

MATB41 - Tutorial 3 (BV355 Fridays 2-3pm)

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Week 1 - Sept. 6th

No tutorials

Week 2 - Sept. 13th

Derivative by definition:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Riemann Sum:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x, \text{ where } \Delta x = \frac{b-a}{n}, x_i = a + i \Delta x$$

$$\text{Useful formulas} \rightarrow \sum_{i=1}^n 1 = n, \sum_{i=1}^n i = \frac{i(i+1)}{2}, \sum_{i=1}^n i^2 = \frac{i(i+1)(2i+1)}{2 \cdot 3}$$

Trig. Identities:

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1 \\ \cos(2x) &= \cos^2 x - \sin^2 x \\ \sin(2x) &= 2\sin x \cos x\end{aligned}$$

Half Angle Identities:

$$\sin\left(\frac{\alpha}{2}\right) = \pm \sqrt{\frac{1-\cos\alpha}{2}} \quad \cos\left(\frac{\alpha}{2}\right) = \pm \sqrt{\frac{1+\cos\alpha}{2}}$$

Trig. Integrals:

$$\int \cos^n(x) \sin^m(x) dx \quad \begin{array}{l} \text{if } n \text{ or } m \text{ are odd} \rightarrow \text{do } u \text{ substitution} \\ \text{if } n \text{ and } m \text{ are even} \rightarrow \text{half angle substitution} \end{array}$$

Ex: Evaluate $\int \frac{\sin^3(\ln x) \cos^3(\ln x)}{x} dx$

$$= \int \sin^3(\ln x) \cos^2(\ln x) \frac{\cos(\ln x)}{x} dx$$

$$= \int \sin^3(\ln x) (1 - \sin^2(\ln x)) \frac{\cos(\ln x)}{x} dx$$

$$= \int u^3 (1-u^2) du$$

$$= \int u^3 - u^5 du$$

$$= \frac{u^4}{4} - \frac{u^6}{6}$$

$$\begin{aligned} \text{Let } u &= \sin(\ln x) \\ du &= \frac{\cos(\ln x)}{x} dx \end{aligned}$$

$$= \frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} + C ,$$

Partial fractions:

Ex: Evaluate $\int \frac{1}{x(x^2-1)} dx$

$$\begin{aligned}\frac{1}{x(x^2-1)} &= \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{A(x^2-1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)} \\ &= \frac{Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx}{x(x-1)(x+1)} = \frac{x^2(A+B+C) + x(B-C) - A}{x(x-1)(x+1)}\end{aligned}$$

$$\left\{ \begin{array}{l} A+B+C=0 \\ B-C=0 \rightarrow B=C \\ -A=1 \rightarrow A=-1 \end{array} \right. \quad \left. \begin{array}{l} -1+2B=0 \\ B=\frac{1}{2}=C \end{array} \right.$$

$$\begin{aligned}\int \frac{1}{x(x^2-1)} dx &= \int \frac{-1}{x} + \frac{1}{2} \cdot \frac{1}{x-1} + \frac{1}{2} \cdot \frac{1}{x+1} dx \\ &= -\ln|x| + \frac{1}{2} \ln|x-1| + \frac{1}{2} \ln|x+1| + C,\end{aligned}$$

Angle between vectors:

$$\cos \theta = \frac{u \cdot w}{\|u\| \|w\|}$$

Cauchy-Schwarz inequality:

$$|u \cdot w| \leq \|u\| \|w\|, \quad u, w \in \mathbb{R}^n$$

Orthogonal:

$$u \cdot v = 0$$

Projection u onto w :

$$\frac{u \cdot w}{\|w\|^2} w$$

Ex: Let $v = (1, -3, 1)$ and $w = (2, 1, 2)$ be vectors in \mathbb{R}^3

$$v \cdot w = (1, -3, 1) \cdot (2, 1, 2) = 2 - 3 + 2 = 1$$

$$\|v\| = \sqrt{1^2 + (-3)^2 + 1^2} = \sqrt{11} \quad \|w\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

