MATB41 - Multivariable Calculus

TA: Angela Zavaleta-Bernuy
Tutorials: TUT0007 (Wed. 11am-12pm)
TUT0006 (Thu. 10am - 11am)
TUT0014 (Thu. 11am - 12pm)
email: angelazb@cs.toronto.edu
website: www.cs.toronto.edu/~angelazb

Week 1 (Sep 14th)

* No tutorials *

Week 2 (Sep 21st)

Curved Lines

Parabola: (with vertex) at the origin

- $y = ax^2$ → open up when $a>0$ and down when $a<0$
- $x = by^2$ → open right when $b>0$ and left when $b<0$

* **Remember:** We can replace x with $x-m$ to shift right and y with $y+n$ to shift up!

Ellipse: (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

intersections $(\pm a, 0)$ and $(0, \pm b)$
When $a=b$, the curve is a circle of radius $a=b$.

Hyperbola: (with center) at the origin

- $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$
- $\left(\frac{y}{a}\right)^2 - \left(\frac{x}{b}\right)^2 = 1$

It has two curves that goes in opposite directions and do not touch the slant asymptote

Ex ① - Sketch the following curves:

a) $x^2 + 3y^2 + 2x - 12y + 10 = 0$

$$(x^2+2x) + 3(y^2-4y)+10 = 0$$
$$(x^2+2x+1) - 1 + 3(y^2-4y+4) - 12 + 10 = 0$$
$$(x+1)^2 + 3(y-2)^2 = 3$$

$$\Rightarrow \frac{(x+1)^2}{3} + (y-2)^2 = 1 \quad \Rightarrow \left(\frac{x+1}{\sqrt{3}}\right)^2 + (y-2)^2 = 1$$

Okie, now what?! Another way to write the ellipse equation would be:

$$\left(\frac{x-m}{a}\right)^2 + \left(\frac{y-n}{b}\right)^2 = 1$$

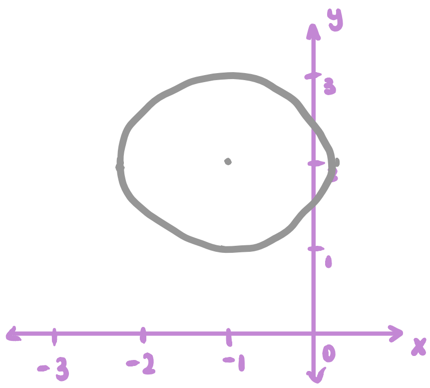
→ we just combined it with ✱

So the new origin will be $(-1, 2)$ as the ellipse will be shifted left by one and up by 2.

Now, let's find the end points

→ The horizontal max/min will be $(-1 \pm \sqrt{3}, 2)$

→ The vertical max/min will be $(-1, 2 \pm 1)$



b) $\frac{(x-2)^2}{4} - \frac{(y+2)^2}{9} = 1$

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 1$$

The origin will be $(2, -2)$, we are shifting right 2 and 2 down.

Check the x-int : $(y=0)$

$$\left(\frac{x-2}{2}\right)^2 - \frac{4}{9} = 1$$

$$\frac{(x-2)^2}{4} = \frac{13}{9}$$

$$(x-2)^2 = \frac{52}{9}$$

$$x-2 = \pm\frac{2\sqrt{13}}{3}$$

$$x = 2 \pm \frac{2\sqrt{13}}{3}$$

Check the y-int: $(x=0)$

$$1 - \left(\frac{y+2}{3}\right)^2 = 1$$

$$y = -2$$

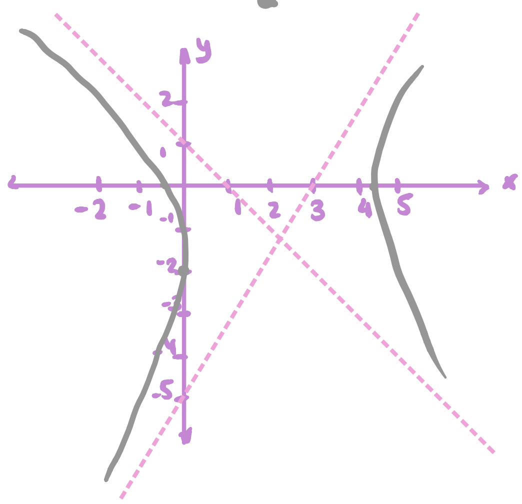
Find the slant asymptote:

