MATB41 - Multivariable Calculus

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Week 1 (Sep 14th)

* No tutorials *

Week 2 (Sep 21st)

Curved Lines

Parabola: (with vertex) at the origin

- $y = ax^2$ → open up when $a>0$ and down when $a<0$
- $x = by^2$ → open right when $b>0$ and left when $b<0$

* Remember: We can replace x with $x-m$ to shift right and y with $y+n$ to shift up!

Ellipse: (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

intersections $(\pm a, 0)$ and $(0, \pm b)$
when $a=b$, the curve is a circle of radius $a=b$.

Hyperbola: (with center) at the origin

- $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ It has two curves that goes in opposite directions and do not touch the slant asymptote
- $\left(\frac{y}{a}\right)^2 - \left(\frac{x}{b}\right)^2 = 1$

Ex ① - Sketch the following curves:

a) $x^2 + 3y^2 + 2x - 12y + 10 = 0$

$$(x^2+2x) + 3(y^2-4y)+10 = 0$$
$$(x^2+2x+1) - 1 + 3(y^2-4y+4) - 12 + 10 = 0$$
$$(x+1)^2 + 3(y-2)^2 = 3$$

$$\Rightarrow \frac{(x+1)^2}{3} + (y-2)^2 = 1 \quad \Rightarrow \left(\frac{x+1}{\sqrt{3}}\right)^2 + (y-2)^2 = 1$$

Okie, now what?! Another way to write the ellipse equation would be:

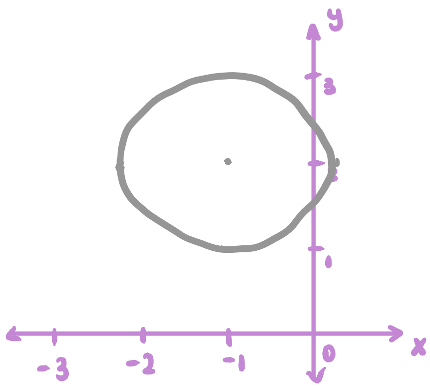
$$\left(\frac{x-m}{a}\right)^2 + \left(\frac{y-n}{b}\right)^2 = 1$$ → we just combined it with *

So the new origin will be $(-1, 2)$ as the ellipse will be shifted left by one and up by 2.

Now, let's find the end points

→ The horizontal max/min will be $(-1 \pm \sqrt{3}, 2)$

→ The vertical max/min will be $(-1, 2 \pm 1)$



b) $\frac{(x-2)^2}{4} - \frac{(y+2)^2}{9} = 1$

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 1$$

The origin will be $(2, -2)$, we are shifting right 2 and 2 down.

Check the x-int : $(y=0)$

$$\left(\frac{x-2}{2}\right)^2 - \frac{4}{9} = 1$$

$$\frac{(x-2)^2}{4} = \frac{13}{9}$$

$$(x-2)^2 = \frac{52}{9}$$

$$x-2 = \pm\frac{2\sqrt{13}}{3}$$

$$x = 2 \pm \frac{2\sqrt{13}}{3}$$

Check the y-int: $(x=0)$

$$1 - \left(\frac{y+2}{3}\right)^2 = 1$$

$$y = -2$$

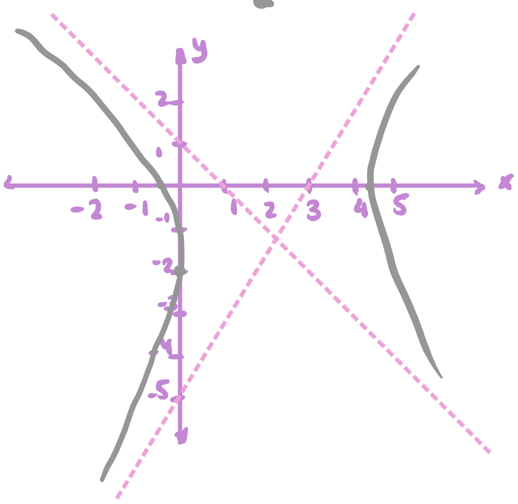
Find the slant asymptote:

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 0$$

$$\frac{x-2}{2} = \frac{y+2}{3}$$
$$3x-6 = 2y+4$$
$$2y = 3x-10$$
$$y = \frac{3x}{2} - 5$$

or

$$\frac{-x+2}{2} = \frac{y+2}{3}$$
$$-3x+6 = 2y+4$$
$$2y = -3x+2$$
$$y = \frac{-3x}{2} + 1$$



Curved Surfaces

3D Sphere: (with center) at the origin of radius R

$$x^2+y^2+z^2=R^2$$

3D Ellipsoid: (with center) at the origin

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$$

3 points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$

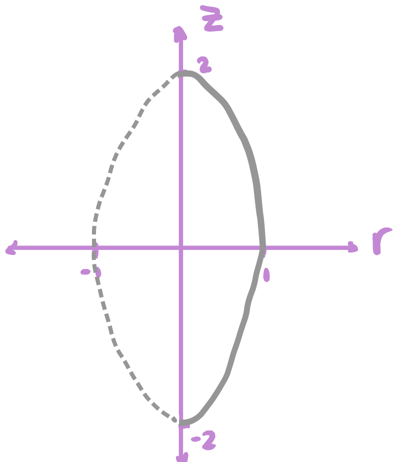
We can replace x^2+y^2 with r^2 to indicate the rotation.

Ex ② - Sketch the following surfaces:

a) $x^2+y^2+\frac{z^2}{4}=1$

$$r^2+\frac{z^2}{4}=1$$
$$4r^2+z^2=4$$
$$r^2+\frac{z^2}{4}=1$$

* Use the ellipsoid formula



$r \geq 0$ because $r=\sqrt{x^2+y^2}$

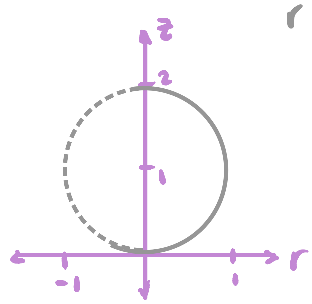
Center at $(0,0)$

r-int : $(\pm 1, 0)$
z-int : $(0, \pm 2)$

b) $x^2+y^2+z^2=2z$

$r^2+z^2-2z=0$
$r^2+z^2-2z+1-1=0$
$r^2+(z-1)^2=1 \rightarrow$ circle!

center $(0,1)$



$r \geq 0$ because $r=\sqrt{x^2+y^2}$

Dot and Cross Product

Let $\vec{u}\cdot\vec{v}=\begin{bmatrix}u_1\\u_2\\\vdots\end{bmatrix}\cdot\begin{bmatrix}v_1\\v_2\\\vdots\end{bmatrix}=u_1v_1+u_2v_2+\cdots$

Let $\vec{u}\times\vec{v}=\begin{vmatrix}i&j&k\\u_1&u_2&u_3\\v_1&v_2&v_3\end{vmatrix}=\left(\begin{vmatrix}u_2&u_3\\v_2&v_3\end{vmatrix},-\begin{vmatrix}u_1&u_3\\v_1&v_3\end{vmatrix},\begin{vmatrix}u_1&u_2\\v_1&v_2\end{vmatrix}\right)$
$=(u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1)$

