

MATB41 - Tutorial 3 (BV355 Fridays 2-3pm)

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Week 1 - Sept. 6th

No tutorials

Week 2 - Sept. 13th

Derivative by definition:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Riemann Sum:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x, \text{ where } \Delta x = \frac{b-a}{n}, x_i = a + i \Delta x$$

$$\text{Useful formulas} \rightarrow \sum_{i=1}^n 1 = n, \sum_{i=1}^n i = \frac{i(i+1)}{2}, \sum_{i=1}^n i^2 = \frac{i(i+1)(2i+1)}{2 \cdot 3}$$

Trig. Identities:

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1 \\ \cos(2x) &= \cos^2 x - \sin^2 x \\ \sin(2x) &= 2\sin x \cos x\end{aligned}$$

Half Angle Identities:

$$\sin\left(\frac{\alpha}{2}\right) = \pm \sqrt{\frac{1-\cos\alpha}{2}} \quad \cos\left(\frac{\alpha}{2}\right) = \pm \sqrt{\frac{1+\cos\alpha}{2}}$$

Trig. Integrals:

$$\int \cos^n(x) \sin^m(x) dx \quad \begin{array}{l} \text{if } n \text{ or } m \text{ are odd} \rightarrow \text{do } u \text{ substitution} \\ \text{if } n \text{ and } m \text{ are even} \rightarrow \text{half angle substitution} \end{array}$$

Ex: Evaluate $\int \frac{\sin^3(\ln x) \cos^3(\ln x)}{x} dx$

$$= \int \sin^3(\ln x) \cos^2(\ln x) \frac{\cos(\ln x)}{x} dx$$

$$= \int \sin^3(\ln x) (1 - \sin^2(\ln x)) \frac{\cos(\ln x)}{x} dx$$

$$= \int u^3 (1-u^2) du$$

$$= \int u^3 - u^5 du$$

$$= \frac{u^4}{4} - \frac{u^6}{6}$$

$$\begin{array}{l} \text{Let } u = \sin(\ln x) \\ du = \frac{\cos(\ln x)}{x} dx \end{array}$$

$$= \frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} + C ,$$

Partial fractions:

Ex: Evaluate $\int \frac{1}{x(x^2-1)} dx$

$$\begin{aligned}\frac{1}{x(x^2-1)} &= \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{A(x^2-1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)} \\ &= \frac{Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx}{x(x-1)(x+1)} = \frac{x^2(A+B+C) + x(B-C) - A}{x(x-1)(x+1)}\end{aligned}$$

$$\left\{ \begin{array}{l} A+B+C=0 \\ B-C=0 \rightarrow B=C \\ -A=1 \rightarrow A=-1 \end{array} \right. \quad \left. \begin{array}{l} -1+2B=0 \\ B=\frac{1}{2}=C \end{array} \right.$$

$$\begin{aligned}\int \frac{1}{x(x^2-1)} dx &= \int \frac{-1}{x} + \frac{1}{2} \cdot \frac{1}{x-1} + \frac{1}{2} \cdot \frac{1}{x+1} dx \\ &= -\ln|x| + \frac{1}{2} \ln|x-1| + \frac{1}{2} \ln|x+1| + C,\end{aligned}$$

Angle between vectors:

$$\cos \theta = \frac{u \cdot w}{\|u\| \|w\|}$$

Cauchy-Schwarz inequality:

$$|u \cdot w| \leq \|u\| \|w\|, \quad u, w \in \mathbb{R}^n$$

Orthogonal:

$$u \cdot v = 0$$

Projection u onto w :

$$\frac{u \cdot w}{\|w\|^2} w$$

Ex: Let $v = (1, -3, 1)$ and $w = (2, 1, 2)$ be vectors in \mathbb{R}^3

$$v \cdot w = (1, -3, 1) \cdot (2, 1, 2) = 2 - 3 + 2 = 1$$

$$\|v\| = \sqrt{1^2 + (-3)^2 + 1^2} = \sqrt{11} \quad \|w\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

(a) Find the angle between v and w

$$\cos \theta = \frac{v \cdot w}{\|v\| \|w\|} = \frac{1}{\sqrt{11} \cdot 3}, \quad \theta = \cos^{-1}\left(\frac{1}{3\sqrt{11}}\right),$$

(b) Verify the Cauchy-Schwarz inequality and the triangle inequality for v and w

$$\begin{aligned}|v \cdot w| &\leq \|v\| \|w\| \\ \Rightarrow 1 &\leq 3\sqrt{11} \quad \checkmark\end{aligned}$$

(c) Find all unit vectors in \mathbb{R}^3 which are orthogonal to both v and w .

Let $u = (a, b, c)$ be orthogonal to both v and w .

$$(1, -3, 1)(a, b, c) = a - 3b + c = 0 \rightarrow a = -c$$

$$(2, 1, 2)(a, b, c) = \underline{2a + b + 2c = 0}$$

$$-5b = 0 \rightarrow b = 0$$

So $u = (k, 0, -k)$, $k \in \mathbb{R}$

$$\|u\| = \sqrt{k^2 + k^2} = \sqrt{2}|k| = 1 \rightarrow k = \frac{1}{\sqrt{2}}$$

The vectors are $(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$

(d) Find the projection of (i) v onto w and (ii) w onto v .

(i) Proj of v onto w : $\frac{v \cdot w}{\|w\|^2} w = \frac{1}{9} (2, 1, 2)$

(ii) Proj of w onto v : $\frac{w \cdot v}{\|v\|^2} v = \frac{1}{11} (1, -3, 1)$

Eigenvalues and eigenvectors:

Values of λ for $\det(A - \lambda I) = 0$ and their vectors respectively.

The matrix is diagonalizable if $P^{-1}AP = D$ is a diagonal matrix where P is the eigenvectors matrix. (or $A = PDP^{-1}$)

Ex: Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A .

Is A diagonalizable?

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -\lambda & 0 \\ 2 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} + 2 \begin{vmatrix} 0 & -\lambda \\ 2 & 0 \end{vmatrix}$$

$$\begin{aligned} &= -\lambda(1-\lambda)(4-\lambda) + 4\lambda \\ &= -\lambda[4 - 5\lambda + \lambda^2] - 4 \\ &= -\lambda(-5\lambda + \lambda^2) \\ &= -\lambda^2(\lambda - 5) = 0 \\ \Rightarrow \lambda_1 &= 0, \lambda_2 = 0, \lambda_3 = 5 \end{aligned}$$

For $\lambda_1 = 0$
 $\lambda_2 = 0$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_1 + 2x_3 = 0 \rightarrow x_1 = -2x_3 \text{ so for } \lambda_1 \text{ we can let } v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

x_2 is a free variable so for λ_2 we can let $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\text{For } \lambda_3=5 \quad \begin{bmatrix} -4 & 0 & 2 \\ 0 & -5 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -4x_1 + 2x_3 &= 0 \\ -5x_2 &= 0 \rightarrow x_2 = 0 \\ 2x_1 - x_3 &= 0 \rightarrow x_3 = 2x_1 \end{aligned} \quad \text{so for } \lambda_3 \text{ we can let } v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad \left[\begin{array}{ccc|ccc} -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 & 0 & 2 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{5} & 0 & \frac{2}{5} \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{5} & 0 & \frac{1}{5} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{5} & 0 & \frac{2}{5} \end{array} \right] = P^{-1}$$

$$\begin{aligned} D &= \begin{bmatrix} -\frac{2}{5} & 0 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \text{ which is a diagonal matrix of } \lambda_1, \lambda_2, \lambda_3. \end{aligned}$$