(a) Find the angle between v and w

$$\cos \theta = \frac{v \cdot w}{\|v\| \|w\|} = \frac{1}{\sqrt{11} \cdot 3}, \quad \theta = \cos^{-1}\left(\frac{1}{3\sqrt{11}}\right),$$

(b) Verify the Cauchy-Schwarz inequality and the triangle inequality for v and w

$$\begin{aligned}|v \cdot w| &\leq \|v\| \|w\| \\ \Rightarrow 1 &\leq 3\sqrt{11} \quad \checkmark\end{aligned}$$

(c) Find all unit vectors in \mathbb{R}^3 which are orthogonal to both v and w .

Let $u = (a, b, c)$ be orthogonal to both v and w .

$$(1, -3, 1)(a, b, c) = a - 3b + c = 0 \rightarrow a = -c$$

$$(2, 1, 2)(a, b, c) = \underline{2a + b + 2c = 0}$$

$$-5b = 0 \rightarrow b = 0$$

So $u = (k, 0, -k)$, $k \in \mathbb{R}$

$$\|u\| = \sqrt{k^2 + k^2} = \sqrt{2}|k| = 1 \rightarrow k = \frac{1}{\sqrt{2}}$$

The vectors are $(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$

(d) Find the projection of (i) v onto w and (ii) w onto v .

(i) Proj of v onto w : $\frac{v \cdot w}{\|w\|^2} w = \frac{1}{9} (2, 1, 2)$

(ii) Proj of w onto v : $\frac{w \cdot v}{\|v\|^2} v = \frac{1}{11} (1, -3, 1)$

Eigenvalues and eigenvectors:

Values of λ for $\det(A - \lambda I) = 0$ and their vectors respectively.

The matrix is diagonalizable if $P^{-1}AP = D$ is a diagonal matrix where P is the eigenvectors matrix. (or $A = PDP^{-1}$)

Ex: Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A .

Is A diagonalizable?

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -\lambda & 0 \\ 2 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} + 2 \begin{vmatrix} 0 & -\lambda \\ 2 & 0 \end{vmatrix}$$

$$\begin{aligned} &= -\lambda(1-\lambda)(4-\lambda) + 4\lambda \\ &= -\lambda[4 - 5\lambda + \lambda^2] - 4 \\ &= -\lambda(-5\lambda + \lambda^2) \\ &= -\lambda^2(\lambda - 5) = 0 \\ \Rightarrow \lambda_1 &= 0, \lambda_2 = 0, \lambda_3 = 5 \end{aligned}$$

For $\lambda_1 = 0$
 $\lambda_2 = 0$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_1 + 2x_3 = 0 \rightarrow x_1 = -2x_3 \text{ so for } \lambda_1 \text{ we can let } v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$2x_1 + 4x_3 = 0 \quad x_2 \text{ is a free variable so for } \lambda_2 \text{ we can let } v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{For } \lambda_3=5 \quad \begin{bmatrix} -4 & 0 & 2 \\ 0 & -5 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -4x_1 + 2x_3 &= 0 \\ -5x_2 &= 0 \rightarrow x_2 = 0 \\ 2x_1 - x_3 &= 0 \rightarrow x_3 = 2x_1 \end{aligned} \quad \text{so for } \lambda_3 \text{ we can let } v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad \left[\begin{array}{ccc|ccc} -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 & 0 & 2 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{5} & 0 & \frac{2}{5} \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{5} & 0 & \frac{1}{5} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{5} & 0 & \frac{2}{5} \end{array} \right] = P^{-1}$$

$$\begin{aligned} D &= \begin{bmatrix} -\frac{2}{5} & 0 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \text{ which is a diagonal matrix of } \lambda_1, \lambda_2, \lambda_3. \end{aligned}$$