

MATB41 - Multivariable Calculus

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Tutorials: TUT9005 (Tuesdays 3-4 pm)

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Week 1 (May 11)

* No tutorials *

Week 2 (May 18)

Surfaces

Curved Lines

Parabola: (with vertex) at the origin

- $y = ax^2$ → open up when $a > 0$ and down when $a < 0$
- $x = by^2$ → open right when $b > 0$ and left when $b < 0$

* Remember: We can replace x with $x-m$ to shift right and y with $y+n$ to shift up!

Ellipse: (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \text{intersections } (\pm a, 0) \text{ and } (0, \pm b)$$

When $a=b$, the curve is a circle of radius $a=b$.

Hyperbola: (with center) at the origin

- $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ It has two curves that goes in opposite directions and do not touch the slant asymptote
- $\left(\frac{y}{a}\right)^2 - \left(\frac{x}{b}\right)^2 = 1$

Ex ① - Sketch the following curves:

a) $x^2 + 3y^2 + 2x - 12y + 10 = 0$

$$\begin{aligned} (x^2 + 2x) + 3(y^2 - 4y) + 10 &= 0 \\ (x^2 + 2x + 1) - 1 + 3(y^2 - 4y + 4) - 12 + 10 &= 0 \\ (x+1)^2 + 3(y-2)^2 &= 3 \end{aligned}$$

$$\Rightarrow \frac{(x+1)^2}{3} + (y-2)^2 = 1 \Rightarrow \left(\frac{x+1}{\sqrt{3}}\right)^2 + (y-2)^2 = 1$$

Okie, now what?! Another way to write the ellipse equation would be:

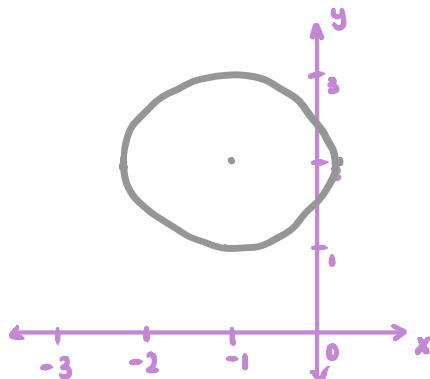
$$\left(\frac{x-m}{a}\right)^2 + \left(\frac{y-n}{b}\right)^2 = 1 \rightarrow \text{we just combined it with *}$$

So the new origin will be $(-1, 2)$ as the ellipse will be shifted left by one and up by 2.

Now, let's find the end points

→ The horizontal max/min will be $(-1 \pm \sqrt{3}, 2)$

→ The vertical max/min will be $(-1, 2 \pm 1)$



b) $\left(\frac{x-2}{4}\right)^2 - \left(\frac{y+2}{9}\right)^2 = 1$

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 1$$

The origin will be $(2, -2)$, we are shifting right 2 and 2 down.

Check the x-int : $(y=0)$

$$\left(\frac{x-2}{2}\right)^2 - \frac{4}{9} = 1$$

$$\left(\frac{x-2}{2}\right)^2 = \frac{13}{9}$$

$$(x-2)^2 = \frac{52}{9}$$

$$x-2 = \pm \frac{2\sqrt{13}}{3}$$

$$x = 2 \pm \frac{2\sqrt{13}}{3}$$

Check the y-int : $(x=0)$

$$1 - \left(\frac{y+2}{3}\right)^2 = 1$$

$$y = -2$$

Find the slant asymptote:

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 0$$

$$\frac{x-2}{2} = \frac{y+2}{3}$$

$$3x-6 = 2y+4$$

$$2y = 3x-10$$

$$y = \frac{3x-10}{2}$$

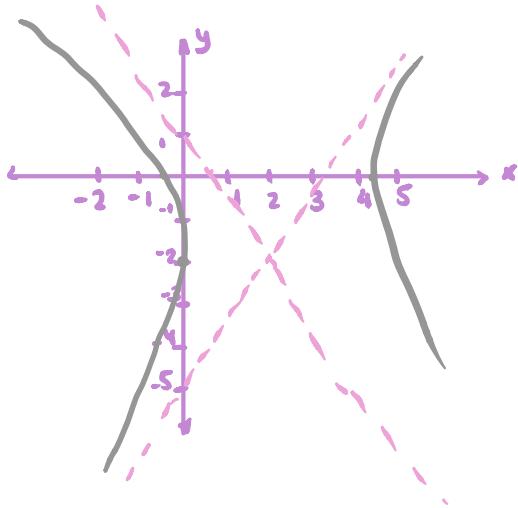
or

$$\frac{-x+2}{2} = \frac{y+2}{3}$$

$$-3x+6 = 2y+4$$

$$2y = -3x+2$$

$$y = \frac{-3x+2}{2}$$



Curved Surfaces

3D Sphere: (with center) at the origin of radius R

$$x^2 + y^2 + z^2 = R^2$$

3D Ellipsoid: (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

3 points $(a, 0, 0), (0, b, 0), (0, 0, c)$

We can replace $x^2 + y^2$ with r^2 to indicate the rotation.

