frac{\partial u}{\partial v} + 3 \frac{\partial u}{\partial w} \right) = 2 \underbrace{\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial v} \right)}_{\text{use } u_x \text{ formula.}} + 3 \underbrace{\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial w} \right)}_{\text{use } u_x \text{ formula.}} \\ &= 2 \left[ 2 \frac{\partial}{\partial v} \left( \frac{\partial u}{\partial v} \right) + 3 \frac{\partial}{\partial w} \left( \frac{\partial u}{\partial v} \right) \right] + 3 \left[ 2 \frac{\partial}{\partial v} \left( \frac{\partial u}{\partial w} \right) + 3 \frac{\partial}{\partial w} \left( \frac{\partial u}{\partial w} \right) \right]\end{aligned}$$

$$= 4 \frac{\partial^2 u}{\partial v^2} + 6 \frac{\partial^2 u}{\partial v \partial w} + 6 \frac{\partial^2 u}{\partial w \partial v} + 9 \frac{\partial^2 u}{\partial w^2}$$

$$= 4 \frac{\partial^2 u}{\partial v^2} + 12 \frac{\partial^2 u}{\partial v \partial w} + 9 \frac{\partial^2 u}{\partial w^2}$$

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial t} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial t} = - \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}$$

$$u_{tt} = \frac{\partial}{\partial t} \left( -\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w} \right) = -\underbrace{\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial v} \right)}_{\text{use } u_t \text{ formula.}} + \underbrace{\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial w} \right)}$$

$$\begin{aligned} &= - \left[ -\frac{\partial}{\partial v} \left( \frac{\partial u}{\partial v} \right) + \frac{\partial}{\partial w} \left( \frac{\partial u}{\partial v} \right) \right] + \left[ -\frac{\partial}{\partial v} \left( \frac{\partial u}{\partial w} \right) + \frac{\partial}{\partial w} \left( \frac{\partial u}{\partial w} \right) \right] \\ &= \frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial w \partial v} - \frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2} \\ &= \frac{\partial^2 u}{\partial v^2} - 2 \frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2} \end{aligned}$$

$$u_{xt} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} \right) = \frac{\partial}{\partial x} \left( -\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w} \right) = -\underbrace{\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial v} \right)}_{\text{use } u_x \text{ formula.}} + \underbrace{\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial w} \right)}$$

$$\begin{aligned} &= - \left[ 2 \frac{\partial}{\partial v} \left( \frac{\partial u}{\partial v} \right) + 3 \frac{\partial}{\partial w} \left( \frac{\partial u}{\partial v} \right) \right] + \left[ 2 \frac{\partial}{\partial v} \left( \frac{\partial u}{\partial w} \right) + 3 \frac{\partial}{\partial w} \left( \frac{\partial u}{\partial w} \right) \right] \\ &= -2 \frac{\partial^2 u}{\partial v^2} - \frac{3 \partial^2 u}{\partial w \partial v} + \frac{2 \partial^2 u}{\partial v \partial w} + 3 \frac{\partial^2 u}{\partial w^2} \\ &= -2 \frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial v \partial w} + \frac{3 \partial^2 u}{\partial w^2} \end{aligned}$$

$$\Rightarrow u_{xx} - u_{xt} - bu_{tt} = 0$$

$$\begin{aligned} \Rightarrow 4 \frac{\partial^2 u}{\partial v^2} + 12 \frac{\partial^2 u}{\partial v \partial w} + 9 \frac{\partial^2 u}{\partial w^2} - &\left[ -2 \frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial v \partial w} + \frac{3 \partial^2 u}{\partial w^2} \right] \\ - 6 \left[ \frac{\partial^2 u}{\partial v^2} - 2 \frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2} \right] &= 0 \end{aligned}$$

$$\Rightarrow 25 \frac{\partial^2 u}{\partial v \partial w} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial v \partial w} = 0$$

$$\int \frac{\partial}{\partial w} \left( \frac{\partial u}{\partial v} \right) dw = \int 0 dw + f(v) \rightarrow \text{integrate over } w \\ v \text{ is a constant.}$$

$$\frac{\partial u}{\partial v} = f(v)$$

$$\int \frac{\partial u}{\partial v} dv = \int f(v) dv + g(w) \rightarrow \text{integrate over } v \\ w \text{ is a constant.}$$

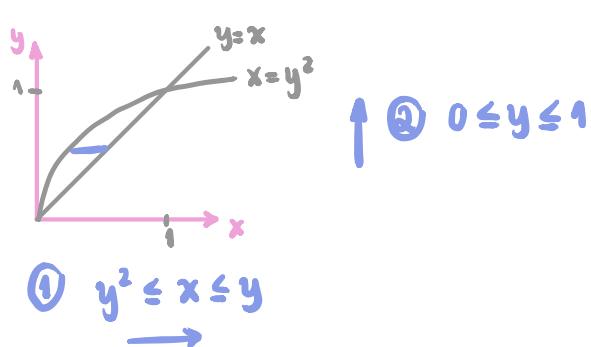
$$u = F(v) + g(w)$$

$$\therefore u(x,t) = F(2x-t) + g(3x+t)$$

## 2D - Integration

**Ex ①** - Give a rough sketch of the regions and evaluate the following integrals

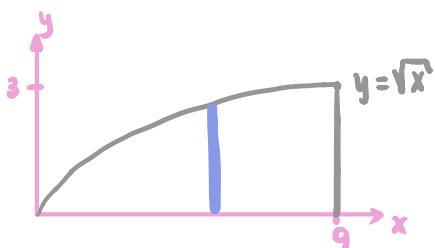
a)  $\iint_D \sqrt{x} dA$  where D is the region bounded by  $x=y$  and  $x=y^2$



$$\begin{aligned}\iint_D \sqrt{x} dA &= \int_0^1 \int_{y^2}^y \sqrt{x} dx dy \\ &= \int_0^1 \frac{2}{3} x^{3/2} \Big|_{y^2}^y dy \\ &= \frac{2}{3} \int_0^1 y^{3/2} - y^3 dy \\ &= \frac{2}{3} \left[ \frac{2}{5} y^{5/2} - \frac{y^4}{4} \right]_0^1 = \frac{1}{10}\end{aligned}$$