Week 3 - Sept. 20th

Vector equation of a line in \mathbb{R}^3 : $\vec{l} = a + t\vec{v}$ $t, a_i, v_i \in \mathbb{R}$
 $\vec{l} = (a_1, a_2, a_3) + t[v_1, v_2, v_3]$ $i=1, 2, 3$

Parametric equation of a line in \mathbb{R}^3 : $x = a_1 + tv_1$,
 $y = a_2 + tv_2$
 $z = a_3 + tv_3$

Ex: For each of the following lines, write its equation in vector and parametric form

(i) The line that passes through the point $p_0 (3, 1, 9)$ in the direction of $\vec{v} = (1, 1, 1)$

V.Eq. $\vec{l} = (3, 1, 9) + t(1, 1, 1)$, $t \in \mathbb{R}$

P.Eq. $x = 3+t$ $y = 1+t$ $z = 9+t$, $t \in \mathbb{R}$

(ii) The line that passes through points $p_0(-1, 1, 2)$ and $p_1(2, 0, -3)$

The direction vector for the line is $(2, 0, -3) - (-1, 1, 2) = (3, -1, -5)$

V.Eq. $\vec{l} = (-1, 1, 2) + t(3, -1, 5)$, $t \in \mathbb{R}$

P.Eq. $x = -1+3t$ $y = 1-t$ $z = 2+5t$, $t \in \mathbb{R}$

(iii) The line that passes through the point $p_0(0, 1, 0)$ and is orthogonal to the plane $10x + 15y + 3z = 11$

The direction vector for this line is a normal vector for the plane
 $\vec{n} = (10, 15, 3)$

V.Eq. $\vec{l} = (0, 1, 0) + t(10, 15, 3)$, $t \in \mathbb{R}$

P.Eq. $x = 10t$ $y = 1+15t$ $z = 3t$, $t \in \mathbb{R}$

Ex: Find an equation of the plane that passes through 3 points $A(-1, 1, 2)$, $B(2, 0, -3)$ and $C(2, -1, 2)$

A pair of direction vectors in the plane are $\vec{v} = (2, 0, -3) - (-1, 1, 2) = (3, -1, -5)$ and $\vec{w} = (2, -1, 2) - (-1, 1, 2) = (3, -2, 0)$

To find the normal of the plane $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & -5 \\ 3 & -2 & 0 \end{vmatrix} = (-10, -15, -3)$

So the plane is $-10x - 15y - 3z = d \rightarrow -10(-1) - 15(1) - 3(2) = -11$

\therefore The equation of the plane is $10x + 15y + 3z = 11$.

Ex: Find an equation of the plane that passes through the origin and contains the line $x = 2+3t$ $y = 1-2t$ $z = 1+t$

The line when $t=0$ passes through the point $(2, 1, 1)$

So the vector $\vec{v} = (0, 0, 0) - (2, 1, 1) = (-2, -1, -1)$ is also on the plane.

We can let $\vec{w} = (3, -2, 2)$

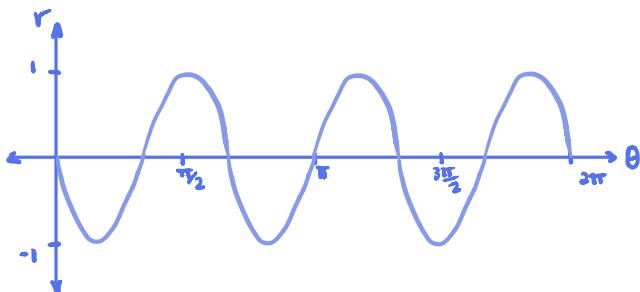
To find the normal of the plane $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & -1 \\ 3 & -2 & 2 \end{vmatrix} = (-4, 1, 7)$

So the plane is $-4x + y + 7z = d \rightarrow -4(0) + (0) + 7(0) = 0$

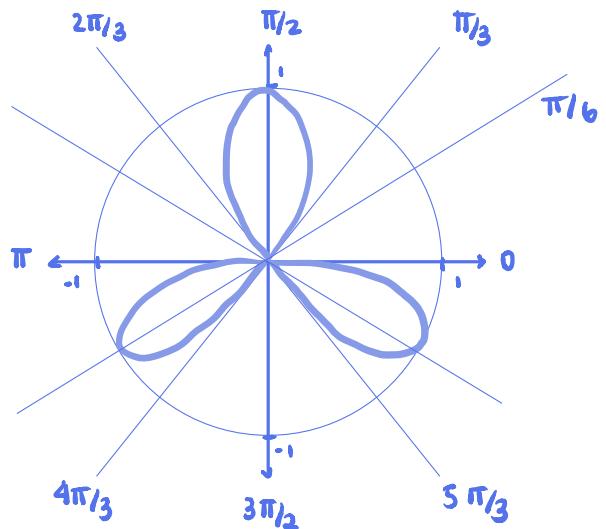
\therefore The equation of the plane is $-4x + y + 7z = 0$

Polar equations : $x = r\cos\theta$ $y = r\sin\theta$
 $x^2 + y^2 = r^2 \cos^2\theta + r^2 \sin^2\theta = r^2 (\cos^2\theta + \sin^2\theta) = r^2$

Ex: Sketch the curve $r = -\sin 3\theta$ in the polar plane

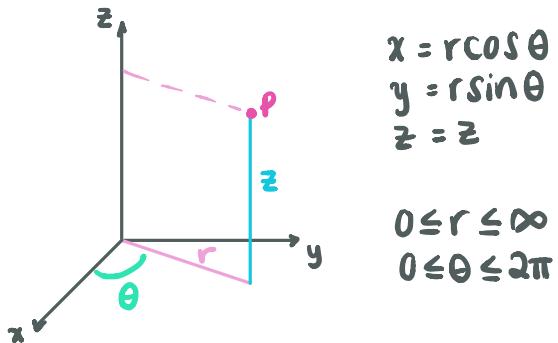


θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	-1	0	1	0	-1	0

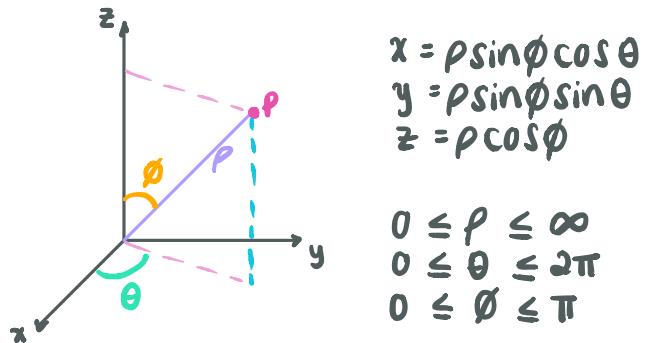


Week 4 - Sept. 27th

Cylindrical Coordinates :



Spherical Coordinates :



Ex: Interpret the equation $1 = 2\cos\theta\sin\theta$ geometrically

Cylindrical :

$$\begin{aligned} &\Rightarrow r^2 = 2r\cos\theta r\sin\theta \\ &\Rightarrow r^2 + y^2 = 2xy \\ &\Rightarrow x^2 - 2xy + y^2 = 0 \\ &\Rightarrow (x-y)^2 = 0 \end{aligned}$$

