MATB41 - Tutorial 3 (BV355 Fridays 2-3pm)

TA: Angela Zavaleta-Bernuy
Office hours: IC404 Fridays 12-12:30 pm
email: angela.zavaletabernuy@mail.utoronto.ca
website: angelazb.github.io

Week 1 - Sept. 6th

No tutorials

Week 2 - Sept. 13th

Derivative by definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

Riemann Sum:

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x, \text{ where } \Delta x = \frac{b-a}{n},\ x_i = a + i\Delta x$$

Useful formulas → $\sum_{i=1}^{n} 1 = n$, $\sum_{i=1}^{n} i = \frac{i(i+1)}{2}$, $\sum_{i=1}^{n} i^2 = \frac{i(i+1)(2i+1)}{2 \cdot 3}$

Trig. Identities:

$\sin^2 x + \cos^2 x = 1$
$\cos(2x) = \cos^2 x - \sin^2 x$
$\sin(2x) = 2\sin a \cos a$

Half Angle Identities:

$$\sin\left(\frac{a}{2}\right) = \pm\sqrt{\frac{1-\cos a}{2}} \qquad \cos\left(\frac{a}{2}\right) = \pm\sqrt{\frac{1+\cos a}{2}}$$

Trig. Integrals:

$\int \cos^n(x)\sin^m(x)\,dx$

if n or m are odd → do u substitution
if n and m are even → half ∡ substitution

Ex: Evaluate $\int \frac{\sin^3(\ln x)\cos^3(\ln x)}{x}\,dx$

$$= \int \sin^3(\ln x)\cos^2(\ln x)\frac{\cos(\ln x)}{x}\,dx$$

$$= \int \sin^3(\ln x)(1-\sin^2(\ln x))\frac{\cos(\ln x)}{x}\,dx$$

let $u = \sin(\ln x)$
$du = \frac{\cos(\ln x)}{x}\,dx$

$$= \int u^3(1-u^2)\,du$$

$$= \int u^3 - u^5\,du$$

$$= \frac{u^4}{4} - \frac{u^6}{6}$$

$$= \frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} + C$$ //

Partial fractions:

Ex: Evaluate $\int \frac{1}{x(x^2-1)}\,dx$

$$\frac{1}{x(x^2-1)} = \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{A(x^2-1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$$= \frac{Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx}{x(x-1)(x+1)} = \frac{x^2(A+B+C) + x(B-C) - A}{x(x-1)(x+1)}$$

$$\begin{cases} A+B+C = 0 \\ B-C = 0 \rightarrow B = C \\ -A = 1 \rightarrow A = -1 \end{cases}$$

$\rightarrow -1 + 2B = 0$

$B = \frac{1}{2} = C$

$$\int \frac{1}{x(x^2-1)}\,dx = \int -\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{x+1}\,dx$$

$$= -\ln|x| + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$$ //

Angle between vectors:

$$\cos\theta = \frac{u\cdot w}{\|u\|\,\|w\|}$$

Cauchy-Schwarz inequality:

$$|u\cdot w| \leq \|u\|\,\|w\|, \quad u, w \in \mathbb{R}^n$$

Orthogonal:

$$u \cdot v = 0$$

Projection u onto w:

$$\frac{u\cdot w}{\|w\|^2}\,w$$

Ex: Let $v = (1,-3,1)$ and $w = (2,1,2)$ be vectors in $\mathbb{R}^3$

$$v\cdot w = (1,-3,1)\cdot(2,1,2) = 2 - 3 + 2 = 1$$

$$\|v\| = \sqrt{1^2 + (-3)^2 + 1^2} = \sqrt{11} \qquad \|w\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

(a) Find the angle between v and w

$$\cos\theta = \frac{v\cdot w}{\|v\|\,\|w\|} = \frac{1}{\sqrt{11}\cdot 3}, \quad \theta = \cos^{-1}\left(\frac{1}{3\sqrt{11}}\right)$$ //

(b) Verify the Cauchy-Schwarz inequality and the triangle inequality for v and w

$$|v\cdot w| \leq \|v\|\,\|w\|$$

$$\Rightarrow \quad 1 < 3\sqrt{11}$$ ✓

(c) Find all unit vectors in $\mathbb{R}^3$ which are orthogonal to both v and w.

Let $u = (a, b, c)$ be orthogonal to both v and w.

$$(1,-3,1)(a,b,c) = a - 3b + c = 0 \rightarrow a = -c$$
$$(2,1,2)(a,b,c) = \underline{2a + b + 2c = 0}$$
$$-5b = 0 \rightarrow b = 0$$

So $u = (k, 0, -k)$, $k \in \mathbb{R}$

$$\|u\| = \sqrt{k^2 + k^2} = \sqrt{2}\,k = 1 \rightarrow k = \frac{1}{\sqrt{2}}$$

The vectors are $\left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$

(d) Find the projection of (i) v onto w and (ii) w onto v.

(i) Proj of v onto w : $\frac{v \cdot w}{\|w\|^2} w = \frac{1}{9}(2,1,2)$

(ii) Proj of w onto v : $\frac{w \cdot v}{\|v\|^2} v = \frac{1}{11}(1,-3,1)$

Eigenvalues and eigenvectors:

Values of λ for $\det(A - \lambda I) = 0$ and their vectors respectively.

The matrix is diagonalizable if $P^{-1}AP = D$ is a diagonal matrix where P is the eigenvectors matrix. (or $A = PDP^{-1}$)

Ex: Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

Is A diagonalizable?

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -\lambda & 0 \\ 2 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} + 2\begin{vmatrix} 0 & -\lambda \\ 2 & 0 \end{vmatrix}$$

$$= -\lambda(1-\lambda)(4-\lambda) + 4\lambda$$
$$= -\lambda[(4 - 5\lambda + \lambda^2) - 4]$$
$$= -\lambda(-5\lambda + \lambda^2)$$
$$= -\lambda^2(\lambda - 5) = 0$$
$$\rightarrow \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda = 5$$

For $\lambda_1 = 0$, $\lambda_2 = 0$
$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_3 = 0 \rightarrow x_1 = -2x_3$ so for λ_1 we can let $v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$2x_1 + 4x_3 = 0$

x_2 is a free variable so for λ_2 we can let $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda_3 = 5$ $\begin{bmatrix} -4 & 0 & 2 \\ 0 & -5 & 0 \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-4x_1 + 2x_3 = 0$

$-5x_2 = 0 \rightarrow x_2 = 0$

$2x_1 - x_3 = 0 \rightarrow x_3 = 2x_1$

so for λ_3 we can let $v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$P = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ $\left[\begin{array}{ccc|ccc} -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 & 0 & 2 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/5 & 0 & 2/5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/5 & 0 & 1/5 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/5 & 0 & 2/5 \end{array}\right] = P^{-1}$

$$D = \begin{bmatrix} -2/5 & 0 & 1/5 \\ 0 & 1 & 0 \\ 1/5 & 0 & 2/5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$, which is a diagonal matrix of $\lambda_1, \lambda_2, \lambda_3$.

Week 3 – Sept. 20th

Vector equation of a line in $\mathbb{R}^3$: $\vec{\ell} = a + t\vec{v}$ $\quad t, a_i, v_i \in \mathbb{R}$

$\vec{\ell} = (a_1, a_2, a_3) + t[v_1, v_2, v_3]$ $\quad i = 1,2,3$

Parametric equation of a line in $\mathbb{R}^3$: $x = a_1 + tv_1$

$y = a_2 + tv_2$

$z = a_3 + tv_3$

Ex: For each of the following lines, write its equation in vector and parametric form

(i) The line that passes through the point P_0 $(3,1,9)$ in the direction of $\vec{v} = (1,1,1)$

V. Eq. $\vec{l} = (3,1,9) + t(1,1,1)$, $t \in \mathbb{R}$

P. Eq. $x = 3+t \quad y = 1+t \quad z = 9+t$, $t \in \mathbb{R}$

(ii) The line that passes through points $P_0(-1,1,2)$ and $P_1(2,0,-3)$

The direction vector for the line is $(2,0,-3) - (-1,1,2) = (3,-1,-5)$

V. Eq. $\vec{l} = (-1,1,2) + t(3,-1,5)$, $t \in \mathbb{R}$

P. Eq. $x = -1+3t \quad y = 1-t \quad z = 2+5t$, $t \in \mathbb{R}$

(iii) The line that passes through the point $P_0(0,1,0)$ and is orthogonal to the plane $10x + 15y + 3z = 11$

The direction vector for this line is a normal vector for the plane $\vec{n} = (10,15,3)$

V. Eq. $\vec{l} = (0,1,0) + t(10,15,3)$, $t \in \mathbb{R}$

P. Eq. $x = 10t \quad y = 1+15t \quad z = 3t$, $t \in \mathbb{R}$

Ex: Find an equation of the plane that passes through 3 points $A(-1,1,2)$, $B(2,0,-3)$ and $C(2,-1,2)$

A pair of direction vectors in the plane are $\vec{v} = (2,0,-3) - (-1,1,2) = (3,-1,-5)$ and $\vec{w} = (2,-1,2) - (-1,1,2) = (3,-2,0)$

To find the normal of the plane $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & -5 \\ 3 & -2 & 0 \end{vmatrix}$

$= (-10,-15,-3)$

So the plane is $-10x - 15y - 3z = d \rightarrow -10(-1) - 15(1) - 3(2) = -11$

$\therefore$ The equation of the plane is $10x + 15y + 3z = 11$.

