# MATB41 - Multivariable Calculus

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## Week 1 (May 11)

* No tutorials *

## Week 2 (May 18)

### Curved Lines

**Parabola:** (with vertex) at the origin

- $y = ax^2$ → open up when $a > 0$ and down when $a < 0$
- $x = by^2$ → open right when $b > 0$ and left when $b < 0$

* **Remember:** We can replace $x$ with $x-m$ to shift right and $y$ with $y+n$ to shift up!

**Ellipse:** (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

intersections $(\pm a, 0)$ and $(0, \pm b)$
when $a = b$, the curve is a circle of radius $a = b$.

**Hyperbola:** (with center) at the origin

- $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$
- $\left(\frac{y}{a}\right)^2 - \left(\frac{x}{b}\right)^2 = 1$

It has two curves that goes in opposite directions and do not touch the slant asymptote

**Ex ① - Sketch the following curves:**

a) $x^2 + 3y^2 + 2x - 12y + 10 = 0$

$$(x^2+2x) + 3(y^2-4y) + 10 = 0$$
$$(x^2+2x+1) - 1 + 3(y^2-4y+4) - 12 + 10 = 0$$
$$(x+1)^2 + 3(y-2)^2 = 3$$

$$\Rightarrow \frac{(x+1)^2}{3} + (y-2)^2 = 1 \quad \Rightarrow \left(\frac{x+1}{\sqrt{3}}\right)^2 + (y-2)^2 = 1$$

Okie, now what?! Another way to write the ellipse equation would be:

$$\left(\frac{x-m}{a}\right)^2 + \left(\frac{y-n}{b}\right)^2 = 1$$

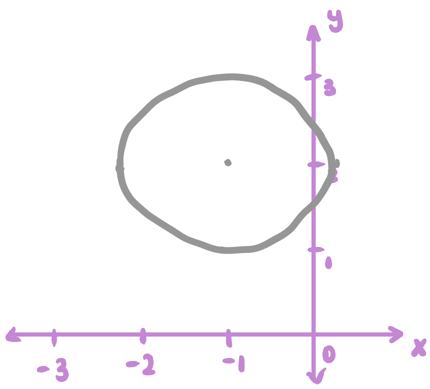
→ we just combined it with *

So the new origin will be $(-1, 2)$ as the ellipse will be shifted left by one and up by 2.

Now, let's find the end points

→ The horizontal max/min will be $(-1 \pm \sqrt{3}, 2)$

→ The vertical max/min will be $(-1, 2 \pm 1)$




b) $\frac{(x-2)^2}{4} - \frac{(y+2)^2}{9} = 1$

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 1$$

The origin will be $(2, -2)$, we are shifting right 2 and 2 down.

Check the x-int: $(y=0)$

$$\left(\frac{x-2}{2}\right)^2 - \frac{4}{9} = 1$$

$$\frac{(x-2)^2}{4} = \frac{13}{9}$$

$$(x-2)^2 = \frac{52}{9}$$

$$x-2 = \pm\frac{2\sqrt{13}}{3}$$

$$x = 2 \pm \frac{2\sqrt{13}}{3}$$

Check the y-int: $(x=0)$

$$1 - \left(\frac{y+2}{3}\right)^2 = 1$$

$$y = -2$$

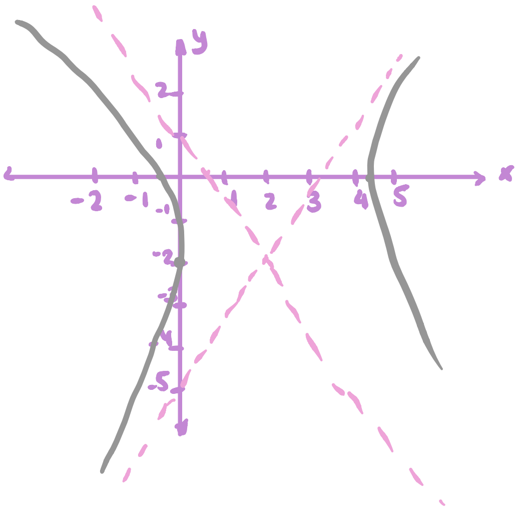
Find the slant asymptote:

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 0$$

$$\frac{x-2}{2} = \frac{y+2}{3}$$
$$3x-6 = 2y+4$$
$$2y = 3x-10$$
$$y = \frac{3x}{2} - 5$$

or

$$\frac{-x+2}{2} = \frac{y+2}{3}$$
$$-3x+6 = 2y+4$$
$$2y = -3x+2$$
$$y = \frac{-3x}{2} + 1$$




## Curved Surfaces

**3D Sphere:** (with center) at the origin of radius R

$$x^2+y^2+z^2 = R^2$$

**3D Ellipsoid:** (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

3 points $(a,0,0), (0,b,0), (0,0,c)$

We can replace $x^2+y^2$ with $r^2$ to indicate the rotation.

**Ex ② - Sketch the following surfaces:**

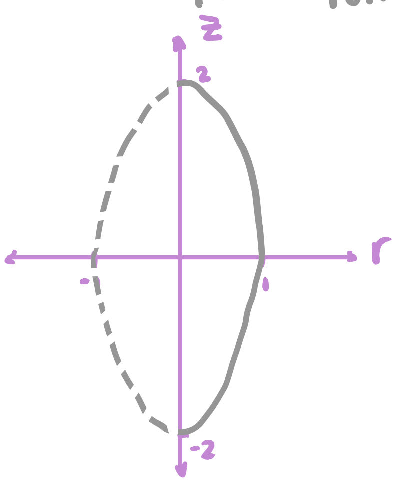
a) $x^2+y^2+\frac{z^2}{4} = 1$

$$r^2 + \frac{z^2}{4} = 1$$
$$4r^2+z^2 = 4$$
$$r^2 + \frac{z^2}{4} = 1$$

Center at $(0,0)$

r-int : $(\pm 1, 0)$
z-int : $(0, \pm 2)$

* Use the ellipsoid formula



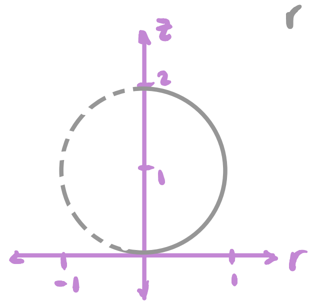

$r \geq 0$ because $r = \sqrt{x^2+y^2}$

b) $x^2+y^2+z^2=2z$

$r^2+z^2-2z=0$
$r^2+z^2-2z+1-1=0$
$r^2+(z-1)^2=1$ → circle!