**Week 9 (Nov 16)**

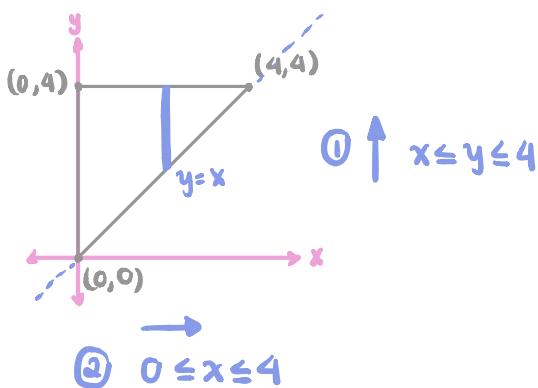
b)  $\iint_D y \sin x^2 dA$  where  $D = \{(x,y) \in \mathbb{R}^2, 0 \leq y \leq \sqrt{x}, 0 \leq x \leq 9\}$



$$\begin{aligned}\iint_D y \sin x^2 dA &= \int_0^9 \int_0^{\sqrt{x}} y \sin x^2 dy dx \\ &= \int_0^9 \frac{y^2}{2} \sin x^2 \Big|_0^{\sqrt{x}} dx \\ &= \frac{1}{2} \int_0^9 x \sin x^2 dx \\ &= -\frac{1}{4} \cos x^2 \Big|_0^9 \\ &= \frac{1}{4} (1 - \cos 81)\end{aligned}$$

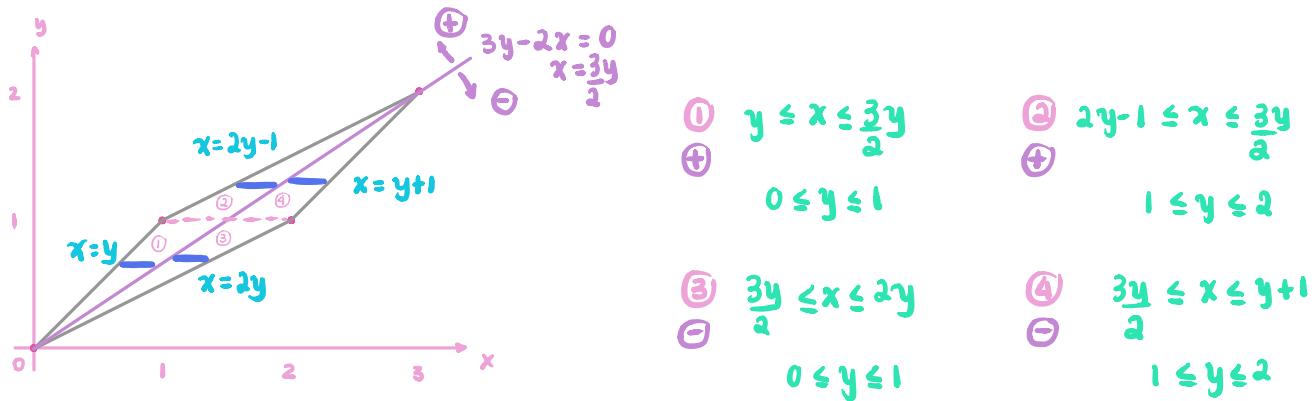
**Ex ②** - Evaluate  $\iint_D f(x,y) dA$  for the functions f and region D

a)  $f(x,y) = \cos y$  and D is the triangle with vertices  $(0,0), (4,4)$  and  $(0,4)$



$$\begin{aligned}\iint_D \cos y dA &= \int_0^4 \int_x^4 \cos y dy dx \\ &= \int_0^4 [\sin y]_x^4 dx \\ &= \int_0^4 \sin 4 - \sin x dx \\ &= [x \sin 4 + \cos x]_0^4 \\ &= 4 \sin 4 + \cos 4 - 1\end{aligned}$$

b)  $f(x,y) = |3y - 2x|$  and D is the parallelogram with vertices  $(0,0), (1,1), (2,1), (3,2)$



$$\int_D |3y - 2x| dA = \textcircled{1} + \textcircled{2} + \textcircled{3} + \textcircled{4}$$

$$= \int_0^1 \int_{\frac{3y}{2}}^{\frac{3y}{2}} (3y - 2x) dx dy + \int_1^2 \int_{2y-1}^{2y} (3y - 2x) dx dy + \int_0^1 \int_{\frac{3y}{2}}^{2y} -(3y - 2x) dx dy + \int_1^2 \int_{\frac{3y}{2}}^{y+1} -(3y - 2x) dx dy$$

$$= \int_0^1 [3yx - x^2]_{\frac{3y}{2}}^{2y} dy + \int_1^2 [3yx - x^2]_{2y-1}^{2y} dy - \int_0^1 [3yx - x^2]_{\frac{3y}{2}}^{2y} dy - \int_1^2 [3yx - x^2]_{\frac{3y}{2}}^{y+1} dy$$

$$= \int_0^1 \left[ \frac{9y^2}{2} - \frac{9y^2}{4} - 3y^2 + y^2 \right] dy + \int_1^2 \left[ \frac{9y^2}{2} - \frac{9y^2}{4} - 6y^2 + 3y + (2y-1)^2 \right] dy$$

$$- \int_0^1 \left[ 6y^2 - 4y^2 - \frac{9y^2}{2} + \frac{9y^2}{4} \right] dy - \int_1^2 \left[ 3y^2 + 3y - (y+1)^2 - \frac{9y^2}{2} + \frac{9y^2}{4} \right] dy$$

$$= \int_0^1 \frac{y^2}{4} dy + \int_1^2 \frac{y^2}{4} - y + 1 dy + \int_0^1 \frac{y^2}{4} dy + \int_1^2 \frac{y^2}{4} - y + 1 dy$$

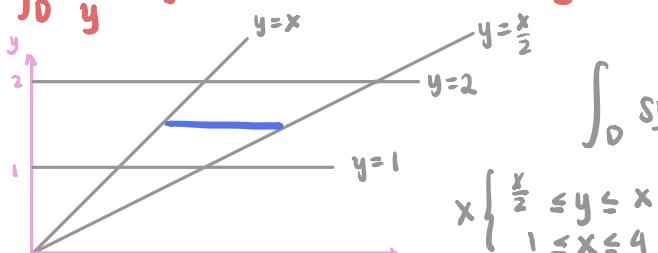
$$= \int_0^1 \frac{y^2}{2} dy + \int_1^2 \frac{y^2}{2} - 2y + 2 dy$$

$$= \frac{y^3}{6} \Big|_0^1 + \left[ \frac{y^3}{6} - y^2 + 2y \right]_1^2$$

$$= \frac{1}{6} + \left[ \frac{4}{3} - 4 + 4 - \frac{1}{6} + 1 - 2 \right] = \frac{1}{3}$$

