MATA02 - The Magic of Numbers

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Tutorials: T2 (HW308 Thursdays 9-10am)
T3 (AA206 Tuesdays 9-10am)
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Week 1 - Jan. 6th

No tutorials

Week 2 - Jan 13th

Today: Greatest Common Divisor (gcd) and Least Common Multiple (lcm)
↳ $\gcd(6,8)=2$ ↳ $\text{lcm}(6,8)=24$

Euclidean Algorithm: If $a=bq+r, 0\leq r<b$ then $\gcd(a,b)=\gcd(b,r)$

* $\gcd(a,0)=a$
* $\gcd(0,b)=b$
* $\text{lcm}(a,b)=\dfrac{ab}{\gcd(a,b)}$

Ex1: Use the Euclidean Algorithm to find $\gcd(51,96)$ and use it to find the $\text{lcm}(51,96)$

$96 = 51\cdot 1 + 45 \quad \rightarrow d=\gcd(96,51)$

$51 = 45\cdot 1 + 6 \quad \rightarrow d=\gcd(51,45)$

$45 = 6\cdot 7 + 3 \quad \rightarrow d=\gcd(45,6)$

$6 = \underline{\underline{3}}\cdot 2 + 0 \quad \rightarrow d=\gcd(6,3)$
↳ stop!

$\therefore \gcd(96,51)=\gcd(51,45)=\gcd(45,6)=\gcd(6,3)=3$

$\therefore \text{lcm}(96,51)=\dfrac{96\times 51}{\gcd(96,51)}=\dfrac{96\times 51}{3}=96\times 17=1632$

* If $a=ms$ and $b=mt$, then $\gcd(a,b)=m\times\gcd(s,t)$

Ex1: (Again!) but using this method.

$\gcd(51,96)=\gcd(3\cdot 17, 3\cdot 32)=3\cdot\gcd(17,32)=3\cdot 1=3$ ✓

Ex2: (T1-Q1) Use the Euclidean Algorithm to find gcd(366,150) and use it to find the lcm(366, 150)

$366 = 150 \cdot 2 + 66$ → $d = \gcd(366,150)$

$150 = 66 \cdot 2 + 18$ → $d = \gcd(150,66)$

$66 = 18 \cdot 3 + 12$ → $d = \gcd(66,18)$

$18 = 12 \cdot 1 + 6$ → $d = \gcd(18,12)$

$12 = \underline{\underline{6}} \cdot 2 + 0$ → $d = \gcd(12,6)$

$\therefore \gcd(366,150) = 6$

$\therefore \operatorname{lcm}(366,150) = \dfrac{366 \times 150}{\gcd(366,150)} = \dfrac{366 \times 150}{6} = 61 \times 150 = 9150$

Ex3: (T1-Q3) Does the equation $12x + 20y = 90$ have a solution of integers x and y?

No :(Because on the left side we can factor out 4 → $4(3x+5y)$ however, 90 is <u>not</u> divisible by 4.

Ex4: Does the equation $11x + 1111 = 121 + 22y$ have a solution of integers x and y?

No :(. Because the right side is divisible by 11 → $11(11+2y)$, but the left side isn't as 1111 is not divisible by 11.

Ex5: (T1-Q4) If a and b are integers, and x is an integer such that $x^2+ax+b=0$. Show that $x \mid b$

↳ b is divisible by x

$x^2+ax+b=0$

$\Rightarrow b = -x^2 - ax$

$\Rightarrow b = -x\underbrace{(x+a)}$

↳ is an integer because x and a are integers

Because b is also an integer, then $x \mid b$. ∎

Ex6: Show that if $a \mid b$ and $b \mid a$, then $a = b$ or $a = -b$

$a \mid b$ means that there exist an $n \in \mathbb{Z}$ such that $an = b$ ①

$b \mid a$ means that there exist an $m \in \mathbb{Z}$ such that $bm = a$ ②

Use ① for ②: $(an)m = a \Rightarrow anm = a \Rightarrow nm = 1$.

