MATB41 - Tutorial 3 (BV355 Fridays 2-3pm)

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Week 1 - Sept. 6th

No tutorials

Week 2 - Sept. 13th

Derivative by definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \qquad f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

Riemann Sum:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x, \text{ where } \Delta x = \frac{b-a}{n},\ x_i = a + i\Delta x$$

Useful formulas $\rightarrow$ $\sum_{i=1}^{n} 1 = n$, $\sum_{i=1}^{n} i = \frac{i(i+1)}{2}$, $\sum_{i=1}^{n} i^2 = \frac{i(i+1)(2i+1)}{2\cdot 3}$

Trig. Identities:

$\sin^2 x + \cos^2 x = 1$
$\cos(2x) = \cos^2 x - \sin^2 x$
$\sin(2x) = 2\sin a \cos a$

Half Angle Identities:

$$\sin\left(\frac{a}{2}\right) = \pm\sqrt{\frac{1-\cos a}{2}} \qquad \cos\left(\frac{a}{2}\right) = \pm\sqrt{\frac{1+\cos a}{2}}$$

Trig. Integrals:

$\int \cos^n(x)\sin^m(x)\,dx$

if n or m are odd $\rightarrow$ do u substitution
if n and m are even $\rightarrow$ half $\measuredangle$ substitution

Ex: Evaluate $\int \frac{\sin^3(\ln x)\cos^3(\ln x)}{x}\,dx$

$$= \int \sin^3(\ln x)\cos^2(\ln x)\,\frac{\cos(\ln x)}{x}\,dx$$

$$= \int \sin^3(\ln x)(1 - \sin^2(\ln x))\,\frac{\cos(\ln x)}{x}\,dx$$

let $u = \sin(\ln x)$
$du = \frac{\cos(\ln x)}{x}\,dx$

$$= \int u^3(1-u^2)\,du$$

$$= \int u^3 - u^5\,du$$

$$= \frac{u^4}{4} - \frac{u^6}{6}$$

$$= \frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} + C$$ //

Partial fractions:

Ex: Evaluate $\int \frac{1}{x(x^2-1)}\,dx$

$$\frac{1}{x(x^2-1)} = \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{A(x^2-1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$$= \frac{Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx}{x(x-1)(x+1)} = \frac{x^2(A+B+C) + x(B-C) - A}{x(x-1)(x+1)}$$

$$\begin{cases} A+B+C = 0 \\ B-C = 0 \rightarrow B = C \\ -A = 1 \rightarrow A = -1 \end{cases}$$

$-1 + 2B = 0$

$B = \frac{1}{2} = C$

$$\int \frac{1}{x(x^2-1)}\,dx = \int \frac{-1}{x} + \frac{1}{2}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{x+1}\,dx$$

$$= -\ln|x| + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$$ //

Angle between vectors:

$$\cos\theta = \frac{u \cdot w}{\|u\|\|w\|}$$

Cauchy-Schwarz inequality:

$$|u \cdot w| \leq \|u\|\|w\|, \quad u, w \in \mathbb{R}^n$$

Orthogonal:

$$u \cdot v = 0$$

Projection u onto w:

$$\frac{u \cdot w}{\|w\|^2} w$$

Ex: Let $v = (1,-3,1)$ and $w = (2,1,2)$ be vectors in $\mathbb{R}^3$

$$v \cdot w = (1,-3,1)\cdot(2,1,2) = 2 - 3 + 2 = 1$$

$$\|v\| = \sqrt{1^2 + (-3)^2 + 1^2} = \sqrt{11} \qquad \|w\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

(a) Find the angle between v and w

$$\cos\theta = \frac{v \cdot w}{\|v\|\|w\|} = \frac{1}{\sqrt{11}\cdot 3}, \quad \theta = \cos^{-1}\left(\frac{1}{3\sqrt{11}}\right)$$ //

(b) Verify the Cauchy-Schwarz inequality and the triangle inequality for v and w

$$|v \cdot w| \leq \|v\|\|w\|$$

$$\Rightarrow \quad 1 < 3\sqrt{11}$$ ✓

(c) Find all unit vectors in $\mathbb{R}^3$ which are orthogonal to both v and w.