### Ex③ - Solve

(a)  $\int_D \frac{\sin y}{y} dy dx$ , where D is the region bounded by  $y=x$ ,  $y=\frac{x}{2}$ ,  $y=1$  and  $y=2$



$$\int_D \frac{\sin y}{y} dy dx = \int_1^2 \int_{\frac{x}{2}}^x \frac{\sin y}{y} dy dx$$

$$= \int_1^2 \int_{\frac{x}{2}}^2 \sin y dy dx$$

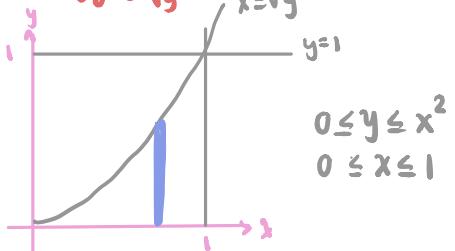
$$\begin{cases} y \leq x \leq 2y \\ 1 \leq y \leq 2 \end{cases}$$

$$= \int_1^2 \frac{\sin y}{y} (2y - y) dy$$

$$= \int_1^2 \sin y dy$$

$$= -\cos y \Big|_1^2 = \cos 1 - \cos 2$$

(b)  $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{2+x^3} dx dy$



$$= \int_0^1 \int_0^{x^2} \sqrt{2+x^3} dy dx$$

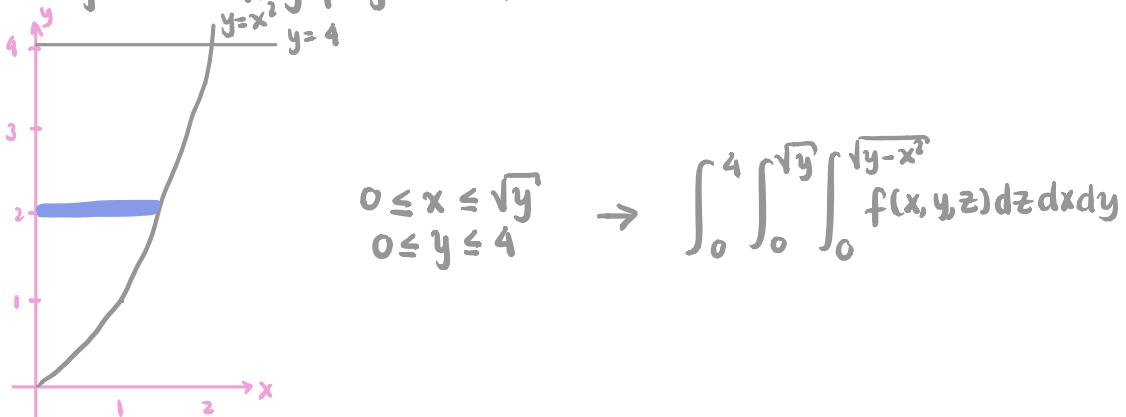
$$= \int_0^1 x^2 \sqrt{2+x^3} dx = \left[ \frac{2}{9} (2+x^3)^{3/2} \right]_0^1 = \frac{2}{9} (3\sqrt{3} - 2\sqrt{2})$$

### 3D-Integration

Ex ④ - Write the integral  $\int_0^2 \int_{x^2}^4 \int_0^{\sqrt{y-x^2}} f(x,y,z) dz dy dx$  in two other orders.

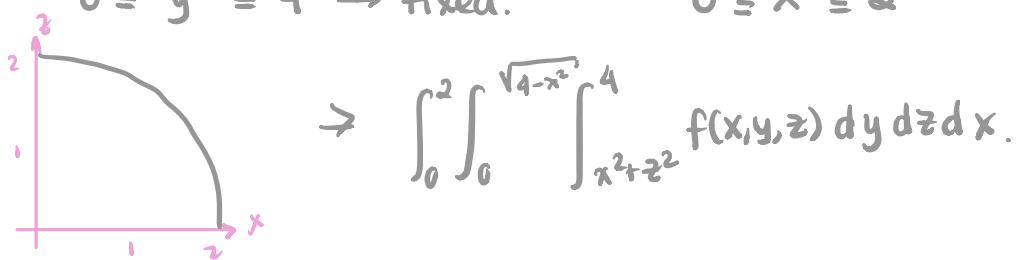
Given  $0 \leq z \leq \sqrt{y-x^2}$   
 $x^2 \leq y \leq 4$   
 $0 \leq x \leq 2$

\* Change the x-y projection.



\* Change the x-z projection.

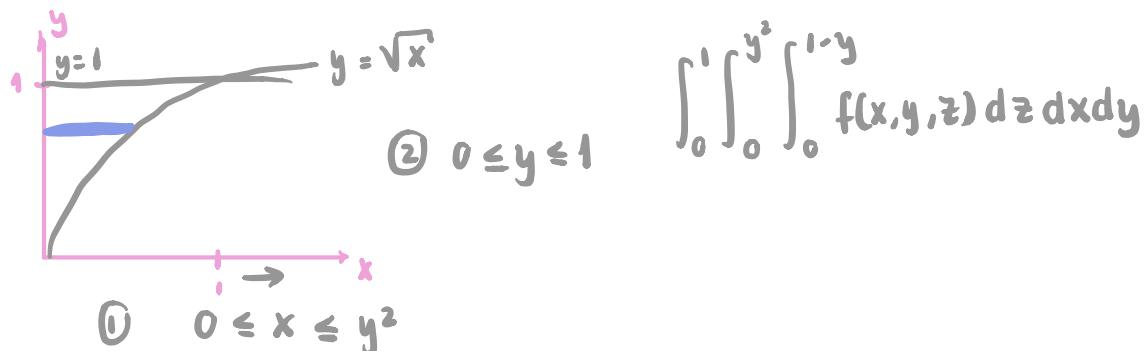
$$\begin{aligned} 0 &\leq z \leq \sqrt{y-x^2} \\ 0 &\leq x \leq \sqrt{y} \\ 0 &\leq y \leq 4 \rightarrow \text{Fixed.} \end{aligned} \quad \rightarrow \quad \begin{aligned} x^2+z^2 &\leq y \leq 4 \\ 0 &\leq z \leq \sqrt{4-x^2} \\ 0 &\leq x \leq 2 \end{aligned}$$



Ex(5) - Write the integral  $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z) dz dy dx$  in two other orders