The only integers n and m that work for $nm = 1$ are $n = m = 1$ or $n = m = -1$

→ When $n = m = 1 \Rightarrow a = b$, by ①

→ when $n = m = -1 \Rightarrow a = -b$, by ②

☒

Week 3 - Jan 20th

Ex 1 (T2 - Q1) Find integers x and y such that $6x + 5y = 4$

let's do Euclid's Algo!

$6 = 5 \cdot 1 + 1$ → But wait! This line here might help:

$5 = 1 \cdot 4 + 1$

$1 = 1 \cdot 1 + 0$

let's write it differently: $6 - 5 \cdot 1 = 1$

Now, we need the equation to equal 4: $(6 - 5 \cdot 1 = 1) \times 4$

$6 \cdot 4 - 5 \cdot 4 = 4$

The signs are different... $6 \cdot \underbrace{4}_{x} + 5 \cdot \underbrace{(-4)}_{y} = 4$

∴ In $6x + 5y = 4$, $x = 4$, $y = -4$

Ex 2 Determine if the equations has integer solutions x and y.

eq 1 $\quad 1015x + 231y = 9$

eq 2 $\quad 1015x + 231y = 28$

① Find the gcd of $(1015, 231)$

$1015 = 231 \times 4 + 91$ (1)

$231 = 91 \times 2 + 49$ (2)

$91 = 49 \times 1 + 42$ (3)

$49 = 42 \times 1 + 7$ (4)

$42 = 7 \times 6 + 0$ → $\gcd(1015, 231) = 7$

so by definition $\exists m, n$ such that $1015m + 231n = 7$

Looking at eq 1 and eq 2, $7 \nmid 9$ and $7 \mid 28$. So only eq 2 has integer solution.

② Write gcd in terms of the values.

$$
\begin{aligned}
7 &= 49 - 42\times 1 && (4)\\
&= 49 - (91 - 49\times 1) && (3)\\
&= 49\times 2 - 91\\
&= (231 - 91\times 2)\times 2 - 91 && (2)\\
&= 231\times 2 - 91\times 4 - 91\\
&= 231\times 2 - 91\times 5\\
&= 231\times 2 - (1015 - 231\times 4)\times 5 && (1)\\
&= 231\times 2 - 1015\times 5 + 231\times 20\\
7 &= 231\times 22 + 1015\times(-5)
\end{aligned}
$$

③
$$
\begin{aligned}
4\times 7 &= 231\times 22\times 4 + 1015\times(-5)\times 4\\
28 &= 231\times \underbrace{88}_{y} + 1015\times \underbrace{(-20)}_{x}
\end{aligned}
$$

④ We want to find a general solution for $28 = 1015X + 231Y$

Theorem: If $d = \gcd(a,b) \mid c$ and x, y is an integer solution of $ax+by=c$, then so is $x + \frac{b}{d}t$, $y - \frac{a}{d}t$ for any integer t.

In this case, $d = 7 \mid 28$ and we have $28 = \underbrace{1015}_{a}x + \underbrace{231}_{b}y$

We know $x = -20$ and $y = 88$ in ③

The general solution has $X = -20 + \frac{231}{7}t = -20 + 33t$

$Y = 88 - \frac{1015}{7}t = 88 - 145t$

$\therefore$ The general solution is $1015(-20+33t) + 231(88-145t) = 28$

<u>Ex 3</u> (T2-Q2) Find all integers x and y such that $30x + 8y = 500$. For which solutions are x and y both positive?

①
$$
\begin{aligned}
30 &= 8\times 3 + 6 && (1)\\
8 &= 6\times 1 + 2 && (2)\\
6 &= 2\times 3 + 0 && \rightarrow \gcd(30,8) = 2
\end{aligned}
$$

②
$$
\begin{aligned}
2 &= 8 - 6\times 1 && (2)\\
&= 8 - (30 - 8\times 3)\times 1 && (1)\\
&= 8 - 30\times 1 + 8\times 3\\
2 &= 8\times 4 + 30\times(-1)
\end{aligned}
$$

