

# MATB44 - Tutorial b (IC 320 Fridays 10-11am)

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## Week 1 - Sept. 6th

No tutorials

## Week 2 - Sept. 13th

ODE:  $F(t, x, x^{(1)}, \dots, x^{(k)}) = 0$

$t$  - independent variable (time)

$x$  - dependent variable (space)

$x$  is a function of  $t$  i.e.  $x(t)$

The order is the highest derivative that appears in  $F$ .

The system is linear if we can rewrite it to separate out the  $x_i^{(j)}$ 's as follows:

$$x_i^{(k)} = g_{ii}(t) + \sum_{l=1}^n \sum_{j=1}^{k-1} f_{i,j,l}(t) x_l^{(j)}$$

Ex:  $\underbrace{(4+t^2)x' + 2tx}_{} = 4t$

By product rule:  $\frac{d}{dt} [(4+t^2)x] = 2tx + (4+t^2)x'$

$$\Rightarrow \frac{d}{dt} [(4+t^2)x] = 4t$$

$$\Rightarrow (4+t^2)x = 2t^2 + C$$

$$\Rightarrow x(t) = \frac{2t^2 + C}{4+t^2}$$

The system is homogeneous if  $g_{ii}(t) = 0$

If there is no direct dependence on  $t$ , the system is autonomous. Form  $x' = f(x)$

Ex: Classify the following equations:

$$* x' + \frac{1}{2}x = \frac{1}{2}e^{t/3} \quad \text{1st order, linear}$$

$$* 3+xx' = t-x \quad \text{1st order, non linear}$$

\*  $x'' + 5x' + 6x = 0$  2nd order, linear, homogeneous, autonomous

\*  $x' = t \sin x$  1st order, non linear

\*  $x'' - x = 0$  2nd order, linear, homogeneous, autonomous

\*  $tx' + 2x = 4t^2$  1st order, linear

\*  $x' - 2x + t = 0$  1st order, linear.

Ex: Solve  $x' = x^2$  with  $x_0 = x(0) > 0$

$$\textcircled{1} \quad \frac{dx}{dt} = x^2 \Rightarrow \frac{dx}{x^2} = dt$$

$$\Rightarrow \int \frac{dx}{x^2} = \int dt$$

$$\Rightarrow -\frac{1}{x} = t + C_1$$

$$\Rightarrow x = \frac{1}{-C_1 - t}$$

$$\text{When } x(0) = -\frac{1}{C_1} = x_0 \text{ so } x = \frac{1}{\frac{1}{x_0} - t}$$

\textcircled{2}  $f(x) = x^2$ , a maximal nonzero interval is  $(x_0, \infty) = (0, \infty)$

$$F(x) = \int_{x_0}^x \frac{dy}{y^2} = -\frac{1}{y} \Big|_{y=x_0}^{y=x} = -\frac{1}{x} + \frac{1}{x_0} \quad \phi(t) = F^{-1}(t)$$

$$T_+ = \lim_{x \rightarrow x_2} F(x) = \lim_{x \rightarrow \infty} -\frac{1}{x} + \frac{1}{x_0} = \frac{1}{x_0}, \text{ so } \phi \text{ is defined for all } t > 0$$

$$t = F(F^{-1}(t)) = -\frac{1}{F^{-1}(t)} + \frac{1}{x_0}$$

$$\Rightarrow \frac{1}{F^{-1}(t)} = \frac{1}{x_0} - t$$

$$\Rightarrow F^{-1}(t) = \frac{1}{\frac{1}{x_0} - t}$$

$$\Rightarrow \phi(t) = \frac{1}{\frac{1}{x_0} - t} \quad \blacksquare$$

## Week 3 - Sept. 20th

Office Hours: Wed. 10-11 am IC404

A particular solution is a solution that has no arbitrary constant. A general solution is a k-parameter family of solutions that contain every particular solution.

A separable first-order ODE is an ODE that can be rewritten as:

$$\dot{x}(t, x) = g(t)f(x) \quad (\text{or } g(t)dt + f(x)dx = 0)$$

$$\Rightarrow \frac{dx}{dt} = g(t)f(x)$$

$$\Rightarrow \frac{dx}{f(x)} = g(t)dt$$

$$\Rightarrow \int \frac{dx}{f(x)} = \int g(t)dt, \quad f(x) \neq 0$$

Ex: Show that the equation  $\dot{x}(t, x) = \frac{t^2}{1-x^2}$  is separable and solve.