Ex: Find an equation of the plane that passes through the origin and contains the line $x = 2+3t \quad y = 1-2t \quad z = 1+2t$

The line when $t=0$ passes through the point $(2,1,1)$
So the vector $\vec{v} = (0,0,0) - (2,1,1) = (-2,-1,-1)$ is also on the plane.
We can let $\vec{w} = (3,-2,2)$

To find the normal of the plane $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & -1 \\ 3 & -2 & 2 \end{vmatrix}$

$= (-4,1,7)$

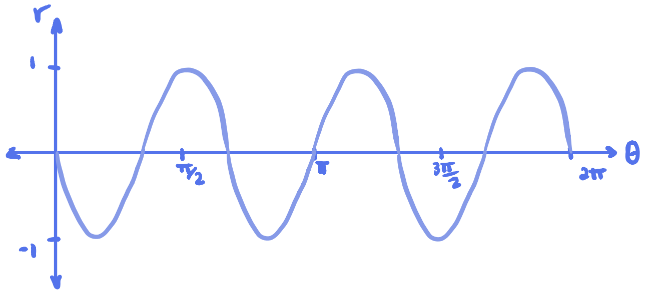
So the plane is $-4x + y + 7z = d \rightarrow -4(0) + (0) + 7(0) = 0$

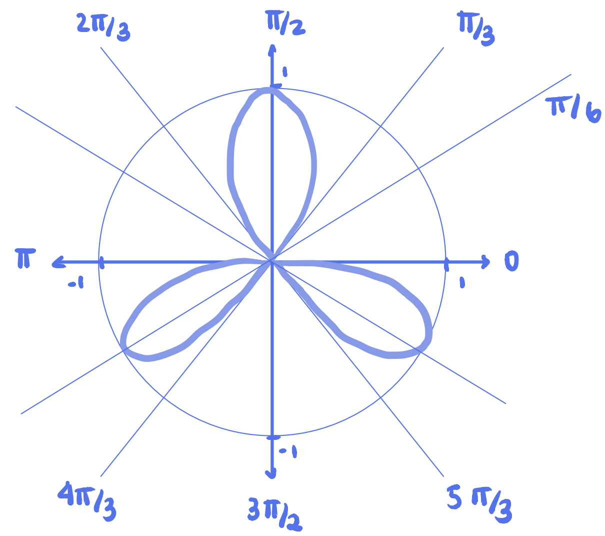
∴ The equation of the plane is $-4x+y+7z=0$

Polar equations : $x = r\cos\theta \quad y = r\sin\theta$

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta+\sin^2\theta) = r^2$$

Ex: Sketch the curve $r=-\sin 3\theta$ in the polar plane

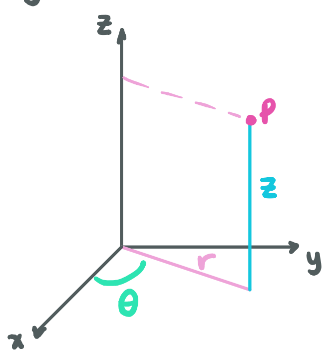




θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	-1	0	1	0	-1	0

Week 4 - Sept. 27th

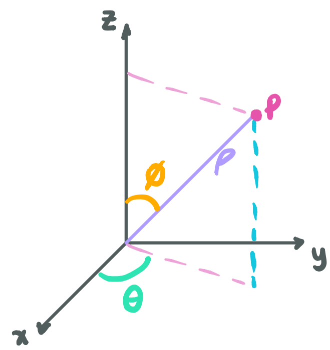
Cylindrical Coordinates:



$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$0 \le r \le \infty$
$0 \le \theta \le 2\pi$

Spherical Coordinates:



$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

$0 \le \rho \le \infty$
$0 \le \theta \le 2\pi$
$0 \le \phi \le \pi$

Ex: Interpret the equation $1 = 2\cos\theta\sin\theta$ geometrically

Cylindrical:

$\Rightarrow \quad r^2 = 2r\cos\theta\, r\sin\theta$
$\Rightarrow \quad x^2+y^2 = 2xy$
$\Rightarrow \quad x^2-2xy+y^2 = 0$
$\Rightarrow \quad (x-y)^2 = 0$

Spherical:

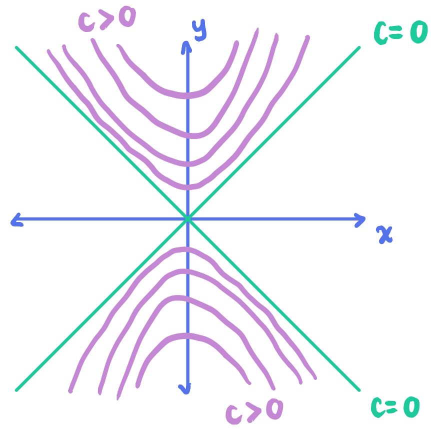
$\Rightarrow \quad \rho^2\sin^2\phi = 2\rho\sin\phi\cos\theta\,\rho\sin\phi\sin\theta$
$\Rightarrow \quad \rho^2\sin^2\phi(\sin^2\theta+\cos^2\theta) = 2\rho\sin\phi\cos\theta\,\rho\sin\phi\sin\theta$
$\Rightarrow \quad \rho^2\sin^2\phi\sin^2\theta + \rho^2\sin^2\phi\cos^2\theta = 2\rho\sin\phi\cos\theta\,\rho\sin\phi\sin\theta$
$\Rightarrow \quad x^2+y^2 = 2xy$
$\Rightarrow \quad x^2-2xy+y^2 = 0$
$\Rightarrow \quad (x-y)^2 = 0$

Ex: Characterize and sketch several level curves of the following functions:

(i) $f(x,y) = \frac{\sqrt{y^2 - x^2}}{2} = c$

$\operatorname{dom} f = \{(x,y) \in \mathbb{R}^2 \mid y^2 \geq x^2\}$, because $y^2 - x^2 \geq 0$

c cannot be -ve, $c \geq 0$ always



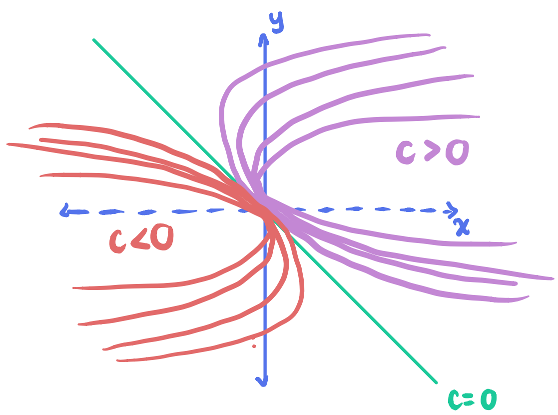
$\underline{c=0}$ $\quad y^2 = x^2$
$\quad y = \pm x \quad (*)$

