MATB44 - Tutorial 6 (IC 320 Fridays 10-11am)

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Week 1 - Sept. 6th

No tutorials

Week 2 - Sept. 13th

ODE: $F(t, x, x^{(1)}, \dots, x^{(k)}) = 0$

t - independent variable (time)
x - dependent variable (space)
x is a function of t i.e. $x(t)$

The order is the highest derivative that appears in F.

The system is linear if we can rewrite it to separate out the $x_\ell^{(j)}$'s as follows:

$$x_i^{(k)} = g_i(t) + \sum_{\ell=1}^{n} \sum_{j=1}^{k-1} f_{i,j,\ell}(t)\, x_\ell^{(j)}$$

Ex: $\underbrace{(4+t^2)x' + 2tx} = 4t$

By product rule: $\frac{d}{dt}\left[(4+t^2)x\right] = 2tx + (4+t^2)x'$

$\Rightarrow \frac{d}{dt}\left[(4+t^2)x\right] = 4t$

$\Rightarrow (4+t^2)x = 2t^2 + C$

$\Rightarrow x(t) = \frac{2t^2 + C}{4+t^2}$

The system is homogeneous if $g_i(t) = 0$

If there is no direct dependance on t, the system is autonomous. Form $x' = f(x)$

Ex: Classify the following equations:

* $x' + \frac{1}{2}x = \frac{1}{2}e^{t/3}$ 1st order, linear

* $3 + xx' = t - x$ 1st order, non linear

* $x''+5x'+6x=0$ — 2nd order, linear, homogeneous, autonomous
* $x'=t\sin x$ — 1st order, non linear
* $x''-x=0$ — 2nd order, linear, homogeneous, autonomous
* $tx'+2x=4t^2$ — 1st order, linear
* $x'-2x+t=4$ — 1st order, linear.

Ex: Solve $x'=x^2$ with $x_0=x(0)>0$

① $\frac{dx}{dt}=x^2 \Rightarrow \frac{dx}{x^2}=dt$

$$\Rightarrow \int \frac{dx}{x^2}=\int dt$$

$$\Rightarrow -\frac{1}{x}=t+C_1$$

$$\Rightarrow x=\frac{1}{-C_1-t}$$

When $x(0)=-\frac{1}{C_1}=x_0$ so $x=\dfrac{1}{\frac{1}{x_0}-t}$

② $f(x)=x^2$, a maximal nonzero interval is $(x_1,x_2)=(0,\infty)$

$$F(x)=\int_{x_0}^{x}\frac{dy}{y^2}=-\frac{1}{y}\Big|_{y=x_0}^{y=x}=-\frac{1}{x}+\frac{1}{x_0}$$

$$\phi(t)=F^{-1}(t)$$

$T_+=\lim_{x\to x_2}F(x)=\lim_{x\to\infty}-\frac{1}{x}+\frac{1}{x_0}=\frac{1}{x_0}$, so ϕ is defined for all $t>0$

$$t=F(F^{-1}(t))=\frac{-1}{F^{-1}(t)}+\frac{1}{x_0}$$

$$\Rightarrow \frac{1}{F^{-1}(t)}=\frac{1}{x_0}-t$$

$$\Rightarrow F^{-1}(t)=\frac{1}{\frac{1}{x_0}-t}$$

$$\Rightarrow \phi(t)=\frac{1}{\frac{1}{x_0}-t}$$

$\blacksquare$

Week 3 - Sept. 20th

Office Hours: Wed. 10-11 am IC404

A particular solution is a solution that has no arbitrary constant. A general solution is a k-parameter family of solutions that contain every particular solution

A separable first-order ODE is an ODE that can be rewritten as:

$$\boxed{\dot{x}(t,x) = g(t)f(x)} \quad (\text{or } g(t)dt + f(x)dx = 0)$$

$$\Rightarrow \quad \frac{dx}{dt} = g(t)f(x)$$

$$\Rightarrow \quad \frac{dx}{f(x)} = g(t)\,dt$$

$$\Rightarrow \quad \boxed{\int \frac{dx}{f(x)} = \int g(t)\,dt}, \quad f(x) \neq 0$$

Ex: Show that the equation $\dot{x}(t,x) = \frac{t^2}{1-x^2}$ is separable and solve.

