

# MATB41 - Multivariable Calculus

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## Week 1 (May 11)

\* No tutorials \*

## Week 2 (May 18)

### Curved Lines

Parabola: (with vertex) at the origin

- $y = ax^2$  → open up when  $a > 0$  and down when  $a < 0$
- $x = by^2$  → open right when  $b > 0$  and left when  $b < 0$

\* Remember: We can replace  $x$  with  $x-m$  to shift right and  $y$  with  $y+n$  to shift up!

Ellipse: (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \text{intersections } (\pm a, 0) \text{ and } (0, \pm b)$$

When  $a=b$ , the curve is a circle of radius  $a=b$ .

Hyperbola: (with center) at the origin

- $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$  It has two curves that goes in opposite directions and do not touch the slant asymptote
- $\left(\frac{y}{a}\right)^2 - \left(\frac{x}{b}\right)^2 = 1$

Ex ① - Sketch the following curves:

a)  $x^2 + 3y^2 + 2x - 12y + 10 = 0$

$$\begin{aligned} (x^2 + 2x) + 3(y^2 - 4y) + 10 &= 0 \\ (x^2 + 2x + 1) - 1 + 3(y^2 - 4y + 4) - 12 + 10 &= 0 \\ (x+1)^2 + 3(y-2)^2 &= 3 \end{aligned}$$

$$\Rightarrow \frac{(x+1)^2}{3} + (y-2)^2 = 1 \Rightarrow \left(\frac{x+1}{\sqrt{3}}\right)^2 + (y-2)^2 = 1$$

Okie, now what?! Another way to write the ellipse equation would be:

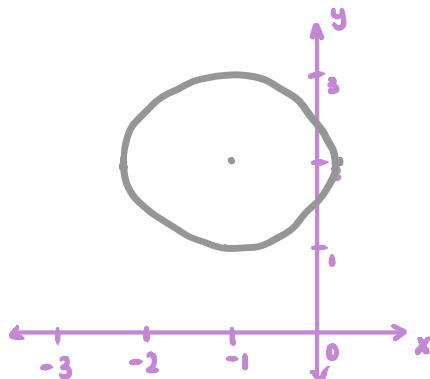
$$\left(\frac{x-m}{a}\right)^2 + \left(\frac{y-n}{b}\right)^2 = 1 \rightarrow \text{we just combined it with *}$$

So the new origin will be  $(-1, 2)$  as the ellipse will be shifted left by one and up by 2.

Now, let's find the end points

→ The horizontal max/min will be  $(-1 \pm \sqrt{3}, 2)$

→ The vertical max/min will be  $(-1, 2 \pm 1)$



b)  $\left(\frac{x-2}{4}\right)^2 - \left(\frac{y+2}{9}\right)^2 = 1$

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 1$$

The origin will be  $(2, -2)$ , we are shifting right 2 and 2 down.

Check the x-int :  $(y=0)$

$$\left(\frac{x-2}{2}\right)^2 - \frac{4}{9} = 1$$

$$\left(\frac{x-2}{2}\right)^2 = \frac{13}{9}$$

$$(x-2)^2 = \frac{52}{9}$$

$$x-2 = \pm \frac{2\sqrt{13}}{3}$$

$$x = 2 \pm \frac{2\sqrt{13}}{3}$$

Check the y-int :  $(x=0)$

$$1 - \left(\frac{y+2}{3}\right)^2 = 1$$

$$y = -2$$

Find the slant asymptote:

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 0$$

$$\frac{x-2}{2} = \frac{y+2}{3}$$

$$3x-6 = 2y+4$$

$$2y = 3x-10$$

$$y = \frac{3x-10}{2}$$

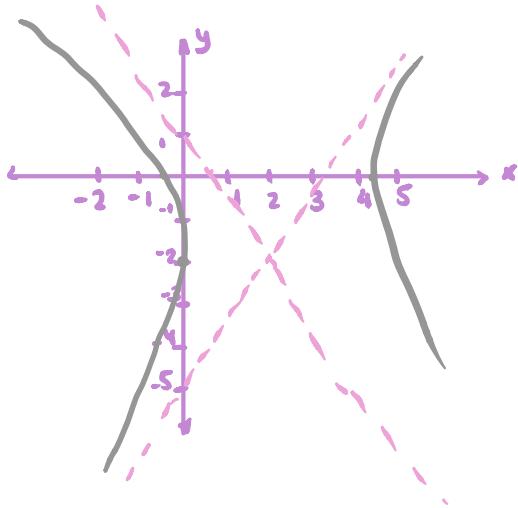
or

$$\frac{-x+2}{2} = \frac{y+2}{3}$$

$$-3x+6 = 2y+4$$

$$2y = -3x+2$$

$$y = \frac{-3x+2}{2}$$



## Curved Surfaces

**3D Sphere:** (with center) at the origin of radius  $R$

$$x^2 + y^2 + z^2 = R^2$$

**3D Ellipsoid:** (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

3 points  $(a, 0, 0)$ ,  $(0, b, 0)$ ,  $(0, 0, c)$

We can replace  $x^2 + y^2$  with  $r^2$  to indicate the rotation.