Center $(0,1)$

$r \geq 0$ because $r=\sqrt{x^2+y^2}$




## Determinant

When $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| = ad-bc$

When $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Ex ③ - Find the determinant of $\begin{bmatrix} -3 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 4 & 3 \end{bmatrix}$

$$\begin{vmatrix} -3 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 4 & 3 \end{vmatrix} = -3\begin{vmatrix} -1 & 2 \\ 4 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix}$$

$= -3(-3-8) - (6-2) + (8+1)$
$= -3 \times -11 - 4 + 9$
$= 33 + 5$
$= 38$

## Dot and Cross Product

Let $\vec{u}\cdot\vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix} = u_1v_1 + u_2v_2 + \dots$

Let $\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left( \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$
$= (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$

You can find the angle between two vectors using dot product:

Let the angle be called $\theta$, then $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

Ex ④ - Find the angle between $\vec{u}=(1,-3,1)$ and $\vec{v}=(2,1,2)$ in $\mathbb{R}^3$

First, we need the dot product and lengths:

$\vec{u}\cdot\vec{v} = (1,-3,1)(2,1,2) = 2-3+2 = 1$

$\|\vec{u}\| = \sqrt{1^2+(-3)^2+1^2} = \sqrt{11}$

$\|\vec{v}\| = \sqrt{2^2+1^2+2^2} = \sqrt{9} = 3$

$\Rightarrow \cos\theta = \dfrac{1}{3\sqrt{11}} \quad \therefore \quad \theta = \cos^{-1}\left(\dfrac{1}{3\sqrt{11}}\right)$

## Line and Planes

Vector equation of a line in $\mathbb{R}^3$: $\vec{\ell} = (a_1, a_2, a_3) + t[v_1, v_2, v_3]$
$\vec{\ell} = a + t\vec{v}$

Parametric equation of a line in $\mathbb{R}^3$:
$x = a_1 + tv_1$
$y = a_2 + tv_2$
$z = a_3 + tv_3$

Ex ⑤ - Find the equation of the line or plane

a) The line through $(1,-1,2)$ and $(3,1,9)$

The direction vector for the line is $(3,1,9)-(1,-1,2) = (2,2,7)$

$\rightarrow$ V. eq. $\vec{\ell} = (1,-1,2) + t(2,2,7)$, $t \in \mathbb{R}$

$\rightarrow$ P. eq. $x = 1+2t$, $y = -1+2t$, $z = 2+7t$, $t \in \mathbb{R}$

b) The plane through $(1,-3,1), (2,1,1), (1,4,0)$

Let's find a pair of directional vectors $\vec{v_1}$ and $\vec{v_2}$

$\vec{v_1} = (2,1,1) - (1,-3,1) = (1,4,0)$

$\vec{v_2} = (1,4,0) - (1,-3,1) = (0,7,-1)$

To get the equation of the plane, we need to find the normal which is $\vec{v_1} \times \vec{v_2}$

$$\vec{n} = \vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 0 \\ 0 & 7 & -1 \end{vmatrix} = (-4, 1, 7)$$

The plane then is $-4x + y + 7z = d$
Let's plug a point: $-4(2) + (1) + 7(1) = -8+1+7 = 0 = d$

$\rightarrow$ The plane is $-4x + y + 7z = 0$

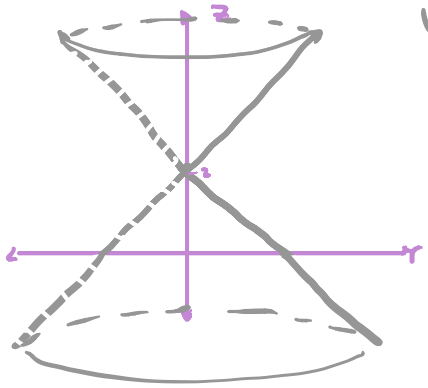
# Week 3 (May 25)

## Level Set

Ex ① - Draw the level set of $f(x,y,z) = z^2 - x^2 - y^2 - 4z$ for $f = -4$

↳ this is in 4D though

$$-4 = z^2 - x^2 - y^2 - 4z$$
$$x^2 + y^2 = z^2 - 4z + 4$$
$$r^2 = (z-2)^2$$
$$r = \pm(z-2)$$




$r \geq 0$

## Limits and Continuity

$f(x,y)$ is continuous at the 2D point $\vec{a}$ if $\lim\limits_{(x,y)\to\vec{a}} f(x,y) = f(\vec{a})$

It's not too hard to show when a limit DNE in $\mathbb{R}^2$ at $(0,0)$:
Try different curves in terms of $x$ or/and $y$, if they approach to different values at $(0,0)$

To show that the limit exists we can use the Squeeze Theorem:

To attain $\lim\limits_{(x,y)\to\vec{a}} f(x,y)$, can try to find $g(x,y)$ and $h(x,y)$ so that:

1. $g(x,y) \leq f(x,y) \leq h(x,y)$ near the point $\vec{a}$
2. $\lim\limits_{(x,y)\to\vec{a}} g(x,y) = L = \lim\limits_{(x,y)\to\vec{a}} h(x,y)$

Then we conclude $\lim\limits_{(x,y)\to\vec{a}} f(x,y) = L$

Ex ② - Decide whether the function has a limit at $(0,0)$

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} = \text{DNE}$

Restrict to $x=0$, $\lim\limits_{(0,y)\to(0,0)} \dfrac{-y^2}{y^2} = -1$

Restrict to $y=0$, $\lim\limits_{(x,0)\to(0,0)} \dfrac{x^2}{x^2} = 1$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{|x|}{\sqrt{x^2+y^2}} = \text{DNE}$

Restrict to $y=0$, $\lim_{(x,0)\to(0,0)} \frac{|x|}{\sqrt{x^2}} = 1$

Restrict to $x=0$, $\lim_{(0,y)\to(0,0)} \frac{|x|}{\sqrt{y}} = \frac{0}{\sqrt{y}} = 0$

c) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \underbrace{\frac{x^3}{x^2+y^2}}_{①} - \underbrace{\frac{y^3}{x^2+y^2}}_{②} = 0$

① $x^2 \le x^2+y^2 \rightarrow 0 \le \frac{x^2}{x^2+y^2} \le 1$

$\rightarrow 0 \cdot x \le \frac{x^3}{x^2+y^2} \le x$

$\rightarrow \lim_{(x,y)\to(0,0)} 0 \le \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} \le \lim_{(x,y)\to(0,0)} x$

$\rightarrow 0 \le \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} \le 0$

$\Rightarrow 0$

② Same as ①

$y^2 \le x^2+y^2 \rightarrow 0 \le \frac{y^2}{x^2+y^2} \le 1$

$\rightarrow 0 \cdot \le \frac{y^3}{x^2+y^2} \le$

$\rightarrow \lim_{(x,y)\to(0,0)} y \cdot 0 \le \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} \le \lim_{(x,y)\to(0,0)} y$

$\rightarrow 0 \le \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} \le 0$

$\Rightarrow 0$

d) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

We know $|\sin\theta| \le 1$ and we saw in class $|\sin\theta| \le \theta$

$\Rightarrow \frac{|\sin(x^2+y^2)|}{x^2+y^2} \le \frac{x^2+y^2}{x^2+y^2} = 1$

Recall $\lim_{t\to 0} \frac{\sin t}{t} = 1$, what if we let $t = x^2+y^2$

$$\Rightarrow \lim_{t\to 0} \frac{\sin t}{t} = 1 = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

Ex ③ - Find if the limit of $f$ exists at $(0,0)$, if it does, find $f(0,0)$ for $f$ to be continuous.

$$f(x,y) = \frac{x^3-x^2-2x^2y+xy^2-y^2-2y^3}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2(x-1-2y)+y^2(x-1-2y)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x-1-2y)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} x-1-2y$$

$$= -1$$

$$\Rightarrow f(0,0) = -1$$

$f(\vec{x})$ is homogeneous of degree $k$ if for every $\vec{x}\in\mathbb{R}^n$ and every scalar $c>0$ we have:

$$f(c\vec{x}) = c^k f(\vec{x})$$

Ex ④ - Show if $f$ is homogeneous

a) $f(x,y) = 8x^2y^2 - 9x^4$

$$\begin{aligned} f(cx,cy) &= 8(cx)^2(cy)^2 - 9(cx)^4 \\ &= 8c^2x^2c^2y^2 - 9c^4x^4 \\ &= 8c^4x^2y^2 - 9c^4x^4 \\ &= c^4(8x^2y^2 - 9x^4) \\ &= c^4 f(x,y) \end{aligned}$$

$\Rightarrow$ $f$ is homogeneous of degree 4.

b) $f(x,y) = \dfrac{x^2}{y^2} + xy + \dfrac{y^2}{x^2}$

$$\begin{aligned} f(cx,cy) &= \frac{(cx)^2}{(cy)^2} + (cx)(cy) + \frac{(cy)^2}{(cx)^2} \\ &= \frac{c^2x^2}{c^2y^2} + c^2xy + \frac{c^2y^2}{c^2x^2} \\ &= \frac{x^2}{y^2} + c^2xy + \frac{y^2}{x^2} \end{aligned}$$ $\Rightarrow$ cannot factor $c$.