$\underline{c>0}$ $\quad y^2 - x^2 = (2c)^2$
$\quad y^2 = (2c)^2 + x^2 \quad (*)$

y-intercepts: $(0, \pm 2c)$

(ii) $f(x,y) = \frac{x+y}{y^2} = c$

$\operatorname{dom} f = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$



$\underline{c=0}$ $\quad x+y = 0$
$\quad y = -x \quad (*)$

$\underline{c\neq 0}$ $\quad x = cy^2 - y$

y-intercepts: $y = cy^2$
$\Rightarrow y = \frac{1}{c}$

$\underline{c>0}$ $\quad x = cy^2 - y$
$\quad x = y(cy-1)$

y-intercepts:
$(0,0), (0, 1/c)$

parabola opening right $(*)$

$\underline{c<0}$ $\quad x = -cy^2 - y$
$\quad x = y(-cy-1)$

y-intercepts:
$(0,0), (0, -1/c)$

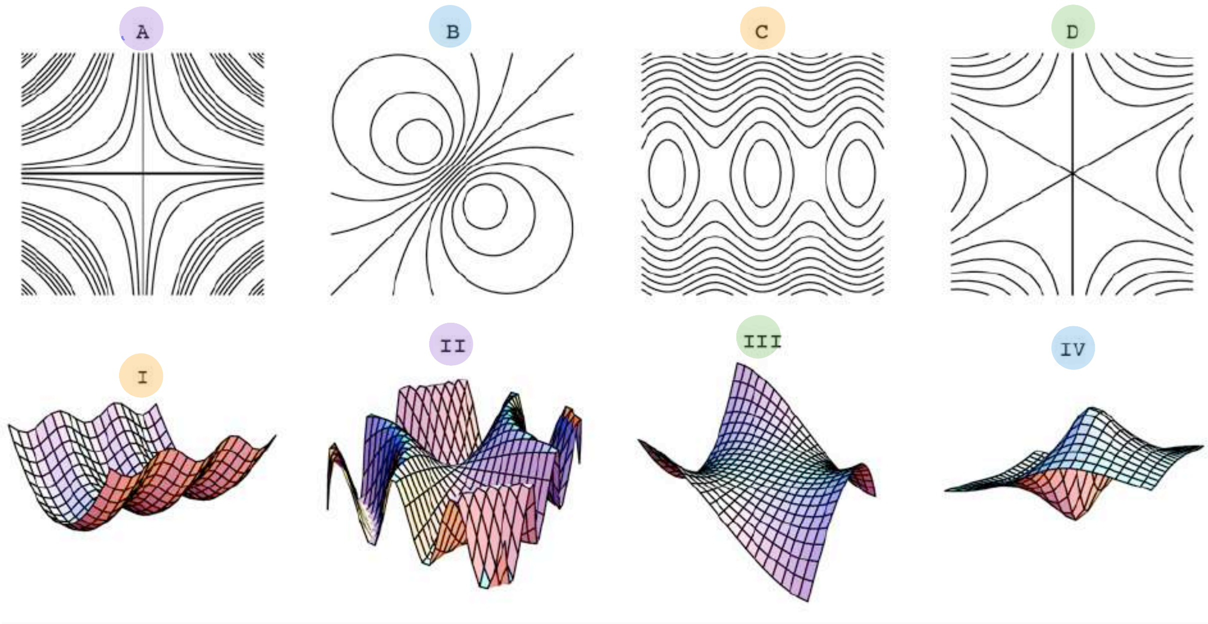
parabola opening left $(*)$

Ex: Give a rough sketch of the surface in $\mathbb{R}^3$ defined by
$3x^2 - 12x - y + 2z^2 + 4z + 9 = 0$

$\Rightarrow y = 3x^2 - 12x + 2z^2 + 4z + 9$
$\Rightarrow y = 3(x^2 - 4x + 4) - 12 + 2(z^2 + 2z + 1) - 2 + 9$
$\Rightarrow y = 3(x-2)^2 + 2(z+1)^2 - 5$

The elliptical paraboloid opening in the positive direction with vertex $(2,-5,-1)$

Ex: Indicate what contour diagram corresponds to each graph.



Week 5 - Oct. 4th

Ex: For each of the following, evaluate the limit or show that the limit does not exist.

(a) $$\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{3x+5y+2} = \frac{e^{(0)}}{3(0)+5(0)+2} = \frac{1}{2}$$

(b) $$\lim_{(x,y)\to(1,-2)} \frac{xy+2x-y-2}{(x^2-1)(y+2)} = \lim_{(x,y)\to(1,-2)} \frac{x(y+2)-(y+2)}{(x^2-1)(y+2)}$$

$$= \lim_{(x,y)\to(1,-2)} \frac{(y+2)(x-1)}{(x-1)(x+1)(y+2)}$$

$$= \lim_{(x,y)\to(1,-2)} \frac{1}{x+1} = \frac{1}{2}$$

(c) $$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

Recall:
$x = r\cos\theta$
$y = r\sin\theta$

$$= \lim_{r\to 0} \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta) = 0$$

(d) $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin(r^2)}{r^2}$$

$$\overset{L'H}{=} \lim_{r\to 0} \frac{2r\cos(r^2)}{2r}$$

$$= \lim_{r\to 0} \cos(r^2) = \cos(0) = 1$$

(e) $$\lim_{(x,y)\to(0,0)} \frac{|x|}{\sqrt{x^2+y^2}}$$

Restrict y-axis: $(y=0)$ $$\lim_{(x,0)\to(0,0)} \frac{|x|}{\sqrt{x^2}} = 1$$

$\neq$

Restrict x-axis: $(x=0)$ $$\lim_{(0,y)\to(0,0)} \frac{0}{\sqrt{y^2}} = 0$$

The limit does not exist.

(f) $$\lim_{(x,y)\to(1,1)} \frac{x^2+y^2-2}{|x-1|+|y-1|} = \lim_{(u,v)\to(0,0)} \frac{(u+1)^2+(v+1)^2-2}{|u|+|v|}$$

We can rewrite this limit letting $u=x-1$ and $v=y-1$

$$= \lim_{(u,v)\to(0,0)} \frac{u^2+2u+v^2+2v}{|u|+|v|}$$

Restrict $v=u$ and $|u|=u$: $$\lim_{u\to 0} \frac{u^2+2u+u^2+2u}{u+u}$$

$$= \lim_{u\to 0} \frac{2u^2+4u}{2u}$$

$$= \lim_{u\to 0} \frac{2u(u+2)}{2u}$$

$$= \lim_{u\to 0} u+2 = 2$$

$\neq$

Restrict $v=-u$ and $|u|=-u$: $$\lim_{u\to 0} \frac{u^2+2u+u^2-2u}{-u-u}$$

$$= \lim_{u\to 0} \frac{2u^2}{-2u}$$

$$= \lim_{u\to 0} u = 0$$

The limit does not exist.

Ex: Find the value of a so that f is continuous at (0,0)

$$f(x,y) = \begin{cases} \dfrac{x^3 - x^2 - 2x^2y + xy^2 - y^2 - 2y^3}{x^2+y^2} & , (x,y) \neq 0 \\ a & , (x,y) = 0 \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2(x-1-2y)+y^2(x-1-2y)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x-1-2y)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} x-1-2y = -1 = a$$

Ex: Determine whether the following functions are continuous throughout their domains.

(a)

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

Restrict y-axis: $(y=0)$ $\quad \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$

$\neq$

Restrict x-axis: $(x=0)$ $\quad \lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1$

The limit does not exist, so it is not continuous at (0,0)

(b)

$$f(x,y) = \begin{cases} \dfrac{2x^3+2xy^2+3x^2+3y^2}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{2x^3+2xy^2+3x^2+3y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{2x(x^2+y^2)+3(x^2+y^2)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(2x+3)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} 2x+3 = 3 \neq 0$$

The function is not continuous at (0,0)

Definition of Limit:

Let $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables x and y defined for all ordered pairs (x,y) in some open disk $D \subseteq \mathbb{R}^2$ centered on a fixed ordered pair (x_0, y_0). except possibly at (x_0, y_0)

We will say that the number $L \in \mathbb{R}$ is the limit of $f(x,y)$ as $(x,y) \in D$ approaches (x_0, y_0) if and only if given any real number $\varepsilon > 0$, we can find a real number $\delta > 0$ (depending on ε) such that $f(x,y)$ satisfies $|f(x,y) - L| < \varepsilon$ whenever the distance between (x,y) and (x_0, y_0) satisfies $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ and we will write:

$$\lim_{(x,y)\to(0,0)} f(x,y) = L \quad \text{or} \quad \lim_{(x,y)\to(0,0)} |f(x,y) - L| = 0$$

Ex: Use the definition of limits to show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$

We need to show that for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 < \sqrt{x^2+y^2} < \delta$, we have $\left|\frac{4xy^2}{x^2+y^2}\right| = 0$

So if $(x,y) \in D$ and $0 < \sqrt{x^2+y^2} < \delta$, we see that you should choose the corresponding positive real number $\delta = \varepsilon/4$ and we get

$$|f(x,y) - 0| = \left|\frac{4xy^2}{x^2+y^2} - 0\right| = 4|x| \cdot \frac{y^2}{x^2+y^2} \leq 4|x| \cdot 1 = 4\sqrt{x^2} \leq 4\sqrt{x^2+y^2} < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$

b/c $y^2 \leq x^2 + y^2$

$$\frac{y^2}{x^2+y^2} \leq 1$$

The number 0 is the limit of the function $f(x,y) = \frac{4xy^2}{x^2+y^2}$ as (x,y) in D approaches $(0,0)$ because for any given number $\varepsilon > 0$, we have shown that we can produce a corresponding number $\delta = \varepsilon/4 > 0$ so that $f(x,y) = \frac{4xy^2}{x^2+y^2}$ satisfies the inequality $\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \varepsilon$ whenever the distance between (x,y) and $(0,0)$ satisfies $0 < \sqrt{x^2+y^2} < \varepsilon/4 = \delta$. So we can write $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$ ∎