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 0$$

$$\frac{x-2}{2} = \frac{y+2}{3}$$
$$3x-6 = 2y+4$$
$$2y = 3x-10$$
$$y = \frac{3x}{2} - 5$$

or

$$\frac{-x+2}{2} = \frac{y+2}{3}$$
$$-3x+6 = 2y+4$$
$$2y = -3x+2$$
$$y = \frac{-3x}{2} + 1$$



Curved Surfaces

3D Sphere: (with center) at the origin of radius R

$$x^2 + y^2 + z^2 = R^2$$

3D Ellipsoid: (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

3 points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$

We can replace x^2+y^2 with r^2 to indicate the rotation.

Ex② - Sketch the following surfaces:

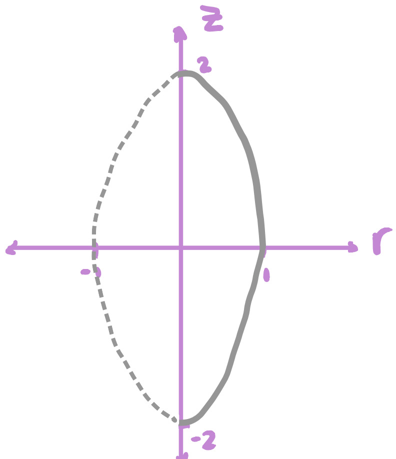
a) $x^2+y^2+\frac{z^2}{4} = 1$

$$r^2 + \frac{z^2}{4} = 1$$
$$4r^2 + z^2 = 4$$
$$r^2 + \frac{z^2}{4} = 1$$

Center at $(0,0)$

r-int : $(\pm 1, 0)$
z-int : $(0, \pm 2)$

* Use the ellipsoid formula

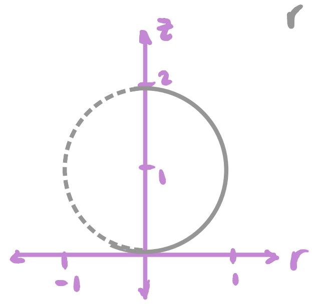


$r \geq 0$ because $r = \sqrt{x^2+y^2}$

b) $x^2+y^2+z^2=2z$

$r^2+z^2-2z=0$
$r^2+z^2-2z+1-1=0$
$r^2+(z-1)^2=1 \rightarrow$ circle!

Center $(0,1)$



$r \geq 0$ because $r=\sqrt{x^2+y^2}$

Dot and Cross Product

Let $\vec{u}\cdot\vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix} = u_1v_1 + u_2v_2 + \ldots$

Let $\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$
$= (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$

You can find the angle between two vectors using dot product:

Let the angle be called θ, then $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

Ex ③ - Find the angle between $\vec{u}=(1,-3,1)$ and $\vec{v}=(2,1,2)$ in $\mathbb{R}^3$

First, we need the dot product and lengths:

$\vec{u}\cdot\vec{v} = (1,-3,1)(2,1,2) = 2-3+2 = 1$

$\|\vec{u}\| = \sqrt{1^2+(-3)^2+1^2} = \sqrt{11}$

$\|\vec{v}\| = \sqrt{2^2+1^2+2^2} = \sqrt{9} = 3$

$\Rightarrow \cos\theta = \dfrac{1}{3\sqrt{11}} \quad \therefore \quad \theta = \cos^{-1}\left(\dfrac{1}{3\sqrt{11}}\right)$

Line and Planes

Vector equation of a line in $\mathbb{R}^3$: $\vec{\ell} = (a_1, a_2, a_3) + t[v_1, v_2, v_3]$
$\vec{\ell} = a + t\vec{v}$

Parametric equation of a line in $\mathbb{R}^3$:
$x = a_1 + tv_1$
$y = a_2 + tv_2$
$z = a_3 + tv_3$