You can find the angle between two vectors using dot product:

Let the angle be called θ, then $\cos\theta=\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

Ex ③ - Find the angle between $\vec{u}=(1,-3,1)$ and $\vec{v}=(2,1,2)$ in $\mathbb{R}^3$

First, we need the dot product and lengths:

$\vec{u}\cdot\vec{v}=(1,-3,1)(2,1,2)=2-3+2=1$

$\|\vec{u}\|=\sqrt{1^2+(-3)^2+1^2}=\sqrt{11}$

$\|\vec{v}\|=\sqrt{2^2+1^2+2^2}=\sqrt{9}=3$

$\Rightarrow \cos\theta=\dfrac{1}{3\sqrt{11}} \quad \therefore \quad \theta=\cos^{-1}\left(\dfrac{1}{3\sqrt{11}}\right)$

Line and Planes

Vector equation of a line in $\mathbb{R}^3$: $\vec{\ell}=(a_1,a_2,a_3)+t[v_1,v_2,v_3]$
$\vec{\ell}=a+t\vec{v}$

Parametric equation of a line in $\mathbb{R}^3$:
$x=a_1+tv_1$
$y=a_2+tv_2$
$z=a_3+tv_3$

Ex ④ - Find the equation of the line or plane

a) The line through $(1,-1,2)$ and $(3,1,9)$

The direction vector for the line is $(3,1,9)-(1,-1,2)=(2,2,7)$

$\rightarrow$ V. eq. $\vec{\ell}=(1,-1,2)+t(2,2,7)$, $t\in\mathbb{R}$

$\rightarrow$ P. eq. $x=1+2t$, $y=-1+2t$, $z=2+7t$, $t\in\mathbb{R}$

b) The plane through $(1,-3,1)$, $(2,1,1)$, $(1,4,0)$

Let's find a pair of directional vectors $\vec{v}_1$ and $\vec{v}_2$

$\vec{v}_1=(2,1,1)-(1,-3,1)=(1,4,0)$

$\vec{v}_2=(1,4,0)-(1,-3,1)=(0,7,-1)$

To get the equation of the plane, we need to find the normal which is $\vec{v}_1\times\vec{v}_2$

$$\vec{n}=\vec{v}_1\times\vec{v}_2=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 4 & 0\\ 0 & 7 & -1\end{vmatrix}=(-4,1,7)$$

The plane then is $-4x+y+7z=d$
Let's plug a point: $-4(2)+(1)+7(1)=-8+1+7=0=d$

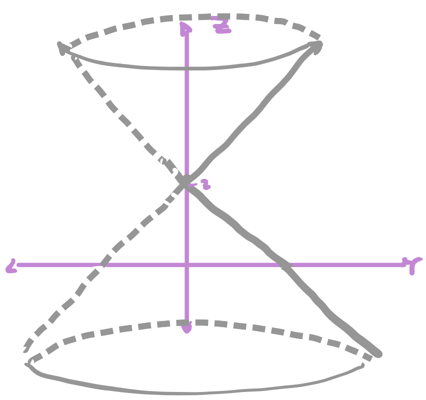
$\rightarrow$ The plane is $-4x+y+7z=0$

Level Set

Ex ⑤ - Draw the level set of $f(x,y,z)=z^2-x^2-y^2-4z$ for $f=-4$

↳ this is in 4D though

$$-4=z^2-x^2-y^2-4z$$
$$x^2+y^2=z^2-4z+4$$
$$r^2=(z-2)^2$$
$$r=\pm(z-2)$$



$r\geq 0$

Week 3 (Sep 28th)

Limits and Continuity

$f(x,y)$ is continuous at the 2D point $\vec{a}$ if $\lim f(x,y)=f(\vec{a})$

$(x,y)\to\vec{a}$

It's not too hard to show when a limit DNE in $\mathbb{R}^2$ at $(0,0)$:
Try different curves in terms of x or/and y, if they approach to different values at $(0,0)$

To show that the limit exists we can use the Squeeze Theorem:

To attain $\lim\limits_{(x,y)\to\vec{a}} f(x,y)$, can try to find $g(x,y)$ and $h(x,y)$ so that:

1. $g(x,y) \le f(x,y) \le h(x,y)$ near the point $\vec{a}$
2. $\lim\limits_{(x,y)\to\vec{a}} g(x,y) = L = \lim\limits_{(x,y)\to\vec{a}} h(x,y)$

Then we conclude $\lim\limits_{(x,y)\to\vec{a}} f(x,y) = L$

Ex 2 - Decide whether the function has a limit at $(0,0)$

a) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \text{DNE}$

Restrict to $x=0$, $\lim\limits_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1$

Restrict to $y=0$, $\lim\limits_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$

b) $\lim\limits_{(x,y)\to(0,0)} \frac{|x|}{\sqrt{x^2+y^2}} = \text{DNE}$

Restrict to $y=0$, $\lim\limits_{(x,0)\to(0,0)} \frac{|x|}{\sqrt{x^2}} = 1$

Restrict to $x=0$, $\lim\limits_{(0,y)\to(0,0)} \frac{|x|}{\sqrt{y}} = \frac{0}{\sqrt{|y|}} = 0$

c) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \underset{①}{\frac{x^3}{x^2+y^2}} - \underset{②}{\frac{y^3}{x^2+y^2}} = 0$

① $x^2 \le x^2+y^2 \rightarrow 0 \le \frac{x^2}{x^2+y^2} \le 1$

$\rightarrow 0\cdot x \le \frac{x^3}{x^2+y^2} \le x$

$$\rightarrow \lim_{(x,y)\to(0,0)} 0 \le \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} \le \lim_{(x,y)\to(0,0)} x$$

$$\rightarrow 0 \le \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} \le 0$$

$$\Rightarrow 0$$

② Same as ①

$$y^2 \le x^2+y^2 \rightarrow 0 \le \frac{y^2}{x^2+y^2} \le 1$$

$$\rightarrow 0. \le \frac{y^3}{x^2+y^2} \le$$

$$\rightarrow \lim_{(x,y)\to(0,0)} y\cdot 0 \le \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} \le \lim_{(x,y)\to(0,0)} y$$

$$\rightarrow 0 \le \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} \le 0$$

$$\Rightarrow 0$$

d) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

We know $|\sin\theta| \le 1$ and we saw in class $|\sin\theta| \le \theta$

$$\Rightarrow \frac{|\sin(x^2+y^2)|}{x^2+y^2} \le \frac{x^2+y^2}{x^2+y^2} = 1$$

Recall $\lim_{t\to 0} \frac{\sin t}{t} = 1$, what if we let $t = x^2+y^2$

$$\Rightarrow \lim_{t\to 0} \frac{\sin t}{t} = 1 = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

Ex③ - Find if the limit of f exists at $(0,0)$, if it does, find $f(0,0)$ for f to be continuous.

