

# MATB41 - Tutorial 3 (BV355 Fridays 2-3pm)

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Week 1 - Sept. 6th

No tutorials

Week 2 - Sept. 13th

Derivative by definition:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Riemann Sum:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x, \text{ where } \Delta x = \frac{b-a}{n}, x_i = a + i \Delta x$$

$$\text{Useful formulas} \rightarrow \sum_{i=1}^n 1 = n, \sum_{i=1}^n i = \frac{i(i+1)}{2}, \sum_{i=1}^n i^2 = \frac{i(i+1)(2i+1)}{2 \cdot 3}$$

Trig. Identities:

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1 \\ \cos(2x) &= \cos^2 x - \sin^2 x \\ \sin(2x) &= 2\sin x \cos x\end{aligned}$$

Half Angle Identities:

$$\sin\left(\frac{\alpha}{2}\right) = \pm \sqrt{\frac{1-\cos\alpha}{2}} \quad \cos\left(\frac{\alpha}{2}\right) = \pm \sqrt{\frac{1+\cos\alpha}{2}}$$

Trig. Integrals:

$$\int \cos^n(x) \sin^m(x) dx \quad \begin{array}{l} \text{if } n \text{ or } m \text{ are odd} \rightarrow \text{do } u \text{ substitution} \\ \text{if } n \text{ and } m \text{ are even} \rightarrow \text{half angle substitution} \end{array}$$

Ex: Evaluate  $\int \frac{\sin^3(\ln x) \cos^3(\ln x)}{x} dx$

$$= \int \sin^3(\ln x) \cos^2(\ln x) \frac{\cos(\ln x)}{x} dx$$

$$= \int \sin^3(\ln x) (1 - \sin^2(\ln x)) \frac{\cos(\ln x)}{x} dx$$

$$= \int u^3 (1-u^2) du$$

$$= \int u^3 - u^5 du$$

$$= \frac{u^4}{4} - \frac{u^6}{6}$$

$$\begin{aligned} \text{Let } u &= \sin(\ln x) \\ du &= \frac{\cos(\ln x)}{x} dx \end{aligned}$$

$$= \frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} + C ,$$

Partial fractions:

Ex: Evaluate  $\int \frac{1}{x(x^2-1)} dx$

$$\begin{aligned}\frac{1}{x(x^2-1)} &= \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{A(x^2-1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)} \\ &= \frac{Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx}{x(x-1)(x+1)} = \frac{x^2(A+B+C) + x(B-C) - A}{x(x-1)(x+1)}\end{aligned}$$

$$\left\{ \begin{array}{l} A+B+C=0 \\ B-C=0 \rightarrow B=C \\ -A=1 \rightarrow A=-1 \end{array} \right. \quad \left. \begin{array}{l} -1+2B=0 \\ B=\frac{1}{2}=C \end{array} \right.$$

$$\begin{aligned}\int \frac{1}{x(x^2-1)} dx &= \int \frac{-1}{x} + \frac{1}{2} \cdot \frac{1}{x-1} + \frac{1}{2} \cdot \frac{1}{x+1} dx \\ &= -\ln|x| + \frac{1}{2} \ln|x-1| + \frac{1}{2} \ln|x+1| + C,\end{aligned}$$

Angle between vectors:

$$\cos \theta = \frac{u \cdot w}{\|u\| \|w\|}$$

Cauchy-Schwarz inequality:

$$|u \cdot w| \leq \|u\| \|w\|, \quad u, w \in \mathbb{R}^n$$

Orthogonal:

$$u \cdot v = 0$$

Projection  $u$  onto  $w$ :

$$\frac{u \cdot w}{\|w\|^2} w$$

Ex: Let  $v = (1, -3, 1)$  and  $w = (2, 1, 2)$  be vectors in  $\mathbb{R}^3$

$$v \cdot w = (1, -3, 1) \cdot (2, 1, 2) = 2 - 3 + 2 = 1$$

$$\|v\| = \sqrt{1^2 + (-3)^2 + 1^2} = \sqrt{11} \quad \|w\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