Ex ④ - Find the equation of the line or plane

a) The line through $(1,-1,2)$ and $(3,1,9)$

The direction vector for the line is $(3,1,9)-(1,-1,2)=(2,2,7)$

$\rightarrow$ V. eq. $\vec{l}=(1,-1,2)+t(2,2,7)$, $t\in\mathbb{R}$

$\rightarrow$ P. eq. $x=1+2t$, $y=-1+2t$, $z=2+7t$, $t\in\mathbb{R}$

b) The plane through $(1,-3,1)$, $(2,1,1)$, $(1,4,0)$

Let's find a pair of directional vectors $\vec{v_1}$ and $\vec{v_2}$

$\vec{v_1}=(2,1,1)-(1,-3,1)=(1,4,0)$

$\vec{v_2}=(1,4,0)-(1,-3,1)=(0,7,-1)$

To get the equation of the plane, we need to find the normal which is $\vec{v_1}\times\vec{v_2}$

$$\vec{n}=\vec{v_1}\times\vec{v_2}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 4 & 0\\ 0 & 7 & -1\end{vmatrix}=(-4,1,7)$$

The plane then is $-4x+y+7z=d$
Let's plug a point: $-4(2)+(1)+7(1)=-8+1+7=0=d$

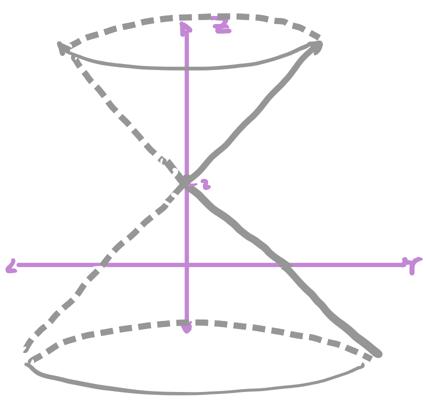
$\rightarrow$ The plane is $-4x+y+7z=0$

Level Set

Ex ⑤ - Draw the level set of $f(x,y,z)=z^2-x^2-y^2-4z$ for $f=-4$

↳ this is in 4D though

$-4=z^2-x^2-y^2-4z$
$x^2+y^2=z^2-4z+4$
$r^2=(z-2)^2$
$r=\pm(z-2)$



$r\geq 0$

Week 3 (Sep 28th)

Limits and Continuity

$f(x,y)$ is continuous at the 2D point $\vec{a}$ if $\lim f(x,y)=f(\vec{a})$

$(x,y) \to \vec{a}$

It's not too hard to show when a limit DNE in $\mathbb{R}^2$ at $(0,0)$: Try different curves in terms of x or/and y, if they approach to different values at $(0,0)$

To show that the limit exists we can use the Squeeze Theorem:

To attain $\lim\limits_{(x,y)\to\vec{a}} f(x,y)$, can try to find $g(x,y)$ and $h(x,y)$ so that:

1. $g(x,y) \le f(x,y) \le h(x,y)$ near the point $\vec{a}$
2. $\lim\limits_{(x,y)\to\vec{a}} g(x,y) = L = \lim\limits_{(x,y)\to\vec{a}} h(x,y)$

Then we conclude $\lim\limits_{(x,y)\to\vec{a}} f(x,y) = L$

Ex 2 - Decide whether the function has a limit at $(0,0)$

a) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \text{DNE}$

Restrict to $x=0$, $\lim\limits_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1$

Restrict to $y=0$, $\lim\limits_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$

b) $\lim\limits_{(x,y)\to(0,0)} \frac{|x|}{\sqrt{x^2+y^2}} = \text{DNE}$

Restrict to $y=0$, $\lim\limits_{(x,0)\to(0,0)} \frac{|x|}{\sqrt{x^2}} = 1$

Restrict to $x=0$, $\lim\limits_{(0,y\to 0,0)} \frac{|x|}{\sqrt{y}} = \frac{0}{\sqrt{y}} = 0$

c) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \underset{1}{\frac{x^3}{x^2+y^2}} - \underset{2}{\frac{y^3}{x^2+y^2}} = 0$