**Ex② - Sketch the following surfaces:**

a)  $x^2 + y^2 + \frac{z^2}{4} = 1$

$$r^2 + \frac{z^2}{4} = 1$$

$$4r^2 + z^2 = 4$$

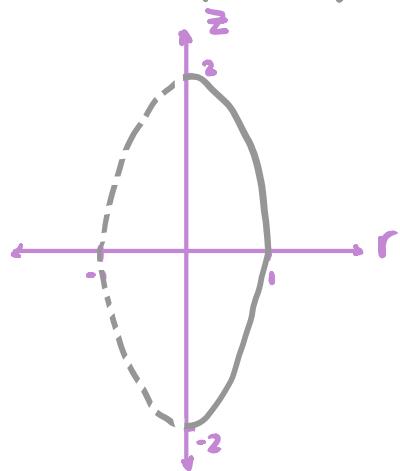
$$r^2 + \frac{z^2}{4} = 1$$

(center at  $(0, 0)$ )

$r$ -int:  $(\pm 1, 0)$

$z$ -int:  $(0, \pm 2)$

\* Use the ellipsoid formula



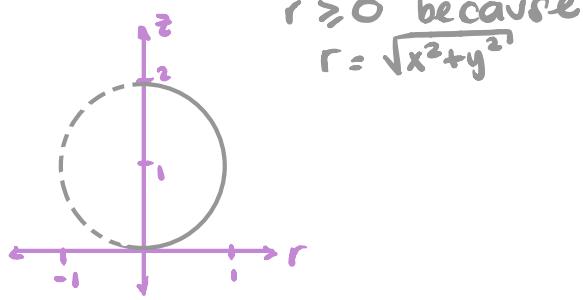
$r \geq 0$  because  
 $r = \sqrt{x^2 + y^2}$

b)  $x^2 + y^2 + z^2 = 2z$

$$\begin{aligned} r^2 + z^2 - 2z &= 0 \\ r^2 + z^2 - 2z + 1 - 1 &= 0 \\ r^2 + (z-1)^2 &= 1 \end{aligned}$$

$\rightarrow$  circle!

(center  $(0, 1)$ )



$r \geq 0$  because  
 $r = \sqrt{x^2 + y^2}$

## Determinant

When  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ,  $|A| = ad - bc$

When  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ ,  $|A| = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Ex③- Find the determinant of  $\begin{bmatrix} -3 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 4 & 3 \end{bmatrix}$

$$\begin{vmatrix} -3 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 4 & 3 \end{vmatrix} = -3 \begin{vmatrix} -1 & 2 \\ 4 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix}$$

$$= -3(-3 - 8) - (6 - 2) + (8 + 1)$$

$$= -3 \times -11 - 4 + 9$$

$$= 33 + 5$$

$$= 38$$

## Dot and Cross Product

let  $\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix} = u_1v_1 + u_2v_2 + \dots$

let  $\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left( \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$

$$= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

You can find the angle between two vectors using dot product:

Let the angle be called  $\theta$ , then  $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$

Ex④- Find the angle between  $\vec{u} = (1, -3, 1)$  and  $\vec{v} = (2, 1, 2)$  in  $\mathbb{R}^3$

First, we need the dot product and lengths:

$$\vec{u} \cdot \vec{v} = (1, -3, 1) \cdot (2, 1, 2) = 2 - 3 + 2 = 1$$

$$\|\vec{u}\| = \sqrt{1^2 + (-3)^2 + 1^2} = \sqrt{11}$$

$$\|\vec{v}\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

$$\Rightarrow \cos \theta = \frac{1}{3\sqrt{11}} \quad \therefore \quad \theta = \cos^{-1}\left(\frac{1}{3\sqrt{11}}\right)$$

## Line and Planes

Vector equation of a line in  $\mathbb{R}^3$ :  $\vec{l} = (a_1, a_2, a_3) + t[v_1, v_2, v_3]$   
 $\vec{l} = \vec{a} + t\vec{v}$

Parametric equation of a line in  $\mathbb{R}^3$ :  $x = a_1 + t v_1$ ,  
 $y = a_2 + t v_2$ ,  
 $z = a_3 + t v_3$

