MATB41 - Multivariable Calculus

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Week 1 (May 11)

* No tutorials *

Week 2 (May 18)

Curved Lines

Parabola: (with vertex) at the origin

- $y = ax^2$ → open up when $a>0$ and down when $a<0$
- $x = by^2$ → open right when $b>0$ and left when $b<0$

* **Remember**: We can replace x with $x-m$ to shift right and y with $y+n$ to shift up!

Ellipse: (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

intersections $(\pm a, 0)$ and $(0, \pm b)$
when $a=b$, the curve is a circle of radius $a=b$.

Hyperbola: (with center) at the origin

- $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$
- $\left(\frac{y}{a}\right)^2 - \left(\frac{x}{b}\right)^2 = 1$

It has two curves that goes in opposite directions and do not touch the slant asymptote

Ex ① - Sketch the following curves:

a) $x^2 + 3y^2 + 2x - 12y + 10 = 0$

$$(x^2+2x) + 3(y^2-4y)+10=0$$
$$(x^2+2x+1)-1+3(y^2-4y+4)-12+10=0$$
$$(x+1)^2+3(y-2)^2=3$$

$$\Rightarrow \frac{(x+1)^2}{3} + (y-2)^2 = 1 \quad \Rightarrow \left(\frac{x+1}{\sqrt{3}}\right)^2 + (y-2)^2 = 1$$

Okie, now what?! Another way to write the ellipse equation would be:

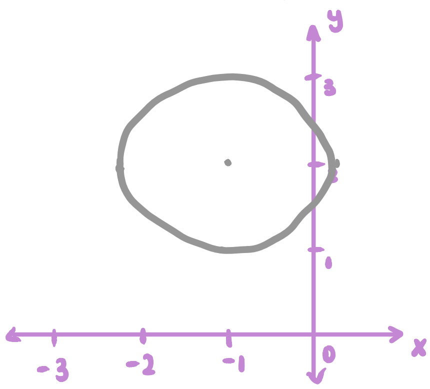
$$\left(\frac{x-m}{a}\right)^2 + \left(\frac{y-n}{b}\right)^2 = 1$$ → we just combined it with ✱

So the new origin will be $(-1, 2)$ as the ellipse will be shifted left by one and up by 2.

Now, let's find the end points

→ The horizontal max/min will be $(-1 \pm \sqrt{3}, 2)$

→ The vertical max/min will be $(-1, 2 \pm 1)$



b) $\frac{(x-2)^2}{4} - \frac{(y+2)^2}{9} = 1$

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 1$$

The origin will be $(2, -2)$, we are shifting right 2 and 2 down.

Check the x-int: $(y=0)$

$$\left(\frac{x-2}{2}\right)^2 - \frac{4}{9} = 1$$

$$\frac{(x-2)^2}{4} = \frac{13}{9}$$

$$(x-2)^2 = \frac{52}{9}$$

$$x-2 = \pm\frac{2\sqrt{13}}{3}$$

$$x = 2 \pm \frac{2\sqrt{13}}{3}$$

Check the y-int: $(x=0)$

$$1 - \left(\frac{y+2}{3}\right)^2 = 1$$

$$y = -2$$

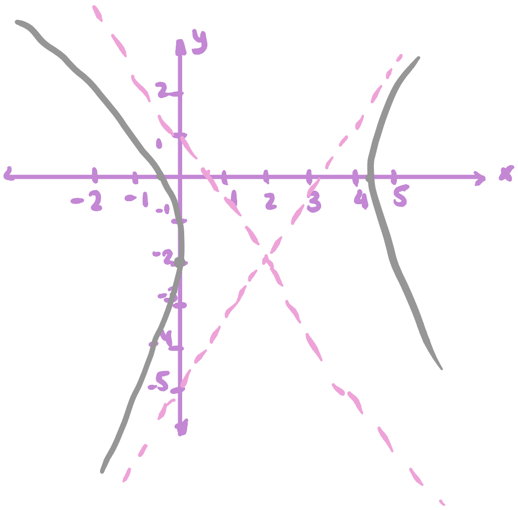
Find the slant asymptote:

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 0$$

$$\frac{x-2}{2} = \frac{y+2}{3}$$
$$3x-6 = 2y+4$$
$$2y = 3x-10$$
$$y = \frac{3x}{2} - 5$$

or

$$\frac{-x+2}{2} = \frac{y+2}{3}$$
$$-3x+6 = 2y+4$$
$$2y = -3x+2$$
$$y = \frac{-3x}{2} + 1$$



Curved Surfaces

3D Sphere: (with center) at the origin of radius R

$$x^2+y^2+z^2 = R^2$$

3D Ellipsoid: (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

3 points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$

We can replace x^2+y^2 with r^2 to indicate the rotation.

Ex② - Sketch the following surfaces:

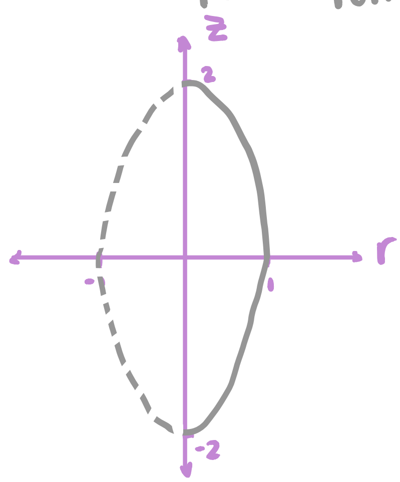
a) $x^2+y^2+\frac{z^2}{4} = 1$

$$r^2 + \frac{z^2}{4} = 1$$
$$4r^2+z^2 = 4$$
$$r^2 + \frac{z^2}{4} = 1$$

Center at $(0,0)$

r-int : $(\pm 1, 0)$
z-int : $(0, \pm 2)$

* Use the ellipsoid formula

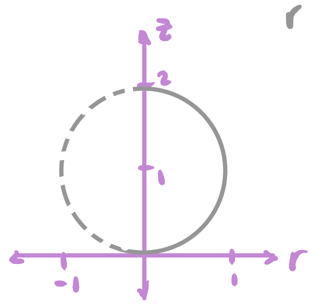


$r \geq 0$ because $r = \sqrt{x^2+y^2}$

b) $x^2+y^2+z^2=2z$

$r^2+z^2-2z=0$
$r^2+z^2-2z+1-1=0$
$r^2+(z-1)^2=1$ → circle!

Center $(0,1)$



$r \geq 0$ because $r=\sqrt{x^2+y^2}$

Determinant

When $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| = ad-bc$

When $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Ex ③ - Find the determinant of $\begin{bmatrix} -3 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 4 & 3 \end{bmatrix}$

$$\begin{vmatrix} -3 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 4 & 3 \end{vmatrix} = -3\begin{vmatrix} -1 & 2 \\ 4 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix}$$

$= -3(-3-8) - (6-2) + (8+1)$
$= -3 \times -11 - 4 + 9$
$= 33+5$
$= 38$

Dot and Cross Product

Let $\vec{u}\cdot\vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix} = u_1v_1 + u_2v_2 + \dots$

Let $\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right)$
$= (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$

You can find the angle between two vectors using dot product:

Let the angle be called θ, then $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

Ex ④ - Find the angle between $\vec{u}=(1,-3,1)$ and $\vec{v}=(2,1,2)$ in $\mathbb{R}^3$

First, we need the dot product and lengths:

$\vec{u}\cdot\vec{v} = (1,-3,1)(2,1,2) = 2-3+2 = 1$

$\|\vec{u}\| = \sqrt{1^2+(-3)^2+1^2} = \sqrt{11}$

$\|\vec{v}\| = \sqrt{2^2+1^2+2^2} = \sqrt{9} = 3$

$\Rightarrow \cos\theta = \dfrac{1}{3\sqrt{11}} \qquad \therefore \quad \theta = \cos^{-1}\left(\dfrac{1}{3\sqrt{11}}\right)$

