

MATB41 - Multivariable Calculus

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Tutorials: TUT0007 (Wed. 11am-12pm)

TUT0006 (Thu. 10am-11am)

TUT0014 (Thu. 11am-12pm)

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Week 1 (Sep 14th)

* No tutorials *

Week 2 (Sep 21st)

Curved Lines

Parabola: (with vertex) at the origin

- $y = ax^2$ → open up when $a > 0$ and down when $a < 0$
- $x = by^2$ → open right when $b > 0$ and left when $b < 0$

* Remember: We can replace x with $x-m$ to shift right and y with $y+n$ to shift up!

Ellipse: (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \text{intersections } (\pm a, 0) \text{ and } (0, \pm b)$$

When $a=b$, the curve is a circle of radius $a=b$.

Hyperbola: (with center) at the origin

$$\cdot \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 \quad \text{It has two curves that goes in opposite directions and do not touch the slant asymptote}$$

$$\cdot \left(\frac{y}{a}\right)^2 - \left(\frac{x}{b}\right)^2 = 1$$

Ex ① - Sketch the following curves:

a) $x^2 + 3y^2 + 2x - 12y + 10 = 0$

$$\begin{aligned} (x^2 + 2x) + 3(y^2 - 4y) + 10 &= 0 \\ (x^2 + 2x + 1) - 1 + 3(y^2 - 4y + 4) - 12 + 10 &= 0 \\ (x+1)^2 + 3(y-2)^2 &= 3 \end{aligned}$$

$$\Rightarrow \left(\frac{x+1}{\sqrt{3}}\right)^2 + (y-2)^2 = 1 \quad \Rightarrow \left(\frac{x+1}{\sqrt{3}}\right)^2 + (y-2)^2 = 1$$

Okie, now what?! Another way to write the ellipse equation would be:

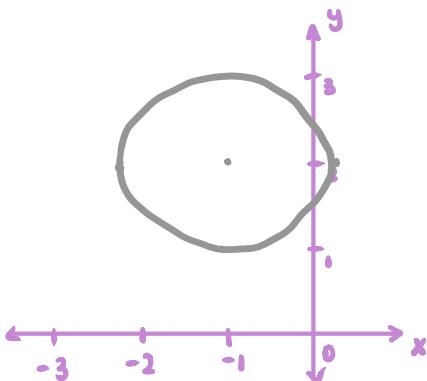
$$\left(\frac{x-m}{a}\right)^2 + \left(\frac{y-n}{b}\right)^2 = 1 \quad \rightarrow \text{we just combined it with *}$$

So the new origin will be $(-1, 2)$ as the ellipse will be shifted left by one and up by 2.

Now, let's find the end points

→ The horizontal max/min will be $(-1 \pm \sqrt{3}, 2)$

→ The vertical max/min will be $(-1, 2 \pm 1)$



b) $\left(\frac{x-2}{4}\right)^2 - \left(\frac{y+2}{9}\right)^2 = 1$

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 1$$

The origin will be $(2, -2)$, we are shifting right 2 and 2 down.

Check the x-int : $(y=0)$

$$\left(\frac{x-2}{2}\right)^2 - \frac{4}{9} = 1$$

$$\left(\frac{x-2}{2}\right)^2 = \frac{13}{9}$$

$$(x-2)^2 = \frac{52}{9}$$

$$x-2 = \pm \frac{2\sqrt{13}}{3}$$

$$x = 2 \pm \frac{2\sqrt{13}}{3}$$

Check the y-int : $(x=0)$

$$1 - \left(\frac{y+2}{3}\right)^2 = 1$$

$$y = -2$$

Find the slant asymptote:

$$\left(\frac{x-2}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 0$$

$$\frac{x-2}{2} = \frac{y+2}{3}$$

$$3x-6 = 2y+4$$

$$2y = 3x - 10$$

$$y = \frac{3x - 10}{2}$$

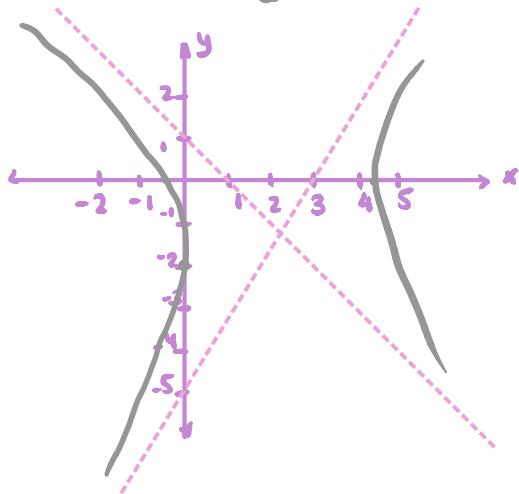
or

$$\frac{-x+2}{2} = \frac{y+2}{3}$$

$$-3x+6 = 2y+4$$

$$2y = -3x + 2$$

$$y = \frac{-3x + 2}{2}$$



Curved Surfaces

3D Sphere: (with center) at the origin of radius R

$$x^2 + y^2 + z^2 = R^2$$

3D Ellipsoid: (with center) at the origin

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \quad \text{3 points } (a, 0, 0), (0, b, 0), (0, 0, c)$$

We can replace $x^2 + y^2$ with r^2 to indicate the rotation.

Ex② - Sketch the following surfaces:

a) $x^2 + y^2 + \frac{z^2}{4} = 1$

$$r^2 + \frac{z^2}{4} = 1$$

$$4r^2 + z^2 = 4$$

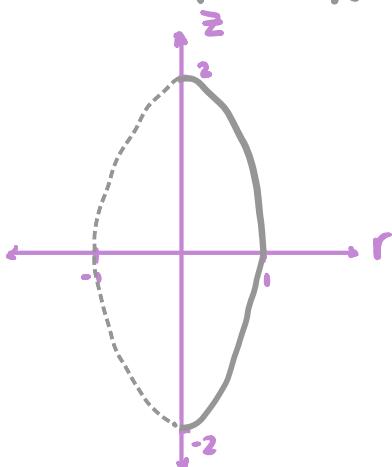
$$r^2 + \frac{z^2}{4} = 1$$

(center at $(0, 0)$)

r -int: $(\pm 1, 0)$

z -int: $(0, \pm 2)$

* Use the ellipsoid formula



$r \geq 0$ because
 $r = \sqrt{x^2 + y^2}$

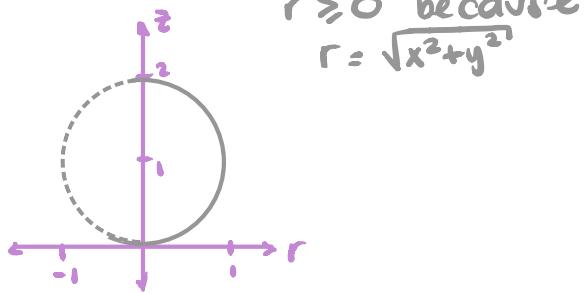
b) $x^2 + y^2 + z^2 = 2z$

$$r^2 + z^2 - 2z = 0$$

$$r^2 + z^2 - 2z + 1 - 1 = 0$$

$$r^2 + (z-1)^2 = 1 \rightarrow \text{circle!}$$

Center $(0, 1)$



$r \geq 0$ because
 $r = \sqrt{x^2 + y^2}$

Dot and Cross Product

let $\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix} = u_1v_1 + u_2v_2 + \dots$

let $\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$
 $= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$

You can find the angle between two vectors using dot product:

Let the angle be called θ , then $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$

Ex ③ - Find the angle between $\vec{u} = (1, -3, 1)$ and $\vec{v} = (2, 1, 2)$ in \mathbb{R}^3

First, we need the dot product and lengths:

$$\vec{u} \cdot \vec{v} = (1, -3, 1) \cdot (2, 1, 2) = 2 - 3 + 2 = 1$$

$$\|\vec{u}\| = \sqrt{1^2 + (-3)^2 + 1^2} = \sqrt{11}$$

$$\|\vec{v}\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

$$\Rightarrow \cos \theta = \frac{1}{3\sqrt{11}} \quad \therefore \quad \theta = \cos^{-1}\left(\frac{1}{3\sqrt{11}}\right)$$

Line and Planes

Vector equation of a line in \mathbb{R}^3 : $\vec{l} = (a_1, a_2, a_3) + t[v_1, v_2, v_3]$
 $\vec{l} = \vec{a} + t\vec{v}$