Ex ⑤ - Find the equation of the line or plane

a) The line through  $(1, -1, 2)$  and  $(3, 1, 9)$

The direction vector for the line is  $(3, 1, 9) - (1, -1, 2) = (2, 2, 7)$

$$\rightarrow \text{V. eq. } \vec{l} = (1, -1, 2) + t(2, 2, 7), \quad t \in \mathbb{R}$$

$$\rightarrow \text{P. eq. } x = 1 + 2t, \quad y = -1 + 2t, \quad z = 2 + 7t, \quad t \in \mathbb{R}$$

b) The plane through  $(1, -3, 1)$ ,  $(2, 1, 1)$ ,  $(1, 4, 0)$

let's find a pair of directional vectors  $\vec{v}_1$  and  $\vec{v}_2$

$$\vec{v}_1 = (2, 1, 1) - (1, -3, 1) = (1, 4, 0)$$

$$\vec{v}_2 = (1, 4, 0) - (1, -3, 1) = (0, 7, -1)$$

To get the equation of the plane, we need to find the normal which is  $\vec{v}_1 \times \vec{v}_2$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 0 \\ 0 & 7 & -1 \end{vmatrix} = (-4, 1, 7)$$

The plane then is  $-4x + y + 7z = d$

$$\text{Let's plug a point: } -4(2) + (1) + 7(1) = -8 + 1 + 7 = 0 = d$$

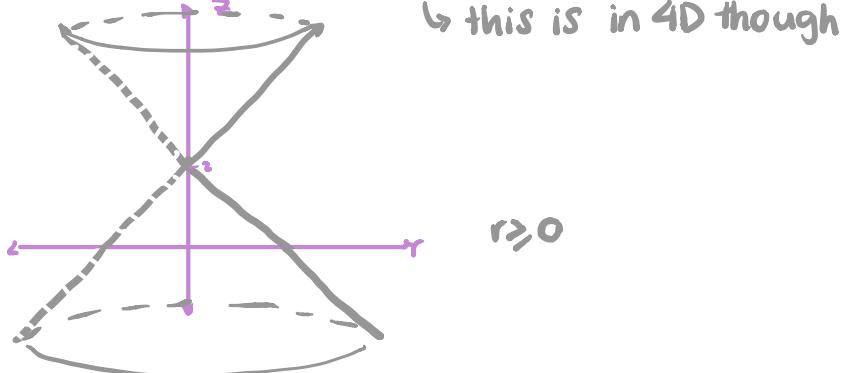
$\rightarrow$  The plane is  $-4x + y + 7z = 0$

## Week 3 (May 25)

### Level Set

Ex ① - Draw the level set of  $f(x,y,z) = z^2 - x^2 - y^2 - 4z$  for  $f = -4$

$$\begin{aligned} -4 &= z^2 - x^2 - y^2 - 4z \\ x^2 + y^2 &= z^2 - 4z + 4 \\ r^2 &= (z-2)^2 \\ r &= \sqrt{(z-2)^2} \end{aligned}$$



### Limits and Continuity

$f(x,y)$  is continuous at the 2D point  $\vec{a}$  if  $\lim_{(x,y) \rightarrow \vec{a}} f(x,y) = f(\vec{a})$

It's not too hard to show when a limit DNE in  $\mathbb{R}^2$  at  $(0,0)$ :  
Try different curves in terms of  $x$  or  $y$ , if they approach to different values at  $(0,0)$

To show that the limit exists we can use the Squeeze Theorem:

To attain  $\lim_{(x,y) \rightarrow \vec{a}} f(x,y)$ , can try to find  $g(x,y)$  and  $h(x,y)$  so that:

1.  $g(x,y) \leq f(x,y) \leq h(x,y)$  near the point  $\vec{a}$

2.  $\lim_{(x,y) \rightarrow \vec{a}} g(x,y) = L = \lim_{(x,y) \rightarrow \vec{a}} h(x,y)$

Then we conclude  $\lim_{(x,y) \rightarrow \vec{a}} f(x,y) = L$

Ex ② - Decide whether the function has a limit at  $(0,0)$

a)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \text{DNE}$

Restrict to  $x=0$ ,  $\lim_{(0,y) \rightarrow (0,0)} \frac{-y^2}{y^2} = -1$

Restrict to  $y=0$ ,  $\lim_{(x,0) \rightarrow (0,0)} \frac{x^2}{x^2} = 1$

b)  $\lim_{(x,y) \rightarrow (0,0)} \frac{|x|}{\sqrt{x^2 + y^2}} = \text{DNE}$

Restrict to  $y=0$ ,  $\lim_{(x,0) \rightarrow (0,0)} \frac{|x|}{\sqrt{x^2}} = 1$

Restrict to  $x=0$ ,  $(0,y \rightarrow 0,0) \lim_{y \rightarrow 0} \frac{|x|}{\sqrt{y}} = \frac{0}{\sqrt{y}} = 0$

c)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2} - \frac{y^3}{x^2 + y^2} = 0$

$$\textcircled{1} \quad x^2 \leq x^2 + y^2 \rightarrow 0 \leq \frac{x^2}{x^2 + y^2} \leq 1$$

$$\rightarrow 0 \cdot x \leq \frac{x^3}{x^2 + y^2} \leq x$$

$$\rightarrow \lim_{(x,y) \rightarrow (0,0)} 0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2} \leq \lim_{(x,y) \rightarrow (0,0)} x$$

$$\rightarrow 0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2} \leq 0 \\ \Rightarrow 0$$