$$\text{Let } f(x) = \frac{1}{1-x^2} \text{ and } g(t) = t^2$$

$$\Rightarrow \int 1-x^2 dx = \int t^2 dt$$

$$x - \frac{x^3}{3} = \frac{t^3}{3} + C_1$$

$$3x - x^3 - t^3 = C, \quad C \in \mathbb{R}$$

Ex: Solve the equation  $\frac{dx}{dt} = \frac{4t-t^3}{4+x^3}$  and find the solution passing through the point  $(0, 1)$

$$\Rightarrow (4+x^3)dx = (4t-t^3)dt$$

$$\int (4+x^3)dx = \int (4t-t^3)dt$$

$$4x + \frac{x^4}{4} = 4t^2 - \frac{t^4}{4} + C_1$$

$$\Rightarrow 16x + x^4 - 8t^2 + t^4 = C, \quad C \in \mathbb{R},$$

$$\text{When } (0, 1) \rightarrow 16+1=17$$

$$\therefore 16x + x^4 - 8t^2 + t^4 = 17$$

Let  $z = f(x, y)$  be a function of  $x$  and  $y$ ,  $f(x, y)$  is homogeneous of order  $n$  if it can be written as  $f(x, y) = x^n g(u)$ , where  $u = \frac{y}{x}$  or  $f(x, y) = y^n g(u)$ , where  $u = \frac{x}{y}$ .

$P(x, y)dx + Q(x, y)dy = 0$  where  $P(x, y)$  and  $Q(x, y)$  are the homogeneous coefficients and can be solved by substituting  $y = ux$ ,  $dy = udx + xdu$

Ex: Find the general solution of  $txx' = t^2 + 2x^2$

$$\Rightarrow \frac{1}{t^2} \left[ tx x' - t^2 - 2x^2 \right]$$

$$\frac{x x'}{t} = 1 + 2 \frac{x^2}{t^2} \quad \text{Let } u = \frac{x}{t} \quad \text{so } x = ut \quad x' = u + tu'$$

$$\Rightarrow u(u + tu') = 1 + 2u^2$$

$$u^2 + tu' u = 1 + 2u^2$$

$$tu' u = 1 + u^2$$

$$\frac{u' u}{1 + u^2} = \frac{1}{t}$$

→ OMG! This is separable!

$$\int \frac{u' u}{1 + u^2} dt = \int \frac{1}{t} dt$$

$$\text{Let } v = 1 + u^2 \quad dv = 2u du$$

$$\frac{1}{2} \int \frac{dv}{v} = \ln|t| + C_1$$

$$\frac{1}{2} \ln|1 + u^2| = \ln|t| + C_1$$

$$\therefore \frac{1}{2} \ln \left| 1 + \frac{x^2}{t^2} \right| - \ln|t| = C$$

A differential expression  $P(x, y)dx + Q(x, y)dy$  is called an exact differential if it is the total differential of some function  $f(x, y)$ .

i.e. if  $P(x, y) = \frac{\partial}{\partial x} f(x, y)$  and  $Q(x, y) = \frac{\partial}{\partial y} f(x, y)$ .

If we can find  $f(x, y)$ ,  $f(x, y) = C$  is the 1-parameter family of solutions.

$P(x, y)dx + Q(x, y)dy = 0$  is exact iff  $\frac{\partial P(x, y)}{\partial y} = \frac{\partial Q(x, y)}{\partial x}$ .

Ex: Find the general solution of  $y' = -\frac{1+2xy^2}{1+2x^2y}$

$$\Rightarrow (1+2x^2y)y' = -(1+2xy^2)$$

$$(1+2xy^2) + (1+2x^2y)y' = 0$$

Is it exact?  $P(x, y) = 1+2xy^2$        $Q(x, y) = 1+2x^2y$

$$\frac{\partial P(x, y)}{\partial y} = 4xy \quad \Leftrightarrow \quad \frac{\partial Q(x, y)}{\partial x} = 4xy \quad \checkmark$$

let's take  $P(x,y)$ , we know  $P(x,y) = \frac{\partial}{\partial x} f(x,y)$

$$\Rightarrow \int P(x,y) dx = f(x,y)$$

$$\int 1 + 2xy^2 dx = x + x^2y^2 + g(y) = f(x,y)$$

$$\Rightarrow \frac{\partial f(x,y)}{\partial y} = 2x^2y + g'(y) = Q(x,y) = 1 + 2x^2y$$

$$\text{So } g'(y) = 1 \rightarrow g(y) = y$$

$$\therefore x + x^2y^2 + y = C$$

Week 4 - Sept. 27th

An integrating factor (IF) will convert an inexact ODE  $P(x,y)dx + Q(x,y)dy = 0$  into an exact ODE  $\text{IF } P(x,y)dx + \text{IF } Q(x,y)dy = 0$