$$f(x,y) = \frac{x^3-x^2-2x^2y+xy^2-y^2-2y^3}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2(x-1-2y)+y^2(x-1-2y)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x-1-2y)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} x-1-2y$$

$$= -1$$

$$\Rightarrow f(0,0) = -1$$

$f(\vec{x})$ is homogeneous of degree k if for every $\vec{x} \in \mathbb{R}^n$ and every scalar $c > 0$ we have:

$$f(c\vec{x}) = c^k f(\vec{x})$$

Ex ④ - Show if f is homogeneous

a) $f(x,y) = 8x^2y^2 - 9x^4$

$$\begin{aligned} f(cx, cy) &= 8(cx)^2(cy)^2 - 9(cx)^4 \\ &= 8c^2x^2c^2y^2 - 9c^4x^4 \\ &= 8c^4x^2y^2 - 9c^4x^4 \\ &= c^4(8x^2y^2 - 9x^4) \\ &= c^4 f(x,y) \end{aligned}$$

$\Rightarrow f$ is homogeneous of degree 4.

b) $f(x,y) = \dfrac{x^2}{y^2} + xy + \dfrac{y^2}{x^2}$

$$\begin{aligned} f(cx, cy) &= \frac{(cx)^2}{(cy)^2} + (cx)(cy) + \frac{(cy)^2}{(cx)^2} \\ &= \frac{c^2x^2}{c^2y^2} + c^2xy + \frac{c^2y^2}{c^2x^2} \\ &= \frac{x^2}{y^2} + c^2xy + \frac{y^2}{x^2} \end{aligned}$$

$\Rightarrow$ cannot factor c.

$\Rightarrow f$ is not homogeneous.

Week 4 (Oct. 5th)

Partial derivatives

Ex ① - Find the partial derivatives of $f(x,y) = y\sin(xy) + xe^{-y^2}$

$$\frac{\partial f}{\partial x} = y^2\cos(xy) + e^{-y^2}$$

$$\frac{\partial f}{\partial y} = \sin(xy) + xy\cos(xy) - 2xye^{-y^2}$$

A function, whose partial derivatives exist and are continuous, is said to be of class C^1

Directional Derivative

The directional derivative towards the direction $\vec{u}$ at the point $\vec{a}$:

$$D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}$$ where $\vec{u}$ is a unit vector.

If f is differentiable, then all directional derivatives exist and

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$ where $\vec{u}$ is a unit vector.

* If f is C^1 at $\vec{a}$, then f is differentiable at $\vec{a}$ (∇f exists)

Ex ② - At the point $(3,1,2)$, find the directional derivative of $f(x,y,z) = xy^3z^2$ along the vector $\vec{u} = (1,3,4)$

f is C^1 → it's partial derivatives exist and are continuous.

$\nabla f(x,y,z) = (y^3z^2, 3xy^2z^2, 2xy^3z)$

$\nabla f(3,1,2) = (4, 36, 12)$

$$\Rightarrow D_{\vec{u}} f(p) = \frac{(4,36,12)(1,3,4)}{\sqrt{1+9+16}} = \frac{4+108+48}{\sqrt{26}} = \frac{160}{\sqrt{26}}$$

Tangent Planes

A tangent plane is given by $\nabla g(a,b,c)((x,y,z)-(a,b,c))=0$

Ex ③ - Compute an equation for the tangent plane at the point p to the graph of the function $z=f(x,y)$

$p = (1,1,1)$ and $xy + yz + zx = 3$

$g(x,y,z) = xy + yz + zx - 3$

$\nabla g(x,y,z) = (y+z, x+z, y+x)$

$\nabla g(1,1,1) = (2,2,2)$

$$\Rightarrow (2,2,2)((x,y,z)-(1,1,1)) = 0$$
$$(2,2,2)(x-1, y-1, z-1) = 0$$
$$2x-2+2y-2+2z-2 = 0$$
$$x+y+z = 3$$

Differentiation

f is differentiable if the gradient vector $\nabla f(\vec{a})$ satisfies

$$\lim_{\vec{h} \to 0} \frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - \nabla f(\vec{a}) \cdot \vec{h}}{|\vec{h}|}$$

Ex ④ - Find all directional derivatives and show whether the function is differentiable at $(0,0)$

$$f(x,y) = \begin{cases} \dfrac{3x^2y+5xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$\vec{a} = (0,0)$
$\vec{u} = (a,b)$

$$D_{\hat{u}} f = \lim_{h\to 0} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$$
$$= \lim_{h\to 0} \frac{f(ha,hb) - 0}{h}$$
$$= \lim_{h\to 0} \frac{1}{h} \frac{3(ha)^2(hb) + 5(ha)(hb)^2}{(ha)^2+(hb)^2}$$
$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{h^3(3a^2b+5ab^2)}{h^2(a^2+b^2)}$$
$$= \lim_{h\to 0} \frac{3a^2b+5ab^2}{a^2+b^2}$$

$\to \vec{u} = (1,0)$, $D_{\hat{u}} f = \dfrac{3(1)(0)+5(1)(0)}{(1)^2+(0)^2} = 0 \to \dfrac{\partial f}{\partial x} = 0$

$\to \vec{u} = (0,1)$, $D_{\hat{u}} f = \dfrac{3(0)(1)+5(0)(1)}{(0)^2+(1)^2} = 0 \to \dfrac{\partial f}{\partial y} = 0$

$$\nabla f(\vec{a}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (0,0)$$

Now, let's apply the definition of differentiability

$$\lim_{\vec{h}\to 0} \frac{f(\vec{a}+\vec{h}) - \overset{0}{f(\vec{a})} - \overset{0}{\nabla f(\vec{a})\cdot\vec{h}}}{|\vec{h}|} = \lim_{\vec{h}\to 0} \frac{f(\vec{h})}{|\vec{h}|}, \text{ let } \vec{h} = (h_1,h_2)$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{1}{\sqrt{h_1^2+h_2^2}} \cdot \frac{3h_1^2h_2 + 5h_1h_2^2}{h_1^2+h_2^2}$$

Restrict $h_1 = 0$

$$\lim_{h_2\to 0} \frac{1}{\sqrt{h_2^2}} \cdot \frac{3(0)h_2 + 5(0)h_2^2}{h_2^2} = 0$$

Restrict $h_1 = h_2$

$$\lim_{h_2\to 0} \frac{1}{\sqrt{2h_2^2}} \cdot \frac{3h_2^3+5h_2^3}{2h_2^2} = \lim_{h_2\to 0} \frac{8h_2^3}{2\sqrt{2}\,h_2^3} = \frac{4}{\sqrt{2}}$$

$\therefore$ f is <u>not</u> differentiable at $(0,0)$

Week 5 (Oct. 19th)

If f is C^2, then $f_{xy} = f_{yx}$ $\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \right]$

Ex ① - At $(x,y) = (0,0)$, find $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$. Are they equal?

Is $f(x,y)$ C^2?

$$f(x,y) = \begin{cases} \frac{2x^4}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Time to find directional derivatives!