② Same as ①

$$y^2 \leq x^2 + y^2 \rightarrow 0 \leq \frac{y^2}{x^2 + y^2} \leq 1$$

$$\rightarrow 0 \cdot \frac{y^3}{x^2 + y^2} \leq$$

$$\rightarrow \lim_{(x,y) \rightarrow (0,0)} 0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{y^3}{x^2 + y^2} \leq \lim_{(x,y) \rightarrow (0,0)} y$$

$$\rightarrow 0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{y^3}{x^2 + y^2} \leq 0 \\ \Rightarrow 0$$

d)  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$

We know  $|\sin \theta| \leq 1$  and we saw in class  $|\sin \theta| \leq \theta$

$$\Rightarrow |\sin(x^2 + y^2)| \leq \frac{x^2 + y^2}{x^2 + y^2} = 1$$

Recall  $\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$ , what if we let  $t = x^2 + y^2$

$$\Rightarrow \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1 = \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

**Ex③ - Find if the limit of  $f$  exists at  $(0,0)$ , if it does, find  $f(0,0)$  for  $f$  to be continuous.**

$$f(x,y) = \frac{x^3 - x^2 - 2x^2y + xy^2 - y^2 - 2y^3}{x^2+y^2}$$

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{x^2(x-1-2y) + y^2(x-1-2y)}{x^2+y^2} &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2+y^2)(x-1-2y)}{x^2+y^2} \\ &= \lim_{(x,y) \rightarrow (0,0)} x-1-2y \\ &= -1 \\ \Rightarrow f(0,0) &= -1 \end{aligned}$$

$f(\vec{x})$  is homogeneous of degree  $k$  if for every  $\vec{x} \in \mathbb{R}^n$  and every scalar  $c > 0$  we have:

$$f(cx) = c^k f(\vec{x})$$

**Ex④ - Show if  $f$  is homogeneous**

a)  $f(x,y) = 8x^2y^2 - 9x^4$

$$\begin{aligned} f(cx, cy) &= 8(cx)^2(cy)^2 - 9(cx)^4 \\ &= 8c^2x^2c^2y^2 - 9c^4x^4 \\ &= 8c^4x^2y^2 - 9c^4x^4 \\ &= c^4(8x^2y^2 - 9x^4) \\ &= c^4 f(x,y) \end{aligned}$$

$\Rightarrow f$  is homogeneous of degree 4.

b)  $f(x,y) = \frac{x^2}{y^2} + xy + \frac{y^2}{x^2}$

$$\begin{aligned} f(cx, cy) &= \frac{(cx)^2}{(cy)^2} + (cx)(cy) + \frac{(cy)^2}{(cx)^2} \\ &= \frac{c^2x^2}{c^2y^2} + c^2xy + \frac{c^2y^2}{c^2x^2} \\ &= \frac{x^2}{y^2} + c^2xy + \frac{y^2}{x^2} \Rightarrow \text{cannot factor } c. \end{aligned}$$

$\Rightarrow f$  is not homogeneous.

## Week 4 (June 1st)

### Partial derivatives

Ex ① - Find the partial derivatives of  $f(x,y) = y \sin(xy) + x e^{-y^2}$

$$\frac{\partial f}{\partial x} = y^2 \cos(xy) + e^{-y^2}$$

$$\frac{\partial f}{\partial y} = \sin(xy) + xy \cos(xy) - 2xye^{-y^2}$$

A function, whose partial derivatives exist and are continuous, is said to be of class C'

### Directional Derivative

The directional derivative towards the direction  $\vec{u}$  at the point  $\vec{a}$ :

$$D_{\vec{u}} f(\vec{a}) = \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h} \text{ where } \vec{u} \text{ is a unit vector.}$$

If  $f$  is differentiable, then all directional derivatives exist and

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} \text{ where } \vec{u} \text{ is a unit vector.}$$

\* If  $f$  is C' at  $\vec{a}$ , then  $f$  is differentiable at  $\vec{a}$  ( $\nabla f$  exists)

Ex ② - At the point (3, 1, 2), find the directional derivative of  $f(x,y,z) = xy^3z^2$  along the vector  $\vec{u} = (1, 3, 4)$

$f$  is C'  $\rightarrow$  it's partial derivatives exist and are continuous.

$$\nabla f(x,y,z) = (y^3z^2, 3xy^2z^2, 2xyz^3)$$

$$\nabla f(3, 1, 2) = (4, 36, 12)$$

$$\Rightarrow D_{\vec{u}} f(p) = \frac{(4, 36, 12)(1, 3, 4)}{\sqrt{1+9+16}} = \frac{4+108+48}{\sqrt{26}} = \frac{160}{\sqrt{26}}$$