Week 6 - Oct. 11th

Ex: Let $f(x,y) = \begin{cases} \dfrac{x^3y - y^3x}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(a) Find $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ for $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{(3x^2y-y^3)(x^2+y^2)-2x(x^3y-y^3x)}{(x^2+y^2)^2} = \frac{3x^4y+3x^2y^3-x^2y^3-y^5-2x^4y+2x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{(x^3-3y^2x)(x^2+y^2)-2y(x^3y-y^3x)}{(x^2+y^2)^2} = \frac{x^5+x^3y^2-3y^2x^3-3y^4x-2x^3y^2+2y^4x}{(x^2+y^2)^2}$$

$$= \frac{x^5-4x^3y^2-y^4x}{(x^2+y^2)^2} = \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}$$

(b) Find $\frac{\partial f}{\partial x}(0,y)$ and $\frac{\partial f}{\partial y}(x,0)$

Using (a) we have

$$\frac{\partial f}{\partial x}(0,y) = \frac{y(-y^4)}{(y^2)^2} = -y \quad , \; y \neq 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(x,0) = \frac{x(x^4)}{(x^2)^2} = x \quad , \; x \neq 0$$

For $y = x = 0$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{0}{h^3}-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{0}{h^3}-0}{h} = 0$$

Directional derivative: $D_v f(p) = \nabla f(p) \cdot \frac{v}{\|v\|}$

Ex: Compute the directional derivative of $f(x,y,z) = xz + y^2z^2$ at the point $(3,-1,2)$ in the direction of the vector $v = (0,-3,4)$

$$\nabla f(x,y,z) = (z, 2yz^2, x+2y^2z)$$
$$\nabla f(3,-1,2) = (2,-8,7)$$

$$D_{(0,-3,4)} f(3,-1,2) = (2,-8,7) \cdot \frac{(0,-3,4)}{\|(0,-3,4)\|} = \frac{8 \cdot 3 + 7 \cdot 4}{\sqrt{9+16}} = \frac{52}{5}$$

Ex: For each of the following evaluate $\frac{\partial f}{\partial y}$ at the point a.

(a) $f(x,y) = y\sin(xy) + xe^{-y^2}$, $a = \left(\frac{\pi}{6}, 2\right)$

$$\frac{\partial f}{\partial y} = \sin(xy) + xy\cos(xy) - 2xy e^{-y^2}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(\frac{\pi}{6},2)} = \sin\frac{\pi}{3} + \frac{\pi}{3}\cos\frac{\pi}{3} - \frac{2\pi}{3}e^{-4} = \frac{\sqrt{3}}{2} + \frac{\pi}{6} - \frac{2\pi}{3e^4}$$

(b) $f(x,y,z) = 2^{\sqrt{x-yz}}$, $a=(3,2,1)$

$$\frac{\partial f}{\partial y} = 2^{\sqrt{x-yz}}(\ln 2)(\sqrt{x-yz})' = \frac{-z(\ln 2)2^{\sqrt{x-yz}}}{2\sqrt{x-yz}}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(3,2,1)} = \frac{(-\ln 2)2^{\sqrt{3-2}}}{2\sqrt{3-2}} = -\ln 2$$

(c) $f(x,y) = \begin{cases} \frac{2x^2y+3y^3}{x^2+y^2} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$, $a=(0,0)$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{3h^3}{h^2}-0}{h} = \lim_{h\to 0} 3 = 3$$

Ex: Let $f:\mathbb{R}^2\to\mathbb{R}^3$ be given by $f(x,y)=(xy^2, x+2y, xy)$ and let $g:\mathbb{R}^3\to\mathbb{R}^3$ be given by $g(x,y,z)=(xy, yz, xz^2)$.

(a) Find Df and Dg. Use the chain rule to find $D(g\circ f)$

$$D(g\circ f)(x,y) = Dg(f(x,y))\,Df(x,y)$$

$$Df = \begin{pmatrix} y^2 & 2xy \\ 1 & 2 \\ y & x \end{pmatrix} \qquad Dg = \begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z^2 & 0 & 2xz \end{pmatrix}$$

$$Dg(f(x,y)) = Dg(xy^2, x+2y, xy) = \begin{pmatrix} x+2y & xy^2 & 0 \\ 0 & xy & x+2y \\ x^2y^2 & 0 & 2x^2y^3 \end{pmatrix}$$

$$D(g\circ f)(x,y) = \begin{pmatrix} x+2y & xy^2 & 0 \\ 0 & xy & x+2y \\ x^2y^2 & 0 & 2x^2y^3 \end{pmatrix}\begin{pmatrix} y^2 & 2xy \\ 1 & 2 \\ y & x \end{pmatrix}$$

$$= \begin{pmatrix} y^2x+2y^3+xy^2 & 2x^2y+4xy^2+2xy^2 \\ xy+xy+2y^2 & 2xy+x^2+2xy \\ x^2y^4+2x^2y^4 & 2x^3y^3+2x^3y^3 \end{pmatrix}$$

$$= \begin{pmatrix} 2y^3+2xy^2 & 2x^2y+6xy^2 \\ 2xy+2y^2 & 4xy+x^2 \\ 3x^2y^4 & 4x^3y^3 \end{pmatrix}$$

(b) Compute $g \circ f$ and $D(g \circ f)$ directly

$$g \circ f(x,y) = g(xy^2, x+2y, xy) = (x^2y^2 + 2xy^3, x^2y + 2xy^2, x^3y^4)$$

$$D(g \circ f)(x,y) = \begin{pmatrix} 2y^3 + 2xy^2 & 2x^2y + 6xy^2 \\ 2xy + 2y^2 & 4xy + x^2 \\ 3x^2y^4 & 4x^3y^3 \end{pmatrix}$$

Ex: If $g(u,v) = f(u^2-v^2, v^2-u^2)$ and f is differentiable, show that g satisfies $v\frac{\partial g}{\partial u} + u\frac{\partial g}{\partial v} = 0$

Let $x = u^2 - v^2$ and $y = v^2 - u^2$, then $g(u,v) = f(x,y)$

Then $\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x}(2u) + \frac{\partial f}{\partial y}(-2u)$

and $\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial f}{\partial x}(-2v) + \frac{\partial f}{\partial y}(2v)$

$$\Rightarrow v\frac{\partial g}{\partial u} + u\frac{\partial g}{\partial v} = v\left[\frac{\partial f}{\partial x}(2u) + \frac{\partial f}{\partial y}(-2u)\right] + u\left[\frac{\partial f}{\partial x}(-2v) + \frac{\partial f}{\partial y}(2v)\right]$$
$$= 2uv\frac{\partial f}{\partial x} - 2uv\frac{\partial f}{\partial y} - 2uv\frac{\partial f}{\partial x} + 2uv\frac{\partial f}{\partial y}$$
$$= 0$$

Week 7 - Oct. 18th

No tutorials

Week 8 - Oct. 25th

The rate of change in f from p_1 to p_2 is the directional derivative $D_v f(p_1)$ where $v = p_2 - p_1$

Ex: Let $f: \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = 2x^2 + 2xz + y^2 + 4y + yz$. What is the rate of change in f if you move from $(1,0,1)$ towards $(1,2,3)$?

$v = (1,2,3) - (1,0,1) = (0,2,2)$

$\nabla f(x,y,z) = (4x + 2z, 2y + 4 + z, 2x + y)$

$$D_{(0,2,2)} f(1,0,1) = \nabla f(1,0,1) \cdot \frac{(0,2,2)}{\|(0,2,2)\|} = \frac{(6,5,2)\cdot(0,2,2)}{\sqrt{4+4}} = \frac{14}{\sqrt{8}}$$
$$= \frac{7}{\sqrt{2}} = \frac{7\sqrt{2}}{2}$$

Ex: Find an equation for the tangent plane to the surface given by $x^3z + x^2y^2 + \sin(yz) = -3$ at the point $(-1, 0, 3)$

$g(x,y,z) = x^3z + x^2y^2 + \sin(yz) + 3 = 0$

$\nabla g(x,y,z) = (3x^2z + 2xy^2,\ 2x^2y + z\cos(yz),\ x^3 + y\cos(yz))$

$\nabla g(-1,0,3) = (9, 3, -1)$ is the normal vector to the surface at $(-1,0,3)$

The equation of the tangent plane is given by

$\nabla g(-1,0,3)((x,y,z) - (-1,0,3)) = 0$ or $(-1)(x-(-1)) + 3(y-0) + (-1)(z-3) = 0$

$\Rightarrow (9, 3, -1)(x+1, y, z-3) = 0$

$\Rightarrow 9x + 9 + 3y - z + 3 = 0$

$\Rightarrow 9x + 3y - z = -12$

Ex: Find the point(s) on the graph of the function $f(x,y) = x^2 + y^2 - 1$ where the tangent plane is parallel to the plane $4x - 8y - z = 3$. What is the equation of the tangent plane at this point?