$\vec{a} = (0,0)$
$\vec{u} = (a,b)$

$$D_{\vec{u}} f = \lim_{h\to 0} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$$
$$= \lim_{h\to 0} \frac{1}{h} f(ha, hb)$$
$$= \lim_{h\to 0} \frac{1}{h} \frac{2(ha)^4}{(ha)^2+(hb)^2}$$
$$= \lim_{h\to 0} \frac{2h^4a^4}{h(h^2a^2+h^2b^2)}$$
$$= \lim_{h\to 0} \frac{2h^4a^4}{h^3(a^2+b^2)}$$
$$= 0$$

$$\Rightarrow \frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$$

Now, let's find the partial derivatives.

$$* \ \frac{\partial f}{\partial x}(x,y) = \frac{8x^3(x^2+y^2) - 2x(2x^4)}{(x^2+y^2)^2} = \frac{8x^5 + 8x^3y^2 - 4x^5}{(x^2+y^2)^2} = \frac{4x^5+8x^3y^2}{(x^2+y^2)^2}$$
$$= \frac{4x^3(x^2+2y^2)}{(x^2+y^2)^2}$$

$$\Rightarrow g(x,y) = \frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{4x^3(x^2+2y^2)}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$* \ \frac{\partial f}{\partial y}(x,y) = \frac{0(x^2+y^2) - 2y(2x^4)}{(x^2+y^2)^2} = \frac{-4x^4y}{(x^2+y^2)^2}$$

$$\Rightarrow k(x,y) = \frac{\partial f}{\partial y}(x,y) = \begin{cases} \frac{-4x^4y}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ \ldots \end{cases}$$

$$\frac{\partial}{\partial y} \quad \begin{cases} \ldots \\ 0 & , \ (x,y) = (0,0) \end{cases}$$

Time to finally find f_{xy} and f_{yx} at $(0,0)$

* We need to find $\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}(0,0)\right)$, aka $\frac{\partial g}{\partial y}(0,0)$

We can find the directional derivative at $(0,0)$ where $\vec{u} = (0,1)$

$$\begin{aligned}
\frac{\partial g}{\partial y}(0,0) &= \lim_{h \to 0} \frac{g(\vec{a} + h\vec{u}) - g(\vec{a})}{h} \\
&= \lim_{h \to 0} \frac{1}{h}\, g(0,h) \\
&= \lim_{h \to 0} \frac{1}{h} \times 0 \\
&= 0
\end{aligned}$$

* We need to find $\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}(0,0)\right)$, aka $\frac{\partial k}{\partial x}(0,0)$

We can find the directional derivative at $(0,0)$ where $\vec{u} = (1,0)$

$$\begin{aligned}
\frac{\partial k}{\partial x}(0,0) &= \lim_{h \to 0} \frac{k(\vec{a} + h\vec{u}) - k(\vec{a})}{h} \\
&= \lim_{h \to 0} \frac{1}{h}\, k(h,0) \\
&= \lim_{h \to 0} \frac{1}{h} \times 0 \\
&= 0
\end{aligned}$$

$$\therefore \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(0,0)\right) = 0 = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(0,0)\right)$$

But, is it C^2? At $(0,0)$, yes!

* $$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{[0(x^2+2y^2) + 4y(4x^3)](x^2+y^2)^2 - 2(2y)(x^2+y^2)4x^3(x^2+2y^2)}{(x^2+y^2)^4}$$

$$= \frac{16x^3y(x^2+y^2) - 16x^3y(x^2+2y^2)}{(x^2+y^2)^3}$$

$$= \frac{16x^3y(x^2+y^2-x^2-2y^2)}{(x^2+y^2)^3} = \frac{-16x^3y^3}{(x^2+y^2)^3}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \begin{cases} \frac{-16x^3y^3}{(x^2+y^2)^3} & , \ (x,y) \neq (0,0) \\ \ldots \end{cases}$$

$$\Big\lfloor \quad 0 \quad , (x,y) = (0,0)$$

$$* \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = \frac{-16x^3y(x^2+y^2)^2 + 4x^4y(2)(2x)(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{-16x^3y(x^2+y^2) + 16x^5y}{(x^2+y^2)^4} = \frac{-16x^3y^3}{(x^2+y^2)^4}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \begin{cases} \dfrac{-16x^3y^3}{(x^2+y^2)^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

So $f(x,y)$ is <u>not</u> C^2

Critical Points

Hessian form: (H_f.)

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) & \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \end{bmatrix}$$

* If H_f. is positive definite, then the Critical Point is a local minimum
* If H_f. is negative definite, then the Critical Point is a local maximum
* Otherwise, the Critical Point is a saddle point.

Let A be nxn and symmetric:

* A is positive definite iff $\det(A_k) > 0$ for $k = 1, 2, \dots, n$.
* A is negative definite iff $(-1)^k \det(A_k) > 0$ for $k = 1, 2, \dots n$

Ex 2 - Find the critical points of $f(x,y) = x^3 - 12xy + 8y^3$

$\frac{\partial f}{\partial x}(x,y) = 3x^2 - 12y$ $\qquad$ $\frac{\partial f}{\partial y}(x,y) = -12x + 24y^2$

$$\begin{cases} 3x^2 - 12y = 0 \\ -12x + 24y^2 = 0 \end{cases} \rightarrow \quad 3x^2 = 12y, \quad y = \frac{x^2}{4}$$

$$\rightarrow -12x + 24\left(\frac{x^2}{4}\right)^2 = 0$$

$$-12x + \frac{24}{16}x^4 = 0$$

$\overline{16}$

$$x\left(\frac{3}{2}x^3 - 12\right) = 0 \rightarrow x = 0 \text{ or } \frac{3}{2}x^3 = 12$$
$$\Rightarrow x^3 = 8 \rightarrow x = 2$$

$\therefore$ CPs are $(0,0)$ and $(2,1)$

$$\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = 6x \qquad \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = -12$$

$$\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = -12 \qquad \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right) = 48y$$

$$\Rightarrow Hf(x,y) = \begin{bmatrix} 6x & -12 \\ -12 & 48y \end{bmatrix}$$

① For $(0,0)$

$$Hf(0,0) = \begin{bmatrix} 0 & -12 \\ -12 & 0 \end{bmatrix}$$

$\rightarrow \det(A_1) = \det([0]) = 0$

$\rightarrow \det(A_2) = \det\left(\begin{bmatrix} 0 & -12 \\ -12 & 0 \end{bmatrix}\right) = -144 < 0$

$\therefore$ saddle point

② For $(2,1)$

$$Hf(2,1) = \begin{bmatrix} 12 & -12 \\ -12 & 48 \end{bmatrix}$$

$\rightarrow \det(A_1) = \det([12]) = 12 > 0$

$\rightarrow \det(A_2) = \det\left(\begin{bmatrix} 12 & -12 \\ -12 & 48 \end{bmatrix}\right) = 576 - 144 = 432 > 0$

$\therefore$ local min.

Ex ③- Find the global extrema for $f(x,y) = 6x^2 - 8x + 2y^2 - 5$ on the closed disk $x^2 + y^2 \leq 1$.