Spherical :

$$\begin{aligned} &\Rightarrow \rho^2 \sin^2\phi = 2\rho\sin\phi\cos\theta\rho\sin\phi\sin\theta \\ &\Rightarrow \rho^2 \sin^2\phi (\sin^2\theta + \cos^2\theta) = 2\rho\sin\phi\cos\theta\rho\sin\phi\sin\theta \\ &\Rightarrow \rho^2 \sin^2\phi \sin^2\theta + \rho^2 \sin^2\phi \cos^2\theta = 2\rho\sin\phi\cos\theta\rho\sin\phi\sin\theta \\ &\Rightarrow x^2 + y^2 = 2xy \\ &\Rightarrow x^2 - 2xy + y^2 = 0 \\ &\Rightarrow (x-y)^2 = 0 \end{aligned}$$

Ex: Characterize and sketch several level curves of the following functions:

$$(i) f(x,y) = \frac{\sqrt{y^2 - x^2}}{2} = c$$

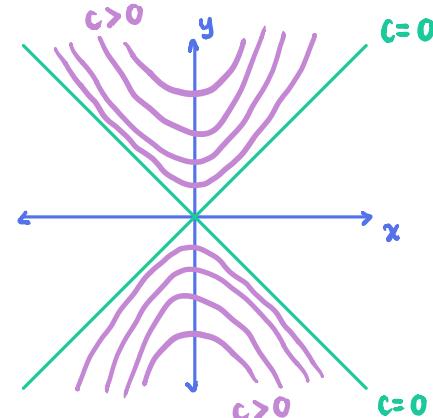
$\text{dom } f = \{(x,y) \in \mathbb{R}^2 \mid y^2 \geq x^2\}$, because $y^2 - x^2 \geq 0$

c cannot be -ve, $c > 0$ always

$$\underline{c=0} \quad y^2 = x^2 \\ y = \pm x \quad (*)$$

$$\underline{c>0} \quad y^2 - x^2 = (2c)^2 \\ y^2 = (2c)^2 + x^2 \quad (*)$$

y -intercepts: $(0, \pm 2c)$



$$(ii) f(x,y) = \frac{x+y}{y^2} = c$$

$\text{dom } f = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$

$$\underline{c=0} \quad x+y=0 \\ y=-x \quad (*)$$

$$\underline{c \neq 0} \quad x = cy^2 - y$$

$$y\text{-intercepts: } y = cy^2 \\ \Rightarrow y = \frac{1}{c}$$

$$\underline{c>0} \quad x = cy^2 - y \\ x = y(cy - 1)$$

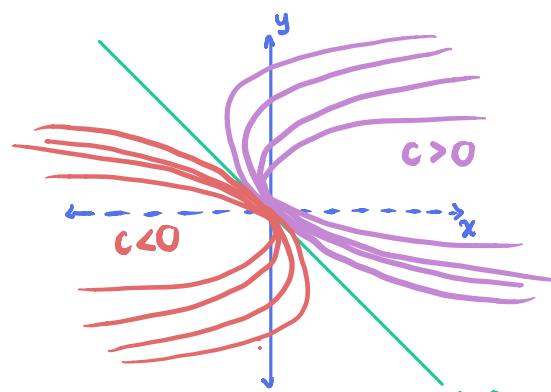
$$\underline{c<0} \quad x = -cy^2 - y \\ x = y(-cy - 1)$$

y -intercepts:
 $(0,0), (0, 1/c)$

y -intercepts:
 $(0,0), (0, -1/c)$

parabola opening
right (*)

parabola opening
left (*)



Ex: Give a rough sketch of the surface in \mathbb{R}^3 defined by
 $3x^2 - 12x - y + 2z^2 + 4z + 9 = 0$

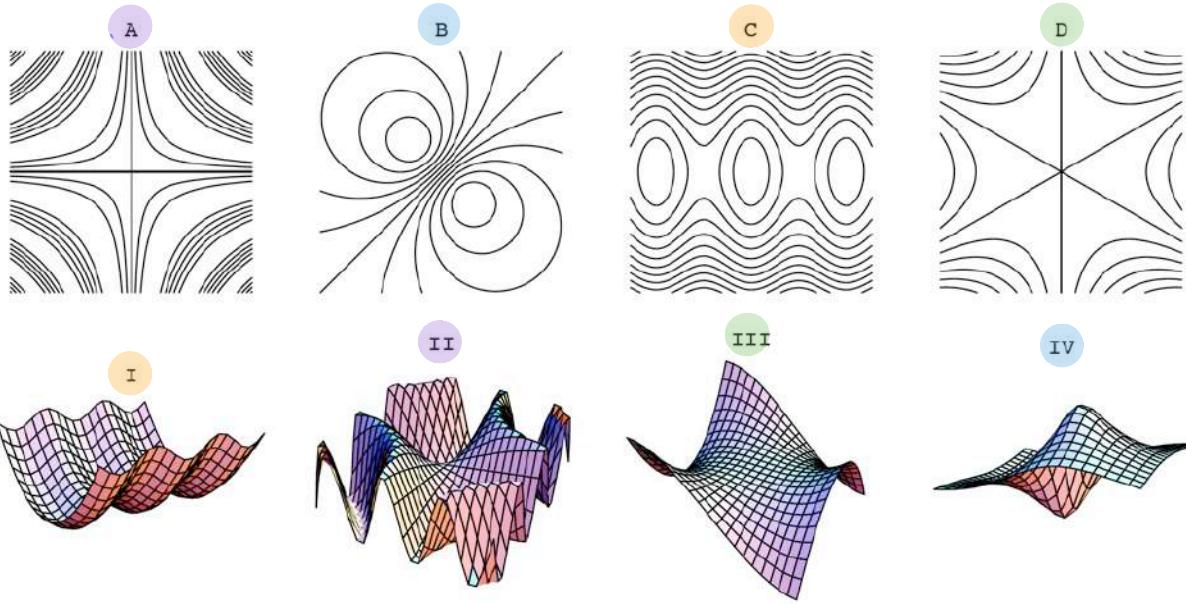
$$\Rightarrow y = 3x^2 - 12x + 2z^2 + 4z + 9$$

$$\Rightarrow y = 3(x^2 - 4x + 4) - 12 + 2(z^2 + 2z + 1) - 2 + 9$$

$$\Rightarrow y = 3(x-2)^2 + 2(z+1)^2 - 5$$

The elliptical paraboloid opening in the positive direction with vertex $(2, -5, -1)$

Ex: Indicate what contour diagram corresponds to each graph.



Week 5 - Oct. 4th

Ex: For each of the following, evaluate the limit or show that the limit does not exist.

$$(a) \lim_{(x,y) \rightarrow (0,0)} \frac{e^{xy}}{3x+5y+2} = \frac{e^{(0)}}{3(0)+5(0)+2} = \frac{1}{2}$$

$$\begin{aligned} (b) \lim_{(x,y) \rightarrow (1,2)} \frac{xy+2x-y-2}{(x^2-1)(y+2)} &= \lim_{(x,y) \rightarrow (1,2)} \frac{x(y+2)-(y+2)}{(x^2-1)(y+2)} \\ &= \lim_{(x,y) \rightarrow (1,2)} \frac{(y+2)(x-1)}{(x-1)(x+1)(y+2)} \\ &= \lim_{(x,y) \rightarrow (1,2)} \frac{1}{x+1} = \frac{1}{2} \end{aligned}$$

$$(c) \lim_{(x,y) \rightarrow (0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^3 \theta - r^3 \sin^3 \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta}$$

$$\begin{aligned} \text{Recall: } x &= r \cos \theta & = \lim_{r \rightarrow 0} \frac{r^3 (\cos^3 \theta - \sin^3 \theta)}{r^2 (\cos^2 \theta + \sin^2 \theta)} \\ y &= r \sin \theta & = \lim_{r \rightarrow 0} r (\cos^3 \theta - \sin^3 \theta) = 0 \end{aligned}$$

$$(d) \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r \rightarrow 0} \frac{\sin(r^2)}{r^2}$$

$$\stackrel{\text{L'H}}{=} \lim_{r \rightarrow 0} \frac{\arcsin(r^2)}{2r}$$

$$= \lim_{r \rightarrow 0} \frac{\cos(r^2)}{2} = \cos(0) = 1$$

$$(e) \lim_{(x,y) \rightarrow (0,0)} \frac{|x|}{\sqrt{x^2+y^2}}$$

Restrict y-axis: $\lim_{(y=0)} \lim_{(x,0) \rightarrow (0,0)} \frac{|x|}{\sqrt{x^2}} = 1$