### Tangent Planes

A tangent plane is given by  $\nabla g(a,b,c) \cdot ((x,y,z) - (a,b,c)) = 0$

Ex ③ - Compute an equation for the tangent plane at the point p to the graph of the function  $z = f(x,y)$

$$p = (1, 1, 1) \text{ and } xy + yz + zx = 3$$

$$g(x, y, z) = xy + yz + zx - 3$$

$$\nabla g(x, y, z) = (y+z, x+z, y+x)$$

$$\nabla g(1, 1, 1) = (2, 2, 2)$$

$$\Rightarrow (2, 2, 2)((x, y, z) - (1, 1, 1)) = 0$$

$$(2, 2, 2)(x-1, y-1, z-1) = 0$$

$$2x-2 + 2y-2 + 2z-2 = 0$$

$$x+y+z = 3$$

## Differentiation

$f$  is differentiable if the gradient vector  $\nabla f(\vec{a})$  satisfies

$$\lim_{\vec{h} \rightarrow 0} \frac{f(\vec{a} + \vec{h}) - f(\vec{a}) - \nabla f(\vec{a}) \cdot \vec{h}}{\|\vec{h}\|}$$

**Ex ④** - Find all directional derivatives and show whether the function is differentiable at  $(0, 0)$

$$f(x, y) = \begin{cases} \frac{3x^2y + 5xy^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$\begin{aligned} \vec{a} &= (0, 0) \\ \vec{u} &= (a, b) \end{aligned}$$

$$\begin{aligned} D_{\vec{u}} f &= \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(ha, hb) - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{3(ha)^2(hb) + 5(ha)(hb)^2}{(ha)^2 + (hb)^2} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot h^3 \frac{(3a^2b + 5ab^2)}{h^2(a^2 + b^2)} \\ &= \lim_{h \rightarrow 0} \frac{3a^2b + 5ab^2}{a^2 + b^2} \end{aligned}$$

$$\rightarrow \vec{u} = (1, 0) , D_{\vec{u}} f = \frac{3(1)(0) + 5(1)(0)}{(1)^2 + (0)^2} = 0 \rightarrow \frac{\partial f}{\partial x} = 0$$

$$\rightarrow \vec{u} = (0, 1) , D_{\vec{u}} f = \frac{3(0)(1) + 5(0)(1)}{(0)^2 + (1)^2} = 0 \rightarrow \frac{\partial f}{\partial y} = 0$$

$$\nabla f(\vec{a}) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (0, 0)$$

Now, let's apply the definition of differentiability

$$\lim_{\vec{h} \rightarrow 0} \frac{f(\vec{a} + \vec{h}) - f(\vec{a}) - \nabla f(\vec{a}) \cdot \vec{h}}{\|\vec{h}\|} = \lim_{\vec{h} \rightarrow 0} \frac{f(\vec{h})}{\|\vec{h}\|}, \text{ let } \vec{h} = (h_1, h_2)$$

$$= \lim_{(h_1, h_2) \rightarrow (0,0)} \frac{1}{\sqrt{h_1^2 + h_2^2}} \cdot \frac{3h_1^2 h_2 + 5h_1 h_2^2}{h_1^2 + h_2^2}$$

Restrict  $h_1 = 0$

$$\lim_{h_2 \rightarrow 0} \frac{1}{\sqrt{h_2^2}} \cdot \frac{3(0)h_2 + 5(0)h_2^2}{h_2^2} = 0$$

Restrict  $h_1 = h_2$

$$\lim_{h_2 \rightarrow 0} \frac{1}{\sqrt{2h_2^2}} \cdot \frac{3h_2^3 + 5h_2^3}{2h_2^2} = \lim_{h_2 \rightarrow 0} \frac{8h_2^3}{2\sqrt{2}h_2^3} = \frac{4}{\sqrt{2}}$$

$\therefore f$  is not differentiable at  $(0,0)$

## Week 5 (June 8th)

If  $f$  is  $C^2$ , then  $f_{xy} = f_{yx}$   $\left[ \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \right]$

Ex① - At  $(x,y) = (0,0)$ , find  $\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$  and  $\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)$ . Are they equal?

IS  $f(x,y)$   $C^2$ ?

$$f(x,y) = \begin{cases} \frac{ax^4}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Time to find directional derivatives?

$$\begin{aligned} \vec{a} &= (0,0) \\ \vec{u} &= (a,b) \end{aligned}$$

$$\begin{aligned} D_{\vec{u}} f &= \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} f(ha, hb) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{\partial(ha)^4}{(ha)^2 + (hb)^2} \\ &= \lim_{h \rightarrow 0} \frac{2h^4 a^4}{h(h^2 a^2 + h^2 b^2)} \\ &= \lim_{h \rightarrow 0} \frac{2h^4 a^4}{h^3(a^2 + b^2)} \\ &= 0 \\ \Rightarrow \frac{\partial f}{\partial x}(0,0) &= 0 = \frac{\partial f}{\partial y}(0,0) \end{aligned}$$

Now, let's find the partial derivatives.

