MATB44 - Tutorial 6 (IC 320 Fridays 10-11am)

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Week 1 - Sept. 6th

No tutorials

Week 2 - Sept. 13th

ODE: $F(t, x, x^{(1)}, \dots, x^{(k)}) = 0$

t - independent variable (time)
x - dependent variable (space)
x is a function of t i.e. $x(t)$

The order is the highest derivative that appears in F.

The system is linear if we can rewrite it to separate out the $x_\ell^{(j)}$'s as follows:

$$x_i^{(k)} = g_i(t) + \sum_{\ell=1}^{n} \sum_{j=1}^{k-1} f_{i,j,\ell}(t)\, x_\ell^{(j)}$$

Ex: $\underbrace{(4+t^2)x' + 2tx}= 4t$

By product rule: $\frac{d}{dt}\left[(4+t^2)x\right] = 2tx + (4+t^2)x'$

$\Rightarrow \frac{d}{dt}\left[(4+t^2)x\right] = 4t$

$\Rightarrow (4+t^2)x = 2t^2 + C$

$\Rightarrow x(t) = \dfrac{2t^2 + C}{4+t^2}$

The system is homogeneous if $g_i(t) = 0$

If there is no direct dependance on t, the system is autonomous. Form $x' = f(x)$

Ex: Classify the following equations:

* $x' + \frac{1}{2}x = \frac{1}{2}e^{t/3}$ 1st order, linear

* $3 + xx' = t - x$ 1st order, non linear

* $x''+5x'+6x=0$ 2nd order, linear, homogeneous, autonomous
* $x'=t\sin x$ 1st order, non linear
* $x''-x=0$ 2nd order, linear, homogeneous, autonomous
* $tx'+2x=4t^2$ 1st order, linear
* $x'-2x+t=4$ 1st order, linear.

Ex: Solve $x'=x^2$ with $x_0=x(0)>0$

① $\frac{dx}{dt}=x^2 \Rightarrow \frac{dx}{x^2}=dt$

$$\Rightarrow \int \frac{dx}{x^2}=\int dt$$

$$\Rightarrow -\frac{1}{x}=t+C_1$$

$$\Rightarrow x=\frac{1}{-C_1-t}$$

When $x(0)=-\frac{1}{C_1}=x_0$ so $x=\frac{1}{\frac{1}{x_0}-t}$

② $f(x)=x^2$, a maximal nonzero interval is $(x_1,x_2)=(0,\infty)$

$$F(x)=\int_{x_0}^{x}\frac{dy}{y^2}=-\frac{1}{y}\Big|_{y=x_0}^{y=x}=-\frac{1}{x}+\frac{1}{x_0}$$

$\phi(t)=F^{-1}(t)$

$T_+=\lim_{x\to x_2}F(x)=\lim_{x\to\infty}-\frac{1}{x}+\frac{1}{x_0}=\frac{1}{x_0}$, so ϕ is defined for all $t>0$

$$t=F(F^{-1}(t))=\frac{-1}{F^{-1}(t)}+\frac{1}{x_0}$$

$$\Rightarrow \frac{1}{F^{-1}(t)}=\frac{1}{x_0}-t$$

$$\Rightarrow F^{-1}(t)=\frac{1}{\frac{1}{x_0}-t}$$

$$\Rightarrow \phi(t)=\frac{1}{\frac{1}{x_0}-t}$$

$\blacksquare$