1 $x^2 \le x^2+y^2 \rightarrow 0 \le \frac{x^2}{x^2+y^2} \le 1$

$\rightarrow 0\cdot x \le \frac{x^3}{x^2+y^2} \le x$

$$\rightarrow \lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} \leq \lim_{(x,y)\to(0,0)} x$$

$$\rightarrow 0 \leq \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} \leq 0$$

$$\Rightarrow 0$$

② Same as ①

$$y^2 \leq x^2+y^2 \rightarrow 0 \leq \frac{y^2}{x^2+y^2} \leq 1$$

$$\rightarrow 0. \leq \frac{y^3}{x^2+y^2} \leq$$

$$\rightarrow \lim_{(x,y)\to(0,0)} y \cdot 0 \leq \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} \leq \lim_{(x,y)\to(0,0)} y$$

$$\rightarrow 0 \leq \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} \leq 0$$

$$\Rightarrow 0$$

d) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

We know $|\sin\theta| \leq 1$ and we saw in class $|\sin\theta| \leq \theta$

$$\Rightarrow \frac{|\sin(x^2+y^2)|}{x^2+y^2} \leq \frac{x^2+y^2}{x^2+y^2} = 1$$

Recall $\lim_{t\to 0} \frac{\sin t}{t} = 1$, what if we let $t = x^2+y^2$

$$\Rightarrow \lim_{t\to 0} \frac{\sin t}{t} = 1 = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

Ex③ - Find if the limit of f exists at $(0,0)$, if it does, find $f(0,0)$ for f to be continuous.

$$f(x,y) = \frac{x^3-x^2-2x^2y+xy^2-y^2-2y^3}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2(x-1-2y)+y^2(x-1-2y)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x-1-2y)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} x-1-2y$$

$$= -1 \qquad \Rightarrow f(0,0) = -1$$

$f(\vec{x})$ is homogeneous of degree k if for every $\vec{x} \in \mathbb{R}^n$ and every scalar $c>0$ we have:

$$f(c\vec{x}) = c^k f(\vec{x})$$

Ex ④ - Show if f is homogeneous

a) $f(x,y) = 8x^2y^2 - 9x^4$

$$\begin{aligned} f(cx, cy) &= 8(cx)^2(cy)^2 - 9(cx)^4 \\ &= 8c^2x^2c^2y^2 - 9c^4x^4 \\ &= 8c^4x^2y^2 - 9c^4x^4 \\ &= c^4(8x^2y^2 - 9x^4) \\ &= c^4 f(x,y) \end{aligned}$$

$\Rightarrow f$ is homogeneous of degree 4.

b) $f(x,y) = \dfrac{x^2}{y^2} + xy + \dfrac{y^2}{x^2}$

$$\begin{aligned} f(cx, cy) &= \frac{(cx)^2}{(cy)^2} + (cx)(cy) + \frac{(cy)^2}{(cx)^2} \\ &= \frac{c^2x^2}{c^2y^2} + c^2xy + \frac{c^2y^2}{c^2x^2} \\ &= \frac{x^2}{y^2} + c^2xy + \frac{y^2}{x^2} \end{aligned}$$ $\Rightarrow$ cannot factor c.

$\Rightarrow f$ is not homogeneous.

Week 4 (Oct. 5th)

Partial derivatives

Ex ① - Find the partial derivatives of $f(x,y) = y\sin(xy) + xe^{-y^2}$

$$\frac{\partial f}{\partial x} = y^2\cos(xy) + e^{-y^2}$$

$$\frac{\partial f}{\partial y} = \sin(xy) + xy\cos(xy) - 2xye^{-y^2}$$

A function, whose partial derivatives exist and are continuous, is said to be of class C^1

Directional Derivative

The directional derivative towards the direction $\vec{u}$ at the point $\vec{a}$:

$$D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$$ where $\vec{u}$ is a unit vector.

If f is differentiable, then all directional derivatives exist and

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$ where $\vec{u}$ is a unit vector.

* If f is C' at $\vec{a}$, then f is differentiable at $\vec{a}$ (∇f exists)

Ex ② - At the point $(3,1,2)$, find the directional derivative of $f(x,y,z) = xy^3z^2$ along the vector $\vec{u} = (1,3,4)$

f is C' → it's partial derivatives exist and are continuous.

