MATB41 - Tutorial 3 (BV355 Fridays 2-3pm)

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Week 1 - Sept. 6th

No tutorials

Week 2 - Sept. 13th

Derivative by definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \qquad f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

Riemann Sum:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x, \text{ where } \Delta x = \frac{b-a}{n}, \; x_i = a + i\Delta x$$

Useful formulas $\rightarrow \sum_{i=1}^{n} 1 = n, \quad \sum_{i=1}^{n} i = \frac{i(i+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{i(i+1)(2i+1)}{2\cdot 3}$

Trig. Identities:

$\sin^2 x + \cos^2 x = 1$
$\cos(2x) = \cos^2 x - \sin^2 x$
$\sin(2x) = 2\sin a \cos a$

Half Angle Identities:

$$\sin\left(\frac{a}{2}\right) = \pm\sqrt{\frac{1-\cos a}{2}} \qquad \cos\left(\frac{a}{2}\right) = \pm\sqrt{\frac{1+\cos a}{2}}$$

Trig. Integrals:

$\int \cos^n(x)\sin^m(x)\,dx$ — if n or m are odd $\rightarrow$ do u substitution; if n and m are even $\rightarrow$ half ∡ substitution

Ex: Evaluate $\int \frac{\sin^3(\ln x)\cos^3(\ln x)}{x}\,dx$

$$= \int \sin^3(\ln x)\cos^2(\ln x)\frac{\cos(\ln x)}{x}\,dx$$

$$= \int \sin^3(\ln x)(1-\sin^2(\ln x))\frac{\cos(\ln x)}{x}\,dx$$

Let $u = \sin(\ln x)$
$du = \frac{\cos(\ln x)}{x}\,dx$

$$= \int u^3(1-u^2)\,du$$

$$= \int u^3 - u^5\,du$$

$$= \frac{u^4}{4} - \frac{u^6}{6}$$

$$= \frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} + C \;//$$

Partial fractions:

Ex: Evaluate $\int \frac{1}{x(x^2-1)}\,dx$

$$\frac{1}{x(x^2-1)} = \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{A(x^2-1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$$= \frac{Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx}{x(x-1)(x+1)} = \frac{x^2(A+B+C) + x(B-C) - A}{x(x-1)(x+1)}$$

$$\begin{cases} A+B+C = 0 \\ B-C = 0 \rightarrow B = C \\ -A = 1 \rightarrow A = -1 \end{cases} \qquad -1 + 2B = 0 \qquad B = \frac{1}{2} = C$$

$$\int \frac{1}{x(x^2-1)}\,dx = \int \frac{-1}{x} + \frac{1}{2}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{x+1}\,dx$$

$$= -\ln|x| + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C \;//$$

Angle between vectors:

$$\cos\theta = \frac{u\cdot w}{\|u\|\|w\|}$$

Cauchy-Schwarz inequality:

$$|u\cdot w| \le \|u\|\|w\| \;,\quad u, w \in \mathbb{R}^n$$

Orthogonal:

$$u\cdot v = 0$$

Projection u onto w:

$$\frac{u\cdot w}{\|w\|^2}\,w$$

Ex: Let $v = (1,-3,1)$ and $w = (2,1,2)$ be vectors in $\mathbb{R}^3$

$$v\cdot w = (1,-3,1)\cdot(2,1,2) = 2 - 3 + 2 = 1$$

$$\|v\| = \sqrt{1^2 + (-3)^2 + 1^2} = \sqrt{11} \qquad \|w\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

(a) Find the angle between v and w

$$\cos\theta = \frac{v\cdot w}{\|v\|\|w\|} = \frac{1}{\sqrt{11}\cdot 3}\;,\quad \theta = \cos^{-1}\left(\frac{1}{3\sqrt{11}}\right) \;//$$

(b) verify the Cauchy-Schwarz inequality and the triangle inequality for v and w

$$|v\cdot w| \le \|v\|\|w\|$$

$$\Rightarrow \quad 1 < 3\sqrt{11} \qquad ✓$$

(c) Find all unit vectors in $\mathbb{R}^3$ which are orthogonal to both v and w.

Let $u = (a, b, c)$ be orthogonal to both v and w.

$$(1,-3,1)(a,b,c) = a - 3b + c = 0 \quad \rightarrow a = -c$$
$$(2,1,2)(a,b,c) = \underline{2a + b + 2c = 0}$$
$$-5b = 0 \rightarrow b = 0$$

So $u = (k, 0, -k)$, $k \in \mathbb{R}$

$$\|u\| = \sqrt{k^2 + k^2} = \sqrt{2}\,k = 1 \quad \rightarrow k = \frac{1}{\sqrt{2}}$$

The vectors are $\left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$

(d) Find the projection of (i) v onto w and (ii) w onto v.

(i) Proj of v onto w : $\frac{v \cdot w}{\|w\|^2} w = \frac{1}{9}(2,1,2)$

(ii) Proj of w onto v : $\frac{w \cdot v}{\|v\|^2} v = \frac{1}{11}(1,-3,1)$

Eigenvalues and eigenvectors:

Values of λ for $\det(A - \lambda I) = 0$ and their vectors respectively.

The matrix is diagonalizable if $P^{-1}AP = D$ is a diagonal matrix where P is the eigenvectors matrix. (or $A = PDP^{-1}$)

Ex: Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

Is A diagonalizable?