(a) Find the angle between  $v$  and  $w$

$$\cos \theta = \frac{v \cdot w}{\|v\| \|w\|} = \frac{1}{\sqrt{11} \cdot 3}, \quad \theta = \cos^{-1}\left(\frac{1}{3\sqrt{11}}\right),$$

(b) Verify the Cauchy-Schwarz inequality and the triangle inequality for  $v$  and  $w$

$$\begin{aligned}|v \cdot w| &\leq \|v\| \|w\| \\ \Rightarrow 1 &\leq 3\sqrt{11} \quad \checkmark\end{aligned}$$

(c) Find all unit vectors in  $\mathbb{R}^3$  which are orthogonal to both  $v$  and  $w$ .

Let  $u = (a, b, c)$  be orthogonal to both  $v$  and  $w$ .

$$(1, -3, 1)(a, b, c) = a - 3b + c = 0 \rightarrow a = -c$$

$$(2, 1, 2)(a, b, c) = \underline{2a + b + 2c = 0}$$

$$-5b = 0 \rightarrow b = 0$$

So  $u = (k, 0, -k)$ ,  $k \in \mathbb{R}$

$$\|u\| = \sqrt{k^2 + k^2} = \sqrt{2}|k| = 1 \rightarrow k = \frac{1}{\sqrt{2}}$$

The vectors are  $(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$  and  $(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$

(d) Find the projection of (i)  $v$  onto  $w$  and (ii)  $w$  onto  $v$ .

(i) Proj of  $v$  onto  $w$ :  $\frac{v \cdot w}{\|w\|^2} w = \frac{1}{9} (2, 1, 2)$

(ii) Proj of  $w$  onto  $v$ :  $\frac{w \cdot v}{\|v\|^2} v = \frac{1}{11} (1, -3, 1)$

Eigenvalues and eigenvectors:

Values of  $\lambda$  for  $\det(A - \lambda I) = 0$  and their vectors respectively.

The matrix is diagonalizable if  $P^{-1}AP = D$  is a diagonal matrix where  $P$  is the eigenvectors matrix. (or  $A = PDP^{-1}$ )

Ex: Let  $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$ . Find the eigenvalues and eigenvectors of  $A$ .

Is  $A$  diagonalizable?

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -\lambda & 0 \\ 2 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} + 2 \begin{vmatrix} 0 & -\lambda \\ 2 & 0 \end{vmatrix}$$

$$\begin{aligned} &= -\lambda(1-\lambda)(4-\lambda) + 4\lambda \\ &= -\lambda[4 - 5\lambda + \lambda^2] - 4 \\ &= -\lambda(-5\lambda + \lambda^2) \\ &= -\lambda^2(\lambda - 5) = 0 \\ \Rightarrow \lambda_1 &= 0, \lambda_2 = 0, \lambda_3 = 5 \end{aligned}$$

For  $\lambda_1 = 0$   
 $\lambda_2 = 0$      $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_1 + 2x_3 = 0 \rightarrow x_1 = -2x_3 \text{ so for } \lambda_1 \text{ we can let } v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$x_2$  is a free variable so for  $\lambda_2$  we can let  $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\text{For } \lambda_3=5 \quad \begin{bmatrix} -4 & 0 & 2 \\ 0 & -5 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -4x_1 + 2x_3 &= 0 \\ -5x_2 &= 0 \rightarrow x_2 = 0 \\ 2x_1 - x_3 &= 0 \rightarrow x_3 = 2x_1 \end{aligned} \quad \text{so for } \lambda_3 \text{ we can let } v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad \left[ \begin{array}{ccc|ccc} -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 & 0 & 2 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{5} & 0 & \frac{2}{5} \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{5} & 0 & \frac{1}{5} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{5} & 0 & \frac{2}{5} \end{array} \right] = P^{-1}$$

$$\begin{aligned} D &= \begin{bmatrix} -\frac{2}{5} & 0 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \text{ which is a diagonal matrix of } \lambda_1, \lambda_2, \lambda_3. \end{aligned}$$

