MATA02 - The Magic of Numbers

TA: Angela Zavaleta-Bernuy
Tutorials: T2 (HW308 Thursdays 9-10am)
T3 (AA206 Tuesdays 9-10am)
T6 (MW160 Thursdays 10-11am)
Office hours: IC404 Mondays 2-4pm
email: angela.zavaletabernuy@mail.utoronto.ca
website: angelazb.github.io

Week 1 - Jan. 6th

No tutorials

Week 2 - Jan 13th

Today: Greatest Common Divisor (gcd) and Least Common Multiple (lcm)

↳ $\gcd(6,8)=2$ ↳ $\text{lcm}(6,8)=24$

Euclidean Algorithm: If $a=bq+r$, $0\le r<b$ then $\gcd(a,b)=\gcd(b,r)$

* $\gcd(a,0)=a$
* $\gcd(0,b)=b$
* $\text{lcm}(a,b)=\dfrac{ab}{\gcd(a,b)}$

Ex 1: Use the Euclidean Algorithm to find $\gcd(51,96)$ and use it to find the $\text{lcm}(51,96)$

$96 = 51\cdot 1 + 45$ → $d=\gcd(96,51)$

$51 = 45\cdot 1 + 6$ → $d=\gcd(51,45)$

$45 = 6\cdot 7 + 3$ → $d=\gcd(45,6)$

$6 = \underline{\underline{3}}\cdot 2 + 0$ ↳ stop! → $d=\gcd(6,3)$

$\therefore \gcd(96,51)=\gcd(51,45)=\gcd(45,6)=\gcd(6,3)=3$

$\therefore \text{lcm}(96,51)=\dfrac{96\times 51}{\gcd(96,51)}=\dfrac{96\times 51}{3}=96\times 17=1632$

* If $a=ms$ and $b=mt$, then $\gcd(a,b)=m\times\gcd(s,t)$

Ex 1: (Again!) but using this method.

$\gcd(51,96)=\gcd(3\cdot 17, 3\cdot 32)=3\cdot\gcd(17,32)=3\cdot 1=3$ ✓

Ex2: (T1-Q1) Use the Euclidean Algorithm to find gcd(366,150) and use it to find the lcm(366, 150)

$366 = 150 \cdot 2 + 66$ → $d = \gcd(366,150)$

$150 = 66 \cdot 2 + 18$ → $d = \gcd(150,66)$

$66 = 18 \cdot 3 + 12$ → $d = \gcd(66,18)$

$18 = 12 \cdot 1 + 6$ → $d = \gcd(18,12)$

$12 = \underline{\underline{6}} \cdot 2 + 0$ → $d = \gcd(12,6)$

$\therefore \gcd(366,150) = 6$

$\therefore \operatorname{lcm}(366,150) = \dfrac{366 \times 150}{\gcd(366,150)} = \dfrac{366 \times 150}{6} = 61 \times 150 = 9150$

Ex3: (T1-Q3) Does the equation $12x + 20y = 90$ have a solution of integers x and y?

No ☹ Because on the left side we can factor out 4 → $4(3x+5y)$ however, 90 is <u>not</u> divisible by 4.

Ex4: Does the equation $11x + 1111 = 121 + 22y$ have a solution of integers x and y?

No ☹. Because the right side is divisible by 11 → $11(11+2y)$, but the left side isn't as 1111 is not divisible by 11.

Ex5: (T1-Q4) If a and b are integers, and x is an integer such that $x^2+ax+b=0$. Show that $x \mid b$
↳ b is divisible by x

$x^2+ax+b=0$
$\Rightarrow b = -x^2 - ax$
$\Rightarrow b = -x\underbrace{(x+a)}$
↳ is an integer because x and a are integers

Because b is also an integer, then $x \mid b$. ■

Ex6: Show that if $a \mid b$ and $b \mid a$, then $a=b$ or $a=-b$

$a \mid b$ means that there exist an $n \in \mathbb{Z}$ such that $an = b$ ①
$b \mid a$ means that there exist an $m \in \mathbb{Z}$ such that $bm = a$ ②

Use ① for ②: $(an)m = a \Rightarrow anm = a \Rightarrow nm = 1$.

The only integers n and m that work for $nm = 1$ are $n = m = 1$ or $n = m = -1$

→ When $n = m = 1 \Rightarrow a = b$, by (1)

→ when $n = m = -1 \Rightarrow a = -b$, by (2)

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