The extrema can be at the interior or boundary.

<u>Interior</u>: $\frac{\partial f}{\partial x}(x,y) = 12x - 8 \qquad \frac{\partial f}{\partial y}(x,y) = 4y$

$$\begin{cases} 12x-8=0 \\ 4y=0 \end{cases} \rightarrow \left(\frac{2}{3}, 0\right) \text{ is the only CP.}$$

Since $\left(\frac{2}{3}\right)^2 + (0)^2 = \frac{4}{9} \leq 1$, the point is inside the disk.

Boundary: it happens when $x^2+y^2=1$. and $-1 \leq x \leq 1$

$y^2 = 1-x^2$, let's substitute this

$$\begin{aligned} g(x) &= 6x^2-8x+2(1-x^2)-5 \\ &= 6x^2-8x+2-2x^2-5 \\ &= 4x^2-8x-3 \end{aligned}$$

$$\begin{aligned} g'(x) = 8x-8 &= 0 \\ 8x &= 8 \\ x &= 1 \end{aligned} \rightarrow y^2 = 1-1=0, \quad (1,0)$$

$$\rightarrow y=0, \quad 0=1-x^2 \rightarrow (-1,0)$$

$\therefore f\left(\frac{2}{3}, 0\right) = -\frac{23}{2}$, global min

$f(-1,0) = 9$, global max

$f(1,0) = -7$

Lagrange Multipliers

Ex①-Use Lagrange multipliers to find the constrained critical points of f subject to the given constraints.

(a) $f(x,y) = xy$, $4x^2+9y^2=32 \rightarrow g(x,y) = 4x^2+9y^2-32$

$h(x,y,\lambda) = xy - \lambda(4x^2+9y^2-32)$

$$\begin{cases} h_x = y-8x\lambda = 0 \rightarrow \lambda = \frac{y}{8x} \\ h_y = x-18y\lambda = 0 \rightarrow \lambda = \frac{x}{18y} \\ h_\lambda = 32-4x^2-9y^2 \end{cases}$$

$\frac{y}{8x} = \frac{x}{18y} \rightarrow 18y^2 = 8x^2 \rightarrow 4x^2 = 9y^2$

or $\lambda = 0$ (contradiction)

$$\begin{aligned} 32-4x^2-4x^2 &= 0 \\ 8x^2 &= 32 \\ x^2 &= 4 \rightarrow 16 = 9y^2 \\ x &= \pm 2 \qquad y = \pm\frac{4}{3} \end{aligned}$$

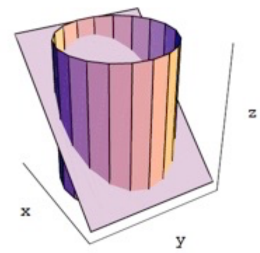
$\therefore$ CPs are $\left(2, \frac{4}{3}\right), \left(2, -\frac{4}{3}\right), \left(-2, \frac{4}{3}\right), \left(-2, -\frac{4}{3}\right)$

Week 6 (Oct 29th)

(b) The cylinder $x^2+y^2=1$ intersects the plane $x+z=1$ in an ellipse. Find the point on that ellipse that is the furthest from the origin.

We need to maximize the distance function (aka $f(x,y,z) = x^2+y^2+z^2$) subject to 2 constraints.

$h(x,y,z,\lambda,\mu) = x^2+y^2+z^2 - \lambda(x^2+y^2-1) - \mu(x+z-1)$



$$\begin{cases} h_x = 2x - 2x\lambda - \mu & ① \\ h_y = 2y - 2y\lambda & ② \\ h_z = 2z - \mu & ③ \\ h_\lambda = 1 - x^2 - y^2 & ④ \\ h_\mu = 1 - x - z & ⑤ \end{cases}$$

② $2y = 2y\lambda$
$\Rightarrow 2y(1-\lambda) = 0$
$y = 0\ ,\ \lambda = 1$

* If $\lambda = 1$:
① $\mu = 2x(1-\lambda) = 0$
③ $2z = \mu = 0 \rightarrow z = 0$
⑤ $x = 1$
④ $1 + y^2 = 1 \rightarrow y = 0$

* If $y = 0$:
④ $x^2 = 1 \rightarrow x = \pm 1$
⑤ $-1 + z = 1 \rightarrow z = 2\ (x = -1)$
$1 + z = 1 \rightarrow z = 0\ (x = 1)$ (we knew)

$\therefore$ CPs $(1,0,0)$ and $(-1,0,2)$

Based on the distance function:

$(1,0,0) \rightarrow \sqrt{1^2} = 1$

$(-1,0,2) \rightarrow \sqrt{(-1)^2+(0)^2+(2)^2} = \sqrt{5}$

$\therefore$ The furtherst point from the origin is $(-1,0,2)$.

Since the distance function is continuous and the intersection is compact in $\mathbb{R}^3$, the Extreme Value Theorem (EVT) ensures the existence of extreme.

Chain Rule

$(f(g(x)))' = f'(g(x))\, g'(x)$

Now, let's say $y = y(t)$ and $t = t(x)$

1D: $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$

2D: $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(\dfrac{dy}{dt} \cdot \dfrac{dt}{dx}\right)$ product rule!

$= \dfrac{d}{dx}\left(\dfrac{dy}{dt}\right)\dfrac{dt}{dx} + \dfrac{dy}{dt} \cdot \dfrac{d}{dx}\left(\dfrac{dt}{dx}\right) = \dfrac{d^2t}{dx^2}$

ODEs - Euler Equation

Second order differential equations:

Homogeneous with constant coefficients

$ay''+by'+cy=0$ where a,b,c are constants.

Characteristic equation $ar^2+br+c=0$, find roots r_1 and r_2

Then $y=c_1e^{r_1t}+c_2e^{r_2t}$ is a general solution

Ex ② - Let $y''+5y'+6y=0$

(a) Find the general solution of the equation

$r^2+5r+6=0=(r+2)(r+3)=0$
$r_1=-2$ and $r_2=-3$

Then $y=c_1e^{-2t}+c_2e^{-3t}$, $c_1,c_2\in\mathbb{R}$

(b) Find the solution of the initial value problem $y(0)=2$, $y'(0)=3$

$t=0$, $y(0)=c_1+c_2=2$

$y'=-2c_1e^{-2t}-3c_2e^{-3t} \rightarrow y'(0)=-2c_1-3c_2=3$

$\begin{cases} c_1+c_2=2 \\ -2c_1-3c_2=3 \end{cases}$ $\quad -c_2=7 \rightarrow c_2=-7$ and $c_1=9$

$\therefore y=9e^{-2t}-7e^{-3t}$

Ex ③ - Find the solution of the initial value problem $4y''-8y'+3y=0$, $y(0)=2, y'(0)=\frac{1}{2}$

$4r^2-8r+3=0=(2r-1)(2r-3)$
$r_1=\frac{1}{2}$, $r_2=\frac{3}{2}$

General sol: $y=c_1e^{t/2}+c_2e^{3t/2}$ $\qquad y'=\frac{c_1}{2}e^{t/2}+\frac{3c_2}{2}e^{3t/2}$