### Week 3 - Sept. 20th

Vector equation of a line in  $\mathbb{R}^3$ :  $\vec{l} = a + t\vec{v}$   $t, a_i, v_i \in \mathbb{R}$   
 $\vec{l} = (a_1, a_2, a_3) + t[v_1, v_2, v_3]$   $i=1, 2, 3$

Parametric equation of a line in  $\mathbb{R}^3$ :  $x = a_1 + tv_1$ ,  
 $y = a_2 + tv_2$   
 $z = a_3 + tv_3$

Ex: For each of the following lines, write its equation in vector and parametric form

(i) The line that passes through the point  $p_0 (3, 1, 9)$  in the direction of  $\vec{v} = (1, 1, 1)$

V.Eq.  $\vec{l} = (3, 1, 9) + t(1, 1, 1)$ ,  $t \in \mathbb{R}$

P.Eq.  $x = 3+t$   $y = 1+t$   $z = 9+t$ ,  $t \in \mathbb{R}$

(ii) The line that passes through points  $p_0(-1, 1, 2)$  and  $p_1(2, 0, -3)$

The direction vector for the line is  $(2, 0, -3) - (-1, 1, 2) = (3, -1, -5)$

V.Eq.  $\vec{l} = (-1, 1, 2) + t(3, -1, 5)$ ,  $t \in \mathbb{R}$

P.Eq.  $x = -1+3t$   $y = 1-t$   $z = 2+5t$ ,  $t \in \mathbb{R}$

(iii) The line that passes through the point  $p_0(0, 1, 0)$  and is orthogonal to the plane  $10x + 15y + 3z = 11$

The direction vector for this line is a normal vector for the plane  
 $\vec{n} = (10, 15, 3)$

V.Eq.  $\vec{l} = (0, 1, 0) + t(10, 15, 3)$ ,  $t \in \mathbb{R}$

P.Eq.  $x = 10t$   $y = 1+15t$   $z = 3t$ ,  $t \in \mathbb{R}$

Ex: Find an equation of the plane that passes through 3 points  $A(-1, 1, 2)$ ,  $B(2, 0, -3)$  and  $C(2, -1, 2)$

A pair of direction vectors in the plane are  $\vec{v} = (2, 0, -3) - (-1, 1, 2) = (3, -1, -5)$  and  $\vec{w} = (2, -1, 2) - (-1, 1, 2) = (3, -2, 0)$

To find the normal of the plane  $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & -5 \\ 3 & -2 & 0 \end{vmatrix} = (-10, -15, -3)$

So the plane is  $-10x - 15y - 3z = d \rightarrow -10(-1) - 15(1) - 3(2) = -11$

$\therefore$  The equation of the plane is  $10x + 15y + 3z = 11$ .

Ex: Find an equation of the plane that passes through the origin and contains the line  $x = 2+3t$   $y = 1-2t$   $z = 1+t$

The line when  $t=0$  passes through the point  $(2, 1, 1)$

So the vector  $\vec{v} = (0, 0, 0) - (2, 1, 1) = (-2, -1, -1)$  is also on the plane.

We can let  $\vec{w} = (3, -2, 2)$

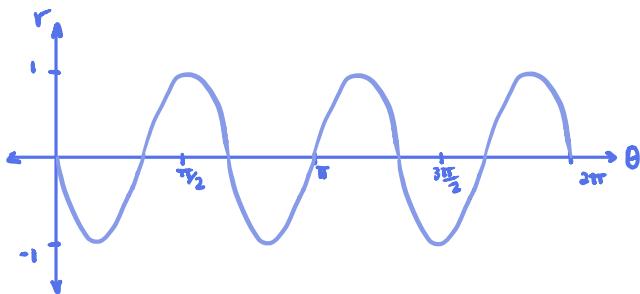
To find the normal of the plane  $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & -1 \\ 3 & -2 & 2 \end{vmatrix} = (-4, 1, 7)$