Given  
 $0 \leq z \leq 1-y$   
 $\sqrt{x} \leq y \leq 1$   
 $0 \leq x \leq 1$

\* Change to  $dz dy dx$



$$\int_0^1 \int_0^{y^2} \int_0^{1-y} f(x, y, z) dz dy dx$$

\* Change to  $dy dx dz$

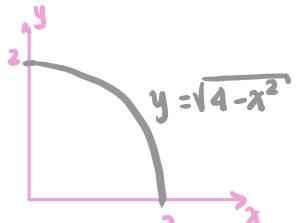
$$\begin{array}{ll} 0 \leq z & z \leq 1-y \\ \sqrt{x} \leq y & y \leq 1 \\ 0 \leq x & x \leq 1 \end{array} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} 1-y \leq 1 \\ x \leq y^2 \end{array} \quad \left. \begin{array}{l} 1-y \leq 1 \\ x \leq y^2 \end{array} \right\} \begin{array}{l} \Rightarrow 0 \leq z \leq 1-y \leq 1 \\ \Rightarrow 0 \leq z \leq 1 \\ \Rightarrow z \leq 1-y \rightarrow y \leq 1-z \\ 0 \leq x \leq (1-z)^2 \end{array}$$

$$\sqrt{x} \leq y \leq 1-z$$

$$\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x, y, z) dy dx dz$$

Week 10 (Nov 23)

Ex(1) - Evaluate  $\iiint_B x dV$  where  $B$  is the first octant solid bounded by the cylinder  $x^2 + y^2 = 4$  and the plane  $2y + z = 4$   $\rightarrow x, y, z \geq 0$



$$\begin{array}{l} 0 \leq z \leq 4 - 2y \\ 0 \leq y \leq \sqrt{4-x^2} \\ 0 \leq x \leq 2 \end{array}$$

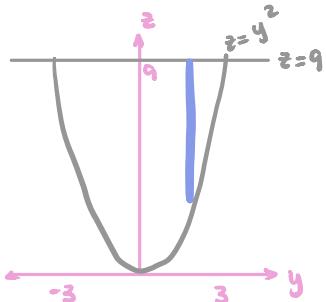
$$\iiint_B x dV = \iint_{\text{proj}} \left( \int_0^{4-2y} x dz \right) dA = \int_0^2 \int_0^{\sqrt{4-x^2}} x(4-2y) dy dx$$

$$= \int_0^2 x \left[ 4y - y^2 \right]_{0}^{\sqrt{4-x^2}} dx = \int_0^2 x \left[ 4\sqrt{4-x^2} - (4-x^2) \right] dx$$

$$\begin{aligned} \text{Let } u &= 4-x^2 \\ du &= -2x dx \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{2} \int 4u^{1/2} - u \, du = -\frac{1}{2} \left[ \frac{8}{3} u^{3/2} - \frac{u^2}{2} \right] = \left[ -\frac{4}{3} (4-x^2)^{3/2} + \frac{(4-x^2)}{2} \right]_0^2 \\
 &= -\frac{4}{3} \left[ (4-4)^{3/2} - 4^{3/2} \right] + \frac{1}{2} \left[ (4-4) - (4-0) \right] \\
 &= \frac{4}{3} - 2 = \frac{32}{3} - \frac{6}{3} = \frac{26}{3}
 \end{aligned}$$

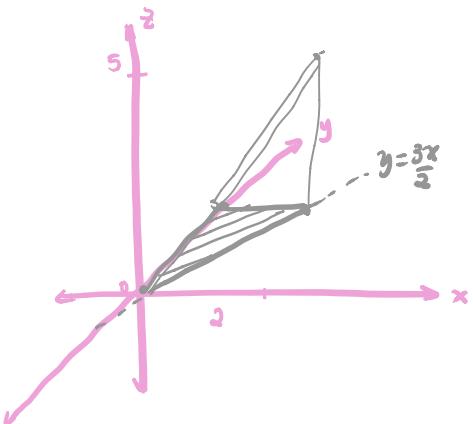
**Ex ②** - Find the volume of the region bounded by  $x=y^2+z^2$ ,  $z=y^2$ ,  $z=9$  and  $x=0$



Fix  $0 \leq x \leq y^2+z^2$   
 $y^2 \leq z \leq 9$   
 $-3 \leq y \leq 3$

$$\begin{aligned}
 V &= \int_B 1 \, dV = \int_{-3}^3 \int_{y^2}^9 \int_0^{y^2+z^2} dx \, dz \, dy = \int_{-3}^3 \int_{y^2}^9 y^2 + z^2 \, dz \, dy \\
 &= \int_{-3}^3 \left[ y^2 z + \frac{z^3}{3} \right]_{y^2}^9 \, dy = \int_{-3}^3 9y^2 + 243 - y^4 - \frac{y^6}{3} \, dy \\
 &= \left[ -\frac{y^7}{21} - \frac{y^5}{5} + 3y^3 + 243y \right]_{-3}^3 = \dots = \frac{46008}{35}
 \end{aligned}$$

**Ex ③** - Find the volume of the tetrahedron with corners at  $(0,0,0)$ ,  $(0,3,0)$ ,  $(2,3,0)$ , and  $(2,3,5)$



$$\begin{aligned}
 &0 \leq z \leq 5x/2 \\
 &\frac{3x}{2} \leq y \leq 3 \\
 &0 \leq x \leq 2 \\
 &\Rightarrow \int_0^2 \int_{\frac{3x}{2}}^3 \int_0^{\frac{5x}{2}} dz \, dy \, dx = \int_0^2 \int_{\frac{3x}{2}}^3 \frac{5x}{2} dy \, dx \\
 &= \int_0^2 \frac{5}{2} x \left( 3 - \frac{3x}{2} \right) dx = \int_0^2 \frac{15x}{2} - \frac{15x^3}{4} dx \\
 &= \frac{15x^2}{4} - \frac{15x^3}{12} \Big|_0^2 = 15 - 10 = 5.
 \end{aligned}$$

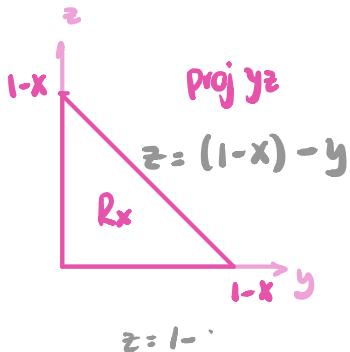
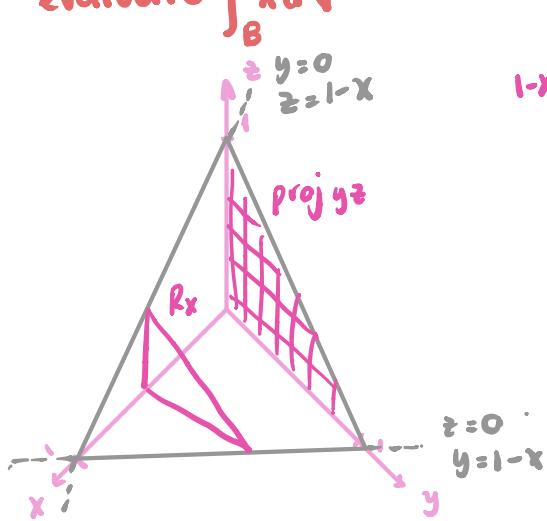
**Ex ④** - Evaluate  $\int_1^2 \int_y^2 \int_0^{\ln(y+z)} e^x \, dx \, dz \, dy$