$\Rightarrow$ $f$ is not homogeneous.

Week 4 (June 1st)

## Partial derivatives

Ex ① - Find the partial derivatives of $f(x,y) = y\sin(xy) + xe^{-y^2}$

$$\frac{\partial f}{\partial x} = y^2\cos(xy) + e^{-y^2}$$

$$\frac{\partial f}{\partial y} = \sin(xy) + xy\cos(xy) - 2xye^{-y^2}$$

A function, whose partial derivatives exist and are continuous, is said to be of class $C^1$

## Directional Derivative

The directional derivative towards the direction $\vec{u}$ at the point $\vec{a}$:

$$D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$$ where $\vec{u}$ is a unit vector.

If $f$ is differentiable, then all directional derivatives exist and

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$ where $\vec{u}$ is a unit vector.

* If $f$ is $C^1$ at $\vec{a}$, then $f$ is differentiable at $\vec{a}$ ($\nabla f$ exists)

Ex ② - At the point $(3,1,2)$, find the directional derivative of $f(x,y,z) = xy^3z^2$ along the vector $\vec{u} = (1,3,4)$

$f$ is $C^1$ → it's partial derivatives exist and are continuous.

$$\nabla f(x,y,z) = (y^3z^2,\ 3xy^2z^2,\ 2xy^3z)$$

$$\nabla f(3,1,2) = (4, 36, 12)$$

$$\Rightarrow D_{\vec{u}} f(p) = \frac{(4,36,12)(1,3,4)}{\sqrt{1+9+16}} = \frac{4+108+48}{\sqrt{26}} = \frac{160}{\sqrt{26}}$$

## Tangent Planes

A tangent plane is given by $\nabla g(a,b,c)((x,y,z)-(a,b,c))=0$

Ex ③ - Compute an equation for the tangent plane at the point $p$ to the graph of the function $z=f(x,y)$

$p=(1,1,1)$ and $xy+yz+zx=3$

$g(x,y,z) = xy + yz + zx - 3$

$\nabla g(x,y,z) = (y+z,\ x+z,\ y+x)$

$\nabla g(1,1,1) = (2,2,2)$

$$\Rightarrow \begin{aligned} (2,2,2)((x,y,z)-(1,1,1)) &= 0 \\ (2,2,2)(x-1,\ y-1,\ z-1) &= 0 \\ 2x-2+2y-2+2z-2 &= 0 \\ x+y+z &= 3 \end{aligned}$$

## Differentiation

$f$ is differentiable if the gradient vector $\nabla f(\vec{a})$ satisfies

$$\lim_{\vec{h}\to 0} \frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - \nabla f(\vec{a})\cdot\vec{h}}{|\vec{h}|}$$

Ex ④ – Find all directional derivatives and show whether the function is differentiable at $(0,0)$

$$f(x,y) = \begin{cases} \dfrac{3x^2y+5xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$\vec{a} = (0,0)$
$\vec{u} = (a,b)$

$$\begin{aligned} D_{\vec{u}}f &= \lim_{h\to 0} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h} \\ &= \lim_{h\to 0} \frac{f(ha,hb) - 0}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\, \frac{3(ha)^2(hb) + 5(ha)(hb)^2}{(ha)^2+(hb)^2} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{h^3(3a^2b+5ab^2)}{h^2(a^2+b^2)} \\ &= \lim_{h\to 0} \frac{3a^2b+5ab^2}{a^2+b^2} \end{aligned}$$

$\rightarrow \vec{u} = (1,0)$, $D_{\vec{u}}f = \dfrac{3(1)(0)+5(1)(0)}{(1)^2+(0)^2} = 0 \rightarrow \dfrac{\partial f}{\partial x} = 0$

$\rightarrow \vec{u} = (0,1)$, $D_{\vec{u}}f = \dfrac{3(0)(1)+5(0)(1)}{(0)^2+(1)^2} = 0 \rightarrow \dfrac{\partial f}{\partial y} = 0$

$\nabla f(\vec{a}) = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right) = (0,0)$

Now, let's apply the definition of differentiability

$$\lim_{\vec{h}\to 0} \frac{f(\vec{a}+\vec{h}) - \overset{\nearrow 0}{f(\vec{a})} - \overset{\nearrow 0}{\nabla f(\vec{a})\cdot\vec{h}}}{|\vec{h}|} = \lim_{\vec{h}\to 0} \frac{f(\vec{h})}{|\vec{h}|}$$ , let $\vec{h}=(h_1,h_2)$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{1}{\sqrt{h_1^2+h_2^2}} \cdot \frac{3h_1^2 h_2 + 5h_1 h_2^2}{h_1^2+h_2^2}$$

Restrict $h_1=0$

$$\lim_{h_2\to 0} \frac{1}{\sqrt{h_2^2}} \cdot \frac{3(0)h_2 + 5(0)h_2^2}{h_2^2} = 0$$

Restrict $h_1=h_2$

$$\lim_{h_2\to 0} \frac{1}{\sqrt{2h_2^2}} \cdot \frac{3h_2^3+5h_2^3}{2h_2^2} = \lim_{h_2\to 0} \frac{8h_2^3}{2\sqrt{2}\,h_2^3} = \frac{4}{\sqrt{2}}$$

$\therefore$ f is not differentiable at (0,0)

## Week 5 (June 8th)

If f is $C^2$, then $f_{xy}=f_{yx}$ $\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right]$

Ex ① - At $(x,y)=(0,0)$, find $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$. Are they equal?

Is $f(x,y)$ $C^2$?

$$f(x,y) = \begin{cases} \frac{2x^4}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

Time to find directional derivatives !

$\vec{a}=(0,0)$
$\vec{u}=(a,b)$

$$D_{\vec{u}}f = \lim_{h\to 0}\frac{f(\vec{a}+h\vec{u})-f(\vec{a})}{h}$$
$$= \lim_{h\to 0}\frac{1}{h} f(ha,hb)$$
$$= \lim_{h\to 0}\frac{1}{h}\frac{2(ha)^4}{(ha)^2+(hb)^2}$$
$$= \lim_{h\to 0}\frac{2h^4a^4}{h(h^2a^2+h^2b^2)}$$
$$= \lim_{h\to 0}\frac{2h^4a^4}{h^3(a^2+b^2)}$$
$$= 0$$

$$\Rightarrow \frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$$

Now, let's find the partial derivatives.