Normal vector of the plane $(4, -8, -1)$ and a normal vector to the tangent plane to $f(x,y)$ is $(2x, 2y, -1)$.

For the planes to be parallel we need $(2x, 2y, -1) = k(4, -8, -1)$ for some k

$2x = 4k \longrightarrow x = 2$

$2y = -8k \longrightarrow y = -4$

$-1 = -k \rightarrow k = 1$

$z = f(x,y) = f(2,-4) = 4 + 16 - 1 = 19$

$4(2) - 8(-4) - 19 = 8 + 32 - 19 = 21$

$\therefore$ The tangent plane passes through $(2, -4, 19)$ and the equation of the tangent plane at this point is $4x - 8y - z = 21$.

Ex: Find the second partial derivatives of

(a) $f(x,y) = 2x^2y^3 + y\sin^2(xy)$

First order:

$$\frac{\partial f}{\partial x} = 4xy^3 + 2y^2\sin(xy)\cos(xy)$$

$$\frac{\partial f}{\partial y} = 6x^2y^2 + \sin^2(xy) + 2xy\sin(xy)\cos(xy)$$

Second order:

$$\frac{\partial^2 f}{\partial x^2} = 4y^3 + 2y^3\cos^2(xy) - 2y^3\sin^2(xy)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 12xy^2 + 4y\sin(xy)\cos(xy) + 2xy^2\cos^2(xy) - 2xy^2\sin^2(xy)$$

$$\frac{\partial^2 f}{\partial y^2} = 12x^2y + 2x\sin(xy)\cos(xy) + 2x\sin(xy)\cos(xy) + 2x^2y\cos^2(xy) - 2x^2y\sin^2(xy)$$

$$= 12x^2y + 4x\sin(xy)\cos(xy) + 2x^2y\cos^2(xy) - 2x^2y\sin^2(xy)$$

(b) $f(x,y,z) = 2^{xz} - \frac{x^3 z}{y^2}$

First order:

$$\frac{\partial f}{\partial x} = z2^{xz}\ln 2 - \frac{3x^2 z}{y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2x^3 z}{y^3}$$

$$\frac{\partial f}{\partial z} = x2^{xz}\ln 2 - \frac{x^3}{y^2}$$

Second order:

$$\frac{\partial^2 f}{\partial x^2} = z^2 2^{xz}\ln^2 2 - \frac{6xz}{y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{6x^2 z}{y^3}$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = 2^{xz}\ln 2 + xz2^{xz}\ln^2 2 - \frac{3x^2}{y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{6x^3 z}{y^4}$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z} = \frac{2x^3}{y^3}$$

$$\frac{\partial^2 f}{\partial z^2} = x^2 2^{xz}\ln^2 2$$

Week 9 - Nov. 1st.

Taylor Series

* (x,y) around $(0,0)$:

$$F(x,y) = F(0,0) + x\frac{\partial F}{\partial x}(0,0) + y\frac{\partial F}{\partial y}(0,0)$$
$$+ \frac{1}{2!}\left[x^2 \frac{\partial^2 F}{\partial x^2}(0,0) + 2xy\frac{\partial^2 F}{\partial x \partial y}(0,0) + y^2 \frac{\partial^2 F}{\partial y^2}(0,0)\right] + \dots$$

* else:

$$F(x,y) = F(a,b) + \frac{\partial F}{\partial x}(x-a) + \frac{\partial F}{\partial y}(y-b)$$
$$+ \frac{1}{2!}\left[\frac{\partial^2 F}{\partial x^2}(x-a)^2 + 2\frac{\partial^2 F}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 F}{\partial y^2}(y-b)^2\right] + \dots$$

Ex: Directly compute the second degree Taylor polynomial about the point $(1,0)$ for $f(x,y) = e^{(x-1)^2}\cos y$

→ $f(1,0) = e^0 \cos(0) = 1$

→ $\frac{\partial f}{\partial x}(x,y) = 2(x-1)e^{(x-1)^2}\cos y$ $\qquad \frac{\partial f}{\partial x}(1,0) = 2(0)e^0\cos(0) = 0$

$\frac{\partial f}{\partial y}(x,y) = -e^{(x-1)^2}\sin y$ $\qquad \frac{\partial f}{\partial y}(1,0) = e^0 \sin(0) = 0$

→ $\frac{\partial^2 f}{\partial x^2}(x,y) = \left(2e^{(x-1)^2} + 4(x-1)^2 e^{(x-1)^2}\right)\cos y$ $\qquad \frac{\partial^2 f}{\partial x}(1,0) = \left(2e^0 + 4(0)e^0\right)\cos(0) = 2$

$\frac{\partial^2 f}{\partial x \partial y}(x,y) = -2(x-1)e^{(x-1)^2}\sin y$ $\qquad \frac{\partial^2 f}{\partial x \partial y}(1,0) = -2(0)e^0\sin(0) = 0$

$\frac{\partial^2 f}{\partial y^2}(x,y) = -e^{(x-1)^2}\cos y$ $\qquad \frac{\partial^2 f}{\partial y^2}(1,0) = -e^0\cos(0) = -1$

$$\therefore T_2 = f(1,0) + \left[\frac{\partial f}{\partial x}(1,0)(x-1) + \frac{\partial f}{\partial y}(1,0)y\right] + \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(1,0)(x-1)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(1,0)(x-1)y + \frac{\partial^2 f}{\partial y^2}y^2\right]$$

$$= 1 + \frac{1}{2!}\left[2(x-1)^2 - y^2\right]$$
$$= 1 + (x-1)^2 - \frac{y^2}{2}$$

Common series around $a=0$ are:

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, $|x| < \infty$ $\qquad \frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k$, $|x| < 1$

$\cos x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$, $|x| < \infty$ $\qquad \ln(1+x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1}$, $|x| < 1$

$\sin x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$, $|x| < \infty$ $\qquad \arctan x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$, $|x| < 1$

Ex: Find the 4^{th} degree Taylor polynomial about the <u>origin</u> of $f(x,y)=(\cos y)\ln(1+xy)$

Recall: $\cos t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!}$, $|t|<\infty$ and $\ln(1+t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{k+1}}{k+1}$, $|t|<1$

$\rightarrow \cos y = 1 - \frac{y^2}{2} + \frac{y^4}{4!} - \cdots$, $|y|<\infty$

$\rightarrow \ln(1+xy) = xy - \frac{x^2y^2}{2} + \frac{x^3y^3}{3} - \cdots$, $|xy|<1$

$$T = \left(1 - \frac{y^2}{2} + \frac{y^4}{4!} - \cdots\right)\left(xy - \frac{x^2y^2}{2} + \frac{x^3y^3}{3} - \cdots\right) = xy - \frac{x^2y^2}{2} + \frac{x^3y^3}{3} - \frac{xy^3}{2} + \frac{x^2y^4}{4} - \cdots$$

$\therefore T_4 = xy - \frac{x^2y^2}{2} - \frac{xy^3}{2}$

Ex: Let π be the plane in $\mathbb{R}^3$ passing through the points $(1,0,4)$, $(2,-1,0)$ and $(3,1,2)$

(a) Find an equation for π.

$v_1 = (2,-1,0) - (1,0,4) = (1,-1,-4)$
$v_2 = (3,1,2) - (1,0,4) = (2,1,-2)$

normal vector of the plane $v_1 \times v_2 = (2+4,\ 2-8,\ 1+2) = (6,-6,3)$

$6x - 6y + 3z = d \rightarrow 6 + 12 = d = 18$

$\therefore 2x - 2y + z = 6$

(b) Give a parametric description for the line ℓ through $(1,1,1)$ and orthogonal to π

normal of π is $(2,-2,1)$ so the line ℓ is $(1,1,1) + t(2,-2,1)$, $t \in \mathbb{R}$.