$$\begin{aligned}
 &= \int_1^2 \int_y^2 e^{\ln(y+z)} - e^0 \, dz \, dy = \int_1^2 \int_y^2 y + z - 1 \, dz \, dy = \int_1^2 yz + \frac{z^2}{2} - z \Big|_y^2 \, dy
 \end{aligned}$$

$$= \int_1^2 y^3 + \frac{y^4}{2} - y^2 - y^2 - \frac{y^2}{2} - y \, dy = \int_1^2 y^3 + \frac{y^4}{4} - \frac{5y^2}{2} - y \, dy$$

$$= \dots = \frac{151}{600}$$

**Ex 5** - If  $B$  is the tetrahedron with vertices  $(0,0,0)$ ,  $(1,0,0)$ ,  $(0,1,0)$ ,  $(0,0,1)$  evaluate  $\int_B x \, dV$



$$\begin{aligned} 0 &\leq z \leq 1-x-y \\ 0 &\leq y \leq 1-x \\ 0 &\leq x \leq 1 \end{aligned}$$

$$\begin{aligned} \iiint_B x \, dV &= \int_0^1 x \int_0^{1-x} \int_0^{1-x-y} dz \, dy \, dx = \int_0^1 x \int_0^{1-x} (1-x-y) \, dy \, dx \\ &= \int_0^1 x \left[ (1-x)y - \frac{y^2}{2} \right]_0^{1-x} dx = \int_0^1 x \left[ (1-x)^2 - \left( \frac{1-x}{2} \right)^2 \right] dx \\ &= \int_0^1 \frac{(1-x)^2}{2} x \, dx = \frac{1}{2} \int_0^1 x - 2x^2 + x^3 \, dx = \frac{1}{2} \left[ \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1 \\ &= \frac{1}{2} \left[ \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] = \frac{1}{24} \end{aligned}$$

## 1D- Parametrization

$$f(t) : \mathbb{R} \rightarrow \mathbb{R}^2 \text{ or } f(t) : \mathbb{R} \rightarrow \mathbb{R}^3$$

for each  $t$  value, there is an  $f(t)$  point in  $\mathbb{R}^2 \mid \mathbb{R}^3$   
If we take  $t$  as time, we take  $f(t)$  to be the position of an object.

**Ex:** Parametrize a circle of radius  $R$

We need to find the coordinate  $(x, y)$  for each  $t$ .

We can use polar coord. so  $f(t) = (R\cos t, R\sin t)$ ,  $t \in [0, 2\pi]$

## 2D- Parametrization .

$$f(u, v) : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

## Coordinate transformation

$$f(u, v) : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ or } f(u, v, w) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

Ex:  $f(r, \theta) = (r\cos\theta, r\sin\theta) = (x, y) \quad r \in [0, R] \quad \theta \in [0, 2\pi]$

Ex① - The path of a particle is given by the following set of parametric eqs.

$$x = 3\cos(2t), \quad y = 1 + \cos^2(2t)$$

$$\Rightarrow x = 3\cos(2t)$$

$$\Rightarrow y = 1 + \cos^2(2t)$$

$$-1 \leq \cos(2t) \leq 1$$

$$0 \leq \cos^2(2t) \leq 1$$

$$-3 \leq 3\cos(2t) \leq 3$$

$$1 \leq 1 + \cos^2(2t) \leq 2$$

$$-3 \leq x \leq 3$$

$$1 \leq y \leq 2$$

$$\rightarrow \cos(2t) = 1 \quad \text{When is } \cos(2t) = 1?$$

$$2t = 0 + 2\pi n$$

$$t = \pi n, \quad n \in \mathbb{Z}$$

$$\rightarrow \cos(2t) = -1 \quad \text{When is } \cos(2t) = -1?$$

$$2t = \pi + 2\pi n$$

$$t = \frac{\pi}{2} + \pi n, \quad n \in \mathbb{Z}$$

$\therefore$  The right end is when  $t = n\pi$  and the left end is when  $t = \frac{\pi}{2} + \pi n$ .

Ex② - For each of the following vector-valued functions, evaluate  $f(0)$ ,  $f(\pi/2)$ ,  $f(2\pi/3)$ . Do any of these functions have domain restriction?

a)  $f(t) = (4\cos t, 3\sin t)$

$$f(0) = (4\cos(0), 3\sin(0)) = (4, 0)$$

$$f(\pi/2) = \left(4\cos\frac{\pi}{2}, 3\sin\frac{\pi}{2}\right) = (0, 3)$$

$$f(2\pi/3) = \left(4\cos\frac{2\pi}{3}, 3\sin\frac{2\pi}{3}\right) = \left(-2, \frac{3\sqrt{3}}{2}\right)$$

b)  $f(t) = (3\tan t, 4\sec t, 5t)$

$$f(0) = (3\tan(0), 4\sec(0), 5(0)) = (0, 4, 0)$$

$$f(\pi/2) = (3\tan(\pi/2), 4\sec(\pi/2) + 5(\pi/2)) \leftarrow \text{DNE}$$

$$f(2\pi/3) = \left(3\tan\frac{2\pi}{3}, 4\sec\frac{2\pi}{3}, 5\left(\frac{2\pi}{3}\right)\right)$$

$$= (-3\sqrt{3}, -8, 10\pi/3)$$

## Derivative Matrix

Ex① - Find  $Df$  given  $f(x,y,z) = (z, 2yz^2, x+2y^2z)$

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad f(x,y,z) = (f_1, f_2, f_3)$$

$$Df = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2z^2 & 4yz \\ 1 & 4yz & 2y^2 \end{pmatrix}$$

Ex② - Find  $Df$  given  $f(x,y) = (xy^2, x+2y, xy)$

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \quad f(x,y) = (f_1, f_2, f_3)$$

$$Df = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \end{pmatrix} = \begin{pmatrix} y^2 & 2xy \\ 1 & 2 \\ y & x \end{pmatrix}$$

## Change of Variable Formula

$$g: A \rightarrow B \quad g(A)=B \quad \int_B f = \int_A f \circ g \cdot |\text{Det}(Dg)|$$

\*Polar Coordinates! YAY ❤

Make the transformation  $g(r, \theta) = (r\cos\theta, r\sin\theta) = (x, y)$

How is it equivalent?  $f \circ g = f(g(r, \theta)) = f(r\cos\theta, r\sin\theta) = f(x, y)$