* $\frac{\partial f}{\partial x}(x,y) = \frac{8x^3(x^2+y^2)-2x(2x^4)}{(x^2+y^2)^2} = \frac{8x^5+8x^3y^2-4x^5}{(x^2+y^2)^2} = \frac{4x^5+8x^3y^2}{(x^2+y^2)^2}$

$= \frac{4x^3(x^2+2y^2)}{(x^2+y^2)^2}$

$$\Rightarrow g(x,y) = \frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{4x^3(x^2+2y^2)}{(x^2+y^2)^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

* $\frac{\partial f}{\partial y}(x,y) = \frac{0(x^2+y^2)-2y(2x^4)}{(x^2+y^2)^2} = \frac{-4x^4y}{(x^2+y^2)^2}$

$$\Rightarrow k(x,y) = \frac{\partial f}{\partial y}(x,y) = \begin{cases} \frac{-4x^4y}{(x^2+y^2)^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

Time to finally find $f_{xy}$ and $f_{yx}$ at $(0,0)$

* We need to find $\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}(0,0)\right)$, aka $\frac{\partial g}{\partial y}(0,0)$

We can find the directional derivative at $(0,0)$ where $\vec{u} = (0,1)$

$$\frac{\partial g}{\partial y}(0,0) = \lim_{h\to 0} \frac{g(\vec{a}+h\vec{u})-g(\vec{a})}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\, g(0,h)$$
$$= \lim_{h\to 0} \frac{1}{h} \times 0$$
$$= 0$$

* We need to find $\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}(0,0)\right)$, aka $\frac{\partial k}{\partial x}(0,0)$

We can find the directional derivative at $(0,0)$ where $\vec{u} = (1,0)$

$$\frac{\partial k}{\partial x}(0,0) = \lim_{h\to 0} \frac{k(\vec{a}+h\vec{u})-k(\vec{a})}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\, k(h,0)$$
$$= \lim_{h\to 0} \frac{1}{h} \times 0$$
$$= 0$$

$$\therefore \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(0,0)\right) = 0 = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(0,0)\right)$$

But, is it $C^2$? At $(0,0)$, yes!

$$* \ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{[0(x^2+2y^2)+4y(4x^3)](x^2+y^2)^2-2(2y)(x^2+y^2)4x^3(x^2+2y^2)}{(x^2+y^2)^4}$$

$$= \frac{16x^3y(x^2+y^2)-16x^3y(x^2+2y^2)}{(x^2+y^2)^3}$$

$$= \frac{16x^3y(x^2+y^2-x^2-2y^2)}{(x^2+y^2)^3} = \frac{-16x^3y^3}{(x^2+y^2)^3}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \begin{cases} \dfrac{-16x^3y^3}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$* \ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = \frac{-16x^3y(x^2+y^2)^2+4x^4y(2)(2x)(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{-16x^3y(x^2+y^2)+16x^5y}{(x^2+y^2)^4} = \frac{-16x^3y^3}{(x^2+y^2)^4}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \begin{cases} \dfrac{-16x^3y^3}{(x^2+y^2)^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

So $f(x,y)$ is <u>not</u> $C^2$

## Critical Points

Hessian form: $(Hf.)$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) & \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \end{bmatrix}$$

* If $Hf.$ is positive definite, then the Critical Point is a local minimum
* If $Hf.$ is negative definite, then the Critical Point is a local maximum
* Otherwise, the Critical Point is a saddle point.

Let A be $n \times n$ and symmetric:

* A is positive definite iff $\det(A_k) > 0$ for $k = 1, 2, \ldots, n$.

* A is negative definite iff $(-1)^k \det(A_k) > 0$ for $k = 1, 2, \ldots n$

Ex ⓐ - Find the critical points of $f(x,y) = x^3 - 12xy + 8y^3$

$\frac{\partial f}{\partial x}(x,y) = 3x^2 - 12y$ $\qquad$ $\frac{\partial f}{\partial y}(x,y) = -12x + 24y^2$

$$\begin{cases} 3x^2 - 12y = 0 \\ -12x + 24y^2 = 0 \end{cases} \rightarrow \quad 3x^2 = 12y, \quad y = \frac{x^2}{4}$$

$$\rightarrow -12x + 24\left(\frac{x^2}{4}\right)^2 = 0$$

$$-12x + \frac{24}{16}x^4 = 0$$

$$x\left(\frac{3}{2}x^3 - 12\right) = 0 \quad \rightarrow \quad x = 0 \text{ or } \frac{3}{2}x^3 = 12$$

$$\Rightarrow x^3 = 8 \rightarrow x = 2$$

$\therefore$ CPs are $(0,0)$ and $(2,1)$

$\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = 6x$ $\qquad$ $\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = -12$

$\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = -12$ $\qquad$ $\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right) = 48y$

$$\Rightarrow Hf(x,y) = \begin{bmatrix} 6x & -12 \\ -12 & 48y \end{bmatrix}$$

① For $(0,0)$

$$Hf(0,0) = \begin{bmatrix} 0 & -12 \\ -12 & 0 \end{bmatrix}$$

$\rightarrow \det(A_1) = \det([0]) = 0$

$\rightarrow \det(A_2) = \det\left(\begin{bmatrix} 0 & -12 \\ -12 & 0 \end{bmatrix}\right) = -144 < 0$

$\therefore$ saddle point

② For $(2,1)$

$$Hf(2,1) = \begin{bmatrix} 12 & -12 \\ -12 & 48 \end{bmatrix}$$

$\rightarrow \det(A_1) = \det([12]) = 12 > 0$

$\rightarrow \det(A_2) = \det\left(\begin{bmatrix} 12 & -12 \\ -12 & 48 \end{bmatrix}\right) = 576 - 144 = 432 > 0$

$\therefore$ local min.

Ex ③- Find the global extrema for $f(x,y) = 6x^2 - 8x + 2y^2 - 5$ on the closed disk $x^2 + y^2 \leq 1$.

The extrema can be at the interior or boundary.

Interior:

$\frac{\partial f}{\partial x}(x,y) = 12x - 8$ $\qquad$ $\frac{\partial f}{\partial y}(x,y) = 4y$

$\begin{cases} 12x - 8 = 0 \\ 4y = 0 \end{cases} \rightarrow \left(\frac{2}{3}, 0\right)$ is the only CP.

Since $\left(\frac{2}{3}\right)^2 + (0)^2 = \frac{4}{9} \leq 1$, the point is inside the disk.

Boundary: it happens when $x^2 + y^2 = 1$. and $-1 \leq x \leq 1$

$y^2 = 1 - x^2$, let's substitute this

$$\begin{aligned} g(x) &= 6x^2 - 8x + 2(1 - x^2) - 5 \\ &= 6x^2 - 8x + 2 - 2x^2 - 5 \\ &= 4x^2 - 8x - 3 \end{aligned}$$

$g'(x) = 8x - 8 = 0$

$8x = 8$

$x = 1 \rightarrow y^2 = 1 - 1 = 0$, $(1,0)$

$\rightarrow y = 0$, $0 = 1 - x^2 \rightarrow (-1, 0)$

$\therefore f\left(\frac{2}{3}, 0\right) = -\frac{23}{2}$, global min

$f(-1,0) = 9$, global max

$f(1,0) = -7$

# Week 6 (June 16th)

## Lagrange Multipliers

Ex ① - Use Lagrange multipliers to find the constrained critical points of $f$ subject to the given constraints.