Let $u = (a, b, c)$ be orthogonal to both v and w.

$$(1,-3,1)(a,b,c) = a - 3b + c = 0 \quad \rightarrow a = -c$$
$$(2,1,2)(a,b,c) = \underline{2a + b + 2c = 0}$$
$$-5b = 0 \rightarrow b = 0$$

So $u = (k, 0, -k)$, $k \in \mathbb{R}$

$$\|u\| = \sqrt{k^2 + k^2} = \sqrt{2}\,k = 1 \quad \rightarrow k = \frac{1}{\sqrt{2}}$$

The vectors are $\left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$

(d) Find the projection of (i) v onto w and (ii) w onto v.

(i) Proj of v onto w : $\frac{v \cdot w}{\|w\|^2} w = \frac{1}{9}(2,1,2)$

(ii) Proj of w onto v : $\frac{w \cdot v}{\|v\|^2} v = \frac{1}{11}(1,-3,1)$

Eigenvalues and eigenvectors:

Values of λ for $\det(A - \lambda I) = 0$ and their vectors respectively.

The matrix is diagonalizable if $P^{-1}AP = D$ is a diagonal matrix where P is the eigenvectors matrix. (or $A = PDP^{-1}$)

Ex: Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A. Is A diagonalizable?

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -\lambda & 0 \\ 2 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} + 2\begin{vmatrix} 0 & -\lambda \\ 2 & 0 \end{vmatrix}$$

$$= -\lambda(1-\lambda)(4-\lambda) + 4\lambda$$
$$= -\lambda[(4 - 5\lambda + \lambda^2) - 4]$$
$$= -\lambda(-5\lambda + \lambda^2)$$
$$= -\lambda^2(\lambda - 5) = 0$$
$$\rightarrow \lambda_1 = 0, \ \lambda_2 = 0, \ \lambda = 5$$

For $\lambda_1 = 0$, $\lambda_2 = 0$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_3 = 0 \rightarrow x_1 = -2x_3$ so for λ_1 we can let $v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$
$2x_1 + 4x_3 = 0$

x_2 is a free variable so for λ_2 we can let $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda_3 = 5$ $\begin{bmatrix} -4 & 0 & 2 \\ 0 & -5 & 0 \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-4x_1 + 2x_3 = 0$
$-5x_2 = 0 \rightarrow x_2 = 0$
$2x_1 - x_3 = 0 \rightarrow x_3 = 2x_1$

so for λ_3 we can let $v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$P = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ $\left[\begin{array}{ccc|ccc} -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 & 0 & 2 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/5 & 0 & 2/5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/5 & 0 & 1/5 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/5 & 0 & 2/5 \end{array}\right] = P^{-1}$

$$D = \begin{bmatrix} -2/5 & 0 & 1/5 \\ 0 & 1 & 0 \\ 1/5 & 0 & 2/5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$, which is a diagonal matrix of $\lambda_1, \lambda_2, \lambda_3$.

Week 3 - Sept. 20th

Vector equation of a line in $\mathbb{R}^3$: $\vec{\ell} = a + t\vec{v}$
$\vec{\ell} = (a_1, a_2, a_3) + t[v_1, v_2, v_3]$ $t, a_i, v_i \in \mathbb{R}$, $i = 1,2,3$

Parametric equation of a line in $\mathbb{R}^3$: $x = a_1 + tv_1$
$y = a_2 + tv_2$
$z = a_3 + tv_3$

Ex: For each of the following lines, write its equation in vector and parametric form

(i) The line that passes through the point P_0 $(3,1,9)$ in the direction of $\vec{v} = (1,1,1)$

V. Eq. $\vec{l} = (3,1,9) + t(1,1,1)$, $t \in \mathbb{R}$

P. Eq. $x = 3+t$ $y = 1+t$ $z = 9+t$, $t \in \mathbb{R}$

(ii) The line that passes through points $P_0(-1,1,2)$ and $P_1(2,0,-3)$

The direction vector for the line is $(2,0,-3) - (-1,1,2) = (3,-1,-5)$

V. Eq. $\vec{l} = (-1,1,2) + t(3,-1,5)$, $t \in \mathbb{R}$

P. Eq. $x = -1+3t$ $y = 1-t$ $z = 2+5t$, $t \in \mathbb{R}$

(iii) The line that passes through the point $P_0(0,1,0)$ and is orthogonal to the plane $10x + 15y + 3z = 11$