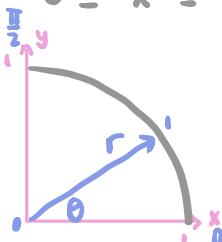
$$Dg = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \quad |\text{Dg}| = r\cos\theta \cdot \cos\theta - (-r)\sin\theta \sin\theta \\ = r\cos^2\theta + r\sin^2\theta \\ = r$$

Ex③ - Evaluate  $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)^{5/2} dy dx$

Change to Polars!

$$0 \leq y \leq \sqrt{1-x^2}$$

$$0 \leq x \leq 1$$



$$0 \leq r \leq 1$$

$$0 \leq \theta \leq \frac{\pi}{2}$$

$$= \int_0^{\pi/2} \int_0^1 (r^2)^{5/2} r dr d\theta$$

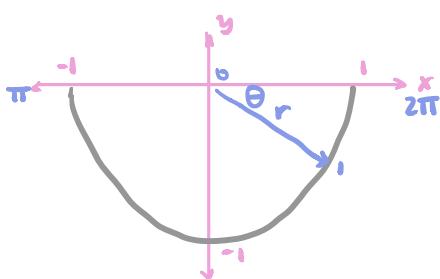
$$= \int_0^{\pi/2} \frac{r^7}{7} \Big|_0^1 d\theta = \frac{1}{7} \int_0^{\pi/2} d\theta = \frac{1}{7} \left(\frac{\pi}{2}\right) = \frac{\pi}{14}$$

Ex ④ - Evaluate  $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 \cos(x^2+y^2) dy dx$

Change to polars!

$$-\sqrt{1-x^2} \leq y \leq 0$$

$$-1 \leq x \leq 1$$



$$0 \leq r \leq 1$$

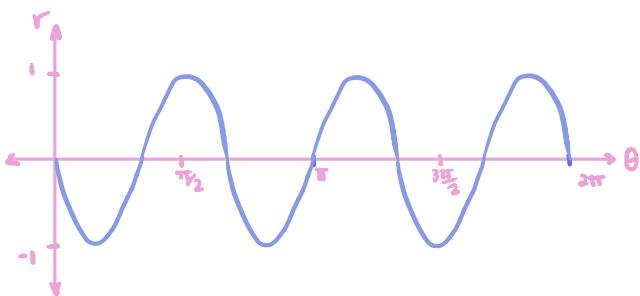
$$\pi \leq \theta \leq 2\pi$$

$$\begin{aligned}
 &= \int_{\pi}^{2\pi} \int_0^1 \cos(r^2) r dr d\theta \\
 &= \int_{\pi}^{2\pi} \frac{1}{2} \sin(r^2) \Big|_0^1 d\theta \\
 &= \int_{\pi}^{2\pi} \frac{1}{2} \sin(1) d\theta \\
 &= \frac{\pi}{2} \sin(1)
 \end{aligned}$$

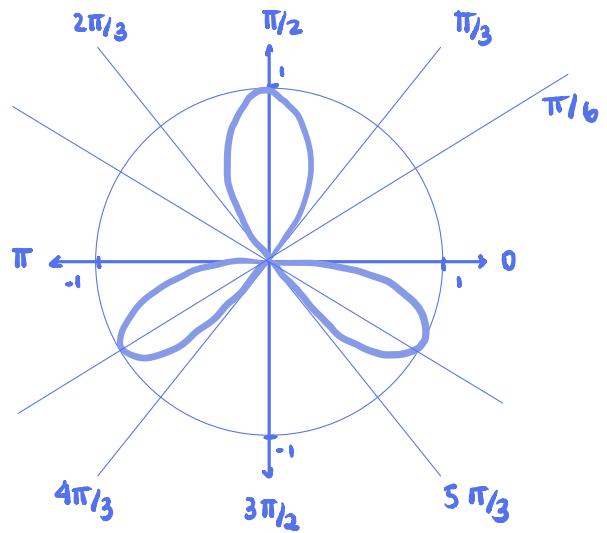
## Week 11 (Nov 30)

### Polar Curves

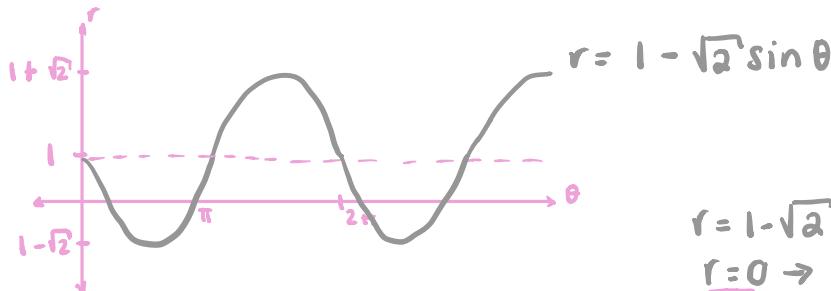
Ex ① - Sketch the curve  $r = -\sin 3\theta$  in the Polar plane



$\theta$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi$
$r$	0	-1	0	1	0	-1	0



Ex ② - Give a rough sketch of  $r = 1 - \sqrt{2} \sin \theta$



$$r = 1 - \sqrt{2} \rightarrow 1 - \sqrt{2} = 1 - \sqrt{2} \sin \theta$$

$$1 = \sin \theta$$

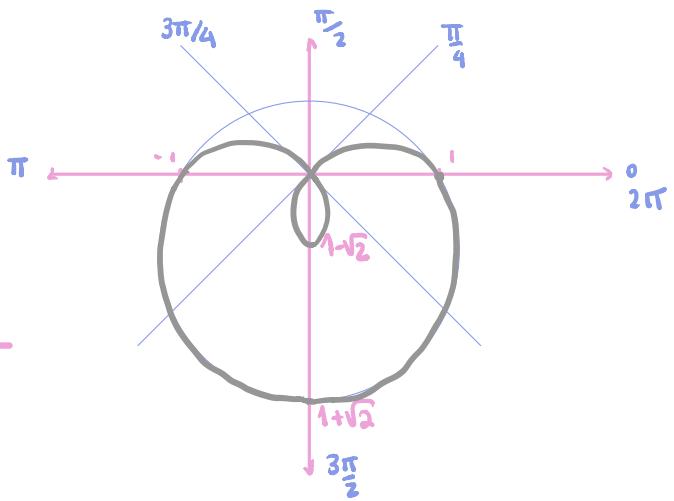
$$\theta = \frac{\pi}{2}$$

$$\begin{aligned}
 r &= 1 - \sqrt{2} \sin \theta \\
 r = 0 &\rightarrow 1 = \sqrt{2} \sin \theta \\
 \sin \theta &= \frac{1}{\sqrt{2}} \\
 \theta &= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \\
 &= \frac{\pi}{4}, \frac{3\pi}{4}
 \end{aligned}$$