Line and Planes

Vector equation of a line in $\mathbb{R}^3$: $\vec{\ell} = (a_1, a_2, a_3) + t[v_1, v_2, v_3]$
$\vec{\ell} = a + t\vec{v}$

Parametric equation of a line in $\mathbb{R}^3$:
$x = a_1 + tv_1$
$y = a_2 + tv_2$
$z = a_3 + tv_3$

Ex ⑤ - Find the equation of the line or plane

a) The line through $(1,-1,2)$ and $(3,1,9)$

The direction vector for the line is $(3,1,9)-(1,-1,2) = (2,2,7)$

$\rightarrow$ V. eq. $\vec{\ell} = (1,-1,2) + t(2,2,7)$, $t \in \mathbb{R}$

$\rightarrow$ P. eq. $x = 1+2t$, $y = -1+2t$, $z = 2+7t$, $t \in \mathbb{R}$

b) The plane through $(1,-3,1)$, $(2,1,1)$, $(1,4,0)$

Let's find a pair of directional vectors $\vec{v_1}$ and $\vec{v_2}$

$\vec{v_1} = (2,1,1) - (1,-3,1) = (1,4,0)$

$\vec{v_2} = (1,4,0) - (1,-3,1) = (0,7,-1)$

To get the equation of the plane, we need to find the normal which is $\vec{v_1} \times \vec{v_2}$

$$\vec{n} = \vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 0 \\ 0 & 7 & -1 \end{vmatrix} = (-4, 1, 7)$$

The plane then is $-4x + y + 7z = d$
Let's plug a point: $-4(2) + (1) + 7(1) = -8+1+7 = 0 = d$

$\rightarrow$ The plane is $-4x + y + 7z = 0$

Week 3 (May 25)

Level Set

Ex ① - Draw the level set of $f(x,y,z) = z^2 - x^2 - y^2 - 4z$ for $f = -4$

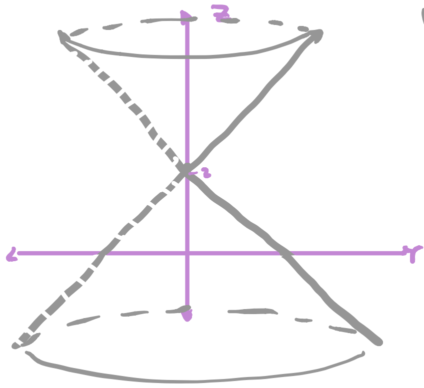
↳ this is in 4D though

$-4 = z^2 - x^2 - y^2 - 4z$

$x^2 + y^2 = z^2 - 4z + 4$

$r^2 = (z-2)^2$

$r = \pm(z-2)$



$r \geq 0$

Limits and Continuity

$f(x,y)$ is continuous at the 2D point $\vec{a}$ if $\lim\limits_{(x,y)\to\vec{a}} f(x,y) = f(\vec{a})$

It's not too hard to show when a limit DNE in $\mathbb{R}^2$ at $(0,0)$:
Try different curves in terms of x or/and y, if they approach to different values at $(0,0)$

To show that the limit exists we can use the Squeeze Theorem:

To attain $\lim\limits_{(x,y)\to\vec{a}} f(x,y)$, can try to find $g(x,y)$ and $h(x,y)$ so that:

1. $g(x,y) \leq f(x,y) \leq h(x,y)$ near the point $\vec{a}$
2. $\lim\limits_{(x,y)\to\vec{a}} g(x,y) = L = \lim\limits_{(x,y)\to\vec{a}} h(x,y)$

Then we conclude $\lim\limits_{(x,y)\to\vec{a}} f(x,y) = L$

Ex ② - Decide whether the function has a limit at $(0,0)$

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2} = \text{DNE}$

Restrict to $x=0$, $\lim\limits_{(0,y)\to(0,0)} \dfrac{-y^2}{y^2} = -1$

Restrict to $y=0$, $\lim\limits_{(x,0)\to(0,0)} \dfrac{x^2}{x^2} = 1$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{|x|}{\sqrt{x^2+y^2}} = \text{DNE}$