(c) Find those points on the ellipsoid $4x^2 + 8y^2 + 4z^2 = 7$ where the tangent plane is parallel to π

The level surface is $g(x,y,z) = 4x^2 + 8y^2 + 4z^2 - 7 \rightarrow \nabla g(x,y,z) = (8x, 16y, 8z)$

$\Rightarrow (8x, 16y, 8z) = k(2,-2,1)$, $k \in \mathbb{R}$ so the tangent plane is parallel

$x = \frac{k}{4}$, $y = -\frac{k}{8}$, $z = \frac{k}{8}$ $\rightarrow 4\left(\frac{k}{4}\right)^2 + 8\left(\frac{k}{8}\right)^2 + 4\left(\frac{k}{8}\right)^2 = 7$

$\rightarrow \frac{k^2}{4} + \frac{k^2}{8} + \frac{k^2}{16} = 7$

$\rightarrow 4k^2 + 2k^2 + k^2 = 112$

$\rightarrow$ $7k^2 = 112$
$\rightarrow$ $k^2 = 16$
$\rightarrow$ $k = \pm 4$

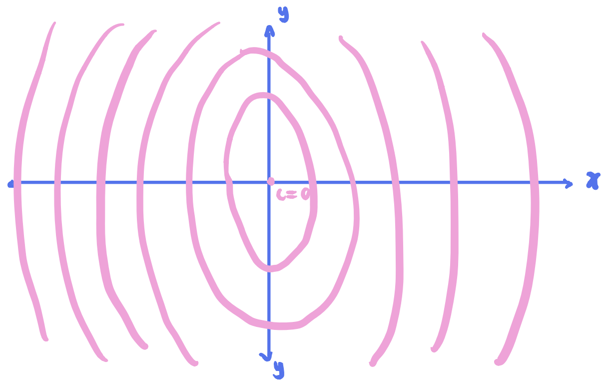
$\therefore$ The points are $\left(\frac{\pm 4}{4}, \frac{\mp 4}{8}, \frac{\pm 4}{8}\right)$:

$(1, -\frac{1}{2}, \frac{1}{2})$ and $(-1, \frac{1}{2}, -\frac{1}{2})$

Ex: Characterize and sketch several level curves of the function $f(x,y) = \sqrt{4x^2+y^2}$

dom f: $\mathbb{R}^2$

$4x^2+y^2 = c^2$
$(2x)^2+y^2 = c^2$



$\underline{c<0}$ $\rightarrow$ never!

$\underline{c=0}$ $\rightarrow$ $(x,y) = (0,0)$

$\underline{c>0}$ $\rightarrow$ ellipses!

x-intercepts: $\left(\frac{\pm c}{2}, 0\right)$

y-intercepts: $(0, \pm c)$

Week 10 - Nov. 8th.

Find eigenvalues of A: Solve $\det(A - \lambda I) = 0$.

Find the eigenvector: For each eigenvalue, find the eigenvectors $(A - \lambda I)\vec{v} = 0$

Let B be the matrix whose rows are the eigenvectors. Show that B is an orthogonal matrix: $BB^T = I_3$.

$\rightsquigarrow$ Hessian form (H.f.)

* If H.f. is positive definite, then the Critical Point is a local minimum
* If H.f. is negative definite, then the Critical Point is a local maximum
* Otherwise, the Critical Point is a saddle point.

Let A be nxn and symmetric:

* A is positive definite iff $\det(A_k) > 0$ for $k = 1, 2, \ldots, n$.
* A is negative definite iff $(-1)^k \det(A_k) > 0$ for $k = 1, 2, \ldots n$

Ex: Find and classify the critical points for the following functions

(a) $f(x,y) = -\frac{x^4}{4} + \frac{2x^3}{3} + 4xy - y^2$

$$\begin{cases} f_x = -x^3 + 2x^2 + 4y = 0 & ① \\ f_y = 4x - 2y = 0 & ② \quad y = 2x \end{cases}$$

Sub ② in ① $\rightarrow -x^3 + 2x^2 + 8x = 0$

$-x(x^2 - 2x - 8) = 0$

$-x(x-4)(x+2) = 0$

$\rightarrow x = 0, 4, -2 \qquad \rightarrow y = 0, 8, -4$

$\therefore$ C.Ps are $(0,0)$, $(4,8)$, $(-2,-4)$

$$Hf = \begin{pmatrix} -3x^2 + 4x & 4 \\ 4 & -2 \end{pmatrix}$$

★ For $(0,0)$:

$$Hf(0,0) = \begin{pmatrix} 0 & 4 \\ 4 & -2 \end{pmatrix}$$

$A_1 = 0 \qquad \det A_2 = \begin{vmatrix} 0 & 4 \\ 4 & -2 \end{vmatrix} = -16 < 0$

$\therefore (0,0)$ is a saddle point

★ For $(4,8)$:

$$Hf(4,8) = \begin{pmatrix} -32 & 4 \\ 4 & -2 \end{pmatrix}$$

$A_1 = -32 < 0 \qquad \det A_2 = \begin{vmatrix} -32 & 4 \\ 4 & -2 \end{vmatrix} = 64 - 16 = 48 > 0$

$\therefore (4,8)$ is a local maximum

★ For $(-2,-4)$:

$$Hf(-2,-4) = \begin{pmatrix} -20 & 4 \\ 4 & -2 \end{pmatrix}$$

$A_1 = -20 < 0 \qquad \det A_2 = \begin{vmatrix} -20 & 4 \\ 4 & -2 \end{vmatrix} = 40 - 16 = 24 > 0$

$\therefore (-2,-4)$ is a local maximum

(b) $f(x,y) = f(x,y) = x+y+\frac{1}{x}+\frac{4}{y}$, $x,y \neq 0$

$$\begin{cases} f_x = 1 - \frac{1}{x^2} = 0 \rightarrow x^2 = 1 \\ f_y = 1 - \frac{4}{y^2} = 0 \rightarrow y^2 = 4 \end{cases} \qquad \begin{cases} x = \pm 1 \\ y = \pm 2 \end{cases}$$

$\therefore$ CPs are $(1,2)$, $(1,-2)$, $(-1,2)$, $(-1,-2)$

$$Hf = \begin{pmatrix} \frac{2}{x^3} & 0 \\ 0 & \frac{8}{y^3} \end{pmatrix}$$

* For $(1,2)$:

$$Hf(1,2) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$A_1 = 2 > 0$ $\quad \det A_2 = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 > 0$

$\therefore (1,2)$ is a local minimum

* For $(1,-2)$:

$$Hf(1,-2) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$A_1 = 2 > 0$ $\quad \det A_2 = \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} = -2 < 0$

$\therefore (1,-2)$ is a saddle point

* For $(-1,2)$:

$$Hf(-1,2) = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$

$A_1 = -2 < 0$ $\quad \det A_2 = \begin{vmatrix} -2 & 0 \\ 0 & 1 \end{vmatrix} = -2 < 0$

$\therefore (-1,2)$ is a saddle point

* For $(-1,-2) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$

$A_1 = \underline{-2 < 0}$ $\quad \det A_2 = \begin{vmatrix} -2 & 0 \\ 0 & -1 \end{vmatrix} = \underline{2 > 0}$

$\therefore (-1,-2)$ is a local maximum

Week 11 - November 15th.

Ex: Use Lagrange multipliers to find the constrained critical points of f subject to the given constraints.

(a) $f(x,y) = 5x + 2y, \quad 5x^2 + 2y^2 = 14$

$h(x,y,\lambda) = 5x + 2y - \lambda(5x^2 + 2y^2 - 14)$

$$\begin{cases} h_x = 5 - 10\lambda x & (1) \\ h_y = 2 - 4\lambda y & (2) \\ h_\lambda = -5x^2 - 2y^2 + 14 & (3) \end{cases}$$

(1) $5 = 10\lambda x$

$\lambda = \frac{x}{2}$

(2) $2 = 4\lambda y$

$\lambda = \frac{y}{2}$

$x = y$

(3) $-5x^2 - 2x^2 + 14 = 0$

$\Rightarrow 7x^2 = 14$

$x^2 = 2$

$x = \pm\sqrt{2} = y$

$\therefore$ CPs: $(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$

(b) $f(x,y,z) = xy + xz + yz - xyz, \quad x+y+z=1, \quad x,y,z \geq 0$

$h(x,y,z,\lambda) = xy + xz + yz - xyz - \lambda(x+y+z-1)$

$$\begin{cases} h_x = y + z - yz - \lambda & (1) \\ h_y = x + z - xz - \lambda & (2) \\ h_z = x + y - xy - \lambda & (3) \\ h_\lambda = -x - y - z + 1 & (4) \end{cases}$$

(1) $y + z - yz = \lambda$

(2) $x + z - xz = \lambda$

(3) $x + y - xy = \lambda$

(4) $x + y + z = 1$

(1) = (2) $\quad y + z - yz = x + z - xz$

$\Rightarrow y(1-z) = x(1-z)$

$\Rightarrow (1-z)(y-x) = 0$

$\Rightarrow z = 1$ or $y = x$

(2) = (3) $\quad x + z - xz = x + y - xy$

$\Rightarrow z(1-x) = y(1-x)$

$\Rightarrow (1-x)(z-y) = 0$

$\Rightarrow x = 1$ or $y = z$

If $z=1$, then either $x=1$ or $y=1$ but either way (4) fails

If $y=x$, then $y=z$ because if $x=1=y$ and (4) will fail again

In (1), (2), (3) $x = y = z$

(4) $x + x + x = 1 \quad x = \frac{1}{3} = y, z \quad \therefore$ CP: $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Ex: The production function for a manufacturer is given by $f(x,y) = 100x^{3/4}y^{1/4}$ where x represents the units of labour (at \$150 per unit) and y represents the units of capital (at \$250 per unit). The total cost of labour and capital is limited to \$50,000. Find the maximum production level for this manufacturer.

