

MATA02 - The Magic of Numbers

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Tutorials: T2 (HW308 Thursdays 9-10 am)

T3 (AA206 Tuesdays 9-10 am)

T6 (MW160 Thursdays 10-11 am)

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Week 1 - Jan. 6th

No tutorials

Week 2 - Jan 13th

Today: Greatest Common Divisor (gcd) and Least Common Multiple (lcm)

$$\hookrightarrow \text{gcd}(6, 8) = 2$$

$$\hookrightarrow \text{lcm}(6, 8) = 24$$

Euclidean Algorithm: If $a = bq + r$, $0 \leq r < b$ then $\text{gcd}(a, b) = \text{gcd}(b, r)$

$$* \text{gcd}(a, 0) = a$$

$$* \text{gcd}(0, b) = b$$

$$* \text{lcm}(a, b) = \frac{ab}{\text{gcd}(a, b)}$$

Ex1: Use the Euclidean Algorithm to find $\text{gcd}(51, 96)$ and use it to find the $\text{lcm}(51, 96)$

$$\begin{aligned} 96 &= 51 \cdot 1 + 45 & \rightarrow d = \text{gcd}(96, 51) \\ 51 &= 45 \cdot 1 + 6 & \rightarrow d = \text{gcd}(51, 45) \\ 45 &= 6 \cdot 7 + 3 & \rightarrow d = \text{gcd}(45, 6) \\ 6 &= 3 \cdot 2 + 0 & \xrightarrow{\text{stop!}} d = \text{gcd}(6, 3) \end{aligned}$$

$$\therefore \text{gcd}(96, 51) = \text{gcd}(51, 45) = \text{gcd}(45, 6) = \text{gcd}(6, 3) = 3$$

$$\therefore \text{lcm}(96, 51) = \frac{96 \times 51}{\text{gcd}(96, 51)} = \frac{96 \times 51}{3} = 96 \times 17 = 1632$$

* If $a = ms$ and $b = mt$, then $\text{gcd}(a, b) = m \times \text{gcd}(s, t)$

Ex1: (Again!) but using this method.

$$\text{gcd}(51, 96) = \text{gcd}(3 \cdot 17, 3 \cdot 32) = 3 \cdot \text{gcd}(17, 32) = 3 \cdot 1 = 3 \quad \checkmark$$

Ex2: (T1-Q1) Use the Euclidean Algorithm to find $\gcd(366, 150)$ and use it to find the $\text{lcm}(366, 150)$

$$366 = 150 \cdot 2 + 66 \rightarrow d = \gcd(366, 150)$$

$$150 = 66 \cdot 2 + 18 \rightarrow d = \gcd(150, 66)$$

$$66 = 18 \cdot 3 + 12 \rightarrow d = \gcd(66, 18)$$

$$18 = 12 \cdot 1 + 6 \rightarrow d = \gcd(18, 12)$$

$$12 = 6 \cdot 2 + 0 \rightarrow d = \gcd(12, 6)$$

$$\therefore \gcd(366, 150) = 6$$

$$\therefore \text{lcm}(366, 150) = \frac{366 \times 150}{\gcd(366, 150)} = \frac{366 \times 150}{6} = 61 \times 150 = 9150$$

Ex3: (T1-Q3) Does the equation $12x+20y=90$ have a solution of integers x and y ?

No \because Because on the left side we can factor out 4 $\rightarrow 4(3x+5y)$ however, 90 is not divisible by 4.

Ex4: Does the equation $11x+1111=121+22y$ have a solution of integers x and y ?

No \because Because the right side is divisible by 11 $\rightarrow 11(11+2y)$, but the left side isn't as 1111 is not divisible by 11.

Ex5: (T1-Q4) If a and b are integers, and x is an integer such that $x^2+ax+b=0$. Show that $x|b$

$\hookrightarrow b$ is divisible by x

$$x^2+ax+b=0$$

$$\Rightarrow b = -x^2 - ax$$

$$\Rightarrow b = -x \underbrace{(x+a)}$$

\hookrightarrow is an integer because x and a are integers

Because b is also an integer, then $x|b$. \square

Ex6: Show that if $a|b$ and $b|a$, then $a=b$ or $a=-b$

$a|b$ means that there exist an $n \in \mathbb{Z}$ such that $an=b$ ①

$b|a$ means that there exist an $m \in \mathbb{Z}$ such that $bm=a$ ②

Use ① for ②: $(an)m=a \Rightarrow anm=a \Rightarrow nm=1$.