Given  $\frac{dy}{dx} + P(x)y = Q(x)$  a known IF is  $e^{\int P(x)dx}$

Ex: Find the general solution of  $\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3}$

$$P(x) = \frac{3}{x}, \quad \text{IF} = e^{\int \frac{3}{x} dx} = e^{3 \ln x} = e^{\ln x^3} = x^3$$

$$\Rightarrow x^3 \frac{dy}{dx} + x^3 \frac{3y}{x} = x^3 \frac{e^x}{x^3}$$

$$\Rightarrow x^3 y' + 3x^2 y = e^x$$

$$\Rightarrow \int (x^3 y' + 3x^2 y) dx = \int e^x dx$$

$$\Rightarrow x^3 y = e^x + C$$

$$\Rightarrow y = \frac{e^x + C}{x^3}$$

Ex: Solve  $6y' - 2y = xy^4, y(0) = -2$ .

$$\Rightarrow 6y^{-4}y' - 2y^{-3} = x \quad \text{let } u = y^{-3} \\ du = -3y^{-4}y' \quad \text{d}u = -3y^{-4}y'$$

$$\Rightarrow -2u' - 2u = x \quad \text{or} \quad u' + u = -\frac{x}{2}$$

$$P(x) = 1 \quad , \quad \text{IF} = e^{\int P dx} = e^x$$

$$\Rightarrow u'e^x + ue^x = -\frac{e^x}{2}$$

$$\Rightarrow \int(u'e^x + ue^x)dx = -\frac{1}{2} \int xe^x dx$$

$$\Rightarrow ue^x = -\frac{1}{2}(xe^x - e^x) + C$$

$$\Rightarrow u = -\frac{1}{2}(x-1) + Ce^{-x}$$

$$\Rightarrow y^{-3} = -\frac{1}{2}(x-1) + Ce^{-x}$$

$$\Rightarrow 2y^{-3} = (1-x) + 2Ce^{-x}$$

$$\Rightarrow y^3 = \frac{2}{(1-x) + 2Ce^{-x}}$$

Side note:

$$\begin{aligned} u &= x & dv &= e^x dx \\ du &= dx & v &= e^x \\ & \int xe^x dx & & = xe^x - \int e^x dx = xe^x - e^x + C. \end{aligned}$$

$$\begin{aligned} &\stackrel{\text{IVP}}{\Rightarrow} (-2)^3 = \frac{2}{1+2C} \\ &\Rightarrow -8 - 16C = 2 \\ &\Rightarrow C = -\frac{5}{8} \\ \Rightarrow y &= \left[ \frac{2}{(1-x) - \frac{5}{4}e^{-x}} \right]^{1/3} \end{aligned}$$

**Bernoulli Equation:** given  $\frac{dy}{dx} + P(x)y = Q(x)y^n$ .

$$\text{multiply by } (1-n)y^{-n} \Rightarrow (1-n)y^{-1}\frac{dy}{dx} + (1-n)y^{1-n}P(x) = (1-n)Q(x)$$

$$\begin{aligned} \text{Substitute } u &= y^{1-n} \\ du &= (1-n)y^{-n}dy \Rightarrow \frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x) \\ &\text{IF } e^{\int (1-n)P(x)dx} \end{aligned}$$

Let  $X$  be a real vector space. A norm on  $X$  is a map  $\|\cdot\|: X \rightarrow [0, \infty)$  s.t.

- (i)  $\|0\| = 0$ ,  $\|x\| > 0$  for  $x \neq 0$
- (ii)  $\|\alpha x\| = |\alpha| \|x\|$  for  $\alpha \in \mathbb{R}$  and  $x \in X$
- (iii)  $\|x+y\| \leq \|x\| + \|y\|$  for  $x, y \in X$

Together  $(X, \|\cdot\|)$  is a normed vector space. A sequence of vectors  $x_n$  converges to  $x$  if  $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$ .

A **contraction** is a mapping  $K: C \subseteq X \rightarrow C$  where there exists a contraction constant  $\theta \in [0, 1)$  s.t.  $\|K(x) - K(y)\| \leq \theta \|x - y\|$ ,  $x, y \in C$ .