(a) $f(x,y) = xy$, $\underrightarrow{4x^2+9y^2=32}$ $g(x,y) = 4x^2+9y^2-32$

$$h(x,y,\lambda) = xy - \lambda(4x^2+9y^2-32)$$

$$\begin{cases} h_x = y - 8x\lambda = 0 \rightarrow \lambda = \dfrac{y}{8x} \\ h_y = x - 18y\lambda = 0 \rightarrow \lambda = \dfrac{x}{18y} \\ h_\lambda = 32 - 4x^2 - 9y^2 \end{cases}$$

$\dfrac{y}{8x} = \dfrac{x}{18y} \rightarrow 18y^2 = 8x^2 \rightarrow 4x^2 = 9y^2$

or $\lambda = 0$ (contradiction)

$32 - 4x^2 - 4x^2 = 0$

$8x^2 = 32$

$x^2 = 4 \longrightarrow 16 = 9y^2$

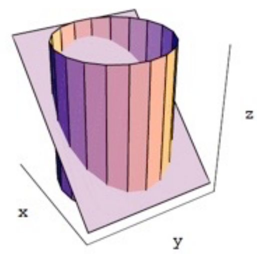
$x = \pm 2$ $\qquad y = \pm\frac{4}{3}$

$\therefore$ CPs are $\left(2, \frac{4}{3}\right)$, $\left(2, -\frac{4}{3}\right)$, $\left(-2, \frac{4}{3}\right)$, $\left(-2, -\frac{4}{3}\right)$

(b) The cylinder $x^2+y^2=1$ intersects the plane $x+z=1$ in an ellipse. Find the point on that ellipse that is the furthest from the origin.

We need to maximize the distance function (aka $f(x,y,z) = x^2+y^2+z^2$) subject to 2 constraints.

$$h(x,y,z,\lambda,\mu) = x^2+y^2+z^2 - \lambda(x^2+y^2-1) - \mu(x+z-1)$$




$$\begin{cases} h_x = 2x - 2x\lambda - \mu & ① \\ h_y = 2y - 2y\lambda & ② \\ h_z = 2z - \mu & ③ \\ h_\lambda = 1 - x^2 - y^2 & ④ \\ h_\mu = 1 - x - z & ⑤ \end{cases}$$

② $2y = 2y\lambda$
$\Rightarrow 2y(1-\lambda) = 0$
$y = 0$, $\lambda = 1$

* If $\lambda = 1$:
① $\mu = 2x(1-\lambda) = 0$
③ $2z = \mu = 0 \rightarrow z = 0$
⑤ $x = 1$
④ $1 + y^2 = 1 \rightarrow y = 0$

* If $y = 0$:
④ $x^2 = 1 \rightarrow x = \pm 1$
⑤ $-1 + z = 1 \rightarrow z = 2$ $(x = -1)$
$1 + z = 1 \rightarrow z = 0$ $(x = 1)$ (we knew)

$\therefore$ CPs $(1,0,0)$ and $(-1,0,2)$

Based on the distance function:

$(1,0,0) \rightarrow \sqrt{1^2} = 1$

$(-1,0,2) \rightarrow \sqrt{(-1)^2+(0)^2+(2)^2} = \sqrt{5}$

$\therefore$ The furtherst point from the origin is $(-1,0,2)$.

Since the distance function is continuous and the intersection is compact in $\mathbb{R}^3$, the Extreme Value Theorem (EVT) ensures the existence of extreme.

## Chain Rule

$$(f(g(x)))' = f'(g(x))\, g'(x)$$

Now, let's say $y=y(t)$ and $t=t(x)$

1D: $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

2D: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt} \cdot \frac{dt}{dx}\right)$ product rule!

$$= \frac{d}{dx}\left(\frac{dy}{dt}\right)\frac{dt}{dx} + \frac{dy}{dt} \cdot \frac{d}{dx}\left(\frac{dt}{dx}\right) = \frac{d^2t}{dx^2}$$

## ODEs - Euler Equation

Second order differential equations:

Homogeneous with constant coefficients

$ay''+by'+cy=0$ where $a,b,c$ are constants.

Characteristic equation $ar^2+br+c=0$, find roots $r_1$ and $r_2$

Then $y=c_1e^{r_1t}+c_2e^{r_2t}$ is a general solution

Ex ② - let $y''+5y'+6y=0$

(a) Find the general solution of the equation

$r^2+5r+6=0=(r+2)(r+3)=0$

$r_1=-2$ and $r_2=-3$

Then $y=c_1e^{-2t}+c_2e^{-3t}$, $c_1,c_2 \in \mathbb{R}$

(b) Find the solution of the initial value problem $y(0)=2,\ y'(0)=3$

$t=0,\quad y(0) = C_1 + C_2 = 2$

$y' = -2C_1e^{-2t} - 3C_2e^{-3t} \rightarrow y'(0) = -2C_1 - 3C_2 = 3$

$\begin{cases} C_1 + C_2 = 2 \\ -2C_1 - 3C_2 = 3 \end{cases}$ $\quad -C_2 = 7 \rightarrow C_2 = -7$ and $C_1 = 9$

$\therefore\ y = 9e^{-2t} - 7e^{-3t}$

Ex ③ - Find the solution of the initial value problem $4y'' - 8y' + 3y = 0$, $y(0) = 2,\ y'(0) = \frac{1}{2}$

$4r^2 - 8r + 3 = 0 = (2r-1)(2r-3)$

$r_1 = \frac{1}{2},\quad r_2 = \frac{3}{2}$

General sol: $y = c_1e^{t/2} + C_2e^{3t/2}$ $\qquad y' = \frac{C_1}{2}e^{t/2} + \frac{3C_2}{2}e^{3t/2}$

$\left.\begin{aligned} y(0) &= C_1 + C_2 = 2 \\ y'(0) &= \frac{C_1}{2} + \frac{3C_2}{2} = \frac{1}{2} \end{aligned}\right\}$ $\begin{aligned} C_1 &= 2 - C_2 \\ C_1 &= 1 - 3C_2 \end{aligned}$ $\qquad C_2 = -\frac{1}{2},\quad C_1 = \frac{5}{2}$

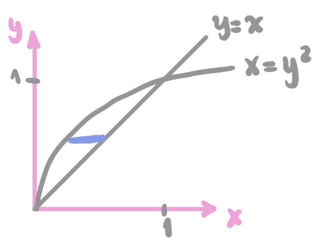
$\therefore\ y = \frac{5}{2}e^{t/2} - \frac{1}{2}e^{3t/2}$

# Week 9 (July 6th)

## 2D - Integration

Ex ① - Give a rough sketch of the regions and evaluate the following integrals

a) $\iint_D \sqrt{x}\,dA$ where D is the region bounded by $x=y$ and $x=y^2$

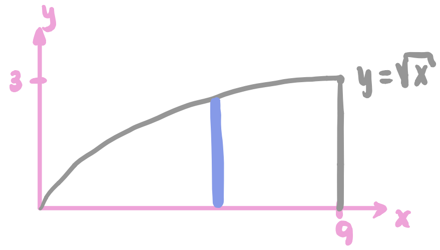



① $y^2 \le x \le y$

② $0 \le y \le 1$

$$\iint_D \sqrt{x}\,dA = \int_0^1\int_{y^2}^{y} \sqrt{x}\,dx\,dy$$
$$= \int_0^1 \frac{2}{3}x^{3/2}\Big|_{y^2}^{y}\,dy$$
$$= \frac{2}{3}\int_0^1 y^{3/2} - y^3\,dy$$
$$= \frac{2}{3}\left[\frac{2}{5}y^{5/2} - \frac{y^4}{4}\right]_0^1 = \frac{1}{10}$$