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -\lambda & 0 \\ 2 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} + 2\begin{vmatrix} 0 & -\lambda \\ 2 & 0 \end{vmatrix}$$

$$= -\lambda(1-\lambda)(4-\lambda) + 4\lambda$$
$$= -\lambda[(4 - 5\lambda + \lambda^2) - 4]$$
$$= -\lambda(-5\lambda + \lambda^2)$$
$$= -\lambda^2(\lambda - 5) = 0$$
$$\rightarrow \lambda_1 = 0, \ \lambda_2 = 0, \ \lambda = 5$$

For $\lambda_1 = 0$, $\lambda_2 = 0$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_3 = 0 \rightarrow x_1 = -2x_3$ so for λ_1 we can let $v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$
$2x_1 + 4x_3 = 0$

x_2 is a free variable so for λ_2 we can let $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda_3 = 5$ $\begin{bmatrix} -4 & 0 & 2 \\ 0 & -5 & 0 \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-4x_1 + 2x_3 = 0$
$-5x_2 = 0 \rightarrow x_2 = 0$
$2x_1 - x_3 = 0 \rightarrow x_3 = 2x_1$

so for λ_3 we can let $v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$P = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 & 0 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/5 & 0 & 2/5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/5 & 0 & 1/5 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/5 & 0 & 2/5 \end{array}\right] = P^{-1}$$

$$D = \begin{bmatrix} -2/5 & 0 & 1/5 \\ 0 & 1 & 0 \\ 1/5 & 0 & 2/5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$
, which is a diagonal matrix of $\lambda_1, \lambda_2, \lambda_3$.

Week 3 - Sept. 20th

Vector equation of a line in $\mathbb{R}^3$: $\vec{\ell} = a + t\vec{v}$ $\quad t, a_i, v_i \in \mathbb{R}$
$\vec{\ell} = (a_1, a_2, a_3) + t[v_1, v_2, v_3]$ $\quad i = 1, 2, 3$

Parametric equation of a line in $\mathbb{R}^3$: $x = a_1 + tv_1$
$y = a_2 + tv_2$
$z = a_3 + tv_3$

Ex: For each of the following lines, write its equation in vector and parametric form

(i) The line that passes through the point P_0 $(3, 1, 9)$ in the direction of $\vec{v} = (1, 1, 1)$

V.Eq. $\vec{l} = (3,1,9) + t(1,1,1)$, $t \in \mathbb{R}$

P.Eq. $x = 3+t$ $y = 1+t$ $z = 9+t$, $t \in \mathbb{R}$

(ii) The line that passes through points $P_0\,(-1,1,2)$ and $P_1\,(2,0,-3)$

The direction vector for the line is $(2,0,-3) - (-1,1,2) = (3,-1,-5)$

V.Eq. $\vec{l} = (-1,1,2) + t(3,-1,5)$, $t \in \mathbb{R}$

P.Eq. $x = -1+3t$ $y = 1-t$ $z = 2+5t$, $t \in \mathbb{R}$

(iii) The line that passes through the point $P_0\,(0,1,0)$ and is orthogonal to the plane $10x + 15y + 3z = 11$

The direction vector for this line is a normal vector for the plane $\vec{n} = (10,15,3)$

V.Eq. $\vec{l} = (0,1,0) + t(10,15,3)$, $t \in \mathbb{R}$

P.Eq. $x = 10t$ $y = 1+15t$ $z = 3t$, $t \in \mathbb{R}$

Ex: Find an equation of the plane that passes through 3 points $A\,(-1,1,2)$, $B\,(2,0,-3)$ and $C\,(2,-1,2)$

A pair of direction vectors in the plane are $\vec{v} = (2,0,-3) - (-1,1,2) = (3,-1,-5)$ and $\vec{w} = (2,-1,2) - (-1,1,2) = (3,-2,0)$

To find the normal of the plane $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & -5 \\ 3 & -2 & 0 \end{vmatrix}$

$= (-10,-15,-3)$

So the plane is $-10x - 15y - 3z = d \rightarrow -10(-1) - 15(1) - 3(2) = -11$

$\therefore$ The equation of the plane is $10x + 15y + 3z = 11$.

Ex: Find an equation of the plane that passes through the origin and contains the line $x = 2+3t$ $y = 1-2t$ $z = 1+2t$

The line when $t=0$ passes through the point $(2,1,1)$
So the vector $\vec{v} = (0,0,0) - (2,1,1) = (-2,-1,-1)$ is also on the plane.
We can let $\vec{w} = (3,-2,2)$

To find the normal of the plane $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & -1 \\ 3 & -2 & 2 \end{vmatrix}$

$= (-4,1,7)$

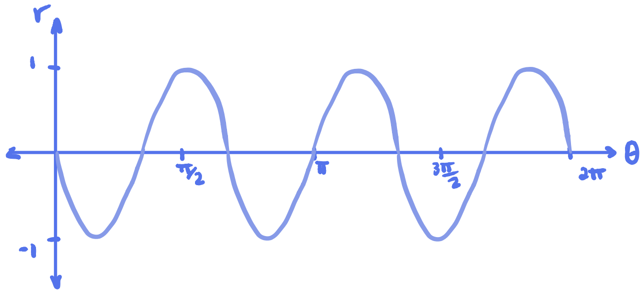
So the plane is $-4x + y + 7z = d \rightarrow -4(0) + (0) + 7(0) = 0$

$\therefore$ The equation of the plane is $-4x + y + 7z = 0$

Polar equations : $x = r\cos\theta \quad y = r\sin\theta$

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

Ex: Sketch the curve $r = -\sin 3\theta$ in the polar plane



θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	-1	0	1	0	-1	0