Restrict x-axis: $\lim_{(x=0)} \lim_{(0,y) \rightarrow (0,0)} \frac{0}{\sqrt{y^2}} = 0$

The limit does not exist.

$$(f) \lim_{(x,y) \rightarrow (1,1)} \frac{x^2+y^2-2}{|x-1|+|y-1|} = \lim_{(u,v) \rightarrow (0,0)} \frac{(u+1)^2+(v+1)^2-2}{|u|+|v|}$$

We can rewrite this limit letting $u=x-1$ and $v=y-1$ $= \lim_{(u,v) \rightarrow (0,0)} \frac{u^2+2u+v^2+2v}{|u|+|v|}$

Restrict $v=u$ and $|u|=u$: $\lim_{u \rightarrow 0} \frac{u^2+2u+u^2+2u}{u+u}$

$$= \lim_{u \rightarrow 0} \frac{2u^2+4u}{2u}$$

$$= \lim_{u \rightarrow 0} \frac{2u(u+2)}{2u}$$

$$= \lim_{u \rightarrow 0} u+2 = 2$$

Restrict $v=-u$ and $|u|=-u$: $\lim_{u \rightarrow 0} \frac{u^2+2u+u^2-2u}{-u-u}$

$$= \lim_{u \rightarrow 0} \frac{2u^2}{-2u}$$

$$= \lim_{u \rightarrow 0} u = 0$$

The limit does not exist.

Ex: Find the value of a so that f is continuous at $(0,0)$

$$f(x,y) = \begin{cases} \frac{x^3 - x^2 - 2x^2y + xy^2 - y^2 - 2y^3}{x^2 + y^2}, & (x,y) \neq 0 \\ a, & (x,y) = 0 \end{cases}$$

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{x^2(x-1-2y) + y^2(x-1-2y)}{x^2+y^2} &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2+y^2)(x-1-2y)}{x^2+y^2} \\ &= \lim_{(x,y) \rightarrow (0,0)} x-1-2y = -1 = a \end{aligned}$$

Ex: Determine whether the following functions are continuous throughout their domains.

$$(a) f(x,y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

$$\text{Restrict } y\text{-axis: } \lim_{(y=0)} \lim_{(x,0) \rightarrow (0,0)} \frac{x^2}{x^2} = 1 \quad \leftarrow \neq$$

$$\text{Restrict } x\text{-axis: } \lim_{(x=0)} \lim_{(0,y) \rightarrow (0,0)} \frac{-y^2}{y^2} = -1 \quad \leftarrow$$

The limit does not exist, so it is not continuous at $(0,0)$

$$(b) f(x,y) = \begin{cases} \frac{2x^3 + 2xy^2 + 3x^2 + 3y^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^3 + 2xy^2 + 3x^2 + 3y^2}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{2x(x^2 + y^2) + 3(x^2 + y^2)}{x^2 + y^2}$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 + y^2)(2x + 3)}{x^2 + y^2}$$

$$= \lim_{(x,y) \rightarrow (0,0)} 2x + 3 = 3 \neq 0$$

The function is not continuous at $(0,0)$

Definition of limit :

Let $f: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function of two variables x and y defined for all ordered pairs (x, y) in some open disk $D \subseteq \mathbb{R}^2$ centered on a fixed ordered pair (x_0, y_0) , except possibly at (x_0, y_0) .

We will say that the number $L \in \mathbb{R}$ is the limit of $f(x, y)$ as $(x, y) \in D$ approaches (x_0, y_0) if and only if given any real number $\epsilon > 0$, we can find a real number $\delta > 0$ (depending on ϵ) such that $f(x, y)$ satisfies $|f(x, y) - L| < \epsilon$ whenever the distance between (x, y) and (x_0, y_0) satisfies $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ and we will write:

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = L \quad \text{or} \quad \lim_{(x,y) \rightarrow (0,0)} |f(x, y) - L| = 0$$

Ex: Use the definition of limits to show that $\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2+y^2} = 0$

We need to show that for any $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < \sqrt{x^2+y^2} < \delta$, we have $\left| \frac{4xy^2}{x^2+y^2} \right| = 0$

So if $(x, y) \in D$ and $0 < \sqrt{x^2+y^2} < \delta$, we see that you should choose the corresponding positive real number $\delta = \epsilon/4$ and we get

$$\begin{aligned} |f(x, y) - 0| &= \left| \frac{4xy^2}{x^2+y^2} - 0 \right| = 4|x| \cdot \frac{y^2}{x^2+y^2} \leq 4|x| \cdot 1 = 4\sqrt{x^2} \leq 4\sqrt{x^2+y^2} < 4\delta \\ &\stackrel{\text{b/c}}{\Rightarrow} y^2 \leq x^2 + y^2 \\ &\frac{y^2}{x^2+y^2} \leq 1 \end{aligned}$$

$$= 4 \left(\frac{\epsilon}{4} \right) = \epsilon$$

The number 0 is the limit of the function $f(x, y) = \frac{4xy^2}{x^2+y^2}$ as (x, y) in D approaches $(0,0)$ because for any given number $\epsilon > 0$, we have shown that we can

produce a corresponding number $\delta = \epsilon/4 > 0$ so that $f(x, y) = \frac{4xy^2}{x^2+y^2}$ satisfies

the inequality $\left| \frac{4xy^2}{x^2+y^2} - 0 \right| < \epsilon$ whenever the distance between (x, y) and $(0,0)$

satisfies $0 < \sqrt{x^2+y^2} < \epsilon/4 = \delta$. So we can write $\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2+y^2} = 0$

□

Week 6 - Oct. 11th

Ex: Let $f(x, y) = \begin{cases} \frac{x^3y - y^3x}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(a) Find $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ for $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{(3x^2y - y^3)(x^2 + y^2) - 2x(x^3y - y^3x)}{(x^2 + y^2)^2} = \frac{3x^4y + 3x^2y^3 - x^2y^3 - y^5 - 2x^4y + 2x^3y^3}{(x^2 + y^2)^2}$$

$$= \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2} = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{(x^3 - 3y^2x)(x^2 + y^2) - 2y(x^3y - y^3x)}{(x^2 + y^2)^2} = \frac{x^5 + x^3y^2 - 3y^2x^3 - 3y^4x - 2x^3y^2 + 2y^4x}{(x^2 + y^2)^2}$$

$$= \frac{x^5 - 4x^3y^2 - y^4x}{(x^2 + y^2)^2} = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

(b) Find $\frac{\partial f}{\partial x}(0,y)$ and $\frac{\partial f}{\partial y}(x,0)$

Using (a) we have

$$\frac{\partial f}{\partial x}(0,y) = \frac{y(-y^4)}{(y^2)^2} = -y \quad , \quad y \neq 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(x,0) = \frac{x(x^4)}{(x^2)^2} = x \quad , \quad x \neq 0$$

For $y=x=0$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{0}{h^3} - 0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \rightarrow 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{0}{h^3} - 0}{h} = 0$$