$$r = \underline{1 + \sqrt{2}} \rightarrow 1 + \sqrt{2} = 1 - \sqrt{2}\sin\theta$$

$$-1 = \sin\theta$$

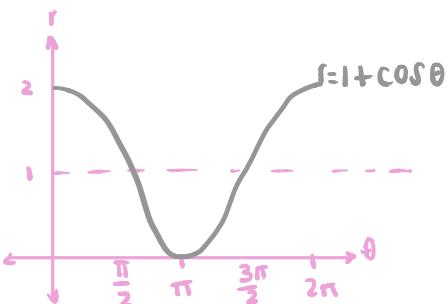
$$\theta = \frac{3\pi}{2}$$



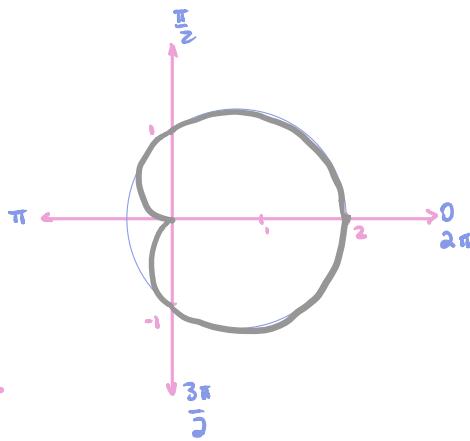
$\theta$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$r$	1	0	$1 - \sqrt{2}$	0	-1	$1 + \sqrt{2}$	1

Ex③ - Find the entire area bounded by the polar graph  $r = 1 + \cos\theta$

$$\begin{aligned} \iint_D dA &= \int_0^{2\pi} \int_0^{1+\cos\theta} r dr d\theta = \int_0^{2\pi} \frac{r^2}{2} \Big|_0^{1+\cos\theta} d\theta = \int_0^{2\pi} \left(1 + \frac{\cos\theta}{2}\right)^2 d\theta \quad \left\{ \cos^2\theta = \frac{\cos 2\theta + 1}{2} \right\} \\ &= \frac{1}{2} \int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta d\theta = \frac{1}{2} \int_0^{2\pi} 1 + 2\cos\theta + \frac{\cos 2\theta + 1}{2} d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2} d\theta = \frac{1}{2} \left[ \frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} \\ &= \frac{1}{2} \times \frac{3(2\pi)}{2} = \frac{3\pi}{2} \end{aligned}$$



$\theta$	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$r$	2	1	0	1	2



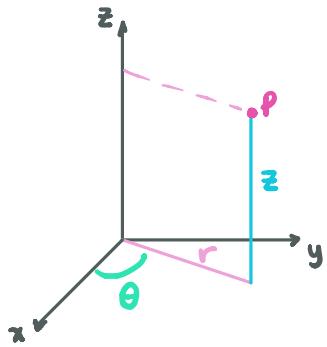
## Cylindrical and Spherical Coordinates

### Cylindrical Coordinates:

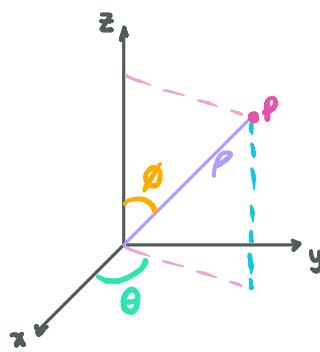
- \* More flexible
- \* When you want to integrate a section of a cylinder

### Spherical Coordinates:

- \* More specific
- \* When you want to integrate a section of a sphere



$$\begin{aligned}x &= r\cos\theta \\y &= r\sin\theta \\z &= z \\0 &\leq r \leq \infty \\0 &\leq \theta \leq 2\pi\end{aligned}$$

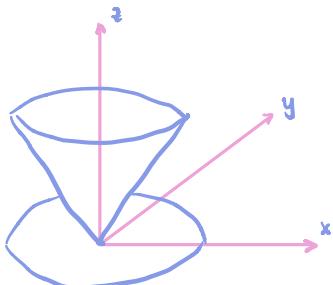


$$\begin{aligned}x &= \rho\sin\phi\cos\theta \\y &= \rho\sin\phi\sin\theta \\z &= \rho\cos\phi \\0 &\leq \rho \leq \infty \\0 &\leq \theta \leq 2\pi \\0 &\leq \phi \leq \pi\end{aligned}$$

**Ex ④** - Evaluate  $\iiint_B z dxdydz$  by using cylindrical coordinates, where B is the region within the cylinder  $x^2+y^2=1$  above the xy-plane and below the cone  $z=(x^2+y^2)^{1/2}$

If we fix x and y, we have  $0 \leq z \leq \sqrt{x^2+y^2}$   
the projection is  $x^2+y^2 \leq 1$ . (unit disk)

\* Recall,  
 $Dg = r$



$$\int_B z dV = \iint_{\text{unit disk}} \int_0^{\sqrt{x^2+y^2}} z dz dA.$$

$$\begin{aligned}\text{SWITCH } D & \quad r^2 = x^2 + y^2 \\& \quad r = \sqrt{x^2 + y^2} \\& \quad \rightarrow 0 \leq z \leq r\end{aligned}\quad \text{unit disk: } 0 \leq r \leq 1 \\0 \leq \theta \leq 2\pi$$

$$\begin{aligned}\int_0^{2\pi} \int_0^1 \int_0^r z r dz dr d\theta &= \frac{1}{2} \int_0^{2\pi} \int_0^1 r \cdot z^2 \Big|_0^r dr d\theta \\&= \frac{1}{2} \int_0^{2\pi} \int_0^1 r^3 dr d\theta = \frac{1}{2} \int_0^{2\pi} \frac{r^4}{4} \Big|_0^1 d\theta \\&= \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4}\end{aligned}$$

**Ex ⑤** - Evaluate  $\iiint_S \frac{dxdydz}{(x^2+y^2+z^2)^{3/2}}$ , where S is the solid bounded by the two spheres  $x^2+y^2+z^2=a^2$  and  $x^2+y^2+z^2=b^2$  where  $0 < a < b$ .