So the plane is  $-4x + y + 7z = d \rightarrow -4(0) + (0) + 7(0) = 0$

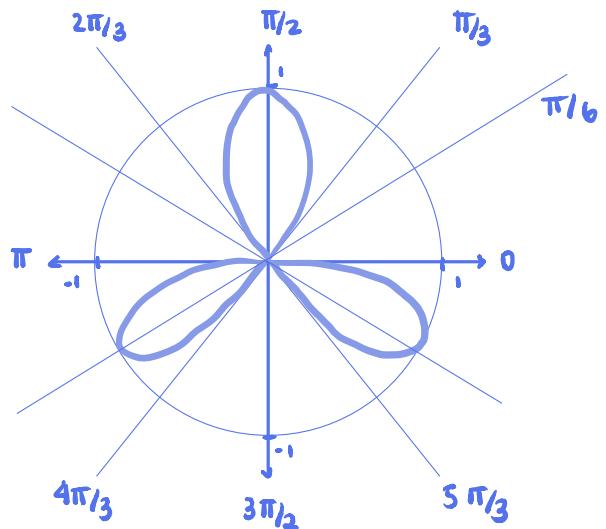
$\therefore$  The equation of the plane is  $-4x + y + 7z = 0$

Polar equations :  $x = r\cos\theta$   $y = r\sin\theta$   
 $x^2 + y^2 = r^2 \cos^2\theta + r^2 \sin^2\theta = r^2 (\cos^2\theta + \sin^2\theta) = r^2$

Ex: Sketch the curve  $r = -\sin 3\theta$  in the polar plane

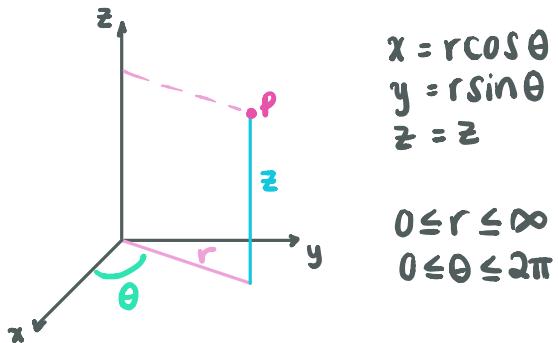


$\theta$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi$
$r$	0	-1	0	1	0	-1	0

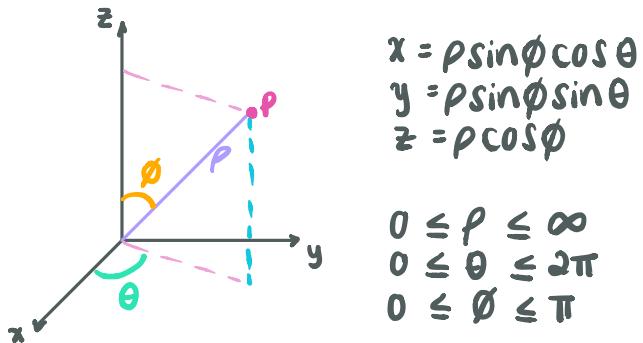


Week 4 - Sept. 27th

Cylindrical Coordinates:



Spherical Coordinates:



Ex: Interpret the equation  $1 = 2\cos\theta\sin\theta$  geometrically

Cylindrical :

$$\Rightarrow r^2 = 2r\cos\theta \cdot r\sin\theta \\ \Rightarrow x^2 + y^2 = 2xy \\ \Rightarrow x^2 - 2xy + y^2 = 0 \\ \Rightarrow (x-y)^2 = 0$$