Note:  $K^n(x) = K(K^{n-1}(x))$      $K^0(x) = x$

Ex: Let  $K(x) = 40 + \frac{x}{3}$ . What is the contraction constant? Is  $K(x)$  a contraction on  $C = [0, 90]$ ? on  $C = [0, 30]$ ?

$$\|K(x) - K(y)\| = \left\| 40 + \frac{x}{3} - 40 - \frac{y}{3} \right\| = \left\| \frac{x}{3} - \frac{y}{3} \right\| = \frac{1}{3} \|x - y\|$$

$$\therefore \theta = \frac{1}{3}$$

$$C[0, 90] \quad K(0) = 40 \in [0, 90]$$

$$C[0, 30] \quad K(0) = 40 \notin [0, 30]$$

Consider an initial value problem (IVP)  $\dot{x} = f(t, x)$ ,  $x(t_0) = x_0$ , where  $x, t \in \mathbb{R}$  and  $f \in C(U, \mathbb{R})$  where  $U \subseteq \mathbb{R}^2$  is an open subset of  $\mathbb{R}^2$  and  $(t_0, x_0) \in U$ .

Let's define **Picard Iteration** by a map  $K: C(U, \mathbb{R}) \rightarrow C(U, \mathbb{R})$

$$K(x)(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds.$$

and the **Picard iterates**.

$$x_0(t) = x_0 \quad (\text{the constant function through the scalar } x_0)$$

$$x_1(t) = K(x_0)(t) = x_0 + \int_{t_0}^t f(s, x_0(s)) ds$$

$$x_2(t) = K^2(x_0)(t) = K(x_1)(t) = x_0 + \int_{t_0}^t f(s, x_1(s)) ds.$$

$\vdots$

$$x_m(t) = K^m(x_0)(t) = K(x_{m-1})(t) = x_0 + \int_{t_0}^t f(s, x_{m-1}(s)) ds$$

Ex: Calculate the Picard iterates  $x_0, x_1, x_2$ . for  $f(t, x) = 3 - 2x$ ,  $x(t_0) = x_0$

$$K(x)(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds$$

$$K^{m+1}(x)(t) = x_0 + \int_{t_0}^t f(s, x_m(s)) ds = x_0 + \int_{t_0}^t (3 - 2x_m(s)) ds.$$

$$\Rightarrow x_0(t) = x_0$$

$$\begin{aligned}\Rightarrow x_1(t) &= K(x_0)(t) = x_0 + \int_{t_0}^t 3 - 2x_0(s) ds \\ &= x_0 + \int_{t_0}^t 3 - 2x_0 ds \\ &= x_0 + (3 - 2x_0)(t - t_0) \\ &= x_0 + t(3 - 2x_0) - t_0(3 - 2x_0)\end{aligned}$$

$$\begin{aligned}\Rightarrow x_2(t) &= K^2(x_0)(t) = x_0 + \int_{t_0}^t 3 - 2x_1(s) ds \\ &= x_0 + \int_{t_0}^t 3 - 2(x_0 + s(3 - 2x_0) - t_0(3 - 2x_0)) ds\end{aligned}$$

$$\begin{aligned}&= x_0 + [3 - 2x_0 + 2t_0(3 - 2x_0)](t - t_0) - 2(3 - 2x_0) \int_{t_0}^t s ds \\ &= x_0 + (3 - 2x_0)(1 + 2t_0)(t - t_0) - (3 - 2x_0) s^2 \Big|_{t_0}^t \\ &= x_0 + (3 - 2x_0)(1 + 2t_0)(t - t_0) - (3 - 2x_0)(t^2 - t_0^2).\end{aligned}$$

## Week 5 - Oct. 4th

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \quad x: \mathbb{R} \rightarrow \mathbb{R}^n$$

let  $K: C(U, \mathbb{R}^n) \rightarrow C(U, \mathbb{R}^n)$ ,  $U \subseteq \mathbb{R}^{n+1}$  an open set

$$K(x)(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds.$$

Then the sequence  $x_0(t) = x_0, x_m = K(x_{m-1})$  converges to the solution  $x$ .