Restrict to $y=0$, $\lim_{(x,0)\to(0,0)} \frac{|x|}{\sqrt{x^2}} = 1$

Restrict to $x=0$, $\lim_{(0,y)\to(0,0)} \frac{|x|}{\sqrt{y}} = \frac{0}{\sqrt{y}} = 0$

c) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \underbrace{\frac{x^3}{x^2+y^2}}_{①} - \underbrace{\frac{y^3}{x^2+y^2}}_{②} = 0$

① $x^2 \le x^2+y^2 \rightarrow 0 \le \frac{x^2}{x^2+y^2} \le 1$

$\rightarrow 0 \cdot x \le \frac{x^3}{x^2+y^2} \le x$

$\rightarrow \lim_{(x,y)\to(0,0)} 0 \le \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} \le \lim_{(x,y)\to(0,0)} x$

$\rightarrow 0 \le \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} \le 0$

$\Rightarrow 0$

② Same as ①

$y^2 \le x^2+y^2 \rightarrow 0 \le \frac{y^2}{x^2+y^2} \le 1$

$\rightarrow 0 \cdot \le \frac{y^3}{x^2+y^2} \le$

$\rightarrow \lim_{(x,y)\to(0,0)} y \cdot 0 \le \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} \le \lim_{(x,y)\to(0,0)} y$

$\rightarrow 0 \le \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} \le 0$

$\Rightarrow 0$

d) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

We know $|\sin\theta| \le 1$ and we saw in class $|\sin\theta| \le \theta$

$\Rightarrow \frac{|\sin(x^2+y^2)|}{x^2+y^2} \le \frac{x^2+y^2}{x^2+y^2} = 1$

Recall $\lim_{t\to 0} \frac{\sin t}{t} = 1$, what if we let $t = x^2+y^2$

$$\Rightarrow \lim_{t\to 0} \frac{\sin t}{t} = 1 = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

Ex ③ - Find if the limit of f exists at $(0,0)$, if it does, find $f(0,0)$ for f to be continuous.

$$f(x,y) = \frac{x^3-x^2-2x^2y+xy^2-y^2-2y^3}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2(x-1-2y)+y^2(x-1-2y)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x-1-2y)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} x-1-2y$$

$$= -1$$

$$\Rightarrow f(0,0) = -1$$

$f(\vec{x})$ is homogeneous of degree k if for every $\vec{x} \in \mathbb{R}^n$ and every scalar $c>0$ we have:

$$f(c\vec{x}) = c^k f(\vec{x})$$

Ex ④ - Show if f is homogeneous

a) $f(x,y) = 8x^2y^2 - 9x^4$

$$\begin{aligned} f(cx, cy) &= 8(cx)^2(cy)^2 - 9(cx)^4 \\ &= 8c^2x^2c^2y^2 - 9c^4x^4 \\ &= 8c^4x^2y^2 - 9c^4x^4 \\ &= c^4(8x^2y^2 - 9x^4) \\ &= c^4 f(x,y) \end{aligned}$$

$\Rightarrow f$ is homogeneous of degree 4.

b) $f(x,y) = \frac{x^2}{y^2} + xy + \frac{y^2}{x^2}$

$$\begin{aligned} f(cx, cy) &= \frac{(cx)^2}{(cy)^2} + (cx)(cy) + \frac{(cy)^2}{(cx)^2} \\ &= \frac{c^2x^2}{c^2y^2} + c^2xy + \frac{c^2y^2}{c^2x^2} \\ &= \frac{x^2}{y^2} + c^2xy + \frac{y^2}{x^2} \end{aligned}$$

$\Rightarrow$ cannot factor c.

$\Rightarrow f$ is not homogeneous.