Ex② - Sketch the following surfaces:

a) $x^2 + y^2 + \frac{z^2}{4} = 1$

$$r^2 + \frac{z^2}{4} = 1$$

$$4r^2 + z^2 = 4$$

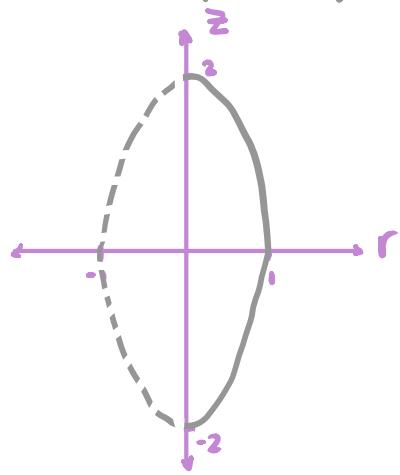
$$r^2 + \frac{z^2}{4} = 1$$

(center at $(0, 0)$)

r -int: $(\pm 1, 0)$

z -int: $(0, \pm 2)$

* Use the ellipsoid formula



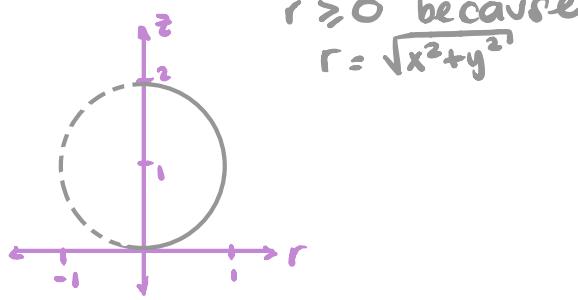
$r \geq 0$ because
 $r = \sqrt{x^2 + y^2}$

b) $x^2 + y^2 + z^2 = 2z$

$$\begin{aligned} r^2 + z^2 - 2z &= 0 \\ r^2 + z^2 - 2z + 1 - 1 &= 0 \\ r^2 + (z-1)^2 &= 1 \end{aligned}$$

\rightarrow circle!

(center $(0, 1)$)



$r \geq 0$ because
 $r = \sqrt{x^2 + y^2}$

Determinant

When $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| = ad - bc$

When $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $|A| = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Ex③- Find the determinant of $\begin{bmatrix} -3 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 4 & 3 \end{bmatrix}$

$$\begin{vmatrix} -3 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 4 & 3 \end{vmatrix} = -3 \begin{vmatrix} -1 & 2 \\ 4 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix}$$

$$= -3(-3 - 8) - (6 - 2) + (8 + 1)$$

$$= -3 \times -11 - 4 + 9$$

$$= 33 + 5$$

$$= 38$$

Dot and Cross Product

let $\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix} = u_1v_1 + u_2v_2 + \dots$

let $\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$

$$= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

You can find the angle between two vectors using dot product:

Let the angle be called θ , then $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$

Ex④- Find the angle between $\vec{u} = (1, -3, 1)$ and $\vec{v} = (2, 1, 2)$ in \mathbb{R}^3

First, we need the dot product and lengths:

$$\vec{u} \cdot \vec{v} = (1, -3, 1) \cdot (2, 1, 2) = 2 - 3 + 2 = 1$$

$$\|\vec{u}\| = \sqrt{1^2 + (-3)^2 + 1^2} = \sqrt{11}$$

$$\|\vec{v}\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

$$\Rightarrow \cos \theta = \frac{1}{3\sqrt{11}} \quad \therefore \quad \theta = \cos^{-1}\left(\frac{1}{3\sqrt{11}}\right)$$

Line and Planes

Vector equation of a line in \mathbb{R}^3 : $\vec{l} = (a_1, a_2, a_3) + t[v_1, v_2, v_3]$
 $\vec{l} = \vec{a} + t\vec{v}$

Parametric equation of a line in \mathbb{R}^3 : $x = a_1 + t v_1$,
 $y = a_2 + t v_2$,
 $z = a_3 + t v_3$

Ex ⑤ - Find the equation of the line or plane

a) The line through $(1, -1, 2)$ and $(3, 1, 9)$

The direction vector for the line is $(3, 1, 9) - (1, -1, 2) = (2, 2, 7)$

$$\rightarrow \text{V. eq. } \vec{l} = (1, -1, 2) + t(2, 2, 7), \quad t \in \mathbb{R}$$

$$\rightarrow \text{P. eq. } x = 1 + 2t, \quad y = -1 + 2t, \quad z = 2 + 7t, \quad t \in \mathbb{R}$$

b) The plane through $(1, -3, 1)$, $(2, 1, 1)$, $(1, 4, 0)$

let's find a pair of directional vectors \vec{v}_1 and \vec{v}_2

$$\vec{v}_1 = (2, 1, 1) - (1, -3, 1) = (1, 4, 0)$$

$$\vec{v}_2 = (1, 4, 0) - (1, -3, 1) = (0, 7, -1)$$

To get the equation of the plane, we need to find the normal which is $\vec{v}_1 \times \vec{v}_2$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 0 \\ 0 & 7 & -1 \end{vmatrix} = (-4, 1, 7)$$

The plane then is $-4x + y + 7z = d$

$$\text{Let's plug a point: } -4(2) + (1) + 7(1) = -8 + 1 + 7 = 0 = d$$

\rightarrow The plane is $-4x + y + 7z = 0$