The only integers n and m that work for $nm=1$ are $n=m=1$ or $n=m=-1$

→ When $n=m=1 \Rightarrow a=b$, by ①

→ When $n=m=-1 \Rightarrow a=-b$, by ②

□

Week 3 - Jan 20th

Ex 1 (T2-Q1) Find integers x and y such that $6x+5y=4$

let's do Euclid's Algo!

$$\begin{aligned} 6 &= 5 \cdot 1 + 1 \\ 5 &= 1 \cdot 4 + 1 \\ 1 &= 1 \cdot 1 + 0 \end{aligned} \rightarrow \text{But wait! This line here might help:}$$

let's write it differently: $6 - 5 \cdot 1 = 1$

Now, we need the equation $(6 - 5 \cdot 1 = 1) \times 4$
to equal 4:

$$\underline{\underline{6}} \cdot \underline{\underline{4}} - \underline{\underline{5}} \cdot \underline{\underline{4}} = \underline{\underline{4}}$$

The signs are different...

$$6 \cdot \underbrace{4}_{x} + 5 \cdot \underbrace{(-4)}_{y} = 4$$

∴ In $6x+5y=4$, $x=4, y=-4$

Ex 2 Determine if the equations has integer solutions x and y .

$$\text{eq 1} \quad 1015x + 231y = 9$$

$$\text{eq 2} \quad 1015x + 231y = 28$$

① Find the gcd of $(1015, 231)$

$$\begin{aligned} 1015 &= 231 \times 4 + 91 \quad (1) \\ 231 &= 91 \times 2 + 49 \quad (2) \\ 91 &= 49 \times 1 + 42 \quad (3) \\ 49 &= 42 \times 1 + 7 \quad (4) \\ 42 &= 7 \times 6 + 0 \end{aligned}$$

$$\rightarrow \gcd(1015, 231) = 7$$

so by definition $\exists m, n$ such that $1015m + 231n = 7$

Looking at eq 1 and eq 2, $7 \nmid 9$ and $7 \mid 28$. So only eq 2 has integer solution.

② Write gcd in terms of the values.

$$\begin{aligned}
 7 &= 49 - 42 \times 1 && (4) \\
 &= 49 - (91 - 49 \times 1) && (3) \\
 &= 49 \times 2 - 91 && \\
 &= (231 - 91 \times 2) \times 2 - 91 && (2) \\
 &= 231 \times 2 - 91 \times 4 - 91 \\
 &= 231 \times 2 - 91 \times 5 \\
 &= 231 \times 2 - (1015 - 231 \times 4) \times 5 && (1) \\
 &= 231 \times 2 - 1015 \times 5 + 231 \times 20 \\
 7 &= 231 \times 22 + 1015 \times (-5)
 \end{aligned}$$

③ $4 \times 7 = 231 \times 22 \times 4 + 1015 \times (-5) \times 4$
 $28 = 231 \times \underbrace{88}_y + 1015 \times \underbrace{(-20)}_x$

④ We want to find a general solution for $28 = 1015X + 231Y$

Theorem: If $d = \gcd(a, b) | c$ and x, y is an integer solution of $ax + by = c$, then so is $x + \frac{b}{d}t, y - \frac{a}{d}t$ for any integer t .

In this case, $d = 7 | 28$ and we have $28 = \underbrace{1015}_a X + \underbrace{231}_b Y$
We know $x = -20$ and $y = 88$ in ③

The general solution has $X = -20 + \frac{231}{7}t = -20 + 33t$

$$Y = 88 - \frac{1015}{7}t = 88 - 145t$$

$$\therefore \text{The general solution is } 1015(-20+33t) + 231(88-145t) = 28$$

Ex3 (T2-Q2) Find all integers x and y such that $30x + 8y = 500$. For which solutions are x and y both positive?

① $30 = 8 \times 3 + 6 \quad (1)$
 $8 = 6 \times 1 + 2 \quad (2)$
 $6 = 2 \times 3 + 0 \quad \rightarrow \gcd(30, 8) = 2$

② $2 = 8 - 6 \times 1 \quad (2)$
 $= 8 - (30 - 8 \times 3) \times 1 \quad (1)$
 $= 8 - 30 \times 1 + 8 \times 3$
 $2 = 8 \times 4 + 30 \times (-1)$

③ $250 \times 2 = 8 \times 4 \times 250 + 30 \times (-1) \times 250$
 $500 = 8 \times \underbrace{1000}_y + 30 \times \underbrace{(-250)}_x$

$$\textcircled{4} \text{ General Solution: } x = -250 + \frac{8}{2} t = -250 + 4t$$

$$y = 1000 - \frac{30}{2} t = 1000 - 15t$$

$$x \geq 0 \rightarrow -250 + 4t \geq 0$$

$$4t \geq 250$$

$$t \geq \frac{125}{2} = 62.5$$

$$y \geq 0 \rightarrow 1000 - 15t \geq 0$$

$$1000 \geq 15t$$

$$\frac{200}{3} \geq t \approx 66.\bar{6}$$

So $62.5 \leq t \leq 66.\bar{6}$, t is an integer

$\therefore t = 63, 64, 65, 66$ to have x and y positive

Ex 4 (T2-Q4) Suppose the sophomores, juniors, and seniors in the tutorial decided to collect money to host a party. If each sophomore contributes \$25, each junior contributes \$18, each senior contributes \$10, \$450 will be collected. If there are 35 students, how many sophomores, juniors, and seniors are there?