$\nabla f(x,y,z) = (y^3z^2, 3xy^2z^2, 2xy^3z)$

$\nabla f(3,1,2) = (4, 36, 12)$

$$\Rightarrow D_{\vec{u}} f(p) = \frac{(4,36,12)(1,3,4)}{\sqrt{1+9+16}} = \frac{4+108+48}{\sqrt{26}} = \frac{160}{\sqrt{26}}$$

Tangent Planes

A tangent plane is given by $\nabla g(a,b,c)((x,y,z)-(a,b,c))=0$

Ex ③ - Compute an equation for the tangent plane at the point p to the graph of the function $z=f(x,y)$

$p = (1,1,1)$ and $xy+yz+zx=3$

$g(x,y,z) = xy+yz+zx-3$

$\nabla g(x,y,z) = (y+z, x+z, y+x)$

$\nabla g(1,1,1) = (2,2,2)$

$$\Rightarrow (2,2,2)((x,y,z)-(1,1,1)) = 0$$
$$(2,2,2)(x-1, y-1, z-1) = 0$$
$$2x-2+2y-2+2z-2 = 0$$
$$x+y+z = 3$$

Differentiation

f is differentiable if the gradient vector $\nabla f(\vec{a})$ satisfies

$$\lim_{\vec{h} \to 0} \frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - \nabla f(\vec{a}) \cdot \vec{h}}{|\vec{h}|}$$

Ex ④ - Find all directional derivatives and show whether the function is differentiable at $(0,0)$

$$f(x,y) = \begin{cases} \dfrac{3x^2y+5xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$\vec{a} = (0,0)$
$\vec{u} = (a,b)$

$$D_{\hat{u}}f = \lim_{h\to 0} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$$

$$= \lim_{h\to 0} \frac{f(ha,hb) - 0}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{3(ha)^2(hb) + 5(ha)(hb)^2}{(ha)^2+(hb)^2}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{h^3(3a^2b+5ab^2)}{h^2(a^2+b^2)}$$

$$= \lim_{h\to 0} \frac{3a^2b+5ab^2}{a^2+b^2}$$

$\rightarrow \vec{u} = (1,0)$, $D_{\hat{u}}f = \dfrac{3(1)(0)+5(1)(0)}{(1)^2+(0)^2} = 0 \rightarrow \dfrac{\partial f}{\partial x} = 0$

$\rightarrow \vec{u} = (0,1)$, $D_{\hat{u}}f = \dfrac{3(0)(1)+5(0)(1)}{(0)^2+(1)^2} = 0 \rightarrow \dfrac{\partial f}{\partial y} = 0$

$$\nabla f(\vec{a}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (0,0)$$

Now, let's apply the definition of differentiability

$$\lim_{\vec{h}\to 0} \frac{f(\vec{a}+\vec{h}) - \overset{0}{f(\vec{a})} - \overset{0}{\nabla f(\vec{a})\cdot\vec{h}}}{|\vec{h}|} = \lim_{\vec{h}\to 0} \frac{f(\vec{h})}{|\vec{h}|} \quad \text{, let } \vec{h} = (h_1,h_2)$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{1}{\sqrt{h_1^2+h_2^2}} \cdot \frac{3h_1^2h_2 + 5h_1h_2^2}{h_1^2+h_2^2}$$

Restrict $h_1 = 0$

$$\lim_{h_2\to 0} \frac{1}{\sqrt{h_2^2}} \cdot \frac{3(0)h_2 + 5(0)h_2^2}{h_2^2} = 0$$

Restrict $h_1 = h_2$

$$\lim_{h_2\to 0} \frac{1}{\sqrt{2h_2^2}} \cdot \frac{3h_2^3 + 5h_2^3}{2h_2^2} = \lim_{h_2\to 0} \frac{8h_2^3}{2\sqrt{2}\, h_2^3} = \frac{4}{\sqrt{2}}$$

$\therefore f$ is <u>not</u> differentiable at $(0,0)$