The direction vector for this line is a normal vector for the plane
$\vec{n} = (10,15,3)$

V. Eq. $\vec{l} = (0,1,0) + t(10,15,3)$, $t \in \mathbb{R}$

P. Eq. $x = 10t$ $y = 1+15t$ $z = 3t$, $t \in \mathbb{R}$

Ex: Find an equation of the plane that passes through 3 points $A(-1,1,2)$, $B(2,0,-3)$ and $C(2,-1,2)$

A pair of direction vectors in the plane are $\vec{v} = (2,0,-3) - (-1,1,2) = (3,-1,-5)$ and $\vec{w} = (2,-1,2) - (-1,1,2) = (3,-2,0)$

To find the normal of the plane $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & -5 \\ 3 & -2 & 0 \end{vmatrix}$

$= (-10,-15,-3)$

So the plane is $-10x - 15y - 3z = d \rightarrow -10(-1) - 15(1) - 3(2) = -11$

$\therefore$ The equation of the plane is $10x + 15y + 3z = 11$.

Ex: Find an equation of the plane that passes through the origin and contains the line $x = 2+3t$ $y = 1-2t$ $z = 1+2t$

The line when $t=0$ passes through the point $(2,1,1)$
So the vector $\vec{v} = (0,0,0) - (2,1,1) = (-2,-1,-1)$ is also on the plane.
We can let $\vec{w} = (3,-2,2)$

To find the normal of the plane $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & -1 \\ 3 & -2 & 2 \end{vmatrix}$

$= (-4,1,7)$

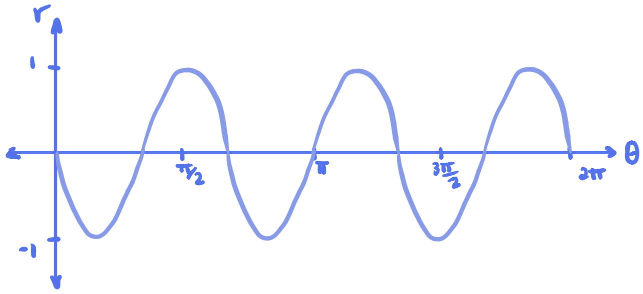
So the plane is $-4x + y + 7z = d \rightarrow -4(0) + (0) + 7(0) = 0$

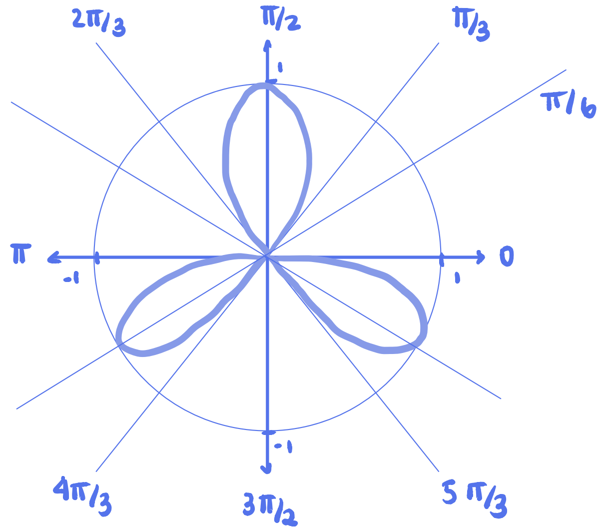
$\therefore$ The equation of the plane is $-4x + y + 7z = 0$

Polar equations : $x = r\cos\theta \quad y = r\sin\theta$

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$

Ex: Sketch the curve $r = -\sin 3\theta$ in the polar plane

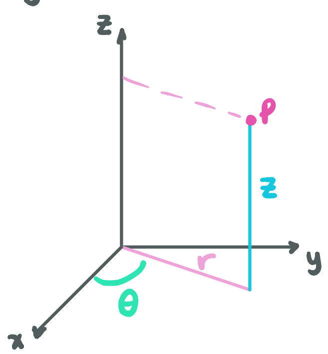




θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	-1	0	1	0	-1	0

Week 4 – Sept. 27th

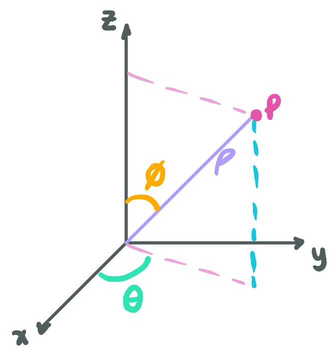
Cylindrical Coordinates:



$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$0 \le r \le \infty$
$0 \le \theta \le 2\pi$

Spherical Coordinates:



$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

$0 \le \rho \le \infty$
$0 \le \theta \le 2\pi$
$0 \le \phi \le \pi$

Ex: Interpret the equation $1 = 2\cos\theta\sin\theta$ geometrically

Cylindrical:

$\Rightarrow r^2 = 2r\cos\theta\, r\sin\theta$
$\Rightarrow x^2 + y^2 = 2xy$
$\Rightarrow x^2 - 2xy + y^2 = 0$
$\Rightarrow (x-y)^2 = 0$

Spherical:

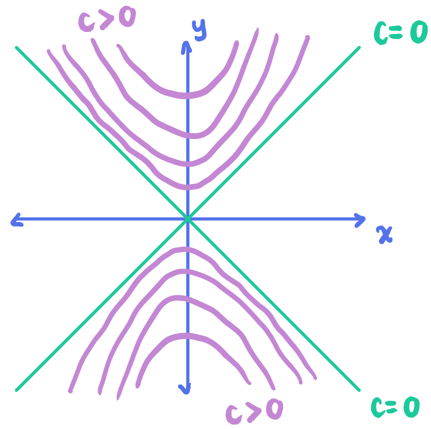
$\Rightarrow \rho^2\sin^2\phi = 2\rho\sin\phi\cos\theta\,\rho\sin\phi\sin\theta$
$\Rightarrow \rho^2\sin^2\phi(\sin^2\theta + \cos^2\theta) = 2\rho\sin\phi\cos\theta\,\rho\sin\phi\sin\theta$
$\Rightarrow \rho^2\sin^2\phi\sin^2\theta + \rho^2\sin^2\phi\cos^2\theta = 2\rho\sin\phi\cos\theta\,\rho\sin\phi\sin\theta$
$\Rightarrow x^2 + y^2 = 2xy$
$\Rightarrow x^2 - 2xy + y^2 = 0$
$\Rightarrow (x-y)^2 = 0$

Ex: Characterize and sketch several level curves of the following functions:

(i) $f(x,y) = \frac{\sqrt{y^2 - x^2}}{2} = c$

$\operatorname{dom} f = \{(x,y) \in \mathbb{R}^2 \mid y^2 \geq x^2\}$, because $y^2 - x^2 \geq 0$

c cannot be -ve, $c \geq 0$ always



$\underline{c=0}$ $\quad y^2 = x^2$

$y = \pm x \qquad (*)$

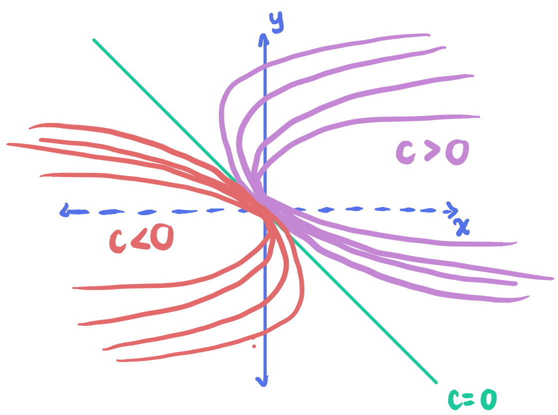
$\underline{c>0}$ $\quad y^2 - x^2 = (2c)^2$

$y^2 = (2c)^2 + x^2 \qquad (*)$

y-intercepts: $(0, \pm 2c)$

(ii) $f(x,y) = \frac{x+y}{y^2} = c$

$\operatorname{dom} f = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$



$\underline{c=0}$ $\quad x + y = 0$

$y = -x \qquad (*)$

$\underline{c \neq 0}$ $\quad x = cy^2 - y$

y-intercepts: $y = cy^2$

$\Rightarrow y = \frac{1}{c}$

$\underline{c>0}$ $\quad x = cy^2 - y$

$x = y(cy - 1)$

y-intercepts:

$(0,0), (0, 1/c)$

parabola opening right $(*)$

$\underline{c<0}$ $\quad x = -cy^2 - y$

$x = y(-cy - 1)$

y-intercepts:

$(0,0), (0, -1/c)$

parabola opening left $(*)$

Ex: Give a rough sketch of the surface in $\mathbb{R}^3$ defined by $3x^2 - 12x - y + 2z^2 + 4z + 9 = 0$

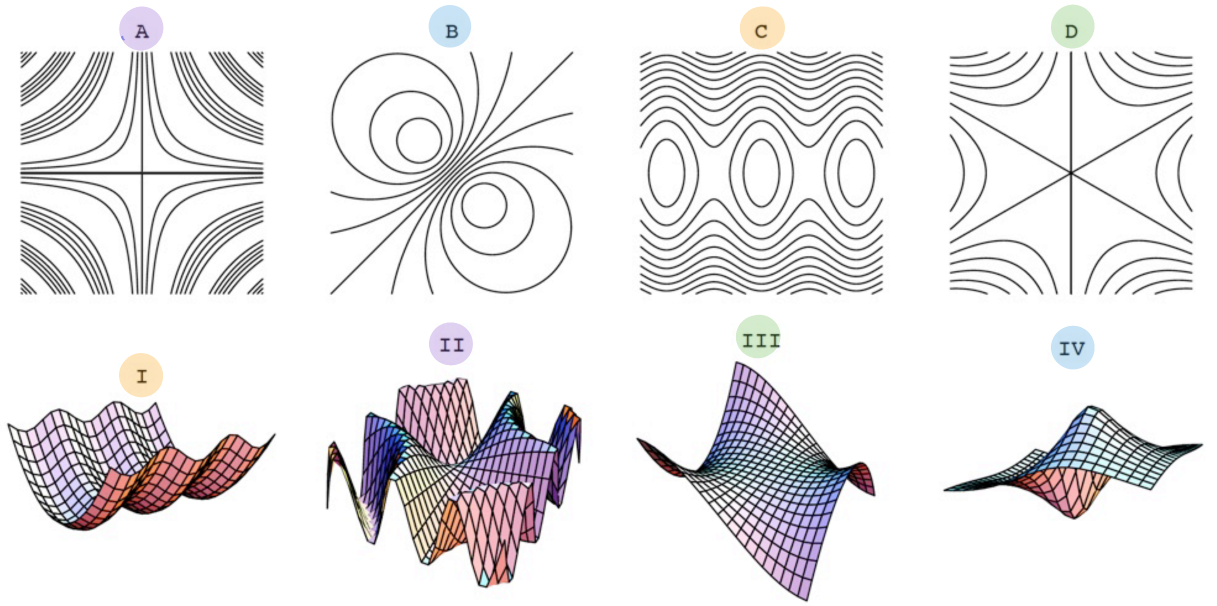
$\Rightarrow y = 3x^2 - 12x + 2z^2 + 4z + 9$

$\Rightarrow y = 3(x^2 - 4x + 4) - 12 + 2(z^2 + 2z + 1) - 2 + 9$

$\Rightarrow y = 3(x-2)^2 + 2(z+1)^2 - 5$

The elliptical paraboloid opening in the positive direction with vertex $(2,-5,-1)$

Ex: Indicate what contour diagram corresponds to each graph.



Week 5 - Oct. 4th

Ex: For each of the following, evaluate the limit or show that the limit does not exist.