Directional derivative: $D_v f(p) = \nabla f(p) \cdot \frac{v}{\|v\|}$

Ex: Compute the directional derivative of $f(x,y,z) = xz + y^2z^2$ at the point $(3,-1,2)$ in the direction of the vector $v=(0,-3,4)$

$$\nabla f(x,y,z) = (z, 2yz^2, x+2y^2z)$$

$$\nabla f(3,-1,2) = (2, -8, 7)$$

$$D_{(0,-3,4)} f(3,-1,2) = (2, -8, 7) \cdot \frac{(0, -3, 4)}{\|(0, -3, 4)\|} = \frac{8 \cdot 3 + 7 \cdot 4}{\sqrt{9+16}} = \frac{52}{5}$$

Ex: For each of the following evaluate $\frac{\partial f}{\partial y}$ at the point a .

$$(a) f(x,y) = y \sin(xy) + x e^{-y^2}, \quad a = \left(\frac{\pi}{6}, 2\right)$$

$$\frac{\partial f}{\partial y} = \sin(xy) + xy \cos(xy) - 2xy e^{-y^2}$$

$$\left. \frac{\partial f}{\partial y} \right|_{(\frac{\pi}{6}, 2)} = \sin \frac{\pi}{3} + \frac{\pi}{3} \cos \frac{\pi}{3} - \frac{2\pi}{3} e^{-4} = \frac{\sqrt{3}}{2} + \frac{\pi}{6} - \frac{2\pi}{3e^4}$$

$$(b) f(x, y, z) = 2^{\sqrt{x-y^2}}, a = (3, 2, 1)$$

$$\frac{\partial f}{\partial y} = 2^{\sqrt{x-y^2}} (\ln 2) (\sqrt{x-y^2})' = -\frac{z(\ln 2) 2^{\sqrt{x-y^2}}}{2\sqrt{x-y^2}}$$

$$\left. \frac{\partial f}{\partial y} \right|_{(3, 2, 1)} = \frac{(-\ln 2) 2^{\sqrt{3-4}}}{2\sqrt{3-4}} = -\ln 2$$

$$(c) f(x, y) = \begin{cases} \frac{2x^2y + 3y^3}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}, a = (0, 0)$$

$$\left. \frac{\partial f}{\partial y} \right|_{(0,0)} = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{3h^3}{h^2} - 0}{h} = \lim_{h \rightarrow 0} 3 = 3$$

Ex: Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by $f(x, y) = (xy^2, x+2y, xy)$ and let $g: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by $g(x, y, z) = (xy, yz, xz^2)$.

(a) Find Df and Dg . Use the chain rule to find $D(g \circ f)$

$$D(g \circ f)(x, y) = Dg(f(x, y)) Df(x, y)$$

$$Df = \begin{pmatrix} y^2 & 2xy \\ 1 & 2 \\ y & x \end{pmatrix} \quad Dg = \begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z^2 & 0 & 2xz \end{pmatrix}$$

$$Dg(f(x, y)) = Dg(xy^2, x+2y, xy) = \begin{pmatrix} x+2y & xy^2 & 0 \\ 0 & xy & x+2y \\ x^2y^2 & 0 & 2x^2y^3 \end{pmatrix}$$

$$D(g \circ f)(x, y) = \begin{pmatrix} x+2y & xy^2 & 0 \\ 0 & xy & x+2y \\ x^2y^2 & 0 & 2x^2y^3 \end{pmatrix} \begin{pmatrix} y^2 & 2xy \\ 1 & 2 \\ y & x \end{pmatrix}$$

$$= \begin{pmatrix} y^3x + 2y^3 + xy^2 & 2x^2y + 4xy^2 + 2xy^2 \\ xy + xy + 2y^2 & 2xy + x^2 + 2xy \\ x^3y^4 + 2x^2y^4 & 2x^3y^3 + 2x^3y^3 \end{pmatrix}$$

$$= \begin{pmatrix} 2y^3 + 2xy^2 & 2x^2y + 6xy^2 \\ 2xy + 2y^2 & 4xy + x^2 \\ 3x^2y^4 & 4x^3y^3 \end{pmatrix}$$

(b) Compute $g \circ f$ and $D(g \circ f)$ directly

$$g \circ f(x, y) = g(xy^2, x+2y, xy) = (x^2y^2 + 2xy^3, x^2y + 2xy^2, x^3y^4)$$

$$D(g \circ f)(x, y) = \begin{pmatrix} 2y^3 + 2xy^2 & 2x^2y + 6xy^2 \\ 2xy + 2y^2 & 4xy + x^2 \\ 3x^2y^4 & 4x^3y^3 \end{pmatrix}$$

Ex: If $g(u, v) = f(u^2 - v^2, v^2 - u^2)$ and f is differentiable, show that g satisfies $v \frac{\partial g}{\partial u} + u \frac{\partial g}{\partial v} = 0$

Let $x = u^2 - v^2$ and $y = v^2 - u^2$, then $g(u, v) = f(x, y)$

$$\text{Then } \frac{\partial g}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial f}{\partial x}(2u) + \frac{\partial f}{\partial y}(-2u)$$

$$\text{and } \frac{\partial g}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{\partial f}{\partial x}(-2v) + \frac{\partial f}{\partial y}(2v)$$

$$\begin{aligned} \Rightarrow v \frac{\partial g}{\partial u} + u \frac{\partial g}{\partial v} &= v \left[\frac{\partial f}{\partial x}(2u) + \frac{\partial f}{\partial y}(-2u) \right] + u \left[\frac{\partial f}{\partial x}(-2v) + \frac{\partial f}{\partial y}(2v) \right] \\ &= 2uv \frac{\partial f}{\partial x} - 2uv \frac{\partial f}{\partial y} - 2uv \frac{\partial f}{\partial x} + 2uv \frac{\partial f}{\partial y} \\ &= 0 \end{aligned}$$

Week 7 - Oct. 18th

No tutorials

Week 8 - Oct. 25th

The rate of change in f from p_1 to p_2 is the directional derivative $D_v f(p_1)$ where $v = p_2 - p_1$

Ex: Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be given by $f(x, y, z) = 2x^2 + 2xz + y^2 + 4y + yz$. What is the rate of change in f if you move from $(1, 0, 1)$ towards $(1, 2, 3)$?

$$v = (1, 2, 3) - (1, 0, 1) = (0, 2, 2)$$

$$\nabla f(x, y, z) = (4x + 2z, 2y + 4 + z, 2x + y)$$

$$\begin{aligned} D_{(0, 2, 2)} f(1, 0, 1) &= \nabla f(1, 0, 1) \cdot \frac{(0, 2, 2)}{\|(0, 2, 2)\|} = \frac{(6, 5, 2) \cdot (0, 2, 2)}{\sqrt{4+4}} = \frac{14}{\sqrt{8}} \\ &= \frac{7}{\sqrt{2}} = \frac{7\sqrt{2}}{2} \end{aligned}$$

Ex: Find an equation for the tangent plane to the surface given by
 $x^3z + x^2y^2 + \sin(yz) = -3$ at the point $(-1, 0, 3)$

$$g(x, y, z) = x^3z + x^2y^2 + \sin(yz) + 3 = 0$$

$$\nabla g(x, y, z) = (3x^2z + 2xy^2, 2x^2y + z\cos(yz), x^3 + y\cos(yz))$$

$\nabla g(-1, 0, 3) = (9, 3, -1)$ is the normal vector to the surface at $(-1, 0, 3)$

The equation of the tangent plane is given by

$$\begin{aligned} \nabla g(-1, 0, 3)((x, y, z) - (-1, 0, 3)) &= 0 \quad \text{or} \quad (-1)(x - (-1)) + 3(y - 0) + (-1)(z - 3) = 0 \\ \Rightarrow (9, 3, -1)(x+1, y, z-3) &= 0 \\ \Rightarrow 9x+9+3y-z+3 &= 0 \\ \Rightarrow 9x+3y-z &= -12 \end{aligned}$$

Ex: Find the point(s) on the graph of the function $f(x, y) = x^2 + y^2 - 1$ where the tangent plane is parallel to the plane $4x - 8y - z = 3$. What is the equation of the tangent plane at this point?