$$\begin{cases} \rho = a \\ \rho = b \end{cases} \Rightarrow a < \rho < b$$

\* Recall,  
 $Dg = \rho^2 \sin\phi$

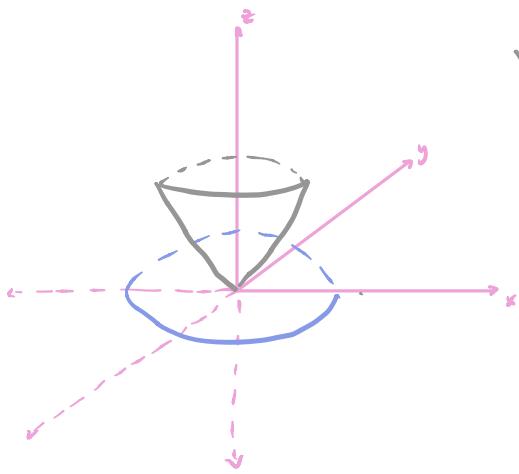
$$\text{So we have } \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{cases}$$

$$\begin{aligned}\iiint_S \frac{dxdydz}{(x^2+y^2+z^2)^{3/2}} &= \int_a^b \int_0^{2\pi} \int_0^\pi \frac{\rho^2 \sin\phi}{(\rho^2)^{3/2}} d\phi d\theta d\rho = \int_a^b \int_0^{2\pi} \frac{1}{\rho} (-\cos\phi) \Big|_0^\pi d\theta d\rho \\&= \int_a^b \int_0^{2\pi} \frac{2}{\rho} d\theta d\rho = 4\pi \int_a^b \frac{1}{\rho} d\rho = 4\pi \ln\rho \Big|_a^b = 4\pi \ln\left(\frac{b}{a}\right)\end{aligned}$$

**Ex ⑥** - Find the volume of the region bounded below by the cone  $z = \sqrt{x^2 + y^2}$  and above by the paraboloid  $z = 2 - x^2 - y^2$ .

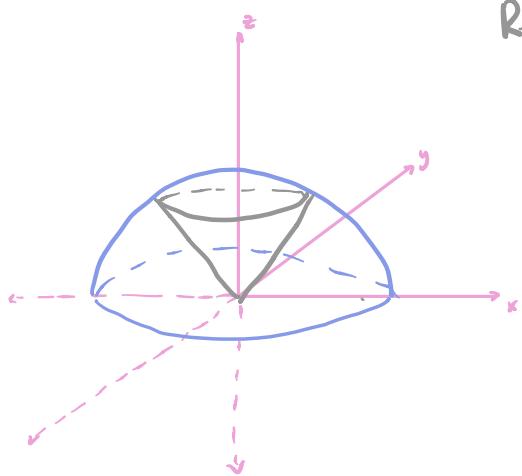
$$\begin{aligned} z &= 2 - (x^2 + y^2) = 2 - r^2 \\ z &= \sqrt{x^2 + y^2} = r \end{aligned}$$

$$\Rightarrow r = 2 - r^2 \Rightarrow r^2 + r - 2 = (r+2)(r-1) = 0 \Rightarrow r = 1$$



$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^1 ((2-r^2) - r) r dr d\theta \\ &= \int_0^{2\pi} \int_0^1 2r - r^2 - r^3 dr d\theta \\ &= \int_0^{2\pi} r^2 - \frac{r^3}{3} - \frac{r^4}{4} \Big|_0^1 d\theta = \int_0^{2\pi} 1 - \frac{1}{3} - \frac{1}{4} d\theta \\ &= \frac{5}{12} (2\pi - 0) = \frac{5\pi}{6} \end{aligned}$$

**Ex ⑦** - Evaluate the volume of the region bounded by the cone  $z = \sqrt{3(x^2 + y^2)}$  and the hemisphere  $z = \sqrt{4 - x^2 - y^2}$



$$\text{Recall } z = \rho \cos \theta$$

$$x^2 = (\rho \sin \theta \cos \theta)^2 = \rho^2 \sin^2 \theta \cos^2 \theta$$

$$y^2 = (\rho \sin \theta \sin \theta)^2 = \rho^2 \sin^2 \theta \sin^2 \theta$$

$$\begin{aligned} \Rightarrow x^2 + y^2 &= \rho^2 (\sin^2 \theta \cos^2 \theta + \sin^2 \theta \sin^2 \theta) \\ &= \rho^2 \sin^2 \theta (\cos^2 \theta + \sin^2 \theta) \\ &= \rho^2 \sin^2 \theta \end{aligned}$$

$$\Rightarrow \rho \cos \theta = \sqrt{3} (\rho^2 \sin^2 \theta)$$

$$\rho \cos \theta = \sqrt{3} \rho \sin \theta$$

$$\frac{1}{\sqrt{3}} = \frac{\sin \theta}{\cos \theta} = \tan \theta \rightarrow \theta = \frac{\pi}{6}$$

$$\Rightarrow \rho \cos \theta = \sqrt{4 - (\rho^2 \sin^2 \theta)}$$

$$\rho^2 \cos^2 \theta = 4 - \rho^2 \sin^2 \theta$$

$$V = \int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin \theta d\rho d\theta d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/6} \frac{\rho^3}{3} \Big|_0^2 \sin \theta d\theta d\theta$$

$$= \int_0^{2\pi} \frac{8}{3} (-\cos \theta) \Big|_0^{\pi/6} d\theta$$

$$= \int_0^{2\pi} \frac{8}{3} (-\sqrt{3} + 1) d\theta$$

$$\rho^2 = 4 \rightarrow \rho = 2$$

$$= \frac{16\pi}{3} \left( -\frac{\sqrt{3}}{2} + 1 \right)$$

Ex ⑧ - Solve  $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2) dz dx dy$

$$\begin{aligned} & 0 \leq y \leq 3 \\ & 0 \leq x \leq \sqrt{9-y^2} \\ & \sqrt{x^2+y^2} \leq z \leq \sqrt{18-x^2-y^2} \end{aligned} \quad \left. \begin{array}{l} \text{First} \\ \text{quadrant} \end{array} \right\} \Rightarrow 0 \leq \theta \leq \frac{\pi}{2}$$

↳ upper half of a cone      ↳ upper half of a sphere.  
 $\Rightarrow \rho = \sqrt{18} \Rightarrow 0 \leq \rho \leq \sqrt{18}$

For  $\emptyset$ , we need to find the intersection of the cone and sphere.

$$\begin{aligned} z &= \sqrt{18 - x^2 - y^2} \\ z^2 &= 18 - (x^2 + y^2) \\ z^2 &= 18 - (\sqrt{x^2 + y^2})^2 \\ z^2 &= 18 - z^2 \end{aligned}$$

$$\begin{aligned} \sqrt{18} \cos \emptyset &= z = 3 \\ \cos \emptyset &= \frac{1}{\sqrt{2}} \\ \emptyset &= \frac{\pi}{4} \end{aligned}$$

$$\Rightarrow 0 \leq \emptyset \leq \frac{\pi}{4}$$

$$\begin{aligned} z^2 &= 9 \\ z &= 3 \end{aligned}$$

$$\Rightarrow \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{\sqrt{18}} \rho^4 \sin \emptyset d\rho d\theta d\emptyset$$