Spherical :

$$\Rightarrow \rho^2 \sin^2\phi = 2\rho\sin\theta\cos\theta \rho\sin\theta\sin\theta \\ \Rightarrow \rho^2 \sin^2\phi (\sin^2\theta + \cos^2\theta) = 2\rho\sin\theta\cos\theta \rho\sin\theta\sin\theta \\ \Rightarrow \rho^2 \sin^2\phi \sin^2\theta + \rho^2 \sin^2\phi \cos^2\theta = 2\rho\sin\theta\cos\theta \rho\sin\theta\sin\theta \\ \Rightarrow x^2 + y^2 = 2xy \\ \Rightarrow x^2 - 2xy + y^2 = 0 \\ \Rightarrow (x-y)^2 = 0$$

Ex: Characterize and sketch several level curves of the following functions:

$$(i) f(x,y) = \frac{\sqrt{y^2 - x^2}}{2} = c$$

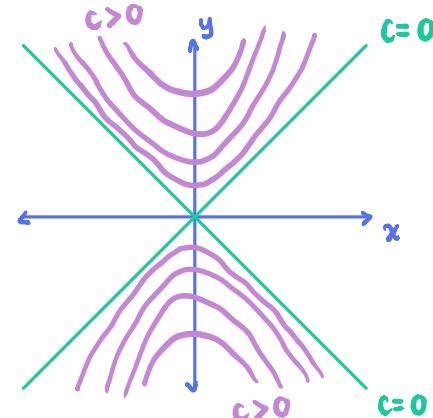
$\text{dom } f = \{(x,y) \in \mathbb{R}^2 \mid y^2 \geq x^2\}$ , because  $y^2 - x^2 \geq 0$

$c$  cannot be -ve,  $c > 0$  always

$$\underline{c=0} \quad y^2 = x^2 \\ y = \pm x \quad (*)$$

$$\underline{c>0} \quad y^2 - x^2 = (2c)^2 \\ y^2 = (2c)^2 + x^2 \quad (*)$$

$y$ -intercepts:  $(0, \pm 2c)$



$$(ii) f(x,y) = \frac{x+y}{y^2} = c$$

$\text{dom } f = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$

$$\underline{c=0} \quad x+y=0 \\ y=-x \quad (*)$$

$$\underline{c \neq 0} \quad x = cy^2 - y$$

$$y\text{-intercepts: } y = cy^2 \\ \Rightarrow y = \frac{1}{c}$$

$$\underline{c>0} \quad x = cy^2 - y \\ x = y(cy - 1)$$

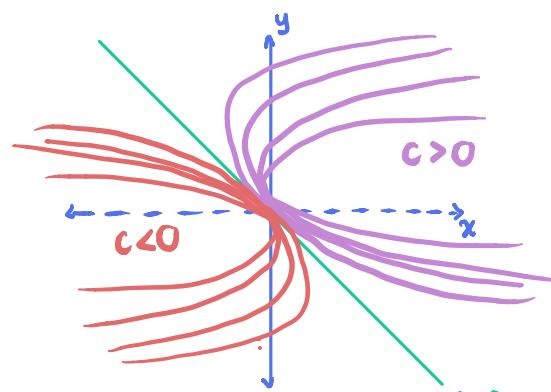
$$\underline{c<0} \quad x = -cy^2 - y \\ x = y(-cy - 1)$$

$y$ -intercepts:  
 $(0,0), (0, 1/c)$

$y$ -intercepts:  
 $(0,0), (0, -1/c)$

parabola opening  
right (\*)

parabola opening  
left (\*)



Ex: Give a rough sketch of the surface in  $\mathbb{R}^3$  defined by  
 $3x^2 - 12x - y + 2z^2 + 4z + 9 = 0$

$$\Rightarrow y = 3x^2 - 12x + 2z^2 + 4z + 9$$

$$\Rightarrow y = 3(x^2 - 4x + 4) - 12 + 2(z^2 + 2z + 1) - 2 + 9$$

$$\Rightarrow y = 3(x-2)^2 + 2(z+1)^2 - 5$$

The elliptical paraboloid opening in the positive direction with vertex  $(2, -5, -1)$

Ex: Indicate what contour diagram corresponds to each graph.