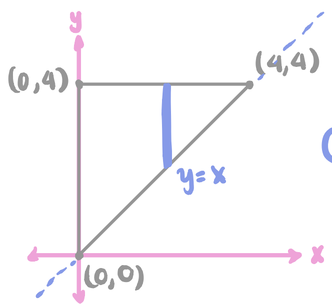
b) $\iint_D y\sin x^2\,dA$ where $D = \{(x,y)\in\mathbb{R}^2,\ 0\le y\le \sqrt{x},\ 0\le x\le 9\}$




$$\iint_D y\sin x^2\,dA = \int_0^9\int_0^{\sqrt{x}} y\sin x^2\,dy\,dx$$
$$= \int_0^9 \frac{y^2}{2}\sin x^2\Big|_0^{\sqrt{x}}dx$$
$$= \frac{1}{2}\int_0^9 x\sin x^2\,dx$$
$$= -\frac{1}{4}\cos x^2\Big|_0^9$$
$$= \frac{1}{4}(1-\cos 81)$$

Ex ② - Evaluate $\iint_D f(x,y)\,dA$ for the functions $f$ and region $D$

a) $f(x,y)=\cos y$ and $D$ is the triangle with vertices $(0,0),(4,4)$ and $(0,4)$

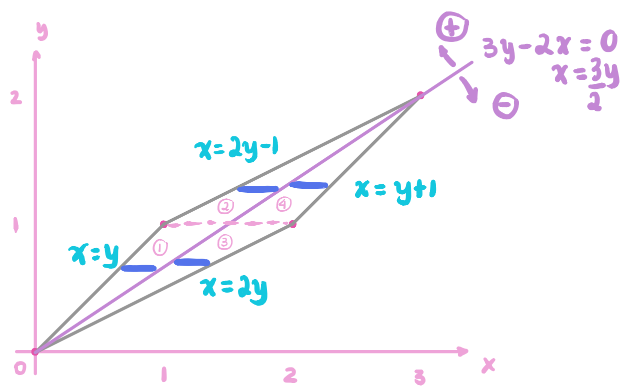



① $x\le y\le 4$

② $0\le x\le 4$

$$\iint_D \cos y\,dA = \int_0^4\int_x^4 \cos y\,dy\,dx$$
$$= \int_0^4 [\sin y]_x^4\,dx$$
$$= \int_0^4 \sin 4 - \sin x\,dx$$
$$= \Big[x\sin 4 + \cos x\Big]_0^4$$
$$= 4\sin 4 + \cos 4 - 1$$

b) $f(x,y)=|3y-2x|$ and $D$ is the parallelogram with vertices $(0,0),(1,1)$
$(2,1),(3,2)$




① ⊕ $y\le x\le \frac{3y}{2}$, $0\le y\le 1$

② ⊕ $2y-1\le x\le \frac{3y}{2}$, $1\le y\le 2$

③ ⊖ $\frac{3y}{2}\le x\le 2y$, $0\le y\le 1$

④ ⊖ $\frac{3y}{2}\le x\le y+1$, $1\le y\le 2$

$$\int_D |3y-2x|\,dA = ① + ② + ③ + ④$$
$$= \int_0^1\int_y^{\frac{3y}{2}}(3y-2x)\,dx\,dy + \int_1^2\int_{2y-1}^{3y/2}(3y-2x)\,dx\,dy + \int_0^1\int_{\frac{3y}{2}}^{2y}-(3y-2x)\,dx\,dy + \int_1^2\int_{\frac{3y}{2}}^{y+1}-(3y-2x)\,dx\,dy$$
$$= \int_0^1\Big[3yx-x^2\Big]_y^{3y/2}dy + \int_1^2\Big[3yx-x^2\Big]_{2y-1}^{3y/2}dy - \int_0^1\Big[3yx-x^2\Big]_{3y/2}^{2y}dy - \int_1^2\Big[3yx-x^2\Big]_{3y/2}^{y+1}dy$$

$$= \int_0^1 \left[\frac{9y^2}{2} - \frac{9y^2}{4} - 3y^2 + y^2\right] dy + \int_1^2 \left[\frac{9y^2}{2} - \frac{9y^2}{4} - 6y^2 + 3y + (2y-1)^2\right] dy$$

$$- \int_0^1 \left[6y^2 - 4y^2 - \frac{9y^2}{2} + \frac{9y^2}{4}\right] dy - \int_1^2 \left[3y^2 + 3y - (y+1)^2 - \frac{9y^2}{2} + \frac{9y^2}{4}\right] dy$$

$$= \int_0^1 \frac{y^2}{4}\, dy + \int_1^2 \frac{y^2}{4} - y + 1\, dy + \int_0^1 \frac{y^2}{4}\, dy + \int_1^2 \frac{y^2}{4} - y + 1\, dy$$

$$= \int_0^1 \frac{y^2}{2}\, dy + \int_1^2 \frac{y^2}{2} - 2y + 2\, dy$$

$$= \left.\frac{y^3}{6}\right|_0^1 + \left[\frac{y^3}{6} - y^2 + 2y\right]_1^2$$

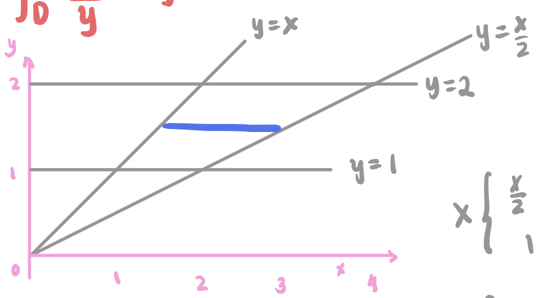
$$= \frac{1}{6} + \left[\frac{4}{3} - 4 + 4 - \frac{1}{6} + 1 - 2\right] = \frac{1}{3}$$

## Week 10 (July 13th)

### 2D - Integration

Ex ① - Solve

(a) $\int_D \frac{\sin y}{y}\, dy\, dx$, where D is the region bounded by $y=x$, $y=\frac{x}{2}$, $y=1$ and $y=2$




$$\int_D \frac{\sin y}{y}\, dy\, dx = \int_1^4 \int_{x/2}^{x} \frac{\sin y}{y}\, dy\, dx$$

$$x\begin{cases} \frac{x}{2} \le y \le x \\ 1 \le x \le 4 \end{cases}$$

$$\begin{cases} y \le x \le 2y \\ 1 \le y \le 2 \end{cases}$$

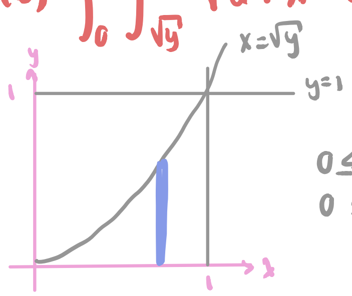
$$= \int_1^2 \int_y^{2y} \frac{\sin y}{y}\, dx\, dy$$

$$= \int_1^2 \frac{\sin y}{y} (2y - y)\, dy$$

$$= \int_1^2 \sin y\, dy$$

$$= -\cos y \Big|_1^2 = \cos 1 - \cos 2$$

(b) $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{2 + x^3}\, dx\, dy$




$$0 \le y \le x^2$$
$$0 \le x \le 1$$

$$= \int_0^1 \int_0^{x^2} \sqrt{2+x^3}\, dy\, dx$$

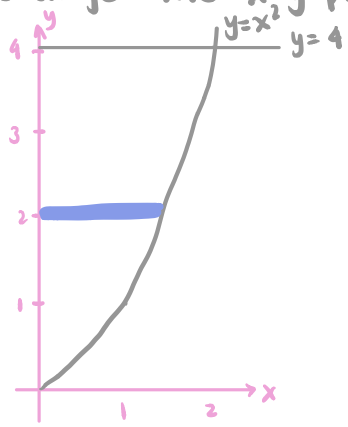
$$= \int_0^1 x^2 \sqrt{2+x^3}\, dx = \left[\frac{2}{9}(2+x^3)^{3/2}\right]_0^1 = \frac{2}{9}(3\sqrt{3} - 2\sqrt{2})$$

# 3D-Integration

Ex ② - Write the integral $\int_0^2 \int_{x^2}^4 \int_0^{\sqrt{y-x^2}} f(x,y,z)\,dz\,dy\,dx$ in two other orders.