$$\text{Ex: } \dot{x}(t) = \begin{pmatrix} t+y \\ y^2+z^2 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow y(0) = 1, \quad z(0) = 0$$

$$x_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left( \begin{array}{l} \int_0^t s + (y(s)) ds \\ \int_0^t (y(s))^2 + (z(s))^2 ds \end{array} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left( \begin{array}{l} \int_0^t s + 1 ds \\ \int_0^t ds \end{array} \right) = \begin{pmatrix} \frac{t^2}{2} + t + 1 \\ t \end{pmatrix}$$

$$x_2(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left( \begin{array}{l} \int_0^t s + \left(\frac{s^2}{2} + s + 1\right) ds \\ \int_0^t \left(\frac{s^2}{2} + s + 1\right)^2 + s^2 ds \end{array} \right)^{(1)} = \begin{pmatrix} 1 + \frac{t^3}{6} + \frac{t^2 + t}{2} \\ \frac{t^5}{20} + \frac{t^4}{4} + t^3 + t^2 + t \end{pmatrix}$$

$$(1) \int_0^t \frac{s^2}{2} + 2s + 1 ds = \left[ \frac{s^3}{6} + s^2 + s \right]_0^t$$

$$(2) \int_0^t \left( \frac{s^4}{4} + s^3 + 2s^2 + 2s + 1 \right) + s^2 ds = \left[ \frac{s^5}{20} + \frac{s^4}{4} + s^3 + s^2 + s \right]_0^t$$

$$\text{Ex: } \dot{x}(t) = \begin{pmatrix} 2z+t^2 \\ t+y \end{pmatrix}, \quad x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow y(1) = 1$$

$$x_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left( \begin{array}{l} \int_1^t 2+z^2 ds \\ \int_1^t s+1 ds \end{array} \right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left( \begin{array}{l} \left[ 2s+\frac{s^3}{3} \right]_1^t \\ \left[ \frac{s^2}{2}+s \right]_1^t \end{array} \right) = \begin{pmatrix} 1+2t+\frac{t^3}{3}-2-\frac{1}{3} \\ 1+\frac{t^2}{2}+t-\frac{1}{2}-1 \end{pmatrix}$$

$$= \begin{pmatrix} 2t+\frac{t^3}{3}-\frac{4}{3} \\ \frac{t^2}{2}+t-\frac{1}{2} \end{pmatrix}$$

$$x_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left( \begin{array}{l} \int_1^t s^2+2s-1+s^2 ds \\ \int_1^t s+2s+\frac{s^3}{3}-\frac{4}{3} ds \end{array} \right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left( \begin{array}{l} \int_1^t 2s^2+2s-1 ds \\ \int_1^t 3s+\frac{s^3}{3}-\frac{4}{3} ds \end{array} \right)$$

$$= \begin{pmatrix} 1+\left[\frac{2s^3}{3}+s^2-s\right]_1^t \\ 1+\left[\frac{3s^2}{2}+\frac{s^4}{12}-\frac{4s}{3}\right]_1^t \end{pmatrix} = \begin{pmatrix} 1+\frac{2t^3}{3}+t^2-t-\frac{2}{3}-1+1 \\ 1+\frac{3t^2}{2}+\frac{t^4}{12}-\frac{4t}{3}-\frac{3}{2}-\frac{1}{12}+\frac{4}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2t^3}{3}+t^2-t+\frac{1}{3} \\ \frac{3t^2}{2}+\frac{t^4}{12}-\frac{4t}{3}+\frac{3}{4} \end{pmatrix}$$

Picard-Lindelöf

$$\text{Ex: } \dot{x} = x^{1/3} + t, \quad x(1) = 0$$

$$V_0 = [0, 2] \times [-1, 1]$$

$$\frac{|f(t, x) - f(t, y)|}{|x-y|} = \frac{|x^{1/3} + t - y^{1/3} - t|}{|x-y|} = \frac{|x^{1/3} - y^{1/3}|}{|x-y|} \rightarrow \infty$$

$$\text{If we let } y=0, \text{ then } \frac{|x^{1/3}|}{|x|} = \frac{1}{|x^{2/3}|} \rightarrow \infty$$

Thus, we cannot show that there exists a unique local solution.

## Complex Numbers:

$\mathbb{C}$  is defined by the numbers defined by  $z = x + yi$ , where  $x, y \in \mathbb{R}$  and  $i = \sqrt{-1}$

$\text{Re}(z)$   $\text{Im}(z)$

A complex number can be represented in polar form  $z = r(\cos \theta + i \sin \theta)$

Modulus of  $z \rightarrow |z| = \sqrt{x^2 + y^2}$

$$e^{iz} = \cos z + i \sin z$$

$$\theta = \arg z = \arctan \frac{y}{x}$$

Ex: Find the modulus and argument of  $z = 1+i$ , then write  $z$  in polar form

$$|z| = \sqrt{1+1} = \sqrt{2}$$

$$\theta = \arg z = \arctan(1) = \frac{\pi}{4}$$

$$z = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \xrightarrow{\text{check!}} = \sqrt{2} \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) = \frac{2}{2} + i \frac{2}{2} = 1+i \quad \text{✓}$$