Normal vector of the plane $(4, -8, -1)$ and a normal vector to the tangent plane to $f(x, y)$ is $(2x, 2y, -1)$.

For the planes to be parallel we need $(2x, 2y, -1) = k(4, -8, -1)$ for some k

$$\begin{array}{ll} 2x = 4k & \longrightarrow x = 2 \\ 2y = -8k & \longrightarrow y = -4 \\ -1 = -k & \rightarrow k = 1 \end{array} \quad \begin{array}{l} z = f(x, y) = f(2, -4) = 4 + 16 - 1 = 19 \\ 4(2) - 8(-4) - 19 = 8 + 32 - 19 = 21 \end{array}$$

\therefore The tangent plane passes through $(2, -4, 19)$ and the equation of the tangent plane at this point is $4x - 8y - z = 21$.

Ex: Find the second partial derivatives of

$$(a) f(x, y) = 2x^2y^3 + y\sin^2(xy)$$

First order:

$$\frac{\partial f}{\partial x} = 4xy^3 + 2y^2\sin(xy)\cos(xy)$$

$$\frac{\partial f}{\partial y} = 6x^2y^2 + \sin^2(xy) + 2xysin(xy)\cos(xy)$$

Second order:

$$\frac{\partial^2 f}{\partial x^2} = 4y^3 + 2y^2\cos^2(xy) - 2y^3\sin^2(xy)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 12xy^2 + 4y \sin(xy) \cos(xy) + 2xy^2 \cos^2(xy) - 2xy^2 \sin^2(xy)$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= 12x^2y + 2x \sin(xy) \cos(xy) + 2x \sin(xy) \cos(xy) + 2x^2y \cos^2(xy) - 2x^2y \sin^2(xy) \\ &= 12x^2y + 4x \sin(xy) \cos(xy) + 2x^2y \cos^2(xy) - 2x^2y \sin^2(xy)\end{aligned}$$

$$(b) f(x, y, z) = 2^{xz} - \frac{x^3z}{y^2}$$

First order:

$$\frac{\partial f}{\partial x} = z 2^{xz} \ln 2 - 3 \frac{x^2 z}{y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2x^3 z}{y^3}$$

$$\frac{\partial f}{\partial z} = x 2^{xz} \ln 2 - \frac{x^3}{y^2}$$

Second order:

$$\frac{\partial^2 f}{\partial x^2} = z^2 2^{xz} \ln^2 2 - 6 \frac{x^2 z}{y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{6x^2 z}{y^3}$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = 2^{xz} \ln 2 + xz 2^{xz} \ln^2 2 - \frac{3x^2}{y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = - \frac{6x^3 z}{y^4}$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z} = \frac{2x^3}{y^3}$$

$$\frac{\partial^2 f}{\partial z^2} = x^2 2^{xz} \ln^2 2$$

Week 9 - Nov. 1st.

Taylor Series

* (x, y) around (0, 0):

$$F(x,y) = F(0,0) + x \frac{\partial F}{\partial x}(0,0) + y \frac{\partial F}{\partial y}(0,0) \\ + \frac{1}{2!} \left[x^2 \frac{\partial^2 F}{\partial x^2}(0,0) + 2xy \frac{\partial^2 F}{\partial x \partial y}(0,0) + y^2 \frac{\partial^2 F}{\partial y^2}(0,0) \right] + \dots$$

* else:

$$F(x,y) = F(a,b) + \frac{\partial F}{\partial x}(x-a) + \frac{\partial F}{\partial y}(y-b) \\ + \frac{1}{2!} \left[\frac{\partial^2 F}{\partial x^2}(x-a)^2 + 2 \frac{\partial^2 F}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 F}{\partial y^2}(y-b)^2 \right] + \dots$$

Ex: Directly compute the second degree Taylor polynomial about the point (1,0) for $f(x,y) = e^{(x-1)^2} \cos y$

$$\rightarrow f(1,0) = e^0 \cos(0) = 1$$

$$\rightarrow \frac{\partial f}{\partial x}(x,y) = 2(x-1)e^{(x-1)^2} \cos y \quad \frac{\partial f}{\partial x}(1,0) = 2(0)e^0 \cos(0) = 0$$

$$\frac{\partial f}{\partial y}(x,y) = -e^{(x-1)^2} \sin y \quad \frac{\partial f}{\partial y}(1,0) = e^0 \sin(0) = 0$$

$$\rightarrow \frac{\partial^2 f}{\partial x^2}(x,y) = (2e^{(x-1)^2} + 4(x-1)^2 e^{(x-1)^2}) \cos y \quad \frac{\partial^2 f}{\partial x^2}(1,0) = (2e^0 + 4(0)e^0) \cos(0) = 2$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = -2(x-1)e^{(x-1)^2} \sin y \quad \frac{\partial^2 f}{\partial x \partial y}(1,0) = -2(0)e^0 \sin(0) = 0$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = -e^{(x-1)^2} \cos y \quad \frac{\partial^2 f}{\partial y^2}(1,0) = -e^0 \cos(0) = -1$$

$$\therefore T_2 = f(1,0) + \left[\frac{\partial f}{\partial x}(1,0)(x-1) + \frac{\partial f}{\partial y}(1,0)y \right] + \frac{1}{2!} \left[\frac{\partial^2 f}{\partial x^2}(1,0)(x-1)^2 + 2 \frac{\partial^2 f}{\partial x \partial y}(1,0)(x-1)y + \frac{\partial^2 f}{\partial y^2}(1,0)y^2 \right]$$

$$= 1 + \frac{1}{2!} \left[2(x-1)^2 - y^2 \right]$$

$$= 1 + (x-1)^2 - \frac{y^2}{2}$$

Common series around $a=0$ are:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, |x| < \infty$$

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k, |x| < 1$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, |x| < \infty$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1}, |x| < 1$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, |x| < \infty$$

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, |x| < 1$$

Ex: Find the 4th degree Taylor polynomial about the origin of
 $f(x,y) = (\cos y) \ln(1+xy)$

Recall: $\cos t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!}$, $|t| < \infty$ and $\ln(1+t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{k+1}}{k+1}$, $|t| < 1$

$$\rightarrow \cos y = 1 - \frac{y^2}{2} + \frac{y^4}{4!} - \dots, |y| < \infty$$

$$\rightarrow \ln(1+xy) = xy - \frac{x^2y^2}{2} + \frac{x^3y^3}{3} - \dots, |xy| < 1$$

$$T = \left(1 - \frac{y^2}{2} + \frac{y^4}{4!} - \dots \right) \left(xy - \frac{x^2y^2}{2} + \frac{x^3y^3}{3} - \dots \right) = \cancel{xy} - \cancel{\frac{x^2y^2}{2}} + \cancel{\frac{x^3y^3}{3}} - \cancel{\frac{xy^3}{2}} + \frac{x^2y^4}{4} - \dots$$

$$\therefore T_4 = xy - \frac{x^2y^2}{2} - \frac{xy^3}{2}$$

Ex: Let π be the plane in \mathbb{R}^3 passing through the points $(1,0,4)$, $(2,-1,0)$ and $(3,1,2)$

(a) Find an equation for π .

$$v_1 = (2, -1, 0) - (1, 0, 4) = (1, -1, -4)$$

$$v_2 = (3, 1, 2) - (1, 0, 4) = (2, 1, -2)$$

$$\text{normal vector of the plane } v_1 \times v_2 = (2+4, 2-8, 1+2) = (6, -6, 3)$$

$$6x - 6y + 3z = d \rightarrow 6 + 12 = d = 18$$

$$\therefore 2x - 2y + z = 6$$

(b) Give a parametric description for the line l through $(1,1,1)$ and orthogonal to π

normal of π is $(2, -2, 1)$ so the line l is $(1, 1, 1) + t(2, -2, 1)$, $t \in \mathbb{R}$.