$$*\frac{\partial f}{\partial x}(x,y) = \frac{8x^3(x^2+y^2) - 2x(2x^4)}{(x^2+y^2)^2} = \frac{8x^5 + 8x^3y^2 - 4x^5}{(x^2+y^2)^2} = \frac{4x^5 + 8x^3y^2}{(x^2+y^2)^2}$$

$$= \frac{4x^3(x^2+2y^2)}{(x^2+y^2)^2}$$

$$\Rightarrow g(x,y) = \frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{4x^3(x^2+2y^2)}{(x^2+y^2)^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$*\frac{\partial f}{\partial y}(x,y) = \frac{0(x^2+y^2) - 2y(2x^4)}{(x^2+y^2)^2} = \frac{-4x^4y}{(x^2+y^2)^2}$$

$$\Rightarrow K(x,y) = \frac{\partial f}{\partial y}(x,y) = \begin{cases} \frac{-4x^4y}{(x^2+y^2)^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

Time to finally find  $f_{xy}$  and  $f_{yx}$  at  $(0,0)$

\* We need to find  $\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}(0,0)\right)$ , aka  $\frac{\partial g}{\partial y}(0,0)$

We can find the directional derivative at  $(0,0)$  where  $\vec{u} = (0,1)$

$$\begin{aligned} \frac{\partial g}{\partial y}(0,0) &= \lim_{h \rightarrow 0} \frac{g(\vec{a}+h\vec{u}) - g(\vec{a})}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} g(0,h) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \times 0 \\ &= 0 \end{aligned}$$

\* We need to find  $\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}(0,0)\right)$ , aka  $\frac{\partial K}{\partial x}(0,0)$

We can find the directional derivative at  $(0,0)$  where  $\vec{u} = (1,0)$

$$\begin{aligned} \frac{\partial K}{\partial x}(0,0) &= \lim_{h \rightarrow 0} \frac{K(\vec{a}+h\vec{u}) - K(\vec{a})}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} K(h,0) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \times 0 \\ &= 0 \end{aligned}$$

$$\therefore \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} (0,0) \right) = 0 = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} (0,0) \right)$$

But, is it  $C^2$ ? At  $(0,0)$ , yes!

$$\begin{aligned} * \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} (x,y) \right) &= \frac{[0(x^2+2y^2) + 4y(4x^3)](x^2+y^2)^2 - 2(2y)(x^2+y^2)4x^3(x^2+2y^2)}{(x^2+y^2)^4} \\ &= \frac{16x^3y(x^2+y^2) - 16x^3y(x^2+2y^2)}{(x^2+y^2)^3} \\ &= \frac{16x^3y(x^2+y^2 - x^2-2y^2)}{(x^2+y^2)^3} = \frac{-16x^3y^3}{(x^2+y^2)^3} \end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} (x,y) \right) = \begin{cases} \frac{-16x^3y^3}{(x^2+y^2)^3} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$\begin{aligned} * \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} (x,y) \right) &= \frac{-16x^3y(x^2+y^2)^2 + 4x^4y(2)(2x)(x^2+y^2)}{(x^2+y^2)^4} \\ &= \frac{-16x^3y(x^2+y^2) + 16x^5y}{(x^2+y^2)^4} = \frac{-16x^3y^3}{(x^2+y^2)^4} \end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} (x,y) \right) = \begin{cases} \frac{-16x^3y^3}{(x^2+y^2)^4} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

So  $f(x,y)$  is not  $C^2$

## Critical Points

Hessian form: ( $H_f$ )

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) & \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \\ \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) & \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) \end{bmatrix}$$

- \* If  $H_f$  is positive definite, then the Critical Point is a local minimum
- \* If  $H_f$  is negative definite, then the Critical Point is a local maximum
- \* Otherwise, the Critical Point is a saddle point.

Let  $A$  be  $n \times n$  and symmetric:

- \*  $A$  is positive definite iff  $\det(A_k) > 0$  for  $k = 1, 2, \dots, n$ .
- \*  $A$  is negative definite iff  $(-1)^k \det(A_k) > 0$  for  $k = 1, 2, \dots, n$

Ex ② - Find the critical points of  $f(x, y) = x^3 - 12xy + 8y^3$

$$\frac{\partial f}{\partial x}(x, y) = 3x^2 - 12y$$

$$\frac{\partial f}{\partial y}(x, y) = -12x + 24y^2$$

$$\begin{cases} 3x^2 - 12y = 0 \\ -12x + 24y^2 = 0 \end{cases} \rightarrow \begin{aligned} 3x^2 &= 12y \\ y &= \frac{x^2}{4} \end{aligned}$$

$$\rightarrow -12x + 24\left(\frac{x^2}{4}\right)^2 = 0$$

$$-12x + \frac{24}{16}x^4 = 0$$

$$x\left(\frac{3}{2}x^3 - 12\right) = 0 \rightarrow x = 0 \text{ or } \frac{3}{2}x^3 = 12$$

$$\Rightarrow x^3 = 8 \rightarrow x = 2$$

∴ CPs are  $(0, 0)$  and  $(2, 1)$

$$\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial x}(x, y)\right) = 6x$$