(a) $$\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{3x+5y+2} = \frac{e^{(0)}}{3(0)+5(0)+2} = \frac{1}{2}$$

(b) $$\lim_{(x,y)\to(1,-2)} \frac{xy+2x-y-2}{(x^2-1)(y+2)} = \lim_{(x,y)\to(1,-2)} \frac{x(y+2)-(y+2)}{(x^2-1)(y+2)}$$

$$= \lim_{(x,y)\to(1,-2)} \frac{(y+2)(x-1)}{(x-1)(x+1)(y+2)}$$

$$= \lim_{(x,y)\to(1,-2)} \frac{1}{x+1} = \frac{1}{2}$$

(c) $$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim_{r\to0} \frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

Recall:
$x = r\cos\theta$
$y = r\sin\theta$

$$= \lim_{r\to0} \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2(\cos^2\theta+\sin^2\theta)}$$

$$= \lim_{r\to0} r(\cos^3\theta - \sin^3\theta) = 0$$

(d) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin(r^2)}{r^2}$

$$\overset{L'H}{=} \lim_{r\to 0} \frac{2r\cos(r^2)}{2r}$$

$$= \lim_{r\to 0} \cos(r^2) = \cos(0) = 1$$

(e) $\lim_{(x,y)\to(0,0)} \frac{|x|}{\sqrt{x^2+y^2}}$

Restrict y-axis: $(y=0)$ $\lim_{(x,0)\to(0,0)} \frac{|x|}{\sqrt{x^2}} = 1$

$\neq$

Restrict x-axis: $(x=0)$ $\lim_{(0,y)\to(0,0)} \frac{0}{\sqrt{y^2}} = 0$

The limit does not exist.

(f) $\lim_{(x,y)\to(1,1)} \frac{x^2+y^2-2}{|x-1|+|y-1|} = \lim_{(u,v)\to(0,0)} \frac{(u+1)^2+(v+1)^2-2}{|u|+|v|}$

We can rewrite this limit letting $u=x-1$ and $v=y-1$ $= \lim_{(u,v)\to(0,0)} \frac{u^2+2u+v^2+2v}{|u|+|v|}$

Restrict $v=u$ and $|u|=u$: $\lim_{u\to 0} \frac{u^2+2u+u^2+2u}{u+u}$

$$= \lim_{u\to 0} \frac{2u^2+4u}{2u}$$

$$= \lim_{u\to 0} \frac{2u(u+2)}{2u}$$

$$= \lim_{u\to 0} u+2 = 2$$

$\neq$

Restrict $v=-u$ and $|u|=-u$: $\lim_{u\to 0} \frac{u^2+2u+u^2-2u}{-u-u}$

$$= \lim_{u\to 0} \frac{2u^2}{-2u}$$

$$= \lim_{u\to 0} u = 0$$

The limit does not exist.

Ex: Find the value of a so that f is continuous at (0,0)

$$f(x,y) = \begin{cases} \dfrac{x^3-x^2-2x^2y+xy^2-y^2-2y^3}{x^2+y^2} & , (x,y) \neq 0 \\ a & , (x,y) = 0 \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2(x-1-2y)+y^2(x-1-2y)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x-1-2y)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} x-1-2y = \underline{\underline{-1 = a}}$$

Ex: Determine whether the following functions are continuous throughout their domains.

(a)

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

Restrict y-axis: $(y=0)$ $\quad \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$

Restrict x-axis: $(x=0)$ $\quad \lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1$

$\neq$

The limit does not exist, so it is not continuous at (0,0)

(b)

$$f(x,y) = \begin{cases} \dfrac{2x^3+2xy^2+3x^2+3y^2}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{2x^3+2xy^2+3x^2+3y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{2x(x^2+y^2)+3(x^2+y^2)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(2x+3)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} 2x+3 = 3 \quad \neq 0$$

The function is not continuous at (0,0)

Definition of Limit:

Let $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables x and y defined for all ordered pairs (x,y) in some open disk $D \subseteq \mathbb{R}^2$ centered on a fixed ordered pair (x_0, y_0). except possibly at (x_0, y_0)

We will say that the number $L \in \mathbb{R}$ is the limit of $f(x,y)$ as $(x,y) \in D$ approaches (x_0, y_0) if and only if given any real number $\varepsilon > 0$, we can find a real number $\delta > 0$ (depending on ε) such that $f(x,y)$ satisfies $|f(x,y) - L| < \varepsilon$ whenever the distance between (x,y) and (x_0,y_0) satisfies $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ and we will write:

$$\lim_{(x,y)\to(0,0)} f(x,y) = L \quad \text{or} \quad \lim_{(x,y)\to(0,0)} |f(x,y) - L| = 0$$