$$\begin{aligned} x + y + z &= 35 & \textcircled{1} \\ 25x + 18y + 10z &= 450 & \textcircled{2} \end{aligned}$$

$$\textcircled{1} \times 10 \rightarrow 10x + 10y + 10z = 350 \quad \textcircled{3}$$

$$\textcircled{2} - \textcircled{3} \rightarrow 15x + 8y = 100$$

let's do Euclid's Alg. to find the gcd of 15 and 8

$$\begin{aligned} \textcircled{1} \quad 15 &= 8 \times 1 + 7 & (1) \\ 8 &= 7 \times 1 + 1 & (2) \\ 7 &= 1 \times 7 + 0 & \rightarrow \gcd(15, 8) = 1 \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad 1 &= 8 - 7 \times 1 & (2) \\ 1 &= 8 - (15 - 8 \times 1) \times 1 & (1) \\ 1 &= 8 \times 2 - 15 \times 1 \\ 1 &= 8 \times \underbrace{2}_{y} + 15 \times \underbrace{(-1)}_{x} \end{aligned}$$

\hookrightarrow but there cannot be a negative solution!

$$\begin{aligned} \textcircled{3} \quad 100 \times 1 &= 8 \times 2 \times 100 + 15 \times (-1) \times 100 \\ 100 &= 8 \times 200 + 15 \times (-100) \end{aligned}$$

④ General solution: $x = -100 + \frac{8t}{1} = -100 + 8t$
 $y = 200 - \frac{15t}{1} = 200 - 15t$

We need to find $x \geq 0, y \geq 0$

$$\begin{aligned} -100 + 8t &\geq 0 & 200 - 15t &\geq 0 \\ 8t &\geq 100 & 200 &\geq 15t \\ t &\geq \frac{25}{2} = 12.5 & 13.3 &\approx \frac{40}{3} \geq t \end{aligned}$$

$\Rightarrow 12.5 \leq t \leq 13.3$, and t is an integer.

$$\begin{aligned} \therefore t=13 &\rightarrow x = -100 + 8 \times 13 = -100 + 104 = 4 \\ &\rightarrow y = 200 - 15 \times 13 = 200 - 195 = 5 \end{aligned}$$

$$\begin{aligned} \rightarrow x+y+z &= 35 \Rightarrow 4+5+z=35 \\ &\Rightarrow z=26 \end{aligned}$$

Ex 5 (T2-Q5) Show there are infinitely primes of the form $4t-1$, where t is an integer

Let's do a proof by contradiction. Which means, we will assume there are only a finite number of primes of the form $4t-1$, we will call them p_1, p_2, \dots, p_r .

Let $N = p_1 p_2 \dots p_r$, and let's consider the number $4N-1$. By our initial assumption, $4N-1$ is not prime, so it has a prime factor!

We claim $4N-1$ has a prime factor of the form $4t-1$.

Now, let's see what happens when we multiply $4x+1$ and $4y+1$

$$\rightarrow (4x+1)(4y+1) = 16xy + 4x + 4y + 1 = 4(4xy + x + y) + 1$$

Which means, $4N-1$ must have a prime factor of form $4t-1$. Since 2 is not a factor.

So there is a p_i such that $p_i | 4N-1$. Then as $p_i | N$, we have $p_i | 1$. Which contradicts!

There are infinitely many primes of the form $4t-1$. \square

Week 4 - Jan 27th

If n is composite, then it has a prime divisor p such that $p \leq \sqrt{n}$.

To determine if n is prime, check whether each of the prime numbers up to \sqrt{n} divide n

Ex1 (T3-Q1) Which of the following numbers are prime?

(a) 407

$\sqrt{407} < 21$ which means that we only need to check for primes up to 20
primes up to 20 = {2, 3, 5, 7, 11, 13, 17, 19}

- $2 \mid 407$? Nope! $407 = 2 \times 203 + 1$
- $3 \mid 407$? Nope! $407 = 3 \times 135 + 2$
- $5 \mid 407$? Nope! $407 = 5 \times 81 + 2$
- $7 \mid 407$? Nope! $407 = 7 \times 58 + 1$
- $11 \mid 407$? Yes $407 = 11 \times 37$

$\therefore 407$ is not prime.