(c) Find those points on the ellipsoid $4x^2 + 8y^2 + 4z^2 = 7$ where the tangent plane is parallel to π

The level surface is $g(x, y, z) = 4x^2 + 8y^2 + 4z^2 - 7 \rightarrow \nabla g(x, y, z) = (8x, 16y, 8z)$

$\Rightarrow (8x, 16y, 8z) = k(2, -2, 1)$, $k \in \mathbb{R}$ so the tangent plane is parallel

$$x = \frac{k}{4}, y = -\frac{k}{8}, z = \frac{k}{8} \rightarrow 4\left(\frac{k}{4}\right)^2 + 8\left(\frac{k}{8}\right)^2 + 4\left(\frac{k}{8}\right)^2 = 7$$

$$\rightarrow \frac{k^2}{4} + \frac{k^2}{8} + \frac{k^2}{16} = 7$$

$$\rightarrow 4k^2 + 2k^2 + k^2 = 112$$

$$\begin{array}{l} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \quad \begin{array}{l} 7k^2 = 112 \\ k^2 = 16 \\ k = \pm 4 \end{array}$$

\therefore The points are $(\pm \frac{4}{4}, \pm \frac{4}{8}, \pm \frac{4}{8})$:

$(1, -\frac{1}{2}, \frac{1}{2})$ and $(-1, \frac{1}{2}, -\frac{1}{2})$

Ex: Characterize and sketch several level curves of the function $f(x,y) = \sqrt{4x^2 + y^2}$

domf: \mathbb{R}^2

$$\begin{aligned} 4x^2 + y^2 &= c^2 \\ (2x)^2 + y^2 &= c^2 \end{aligned}$$

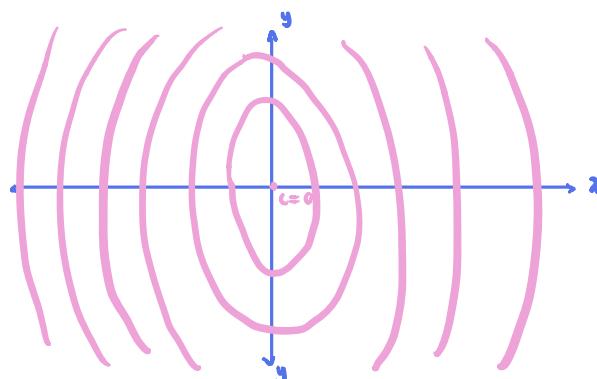
$c < 0$ \rightarrow never!

$c = 0$ $\rightarrow (x,y) = (0,0)$

$c > 0$ \rightarrow ellipses!

x-intercepts: $(\pm \frac{c}{2}, 0)$

y-intercepts: $(0, \pm c)$



Week 10 - Nov. 8th.

Find eigenvalues of A: Solve $\det(A - \lambda I) = 0$.

Find the eigenvector: For each eigenvalue, find the eigenvectors $(A - \lambda I)\vec{v} = 0$

Let B be the matrix whose rows are the eigenvectors. Show that B is an orthogonal matrix: $BB^T = I_3$.

~ Hessian form (H.f.)

- * If H.f. is positive definite, then the Critical Point is a local minimum
- * If H.f. is negative definite, then the Critical Point is a local maximum
- * Otherwise, the Critical Point is a saddle point.

Let A be $n \times n$ and symmetric:

- * A is positive definite iff $\det(A_K) > 0$ for $K = 1, 2, \dots, n$.
- * A is negative definite iff $(-1)^K \det(A_K) > 0$ for $K = 1, 2, \dots, n$

Ex: Find and classify the critical points for the following functions

(a) $f(x,y) = -\frac{x^4}{4} + \frac{2x^3}{3} + 4xy - y^2$

$$\begin{cases} f_x = -x^3 + 2x^2 + 4y = 0 \quad ① \\ f_y = 4x - 2y = 0 \quad ② \quad y = 2x \end{cases}$$

$$\begin{aligned} \text{Sub } ② \text{ in } ① &\rightarrow -x^3 + 2x^2 + 8x = 0 \\ &-x(x^2 - 2x - 8) = 0 \\ &-x(x-4)(x+2) = 0 \\ &\rightarrow x = 0, 4, -2 \quad \rightarrow y = 0, 8, -4 \end{aligned}$$

∴ C.Ps are $(0,0), (4,8), (-2,-4)$

$$Hf = \begin{pmatrix} -3x^2 + 4x & 4 \\ 4 & -2 \end{pmatrix}$$

* For $(0,0)$:

$$Hf(0,0) = \begin{pmatrix} 0 & 4 \\ 4 & -2 \end{pmatrix}$$

$$A_1 = 0 \quad \det A_2 = \begin{vmatrix} 0 & 4 \\ 4 & -2 \end{vmatrix} = -16 < 0$$

∴ $(0,0)$ is a saddle point

* For $(4,8)$:

$$Hf(4,8) = \begin{pmatrix} -32 & 4 \\ 4 & -2 \end{pmatrix}$$

$$A_1 = -32 < 0 \quad \det A_2 = \begin{vmatrix} -32 & 4 \\ 4 & -2 \end{vmatrix} = 64 - 16 = 48 > 0$$

∴ $(4,8)$ is a local maximum

* For $(-2,-4)$:

$$Hf(-2,-4) = \begin{pmatrix} -20 & 4 \\ 4 & -2 \end{pmatrix}$$

$$A_1 = -20 < 0 \quad \det A_2 = \begin{vmatrix} -20 & 4 \\ 4 & -2 \end{vmatrix} = 40 - 16 = 24 > 0$$

∴ $(-2,-4)$ is a local maximum

$$(b) f(x,y) = f(x,y) = x+y + \frac{1}{x} + \frac{4}{y}, \quad x,y \neq 0$$

$$\begin{cases} f_x = 1 - \frac{1}{x^2} = 0 & \rightarrow x^2 = 1 \\ f_y = 1 - \frac{4}{y^2} = 0 & \rightarrow y^2 = 4 \end{cases} \quad \begin{cases} x = \pm 1 \\ y = \pm 2 \end{cases}$$

\therefore CPs are $(1,2), (1,-2), (-1,2), (-1,-2)$

$$Hf = \begin{pmatrix} \frac{2}{x^3} & 0 \\ 0 & \frac{8}{y^3} \end{pmatrix}$$

* For $(1,2)$:

$$Hf(1,2) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A_1 = \underline{2 > 0} \quad \det A_2 = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = \underline{2 > 0}$$

$\therefore (1,2)$ is a local minimum

* For $(1,-2)$:

$$Hf(1,-2) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A_1 = \underline{2 > 0} \quad \det A_2 = \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} = \underline{-2 < 0}$$

$\therefore (1,-2)$ is a saddle point

* For $(-1,2)$:

$$Hf(-1,2) = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A_1 = \underline{-2 < 0} \quad \det A_2 = \begin{vmatrix} -2 & 0 \\ 0 & 1 \end{vmatrix} = \underline{-2 < 0}$$

$\therefore (-1,2)$ is a saddle point

$$\star \text{ For } (-1, -2) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A_1 = \underline{-2 < 0} \quad \det A_2 = \begin{vmatrix} -2 & 0 \\ 0 & -1 \end{vmatrix} = \underline{2 > 0}$$

$\therefore (-1, -2)$ is a local maximum

Week 11 - November 15th.