$\left.\begin{aligned} y(0)&=c_1+c_2=2 \\ y'(0)&=\frac{c_1}{2}+\frac{3c_2}{2}=\frac{1}{2} \end{aligned}\right\}$ $\begin{aligned} c_1&=2-c_2 \\ c_1&=1-3c_2 \end{aligned}$ $\qquad c_2=-\frac{1}{2}$, $c_1=\frac{5}{2}$

$\therefore y=\frac{5}{2}e^{t/2}-\frac{1}{2}e^{3t/2}$

Week 7 (Nov 2nd)

Ex ① - Consider $u = u(x,y)$. Find the general solution to the PDE $2u_x - u_y = 10$ with the initial condition $u(x,0) = 5x$ $\qquad a=2,\ b=-1$

Let $s = ax + by$, $t = bx - ay$

$s = 2x - y$, $t = -x - 2y$

$$\begin{cases} \frac{\partial s}{\partial x} = 2 & \frac{\partial s}{\partial y} = -1 \\ \frac{\partial t}{\partial x} = -1 & \frac{\partial t}{\partial y} = -2 \end{cases}$$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = 2\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = -\frac{\partial u}{\partial s} - 2\frac{\partial u}{\partial t}$$

$$\Rightarrow \quad 2u_x - u_y = 10$$

$$2\left(2\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t}\right) - \left(-\frac{\partial u}{\partial s} - 2\frac{\partial u}{\partial t}\right) = 10$$

$$5\frac{\partial u}{\partial s} = 10$$

$$\frac{\partial u}{\partial s} = 2$$

$$\int \partial u = \int 2\,\partial s$$

$$u = 2s + f(t)$$

$$\Rightarrow u(x,y) = 2(2x-y) + f(-x-2y) \qquad \text{G.S.}$$

IVP. Let $u(x,0) = 5x$

$$\Rightarrow u(x,0) = 4x + f(-x) = 5x$$

$$f(-x) = x \quad \rightarrow \quad f(-x-2y) = x + 2y$$

$$\therefore\ u(x,y) = 2(2x-y) + x + 2y = 5x$$

Ex ② - Consider $u = u(x,y)$. Find the general solution to the PDE $u_x - u_y + u^2 = 0$ with the initial condition $u(x,0) = \frac{1}{3x}$ $\qquad a=1,\ b=-1$

Let $s = x - y$, $t = -x - y$

$$\begin{cases} \frac{\partial s}{\partial x} = 1 & \frac{\partial s}{\partial y} = -1 \\ \frac{\partial t}{\partial x} = -1 & \frac{\partial t}{\partial y} = -1 \end{cases}$$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = -\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t}$$

$\Rightarrow$ $u_x - u_y + u^2 = 0$

$$\left(\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t}\right) - \left(-\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t}\right) + u^2 = 0$$

$$2\frac{\partial u}{\partial s} = -u^2$$

$$\int \frac{-2\,\partial u}{u^2} = \int \partial s$$

$$\frac{2}{u} = s + f(t)$$

$$u = \frac{2}{s + f(t)}$$

$$\Rightarrow \quad u(x,y) = \frac{2}{(x-y) + f(-x-y)} \qquad \text{G.S.}$$

IVP. Let $u(x,0) = \frac{1}{3x}$

$$u(x,0) = \frac{2}{x + f(-x)} = \frac{1}{3x}$$

$$6x = x + f(-x)$$
$$f(-x) = 5x$$
$$f(x) = -5x$$

$$u(x,y) = \frac{2}{x - y - 5(-x-y)} = \frac{2}{6x + 4y} = \frac{1}{3x + 2y}$$

Ex 3 - Let $u = u(x,y)$, $x = s^2 + 3t$, $y = s^2 t$. Find $\frac{\partial^2 u}{\partial s^2}$

$$\boxed{\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}}$$

$$\begin{cases} \frac{\partial x}{\partial s} = 2s & \frac{\partial y}{\partial s} = 2st \\ \frac{\partial x}{\partial t} = 3 & \frac{\partial y}{\partial t} = s^2 \end{cases}$$

$$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}\right)$$

$$= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial s} + \frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial s}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial}{\partial s}\left(\frac{\partial y}{\partial s}\right)$$

* $$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial s} = 2s\,u_{xx} + 2st\,u_{xy}$$

* $$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial s} = 2s\,u_{xy} + 2st\,u_{yy}$$

$$\Rightarrow \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial s}\right) = (2su_{xx} + 2st\,u_{xy})(2s) + u_x(2) + (2su_{xy} + 2stu_{yy})(2st) + u_y(2t)$$

$$= 4s^2u_{xx} + 4st\,u_{xy} + 2u_x + 4s^2t\,u_{xy} + 4s^2t^2u_{yy} + 2tu_y$$

$$= 4s^2u_{xx} + 4st\,u_{xy}(1+s) + 4s^2t^2u_{yy} + 2u_x + 2tu_y$$

Week 8 (Nov 9th)

Ex① - Find the general solution $u = u(x,t)$ to the PDE.
$u_{xx} - u_{xt} - 6u_{tt} = 0$ with the change of variables: $v = 2x - t$, $w = 3x + t$

$u = u(v,w)$ and $v = v(x,t)$, $w = w(x,t)$

$$\begin{cases} \frac{\partial v}{\partial x} = 2 & \frac{\partial w}{\partial x} = 3 \\ \frac{\partial v}{\partial t} = -1 & \frac{\partial w}{\partial t} = 1 \end{cases}$$

By Chain Rule:

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x} = 2\frac{\partial u}{\partial v} + 3\frac{\partial u}{\partial w}$$

$$u_{xx} = \frac{\partial}{\partial x}\left(2\frac{\partial u}{\partial v} + 3\frac{\partial u}{\partial w}\right) = \underbrace{2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v}\right) + 3\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial w}\right)}_{\text{use } u_x \text{ formula.}}$$

$$= 2\left[2\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v}\right) + 3\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v}\right)\right] + 3\left[2\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial w}\right) + 3\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial w}\right)\right]$$

$$= 4\frac{\partial^2 u}{\partial v^2} + 6\frac{\partial^2 u}{\partial v\partial w} + 6\frac{\partial^2 u}{\partial v\partial w} + 9\frac{\partial^2 u}{\partial w^2}$$

$$= 4\frac{\partial^2 u}{\partial v^2} + 12\frac{\partial^2 u}{\partial v\partial w} + 9\frac{\partial^2 u}{\partial w^2}$$