Ex: Use the definition of limits to show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$

We need to show that for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 < \sqrt{x^2+y^2} < \delta$, we have $\left|\frac{4xy^2}{x^2+y^2}\right| = 0$

So if $(x,y) \in D$ and $0 < \sqrt{x^2+y^2} < \delta$, we see that you should choose the corresponding positive real number $\delta = \varepsilon/4$ and we get

$$|f(x,y) - 0| = \left|\frac{4xy^2}{x^2+y^2} - 0\right| = 4|x| \cdot \frac{y^2}{x^2+y^2} \leq 4|x| \cdot 1 = 4\sqrt{x^2} \leq 4\sqrt{x^2+y^2} < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$

b/c $y^2 \leq x^2 + y^2$

$\frac{y^2}{x^2+y^2} \leq 1$

The number 0 is the limit of the function $f(x,y) = \frac{4xy^2}{x^2+y^2}$ as (x,y) in D approaches $(0,0)$ because for any given number $\varepsilon > 0$, we have shown that we can produce a corresponding number $\delta = \varepsilon/4 > 0$ so that $f(x,y) = \frac{4xy^2}{x^2+y^2}$ satisfies the inequality $\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \varepsilon$ whenever the distance between (x,y) and $(0,0)$ satisfies $0 < \sqrt{x^2+y^2} < \varepsilon/4 = \delta$. So we can write $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$ $\blacksquare$

Week 6 - Oct. 11th

Ex: Let $f(x,y) = \begin{cases} \frac{x^3y - y^3x}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(a) Find $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ for $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{(3x^2y-y^3)(x^2+y^2)-2x(x^3y-y^3x)}{(x^2+y^2)^2} = \frac{3x^4y+3x^2y^3-x^2y^3-y^5-2x^4y+2x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{(x^3-3y^2x)(x^2+y^2)-2y(x^3y-y^3x)}{(x^2+y^2)^2} = \frac{x^5+x^3y^2-3y^2x^3-3y^4x-2x^3y^2+2y^4x}{(x^2+y^2)^2}$$

$$= \frac{x^5-4x^3y^2-y^4x}{(x^2+y^2)^2} = \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}$$

(b) Find $\frac{\partial f}{\partial x}(0,y)$ and $\frac{\partial f}{\partial y}(x,0)$

Using (a) we have

$$\frac{\partial f}{\partial x}(0,y) = \frac{y(-y^4)}{(y^2)^2} = -y \quad , y \neq 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(x,0) = \frac{x(x^4)}{(x^2)^2} = x \quad , \quad x \neq 0$$

For $y=x=0$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0}{h^3}-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0}{h^3}-0}{h} = 0$$

Directional derivative: $D_v f(p) = \nabla f(p) \cdot \frac{v}{\|v\|}$

Ex: Compute the directional derivative of $f(x,y,z) = xz + y^2z^2$ at the point $(3,-1,2)$ in the direction of the vector $v=(0,-3,4)$

$\nabla f(x,y,z) = (z, 2yz^2, x+2y^2z)$
$\nabla f(3,-1,2) = (2,-8,7)$

$$D_{(0,-3,4)} f(3,-1,2) = (2,-8,7) \cdot \frac{(0,-3,4)}{\|(0,-3,4)\|} = \frac{8\cdot 3 + 7\cdot 4}{\sqrt{9+16}} = \frac{52}{5}$$

Ex: For each of the following evaluate $\frac{\partial f}{\partial y}$ at the point a.

(a) $f(x,y) = y\sin(xy) + xe^{-y^2}$, $a = \left(\frac{\pi}{6}, 2\right)$

$$\frac{\partial f}{\partial y} = \sin(xy) + xy\cos(xy) - 2xy e^{-y^2}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(\frac{\pi}{6},2)} = \sin\frac{\pi}{3} + \frac{\pi}{3}\cos\frac{\pi}{3} - \frac{2\pi}{3}e^{-4} = \frac{\sqrt{3}}{2} + \frac{\pi}{6} - \frac{2\pi}{3e^4}$$