Let $f(x) = \frac{1}{1-x^2}$ and $g(t) = t^2$

$$\Rightarrow \int 1-x^2\,dx = \int t^2\,dt$$

$$x - \frac{x^3}{3} = \frac{t^3}{3} + C_1$$

$$3x - x^3 - t^3 = C, \quad C \in \mathbb{R}$$

Ex: Solve the equation $\frac{dx}{dt} = \frac{4t-t^3}{4+x^3}$ and find the solution passing through the point $(0,1)$

$$\Rightarrow (4+x^3)\,dx = (4t-t^3)\,dt$$

$$\int (4+x^3)\,dx = \int (4t-t^3)\,dt$$

$$4x + \frac{x^4}{4} = 2t^2 - \frac{t^4}{4} + C_1$$

$$\Rightarrow 16x + x^4 - 8t^2 + t^4 = C, \quad C \in \mathbb{R},$$

When $(0,1) \to 16+1 = 17$

$$\therefore 16x + x^4 - 8t^2 + t^4 = 17$$

Let $z=f(x,y)$ be a function of x and y. $f(x,y)$ is homogeneous of order n if it can be written as $\boxed{f(x,y)=x^n g(u)}$, where $u=\frac{y}{x}$ or $f(x,y)=y^n g(u)$, where $u=\frac{x}{y}$.

$\boxed{P(x,y)dx+Q(x,y)dy=0}$ where $P(x,y)$ and $Q(x,y)$ are the homogeneous coefficients and can be solved by substituting $y=ux$, $dy=udx+xdu$

Ex: Find the general solution of $txx'=t^2+2x^2$

$$\Rightarrow \frac{1}{t^2}\left[txx'=t^2+2x^2\right]$$

$$\frac{x}{t}x'=1+2\frac{x^2}{t^2} \qquad \text{Let } u=\frac{x}{t} \text{ so } x=ut,\ x'=u+tu'$$

$$\Rightarrow u(u+tu')=1+2u^2$$
$$u^2+tu'u=1+2u^2$$
$$tu'u=1+u^2$$
$$\frac{u'u}{1+u^2}=\frac{1}{t} \quad \rightarrow \text{OMG! This is separable!}$$

$$\int\frac{uu'}{1+u^2}=\int\frac{1}{t}dt \qquad \text{let } v=1+u^2,\ dv=2udu$$

$$\frac{1}{2}\int\frac{dv}{v}=\ln|t|+C_1$$
$$\frac{1}{2}\ln|1+u^2|=\ln|t|+C$$
$$\therefore \frac{1}{2}\ln\left|1+\frac{x^2}{t^2}\right|-\ln|t|=C$$

A differential expression $\boxed{P(x,y)dx+Q(x,y)dy}$ is called an exact differential if it is the total differential of some function $f(x,y)$.
i.e. if $\boxed{P(x,y)=\frac{\partial}{\partial x}f(x,y)}$ and $\boxed{Q(x,y)=\frac{\partial}{\partial y}f(x,y)}$.
If we can find $f(x,y)$, $f(x,y)=C$ is the 1-parameter family of solutions.

$P(x,y)dx+Q(x,y)dy=0$ is exact iff $\frac{\partial P(x,y)}{\partial y}=\frac{\partial Q(x,y)}{\partial x}$.

Ex: Find the general solution of $y'=-\frac{1+2xy^2}{1+2x^2y}$

$$\Rightarrow (1+2x^2y)y'=-(1+2xy^2)$$
$$(1+2xy^2)+(1+2x^2y)y'=0$$

Is it exact? $\quad P(x,y)=1+2xy^2 \qquad Q(x,y)=1+2x^2y$

$$\frac{\partial P(x,y)}{\partial y}=4xy \iff \frac{\partial Q(x,y)}{\partial x}=4xy \quad \checkmark$$

let's take $P(x,y)$, we know $P(x,y) = \frac{\partial}{\partial x} f(x,y)$

$\Rightarrow \int P(x,y)\,dx = f(x,y)$

$\int 1+2xy^2\,dx = x + x^2y^2 + g(y) = f(x,y)$

$\Rightarrow \frac{\partial}{\partial y} f(x,y) = 2x^2y + g'(y) = Q(x,y) = 1 + 2x^2y$

So $g'(y) = 1 \rightarrow g(y) = y$

$\therefore x + x^2y^2 + y = C$