(b) 463

$\sqrt{463} < 22$ which means that we only need to check for primes up to 21
primes up to 21 = {2, 3, 5, 7, 11, 13, 17, 19}

- $2 \mid 463$? Nope! $463 = 2 \times 231 + 1$
- $3 \mid 463$? Nope! $463 = 3 \times 154 + 1$
- $5 \mid 463$? Nope! $463 = 5 \times 92 + 3$
- $7 \mid 463$? Nope! $463 = 7 \times 66 + 1$
- $11 \mid 463$? Nope! $463 = 11 \times 42 + 1$
- $13 \mid 463$? Nope! $463 = 13 \times 35 + 8$
- $17 \mid 463$? Nope! $463 = 17 \times 27 + 4$
- $19 \mid 463$? Nope! $463 = 19 \times 24 + 7$

$\therefore 463$ is prime.

Sieve of Eratosthenes: To find all prime numbers between a and b ($a \leq b$), for each prime p up to \sqrt{b} , cross out every multiple of p . The remaining numbers are primes.

Ex2 Find all primes between 235 and 265

$\sqrt{265} < 17$ which means that we only need to check for primes up to 16

Primes up to 16 = {2, 3, 5, 7, 11, 13}

235	236	237	238	239	240	241	242	243	244
245	246	247	248	249	250	251	252	253	254
255	256	257	258	259	260	261	262	263	264
265									

Step #1: Cross out all #'s that are divisible by 2

Step #2: Cross out all #'s that are divisible by 3

Step #3: Cross out all #'s that are divisible by 5

Step #4: Cross out all #'s that are divisible by 7

Step #5: Cross out all #'s that are divisible by 11

Step #6: Cross out all #'s that are divisible by 13

\therefore The primes in between 235 and 265 are {239, 241, 251, 257, 263}

Ex3 (T3-Q2) Find all primes between 201 and 250

$\sqrt{250} < 16$ which means that we only need to check for primes up to 15

Primes up to 15 = {2, 3, 5, 7, 11, 13}

201	202	203	204	205	206	207	208	209	210
211	212	213	214	215	216	217	218	219	220
221	222	223	224	225	226	227	228	229	230
231	232	233	234	235	236	237	238	239	240
241	242	243	244	245	246	247	248	249	250

\therefore The primes in between 201 and 250 are {211, 223, 227, 229, 233, 239, 241}

Ex4 (T3-Q3) Find the prime-power decomposition for each of the following.

(a) $437 = 19 \times 23$

(b) $709 = 709$

(c) $876 = 2^2 \times 3 \times 73$

- * An element x of a set is called a **unit** if there exists an element y such that $xy=1$
- * An **irreducible** is a non-unit element of A whose only positive divisors are itself, units, and products of itself and units.

Ex: Can we write a # in two different products of irreducibles?
 $\Rightarrow 100 = 4 \times 25 = 10 \times 10$

Ex5 (T3-Q4) Consider the set containing positive integers of the form $5t+1$. Find an element of B with two different irreducible decompositions in B .

$$t=1 \rightarrow 6 = 2 \times 3$$

$$t=2 \rightarrow 11 = 11$$

$$t=3 \rightarrow 16 = 2^4$$

$$t=4 \rightarrow 21 = 3 \times 7$$

$$t=5 \rightarrow 26 = 2 \times 13$$

$$t=6 \rightarrow 31 = 31$$

$$t=7 \rightarrow 36 = 2^2 \times 3^2$$

$$t=8 \rightarrow 41 = 41$$

$$t=9 \rightarrow 46 = 2 \times 23$$

$$t=10 \rightarrow 51 = 3 \times 17$$

$$t=11 \rightarrow 56 = 2^3 \times 7$$

:

Any common factors? $2^4, 3, 7$

$$2^4 \times 3 \times 7 = 336 = 335 + 1 = 5 \times 67 + 1$$

How can we decompose 336? Remember they need to be in B ($5t+1$ form)

$$\overbrace{2 \times 2 \times 2}^{\textcircled{1}} \times \overbrace{2 \times 3 \times 7}^{\textcircled{2}}$$

$$336 = 56 \times 6 = 16 \times 21$$

→ both are irreducible.

Ex5 (T3-Q5) Suppose n is composite, and let p be the smallest prime divisor of n . If $p > \sqrt[3]{n}$, show that n/p is prime.

Suppose n/p is not prime. Then $n/p = q \times r$ where $q, r \in \mathbb{Z}$. Since p is the smallest prime divisor of n , then $p \leq q$ and $p \leq r$.

$$\Rightarrow n = p \times q \times r \geq p^3 > (\sqrt[3]{n})^3 = n$$

↳ $n > n$?! impossible!

$\therefore n/p$ is prime