(b) $f(x,y,z) = 2^{\sqrt{x-yz}}$, $a=(3,2,1)$

$$\frac{\partial f}{\partial y} = 2^{\sqrt{x-yz}}(\ln 2)(\sqrt{x-yz})' = \frac{-z(\ln 2)2^{\sqrt{x-yz}}}{2\sqrt{x-yz}}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(3,2,1)} = \frac{(-\ln 2)2^{\sqrt{3-2}}}{2\sqrt{3-2}} = -\ln 2$$

(c) $f(x,y) = \begin{cases} \dfrac{2x^2y+3y^3}{x^2+y^2} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$, $a=(0,0)$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{3h^3}{h^2}-0}{h} = \lim_{h\to 0} 3 = 3$$

Ex: Let $f:\mathbb{R}^2\to\mathbb{R}^3$ be given by $f(x,y)=(xy^2, x+2y, xy)$ and let $g:\mathbb{R}^3\to\mathbb{R}^3$ be given by $g(x,y,z)=(xy, yz, xz^2)$.

(a) Find Df and Dg. Use the chain rule to find $D(g\circ f)$

$$D(g\circ f)(x,y) = Dg(f(x,y))\,Df(x,y)$$

$$Df = \begin{pmatrix} y^2 & 2xy \\ 1 & 2 \\ y & x \end{pmatrix} \qquad Dg = \begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z^2 & 0 & 2xz \end{pmatrix}$$

$$Dg(f(x,y)) = Dg(xy^2, x+2y, xy) = \begin{pmatrix} x+2y & xy^2 & 0 \\ 0 & xy & x+2y \\ x^2y^2 & 0 & 2x^2y^3 \end{pmatrix}$$

$$D(g\circ f)(x,y) = \begin{pmatrix} x+2y & xy^2 & 0 \\ 0 & xy & x+2y \\ x^2y^2 & 0 & 2x^2y^3 \end{pmatrix}\begin{pmatrix} y^2 & 2xy \\ 1 & 2 \\ y & x \end{pmatrix}$$

$$= \begin{pmatrix} y^2x+2y^3+xy^2 & 2x^2y+4xy^2+2xy^2 \\ xy+xy+2y^2 & 2xy+x^2+2xy \\ x^2y^4+2x^2y^4 & 2x^3y^3+2x^3y^3 \end{pmatrix}$$

$$= \begin{pmatrix} 2y^3+2xy^2 & 2x^2y+6xy^2 \\ 2xy+2y^2 & 4xy+x^2 \\ 3x^2y^4 & 4x^3y^3 \end{pmatrix}$$

(b) Compute $g \circ f$ and $D(g \circ f)$ directly

$$g \circ f(x,y) = g(xy^2, x+2y, xy) = (x^2y^2 + 2xy^3,\ x^2y + 2xy^2,\ x^3y^4)$$

$$D(g \circ f)(x,y) = \begin{pmatrix} 2y^3 + 2xy^2 & 2x^2y + 6xy^2 \\ 2xy + 2y^2 & 4xy + x^2 \\ 3x^2y^4 & 4x^3y^3 \end{pmatrix}$$

Ex: If $g(u,v) = f(u^2 - v^2, v^2 - u^2)$ and f is differentiable, show that g satisfies $v\frac{\partial g}{\partial u} + u\frac{\partial g}{\partial v} = 0$

Let $x = u^2 - v^2$ and $y = v^2 - u^2$, then $g(u,v) = f(x,y)$

Then $\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial f}{\partial x}(2u) + \frac{\partial f}{\partial y}(-2u)$

and $\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{\partial f}{\partial x}(-2v) + \frac{\partial f}{\partial y}(2v)$

$$\Rightarrow v\frac{\partial g}{\partial u} + u\frac{\partial g}{\partial v} = v\left[\frac{\partial f}{\partial x}(2u) + \frac{\partial f}{\partial y}(-2u)\right] + u\left[\frac{\partial f}{\partial x}(-2v) + \frac{\partial f}{\partial y}(2v)\right]$$

$$= 2uv\frac{\partial f}{\partial x} - 2uv\frac{\partial f}{\partial y} - 2uv\frac{\partial f}{\partial x} + 2uv\frac{\partial f}{\partial y}$$

$$= 0$$