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial t} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial t} = -\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}$$

$$u_{tt} = \frac{\partial}{\partial t}\left(-\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) = \underbrace{-\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial v}\right) + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial w}\right)}_{\text{use } u_t \text{ formula.}}$$

$$= -\left[-\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v}\right)\right] + \left[-\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial w}\right) + \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial w}\right)\right]$$

$$= \frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial w \partial v} - \frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2}$$

$$= \boxed{\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2}}$$

$$u_{xt} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) = \underbrace{-\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial w}\right)}_{\text{use } u_x \text{ formula.}}$$

$$= -\left[2\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v}\right) + 3\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v}\right)\right] + \left[2\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial w}\right) + 3\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial w}\right)\right]$$

$$= -2\frac{\partial^2 u}{\partial v^2} - \frac{3\partial^2 u}{\partial w \partial v} + \frac{2\partial^2 u}{\partial v \partial w} + \frac{3\partial^2 u}{\partial w^2}$$

$$= \boxed{-2\frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial v \partial w} + \frac{3\partial^2 u}{\partial w^2}}$$

$$\Rightarrow u_{xx} - u_{xt} - 6u_{tt} = 0$$

$$\Rightarrow 4\frac{\partial^2 u}{\partial v^2} + \frac{12\partial^2 u}{\partial v \partial w} + \frac{9\partial^2 u}{\partial w^2} - \left[-2\frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial v \partial w} + \frac{3\partial^2 u}{\partial w^2}\right]$$
$$-6\left[\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2}\right] = 0$$

$$\Rightarrow 25\frac{\partial^2 u}{\partial v \partial w} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial v \partial w} = 0$$

$$\int \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v}\right) dw = \int 0 \, dw + f(v)$$ → integrate over w
v is a constant.

$$\frac{\partial u}{\partial v} = f(v)$$

$$\int \frac{\partial u}{\partial v} dv = \int f(v) dv + g(w)$$ → integrate over v
w is a constant

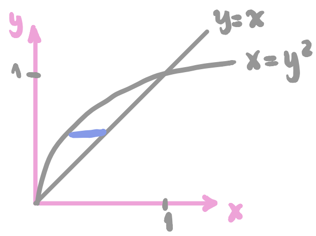
$$u = F(v) + g(w)$$

$$\therefore \quad u(x,t) = F(2x-t) + g(3x+t)$$

2D - Integration

Ex ① - Give a rough sketch of the regions and evaluate the following integrals

a) $\iint_D \sqrt{x}\, dA$ where D is the region bounded by $x = y$ and $x = y^2$



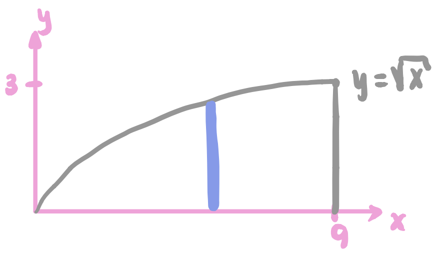
① $y^2 \leq x \leq y$

② $0 \leq y \leq 1$

$$\iint_D \sqrt{x}\, dA = \int_0^1 \int_{y^2}^{y} \sqrt{x}\, dx\, dy$$
$$= \int_0^1 \frac{2}{3} x^{3/2} \Big|_{y^2}^{y} dy$$
$$= \frac{2}{3} \int_0^1 y^{3/2} - y^3 dy$$
$$= \frac{2}{3} \left[\frac{2}{5} y^{5/2} - \frac{y^4}{4} \right]_0^1 = \frac{1}{10}$$

Week 9 (Nov 16)

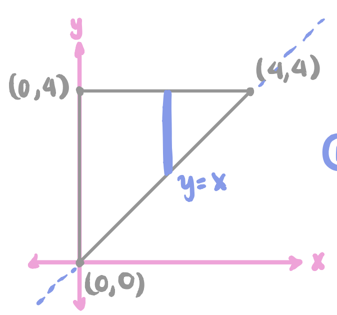
b) $\iint_D y \sin x^2\, dA$ where $D = \{(x,y) \in \mathbb{R}^2, 0 \leq y \leq \sqrt{x}, 0 \leq x \leq 9\}$



$$\iint_D y \sin x^2 dA = \int_0^9 \int_0^{\sqrt{x}} y \sin x^2 dy\, dx$$
$$= \int_0^9 \frac{y^2}{2} \sin x^2 \Big|_0^{\sqrt{x}} dx$$
$$= \frac{1}{2} \int_0^9 x \sin x^2 dx$$
$$= -\frac{1}{4} \cos x^2 \Big|_0^9$$
$$= \frac{1}{4}(1 - \cos 81)$$

Ex ② - Evaluate $\iint_D f(x,y)\, dA$ for the functions f and region D

a) $f(x,y) = \cos y$ and D is the triangle with vertices (0,0), (4,4) and (0,4)

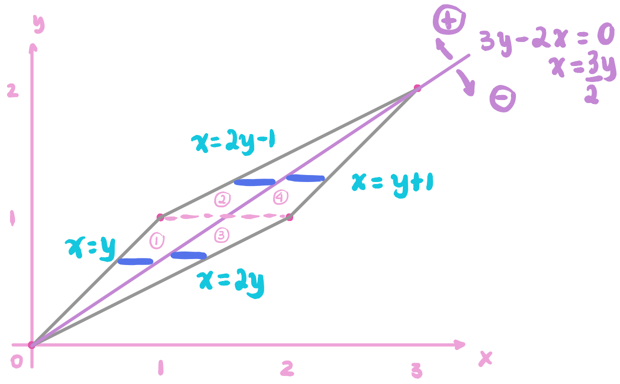


① $x \leq y \leq 4$

② $0 \leq x \leq 4$

$$\iint_D \cos y\, dA = \int_0^4 \int_x^4 \cos y\, dy\, dx$$
$$= \int_0^4 [\sin y]_x^4 dx$$
$$= \int_0^4 \sin 4 - \sin x\, dx$$
$$= [x \sin 4 + \cos x]_0^4$$
$$= 4 \sin 4 + \cos 4 - 1$$

b) $f(x,y) = |3y-2x|$ and D is the parallelogram with vertices $(0,0), (1,1)$ $(2,1), (3,2)$



① $y \le x \le \frac{3y}{2}$, $0 \le y \le 1$ (+)

② $2y-1 \le x \le \frac{3y}{2}$, $1 \le y \le 2$ (+)

③ $\frac{3y}{2} \le x \le 2y$, $0 \le y \le 1$ (−)