Parametric equation of a line in \mathbb{R}^3 : $x = a_1 + t v_1$
 $y = a_2 + t v_2$
 $z = a_3 + t v_3$

Ex ④ - Find the equation of the line or plane

a) The line through $(1, -1, 2)$ and $(3, 1, 9)$

The direction vector for the line is $(3, 1, 9) - (1, -1, 2) = (2, 2, 7)$

$$\rightarrow \text{v. eq. } \vec{l} = (1, -1, 2) + t(2, 2, 7), t \in \mathbb{R}$$

$$\rightarrow \text{P. eq. } x = 1 + 2t, y = -1 + 2t, z = 2 + 7t, t \in \mathbb{R}$$

b) The plane through $(1, -3, 1)$, $(2, 1, 1)$, $(1, 4, 0)$

let's find a pair of directional vectors \vec{v}_1 and \vec{v}_2

$$\vec{v}_1 = (2, 1, 1) - (1, -3, 1) = (1, 4, 0)$$

$$\vec{v}_2 = (1, 4, 0) - (1, -3, 1) = (0, 7, -1)$$

To get the equation of the plane, we need to find the normal which is $\vec{v}_1 \times \vec{v}_2$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 0 \\ 0 & 7 & -1 \end{vmatrix} = (-4, 1, 7)$$

The plane then is $-4x + y + 7z = d$

$$\text{Let's plug a point: } -4(2) + (1) + 7(1) = -8 + 1 + 7 = 0 = d$$

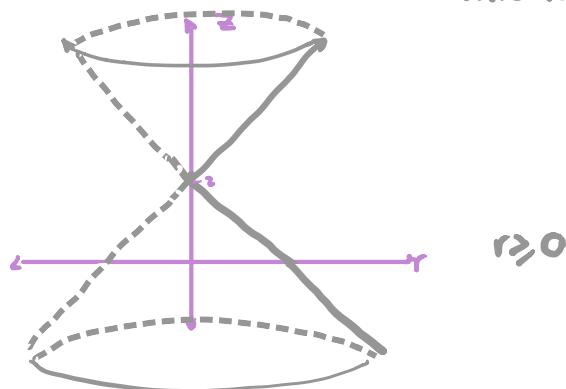
\rightarrow The plane is $-4x + y + 7z = 0$

Level Set

Ex ⑤ - Draw the level set of $f(x, y, z) = z^2 - x^2 - y^2 - 4z$ for $f = -4$

\hookrightarrow this is in 4D though

$$\begin{aligned} -4 &= z^2 - x^2 - y^2 - 4z \\ x^2 + y^2 &= z^2 - 4z + 4 \\ r^2 &= (z-2)^2 \\ r &= \sqrt{|z-2|} \end{aligned}$$



Week 3 (Sep 28th)

Limits and Continuity

$f(x, y)$ is continuous at the 2D point \vec{a} if $\lim_{(x,y) \rightarrow \vec{a}} f(x, y) = f(\vec{a})$

$(x,y) \rightarrow \vec{a}$

It's not too hard to show when a limit DNE in \mathbb{R}^2 at $(0,0)$:
 Try different curves in terms of x or y , if they approach to different values at $(0,0)$

To show that the limit exists we can use the Squeeze Theorem:

To attain $\lim_{(x,y) \rightarrow \vec{a}} f(x,y)$, can try to find $g(x,y)$ and $h(x,y)$ so that:

$$1. g(x,y) \leq f(x,y) \leq h(x,y) \text{ near the point } \vec{a}$$

$$2. \lim_{(x,y) \rightarrow \vec{a}} g(x,y) = L = \lim_{(x,y) \rightarrow \vec{a}} h(x,y)$$