Ex: Use Lagrange multipliers to find the constrained critical points of f subject to the given constraints.

$$(a) f(x, y) = 5x + 2y, \quad 5x^2 + 2y^2 = 14$$

$$h(x, y, \lambda) = 5x + 2y - \lambda(5x^2 + 2y^2 - 14)$$

$$\begin{cases} h_x = 5 - 10\lambda x \\ h_y = 2 - 4\lambda y \\ h_\lambda = -5x^2 - 2y^2 + 14 \end{cases} \quad \begin{matrix} ① \\ ② \\ ③ \end{matrix}$$

$$① 5 = 10\lambda x$$

$$\lambda = \frac{x}{2}$$

$$② 2 = 4\lambda y$$

$$\lambda = \frac{y}{2}$$

$$x=y$$

$$③ -5x^2 - 2y^2 + 14 = 0$$

$$\Rightarrow 7x^2 = 14$$

$$x^2 = 2$$

$$x = \pm\sqrt{2} = y$$

$$\therefore \text{CPs: } (\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$$

$$(b) f(x, y, z) = xy + xz + yz - xyz, \quad x+y+z=1, \quad x, y, z \geq 0$$

$$h(x, y, z, \lambda) = xy + xz + yz - xyz - \lambda(x+y+z-1)$$

$$\begin{cases} h_x = y+z-yz-\lambda \\ h_y = x+z-xz-\lambda \\ h_z = x+y-xy-\lambda \\ h_\lambda = -x-y-z+1 \end{cases} \quad \begin{matrix} ① \\ ② \\ ③ \\ ④ \end{matrix}$$

$$① y+z-yz=\lambda$$

$$① = ② \quad y+z-yz = x+z-xz \quad ④ = ③ \quad x+z-xz = x+y-xz$$

$$\Rightarrow y(1-z) = x(1-z)$$

$$\Rightarrow z(1-x) = y(1-x)$$

$$② x+z-xz=\lambda$$

$$\Rightarrow (1-z)(y-x)=0$$

$$\Rightarrow (1-x)(z-y)=0$$

$$③ x+y-xy=\lambda$$

$$\Rightarrow z=1 \text{ or } y=x$$

$$\Rightarrow x=1 \text{ or } y=z$$

$$④ x+y+z=1$$

If $z=1$, then either $x=1$ or $y=1$ but either way ④ fails
If $y=x$, then $y=z$ because if $x=1=y$ and ④ will fail again

$$\text{In } ①, ②, ③ \quad x=y=z$$

$$④ x+x+x=1 \quad x = \frac{1}{3} = y, z$$

$$\therefore \text{CP: } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Ex: The production function for a manufacturer is given by $f(x,y) = 100x^{3/4}y^{1/4}$, where x represents the units of labour (at \$150 per unit) and y represents the units of capital (at \$250 per unit). The total cost of labour and capital is limited to \$50,000. Find the maximum production level for this manufacturer.

Constraint is $150x + 250y = 50000$, to maximize $f(x,y)$ we can use Lagrange multipliers.

$$h(x,y,\lambda) = 100x^{3/4}y^{1/4} - \lambda(150x + 250y - 50000)$$

$$\begin{cases} h_x = 75x^{-1/4}y^{1/4} - 150\lambda & \textcircled{1} \\ h_y = 25x^{3/4}y^{-3/4} - 250\lambda & \textcircled{2} \\ h_\lambda = -150x - 250y + 50000 & \textcircled{3} \end{cases}$$

$$\textcircled{1} \quad 75x^{-1/4}y^{1/4} = 150\lambda$$

$$\lambda = \frac{x^{-1/4}y^{1/4}}{2} = \frac{y^{1/4}}{2x^{1/4}}$$

$$\frac{y^{1/4}}{2x^{1/4}} = \frac{x^{3/4}}{10y^{3/4}}$$

$$10y = 2x$$

$$x = 5y$$

$$\textcircled{2} \quad 25x^{3/4}y^{-3/4} = 250\lambda$$

$$\lambda = \frac{x^{3/4}y^{-3/4}}{10} = \frac{x^{3/4}}{10y^{3/4}}$$

$$\textcircled{3} \quad 150(5y) + 250y = 50000$$

$$75y + 25y = 5000$$

$$100y = 5000$$

$$y = 50 \rightarrow x = 250$$

Which means that the maximum production level is when $x=250$ and $y=50$

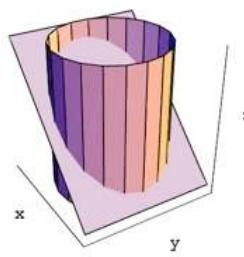
$f(250, 50) \approx 16718.5076$, so the maximum production units are 16718.

Ex: The cylinder $x^2 + y^2 = 1$ intersects the plane $x+z=1$ in an ellipse. Find the point on that ellipse that is the furthest from the origin.

We need to maximize the distance function $(x^2 + y^2 + z^2)$ subject to 2 constraints.

$$h(x,y,z,\lambda, \mu) = x^2 + y^2 + z^2 - \lambda(x^2 + y^2 - 1) - \mu(x + z - 1)$$

$$\begin{cases} h_x = 2x - 2x\lambda - \mu & \textcircled{1} \\ h_y = 2y - 2y\lambda & \textcircled{2} \\ h_z = 2z - \mu & \textcircled{3} \\ h_\lambda = 1 - x^2 - y^2 & \textcircled{4} \\ h_\mu = 1 - x - z & \textcircled{5} \end{cases}$$



$$\begin{aligned} \textcircled{2} \quad 2y &= 2y\lambda \\ \Rightarrow 2y(1-\lambda) &= 0 \\ y &= 0, \lambda = 1 \end{aligned}$$

* If $\lambda = 1$:

$$\begin{aligned} \textcircled{1} \quad u &= 2x(1-\lambda) = 0 \\ \textcircled{3} \quad 2z &= u = 0 \rightarrow z = 0 \\ \textcircled{5} \quad x &= 1 \\ \textcircled{4} \quad 1+y^2 &= 1 \rightarrow y = 0 \end{aligned}$$

* If $y = 0$:

$$\begin{aligned} \textcircled{4} \quad x^2 &= 1 \rightarrow x = \pm 1 \\ \textcircled{5} \quad -1+z &= 1 \rightarrow z = 2 \quad (x = -1) \\ 1+z &= 1 \rightarrow z = 0 \quad (x = 1) \quad (\text{We know}) \end{aligned}$$

\therefore CPs $(1, 0, 0)$ and $(-1, 0, 2)$

Based on the distance function:

$$(1, 0, 0) \rightarrow \sqrt{1^2} = 1$$

$$(-1, 0, 2) \rightarrow \sqrt{(-1)^2 + (0)^2 + (2)^2} = \sqrt{5}$$

\therefore The furthest point from the origin is $(-1, 0, 2)$.

Since the distance function is continuous and the intersection is compact in \mathbb{R}^3 , the Extreme Value Theorem (EVT) ensures the existence of extreme.