Given
$$0 \le z \le \sqrt{y-x^2}$$
$$x^2 \le y \le 4$$
$$0 \le x \le 2$$

* Change the x-y projection.

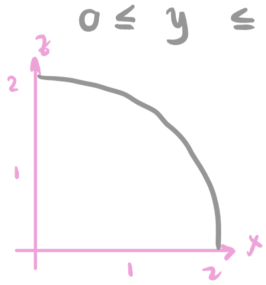



$$0 \le x \le \sqrt{y}$$
$$0 \le y \le 4$$
$$\rightarrow \int_0^4 \int_0^{\sqrt{y}} \int_0^{\sqrt{y-x^2}} f(x,y,z)\,dz\,dx\,dy$$

* Change the x-z projection.

$$0 \le z \le \sqrt{y-x^2}$$
$$0 \le x \le \sqrt{y}$$
$$0 \le y \le 4 \rightarrow \text{Fixed.}$$

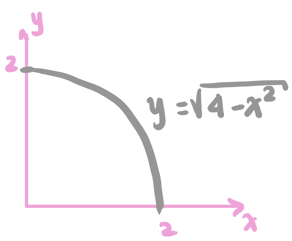
$$x^2+z^2 \le y \le 4$$
$$0 \le z \le \sqrt{4-x^2}$$
$$0 \le x \le 2$$




$$\rightarrow \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+z^2}^4 f(x,y,z)\,dy\,dz\,dx.$$

Ex ③ - Evaluate $\iiint_B x\,dV$ where B is the first octant solid bounded by the cylinder $x^2+y^2=4$ and the plane $2y+z=4$

$\hookrightarrow x,y,z \ge 0$




$$0 \le z \le 4-2y$$
$$0 \le y \le \sqrt{4-x^2}$$
$$0 \le x \le 2$$

$$\iiint_B x\,dV = \iint_{proj}\left(\int_0^{4-2y} x\,dz\right)dA = \int_0^2 \int_0^{\sqrt{4-x^2}} x(4-2y)\,dy\,dx$$

$$= \int_0^2 x\left[4y-y^2\right]_0^{\sqrt{4-x^2}} dx = \int_0^2 x\left[4\sqrt{4-x^2}-(4-x^2)\right]dx$$

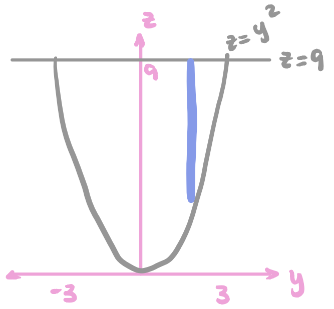
Let $u=4-x^2$
$du=-2x\,dx$

$$= \frac{-1}{2}\int 4u^{1/2} - u \, du = -\frac{1}{2}\left[\frac{8}{3}u^{3/2} - \frac{u^2}{2}\right] = \left[-\frac{4}{3}(4-x^2)^{3/2} + \frac{(4-x^2)}{2}\right]_0^2$$

$$= -\frac{4}{3}\left[(4-4)^{3/2} - 4^{3/2}\right] + \frac{1}{2}\left[(4-4)-(4-0)\right]$$

$$= \frac{4^{5/2}}{3} - 2 = \frac{32}{3} - \frac{6}{3} = \frac{26}{3}$$

Ex ④ - Find the volume of the region bounded by $x = y^2 + z^2$, $z = y^2$, $z = 9$ and $x = 0$




Fix $0 \le x \le y^2 + z^2$
$y^2 \le z \le 9$
$-3 \le y \le 3$

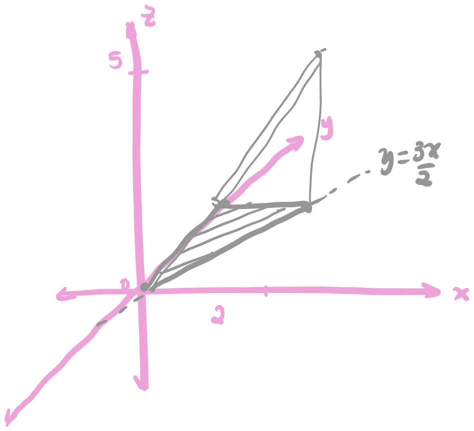
$$V = \int_B 1 \, dV = \int_{-3}^{3}\int_{y^2}^{9}\int_0^{y^2+z^2} dx \, dz \, dy = \int_{-3}^{3}\int_{y^2}^{9} y^2 + z^2 \, dz \, dy$$

$$= \int_{-3}^{3}\left[y^2 z + \frac{z^3}{3}\right]_{y^2}^{9} dy = \int_{-3}^{3} 9y^2 + 243 - y^4 - \frac{y^6}{3} \, dy$$

$$= \left[-\frac{y^7}{21} - \frac{y^5}{5} + 3y^3 + 243y\right]_{-3}^{3} = \dots = \frac{46008}{35}$$

# Week 11 (July 20th)

Ex ① - Find the volume of the tetrahedron with corners at $(0,0,0)$, $(0,3,0)$, $(2,3,0)$, and $(2,3,5)$




$0 \le z \le 5x/2$
$\frac{3x}{2} \le y \le 3$
$0 \le x \le 2$

$$\Rightarrow \int_0^2\int_{3x/2}^{3}\int_0^{5x/2} dz \, dy \, dx = \int_0^2\int_{3x/2}^{3} \frac{5x}{2} \, dy \, dx$$

$$= \int_0^2 \frac{5}{2}x\left(3 - \frac{3x}{2}\right) dx = \int_0^2 \frac{15x}{2} - \frac{15x^2}{4} \, dx$$

$$= \frac{15x^2}{4} - \frac{15x^3}{12}\Big|_0^2 = 15 - 10 = 5.$$

Ex ② - Evaluate $\int_1^2\int_y^{y^2}\int_0^{\ln(y+z)} e^x \, dx \, dz \, dy$

$$= \int_1^2\int_y^{y^2} e^{\ln(y+z)} - e^0 \, dz \, dy = \int_1^2\int_y^{y^2} y + z - 1 \, dz \, dy = \int_1^2 yz + \frac{z^2}{2} - z \Big|_y^{y^2} dy$$

$$= \int_1^2 y^3 + \frac{y^4}{2} - y^2 - y^2 - \frac{y^2}{2} - y \, dy = \int_1^2 y^3 + \frac{y^4}{4} - \frac{5y^2}{2} - y \, dy$$

$$= \dots = \frac{151}{600}$$

## 1D- Parametrization

$f(t): \mathbb{R} \to \mathbb{R}^2$ or $f(t): \mathbb{R} \to \mathbb{R}^3$

for each $t$ value, there is an $f(t)$ point in $\mathbb{R}^2 / \mathbb{R}^3$
If we take $t$ as time, we take $f(t)$ to be the position of an object.