$$\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}(x, y)\right) = -12$$

$$\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}(x, y)\right) = -12$$

$$\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial y}(x, y)\right) = 48y$$

$$\Rightarrow H_f(x, y) = \begin{bmatrix} 6x & -12 \\ -12 & 48y \end{bmatrix}$$

① For  $(0, 0)$

$$H_f(0, 0) = \begin{bmatrix} 0 & -12 \\ -12 & 0 \end{bmatrix}$$

$$\rightarrow \det(A_1) = \det([0]) = 0$$

$$\rightarrow \det(A_2) = \det\left(\begin{bmatrix} 0 & -12 \\ -12 & 0 \end{bmatrix}\right) = -144 < 0$$

∴ saddle point

② For  $(2, 1)$

$$H_f(2, 1) = \begin{bmatrix} 12 & -12 \\ -12 & 48 \end{bmatrix}$$

$$\rightarrow \det(A_1) = \det([12]) = 12 > 0$$

$$\rightarrow \det(A_2) = \det\left(\begin{bmatrix} 12 & -12 \\ -12 & 48 \end{bmatrix}\right) = 576 - 144 = 432 > 0$$

$\therefore$  local min.

Ex ③- Find the global extrema for  $f(x, y) = 6x^2 - 8x + 2y^2 - 5$  on the closed disk  $x^2 + y^2 \leq 1$ .

The extrema can be at the interior or boundary.

Interior:

$$\frac{\partial f}{\partial x}(x, y) = 12x - 8 \quad \frac{\partial f}{\partial y}(x, y) = 4y$$

$$\begin{cases} 12x - 8 = 0 \\ 4y = 0 \end{cases} \rightarrow \left(\frac{2}{3}, 0\right) \text{ is the only CP.}$$

Since  $\left(\frac{2}{3}\right)^2 + (0)^2 = \frac{4}{9} \leq 1$ , the point is inside the disk.

Boundary: it happens when  $x^2 + y^2 = 1$ . and  $-1 \leq x \leq 1$   
 $y^2 = 1 - x^2$ , let's substitute this

$$\begin{aligned} g(x) &= 6x^2 - 8x + 2(1 - x^2) - 5 \\ &= 6x^2 - 8x + 2 - 2x^2 - 5 \\ &= 4x^2 - 8x - 3 \end{aligned}$$

$$\begin{aligned} g'(x) &= 8x - 8 = 0 \\ 8x &= 8 \\ x &= 1 \quad \rightarrow y^2 = 1 - 1 = 0, (1, 0) \\ &\rightarrow y = 0, 0 = 1 - x^2 \rightarrow (-1, 0) \end{aligned}$$

$$\therefore f\left(\frac{2}{3}, 0\right) = -\frac{23}{2}, \text{ global min}$$

$$f(-1, 0) = 9, \text{ global max}$$

$$f(1, 0) = -7$$

Week 6 (June 16th)

## Lagrange Multipliers

**Ex ①** Use Lagrange multipliers to find the constrained critical points of  $f$  subject to the given constraints.

(a)  $f(x,y) = xy$ ,  $4x^2 + 9y^2 = 32$ ,  $g(x,y) = 4x^2 + 9y^2 - 32$

$$h(x,y,\lambda) = xy - \lambda(4x^2 + 9y^2 - 32)$$

$$\begin{cases} h_x = y - 8x\lambda = 0 \rightarrow \lambda = \frac{y}{8x} \\ h_y = x - 18y\lambda = 0 \rightarrow \lambda = \frac{x}{18y} \\ h_\lambda = 32 - 4x^2 - 9y^2 = 0 \end{cases} \quad \left[ \begin{array}{l} \frac{y}{8x} = \frac{x}{18y} \rightarrow 18y^2 = 8x^2 \rightarrow 4x^2 = 9y^2 \\ \text{or } \lambda = 0 \text{ (contradiction)} \end{array} \right]$$

$$8x^2 = 32 \rightarrow x^2 = 4 \rightarrow x = \pm 2$$

$$y = \pm \frac{4}{3}$$

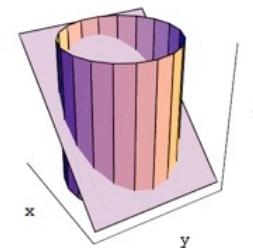
$$\therefore \text{CPs are } \left(2, \frac{4}{3}\right), \left(2, -\frac{4}{3}\right), \left(-2, \frac{4}{3}\right), \left(-2, -\frac{4}{3}\right)$$

(b) The cylinder  $x^2 + y^2 = 1$  intersects the plane  $x+z=1$  in an ellipse. Find the point on that ellipse that is the furthest from the origin.