④ $\frac{3y}{2} \le x \le y+1$, $1 \le y \le 2$ (−)

$$\int_D |3y-2x|\, dA = ① + ② + ③ + ④$$

$$= \int_0^1 \int_y^{\frac{3y}{2}} (3y-2x)\,dx\,dy + \int_1^2 \int_{2y-1}^{3y/2} (3y-2x)\,dx\,dy + \int_0^1 \int_{\frac{3y}{2}}^{2y} -(3y-2x)\,dx\,dy + \int_1^2 \int_{\frac{3y}{2}}^{y+1} -(3y-2x)\,dx\,dy$$

$$= \int_0^1 \left[3yx - x^2\right]_y^{3y/2} dy + \int_1^2 \left[3yx - x^2\right]_{2y-1}^{3y/2} dy - \int_0^1 \left[3yx - x^2\right]_{3y/2}^{2y} dy - \int_1^2 \left[3yx - x^2\right]_{3y/2}^{y+1} dy$$

$$= \int_0^1 \left[\frac{9y^2}{2} - \frac{9y^2}{4} - 3y^2 + y^2\right] dy + \int_1^2 \left[\frac{9y^2}{2} - \frac{9y^2}{4} - 6y^2 + 3y + (2y-1)^2\right] dy$$

$$- \int_0^1 \left[6y^2 - 4y^2 - \frac{9y^2}{2} + \frac{9y^2}{4}\right] dy - \int_1^2 \left[3y^2 + 3y - (y+1)^2 - \frac{9y^2}{2} + \frac{9y^2}{4}\right] dy$$

$$= \int_0^1 \frac{y^2}{4}\,dy + \int_1^2 \frac{y^2}{4} - y + 1\,dy + \int_0^1 \frac{y^2}{4}\,dy + \int_1^2 \frac{y^2}{4} - y + 1\,dy$$

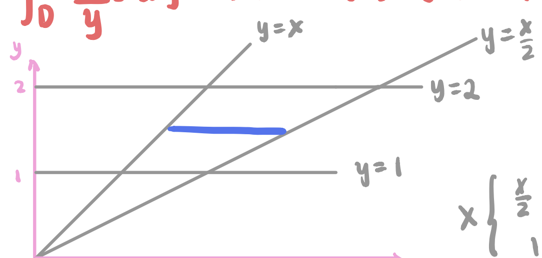
$$= \int_0^1 \frac{y^2}{2}\,dy + \int_1^2 \frac{y^2}{2} - 2y + 2\,dy$$

$$= \left.\frac{y^3}{6}\right|_0^1 + \left[\frac{y^3}{6} - y^2 + 2y\right]_1^2$$

$$= \frac{1}{6} + \left[\frac{4}{3} - 4 + 4 - \frac{1}{6} + 1 - 2\right] = \frac{1}{3}$$

Ex ③ - Solve

(a) $\int_D \frac{\sin y}{y}\,dy\,dx$, where D is the region bounded by $y = x$, $y = \frac{x}{2}$, $y = 1$ and $y = 2$



$$\int_D \frac{\sin y}{y}\,dy\,dx = \int_1^4 \int_{x/2}^{x} \frac{\sin y}{y}\,dy\,dx$$

$x \begin{cases} \frac{x}{2} \le y \le x \\ 1 \le x \le 4 \end{cases}$

$$= \int_1^2 \int^{2y} \frac{\sin y}{y}\,dx\,dy$$

$$\begin{cases} y \le x \le 2y \\ 1 \le y \le 2 \end{cases}$$

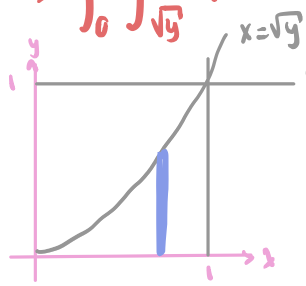
$$\int_1 \int_y \frac{\ }{y}$$

$$= \int_1^2 \frac{\sin y}{y}(2y-y)\,dy$$

$$= \int_1^2 \sin y\,dy$$

$$= -\cos y \Big|_1^2 = \cos 1 - \cos 2$$

(b) $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{2+x^3}\,dx\,dy$



$$0 \le y \le x^2$$
$$0 \le x \le 1$$

$$= \int_0^1 \int_0^{x^2} \sqrt{2+x^3}\,dy\,dx$$

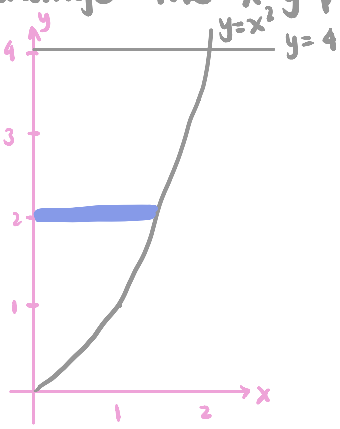
$$= \int_0^1 x^2\sqrt{2+x^3}\,dx = \left[\frac{2}{9}(2+x^3)^{3/2}\right]_0^1 = \frac{2}{9}(3\sqrt{3}-2\sqrt{2})$$

3D-Integration

Ex ④ - Write the integral $\int_0^2 \int_{x^2}^4 \int_0^{\sqrt{y-x^2}} f(x,y,z)\,dz\,dy\,dx$ in two other orders.

Given
$$0 \le z \le \sqrt{y-x^2}$$
$$x^2 \le y \le 4$$
$$0 \le x \le 2$$

* Change the x-y projection.



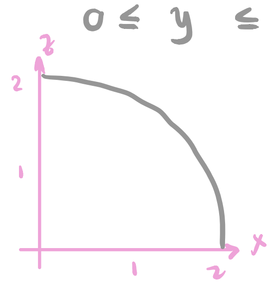
$$0 \le x \le \sqrt{y}$$
$$0 \le y \le 4$$

$$\rightarrow \int_0^4 \int_0^{\sqrt{y}} \int_0^{\sqrt{y-x^2}} f(x,y,z)\,dz\,dx\,dy$$

* Change the x-z projection.

$$0 \le z \le \sqrt{y-x^2}$$
$$0 \le x \le \sqrt{y}$$
$$0 \le y \le 4 \rightarrow \text{Fixed.}$$

$$x^2+z^2 \le y \le 4$$
$$0 \le z \le \sqrt{4-x^2}$$
$$0 \le x \le 2$$

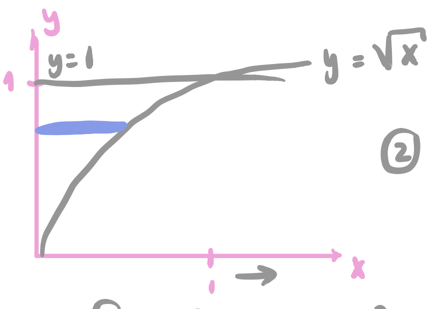


$$\rightarrow \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+z^2}^4 f(x,y,z)\,dy\,dz\,dx.$$

Ex⑤ - Write the integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx$ in two other orders

Given
$$0 \le z \le 1-y$$
$$\sqrt{x} \le y \le 1$$
$$0 \le x \le 1$$

(*) Change to dzdxdy



② $0 \le y \le 1$

$$\int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)\, dz\, dx\, dy$$

① $0 \le x \le y^2$

(*) Change to dydxdz

$0 \le z$ $\quad z \le 1-y$ $\quad \} \quad 1-y \le 1 \quad \Rightarrow \quad 0 \le z \le 1-y \le 1$

$\sqrt{x} \le y$ $\quad y \le 1$ $\quad \Rightarrow \quad 0 \le z \le 1$

$0 \le x$ $\quad x \le 1$ $\quad \} \quad x \le y^2 \quad \Rightarrow \quad z \le 1-y \to y \le 1-z$

$0 \le x \le (1-z)^2$

$\sqrt{x} \le y \le 1-z$

$$\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\, dy\, dx\, dz$$