Ex: Parametrize a circle of radius $R$

We need to find the coordinate $(x,y)$ for each $t$.

We can use polar coord. so $f(t) = (R\cos t, R\sin t)$, $t \in [0, 2\pi]$

## 2D- Parametrization.

$f(u,v): \mathbb{R}^2 \to \mathbb{R}^3$

## Coordinate transformation

$f(u,v): \mathbb{R}^2 \to \mathbb{R}^2$ or $f(u,v,w): \mathbb{R}^3 \to \mathbb{R}^3$

Ex: $f(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y)$ $\quad r \in [0,R]$ $\theta \in [0, 2\pi]$

Ex③ - The path of a particle is given by the following set of parametric eqs.
$x = 3\cos(2t)$, $y = 1 + \cos^2(2t)$

$\Rightarrow x = 3\cos(2t)$

$$-1 \leq \cos(2t) \leq 1$$
$$-3 \leq 3\cos(2t) \leq 3$$
$$-3 \leq x \leq 3$$

$\Rightarrow y = 1 + \cos^2(2t)$

$$0 \leq \cos^2(2t) \leq 1$$
$$1 \leq 1 + \cos^2(2t) \leq 2$$
$$1 \leq y \leq 2$$

$\to \cos(2t) = 1$ $\quad$ When is $\cos(2t) = 1$?

$$2t = 0 + 2\pi n$$
$$t = \pi n, \quad n \in \mathbb{Z}$$

$\to \cos(2t) = -1$ $\quad$ When is $\cos(2t) = -1$?

$$2t = \pi + 2\pi n$$
$$t = \frac{\pi}{2} + \pi n, \quad n \in \mathbb{Z}$$

$\therefore$ The right end is when $t = n\pi$ and the left end is when $t = \frac{\pi}{2} + \pi n$.

Ex ④ - For each of the following vector-valued functions, evaluate $f(0)$, $f(\pi/2)$, $f(2\pi/3)$. Do any of these functions have domain restriction?

a) $f(t) = (4\cos t, 3\sin t)$

$$f(0) = (4\cos(0), 3\sin(0)) = (4, 0)$$

$$f(\pi/2) = \left(4\cos\frac{\pi}{2}, 3\sin\frac{\pi}{2}\right) = (0, 3)$$

$$f(2\pi/3) = \left(4\cos\frac{2\pi}{3}, 3\sin\frac{2\pi}{3}\right) = \left(-2, \frac{3\sqrt{3}}{2}\right)$$

b) $f(t) = (3\tan t, 4\sec t, 5t)$

$$f(0) = (3\tan(0), 4\sec(0), 5(0)) = (0, 4, 0)$$

$$f(\pi/2) = (3\tan(\pi/2), 4\sec(\pi/2) + 5(\pi/2)k \quad , \text{DNE}$$

$$f(2\pi/3) = \left(3\tan\frac{2\pi}{3}, 4\sec\frac{2\pi}{3}, 5\left(\frac{2}{3}\pi\right)\right)$$

$$= (-3\sqrt{3}, -8, 10\pi/3)$$

## Week 12 (July 27th)

### Derivative Matrix

Ex ① - Find $Df$ given $f(x,y,z) = (z, 2yz^2, x + 2y^2z)$

$f: \mathbb{R}^3 \to \mathbb{R}^3$ $\qquad f(x,y,z) = (f_1, f_2, f_3)$

$$Df = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2z^2 & 4yz \\ 1 & 4yz & 2y^2 \end{pmatrix}$$

Ex ② - Find $Df$ given $f(x,y) = (xy^2, x + 2y, xy)$

$f: \mathbb{R}^2 \to \mathbb{R}^3$ $\qquad f(x,y) = (f_1, f_2, f_3)$

$$Df = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \end{pmatrix} = \begin{pmatrix} y^2 & 2xy \\ 1 & 2 \\ y & x \end{pmatrix}$$

### Change of Variable Formula

$g: A \to B \quad g(A) = B \qquad \int_B f = \int_A f \circ g \cdot |\mathrm{Det}(Dg)|$

# *Polar Coordinates! YAY ♡

Make the transformation $g(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y)$

How is it equivalent? $f \circ g = f(g(r,\theta)) = f(r\cos\theta, r\sin\theta) = f(x,y)$

$$Dg = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

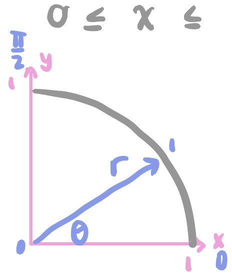
$$\begin{aligned} |Dg| &= r\cos\theta\cdot\cos\theta - (-r)\sin\theta\sin\theta \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r \end{aligned}$$

Ex ③ - Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)^{5/2}\,dy\,dx$

Change to Polars!

$0 \le y \le \sqrt{1-x^2}$

$0 \le x \le 1$




$0 \le r \le 1$

$0 \le \theta \le \frac{\pi}{2}$

$$= \int_0^{\pi/2}\int_0^1 (r^2)^{5/2}\, r\,dr\,d\theta$$

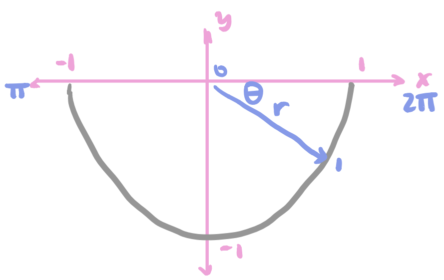
$$= \int_0^{\pi/2} \frac{r^7}{7}\Big|_0^1 d\theta = \frac{1}{7}\int_0^{\pi/2} d\theta = \frac{1}{7}\left(\frac{\pi}{2}\right) = \frac{\pi}{14}$$

Ex ④ - Evaluate $\displaystyle\int_{-1}^1\int_{-\sqrt{1-x^2}}^0 \cos(x^2+y^2)\,dy\,dx$

Change to polars!

$-\sqrt{1-x^2} \le y \le 0$

$-1 \le x \le 1$




$0 \le r \le 1$

$\pi \le \theta \le 2\pi$

$$= \int_\pi^{2\pi}\int_0^1 \cos(r^2)\, r\,dr\,d\theta$$

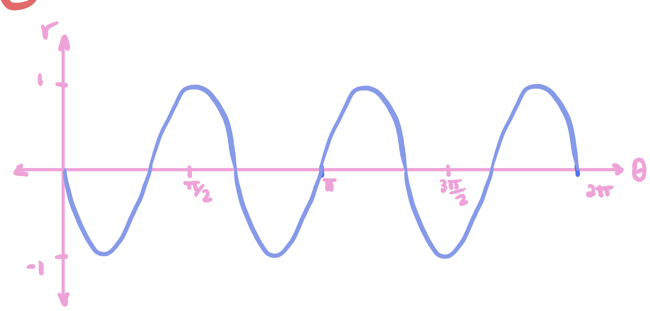
$$= \int_\pi^{2\pi} \frac{1}{2}\sin(r^2)\Big|_0^1 d\theta$$

$$= \int_\pi^{2\pi} \frac{1}{2}\sin(1)\,d\theta$$

$$= \frac{\pi}{2}\sin(1)$$

## Polar curves

Ex ⑤ - Sketch the curve $r = -\sin 3\theta$ in the polar plane



| $\theta$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|
| $r$ | $0$ | $-1$ | $0$ | $1$ | $0$ | $-1$ | $0$ |