Constraint is $150x+250y=50000$, to maximize $f(x,y)$ we can use lagrange multipliers.

$h(x,y,\lambda) = 100x^{3/4}y^{1/4} - \lambda(150x+250y-50000)$

$$\begin{cases} h_x = 75x^{-1/4}y^{1/4} - 150\lambda & ① \\ h_y = 25x^{3/4}y^{-3/4} - 250\lambda & ② \\ h_\lambda = -150x - 250y + 50000 & ③ \end{cases}$$

① $75x^{-1/4}y^{1/4} = 150\lambda$

$\lambda = \dfrac{x^{-1/4}y^{1/4}}{2} = \dfrac{y^{1/4}}{2x^{1/4}}$

② $25x^{3/4}y^{-3/4} = 250\lambda$

$\lambda = \dfrac{x^{3/4}y^{-3/4}}{10} = \dfrac{x^{3/4}}{10y^{3/4}}$

$\dfrac{y^{1/4}}{2x^{1/4}} = \dfrac{x^{3/4}}{10y^{3/4}}$

$10y = 2x$

$x = 5y$

③ $15\cancel{0}(5y) + 25\cancel{0}y = 5000\cancel{0}$

$75y + 25y = 5000$

$100y = 5000$

$y = 50 \rightarrow x = 250$

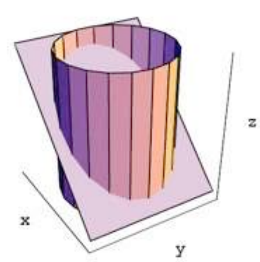
Which means that the maximum production level is when $x=250$ and $y=50$

$f(250,50) \approx 16718.5076$, so the maximum production units are 16718.

Ex: The cylinder $x^2+y^2=1$ intersects the plane $x+z=1$ in an ellipse. Find the point on that ellipse that is the furthest from the origin.

We need to maximize the distance function $(x^2+y^2+z^2)$ subject to 2 constraints.

$h(x,y,z,\lambda,\mu) = x^2+y^2+z^2 - \lambda(x^2+y^2-1) - \mu(x+z-1)$



$$\begin{cases} h_x = 2x - 2x\lambda - \mu & ① \\ h_y = 2y - 2y\lambda & ② \\ h_z = 2z - \mu & ③ \\ h_\lambda = 1 - x^2 - y^2 & ④ \\ h_\mu = 1 - x - z & ⑤ \end{cases}$$

② $2y = 2y\lambda$
$\Rightarrow 2y(1-\lambda) = 0$
$y = 0\ ,\ \lambda = 1$

* If $\lambda = 1$:
① $\mu = 2x(1-\lambda) = 0$
③ $2z = \mu = 0 \rightarrow z = 0$
⑤ $x = 1$
④ $1 + y^2 = 1 \rightarrow y = 0$

* If $y = 0$:
④ $x^2 = 1 \rightarrow x = \pm 1$
⑤ $-1 + z = 1 \rightarrow z = 2\ (x = -1)$
$1 + z = 1 \rightarrow z = 0\ (x = 1)$ (we knew)

$\therefore$ CPs $(1,0,0)$ and $(-1,0,2)$

Based on the distance function:

$(1,0,0) \rightarrow \sqrt{1^2} = 1$

$(-1,0,2) \rightarrow \sqrt{(-1)^2 + (0)^2 + (2)^2} = \sqrt{5}$

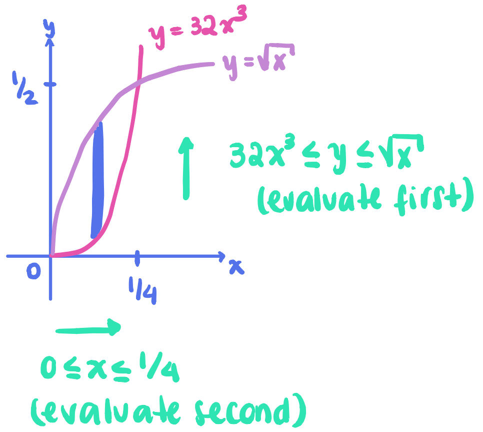
$\therefore$ The furtherst point from the origin is $(-1,0,2)$.

Since the distance function is continuous and the intersection is compact in $\mathbb{R}^3$, the Extreme Value Theorem (EVT) ensures the existence of extreme.

Week 12 – November 22nd

Give a rough sketch of the region and evaluate the following integrals

Ex: $\int_D 3xy\, dA$, where D is the region bounded by $y = 32x^3$ and $y = \sqrt{x}$



$$\int_D 3xy\, dA = \int_0^{1/4} \int_{32x^3}^{\sqrt{x}} 3xy\, dy\, dx = 3\int_0^{1/4} x \int_{32x^3}^{\sqrt{x}} y\, dy\, dx$$

$$= 3\int_0^{1/4} x \left[\frac{y^2}{2}\right]_{32x^3}^{\sqrt{x}} dx$$

$$= \frac{3}{2}\int_0^{1/4} x\left[\sqrt{x}^2 - (32x^3)^2\right] dx$$

$$= \frac{3}{2}\int_0^{1/4} x\left[x - 1024x^6\right] dx$$

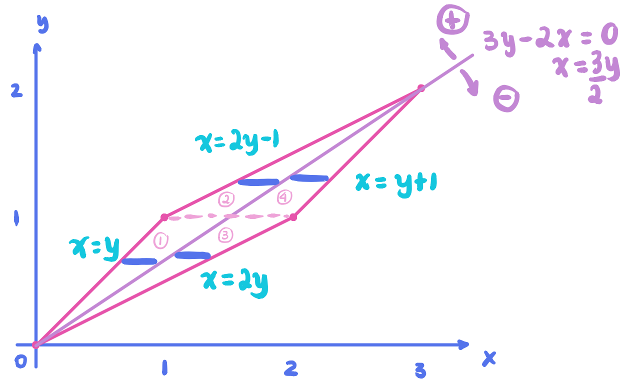
$$= \int_0^{1/4} \frac{3}{2}x^2 - 1536x^7\, dx$$

$$= \left[\frac{x^3}{2} - 192x^8\right]_0^{1/4}$$

$$= \frac{(1/4)^3}{2} - 192\left(\frac{1}{4}\right)^8$$

$$= \frac{1}{128} - \frac{3}{1024} = \frac{8-3}{1024} = \frac{5}{1024}$$

Ex: $\int_D |3y-2x|\,dA$, where D is the parallelogram with vertices $(0,0)$, $(1,1)$, $(2,1)$, $(3,2)$.



① ⊕ $y \le x \le \frac{3y}{2}$, $0 \le y \le 1$

② ⊕ $2y-1 \le x \le \frac{3y}{2}$, $1 \le y \le 2$

③ ⊖ $\frac{3y}{2} \le x \le 2y$, $0 \le y \le 1$

④ ⊖ $\frac{3y}{2} \le x \le y+1$, $1 \le y \le 2$

$$\int_D |3y-2x|\,dA = ① + ② + ③ + ④$$

$$= \int_0^1\int_y^{\frac{3y}{2}} (3y-2x)\,dx\,dy + \int_1^2\int_{2y-1}^{3y/2} (3y-2x)\,dx\,dy + \int_0^1\int_{\frac{3y}{2}}^{2y} -(3y-2x)\,dx\,dy + \int_1^2\int_{\frac{3y}{2}}^{y+1} -(3y-2x)\,dx\,dy$$

$$= \int_0^1 \left[3yx - x^2\right]_y^{3y/2} dy + \int_1^2 \left[3yx-x^2\right]_{2y-1}^{3y/2} dy - \int_0^1 \left[3yx-x^2\right]_{3y/2}^{2y} dy - \int_1^2 \left[3yx-x^2\right]_{3y/2}^{y+1} dy$$

$$= \int_0^1 \left[\frac{9y^2}{2} - \frac{9y^2}{4} - 3y^2 + y^2\right] dy + \int_1^2 \left[\frac{9y^2}{2} - \frac{9y^2}{4} - 6y^2 + 3y + (2y-1)^2\right] dy$$

$$- \int_0^1 \left[6y^2 - 4y^2 - \frac{9y^2}{2} + \frac{9y^2}{4}\right] dy - \int_1^2 \left[3y^2 + 3y - (y+1)^2 - \frac{9y^2}{2} + \frac{9y^2}{4}\right] dy$$