③
$$
\begin{aligned}
250\times 2 &= 8\times 4\times 250 + 30\times(-1)\times 250\\
500 &= 8\times \underbrace{1000}_{y} + 30\times \underbrace{(-250)}_{x}
\end{aligned}
$$

④ General solution: $x = -250 + \frac{8}{2}t = -250 + 4t$

$y = 1000 - \frac{30}{2}t = 1000 - 15t$

$x \geq 0 \rightarrow -250 + 4t \geq 0$
$4t \geq 250$
$t \geq \frac{125}{2} = 62.5$

$y \geq 0 \rightarrow 1000 - 15t \geq 0$
$1000 \geq 15t$
$\frac{200}{3} \geq t \approx 66.\bar{6}$

So $62.5 \leq t \leq 66.\bar{6}$, t is an integer

$\therefore$ $t = 63, 64, 65, 66$ to have x and y positive

Ex 4 (T2-Q4) Suppose the sophomores, juniors, and seniors in the tutorial decided to collect money to host a party. If each sophomore contributes \$25, each junior contributes \$18, each senior contributes \$10, \$450 will be collected. If there are 35 students, how many sophomores, juniors, and seniors are there?

$x + y + z = 35$ ①
$25x + 18y + 10z = 450$ ②

① $\times 10 \rightarrow 10x + 10y + 10z = 350$ ③

② - ③ $\rightarrow 15x + 8y = 100$

let's do Euclid's Alg. to find the gcd of 15 and 8

①
$15 = 8 \times 1 + 7$ (1)
$8 = 7 \times 1 + 1$ (2)
$7 = 1 \times 7 + 0$ $\rightarrow \gcd(15, 8) = 1$

②
$1 = 8 - 7 \times 1$ (2)
$1 = 8 - (15 - 8 \times 1) \times 1$ (1)
$1 = 8 \times 2 - 15 \times 1$
$1 = 8 \times \underbrace{2}_{y} + 15 \times \underbrace{(-1)}_{x}$

↳ but there cannot be a negative solution!

③
$100 \times 1 = 8 \times 2 \times 100 + 15 \times (-1) \times 100$
$100 = 8 \times 200 + 15 \times (-100)$

④ General solution: $x = -100 + \frac{8t}{1} = -100 + 8t$

$y = 200 - \frac{15t}{1} = 200 - 15t$

We need to find $x \geq 0$, $y \geq 0$

$$-100 + 8t \geq 0 \qquad\qquad 200 - 15t \geq 0$$
$$8t \geq 100 \qquad\qquad 200 \geq 15t$$
$$t \geq \frac{25}{2} = 12.5 \qquad\qquad 13.\bar{3} \approx \frac{40}{3} \geq t$$

$\Rightarrow 12.5 \leq t \leq 13.\bar{3}$, and t is an integer.

$\therefore\ t = 13 \rightarrow x = -100 + 8 \times 13 = -100 + 104 = 4$
$\rightarrow y = 200 - 15 \times 13 = 200 - 195 = 5$

$\rightarrow x + y + z = 35 \Rightarrow 4 + 5 + z = 35$
$\Rightarrow z = 26$

Ex 5 (T2-Q5) Show there are infinitely primes of the form $4t-1$, where t is an integer

Let's do a proof by contradiction. Which means, we will assume there are only a finite number of primes of the form $4t-1$, we will call them $p_1, p_2, \cdots, p_r$.

Let $N = p_1 p_2 \cdots p_r$, and let's consider the number $4N-1$. By our initial assumption, $4N-1$ is not prime, so it has a prime factor!

We claim $4N-1$ has a prime factor of the form $4t-1$.

Now, let's see what happens when we multiply $4x+1$ and $4y+1$

$\rightarrow (4x+1)(4y+1) = 16xy + 4x + 4y + 1 = 4(4xy + x + y) + 1$

Which means, $4N-1$ must have a prime factor of form $4t-1$. Since 2 is not a factor.

So there is a p_i such that $p_i \mid 4N-1$. Then as $p_i \mid N$, we have $p_i \mid 1$. Which contradicts!

There are infinitely many primes of the form $4t-1$. ∎