We need to maximize the distance function (aka  $f(x,y,z) = x^2 + y^2 + z^2$ ) subject to 2 constraints.

$$h(x,y,z,\lambda, \mu) = x^2 + y^2 + z^2 - \lambda(x^2 + y^2 - 1) - \mu(x + z - 1)$$

$$\begin{cases} h_x = 2x - 2x\lambda - \mu & ① \\ h_y = 2y - 2y\lambda & ② \\ h_z = 2z - \mu & ③ \\ h_\lambda = 1 - x^2 - y^2 & ④ \\ h_\mu = 1 - x - z & ⑤ \end{cases}$$



$$\begin{aligned} ② \quad 2y &= 2y\lambda \\ \Rightarrow 2y(1-\lambda) &= 0 \\ y &= 0, \lambda = 1 \end{aligned}$$

$$\begin{aligned} * \text{ If } \lambda &= 1: \\ ① \quad \mu &= 2x(1-\lambda) = 0 \\ ③ \quad 2z &= \mu = 0 \rightarrow z = 0 \\ ⑤ \quad x &= 1 \\ ④ \quad 1+y^2 &= 1 \rightarrow y = 0 \end{aligned}$$

$$\begin{aligned} * \text{ If } y &= 0: \\ ④ \quad x^2 &= 1 \rightarrow x = \pm 1 \\ ⑤ \quad -1+z &= 1 \rightarrow z = 2 \quad (x = -1) \\ 1+z &= 1 \rightarrow z = 0 \quad (x = 1) \quad (\text{We know}) \end{aligned}$$

$$\therefore \text{CPs } (1, 0, 0) \text{ and } (-1, 0, 2)$$

Based on the distance function:

$$(1, 0, 0) \rightarrow \sqrt{1^2} = 1$$

$$(-1, 0, 2) \rightarrow \sqrt{(-1)^2 + (0)^2 + (2)^2} = \sqrt{5}$$

∴ The furthest point from the origin is  $(-1, 0, 2)$ .

Since the distance function is continuous and the intersection is compact in  $\mathbb{R}^3$ , the Extreme Value Theorem (EVT) ensures the existence of extreme.

### Chain Rule

$$(f(g(x)))' = f'(g(x)) g'(x)$$

Now, let's say  $y = y(t)$  and  $t = t(x)$

$$\text{1D: } \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\text{2D: } \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{dy}{dt} \cdot \frac{dt}{dx} \right) \quad \text{product rule!}$$

$$= \frac{d}{dx} \left( \frac{dy}{dt} \right) \frac{dt}{dx} + \frac{dy}{dt} \cdot \frac{d}{dx} \left( \frac{dt}{dx} \right) = \frac{d^2t}{dx^2}$$

### ODEs - Euler Equation

Second Order differential equations:

Homogeneous with constant coefficients

$$ay'' + by' + cy = 0 \quad \text{where } a, b, c \text{ are constants.}$$

Characteristic equation  $ar^2 + br + c = 0$ , find roots  $r_1$  and  $r_2$

then  $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$  is a general solution

Ex ② - let  $y'' + 5y' + 6y = 0$

(a) Find the general solution of the equation

$$r^2 + 5r + 6 = 0 = (r+2)(r+3) = 0$$

$$r_1 = -2 \text{ and } r_2 = -3$$

$$\text{Then } y = c_1 e^{-2t} + c_2 e^{-3t}, c_1, c_2 \in \mathbb{R}$$

(b) Find the solution of the initial value problem  $y(0)=2$ ,  $y'(0)=3$

$$t=0, y(0) = C_1 + C_2 = 2$$

$$y' = -2C_1 e^{-2t} - 3C_2 e^{-3t} \rightarrow y'(0) = -2C_1 - 3C_2 = 3$$

$$\begin{cases} C_1 + C_2 = 2 \\ -2C_1 - 3C_2 = 3 \end{cases} \rightarrow C_2 = -7 \text{ and } C_1 = 9$$

$$\therefore y = 9e^{-2t} - 7e^{-3t}$$

Ex ③ - Find the solution of the initial value problem  $4y'' - 8y' + 3y = 0$ ,

$$y(0) = 2, y'(0) = \frac{1}{2}$$

$$4r^2 - 8r + 3 = 0 = (2r-1)(2r-3)$$

$$r_1 = \frac{1}{2}, r_2 = \frac{3}{2}$$

$$\text{General Sol : } y = C_1 e^{t/2} + C_2 e^{3t/2} \quad y' = \frac{C_1}{2} e^{t/2} + \frac{3C_2}{2} e^{3t/2}$$

$$\begin{aligned} y(0) &= C_1 + C_2 = 2 \\ y'(0) &= \frac{C_1}{2} + \frac{3C_2}{2} = \frac{1}{2} \end{aligned} \quad \left\{ \begin{array}{l} C_1 = 2 - C_2 \\ C_1 = 1 - 3C_2 \end{array} \right. \quad C_2 = -\frac{1}{2}, C_1 = \frac{5}{2}$$

$$\therefore y = \frac{5}{2} e^{t/2} - \frac{1}{2} e^{3t/2}$$