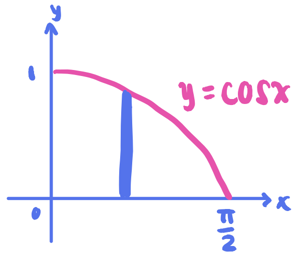
$$= \int_0^1 \frac{y^2}{4}\,dy + \int_1^2 \frac{y^2}{4} - y + 1\,dy + \int_0^1 \frac{y^2}{4}\,dy + \int_1^2 \frac{y^2}{4} - y + 1\,dy$$

$$= \int_0^1 \frac{y^2}{2}\,dy + \int_1^2 \frac{y^2}{2} - 2y + 2\,dy$$

$$= \frac{y^3}{6}\Big|_0^1 + \left[\frac{y^3}{6} - y^2 + 2y\right]_1^2$$

$$= \frac{1}{6} + \left[\frac{4}{3} - 4 + 4 - \frac{1}{6} + 1 - 2\right] = \frac{1}{3}$$

Ex: $\int_0^{\pi/2}\int_0^{\cos x} y \sin x\,dy\,dx \rightarrow$ $0 \le y \le \cos x$, $0 \le x \le \frac{\pi}{2}$



$$\int_0^{\pi/2}\int_0^{\cos x} y \sin x\,dy\,dx = \int_0^{\pi/2} \sin x \left[\frac{y^2}{2}\right]_0^{\cos x} dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin x \cos^2 x\,dx$$

let $u = \cos x$
$du = -\sin x\,dx$

$$= -\frac{1}{2}\int u^2\,du$$

$$= -\frac{1}{2}\left[\frac{u^3}{3}\right]$$

$$= -\frac{1}{6}\cos^3 x \Big|_0^{\pi/2}$$

$$= -\frac{1}{6}\left[\cos^3\frac{\pi}{2} - \cos^3(0)\right] = \frac{1}{6}$$

Week 13 – November 29th

Ex: Write two different triple integral iterated integrals for the volume of B. Let B be the region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$. Evaluate the integral

Fix x and y $\rightarrow$ $x^2 + y^2 \le z \le 8 - x^2 - y^2$ $\rightarrow$ $x^2 + y^2 \le 8 - x^2 - y^2$

$2x^2 + 2y^2 \le 8$

$x^2 + y^2 \le 4 \rightarrow \{(x,y) \mid x^2 + y^2 \le 4\}$

$r = 2$

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$ or $-\sqrt{4-y^2} \le x \le \sqrt{4-y^2}$

$-2 \le x \le 2$ $\qquad$ $-2 \le y \le 2$

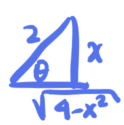
① $$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} 1\,dz\,dy\,dx$$

② $$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{x^2+y^2}^{8-x^2-y^2} 1\,dz\,dx\,dy$$

$$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 8 - x^2 - y^2 - x^2 - y^2\,dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 8 - 2x^2 - 2y^2\,dy\,dx$$

$$= \int_{-2}^{2}\left[(8-2x^2)y - \frac{2y^3}{3}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2}\left[(8-2x^2)(\sqrt{4-x^2} + \sqrt{4-x^2}) - \frac{2}{3}\left(\sqrt{4-x^2}^{\,3} + \sqrt{4-x^2}^{\,3}\right)\right] dx$$



$$= \int_{-2}^{2}\left[4(4-x^2)(\sqrt{4-x^2}) - \frac{4}{3}\sqrt{4-x^2}^{\,3}\right] dx$$

let $x = 2\sin\theta$
$dx = 2\cos\theta\,d\theta$

$$= \int\left[4(4-4\sin^2\theta)\sqrt{4-4\sin^2\theta} - \frac{4}{3}\sqrt{4-4\sin^2\theta}^{\,3}\right](2\cos\theta\,d\theta)$$

$$= \int \left[4(4\cos^2\theta)\,2\cos\theta - \frac{4}{3}(2\cos\theta)^3\right](2\cos\theta\,d\theta)$$

$$= \int \left[32\cos^3\theta - \frac{32}{3}\cos^3\theta\right](2\cos\theta\,d\theta)$$

$$= \int \frac{128}{3}\cos^4\theta\,d\theta$$

$$= \frac{128}{3}\int \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{32}{3}\int 1 + 2\cos 2\theta + \cos^2 2\theta\,d\theta$$

$$= \frac{32}{3}\left[\theta + \sin 2\theta + \int \frac{1+\cos 4\theta}{2}\,d\theta\right]$$

$$= \frac{32}{3}\left[\arcsin\frac{x}{2} + \sin\left(2\arcsin\frac{x}{2}\right) + \frac{1}{2}\left(\theta + \frac{\sin 4\theta}{4}\right)\right]$$

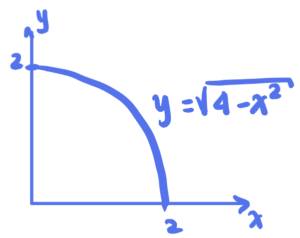
$$= \frac{32}{3}\left[\frac{3}{2}\arcsin\frac{x}{2} + \sin\left(2\arcsin\frac{x}{2}\right) + \frac{1}{8}\sin\left(4\arcsin\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= 32\left(\frac{\pi}{2}+\frac{\pi}{2}\right) + \frac{32}{3}\left(\sin\pi - \sin(-\pi)\right) + \frac{4}{3}\left(\sin(2\pi) - \sin(-2\pi)\right)$$

$$= 16\pi$$

Ex: Evaluate $\iiint_B x\,dV$ where B is the first octant solid bounded by the cylinder $x^2+y^2=4$ and the plane $2y+z=4$

$\underline{\text{first octant}} \to x, y, z \geq 0$



$0 \leq z \leq 4-2y$
$0 \leq y \leq \sqrt{4-x^2}$
$0 \leq x \leq 2$

$$\iiint_B x\,dV = \iint_{proj}\left(\int_0^{4-2y} x\,dz\right)dA = \int_0^2\int_0^{\sqrt{4-x^2}} x(4-2y)\,dy\,dx$$

$$= \int_0^2 x\left[4y - y^2\right]_0^{\sqrt{4-x^2}}dx = \int_0^2 x\left[4\sqrt{4-x^2} - (4-x^2)\right]dx$$

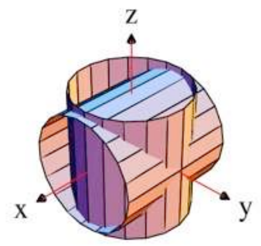
let $u = 4-x^2$
$du = -2x\,dx$

$$= -\frac{1}{2}\int 4u^{1/2} - u\,du = -\frac{1}{2}\left[\frac{8}{3}u^{3/2} - \frac{u^2}{2}\right] = \left[-\frac{4}{3}(4-x^2)^{3/2} + \frac{(4-x^2)^2}{2}\right]_0^2$$

$$= -\frac{4}{3}\left[(4-4)^{3/2} - 4^{3/2}\right] + \frac{1}{2}\left[(4-4) - (4-0)\right]$$

$$= \frac{4^{5/2}}{3} - 2 = \frac{32}{3} - \frac{6}{3} = \frac{26}{3}$$

Ex: Find the volume of the region which lies inside both $x^2+y^2=r^2$ and $y^2+z^2=r^2$



We can fix y for $-r \le y \le r$.

Then we have:
$$\left.\begin{aligned} -\sqrt{r^2-y^2} \le x \le \sqrt{r^2-y^2} \\ -\sqrt{r^2-y^2} \le z \le \sqrt{r^2-y^2} \end{aligned}\right\} R_y$$

Hence the volume is

$$\int_B 1\,dV = \int_{-r}^{r}\left(\iint_{R_y} 1\,dA\right)dy = \int_{-r}^{r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dx\,dz\,dy$$

omg all of these are symmetric!

$$= 2\cdot 2\cdot 2\int_0^r\int_0^{\sqrt{r^2-y^2}}\int_0^{\sqrt{r^2-y^2}} dx\,dz\,dy$$

$$= 8\int_0^r\int_0^{\sqrt{r^2-y^2}} \sqrt{r^2-y^2}\,dz\,dy$$

$$= 8\int_0^r r^2-y^2\,dy$$

$$= 8\left[r^2y - \frac{y^3}{3}\right]_0^r = 8\left[r^3 - \frac{r^3}{3}\right] = \frac{16}{3}r^3$$