Then we conclude $\lim_{(x,y) \rightarrow \vec{a}} f(x,y) = L$

Ex ② - Decide whether the function has a limit at $(0,0)$

$$a) \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \text{DNE}$$

$$\text{Restrict to } x=0, \lim_{(0,y) \rightarrow (0,0)} \frac{-y^2}{y^2} = -1$$

$$\text{Restrict to } y=0, \lim_{(x,0) \rightarrow (0,0)} \frac{x^2}{x^2} = 1$$

$$b) \lim_{(x,y) \rightarrow (0,0)} \frac{|x|}{\sqrt{x^2+y^2}} = \text{DNE}$$

$$\text{Restrict to } y=0, \lim_{(x,0) \rightarrow (0,0)} \frac{|x|}{\sqrt{x^2}} = 1$$

$$\text{Restrict to } x=0, \lim_{(0,y) \rightarrow (0,0)} \frac{|x|}{\sqrt{y}} = \frac{0}{\sqrt{y}} = 0$$

$$c) \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{\underset{\textcircled{1}}{x^3}}{\underset{\textcircled{2}}{x^2 + y^2}} - \frac{\underset{\textcircled{2}}{y^3}}{\underset{\textcircled{2}}{x^2 + y^2}} = 0$$

$$\textcircled{1} \quad x^2 \leq x^2 + y^2 \rightarrow 0 \leq \frac{x^2}{x^2 + y^2} \leq 1$$

$$\rightarrow 0 \cdot x \leq \frac{x^3}{x^2 + y^2} \leq x$$

$$\rightarrow \lim_{(x,y) \rightarrow (0,0)} 0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2+y^2} \leq \lim_{(x,y) \rightarrow (0,0)} x$$

$$\rightarrow 0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2+y^2} \leq 0 \\ \Rightarrow 0$$

② Same as ①

$$y^2 \leq x^2+y^2 \rightarrow 0 \leq \frac{y^2}{x^2+y^2} \leq 1$$

$$\rightarrow 0 \cdot \leq \frac{y^3}{x^2+y^2} \leq$$

$$\rightarrow \lim_{(x,y) \rightarrow (0,0)} y \cdot 0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{y^3}{x^2+y^2} \leq \lim_{(x,y) \rightarrow (0,0)} y$$

$$\rightarrow 0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{y^3}{x^2+y^2} \leq 0 \\ \Rightarrow 0$$

d) $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

We know $|\sin \theta| \leq 1$ and we saw in class $|\sin \theta| \leq \theta$

$$\Rightarrow |\frac{\sin(x^2+y^2)}{x^2+y^2}| \leq \frac{x^2+y^2}{x^2+y^2} = 1$$

Recall $\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$, what if we let $t = x^2+y^2$

$$\Rightarrow \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1 = \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

Ex③ - Find if the limit of f exists at $(0,0)$, if it does, find $f(0,0)$ for f to be continuous.

$$f(x,y) = \frac{x^3-x^2-2x^2y+xy^2-y^2-2y^3}{x^2+y^2}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2(x-1-2y)+y^2(x-1-2y)}{x^2+y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2+y^2)(x-1-2y)}{x^2+y^2}$$

$$= \lim_{(x,y) \rightarrow (0,0)} x-1-2y \\ = -1$$

$$\Rightarrow f(0,0) = -1$$

$f(\vec{x})$ is homogeneous of degree k if for every $\vec{x} \in \mathbb{R}^n$ and every scalar $c > 0$ we have:

$$f(c\vec{x}) = c^k f(\vec{x})$$

Ex ④ - Show if f is homogeneous

a) $f(x, y) = 8x^2y^2 - 9x^4$

$$\begin{aligned} f(cx, cy) &= 8(cx)^2(cy)^2 - 9(cx)^4 \\ &= 8c^2x^2c^2y^2 - 9c^4x^4 \\ &= 8c^4x^2y^2 - 9c^4x^4 \\ &= c^4(8x^2y^2 - 9x^4) \\ &= c^4 f(x, y) \end{aligned}$$

$\Rightarrow f$ is homogeneous of degree 4.

b) $f(x, y) = \frac{x^2}{y^2} + xy + \frac{y^2}{x^2}$

$$\begin{aligned} f(cx, cy) &= \frac{(cx)^2}{(cy)^2} + (cx)(cy) + \frac{(cy)^2}{(cx)^2} \\ &= \frac{c^2x^2}{c^2y^2} + c^2xy + \frac{c^2y^2}{c^2x^2} \\ &= \frac{x^2}{y^2} + c^2xy + \frac{y^2}{x^2} \quad \Rightarrow \text{cannot factor } c. \end{aligned}$$

$\Rightarrow f$ is not homogeneous.